ON THE STRUCTURE OF WEAK SOLUTIONS TO SCALAR CONSERVATION LAWS WITH FINITE ENTROPY PRODUCTION

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Abstract. We consider weak solutions with finite entropy production to the scalar conservation law

$$\partial_t u + \mathrm{div}_x F(u) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d.$$

Building on the kinetic formulation we prove under suitable nonlinearity assumption on f that the set of non Lebesgue points of u has Hausdorff dimension at most d. A notion of Lagrangian representation for this class of solutions is introduced and this allows for a new interpretation of the entropy dissipation measure.

1. Introduction

We study the structure of weak solutions to the scalar conservation law

$$\partial_t u + \mathrm{div}_x F(u) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d. \tag{1.1}$$

It is well-known that uniqueness for the Cauchy problem associated to (1.1) fails in the class of weak solutions and it is restored by requiring the dissipation of every convex entropy:

$$\mu_\eta := \partial_t \eta(u) + \mathrm{div}_x Q(u) \le 0 \quad \text{in } \mathcal{D}', \tag{1.2}$$

where $\eta''(v) \ge 0$ and $Q'(v) = \eta'(v)F'(v)$ for every $v \in \mathbb{R}$. Bounded solutions satisfying (1.2) for every convex entropy η and corresponding flux Q are called entropy solutions and the Cauchy problem associated to (1.1) is well-posed in this class [Kru70]. A rich literature investigates the regularizing effect and the fine properties that the nonlinearity of the flux function F coupled with (1.2) induces on weak solutions.

The one space dimensional case is special and this regularizing effect is now quite well understood. Starting from the celebrated one sided Lipschitz estimate in [Ole63] for uniformly convex fluxes several regularity results have been obtained, even for more general nonlinear fluxes [ADL04, Daf85, Che86, Mar18]. The arguments essentially rely on the structure of characteristics: although the classical method of characteristics is not available for nonsmooth solutions, some rigidity is preserved. In one space dimension two trajectories with different speed typically intersect and the interaction of the characteristics provides the regularization. Of course this is not the case for several space dimensions.

A widely used tool to study entropy solutions to (1.1) is the kinetic formulation introduced in [LPT94], where also the first regularity results in terms of fractional Sobolev spaces have been obtained by means of velocity averaging lemmas. For further developments see [TT07, GP13]. Being the sign of μ_η not relevant in the kinetic formulation, most of the available results in this direction hold in the more general setting where (1.2) is replaced by

$$\mu_\eta := \partial_t \eta(u) + \mathrm{div}_x Q(u) \in \mathcal{M}\left((0,T)\times\mathbb{R}^d\right), \tag{1.3}$$

i.e. the entropy production measure μ_η is required to be locally finite but without constraints on its sign. We refer to these solutions as *weak solutions with finite entropy production*. The first example where the sign of the entropy production is used to improve the available regularity results in the kinetic framework is [GL19]. In [DLOW03] (see also the presentation in [COW08] and the previous work [AKLR02] for a fully nonlinear conservation law with $d = 1$) the authors proved that under mild nonlinearity assumptions on f, bounded weak solutions with finite entropy production share several fine properties with BV functions: more in details they proved that there exists a rectifiable set J of dimension d such that

1. u has vanishing mean oscillation at every $(t,x) \notin J$;
2. u has left and right traces on J;

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(3) $\mu_\eta \llcorner J = ((\eta(u^+), q(u^+)) - (\eta(u^-), q(u^-))) \cdot \mathbf{n} \llcorner J$, where $u^\pm$ denotes the traces on J and $\mathbf{n}$ denotes the normal to J.

For BV solutions (1) and (3) can be improved to

(1') every $(t,x) \notin J$ is a Lebesgue point;
(3') $\mu_\eta = ((\eta(u^+), q(u^+)) - (\eta(u^-), q(u^-))) \cdot \mathbf{n} \llcorner J$.

In [Sil19] the author considered the case of entropy solutions to (1.1) with a power-type nonlinearity assumption on f (see Assumption 4.1): in this setting he proved that every point $(t,x) \notin J$ is actually a continuity point, providing therefore a positive answer about (1'). Moreover he showed that $\mu = \mu \llcorner \bar{J}$, where $\bar{J}$ denotes the topological closure of J, partially answering about (3').

It is also worth to mention that both questions have affirmative answer for entropy solutions in one space dimension. Property (1') is valid under the milder nonlinearity assumption that $\{v : f''(v) \neq 0\}$ is dense in $\mathbb{R}$: see [BM17], where it is also proved that Property (3') holds for general smooth fluxes, see also [DLR03] for an earlier proof in the case of fluxes with finitely many nondegenerate inflection points. Moreover in [Daf06] it is proved that μ_η vanishes for continuous weak solutions, without a priori requiring that they are entropic.

Entropy solutions are of course the most relevant in the theory of scalar conservation laws, nevertheless weak solutions that are not entropic arise naturally together with (1.3) in certain situations: in [BBMN10, Mar10] they arise in the study of large deviations for stochastic conservation laws. We refer to [LO18] and the reference therein for more motivations.

Property (1') has been addressed for the first time out of the entropic setting in [LO18] for the Burgers equation:

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{1.4}$$

The authors proved that the set of non Lebesgue points of any weak solution to (1.4) with finite entropy production has Hausdorff dimension at most 1. It is also remarkable that they only need (1.3) to be satisfied for the entropy $\bar{\eta}(u) = u^2/2$. On the other hand their argument relies on the link between Burgers equation and Hamilton-Jacobi equation, therefore it seems limited to the one dimensional case.

In this work we obtain the analogous result in the general setting, thus providing a partial answer about (1') in the affermative.

Theorem 1.1. *Let u be a bounded weak solution to (1.1) with finite entropy production. Under a power-like nonlinearity assumption on the flux f, the set of non Lebesgue points of u has Hausdorff dimension at most d.*

We remark here that this theorem as well as the result in [DLOW03] is proved in the more general setting of quasi-solutions (see Section 4).

The proof of Theorem 1.1 is based on a new estimate which relates the solution to the free motion in the kinetic formulation. As another byproduct of this estimate we get the following theorem, which extends the case of bounded entropy solutions studied in [BBM17].

Theorem 1.2. *Let u be a bounded nonnegative weak solution with finite entropy production to (1.1), with $u_0 \in L^1(\mathbb{R}^d)$. Then u admits a* Lagrangian representation, *i.e. there exists a nonnegative bounded measure ω on*

$$\Gamma := \left\{ \gamma = (\gamma^1, \gamma^2) \in \mathrm{BV}\left([0,T), \mathbb{R}^d \times [0,+\infty)\right) : \gamma^1 \text{ is Lipschitz} \right\}$$

which satisfies the following conditions:

(1) for every $t \in [0,T)$ it holds

$$(e_t)_\sharp \omega = \mathscr{L}^{d+1} \llcorner \left\{ (x,v) \in \mathbb{R}^d \times [0,+\infty) : v < u(t,x) \right\},$$

where $e_t(\gamma) = \lim_{s \to t^+} \gamma(s)$ is the evaluation map;

(2) the measure ω is concentrated on characteristics, i.e. on curves $\gamma \in \Gamma$ such that

$$\dot{\gamma}^1(t) = f'(\gamma^2(t)) \quad \text{for a.e. } t \in (0,T);$$

(3) the following integral estimate holds:

$$\int_\Gamma \mathrm{Tot.Var.}_{[0,T)} \gamma^2 d\omega < \infty.$$

The Lagrangian representation is of course strictly related to the kinetic formulation and it is also reminiscent of the superposition principle for nonnegative measure valued solutions to the continuity equation [Amb04]. By means of this notion we provide a representation of the entropy production measure μ_η. We notice that the existence of a Lagrangian representation, even if in a different form, has been a crucial ingredient to prove the optimal regularizing effect [Mar18] and the structure of entropy solutions [BM17] in the case of a single space dimension. Moreover it is the crucial ingredient to prove in [BBM17] that μ_η vanishes for bounded and continuous entropy solutions (see also [Sil19] for a similar proof of this result).

Structure of the paper. In Section 2 we introduce the notion of quasi solutions and the corresponding kinetic formulation. We moreover recall a few results from the theory of L^1 optimal transport that will be relevant in the construction of the Lagrangian representation. The short Section 3 is devoted to the proof of the kinetic estimate. As a first consequence we deduce Theorem 1.1 in Section 4. Theorem 1.2 is proved instead in Section 5.

2. Preliminaries and setting

In this section we introduce the setting of quasi-solutions following [DLOW03] and we provide the notion of kinetic formulation. Moreover we recall a few facts from L^1-Optimal Transport that will be useful in the second part of the work.

2.1. Quasi-solutions.

We consider flux functions $f \in C^2(\mathbb{R}, \mathbb{R}^d)$.

Definition 2.1. Let $\mathscr{E}_+$ denote the set of all $q \in C(\mathbb{R}, \mathbb{R}^d)$ for which there exists an η with

$$q'(v) = \eta'(v) f'(v) \qquad \text{and} \qquad \eta''(v) \geq 0 \quad \text{in} \quad \mathcal{D}'_v.$$

We call a measurable function $u : \mathbb{R}^d \to (0,1)$ a *quasi-solution* if

$$\mu_q := -\mathrm{div}_x q(u) \in \mathcal{M}(\mathbb{R}^d) \quad \text{for all } q \in \mathscr{E}_+. \tag{2.1}$$

Remark 2.2. We observe that we can recover (1.3) from (2.1) considering $f = (\mathbf{I}, F)$. Conversely quasi-solutions can be interpreted as time-independent functions satisfying (1.3). Notice moreover that (2.1) does not imply $\mathrm{div}_x f(u) = 0$. In particular this setting is more general than (1.1), (1.3) and it allows to consider suitable sources. Finally observe that considering quasi-solutions taking values in $(0,1)$ is not restrictive, up to translations and rescaling of the flux f.

In the following Proposition we introduce the kinetic formulation for quasi-solutions.

Proposition 2.3. *Let u be a quasi-solution and let $\chi : \mathbb{R}^d \times [0,+\infty) \to \{0,1\}$ be*

$$\chi(x,v) := \begin{cases} 1 & \text{if } 0 < v \leq u(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a locally finite Radon measure $\mu \in \mathcal{M}(\mathbb{R}_v \times \mathbb{R}^d_x)$ such that

$$f'(v) \cdot \nabla_x \chi(v,x) = \partial_v \mu \quad \text{in } \mathcal{D}'_{v,x}.$$

In the following we will denote by ν the x- marginal of the total variation $|\mu|$ of μ:

$$\nu(A) := |\mu|(\mathbb{R} \times A) \quad \text{for every Borel set } A \subset \mathbb{R}^d.$$

2.2. Duality for L^1 optimal transport.

Definition 2.4. Let (X,d) be a Polish metric space and let μ_1, μ_2 be two probability measures on X. The Wasserstein distance of order 1 between μ_1 and μ_2 is defined by

$$W_1(\mu_1,\mu_2) := \inf_{\pi \in \Pi(\mu_1,\mu_2)} \int_X d(x,y) d\pi(x,y), \tag{2.2}$$

where $\Pi(\mu_1,\mu_2)$ is the set of transport plans from μ_1 to μ_2, i.e.

$$\Pi(\mu_1,\mu_2) := \left\{ \omega \in \mathcal{P}(X^2) : \pi_{1\sharp}\omega = \mu_1, \pi_{2\sharp}\omega = \mu_2 \right\},$$

denoting by $\pi_1, \pi_2 : X^2 \to X$ the two natural projections.

Notice that W_1 can take value $+\infty$.

In order to prove the existence of a Lagrangian representation for weak solutions with finite entropy production the following duality formula will be useful (see for example [Vil09]).

Proposition 2.5. *For any $\mu_1, \mu_2 \in \mathcal{P}(X)$, it hods*

$$W_1(\mu_1, \mu_2) = \sup_{\phi \in L^1(\mu_1), \|\phi\|_{\mathrm{Lip}} \le 1} \left(\int_X \phi d\mu_1 - \int_X \phi d\mu_2 \right).$$

The next theorem from [BD18] provides the existence of an L^1-optimal map with respect to quite general distances on $\mathbb{R}^N$.

Theorem 2.6. *Let $X = \mathbb{R}^N$ with $N \in \mathbb{N}$ be the euclidean space equipped with the distance induced by a convex norm $|\cdot|_{D*}$. Let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^N)$ be such that $\mu_1 \ll \mathscr{L}^N$ and the infimum in (2.2) is finite. Then there exists an optimal plan π in (2.2) induced by a map, i.e. there exists a measurable map $T : \mathbb{R}^N \to \mathbb{R}^N$ such that $T_\sharp \mu_1 = \mu_2$ and*

$$W_1(\mu_1, \mu_2) = \int_X |T(x) - x|_{D*} d\mu_1(x).$$

We remark that the result above would be much easier requiring only the existence of an optimal plan instead of an optimal map. This would be sufficient for our goal but this result allows for a more transparent construction in Section 5.

