HEIGHT ESTIMATE AND LIPSCHITZ APPROXIMATION FOR GEODESICS IN CARNOT GROUPS

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ABSTRACT. In [11] it is proved that length-minimizing curves in Carnot groups have infinitesimal excess at any point, for a suitable sequence of scales. In this paper we prove some results dealing with the small excess regime. We prove a height-estimate for horizontal curves and an approximation of geodesics with Lipschitz graphs along the direction where excess is small. The setting is that of free Carnot groups.

1. INTRODUCTION

The most important open problem in sub-Riemannian geometry is the regularity of length minimizing curves, see [9, 13, 10]. The difficulty of the problem is due to the existence of singular extremal that can be length minimizing [7]. For these extremals the classical tools of geometric control theory do not provide any further regularity beyond the Lipschitz continuity. Recently, there was some progress on the problem based on techniques inspired by geometric measure theory, see [6, 8, 4, 11] and also [5, 2, 12].

In particular, in [11] it is proved that, in the setting of Carnot groups, for any point in the support of a length-minimizing curve there exists an infinitesimal sequence of scales such that the excess of the curve is infinitesimal. In fact, this implies that there is a line in the tangent cone of the curve at that point.

In this paper, we study length minimizing curves in the small excess regime. We first prove a height estimate and then an approximation of the curve by means of Lipschitz graphs. These results were announced in [10]. In the theory of minimal surfaces, the Lipschitz approximation of a minimal current is the first step in the regularity theory and paves the way to the so-called harmonic approximation and to the improved excess-decay lemma.

Given integers $m, s \geq 2$, we denote by $F_{m,s}$ the real Lie algebra generated by m elements that is nilpotent with step s. This Lie algebra can be realized as a Lie algebra of left-invariant vector fields in $\mathbb{R}^n$, where $n \geq 3$ is the dimension of $F_{m,s}$ as a vector space. We denote the m vector fields generating the Lie algebra by

$$X_1, \dots, X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

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The Campbell-Hausdorff-Beker formula gives $F_{m,s}$ the structure of a Lie group that we denote by $G_{m,s}$. The group operation is denoted by a dot $\cdot$. The underlying manifold of $G_{m,s}$ is again $\mathbb{R}^n$.

We call $V_1 = \mathrm{span}\{X_1,\dots,X_m\}$ the generating layer and we fix on V_1 the scalar product $\langle\cdot,\cdot\rangle$ that makes $X_1,\dots,X_m$ orthonormal. We denote by $|\cdot| = \langle\cdot,\cdot\rangle^{1/2}$ the corresponding norm. The Lie algebra $F_{m,s}$ has the grading

$$F_{m,s} = V_1 \oplus \ldots \oplus V_s,$$

where $V_{i+1} = [V_1, V_i]$ and $V_{s+1} = \{0\}$. To the stratum V_i we assign the weight i and for each $\lambda > 0$ the map defined by $\delta_\lambda(X) = \lambda^i X$ if and only if $X \in V_i$, linearly extends to an automorphism of $F_{m,s}$. We identify $G_{m,s}$ with $\mathbb{R}^n$ using exponential coordinates. We complete $X_1,\dots,X_m$ to a basis $X_1,\dots,X_n$ of $F_{m,s}$ ordered by the grading and we assume that

$$x = (x_1,\dots,x_n) = \exp\bigg(\sum_{i=1}^n x_i X_i\bigg).$$

We shall work with vector fields $X_1,\dots,X_n$ given by the Hall basis construction, see Section 2.

To the jth coordinate we assign the weight $w_j = i$ if and only if the element $e_j = (0,\dots,1,\dots 0)$, with 1 at the jth position, satisfies $e_j \in \exp(V_i)$. Then for any $\lambda > 0$ the dilations

$$\delta_\lambda(x) = (\lambda^{w_1}x_1,\dots,\lambda^{w_j}x_j,\dots,\lambda^{w_n}x_n)$$

are automorphisms of $G_{m,s}$.

A Lipschitz continuous curve $\gamma : [0,1] \to G_{m,s}$ is admissible if $\dot\gamma \in V_1(\gamma)$ a.e., that is if $\dot\gamma = \sum_{j=1}^m h_j X_j(\gamma)$ for uniquely determined functions $h_j \in L^\infty(0,1)$, $j = 1,\dots,m$. The length of γ is

$$L(\gamma) = \int_0^1 |\dot\gamma(t)|dt = \int_0^1 |h(t)|dt.$$

There is always a reparameterization of γ by arc-length, i.e., such that $|h(t)| = (h_1(t)^2 + \ldots + h_m(t)^2)^{1/2} = 1$ for a.e. $t \in [0, L(\gamma)]$.

The Carnot-Carathéodory distance d between two points $x, y \in G_{m,r}$ is the infimum (minimum) of $L(\gamma)$ among all admissible curves γ such that $\gamma(0) = x$ and $\gamma(1) = y$. This distance is left-invariant and homogeneous with respect to dilations:

i) $d(z\cdot x, z\cdot y) = d(x,y)$ for all $x,y,z \in G_{m,s}$;

ii) $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y)$ for all $x,y \in G_{m,s}$ and $\lambda > 0$.

An admissible curve $\gamma : [0,1] \to G_{m,s}$ is a geodesic (i.e., a length minimizing curve) if $d(\gamma(0),\gamma(1)) = L(\gamma)$. One of the major open problems in sub-Riemannian geometry is the regularity of length minimizing curves, even in the setting of Lie groups of Carnot type.

Our first result is the so-called height estimate. It states that an admissible curve is contained in a thin tube around a fixed direction, provided that the excess of the curve in this direction is small. Without loss of generality, we assume that this direction is the one given by the first vector field X_1.

Definition 1.1. The *parametric excess* of an admissible curve $\gamma : [0,1] \to G_{m,s}$ in direction X_1, at $\eta \in [0,1]$ and at a scale $r > 0$ such that $\eta + r \le 1$ is

$$E(\gamma;\eta;r;X_1) := \frac{1}{r}\int_{\eta}^{\eta+r} |\dot\gamma - X_1(\gamma)|^2 dt.$$

Theorem 1.2 (Height estimate)**.** Let $\gamma : [0,1] \to G_{m,s}$ be an admissible curve parameterized by arc-length with $\gamma(0) = 0$ and let $0 < r \le 1$. Then for all $i = 2,\dots,n$ there exist positive integers $\alpha_i, \beta_i \in \mathbb{N}$ such that:

i) $\alpha_i + \beta_i + 1 = w_i$, the weight of the ith coordinate;

ii) $\left(\dfrac{|\gamma_i(t)|}{|t|^{\alpha_i}}\right)^{\frac{1}{\beta_i+1}} \le t\sqrt{E(\gamma;0;r;X_1)}$ for all $0 < t \le r$.

Above, γ_i is the ith coordinate of γ in exponential coordinates.

Theorem 1.2 is proved in Section 3. Our second result is the approximation of length-minimizing curves by Lipschitz graphs along a fixed direction where the excess is small. This result is better formulated in terms of a geometric notion of excess.

Let $\gamma : [-1,1] \to G_{m,s}$ be an admissible injective curve and let $\Gamma = \gamma([-1,1])$ be its support. The curve γ can be assumed to be parameterized by arc-length and so the tangent $\dot\gamma(t) \in V_1(\gamma(t))$ exists for a.e. $t \in [-1,1]$ and satisfies $|\dot\gamma(t)| = 1$. We denote by $\mathscr{H}^1$ the 1-dimensional Hausdorff measure in $G_{m,s} = \mathbb{R}^n$ defined using the Carnot-Carathéodory metric d. Then for $\mathscr{H}^1$-a.e. $x \in \Gamma$ we can define the unit tangent vector $\tau_\Gamma(x) = \dot\gamma(t)$ where $t \in [-1,1]$ is such that $\gamma(t) = x$.

Definition 1.3. Let Γ be the support of an admissible curve γ, oriented by the unit tangent τ_Γ. The *geometric excess* of Γ in direction X_1, at $x \in \Gamma$ and at scale $r > 0$ is

$$E(\Gamma;x;r;X_1) = \fint_{\Gamma\cap B_r(x)} |\tau_\Gamma - X_1|^2 d\mathscr{H}^1,$$

where $B_r(x)$ is a ball in the Carnot-Carathéodory metric.

We denote by $\pi : G_{m,s} = \mathbb{R}^n \to \mathbb{R}$, $\pi(x) = \pi(x_1,\dots,x_n) = x_1$, the projection onto the first coordinate. We denote by $B_r = B_r(0)$ Carnot-Carathéodory balls centered at 0.