3. A weak estimate

Relying on the kinetic formulation we prove in this short section the main estimate of this work. For every $u : \mathbb{R}^d \to [0, +\infty)$ we denote its subgraph by

$$E_u := \left\{ (x, v) \in \mathbb{R}^d \times [0, +\infty) : v \le u(x) \right\},$$

For every measurable set $E \subset \mathbb{R}^d \times [0, +\infty)$ we denote by $\chi_E : \mathbb{R}^d \times [0, +\infty) \to \{0, 1\}$ the associated characteristic function:

$$\chi_E(x, v) := \begin{cases} 1 & \text{if } (x, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We moreover consider the *free-transport* operator introduced in [Bre84]: given E as above and $s \in \mathbb{R}$ let

$$\mathrm{FT}(E, s) := \left\{ (x, v) \in \mathbb{R}^d \times [0, +\infty) : (x - f'(v)s, v) \in E \right\}. \tag{3.1}$$

This terminology is motivated by the following lemma:

Lemma 3.1. *Let $E \subset \mathbb{R}^d \times [0, +\infty)$ be a measurable set and denote by $\chi^1 : \mathbb{R} \times \mathbb{R}^d \times [0, +\infty)$ the function defined by*

$$\chi^1(s, x, v) := \chi_{\mathrm{FT}(E_u, s)}(x, v). \tag{3.2}$$

Then χ^1 solves the linear transport equation

$$\partial_s \chi^1 + f'(v) \cdot \nabla_x \chi^1 = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d \times [0, +\infty)).$$

Proof. For any $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d \times [0, +\infty))$ we consider

$$\tilde{\varphi}(s, x, v) := \varphi(s, x - sf'(v), v)$$

so that by (3.1), it holds

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}^d \times [0, +\infty)} (\partial_s \varphi + f'(v) \cdot \nabla_x \varphi) \chi^1 ds dx dv &= \int_{\mathbb{R} \times \mathbb{R}^d \times [0, +\infty)} \partial_s \tilde{\varphi}(s, x, v) \chi_{E_u}(x, v) ds dx dv \\ &= 0, \end{aligned}$$

where, in the last equality, we used that $\chi_{E_u}(x, v)$ does not depend on the variable s and $\tilde{\varphi}$ has compact support. □

For any $R > 0$ and $\bar{x} \in \mathbb{R}^d$ denote by $B_R(\bar{x})$ the ball of radius $R > 0$ and center $\bar{x}$. Moreover we set $\pi_x : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}^d$ the first projection.

Theorem 3.2. *Let u be a quasi-solution and $\bar{s}>0$. Let $\phi \in C_c^1(\mathbb{R}^d \times (0,+\infty))$ be such that $\pi_x(\operatorname{supp}\phi) \subset B_R(\bar{x})$. Then*

$$\int_{\mathbb{R}^d\times[0,+\infty)} \phi(x,v)(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\bar{s})})dxdv \le \left(\bar{s}\|\partial_v\phi\|_{L^\infty} + \frac{\bar{s}^2}{2}\|f''\|_{L^\infty}\|\nabla_x\phi\|_{L^\infty}\right)\nu(B_{R+\|f'\|_\infty\bar{s}}(\bar{x})). \tag{3.3}$$

Proof. For every $s\in[0,\bar{s}]$ let χ^1 be defined as in (3.2) and

$$\chi^2(s,\cdot,\cdot) := \chi_{E_u}.$$

By Proposition 2.3 and Lemma 3.1 we have

$$\begin{aligned} \partial_s\chi^1 + f'(v)\cdot\nabla_x\chi^1 &= 0 \qquad \text{in } \mathcal{D}',\\ \partial_s\chi^2 + f'(v)\cdot\nabla_x\chi^2 &= \partial_v\tilde{\mu} \qquad \text{in } \mathcal{D}', \end{aligned}$$

where $\tilde{\mu} = \mathscr{L}^1\times\mu$. Let $\tilde{\chi} := \chi^2-\chi^1$ and $\psi(s,x,v) := \phi(x+f'(v)(\bar{s}-s),v)$. Then by a straightforward computation it follows that

$$\partial_s(\tilde{\chi}\psi) + f'(v)\cdot\nabla_x(\tilde{\chi}\psi) = \psi\partial_v\tilde{\mu} \tag{3.4}$$

holds in the sense of distributions. Let $g:[0,\bar{s}]\to\mathbb{R}$ be defined by

$$g(s) = \int_{\mathbb{R}^d\times(0,+\infty)} \tilde{\chi}(s)\psi(s)dxdv$$

and notice that g interpolates between

$$g(0)=0 \qquad \text{and} \qquad g(\bar{s}) = \int_{\mathbb{R}^d\times[0,+\infty)} \phi(x,v)(\chi_{E_u}-\chi_{\mathrm{FT}(E_u,\bar{s})})dxdv.$$

It follows from (3.4) and the definition of $\tilde{\mu}$ that

$$g'(s) = -\int_{\mathbb{R}^d\times[0,+\infty)} \partial_v\psi(s)d\mu$$

holds in the sense of distributions. Therefore $g\in C^1([0,\bar{s}])$ and we have

$$\begin{aligned} \int_{\mathbb{R}^d\times[0,+\infty)} \phi(\chi_{E_u}-\chi_{\mathrm{FT}(E_u,\bar{s})})dxdv &= g(\bar{s})-g(0)\\ &= \int_0^{\bar{s}} g'(s)ds\\ &= -\int_0^{\bar{s}}\int_{\mathbb{R}^d\times[0,+\infty)} \partial_v\psi(s)d\mu ds\\ &= -\int_0^{\bar{s}}\int_{\mathbb{R}^d\times[0,+\infty)} \left(\partial_v\phi + f''(v)\cdot\nabla_x\phi\cdot(\bar{s}-s)\right)d\mu ds\\ &\le \left(\bar{s}\|\partial_v\phi\|_{L^\infty} + \frac{\bar{s}^2}{2}\|f''\|_{L^\infty}\|\nabla_x\phi\|_{L^\infty}\right)\nu(B_{R+\|f'\|_\infty\bar{s}}(\bar{x})), \end{aligned}$$

which is our goal. □

Remark 3.3. In view of Theorem 2.6, the estimate (3.3) can be interpreted as follows: the total variation of the kinetic measure μ controls the 1-Wasserstein distance between the subgraph E_u of u and its free transport $\mathrm{FT}(E_u,\bar{s})$ for some $\bar{s}>0$. This point of view will be exploited in Section 5 in the time-dependent setting.

4. Structure of quasi-solutions

In this section we assume the following quantitative nonlinearity estimate on the flux function f.

Assumption 4.1. There exists $\alpha\in(0,1]$ and $C>0$ so that for every $\xi\in\mathbb{R}^d$ with $|\xi|=1$, and any $\delta>0$, we have

$$\mathscr{L}^1\big(\{v\in[0,1] : |f'(v)\cdot\xi|<\delta\}\big) \le C\delta^\alpha. \tag{4.1}$$

We recall that this is the assumption that provides a fractional Sobolev regularity of the entropy solutions in [LPT94] and it is used in [Sil19] to prove Property (1') of the introduction in the entropic setting. The next technical lemma makes rigorous the following observation: (4.1) is a quantitative formulation of the weaker condition

$$\operatorname{span}\{f'(v) : v \in (a,b)\} = \mathbb{R}^d \qquad \text{for all} \quad 0 < a < b < 1.$$

Lemma 4.2. *Let f be such that Assumption 4.1 holds. Then there exists $\tilde{c} > 0$ depending on $d, \|f'\|_{L^\infty}$ and C from Assumption 4.1 such that for every $\bar{v}, \bar{h} > 0$ for which $\bar{v} + \bar{h} \le 1$ there exist $v_1 < \ldots < v_d \in [\bar{v}, \bar{v} + \bar{h}]$ enjoying the following property: for every $a \in \mathbb{R}^d$ there exists $a_1, \ldots, a_d \in \mathbb{R}$ so that*

$$a = \sum_{i=1}^d a_i f'(v_i), \qquad |a_i| \le \tilde{c}\frac{|a|}{\bar{h}^{d/\alpha}}, \qquad \textit{and} \qquad |v_{i+1} - v_i| \ge \frac{\bar{h}}{2d}. \tag{4.2}$$

Proof. For $j = 1, \ldots, 2d$ let

$$I_j := \left[\bar{v} + (j-1)\frac{\bar{h}}{2d}, \bar{v} + j\frac{\bar{h}}{2d}\right].$$

Let $\delta > 0$ be such that $(1 \vee C)\delta^\alpha = \frac{\bar{h}}{4d} < \frac{\bar{h}}{2d}$, where C is the constant in Assumption 4.1. In particular $\delta \in (0,1)$. Then we have that there exists $v_1 \in I_1$ such that $|f'(v_1)| \ge \delta$. We denote by $\xi_1 = \frac{f'(v_1)}{|f'(v_1)|}$. By Assumption 4.1 we can define inductively for $i = 2, \ldots, d$ a vector $\xi_i \in S^{d-1}$ such that $\xi_i \perp \operatorname{span}_{j \le i-1}\{f'(v_j)\}$ and a value $v_i \in I_{2i-1}$ such that $|f'(v_i) \cdot \xi_i| \ge \delta$. Proceeding in this way we can choose $v_1, \ldots, v_d$ to satisfy $|v_{i+1} - v_i| \ge \frac{\bar{h}}{2d}$ for every $i = 1, \ldots, d-1$ and $\operatorname{span}_{i \le d}\{f'(v_i)\} = \mathbb{R}^d$. In particular for every $a \in \mathbb{R}^d$ there exists $a_1, \ldots, a_d$ such that

$$a = \sum_{i=1}^d a_i f'(v_i).$$

Now we estimate the size of the coefficients a_i: inductively we prove that there exists a constant c_i depending on i and $\|f'\|_\infty$ such that for every $i = 1, \ldots, d$ it holds $|a_{d+1-i}| \le c_i \frac{|a|}{\delta^i}$. By the choice of ξ_i we have that

$$a \cdot \xi_{d+1-i} = \sum_{j=1}^i a_{d+1-j} f'(v_{d+1-j}) \cdot \xi_{d+1-i}$$

therefore

$$|a_{d+1-i}| \le \frac{|a \cdot \xi_{d+1-i}|}{|f'(v_{d+1-i}) \cdot \xi_{d+1-i}|} + \sum_{j=1}^{i-1} |a_{d+1-j}| \frac{|f'(v_{d+1-j}) \cdot \xi_{d+1-i}|}{|f'(v_{d+1-i}) \cdot \xi_{d+1-i}|}. \tag{4.3}$$

For $i = 1$ the estimate (4.3) says

$$|a_d| \le \frac{|a \cdot \xi_d|}{|f'(v_d) \cdot \xi_d|} \le \frac{|a|}{\delta},$$

so that the claim is satisfied with $c_1 = 1$. For $i = 2, \ldots, d$ we get

$$|a_{d+1-i}| \le \frac{|a|}{\delta} + \sum_{j=1}^{i-1} |a_{d+1-j}| \frac{\|f'\|_{L^\infty}}{\delta} \le \frac{|a|}{\delta} + \sum_{j=1}^{i-1} \frac{c_j |a|}{\delta^j} \cdot \frac{\|f'\|_{L^\infty}}{\delta},$$

therefore the claim is satisfied with $c_i = 1 + \|f'\|_{L^\infty} \sum_{j=1}^{i-1} c_j$. Letting $\bar{c} = \max_i c_i$ and exploiting the choice of δ we get that there exists $\tilde{c} > 0$ as in the statement such that (4.2) holds and this concludes the proof. □

Let us now fix some notation: for every $x \in \mathbb{R}^d$ and any $r > 0$, denote by

$$(u)_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u \qquad \text{and} \qquad 2\bar{h}_r(x) = \max_{y_1, y_2 \in \bar{B}_{2r}(x)} \left((u)_{B_r(y_1)} - (u)_{B_r(y_2)}\right). \tag{4.4}$$

In the following lemma we prove that $\bar{h}_r(x)$ can be estimated in terms of the difference between E_u and an appropriate free transport of itself.