Theorem 1.4 (Lipschitz approximation)**.** Let $\gamma : [-1,1] \to G_{m,s}$ be a geodesic parameterized by arc-length, with $\gamma(0) = 0$ and support Γ. For any $\varepsilon > 0$ there exist a closed set $I \subset \pi(\Gamma \cap B_{1/4}) \subset \mathbb{R}$ and a curve $\bar\gamma : I \to G_{m,s}$ with support $\bar\Gamma$ such that:

i) $\bar\Gamma \subset \Gamma$;

ii) $\bar\gamma_1(t) = t$ for $t \in I$, i.e., $\bar\gamma$ is a graph along X_1;

iii) $\left|\left(\bar\gamma(s)^{-1}\cdot\bar\gamma(t)\right)_i\right|^{1/w_i} \le \varepsilon|t-s|$ for $s,t \in I$ and $i = 2,\dots,n$;

iv) $\mathscr{H}^1\left(B_{1/4}\cap\bar\Gamma\smallsetminus\Gamma\right)\le C(\varepsilon,\alpha_i,\beta_i)E(\Gamma;0;1;X_1)$;
v) $\mathscr{L}^1\left(\pi\left(\Gamma\cap B_{1/4}\right)\smallsetminus I\right)\le C(\varepsilon,\alpha_i,\beta_i)E(\Gamma;0;1;X_1)$.

Above, $\mathscr{L}^1$ is the Lebesgue measure on $\mathbb{R}$ and $C(\varepsilon,\alpha_i,\beta_i)$ is a constant depending on ε and on the integers α_i,β_i, $i=2,\dots,n$, given by Theorem 1.2. We comment on iii). In $G_{m,s}$ we can define the pseudo-norm

$$\|x\| = \max\left\{|x_i|^{1/w_i} : i=1,\dots,n\right\}.$$

Then there is a constant $C_1>0$ such that for all $x,y\in G_{m,s}$

$$\frac{1}{C_1}d(x,y)\le\|y^{-1}\cdot x\|\le C_1 d(x,y). \tag{1.1}$$

Condition iii) asserts that the graph $\bar\gamma$ is Lipschitz for the Carnot-Carathéodory metric, with Lipschitz constant proportional to ε.

Theorem 1.4 is proved in Section 4. The assumption that γ be a geodesic can be weakened. A sufficient condition for the validity of the Lipschitz approximation is the assumption that Γ is 1-Ahlfors regular, i.e., the assumption that there exist constants $0<c_1<c_2$ such that for $0\le r\le 1$

$$c_1 r\le\mathscr{H}^1(\Gamma\cap B_r(x))\le c_2 r,$$

for any point $x\in\Gamma$. The lenght minimality implies these density estimates.

2. Hall Basis of free vector fields

In this section we review Grayson and Grossmann's method to construct a basis of vector fields in $\mathbb{R}^n$ that span a free Lie algebra. We will use the explicit formulas for these vector fields in order to get the integers α_i and β_i in Theorem 1.2. We refer to [3] for more details.

Let $E_1,\dots,E_m$ be the m generators of the free Lie algebra $F_{m,s}$. We assign to them the weight 1. We complete these elements to a basis of $F_{m,s}$ in a recursive way. If we have already defined basis elements of weights $1,\dots,r-1$, they are ordered so that $E<F$ if weight(E) <weight(F). Also, if weight$(E)=q$ and weight$(F)=t$ and $r=q+t$, then $[E,F]$ is a basis element of weight r if:

(1) E and F are basis elements and $E>F$;
(2) if $E=[G,H]$, then $F\ge H$.

The resulting basis is called a Hall basis.

We number the basis elements for the Lie algebra by ordering them as explained above, i.e., $E_{m+1}=[E_2,E_1], E_{m+2}=[E_3,E_1], E_{m+3}=[E_3,E_2], E_{m+4}=[E_4,E_1]$, etc. Consider a basis element E_i and write it as a bracket of lower order basis elements,

$E_i = [E_{j_1}, E_{k_1}]$, where $j_1 > k_1$. Repeat this process of writing the left-most element as a bracket of lower basis elements, until we obtain

$$E_i = [[\cdots[[E_{j_p}, E_{k_p}]E_{k_{p-1}}], \cdots, E_{k_2}], E_{k_1}], \tag{2.2}$$

where $k_p < j_p \le m$, and $k_{l+1} \le k_l$ for $1 \le l \le p-1$. This expansion involves p brackets, and we write $\ell(i) = p$ and define $\ell(1) = \ldots = \ell(m) = 0$. We associate to this expansion a multi-index $I(i) = (a_1, \ldots, a_n)$, with a_q defined by $a_q = \#\{t : k_t = q\}$. For the first m basis elements, the associated multi-index is $(0, \ldots, 0)$. We say that E_i is a *direct descendant* of each E_{j_t}, and we indicate this by writing $j_t \prec i$. Moreover, to any index i we can associate another index $\Lambda_i \in \{1, \ldots, m\}$, being the index of the (unique) generator that has i as a direct descendant, that is $\Lambda_i \prec i$; if $i \in \{1, \ldots, m\}$ already, then set $\Lambda_i = i$. If $E_i = [E_j, E_k]$, then $\Lambda_i = \Lambda_j$, $\ell(i) = \ell(j) + 1$ and each entry in $I(i)$ is at least as large as the corresponding entry in $I(j)$.

For every pair i and j with $j \prec i$, we define the monomial $p_{i,j}$ in $\mathbb{R}^n$ by

$$p_{i,j}(x) = \frac{(-1)^{\ell(i)-\ell(j)}}{(I(i) - I(j))!} x^{I(i)-I(j)}. \tag{2.3}$$

Lemma 2.1. Consider the Hall basis $E_1, \ldots, E_n$ and suppose that for $i \in \{m+1, \ldots, n\}$, the corresponding basis element E_i is of the form $E_i = [E_j, E_q]$ for some $1 \le q < j < i$. Then

$$p_{i,\Lambda_i}(x) = -\frac{p_{j,\Lambda_i}(x) x_q}{I(i)_q} \tag{2.4}$$

and in particular $\left|p_{i,\Lambda_i}(x)\right| \le \left|p_{j,\Lambda_i}(x) x_q\right|$.

Proof. Indeed if we consider $E_i = [E_j, E_q]$ and we remember its decomposition as in (2.2), we have that

$$E_i = [[\cdots[[E_{j_p}, E_{k_p}]E_{k_{p-1}}], \cdots, E_{k_2}], E_{k_1}],$$

therefore $E_q = E_{k_1}$ and $E_j = [[\cdots[[E_{j_p}, E_{k_p}]E_{k_{p-1}}], \cdots, E_{k_2}]$. Moreover $\Lambda_i = \Lambda_j$, $\ell(i) = \ell(j) + 1$ and all entries of $I(i)$ are equal to those of $I(j)$ except for the q-th one, where we have that $I(i)_q = I(j)_q + 1$. Notice that in particular $I(i)_q \ge 1$. The thesis now follows immediately from (2.3). □

The next theorem gives the connection between the abstract Lie algebra $F_{m,s}$ and the vector space $\mathbb{R}^n$, and it will be the starting point of our computations.

Theorem 2.2 (Grayson-Grossman)**.** Fix $s \ge 1$ and $m \ge 2$ and let n be the dimension of $F_{m,s}$. The vector fields in $\mathbb{R}^n$

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + \sum_{j \succ 2} p_{j,2}(x) \frac{\partial}{\partial x_j}, \quad \ldots \quad X_m = \frac{\partial}{\partial x_m} + \sum_{j \succ m} p_{j,m}(x) \frac{\partial}{\partial x_j} \tag{2.5}$$

generate a Lie algebra isomorphic to $F_{m,s}$.

3. Proof of Theorem 1.2

As explained above, we identify $G_{m,s}$ with $\mathbb{R}^n$ and we fix the vector fields $X_1, \dots, X_m$ as in (2.5). Let $\gamma : [0,1] \to (\mathbb{R}^n, d)$ be an admissible curve parameterized by arc-length, where d is the Carnot-Carathéodory distance associated to $X_1, \dots, X_m$. Thus for a.e. $\tau \in [0,1]$ we have

$$\dot{\gamma}(\tau) = \sum_{i=1}^{m} h_i(\tau) X_i(\gamma(\tau)),$$

where $h_1, \dots, h_m \in L^\infty(0,1)$ satisfy

$$h_1^2(\tau) + \dots + h_m^2(\tau) = 1 \quad \text{for a.e } \tau \in [0,1]. \tag{3.6}$$