Lemma 4.3. *Let $\tilde c > 0$ be as in Lemma 4.2. Then there exists $c_1 = c_1(d, f, \tilde c) > 0$ such that for every $\bar x \in \mathbb{R}^d, r > 0$ and every $\Delta v \in \left[0, c_1 \bar h_r(\bar x)^{\frac{d}{\alpha}+1}\right]$ there exists $\bar y \in \mathbb{R}^d$, $\tilde v \in (0,1)$ and $\tilde a \in \mathbb{R}$ such that*

$$|\tilde a| \le \tilde c \frac{4r}{\bar h_r(\bar x)^{d/\alpha}}, \qquad |\bar y - \bar x| \le 2r + d\|f'\|_\infty \tilde c \frac{4r}{\bar h_r(\bar x)^{d/\alpha}} \tag{4.5}$$

and

$$\mathcal{L}^{d+1}(\mathcal{C} \cap E_u) - \mathcal{L}^{d+1}(\mathrm{FT}(E_u, \tilde a) \cap \mathcal{C}) \ge \bar h_r(\bar x)|B_r| \frac{\Delta v}{2d}, \tag{4.6}$$

where

$$\mathcal{C} := B_r(\bar y) \times [\tilde v, \tilde v + \Delta v]. \tag{4.7}$$

Proof. For any $x \in \mathbb{R}^d$, $v \in [0,1]$ and $r > 0$ let us set

$$m_r(x, v) := \mathscr{L}^d\big(\{y \in B_r(x) : u(y) > v\}\big) \tag{4.8}$$

and notice that by a simple application of Fubini theorem we have

$$(u)_{B_r(x)} = \frac{1}{|B_r(x)|} \int_0^1 m_r(x, v) dv.$$

We fix $\bar x \in \mathbb{R}^d$ and $r > 0$ and in the remaining part of the proof we simply denote $\bar h_r(\bar x)$ by $\bar h$ and m_r by m. Let $y_1, y_2 \in B_{2r}(\bar x)$ be realizing the maximum in (4.4). Therefore

$$\begin{aligned}
2\bar h|B_r| &= \int_0^1 (m(y_1, v) - m(y_2, v))\, dv \\
&= \int_0^{1-\bar h} \big(m(y_1, v + \bar h) - m(y_2, v)\big)\, dv + \int_0^{\bar h} m(y_1, v) dv - \int_{1-\bar h}^1 m(y_2, v) dv \\
&\le \int_1^{1-\bar h} \big(m(y_1, v + \bar h) - m(y_2, v)\big)\, dv + \bar h|B_r|,
\end{aligned}$$

where in the last inequality we used (4.8) to deduce that $m(y_1, v) \le |B_r|$ for every $v \in (0, \bar h)$ and $m(y_2, v) \ge 0$ for every $v \in (1 - \bar h, 1)$. In particular there exists $\bar v \in [0, 1 - \bar h]$ such that

$$m\left(y_1, \bar v + \bar h\right) - m(y_2, \bar v) \ge \bar h|B_r|. \tag{4.9}$$

Now we are in position to apply Lemma 4.2 with the choice of $\bar v$ and $\bar h$ as above and with $a = y_1 - y_2$. Let $v_1, \dots, v_d$ be from Lemma 4.2 and set $x_0 := y_2$ and inductively $x_i := x_{i-1} + a_i f'(v_i)$ for $i = 1, \dots, d$. We moreover set $v_0 = \bar v$ and $v_{d+1} = \bar v + \bar h$. By construction $y_1 = x_d$ therefore by (4.9) we have that $m(x_d, v_{d+1}) - m(x_0, v_0) \ge \bar h|B_r|$. Since $v_0 \le v_1$ it holds $m(x_0, v_1) \le m(x_0, v_0)$ therefore

$$\bar h|B_r| \le m(x_d, v_{d+1}) - m(x_0, v_1) = \sum_{i=1}^d m(x_i, v_{i+1}) - m(x_{i-1}, v_i).$$

In particular there exists $l \in \{1, \dots, d\}$ such that

$$m(x_l, v_{l+1}) - m(x_{l-1}, v_l) \ge \frac{\bar h|B_r|}{d}. \tag{4.10}$$

We set

$$\tilde a := a_l, \qquad \tilde v := v_l \qquad \text{and} \qquad \bar y := x_l. \tag{4.11}$$

The first estimate in (4.5) directly follows from (4.2) in Lemma 4.2. In order to prove the second estimate in (4.5), observe that by construction

$$\bar y = y_2 + \sum_{i=1}^l a_i f'(v_i), \qquad \text{and} \qquad |\bar x - y_2| \le 2r.$$

Therefore

$$|\bar x - \bar y| \le 2r + \|f'\|_{L^\infty} \sup_{i=1,\dots,l} |a_i|.$$

Estimating as above $\sup |a_i|$ by (4.2), this proves the second estimate in (4.5).

In the remaining part of the proof we show (4.6). Given $x \in \mathbb{R}^d$, $v \in [0,1]$ and $s \in \mathbb{R}$ set

$$m'(x,v,s) := m(x - f'(v)s, v) = \mathscr{L}^d\big(\{y \in B_r(x) : (y,v) \in \mathrm{FT}(E_u, s)\}\big). \tag{4.12}$$

Let us prove first the following claim.

Claim. There exists a constant $c_1 = c_1(d,f,\tilde{c}) > 0$ such that if $0 \le \Delta v \le c_1 \bar{h}^{\frac{d}{\alpha}+1}$, then for every $w_1 \in [v_l, v_l + \Delta v]$ and any $w_2 \in [v_{l+1} - \Delta v, v_{l+1}]$ it holds

$$m(x_l, w_2) - m'(x_l, w_1, a_l) \ge \frac{\bar{h}|B_r|}{2d}.$$

Proof of the claim. By the monotonicity of m with respect to v and (4.10) it holds

$$\begin{aligned} m(x_l,w_2) - m'(x_l,w_1,a_l) &= m(x_l,w_2) - m(x_l,v_{l+1}) + m(x_l,v_{l+1}) - m(x_{l-1},v_l) + m(x_{l-1},v_l) - m'(x_l,w_1,a_l) \\ &\ge \frac{\bar{h}|B_r|}{d} + m(x_l - a_l f'(v_l), v_l) - m(x_l - a_l f'(w_1), w_1) \\ &\ge \frac{\bar{h}|B_r|}{d} + m(x_l - a_l f'(v_l), v_l) - m(x_l - a_l f'(w_1), v_l). \end{aligned}$$

Therefore it is sufficient to prove that there exists c_1 as in the statement such that if $0 \le \Delta v \le c_1 \bar{h}^{\frac{d}{\alpha}+1}$, then

$$m(x_l - a_l f'(w_1), v_l) - m(x_l - a_l f'(v_l), v_l) \le \frac{\bar{h}|B_r|}{2d}. \tag{4.13}$$

Clearly it holds

$$\begin{aligned} m(x_l - a_l f'(w_1), v_l) - m(x_l - a_l f'(v_l), v_l) \le{}& |B_r(x_l - a_l f'(w_1)) \Delta B_r(x_l - a_l f'(v_l))| \\ \le{}& C_d r^{d-1} |a_l| \|f''\|_{L^\infty} \Delta v. \end{aligned}$$

Since by Lemma 4.2 it holds $|a_l| \le \tilde{c} \frac{|a|}{\bar{h}^{\frac{d}{\alpha}}}$ and $|a| = |y_2 - y_1| \le 4r$, we get

$$m(x_l - a_l f'(w_1), v_l) - m(x_l - a_l f'(v_l), v_l) \le \frac{4 C_d r^d \tilde{c} \|f''\|_{L^\infty}}{\bar{h}^{\frac{d}{\alpha}}} \Delta v,$$

which implies (4.13) if $0 \le \Delta v \le c_1 \bar{h}^{\frac{d}{\alpha}+1}$ with $c_1 = \omega_d \left(8 d C_d \tilde{c} \|f''\|_{L^\infty}\right)^{-1}$. This concludes the proof of the claim.

Let us now consider the cylinders $\mathcal{C}$ defined in (4.7) and

$$\tilde{\mathcal{C}} := B_r(x_l) \times [v_{l+1} - \Delta v, v_{l+1}].$$

By the previous claim and the monotonicity of m with respect to v we have

$$\begin{aligned} \mathcal{L}^{d+1}(\mathcal{C} \cap E_u) - \mathcal{L}^{d+1}(\mathrm{FT}(E_u, a_l) \cap \mathcal{C}) &\ge \mathcal{L}^{d+1}\left(\tilde{\mathcal{C}} \cap E_u\right) - \mathcal{L}^{d+1}(\mathrm{FT}(E_u, a_l) \cap \mathcal{C}) \\ &= \int_0^{\Delta v} m(x_l, v_{l+1} - \Delta v + v) - m'(x_l, v_l + v, a_l) dv \\ &\ge \frac{\bar{h}|B_r|}{2d} \Delta v, \end{aligned}$$

where in the equality we used the definition of m' in (4.12) and that with the choice in (4.11) we have $\mathcal{C} = B_r(x_l) \times [v_l, v_l + \Delta v]$. This concludes the proof of the lemma. □

In the above lemma we showed the existence of a cylinder $\mathcal{C} \subset \mathbb{R}^d \times [0,1]$, where the difference of E_u and $\mathrm{FT}(E_u, \tilde{a})$ is bounded from below in terms of $\bar{h}_r(\bar{x})$. In the next proposition we build a suitable approximation ϕ of the indicator function of the cylinder $\mathcal{C}$ and we use it as a test function in Theorem 3.2 to estimate $\bar{h}_r(\bar{x})$ in terms of the dissipation measure ν.

Proposition 4.4. *Let $\bar{x} \in \mathbb{R}^d$. Then there exist $C_1 = C_1(d,f) > 0$, $C_2 = C_2(d,f,c_1,\tilde{c}) > 0$ and $\gamma = \gamma(d,\alpha) > 0$ such that for every $r > 0$ there exists $r_2 \in [r, C_1 r / \bar{h}_r(\bar{x})^{d/\alpha}]$ for which*

$$\nu(B_{r_2}(\bar{x})) \ge C_2 \bar{h}_r(\bar{x})^\gamma r_2^{d-1}.$$

Proof. Let $a, b > 0$ and $\psi_{a,b} : [0,+\infty) \to [0,1]$ be a smooth function such that

(1) $\psi_{a,b}(t) = 1$ for every $t \in [0,a]$,
(2) $\psi_{a,b}(t) = 0$ for every $t \ge a + b$,

(3) $|\psi'_{a,b}(t)| \le \frac{2}{b}$ for every $t > 0$.

Let $r', v' > 0$ be two parameters that will be fixed later and let $\bar{y}, \tilde{v}, \tilde{a}, \Delta v, \mathcal{C}$ be as in Lemma 4.3. Consider the function $\phi : \mathbb{R}^d_x \times [0, +\infty)_v \to \mathbb{R}$ defined by

$$\phi(x,v) := \psi_{r,r'}(|x-\bar{y}|)\psi_{\Delta v/2, v'}\left(\left|v - \left(\tilde{v} + \frac{\Delta v}{2}\right)\right|\right).$$

Then the following estimate holds:

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,+\infty)} \phi(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\tilde{a})})dxdv &= \int_{\mathcal{C}} \phi(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\tilde{a})})dxdv + \int_{(\operatorname{supp}\phi)\setminus\mathcal{C}} \phi(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\tilde{a})})dxdv \\ &= \mathcal{L}^{d+1}(\mathcal{C}\cap E_u) - \mathcal{L}^{d+1}(\mathcal{C}\cap \mathrm{FT}(E_u,\tilde{a})) + \int_{(\operatorname{supp}\phi)\setminus\mathcal{C}} \phi(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\tilde{a})})dxdv \\ &\ge \frac{\bar{h}|B_r|}{2d}\Delta v - \mathcal{L}^{d+1}\left(((\operatorname{supp}\phi)\setminus\mathcal{C})\cap(\mathbb{R}^d\times[0,1])\right),\end{aligned} \tag{4.14}$$

where in the last inequality we used that u takes values in $[0,1]$, so that $\chi_{E_u} - \chi_{\mathrm{FT}(E_u,\tilde{a})}$ is identically zero in the complement of $\mathbb{R}^d \times [0,1]$. By an elementary geometric consideration and assuming $r' \le r$ we get

$$\begin{aligned}\mathcal{L}^{d+1}\left(((\operatorname{supp}\phi)\setminus\mathcal{C})\cap(\mathbb{R}^d\times[0,1])\right) &\le \omega_d(r+r')^d((\Delta v + 2v')\wedge 1) - \omega_d r^d \Delta v \\ &\le c_d r^{d-1} r' + 2\omega_d r^d v',\end{aligned} \tag{4.15}$$

where c_d is a geometric constant depending only on the dimension d. It follows from (4.14) and (4.15) that under the constraints

$$c_d r^{d-1} r' \le \frac{\bar{h}_r(\bar{x})|B_r|}{8d}\Delta v \qquad \text{and} \qquad 2\omega_d r^d v' \le \frac{\bar{h}_r(\bar{x})|B_r|}{8d}\Delta v \tag{4.16}$$

it holds

$$\int_{\mathbb{R}^d\times[0,+\infty)} \phi(\chi_{E_u} - \chi_{\mathrm{FT}(E_u,a_l)})dxdv \ge \frac{\bar{h}_r(\bar{x})|B_r|}{4d}\Delta v.$$

By (3.3) we deduce

$$\frac{\bar{h}_r(\bar{x})|B_r|}{4d}\Delta v \le \left(\frac{2|\tilde{a}|}{v'} + \frac{\tilde{a}^2}{r'}\|f''\|_{L^\infty}\right)\nu(B_{r+r'+\|f'\|_{L^\infty}|a_l|}(\bar{y})). \tag{4.17}$$

Setting $r_2 := r + r' + \|f'\|_{L^\infty}|a_l| + |\bar{x}-\bar{y}|$, we have $B_{r+r'+\|f'\|_{L^\infty}|a_l|}(\bar{y}) \subset B_{r_2}(\bar{x})$, so that it follows by (4.17) that

$$\frac{\bar{h}_r(\bar{x})|B_r|}{4d}\Delta v \le \left(\frac{2|\tilde{a}|}{v'} + \frac{\tilde{a}^2}{r'}\|f''\|_{L^\infty}\right)\nu(B_{r_2}(\bar{x})). \tag{4.18}$$

By (4.5) and assuming $r' \le r$ we have that

$$r_2 \le \|f'\|_{L^\infty}|a_l| + |\bar{x}-\bar{y}| \le (2 + 4\tilde{c}(d+1)\|f'\|_{L^\infty})\frac{r}{\bar{h}_r(\bar{x})^{d/\alpha}}. \tag{4.19}$$