First of all notice that

$$|\dot{\gamma}(\tau) - X_1|^2 = |\dot{\gamma}(\tau)|^2 - 2\langle \dot{\gamma}(\tau), X_1 \rangle + |X_1|^2 = 2(1 - h_1(\tau)).$$

From (3.6) we deduce that

a) $|h_i| \le 1$ for all $i = 1, \dots, m$;
b) for all $i \ne 1$, $h_i^2 \le 1 - h_1^2 = (1 - h_1)(1 + h_1) \le 2(1 - h_1)$;
c) for $t \in [0,1]$ and for all $i \ne 1$, by Hölder's inequality we have

$$\int_0^t |h_i(\tau)| d\tau \le t \sqrt{\frac{1}{t} \int_0^t h_i(\tau)^2 d\tau} \le t \sqrt{\frac{1}{t} \int_0^t 2(1 - h_1(\tau)) d\tau} = t \sqrt{E(\gamma; 0; t; X_1)}.$$

For semplicity, we shall use the notation $E(t) = E(\gamma; 0; t; X_1)$. We will prove the existence of integers α_i and β_i such that the claims i) and ii) in Theorem 1.2 and such that

$$\left| p_{i,\Lambda_i}(\gamma(\tau)) \right| \le t^{\alpha_i + \beta_i} E(t)^{\beta_i/2} \qquad \text{for all } 0 \le \tau \le t \tag{3.7}$$

hold for every $i \in \{2, \dots, n\}$. The proof is by induction on the weight of i.

The initial step is with $i = 2, \dots, m$. In this case we have $w_i = 1$ and we choose $\alpha_i = \beta_i = 0$. Then i) holds and also

$$\left| \gamma_i(t) \right| \le \int_0^t \left| h_i(\tau) \right| d\tau \le t \sqrt{E(t)},$$

which is ii). Condition (3.7) also holds because for $i \in \{1, \dots, m\}$ we have $p_{i,\Lambda_i} = p_{i,i} = 1$.

We now prove the inductive step. Let i be of weight $w_i \ge 2$. Following the Hall basis construction, X_i will be of the form $X_i = [X_j, X_q]$ for some $1 \le q < j < i$ with weights w_j and w_q such that $w_j + w_q = w_i$. Λ_i is the (unique) index in $\{2, \dots, m\}$ that has i as a direct descendant. Notice that Λ_i can't be 1 due to the construction of the Hall basis, and moreover $\Lambda_i = \Lambda_j$.

By lemma 2.1 we have $\left| p_{i,\Lambda_i}(x) \right| \le |p_{j,\Lambda_i}(x) x_q|$. Therefore, by the inductive assumption ii) of Theorem 1.2 on γ_q and by the inductive assumption (3.7) on p_{j,Λ_i}, there exist positive integers $\alpha_q, \beta_q, \alpha_j, \beta_j$, with $\alpha_j + \beta_j + 1 = w_j$ and $\alpha_q + \beta_q + 1 = w_q$, such that

$$\begin{aligned}
\left|p_{i,\Lambda_i}(\gamma(\tau))\right| &\leq \left|p_{j,\Lambda_i}(\gamma(\tau))\right|\left|\gamma_q(\tau)\right| \\
&\leq \begin{cases} t^{\alpha_j}\left(t\sqrt{E(t)}\right)^{\beta_j} t^{\alpha_q}\left(t\sqrt{E(t)}\right)^{\beta_q+1} & \text{if } q > 1 \\ t^{\alpha_j}\left(t\sqrt{E(t)}\right)^{\beta_j} t & \text{if } q = 1 \end{cases} \\
&\leq t^{\alpha_i}\left(t\sqrt{E(t)}\right)^{\beta_i},
\end{aligned}$$

where in the first case we set $\alpha_i := \alpha_j + \alpha_q$ and $\beta_i := \beta_j + \beta_q + 1$, and in the second one $\alpha_i := \alpha_j + 1$ and $\beta_i := \beta_j$. Notice that in both cases $\alpha_i + \beta_i + 1 = w_j + w_q = w_i$, as we wanted. This concluds the proof of the induction step for (3.7).

At this point we have that $\dot{\gamma}_i = h_{\Lambda_i} p_{i,\Lambda_i}$ and so

$$\gamma_i(t) = \int_0^t h_{\Lambda_i}(\tau) p_{i,\Lambda_i}(\gamma(\tau)) d\tau.$$

Hence using estimate (3.7) (that we already proved to be true at this step) we obtain

$$\begin{aligned}
\left|\gamma_i(t)\right| &\leq t^{\alpha_i}\left(t\sqrt{E(t)}\right)^{\beta_i} \int_0^t \left|h_{\Lambda_i}(\tau)\right| d\tau \\
&\leq t^{\alpha_i}\left(t\sqrt{E(t)}\right)^{\beta_i} t\sqrt{E(t)} \qquad (3.8) \\
&= t^{\alpha_i}\left(t\sqrt{E(t)}\right)^{\beta_i+1},
\end{aligned}$$

that becomes

$$\left(\frac{\left|\gamma_i(t)\right|}{t^{\alpha_i}}\right)^{\frac{1}{\beta_i+1}} \leq t\sqrt{E(t)}.$$

Thus we proved the point ii) of Theorem 1.2 for the index i with general weight, and this concludes the proof. □

Now we compare the parametric and the geometric definitions of excess. We start by recalling the construction of the 1-Hausdorff measure in the metric space $(\mathbb{R}^n, d)$. We refer to [1] for more details. For any subset $U \subseteq \mathbb{R}^n$ we call

$$\operatorname{diam}(U) = \sup\{d(x,y) : x, y \in U\}$$

the *diameter* of U, where d is the Carnot-Carathéodory metric. By definition, we set $\operatorname{diam}(\varnothing) = 0$. Let S be a subset of $\mathbb{R}^n$ and $\delta > 0$ a real number, and define

$$\mathscr{H}^1_\delta(S) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i) : S \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) < \delta\right\}.$$

It can be proved that each $\mathscr{H}^1_\delta$ is an outer measure. Since the map $\delta \mapsto \mathscr{H}^1_\delta(S)$ is decreasing, the limit

$$\mathscr{H}^1(S) := \lim_{\delta \downarrow 0} \mathscr{H}^1_\delta(S) = \sup_{\delta > 0} \mathscr{H}^1_\delta(S)$$

exists (although it may be infinite). $\mathscr{H}^1$ is a Borel measure in $\mathbb{R}^n$.

Now denote the support of γ by $\Gamma = \gamma([0,1]) \subset \mathbb{R}^n = G_{m,s}$, and its unit tangent vector by $\tau_\Gamma = \dot{\gamma} \in \operatorname{span}\{X_1,\dots,X_m\}$. As above, γ is parameterized by arc-length.

Using the definition of $\mathscr{H}^1$ it is not difficult to see that

$$\mathscr{H}^1(\Gamma) \le \operatorname{Var}(\gamma) := \sup\left\{\sum_{i=0}^{k-1} d(\gamma(t_{i+1}),\gamma(t_i)) \,:\, 0 \le t_0 < t_1 < \ldots < t_k \le 1\right\}. \tag{3.9}$$

If γ is injective, than we have the equality $\mathscr{H}^1(\Gamma) = \operatorname{Var}(\gamma)$. This can be proved in the following way. It is easy to show that for any $a,b \in [0,1]$ we have

$$\mathscr{H}^1\left(\gamma([a,b])\right) \ge d\left(\gamma(a),\gamma(b)\right).$$

Now take $0 \le t_0 < \cdots < t_k \le 1$. We have

$$\sum_{i=0}^{k-1} d\left(\gamma(t_{i+1},\gamma(t_i)\right) \le \sum_{i=0}^{k-1} \mathscr{H}^1\left(\gamma([t_i,t_{i+1}])\right) \le \mathscr{H}^1\left(\Gamma\right),$$

where the last inequality relies on the injectivity of γ and on the additivity of the Hausdorff measure. This shows that $\mathscr{H}^1(\Gamma) \ge \operatorname{Var}(\gamma)$.

On the other hand we have the following result, see [8, page 26]:

Theorem 3.1. Let $\gamma : [0,1] \to (\mathbb{R}^n, d)$ be a Lipschitz curve with controls $h \in L^\infty(0,1)^m$, i.e., $\dot{\gamma} = \sum_{j=1}^m h_j X_j(\gamma)$. Then we have

$$\operatorname{Var}(\gamma) = \int_0^1 \left|h(t)\right| dt. \tag{3.10}$$

From the previous discussion we deduce that if $\gamma : [0,1] \to G_{m,s}$ is an injective admissible curve then for any compact set $K \subset \Gamma$ we have

$$\mathscr{H}^1(K) = \int_{\gamma^{-1}(K)} \left|h(\tau)\right| d\tau.$$

Theorem 1.2 can now be rephrased in the following way.