By Lemma 4.3 we can choose $\Delta v = c_1 \bar{h}_r(\bar{x})^{\frac{d}{\alpha}+1}$ in (4.18); making this choice and plugging the estimate on $|\tilde{a}|$ from (4.5) in the inequality (4.18), we get that for every $r' \le r$ and v' satisfying (4.16) it holds

$$\frac{\bar{h}_r(\bar{x})|B_r|}{4d} c_1 \bar{h}_r(\bar{x})^{\frac{d}{\alpha}+1} \le \left(\frac{8\tilde{c}r}{\bar{h}_r(\bar{x})^{d/\alpha}v'} + \frac{16\tilde{c}^2\|f''\|_{L^\infty}r^2}{\bar{h}_r(\bar{x})^{2d/\alpha}r'}\right)\nu(B_{r_2}(\bar{x})).$$

In particular at least one of the two addends in the right hand side must be bigger than half the left hand side, i.e. at least one of the following inequalities holds:

$$\frac{\omega_d r^{d-1} c_1 v' \bar{h}_r(\bar{x})^{\frac{2d}{\alpha}+2}}{32d\tilde{c}} \le \nu(B_{r_2}(\bar{x})), \qquad \frac{\bar{h}_r(\bar{x})^{\frac{3d}{\alpha}+2}\omega_d r^{d-2} c_1 r'}{64d\tilde{c}^2\|f''\|_{L^\infty}} \le \nu(B_{r_2}(\bar{x})). \tag{4.20}$$

We choose now v' and r' as follows:

$$v' := \frac{\bar{h}_r(\bar{x})\Delta v}{16d} = \frac{c_1\bar{h}_r(\bar{x})^{\frac{d}{\alpha}+2}}{16d} \qquad \text{and} \qquad r' := r \wedge \left(\frac{\bar{h}_r(\bar{x})|B_r|\Delta v}{8dc_d r^{d-1}}\right) = r \wedge \left(\frac{c_1\bar{h}_r(\bar{x})^{\frac{d}{\alpha}+2}\omega_d r}{8dc_d}\right). \tag{4.21}$$

In particular the constraints $r' \le r$ and (4.16) are satisfied. The first inequality in (4.20) reads

$$\frac{\omega_d c_1^2}{256 d^2 \tilde{c}} \bar{h}_r(\bar{x})^{\frac{3d}{\alpha}+4} r^{d-1} \le \nu(B_{r_2}(\bar{x})). \tag{4.22}$$

The second inequality in (4.20) implies

$$\frac{\omega_d c_1}{64 d \tilde{c}^2 \|f''\|_{L^\infty}} \bar{h}_r(\bar{x})^{\frac{3d}{\alpha}+2} r^{d-1} \le \nu(B_{r_2}(\bar{x})) \qquad \text{or} \qquad \frac{\omega_d^2 c_1^2}{512 d^2 c_d \tilde{c}^2 \|f''\|_{L^\infty}} \bar{h}_r(\bar{x})^{\frac{4d}{\alpha}+4} r^{d-1} \le \nu(B_{r_2}(\bar{x})), \tag{4.23}$$

depending on in which terms the minimum in (4.21) is attained. So we have that at least one of the three inequalities from (4.22) and (4.23) holds true. Therefore there exists $\tilde{c}_1 > 0$ depending on $d, c_1, \tilde{c}, f$ such that

$$\tilde{c}_1 \bar{h}_r(\bar{x})^{\frac{4d}{\alpha}+4} r^{d-1} \le \nu(B_{r_2}(\bar{x})). \tag{4.24}$$

The last step is to replace r with r_2 in (4.24). From (4.19) we have that

$$r^{d-1} \ge r_2^{d-1} \bar{h}_r(\bar{x})^{\frac{d(d-1)}{\alpha}} \frac{1}{(2 + 4\tilde{c}(d+1)\|f'\|_{L^\infty})^{d-1}}.$$

Therefore we get from (4.24) that the statement holds true with

$$\gamma = \frac{d(d+3)}{\alpha} + 4 \qquad \text{and} \qquad C_2 = \frac{\tilde{c}_1}{(2 + 4\tilde{c}(d+1)\|f'\|_{L^\infty})^{d-1}}$$

and this concludes the proof. □

In the following corollary we deduce a power decay of $\bar{h}_r(\bar{x})$ from the power decay of $\nu(B_r(\bar{x}))r^{d-1}$.

Corollary 4.5. *Let $\bar{x} \in \mathbb{R}^d$, $R, C > 0$ and $\beta > 0$ be such that $\nu(B_r(\bar{x})) \le C\left(r^{d-1+\beta}\right)$ for $r \in (0, R)$. Then there exists $\gamma' > 0$ such that*

$$\bar{h}_r(\bar{x}) = O\left(r^{\gamma'}\right) \qquad \text{as } r \to 0.$$

Proof. Let

$$\gamma' = \frac{\beta}{\gamma}\left(1 + \frac{d\beta}{\alpha\gamma}\right)^{-1}. \tag{4.25}$$

Assume that $\bar{h}_r(\bar{x}) \ge r^{\gamma'}$. Then by Proposition 4.4 there exists $r_2 \in [r, C_1 \frac{r}{\bar{h}_r^{d/\alpha}(\bar{x})}]$ such that

$$\bar{h}_r(\bar{x}) \le \left(\frac{\nu(B_{r_2}(\bar{x}))}{C_2 r_2^{d-1}}\right)^{\frac{1}{\gamma}} \le \left(\frac{C}{C_2}\right)^{\frac{1}{\gamma}} r_2^{\beta/\gamma} \le \left(\frac{CC_1^\beta}{C_2}\right)^{\frac{1}{\gamma}} \frac{r^{\beta/\gamma}}{\bar{h}_r^{d\beta/\alpha\gamma}(\bar{x})} \le \left(\frac{CC_1^\beta}{C_2}\right)^{\frac{1}{\gamma}} \frac{r^{\beta/\gamma}}{r^{\gamma' d\beta/\alpha\gamma}} = \left(\frac{CC_1^\beta}{C_2}\right)^{\frac{1}{\gamma}} r^{\gamma'},$$

where in the last equality we used the definition of γ' in (4.25). This proves that for sufficiently small r it holds

$$\bar{h}_r(\bar{x}) \le \left(1 \vee \left(\frac{CC_1^\beta}{C_2}\right)^{\frac{1}{\gamma}}\right) r^{\gamma'}.$$

□

In the following lemma we prove that a power decay of $\bar{h}_r(\bar{x})$ as $r \to 0$ is sufficient to establish that $\bar{x}$ is a Lebesgue point of u; on the other hand $\bar{h}_r(\bar{x}) = o(1)$ is not sufficient and indeed it is equivalent to require that $\bar{x}$ has vanishing mean oscillation. This explains the need Assumption 4.1 in our proof.

Lemma 4.6. *Let $\bar{x} \in \mathbb{R}^d$ be a point of vanishing mean oscillation of u such that $\exists \gamma' > 0$ for which*

$$\bar{h}_r(\bar{x}) = O\left(r^{\gamma'}\right) \qquad \text{as } r \to 0.$$

Then $\bar{x}$ is a Lebesgue point of u.

Proof. It is sufficient to prove that there exists $\lim_{r\to 0}(u)_{B_r(\bar{x})}$. In the following of this proof we will not specify the center $\bar{x}$ and we will write u_r for $(u)_{B_r(\bar{x})}$.

We assume the following elementary fact which follows from Fubini theorem: for every $r > 0$ there exists $r' \in [2r, 3r]$ such that

$$|u_{r'} - u_r| \le \bar{h}_r.$$

For $r > 0$ denote by

$$\varepsilon(r) := \fint_{B_r} |u - u_r|.$$

We notice that if $r \in [r'/3, r']$, then $|u_r - u_{r'}| \le 3^d \varepsilon(r')$. In fact

$$\begin{aligned}\varepsilon(r') &= \fint_{B_{r'}} |u - u_{r'}| \\ &\ge \frac{|B_r|}{|B_{r'}|} \fint_{B_r} |u - u_{r'}| \\ &\ge \frac{|B_r|}{|B_{r'}|} |u_r - u_{r'}| \\ &= 3^{-d} |u_r - u_{r'}|.\end{aligned}$$

We prove that $(u)_{B_r}$ is a Cauchy sequence as $r \to 0$. Let $0 < r < R$. If $r > R/3$ then $|u_r - u_R| \le 3^d \varepsilon(R)$, otherwise let $r_1 \in [2r, 3r]$ be such that $|u_r - u_{r_1}| \le \bar{h}_r$. Iterating this argument we have that there exist $n \in \mathbb{N}$ and $r_1, \dots, r_n$ such that $|u_{r_i} - u_{r+1}| \le \bar{h}_{r_i}$, $r_i \in [2^i r, 3^i r]$ and $u_{r_n} \in [R/3, R]$. So we have

$$\begin{aligned}|u_r - u_R| &\le |u_r - u_{r_1}| + \sum_{i=1}^{n-1} |u_{r_i} - u_{r_{i+1}}| + |u_{r_n} - u_R| \\ &\le \bar{h}(r) + \sum_{i=1}^{n-1} \bar{h}(r_i) + \varepsilon(R) \\ &\le C \left(r^{\gamma'} + \sum_{i=1}^{n-1} r_i^{\gamma'} \right) + \varepsilon(R) \le C_{\gamma'} R^{\gamma'} + \varepsilon(R),\end{aligned}$$

which converges to 0 as $R \to 0$ and this proves the lemma. □

Let $J \subset \tilde{J} \subset \mathbb{R}^d$ defined by:

$$J^c := \left\{ x \in \mathbb{R}^d : \lim_{r \to 0} \frac{\nu(B_r(x))}{r^{d-1}} = 0 \right\},$$
$$\tilde{J}^c := \left\{ x \in \mathbb{R}^d : \limsup_{r \to 0} \frac{\nu(B_r(x))}{r^{d-1+\alpha}} < +\infty \text{ for some } \alpha > 0 \right\}.$$

Let moreover $R \subset \mathbb{R}^d$ the set of Lebesgue points of u. In [DLOW03] is proved that every point in J^c is a vanishing mean oscillation point of u. Therefore it immediately follows by Corollary 4.5 and Lemma 4.6 that each point in $\tilde{J}^c$ is a Lebesgue point of u so that

$$\tilde{J}^c \subset R \subset J^c. \tag{4.26}$$

We state the main result in the following theorem. At this point the argument of the proof is the same as in [LO18]. We sketch it here for completeness.

Theorem 4.7. *Let f be a flux satisfying Assumption 4.1 and let u be a quasi-solution as in Definition 2.1. Then the set R^c of non Lebesgue points of u has Hausdorff dimension at most $d-1$.*

Proof. In view of (4.26) it is sufficient to check we check that $\tilde{J}$ has Hausdorff dimension at most $d-1$: let $\alpha, K, R > 0$ and set

$$E_{\alpha,K,R} := \left\{ x \in B_R \subset \mathbb{R}^d : \nu(B_r(x)) \le K r^{d-1+\alpha} \, \forall r \in (0,1) \right\}.$$

By Vitali covering theorem it follows that $\mathscr{H}^{d-1+\alpha}(B_R \setminus E_{\alpha,K,R}) \lesssim K^{-1} \nu(B_{R+1})$. Therefore setting $E_{\alpha,K} := \bigcup_{R>0} E_{\alpha,K,R}$ we have that $E^c_{\alpha,K}$ has Hausdorff dimension at most $d-1+\alpha$. Being

$$\tilde{J} = \bigcap_{\alpha, K > 0} E^c_{\alpha,K}$$

it has Hausdorff dimension at most $d-1$. □

In the last part of this section we notice with a simple example that the inclusion $R \subset J^c$ can be strict: in particular we provide a quasi-solution to the Burgers equation (4.27) on $\mathbb{R}^2$ for which the origin does not belong to J and it is not a Lebesgue point of u:

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0. \tag{4.27}$$

This shows that Property (1') in the introduction is not true in general, the condition $\mathscr{H}^{d-1}(R^c \setminus J) = 0$ would be satisfactory as well, but we cannot prove it or disprove it here.

Example. Let $u_0 \in L^\infty(\mathbb{R})$ be such that 0 is a vanishing mean oscillation point of u_0 but not a Lebesgue point: consider for example $u_0(x) = \sin(\log|\log|x||)$ and let $u_1 : [0,+\infty) \times \mathbb{R}$ be the entropy solution to the Cauchy problem for (4.27) with initial datum u_0. Moreover let $u_2 : [0,+\infty) \times \mathbb{R}$ be the entropy solution of the Cauchy problem

$$\begin{cases} \partial_t u - \partial_x \left(\frac{u^2}{2}\right) = 0, \\ u(0,\cdot) = u_0 \end{cases} \tag{4.28}$$

and set

$$u(t,x) = \begin{cases} u_1(t,x) & \text{if } t \geq 0; \\ u_2(-t,x) & \text{if } t < 0. \end{cases}$$

Being $u_1, u_2 \in C([0,+\infty); L^1_{\text{loc}}(\mathbb{R}))$ it is straightforward to check that u is a quasi solution on the whole $\mathbb{R}^2$. We now check that the origin does not belong to J and that it is not a Lebesgue point in the two variables (t,x). Let us denote by

$$(u_0)_r := \frac{1}{2r}\int_{-r}^{r} u_0(x)dx, \qquad \varepsilon(u_0,r) := \frac{1}{2r}\int_{-r}^{r} |u_0(x) - (u_0)_r| dx.$$

By Kruzkov contraction estimate in $L^1(\mathbb{R})$ we have that for any $r > 0$ and any $t \in [0,2r]$ it holds

$$\frac{1}{4r}\int_{-2r}^{2r} |u(t,x) - (u_0)_{4r}| dx \leq \frac{1}{4r}\int_{-4r}^{4r} |u_0(x) - (u_0)_{4r}| dx = 2\varepsilon(u_0,4r), \tag{4.29}$$

so that integrating for $t \in [0,2r]$ we get

$$\int_0^{2r}\int_{-2r}^{2r} |u(t,x) - (u_0)_{4r}| dxdt \leq 16\varepsilon(u_0,4r)r^2. \tag{4.30}$$

Since $(u_0)_r$ is not converging as $r \to 0$ and $\varepsilon(u_0,r) \to 0$ as $r \to 0$ this proves that the origin is not a Lebesgue point of u. Moreover it follows from (4.30) by Fubini theorem that there exist $s_1 \in [-2r,-r]$, $s_2 \in [r,2r]$ such that

$$\int_0^{2r} |u(t,s_1) - (u_0)_{4r}| dt \leq 16\varepsilon(u_0,4r)r, \qquad \int_0^{2r} |u(t,s_2) - (u_0)_{4r}| dt \leq 16\varepsilon(u_0,4r)r. \tag{4.31}$$

Computing the balance for the entropy $\bar\eta(u) = u^2/2$ on the domain $D := (0,2r) \times (s_1,s_2)$ we get by (4.29) and (4.31)

$$\begin{aligned} |\mu_{\bar\eta}(D)| \leq & \int_{s_1}^{s_2} |\bar\eta(u(2r,x)) - \bar\eta(u_0(x))| dx + \int_0^{2r} |\bar q(u(t,s_2)) - \bar q(u(t,s_1))| dt \\ \leq & \int_{s_1}^{s_2} \left(|\bar\eta(u(2r,x)) - \bar\eta((u_0)_{4r})| + |\bar\eta((u_0)_{4r}) - \bar\eta(u_0(x))|\right) dx \\ & + \int_0^{2r} \left(|\bar q(u(t,s_2)) - \bar q((u_0)_{4r})| + |\bar q((u_0)_{4r}) - \bar q(u(t,s_1))|\right) dt \\ \leq & \ 48\varepsilon(u_0,4r)r, \end{aligned}$$

being $\bar\eta$ and $\bar q(u) = u^3/3$ both 1-Lipschitz functions on $[-1,1]$. The same computation holds for $t < 0$ and since for entropy solutions to Burgers equation and to (4.28) it holds $-\mu_{\bar\eta} = |\mu_{\bar\eta}| = \nu$ this proves that $\nu(B_r(0)) = o(r)$, i.e. $0 \notin J$.

5. Lagrangian representation for the time dependent case

In this section we consider the Cauchy problem for the scalar conservation law:

$$\begin{cases} u_t + \mathrm{div}_x f(u) = 0, \\ u(0,\cdot) = u_0, \end{cases} \tag{5.1}$$

with $u : [0,T)\times\mathbb{R}^d \to \mathbb{R}$ a measurable function for some $T > 0$, $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $f \in C^2(\mathbb{R},\mathbb{R}^d)$.

Definition 5.1. We say that $u \in C([0,T]; L^1(\mathbb{R}^d))$ is a *weak solution with finite entropy production* if it solves (5.1) in the sense of distributions and for every entropy $\eta \in C^2(\mathbb{R})$ such that $\eta''(v) \geq 0$ and corresponding flux $q : \mathbb{R} \to \mathbb{R}^d$ satisfying $q' = \eta' f'$ the distribution

$$\mu_\eta := \eta(u)_t + \mathrm{div}_x q(u) \tag{5.2}$$

is a finite Radon measure in $[0,T] \times \mathbb{R}^d$.

Remark 5.2. We notice that the existence of an L^1 continuous representative in time of a weak solution u with finite entropy production can be deduced from (5.2) under the assumption of genuine nonlinearity of the flux f (see [Daf16]).

In order to keep the presentation simpler we restrict our attention to the following class of solutions.

Assumption 5.3. The weak solution $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ to (5.1) is bounded, nonnegative and $\|u_0\|_{L^1(\mathbb{R}^d)} = 1$.

In this context the kinetic formulation has the following form:

Proposition 5.4. *Let u be a weak solution with finite entropy production and let $\chi : [0,T) \times \mathbb{R}^d \times [0,+\infty) \to \{0,1\}$ be*

$$\chi(t,x,v) := \begin{cases} 1 & \text{if } 0 < v \leq u(t,x), \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Then there exists a finite Radon measure $\mu \in \mathcal{M}([0,T) \times \mathbb{R}^d \times \mathbb{R})$ such that

$$\partial_t \chi + f'(v) \cdot \nabla_x \chi = \partial_v \mu \quad \text{in } \mathcal{D}'_{t,x,v}. \tag{5.4}$$

We denote by ν the projection on $[0,T) \times \mathbb{R}^d$ of the total variation $|\mu|$ of μ and we notice that the L^1 continuity in time of u implies that $(\pi_t)_\sharp \nu \in \mathcal{M}([0,T))$ has no atoms.

In order to introduce the notion of Lagrangian representation we set some notation: we denote by

$$\tilde{\Gamma} := \mathrm{BV}\left([0,T); \mathbb{R}^d \times [0,+\infty)\right),$$
$$\Gamma := \left\{\gamma = (\gamma^1,\gamma^2) \in \tilde{\Gamma} : \gamma^1 \text{ is Lipschitz}\right\}.$$

In order to fix a representative we will also assume that γ is continuous from the right. We will consider on Γ and $\tilde{\Gamma}$ the topology τ induced by the distance

$$d_\tau(\gamma,\bar{\gamma}) := \sup_{t\in[0,T)} \left|\gamma^1(t) - \bar{\gamma}^1(t)\right| + \int_0^T \left|\gamma^2(t) - \bar{\gamma}^2(t)\right| dt.$$

For every $t \in [0,T)$ let $e_t : \tilde{\Gamma} \to \mathbb{R}^d \times [0,+\infty)$ be the evaluation map:

$$e_t(\gamma) := \lim_{s\to t^+} \gamma(s) = \gamma(t).$$

Definition 5.5. Let u be a weak solution to (5.1) with finite entropy production satisfying Assumption 5.3. We say that $\omega \in \mathcal{M}(\Gamma)$ is a *Lagrangian representation* of u if the following conditions hold:

(1) for every $t \in [0,T)$ it holds

$$(e_t)_\sharp \omega = \mathscr{L}^{d+1} \llcorner E_{u(t)}, \tag{5.5}$$

where $E_{u(t)} := \{(x,v) \in \mathbb{R}^d \times [0,+\infty) : v \leq u(t,x)\}$.

(2) the measure ω is concentrated on the set of curves $\gamma \in \Gamma$ such that

$$\dot{\gamma}^1(t) = f'(\gamma^2(t)) \quad \text{for a.e. } t \in [0,T); \tag{5.6}$$

(3)

$$\int_{\Gamma} \mathrm{Tot.Var.}_{[0,T)} \gamma^2 d\omega(\gamma) < \infty. \tag{5.7}$$

A few comments are in order: the condition (5.5) encodes the link between the measure ω and the weak solution u, while (5.6) says that the mass is transported with the characteristic speed. Finally (5.7) is just a regularity requirement and it is related to the finiteness of the entropy production. This connection will be made more explicit in the propositions 5.11 and 5.12.

With the same notation as in Section 3 we can state in this setting the analogous of Theorem 3.2, exploiting the special role of the variable t and the conservation of $\|u(t)\|_{L^1(\mathbb{R}^d)}$. Notice that in this case the variable t plays the role of the artificial variable s in the proof of Theorem 3.2.

Proposition 5.6. *Let u be a weak solution to (5.1) with finite entropy production satisfying Assumption 5.3. Let moreover $\bar s > 0$ and $\phi \in C^1_c(\mathbb{R}^d \times (0,+\infty))$ be such that $\pi_x(\operatorname{supp}\phi) \subset B_R(\bar x)$ for some $\bar x \in \mathbb{R}^d$ and $R > 0$. Then for every $\bar t > 0$ it holds*

$$\int_{\mathbb{R}^d\times[0,+\infty)} \phi(x,v)(\chi_{E_{u(\bar t+\bar s)}} - \chi_{\mathrm{FT}(E_{u(\bar t)},\bar s)})dxdv \le (\|\partial_v\phi\|_{L^\infty} + \bar s\|f''\|_{L^\infty}\|\nabla_x\phi\|_{L^\infty})\,\nu((\bar t,\bar t+\bar s)\times B_{R+\|f'\|_\infty \bar s}(\bar x)), \tag{5.8}$$

where $\mathrm{FT}(E_{u(\bar t)},\bar s) := \left\{(x,v) \in \mathbb{R}^d \times [0,+\infty) : u(\bar t, x - f'(v)\bar s) \ge v\right\}$.

Proof. The proof is a slight modification of the proof of Theorem 3.2, we report the main steps here for convenience of the reader. For every $t \in \mathbb{R}$, let

$$\chi^1(t,x,v) := \chi_{\mathrm{FT}(E_{u(\bar t)},t-\bar t)},$$

so that, by a straightforward adaptation of Lemma 3.1, χ^1 solves

$$\partial_t\chi^1 + f'(v)\cdot\nabla_x\chi^1 = 0, \qquad \text{in } \mathcal{D}'\left(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\right). \tag{5.9}$$

Moreover let $\chi^2 := \chi$ as in (5.3) so that it follows from (5.4) and (5.9) that $\tilde\chi := \chi^2 - \chi^1$ solves

$$\partial_t\tilde\chi + f'(v)\cdot\nabla_x\tilde\chi = \partial_v\mu, \qquad \text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^d\times\mathbb{R}\right). \tag{5.10}$$

Given ϕ as in the statement, we set $\psi(t,x,v) := \phi(x - f'(v)(t-\bar t), v)$. Then it follows from (5.10) that

$$\partial_t(\tilde\chi\psi) + f'(v)\cdot\nabla_x(\tilde\chi\psi) = \psi\partial_v\mu, \qquad \text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^d\times\mathbb{R}\right). \tag{5.11}$$

For every $t \in [0,T)$ we set

$$g(t) := \int_{\mathbb{R}^d\times[0,+\infty)} \tilde\chi(t)\psi(t)dxdv,$$

and we observe that

$$g(\bar t) = 0 \qquad \text{and} \qquad g(\bar t+\bar s) = \int_{\mathbb{R}^d\times[0,+\infty)} \phi(x,v)(\chi_{E_{u(\bar t+\bar s)}} - \chi_{\mathrm{FT}(E_{u(\bar t)},\bar s)})dxdv. \tag{5.12}$$

By (5.11) and (5.12) we have that

$$\begin{aligned} g(\bar t+\bar s) &= -\int_{(\bar t,\bar t+\bar s)\times\mathbb{R}^d\times[0,+\infty)} \partial_v\psi d\mu \\ &= -\int_{(\bar t,\bar t+\bar s)\times\mathbb{R}^d\times[0,+\infty)} \big(\partial_v\phi(x-f'(v)(t-\bar t),v) - f''(v)\cdot\nabla_x\phi(x-f'(v)(t-\bar t),v)\big)d\mu \\ &\le (\|\partial_v\phi\|_{L^\infty} + \bar s\|f''\|_{L^\infty}\|\nabla_x\phi\|_{L^\infty})\,|\mu|\left((\bar t,\bar t+\bar s)\times\mathbb{R}^d\times[0,+\infty)\right), \end{aligned}$$

which, by definition of ν, is what we wanted to prove. □

We are going to consider (5.8) for small $\bar s$. In view of our interpretation through Proposition 2.5 the additional factor $\bar s$ in the second term of the right hand side corresponds to a different behavior of the horizontal and vertical displacements. This is why we are going to consider anisotropic distances.

Let $L > 0$; we denote by

$$d_L((x_1,v_1),(x_2,v_2)) := L|x_1-x_2| + |v_1-v_2|.$$

We set $X = \mathbb{R}^d \times [0,+\infty)$ and we denote by W^L_1 the Wasserstein distance on $\mathcal{P}((X,d_L))$.