Corollary 3.2. Let $\gamma : [0,1] \to G_{m,s}$ be a geodesic parameterized by arc-length, with $\gamma(0) = 0$ and support Γ. Let $0 < r \le 1$. Then for all $i = 2,\dots,n$ there exist positive integers α_i, β_i with $\alpha_1 + \beta_i + 1 = w_i$ and such that:

$$\left(\frac{\left|\gamma_i(t)\right|}{|t|^{\alpha_i}}\right)^{\frac{1}{\beta_i+1}} \le t\sqrt{E(\Gamma;0;r;X_1)} \quad \text{for all } 0 < t \le r. \tag{3.11}$$

Proof. Observe that if Γ is a length minimizer and $d(0,\gamma(1)) \ge r$, then $\mathscr{H}^1(\Gamma \cap B_r(x)) = r$. Using this observation one can see that the two definitions of excess coincide, and then conclude. □

Remark 3.3. The proof of the last corollary shows that for the validity of the result it is enough that Γ satisfies certain density estimates, without necessarily being a length minimizer; if there exist two constants $0 < c_1 \le c_2$ such that

$$c_1 r \le \mathscr{H}^1(\Gamma \cap B_r(x)) \le c_2 r,$$

then (3.11) holds with an appropriate constant in the right hand-side of the inequality.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We start with some elementary properties of the projection $\pi : G_{m,s} = \mathbb{R}^n \to \mathbb{R}$ defined by $\pi(x) = x_1$, where $x = (x_1, \dots, x_n)$ are the exponential coordinates associated with the vector fields given by the Hall basis. It is well-known that $\pi : (G_{m,s}, \cdot) \to (\mathbb{R}, +)$ is a group homomorphism, i.e., $\pi(x \cdot y) = (x \cdot y)_1 = x_1 + y_1 = \pi(x) + \pi(y)$. Moreover, we have

$$\left|\pi(x) - \pi(y)\right| = |x_1 - y_1| \le d(x, y),$$

i.e. π is 1-Lipschitz from $(\mathbb{R}^n, d)$ to $\mathbb{R}$.

Let $B_{1/4} = \{x \in \mathbb{R}^n : d(x, 0) < 1/4\}$ and for $\eta > 0$ consider the set

$$\bar{\Gamma} = \left\{x \in \Gamma \cap B_{1/4} : E(\Gamma; x; r; X_1) \le \eta \text{ for all } 0 \le r \le 1/2\right\} \subset \Gamma.$$

Take points $x \in \Gamma \cap B_{1/4}$ and $y \in \bar{\Gamma}$, with $x \neq y$, and define $\lambda = d(x, y) > 0$. By the triangle inequality we have $\lambda \le 1/2$. The set

$$\Gamma_\lambda = \delta_{\frac{1}{\lambda}}(y^{-1} \cdot \Gamma)$$

is the support of a length-minimizing curve, because left-translations and dilations take geodesics to geodesics. Moreover, we have $0 \in \Gamma_\lambda$.

The point $z = \delta_{1/\lambda}(y^{-1} \cdot x)$ is in Γ_λ and by the invariance properties of the Carnot-Carathéodory distance we have $d(z, 0) = \frac{1}{\lambda} d(x, y) = 1$. By the height-estimate (3.11), we have that for any $i \ge 2$

$$|z_i|^{\frac{1}{\beta_i + 1}} = \left(\frac{|z_i|}{d(z, 0)^{\alpha_i}}\right)^{\frac{1}{\beta_i + 1}} \le \sqrt{E(\Gamma_\lambda; 0; 1; X_1)} = \sqrt{E(\Gamma; y; \lambda; X_1)} \le \sqrt{\eta}.$$

We used the elementary invariance properties of excess $E(\Gamma_\lambda; 0; 1; X_1) = E(\Gamma; y; \lambda; X_1)$. By (1.1), this in turn gives

$$\left|(y^{-1} \cdot x)_i\right| = \left|\left(\delta_\lambda(z)\right)_i\right| = \lambda^{w_i} |z_i| \le \eta^{\beta_i/2 + 1/2} d(x, y)^{w_i} \le C_1^{w_i}\, \eta^{\beta_i/2 + 1/2} \left\|y^{-1} \cdot x\right\|^{w_i}. \tag{4.12}$$