Corollary 5.7. *Let u be a weak solution to (5.1) with finite entropy production satisfying Assumption 5.3. Let moreover $L>0$ and $t,\bar{s}\geq 0$ be such that*
$$\bar{s}\leq \frac{1}{\|f''\|_{L^\infty}L^2}. \tag{5.13}$$
Then there exists $T=(T^1,T^2):\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}^d\times[0,+\infty)$ such that
$$\begin{aligned}&T_\sharp\left(\mathscr{L}^{d+1}\llcorner \mathrm{FT}(E_{u(t)},\bar{s})\right)=\mathscr{L}^{d+1}\llcorner E_{u(t+\bar{s})},\\ &\int_{\mathrm{FT}(E_{u(t)},\bar{s})}\left(L|T^1(x,v)-x|+|T^2(x,v)-v|\right)dxdv\leq\left(1+\frac{1}{L}\right)\nu\left((t,t+\bar{s})\times\mathbb{R}^d\right).\end{aligned}\tag{5.14}$$

Proof. Let $\phi\in C^1(\mathbb{R}^d\times(0,+\infty))$ be 1-Lipschitz with respect to d_L. This is equivalent to require that $\|\nabla_x\phi\|_{L^\infty}\leq L$ and $\|\partial_v\phi\|_{L^\infty}\leq 1$. From (5.8) it follows that
$$\sup_{\mathrm{Lip}_{d_L}(\phi)\leq 1}\int_{\mathbb{R}^d\times[0,+\infty)}\phi(x,v)(\chi_{E_{u(t+\bar{s})}}-\chi_{\mathrm{FT}(E_{u(t)},\bar{s})})dxdv\leq(1+\bar{s}L\|f''\|_{L^\infty})\,\nu\left((t,t+\bar{s})\times\mathbb{R}^d\right).$$
By Proposition 2.5 and (5.13) it follows that
$$W_1^L(\mathrm{FT}(E_{u(t)},\bar{s}),E_{u(t+\bar{s})})\leq\left(1+\frac{1}{L}\right)\nu\left((t,t+\bar{s})\times\mathbb{R}^d\right).$$
The conclusion follows from Theorem 2.6. □

5.1. **Approximation scheme.** In this part we build an approximate Lagrangian representation by means of the free transport operator and Corollary 5.7.

Given $T>0$ and $n\in\mathbb{N}$ we set $\bar{s}_n=2^{-n}T$ and $L_n=(\bar{s}_n\|f''\|_{L^\infty})^{-1/2}$ so that $\bar{s}_n$ and L_n satisfy (5.13). For every $k=1,\dots,2^n-1$ let T_k be an optimal transport map from $\mathscr{L}^{d+1}\llcorner\mathrm{FT}(E_{u((k-1)\bar{s}_n)},\bar{s}_n)$ to $\mathscr{L}^{d+1}\llcorner E_{u(k\bar{s}_n)}$ given by Corollary 5.7. For every $(x,v)\in E_{u_0}$ we build a trajectory $\gamma_{(x,v)}=(\gamma^1_{(x,v)},\gamma^2_{(x,v)}):[0,T)\to\mathbb{R}^d\times[0,+\infty)$. First we define inductively $\gamma_{(x,v)}(k\bar{s}_n)$ for $k=0,\dots,2^n-1$. We set
$$\begin{cases}\gamma_{(x,v)}(0)=(x,v),\\ \gamma_{(x,v)}(k\bar{s}_n)=T_k\Big(\gamma_{(x,v)}\big((k-1)\bar{s}_n\big)+\big(\bar{s}_nf'(\gamma^2_{(x,v)}((k-1)\bar{s}_n)),0\big)\Big)\quad\text{for } k=1,\dots,2^n-1.\end{cases}\tag{5.15}$$
Next we set for $t\in(k\bar{s}_n,(k+1)\bar{s}_n)$ and $k=0,\dots,2^n-1$
$$\gamma_{(x,v)}(t)=\gamma_{(x,v)}(k\bar{s}_n)+\Big((t-k\bar{s}_n)f'\big(\gamma^2_{(x,v)}(k\bar{s}_n)\big),0\Big).\tag{5.16}$$
We now define $\omega_n\in\mathcal{P}(\tilde{\Gamma})$ by
$$\omega_n:=\int_{E_{u_0}}\delta_{\gamma_{(x,v)}}dxdv,\tag{5.17}$$
where $\gamma_{(x,v)}$ is defined by (5.15) and (5.16).

By construction we have the following properties:

(1) for every $k=1,\dots,2^n$ it holds
$$\mathscr{L}^{d+1}\llcorner\mathrm{FT}(E_{u((k-1)\bar{s}_n)},\bar{s}_n)=(e_{k\bar{s}_n-})_\sharp\,\omega_n,$$
where we denoted by $e_{t-}:\tilde{\Gamma}\to\mathbb{R}^d\times[0,+\infty)$ the map defined by $e_{t-}(\gamma)=\lim_{s\to t-}\gamma(s)$;

(2) for every $k=0,\dots,2^n-1$ it holds
$$\mathscr{L}^{d+1}\llcorner E_{u(k\bar{s}_n)}=(e_{k\bar{s}_n})_\sharp\,\omega_n.$$

In particular T_k is an optimal transport map between $(e_{k\bar{s}_n-})_\sharp\,\omega_n$ and $(e_{k\bar{s}_n})_\sharp\,\omega_n$ for every $k=1,\dots,2^n-1$.

In the following lemma we prove some uniform integral estimates on the trajectories $\gamma_{(x,v)}$ that will be useful to establish the tightness of the measures ω_n.

Lemma 5.8. *Let ω_n be defined in (5.17). Then the following estimates hold:*

(1)
$$\int_{\tilde{\Gamma}}\sup_{t\in[0,T)}\left|\gamma^1(t)-\gamma^1(0)-\int_0^tf'(\gamma^2(s))ds\right|d\omega_n(\gamma)\leq\frac{1}{L_n}\left(1+\frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right).\tag{5.18}$$

(2)

$$\int_{\tilde{\Gamma}} \text{Tot.Var.}_{[0,T)}\gamma^2 d\omega_n(\gamma) \leq \left(1+\frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right). \tag{5.19}$$

Proof. For every $t\in[0,T)$ and $(x,v)\in E_{u_0}$, by the construction of the curves $\gamma_{(x,v)}$, it holds

$$\begin{aligned}\left|\gamma^1_{(x,v)}(t)-\gamma^1_{(x,v)}(0)-\int_0^t f'(\gamma^2_{(x,v)}(s))ds\right| &\leq \sum_{k=1}^{2^n-1}\left|\gamma^1_{(x,v)}(k\bar{s}_n)-\gamma^1_{(x,v)}(k\bar{s}_n-)\right| \\ &= \sum_{k=1}^{2^n-1}\left|T^1_k(\gamma_{(x,v)}(k\bar{s}_n-))-\gamma^1_{(x,v)}(k\bar{s}_n-)\right|,\end{aligned}$$

where we denoted by $\gamma^1_{(x,v)}(k\bar{s}_n-):=\lim_{s\to k\bar{s}_n-}\gamma^1_{(x,v)}(s)$. Therefore, from the definition of ω_n, recalling that T_k is an optimal transport map between $(e_{k\bar{s}_n-})_\sharp\,\omega_n$ and $(e_{k\bar{s}_n})_\sharp\,\omega_n$ for every $k=1,\dots,2^n-1$, and by the estimate (5.14) it holds

$$\begin{aligned}\int_{\tilde{\Gamma}}\sup_{t\in[0,T)}&\left|\gamma^1(t)-\gamma^1(0)-\int_0^t f'(\gamma^2(s))ds\right|d\omega_n(\gamma)= \\ &=\int_{E_{u_0}}\sup_{t\in[0,T)}\left|\gamma^1_{(x,v)}(t)-\gamma^1_{(x,v)}(0)-\int_0^t f'(\gamma^2_{(x,v)}(s))ds\right|dxdv \\ &\leq\sum_{k=1}^{2^n-1}\int_{E_{u_0}}\left|T^1_k(\gamma_{(x,v)}(k\bar{s}_n-))-\gamma^1_{(x,v)}(k\bar{s}_n-)\right|dxdv \\ &\leq\sum_{k=1}^{2^n-1}\frac{1}{L_n}\left(1+\frac{1}{L_n}\right)\nu\left(((k-1)\bar{s}_n,k\bar{s}_n)\times\mathbb{R}^d\right) \\ &\leq\frac{1}{L_n}\left(1+\frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right),\end{aligned}\tag{5.20}$$

where in the last inequality we used that

$$\sum_{k=1}^{2^n-1}\nu\left(((k-1)\bar{s}_n,k\bar{s}_n)\times\mathbb{R}^d\right)=\nu\left(\bigcup_{k=1}^{2^n-1}((k-1)\bar{s}_n,k\bar{s}_n)\times\mathbb{R}^d\right)\leq\nu\left((0,T)\times\mathbb{R}^d\right).$$

This proves (5.18); we now use a similar argument to prove (5.19): for every $(x,v)\in E_{u_0}$, by construction of the curves $\gamma_{(x,v)}$, it holds

$$\begin{aligned}\text{Tot.Var.}_{[0,T)}\gamma^2_{(x,v)} &= \sum_{k=1}^{2^n-1}\left|\gamma^2_{(x,v)}(k\bar{s}_n)-\gamma^2_{(x,v)}(k\bar{s}_n-)\right| \\ &= \sum_{k=1}^{2^n-1}\left|T^2_k(\gamma_{(x,v)}(k\bar{s}_n-))-\gamma^2_{(x,v)}(k\bar{s}_n-)\right|.\end{aligned}$$

Therefore, from the definition of ω_n, recalling that T_k is an optimal transport map between $(e_{k\bar{s}_n-})_\sharp\,\omega_n$ and $(e_{k\bar{s}_n})_\sharp\,\omega_n$ for every $k=1,\dots,2^n-1$, and by the estimate (5.14) it holds

$$\begin{aligned}\int_{\tilde{\Gamma}}\text{Tot.Var.}_{[0,T)}\gamma^2 d\omega_n(\gamma) &= \int_{E_{u_0}}\text{Tot.Var.}_{[0,T)}\gamma^2_{(x,v)}dxdv \\ &= \sum_{k=1}^{2^n-1}\int_{E_{u_0}}\left|T^2_k(\gamma_{(x,v)}(k\bar{s}_n-))-\gamma^2_{(x,v)}(k\bar{s}_n-)\right|dxdv \\ &\leq \sum_{k=1}^{2^n-1}\left(1+\frac{1}{L_n}\right)\nu\left(((k-1)\bar{s}_n,k\bar{s}_n)\times\mathbb{R}^d\right) \\ &\leq \left(1+\frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right).\end{aligned}\tag{5.21}$$

This completes the proof of (5.19) and of the lemma. □

Lemma 5.9. *For every $0 \leq s \leq t < T$ it holds*

$$W_1((e_t)_\sharp\omega_n,(e_s)_\sharp\omega_n) \leq \|f'\|_{L^\infty}|t-s| + 2\nu\left(\left(s-\frac{T}{2^n},t\right)\times\mathbb{R}^d\right).$$

Proof. Let $k_s, k_t \in [0, 2^n-1]\cap\mathbb{Z}$ be such that

$$k_s\bar{s}_n \leq s < (k_s+1)\bar{s}_n \qquad \text{and} \qquad k_t\bar{s}_n \leq t < (k_t+1)\bar{s}_n.$$

If $k_s = k_t$ then we have

$$(e_s)_\sharp\omega_n = \mathscr{L}^{d+1}\llcorner \mathrm{FT}(E_{u(k_s\bar{s}_n)}, s-k_s\bar{s}_n)$$

and

$$\begin{aligned}(e_t)_\sharp\omega_n &= \mathscr{L}^{d+1}\llcorner \mathrm{FT}(E_{u(k_s\bar{s}_n)}, t-k_s\bar{s}_n)\\ &= \mathscr{L}^{d+1}\llcorner \mathrm{FT}(\mathrm{FT}(E_{u(k_s\bar{s}_n)}, s-k_s\bar{s}_n), t-s).\end{aligned}$$

Therefore the map $T:\mathbb{R}^d\times[0,+\infty)\to\mathbb{R}^d\times[0,+\infty)$ defined by

$$T(x,v) = \big(x+f'(v)(t-s), v\big)$$

satisfies the constraint $T_\sharp((e_s)_\sharp\omega_n) = (e_t)_\sharp\omega_n$ and this proves that

$$\begin{aligned}W_1((e_t)_\sharp\omega_n,(e_s)_\sharp\omega_n) &\leq \int_{\mathbb{R}^d\times[0,+\infty)} |T(x,v)-(x,v)|d(e_s)_\sharp\omega_n\\ &= \int_{\mathbb{R}^d\times[0,+\infty)} |f'(v)(t-s)|d(e_s)_\sharp\omega_n\\ &\leq \|f'\|_{L^\infty}|t-s|\omega_n\left(\tilde{\Gamma}\right)\\ &= \|f'\|_{L^\infty}|t-s|.\end{aligned}$$

Otherwise it holds $k_s < k_t$ and we estimate by the triangular inequality

$$\begin{aligned}W_1((e_t)_\sharp\omega_n,(e_s)_\sharp\omega_n) \leq{}& W_1\left((e_s)_\sharp\omega_n, \lim_{r\to(k_s+1)\bar{s}_n^-}(e_r)_\sharp\omega_n\right) + \sum_{k=k_s+1}^{k_t} W_1\left(\lim_{r\to k\bar{s}_n^-}(e_r)_\sharp\omega_n,(e_{k\bar{s}_n})_\sharp\omega_n\right)\\ &+\sum_{k=k_s+1}^{k_t-1} W_1\left((e_{k\bar{s}_n})_\sharp\omega_n, \lim_{r\to(k+1)\bar{s}_n^-}(e_r)_\sharp\omega_n\right) + W_1\left((e_{k_t\bar{s}_n})_\sharp\omega_n,(e_t)_\sharp\omega_n\right)\\ \leq{}& \|f'\|_{L^\infty}((k_s+1)\bar{s}_n - s) + \sum_{k=k_s+1}^{k_t} W_1\left(\lim_{r\to k\bar{s}_n^-}(e_r)_\sharp\omega_n,(e_{k\bar{s}_n})_\sharp\omega_n\right)\\ &+(k_t-k_s-1)\|f'\|_{L^\infty}\bar{s}_n + \|f'\|_{L^\infty}(t-k_t\bar{s}_n),\end{aligned}$$

where the second inequality easily follows by the definition of ω_n and by the case $k_s = k_t$. Assuming $L_n \geq 1$, which trivially holds for n large enough, we get from (5.14) that for every $k = k_s+1,\dots,k_t$,

$$\begin{aligned}W_1\left(\lim_{r\to k\bar{s}_n^-}(e_r)_\sharp\omega_n,(e_{k\bar{s}_n})_\sharp\omega_n\right) &= \int_{\mathrm{FT}(E_{u((k-1)\bar{s}_n)},\bar{s}_n)} |T_k(x,v)-(x,v)|dxdv\\ &\leq 2\nu\left(((k-1)\bar{s}_n, k\bar{s}_n)\times\mathbb{R}^d\right).\end{aligned}$$

Finally we have

$$W_1((e_t)_\sharp\omega_n,(e_s)_\sharp\omega_n) \leq \|f'\|_{L^\infty}|t-s| + 2\nu\left((k_s\bar{s}_n,k_t\bar{s}_n)\times\mathbb{R}^d\right).$$

The conclusion follows since

$$(k_s\bar{s}_n,k_t\bar{s}_n)\subset\left(s-\frac{T}{2^n},t\right).$$ □

The following theorem is the main result of this section.

Theorem 5.10. *The sequence $\omega_n \in \mathcal{P}(\tilde{\Gamma})$ defined in (5.17) is tight and any of its limit points is a Lagrangian representation of u. In particular any weak solution to (5.1) with finite entropy production satisfying Assumption 5.3 admits a Lagrangian representation.*

Proof. Step 1. The sequence $(\omega_n)_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\tilde\Gamma)$. For every $n \in \mathbb{N}$ and $M, R > 0$ we denote by $\tilde\Gamma_{n,M,R} \subset \tilde\Gamma$ the set of curves γ such that the following conditions hold:

(1) $\gamma(0) \in B_R(0) \times [0, \|u\|_{L\infty}]$;
(2) for every $k = 1, \dots, 2^n - 1$

$$\mathrm{Lip}(\gamma\llcorner[(k-1)2^{-n}T, k2^{-n}T)) \le \|f'\|_{L^\infty};$$

(3) $\mathrm{Tot.Var.}_{[0,T)}\gamma^2 \le M$;
(4)

$$\sum_{k=1}^{2^n-1} |\gamma^1(2^{-n}Tk) - \gamma^1(2^{-n}Tk-)| \le M2^{-n/2}. \tag{5.22}$$

Let moreover

$$\Gamma_{M,R} := \left\{\gamma \in \Gamma : \gamma(0) \in B_R \times [0, \|u\|_\infty], \mathrm{Lip}(\gamma^1) \le \|f'\|_{L^\infty}, \mathrm{Tot.Var.}\gamma^2 \le M\right\}.$$

For every $M, R > 0$ we claim that the set

$$\tilde\Gamma_{M,R} := \Gamma_{M,R} \cup \bigcup_{n=1}^{\infty} \tilde\Gamma_{n,M,R}$$

is compact in $\tilde\Gamma$ with the topology τ, namely $\left\{\gamma^2 : \gamma = (\gamma^1,\gamma^2) \in \tilde\Gamma_{M,R}\right\}$ is compact in $L^1([0,T);[0,+\infty))$ and $\left\{\gamma^1 : \gamma \in \tilde\Gamma_{M,R}\right\}$ is compact with respect to the sup norm in $\mathrm{BV}([0,T);\mathbb{R}^d)$. The compactness of $\left\{\gamma^2 : \gamma \in \tilde\Gamma_{M,R}\right\}$ follows from the uniform bound on the total variation

$$\mathrm{Tot.Var.}\gamma^2 \le M \qquad \text{for every } \gamma \in \tilde\Gamma_{M,R}.$$

By the uniform bound on the Lipschitz constant of γ^1 in every interval of the form $(k2^{-n}T, (k+1)2^{-n}T)$ we deduce the compactness of $\left\{\gamma^1 : \gamma \in \tilde\Gamma_{n,M,R}\right\}$ for every $n \in \mathbb{N}$ and the compactness of $\Gamma_{M,R}$. In order to prove the compactness of $\left\{\gamma^1 : \gamma \in \tilde\Gamma_{M,R}\right\}$ we notice that if γ_k is a sequence with $\gamma_k \in \Gamma_{n,M,R}$ for some $n = n(k) \to \infty$ as $k \to \infty$, then it follows from (5.22) that γ^1_k has a subsequence converging to an element of $\Gamma_{M,R}$.

We now prove that for every $\varepsilon > 0$ there exist $M, R > 0$ such that for every $n \in \mathbb{N}$ it holds $\omega_n(\tilde\Gamma^c_{M,R}) \le \varepsilon$.

By construction ω_n is concentrated on

$$\bigcup_{M,R>0} \tilde\Gamma_{n,M,R},$$

therefore in order to prove the claim it is sufficient to show that there exists $M, R > 0$ such that for every $n \in \mathbb{N}$ it holds $\omega_n(\tilde\Gamma^c_{n,M,R}) \le \varepsilon$. We choose R such that

$$\int_{B_R(0)^c} |u_0|(x)dx < \frac{\varepsilon}{3}. \tag{5.23}$$

Since $(e_0)_\sharp\omega_n = \mathscr{L}^{d+1}\llcorner E_{u_0}$, then

$$\omega_n\left(\left\{\gamma \in \tilde\Gamma : \gamma^1(0) \notin B_R(0)\right\}\right) < \frac{\varepsilon}{3}. \tag{5.24}$$

By Chebychev inequality and (5.19) it follows

$$\omega_n\left(\left\{\gamma \in \tilde\Gamma : \mathrm{Tot.Var.}\gamma^2 > M\right\}\right) \le \frac{1}{M}\left(1 + \frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right).$$

By definition we have $L_n = 2^{n/2}\left(T\|f''\|_{L^\infty}\right)^{-1/2} \ge L_1$, therefore for every $n \in \mathbb{N}$ and

$$M \ge \frac{1}{3\varepsilon}\left(1 + \frac{1}{L_1}\right)\nu((0,T)\times\mathbb{R}^d) =: M_1$$

it holds

$$\omega_n\left(\left\{\gamma \in \tilde\Gamma : \mathrm{Tot.Var.}\gamma^2 > M\right\}\right) \le \frac{\varepsilon}{3}. \tag{5.25}$$

Similarly by (5.18) it follows

$$\omega_n\left(\left\{\gamma\in\tilde\Gamma:\sum_{k=1}^{2^n-1}|\gamma^1(2^{-n}Tk)-\gamma^1(2^{-n}Tk-)|>M2^{-n/2}\right\}\right)\le\frac{2^{n/2}}{ML_n}\left(1+\frac{1}{L_n}\right)\nu\left((0,T)\times\mathbb{R}^d\right).$$

Therefore for every $n\in\mathbb{N}$ and every

$$M\ge\frac{(T\|f''\|_{L^\infty})^{1/2}}{3\varepsilon}\left(1+\frac{1}{L_1}\right)\nu((0,T)\times\mathbb{R}^d)=:M_2$$

it holds

$$\omega_n\left(\left\{\gamma\in\tilde\Gamma:\sum_{k=1}^{2^n-1}|\gamma^1(2^{-n}Tk)-\gamma^1(2^{-n}Tk-)|>M2^{-n/2}\right\}\right)\le\frac{\varepsilon}{3}. \tag{5.26}$$

Finally, choosing $M=\max\{M_1,M_2\}$ and R so that (5.23) is satisfied it follows as a result of (5.24), (5.25), and (5.26) that $\omega_n\left(\tilde\Gamma^c_{n,M,R}\right)\le\varepsilon$. This concludes the proof of the tightness of the sequence of measures ω_n.

Step 2. Let ω be a limit point of the sequence $(\omega_n)_{n\in\mathbb{N}}$. Then (5.5) holds true. We prove this by showing separately the two convergences in the sense of distributions:

(1) for every $t\in[0,T)$

$$\lim_{n\to\infty}(e_t)_\sharp\omega_n=\mathscr{L}^{d+1}\llcorner E_{u(t)}; \tag{5.27}$$

(2) for every $t\in[0,T)$

$$\lim_{n\to\infty}(e_t)_\sharp\omega_n=(e_t)_\sharp\omega. \tag{5.28}$$

We first notice that (5.27) is trivially true for every $t\in[0,T)$ of the form $t=kT2^{-N}$ with $k,N\in\mathbb{N}$ since in this case it holds $(e_t)_\sharp\omega_n=\mathscr{L}^{d+1}\llcorner E_{u(t)}$ for every $n\ge N$. We observe that u is continuous in $L^1(\mathbb{R}^d)$ with respect to t by assumption and that since $(\pi_t)_\sharp\nu$ has no atoms the sequence of curves $t\mapsto(e_t)_\sharp\omega_n$ is converging uniformly to a continuous limit with respect to the W_1 distance by Lemma 5.9. Therefore (5.27) holds for every $t\in[0,T)$ by continuity.

Let ω_{n_k} be a weakly convergent subsequence and denote by ω its limit. Notice that e_t is not continuous on $\tilde\Gamma$ endowed with the topology τ introduced above, so we cannot directly deduce (5.28) from the weak convergence of ω_{n_k} to ω. This motivates the following definition: let $\Delta t>0$ and consider $\tilde e_t^{\Delta t}:\tilde\Gamma\to\mathbb{R}^d\times[0,+\infty)$ defined by

$$\tilde e_t^{\Delta t}(\gamma):=\frac{1}{\Delta t}\int_t^{t+\Delta t}e_s(\gamma)ds.$$

We prove first that for every $t\in(0,T-\Delta t)$ it holds

$$\lim_{k\to\infty}(\tilde e_t^{\Delta t})_\sharp\omega_{n_k}=(\tilde e_t^{\Delta t})_\sharp\omega. \tag{5.29}$$

For every $\phi\in C^\infty_c(\mathbb{R}^d\times[0,+\infty))$ let $\Phi:\tilde\Gamma\to\mathbb{R}$ be defined by

$$\Phi(\gamma)=\frac{1}{\Delta t}\int_t^{t+\Delta t}\phi(\gamma(s))ds.$$

The function Φ is continuous with respect to the topology τ, therefore

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^d\times[0,+\infty)}\phi(x,v)d(\tilde e_t^{\Delta t})_\sharp\omega_{n_k}&=\lim_{k\to\infty}\int_{\tilde\Gamma}\int_t^{t+\Delta t}\phi(\gamma(s))dsd\omega_{n_k}(\gamma)\\&=\lim_{k\to\infty}\int_{\tilde\Gamma}\Phi(\gamma)d\omega_{n_k}(\gamma)\\&=\int_{\tilde\Gamma}\Phi(\gamma)d\omega(\gamma)\\&=\int_{\mathbb{R}^d\times[0,+\infty)}\phi(x,v)d(\tilde e_t^{\Delta t})_\sharp\omega,\end{aligned}$$

and this proves (5.29).

On the other hand by (5.27) it holds

$$\lim_{k\to\infty}(\tilde{e}_t^{\Delta t})_\sharp \omega_{n_k} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathscr{L}^{d+1} \llcorner E_{u(s)} ds. \tag{5.30}$$

From (5.29) and (5.30) we get that (5.28) holds for $\mathscr{L}^1$-a.e. $t\in[0,T)$. The equality actually holds for every $t\in[0,T)$ since $(e_t)_\sharp\omega$ is continuous with respect to t.

Step 3. The measure ω is concentrated on characteristic curves and (5.7) holds true. Notice that the function $g:\tilde{\Gamma}\to\mathbb{R}$ defined by

$$g(\gamma) := \sup_{t\in[0,T)} \left|\gamma^1(t)-\gamma^1(0)-\int_0^t f'(\gamma^2(s))ds\right|$$

is lower semicontinuous, therefore

$$\int_{\tilde{\Gamma}} g(\gamma)d\omega \le \lim_{n\to\infty}\int_{\tilde{\Gamma}} g(\gamma)d\omega_n,$$

which is actually equal to 0 by (5.18), since $L_n\to\infty$ as $n\to\infty$. Similarly (5.19) implies (5.7). □

In the last part of this section we show how it is possible to decompose the entropy production measures μ^η of any entropy η along the characteristic curves.