Now we take $\varepsilon > 0$ such that $\varepsilon^{w_i} < 1/2$ for all i and we choose $\eta = \eta(\varepsilon, \alpha_i, \beta_i) > 0$ such that for all $i = 2, \dots, n$ we have

$$C_1^{w_i}\, \eta^{\beta_i/2 + 1/2} \le \min\left\{\varepsilon^{w_i}, \frac{1}{2}\right\} = \varepsilon^{w_i}. \tag{4.13}$$

In this way, the maximum norm is given by

$$\|y^{-1}\cdot x\| = \max_{j=1,\dots,n}\left|(y^{-1}\cdot x)_j\right|^{1/w_j} = \left|(y^{-1}\cdot x)_1\right|^{1/w_1} = |x_1 - y_1|,$$

and (4.12) becomes

$$\left|(y^{-1}\cdot x)_i\right|^{1/w_i} \le \varepsilon |x_1 - y_1|, \quad i = 2,\dots,n. \tag{4.14}$$

The projection $\pi : \bar\Gamma \to \mathbb{R}$ is injective because $\pi(x) = \pi(y)$ means $x_1 = y_1$ and thus, by (4.14), we have $\left|(y^{-1}\cdot x)_i\right| = 0$ for all $i \ge 2$. This implies $y^{-1}\cdot x = 0$ and so $x = y$. Let $I = \pi\left(\bar\Gamma\right)$ and denote by $\pi^{-1} : I \to \bar\Gamma$ the inverse of the projection. We define the curve $\bar\gamma : I \to \mathbb{R}^n$ letting

$$\bar\gamma(t) = \pi^{-1}(t), \quad t \in I.$$

The support of $\bar\gamma$ is $\bar\Gamma \subset \Gamma$. This is claim i) in Theorem 1.4.

Then we have $\bar\gamma_1(t) = \pi\left(\pi^{-1}(t)\right) = t$ for all $t \in I$. This is claim ii). Claim iii) follows from (4.14).

Next, we prove claim iv). For any point $x \in B_{1/4} \cap \Gamma \smallsetminus \bar\Gamma$ there exists a radius $0 < r_x \le 1/2$ such that

$$\frac{1}{2r_x}\int_{\Gamma\cap B_{r_x}(x)} |\tau_\Gamma - X_1|^2 \, d\mathscr{H}^1 = E(\Gamma; x; r_x; X_1) > \eta.$$

Moreover, since

$$B_{1/4} \cap \Gamma \smallsetminus \bar\Gamma \subset \bigcup_{x \in B_{1/4}\cap\Gamma\smallsetminus\bar\Gamma} B_{r_x/5}(x) \cap \Gamma,$$

using the 5-covering lemma, there exists a sequence of points $x_k \in B_{1/4} \cap \Gamma \smallsetminus \bar\Gamma$ such that, letting $r_k = r_{x_k}$, we have

$$B_{1/4} \cap \Gamma \smallsetminus \bar\Gamma \subset \bigcup_{k\in\mathbb{N}} B_{r_k}(x_k) \cap \Gamma,$$

where the balls $B_{r_k/5}(x_k)$ are pair-wise disjoint. Thus we obtain

$$\begin{aligned}\mathscr{H}^1\left(B_{1/4} \cap \Gamma \smallsetminus \bar\Gamma\right) &\le \sum_{k\in\mathbb{N}} \mathscr{H}^1\left(B_{r_k}(x_k) \cap \Gamma\right) = \sum_{k\in\mathbb{N}} 2r_k \\ &\le \sum_{k\in\mathbb{N}} \frac{1}{\eta}\int_{\Gamma\cap B_{r_k}(x_k)} |\tau_\Gamma - X_1|^2 \, d\mathscr{H}^1 \\ &\le \frac{1}{\eta}\int_{\Gamma\cap B_1} |\tau_\Gamma - X_1|^2 \, d\mathscr{H}^1 = \frac{2}{\eta}E(\Gamma; 0; 1; X_1).\end{aligned}$$

Finally, claim v) follows from iv) and from the fact that the projection π is 1-Lipschitz. The set I may be assumed to be closed, because all the claims are stable passing to the closure. □

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