Let η be a smooth and convex entropy and set

$$\mu^\eta_\gamma = (\mathbb{I},\gamma)_\sharp\Big(\big(\eta''\circ\gamma^2\big)\tilde{D}\gamma^2\Big) + \eta''(v)\left(\mathscr{H}^1\llcorner E^+_\gamma - \mathscr{H}^1\llcorner E^-_\gamma\right), \tag{5.31}$$

where

$$\begin{aligned} E^+_\gamma &:= \{(t,x,v):\gamma^1(t)=x, \gamma^2(t-)<\gamma^2(t+), v\in(\gamma^2(t-),\gamma^2(t+))\},\\ E^-_\gamma &:= \{(t,x,v):\gamma^1(t)=x, \gamma^2(t+)<\gamma^2(t-), v\in(\gamma^2(t+),\gamma^2(t-))\}. \end{aligned} \tag{5.32}$$

Notice that for every $\gamma\in\Gamma$ the measure μ^η_γ in (5.31) is well-defined: since every $\gamma\in\Gamma$ has bounded variation, then $\left(\mathscr{H}^1\llcorner E^+_\gamma - \mathscr{H}^1\llcorner E^-_\gamma\right)$ is a finite measure with compact support in $[0,T]\times\mathbb{R}^d\times[0,+\infty)$, and $(t,x,v)\mapsto\eta''(v)\in C^\infty\left([0,T]\times\mathbb{R}^d\times[0,+\infty)\right)$. Accordingly we define

$$\bar{\mu}^\eta := \int_\Gamma \mu^\eta_\gamma\, d\omega.$$

In the next proposition we exploit the relation between the measures μ in Proposition 5.4 and $\bar{\mu}^\eta$.

Proposition 5.11. *Let $\bar{\eta}(u)=u^2/2$ and $\omega\in\mathcal{P}_1(\tilde{\Gamma})$ as in Theorem 5.10. Then*

$$\mu = \int_\Gamma \mu^{\bar{\eta}}_\gamma d\omega(\gamma) \qquad \text{and} \qquad |\mu| = \int_\Gamma \left|\mu^{\bar{\eta}}_\gamma\right| d\omega(\gamma). \tag{5.33}$$

Proof. Let $\phi\in C^1_c((0,T)\times\mathbb{R}^d\times(0,+\infty))$. Testing (5.4) with $\bar{\phi}:=\phi\eta'(v)$ we get

$$\int \partial_v\bar{\phi}d\mu = \int_{E_u}\eta'(v)\left(\partial_t\phi + f'(v)\cdot\nabla_x\phi\right)dtdxdv. \tag{5.34}$$

Being ω a Lagrangian representation of u it holds

$$\begin{aligned} \int_{E_u}\eta'(v)\left(\partial_t\phi+f'(v)\cdot\nabla_x\phi\right)dtdxdv &= \int_\mathbb{R}\int_\Gamma \eta'(\gamma^2(t))\left(\partial_t\phi(t,\gamma(t))+f'(\gamma^2(t))\cdot\nabla_x\phi(t,\gamma(t))\right)d\omega dt\\ &= \int_\Gamma\int_{(0,T)}\eta'(\gamma^2(t))\left(\partial_t\phi(t,\gamma(t))+\dot{\gamma}^1(t)\cdot\nabla_x\phi(t,\gamma(t))\right)dtd\omega\\ &=: \mathrm{A}. \end{aligned} \tag{5.35}$$

Set $\phi_\gamma:(0,T)\to\mathbb{R}$ be defined by $\phi_\gamma(t)=\phi(t,\gamma(t))$. By (5.7) for ω-a.e. $\gamma\in\mathcal{P}(\Gamma)$ the function ϕ_γ has bounded variation and by the chain rule for BV functions the following equality between measures holds:

$$\begin{aligned} D_t(\eta'\circ\gamma^2\phi_\gamma) &= \eta'(\gamma^2(t))\left(\partial_t\phi(t,\gamma(t))+\dot{\gamma}^1(t)\cdot\nabla_x\phi(t,\gamma(t))\right)\mathscr{L}^1 + \eta'(\gamma^2(t))\partial_v\phi(t,\gamma(t))\tilde{D}_t\gamma^2\\ &\quad + \phi_\gamma(t)\eta''(\gamma^2(t))\tilde{D}_t\gamma^2 + \sum_{t_j\in J_\gamma}\left(\eta'(\gamma^2(t_j+))\phi_\gamma(t_j+)-\eta'(\gamma^2(t_j-))\phi_\gamma(t_j-)\right)\delta_{t_j}, \end{aligned}$$

where J_γ denotes the jump set of γ and $\tilde{D}_t\gamma^2$ denotes the diffuse part of the measure $D_t\gamma^2$, i.e. the absolutely continuous part plus the Cantor part (see [AFP00]). Plugging it into (5.35) we get

$$\begin{aligned}
\mathrm{A} = & -\int_\Gamma\int_{(0,T)} \big(\eta'(\gamma^2(t))\partial_v\phi(t,\gamma(t)) + \eta''(\gamma^2(t))\phi(t,\gamma(t))\big)\, d\tilde{D}_t\gamma^2(t)d\omega(\gamma) \\
& -\int_\Gamma \sum_{t_j\in J_\gamma} \big(\eta'(\gamma^2(t_j+))\phi_\gamma(t_j+) - \eta'(\gamma^2(t_j-))\phi_\gamma(t_j-)\big)\, d\omega(\gamma) \\
= & -\int_\Gamma\int_{(0,T)} \big(\partial_v\bar{\phi}(t,\gamma(t))\big)\, d\tilde{D}_t\gamma^2(t)d\omega(\gamma) \\
& -\int_\Gamma \left(\int_{\mathbb{R}^{d+2}} \partial_v\bar{\phi} d[\mathscr{H}^1 \llcorner E_\gamma^+ - \mathscr{H}^1 \llcorner E_\gamma^-]\right) d\omega(\gamma) \\
= & -\int_\Gamma\int_{\mathbb{R}^{d+2}} \partial_v\bar{\phi} d\mu_\gamma^{\bar{\eta}} d\omega(\gamma),
\end{aligned} \tag{5.36}$$

where in the second equality we used the relation $\partial_v\bar{\phi} = \eta'(v)\partial_v\phi + \eta''(v)\phi$ and the definition of $E_\gamma^\pm$ in (5.32) and in the last equality we used the definition of $\mu_\gamma^{\bar{\eta}}$, namely (5.31) specified for $\eta = \bar{\eta}$, for which we have $\bar{\eta}'' \equiv 1$. Comparing this with (5.34) we get the first expression in (5.33).

In order to prove the second part of the statement notice that the following inequality between measures trivially holds:

$$|\mu| \le \int_\Gamma \left|\mu_\gamma^{\bar{\eta}}\right| d\omega(\gamma).$$

In order to conclude that the equality holds it is actually enough to check that

$$\left(\int_\Gamma \left|\mu_\gamma^{\bar{\eta}}\right| d\omega(\gamma)\right) \left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right) \le |\mu| \left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right).$$

Being $|\mu_\gamma^{\bar{\eta}}|((0,T)\times\mathbb{R}^d\times(0,+\infty)) = \mathrm{Tot.Var.}_{(0,T)}\gamma^2$ lower semicontinuous with respecto to γ it holds

$$\left(\int_\Gamma \left|\mu_\gamma^{\bar{\eta}}\right| d\omega(\gamma)\right) \left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right) \le \liminf_{n\to\infty} \left(\int_\Gamma \mathrm{Tot.Var.}_{(0,T)}(\gamma^2) d\omega_n(\gamma)\right) \left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right).$$

By Lemma 5.8 we finally have that

$$\begin{aligned}
\liminf_{n\to\infty} \left(\int_\Gamma \mathrm{Tot.Var.}_{(0,T)}(\gamma^2) d\omega_n(\gamma)\right) \left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right) &\le \nu\left((0,T)\times\mathbb{R}^d\right) \\
&= |\mu|\left((0,T)\times\mathbb{R}^d\times(0,+\infty)\right)
\end{aligned}$$

and this concludes the proof. ☐

Finally we exploit the well-known relation between the measure μ and the entropy dissipation measures μ^η to decompose them along the characteristic curves.

Proposition 5.12. *For every smooth convex entropy η the following representation formula holds:*

$$(\pi_{t,x})_\sharp \left(\int_\Gamma \mu_\gamma^\eta d\omega\right) = \mu^\eta. \tag{5.37}$$

Proof. Assume without loss of generality that $\eta(0) = 0$ and $q(0) = 0$. Given $\varphi \in C_c^1((0,T)\times\mathbb{R}^d)$ and by means of the elementary identities

$$u(t,x) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(v)\, dv$$

and

$$\eta(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(v)\eta'(v)\, dv, \qquad q(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(v)q'(v)\, dv$$

we easily get

$$-\langle \mu^\eta, \varphi\rangle := -\langle \eta(u)_t + \mathrm{div}_x q(u), \varphi\rangle = \int_{E_u} \eta'(v)\big(\varphi_t(t,x) + f'(v)\cdot\nabla_x\varphi(t,x)\big)\, dtdxdv. \tag{5.38}$$

The result of (5.35) and (5.36) specifying $\phi(t,x,v) = \varphi(t,x)$, and therefore $\bar{\phi}(t,x,v) = \eta'(v)\varphi(t,x)$, is

$$\int_{E_u} \eta'(v)\big(\varphi_t(t,x) + f'(v)\cdot\nabla_x\varphi(t,x)\big)\,dtdxdv = -\int_\Gamma \int \eta''(v)\varphi(t,x)d\mu_\gamma^{\bar{\eta}}d\omega. \tag{5.39}$$

Recalling that $\bar{\eta}''(v) = 1$ for every $v \in \mathbb{R}$, it follows from the definition of μ_γ^η in (5.31) that

$$\mu_\gamma^\eta = \eta''(v)\mu_\gamma^{\bar{\eta}}. \tag{5.40}$$

The representation formula (5.37) in the statement follows from (5.38), (5.39) and (5.40). □

Remark 5.13. A strictly related statement to Property (3') in the Introduction is the following claim: a Lagrangian representation ω is concentrated on a set of curves $\gamma \in \Gamma$ such that $D_t\gamma^2$ is purely atomic. Notice that this formulation is natural for general smooth fluxes, even without any nonlinearity assumption. As already mentioned in the introduction this claim has been proved in several space dimension only for continuous entropy solutions in [BBM17], where actually $D_t\gamma^2 = 0$ for ω-a.e. $\gamma \in \Gamma$.

References

[ADL04] L. Ambrosio and C. De Lellis. A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations. *J. Hyperbolic Differ. Equ.*, 1(4):813–826, 2004.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Science Publications. Clarendon Press, 2000.

[AKLR02] L. Ambrosio, B. Kirchheim, M. Lecumberry, and T. Rivière. On the rectifiability of defect measures arising in a micromagnetics model. Nonlinear problems in mathematical physics and related topics, II, 29?60, Int. Math. Ser. (N. Y.), 2, Kluwer/Plenum, New York, 2002.

[Amb04] Luigi Ambrosio. Transport equation and Cauchy problem for *BV* vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[BBM17] S. Bianchini, P. Bonicatto, and E. Marconi. A lagrangian approach to multidimensional conservation laws. *preprint SISSA 36/MATE*, 2017.

[BBMN10] G. Bellettini, L. Bertini, M. Mariani, and M. Novaga. Γ-entropy cost for scalar conservation laws. *Archive for Rational Mechanics and Analysis*, 195:261–309, 2010.

[BD18] S. Bianchini and S. Daneri. On Sudakov's type decomposition of transference plans with norm costs. *Mem. Amer. Math. Soc.*, 251(1197):vi+112, 2018.

[BM17] S. Bianchini and E. Marconi. On the structure of L^∞ entropy solutions to scalar conservation laws in one-space dimension-entropy solutions to scalar conservation laws in one-space dimension. *Archive for Rational Mechanics and Analysis*, Jun 2017.

[Bre84] Y. Brenier. Averaged multivalued solutions for scalar conservation laws. *SIAM J. Numer. Anal.*, 21(6):1013–1037, 1984.

[Che86] K. S. Cheng. A regularity theorem for a nonconvex scalar conservation law. *J. Differential Equations*, 61(1):79–127, 1986.

[COW08] G. Crippa, F. Otto, and M. Westdickenberg. *Regularizing Effect of Nonlinearity in Multidimensional Scalar Conservation Laws*, volume 5 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2008.

[Daf85] C. M. Dafermos. Regularity and large time behaviour of solutions of a conservation law without convexity. *Proc. Roy. Soc. Edinburgh Sect. A*, 99(3-4):201–239, 1985.

[Daf06] C. M. Dafermos. Continuous solutions for balance laws. *Ric. Mat.*, 55(1):79–91, 2006.

[Daf16] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.

[DLOW03] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, 170(2):137–184, 2003.

[DLR03] C. De Lellis and T. Rivière. The rectifiability of entropy measures in one space dimension. *J. Math. Pures Appl. (9)*, 82(10):1343–1367, 2003.

[GL19] B. Gess and X. Lamy. Regularity of solutions to scalar conservation laws with a force. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(2):505–521, 2019.

[GP13] F. Golse and B. Perthame. Optimal regularizing effect for scalar conservation laws. *Rev. Mat. Iberoam.*, 29(4):1477–1504, 2013.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[LO18] X. Lamy and F. Otto. On the regularity of weak solutions to Burgers' equation with finite entropy production. *Calc. Var. Partial Differential Equations*, 57(4):Art. 94, 19, 2018.

[LPT94] P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191, 1994.

[Mar10] M. Mariani. Large deviations principles for stochastic scalar conservation laws. *Probab. Theory Related Fields*, 147(3-4):607–648, 2010.

[Mar18] E. Marconi. Regularity estimates for scalar conservation laws in one space dimension. *J. Hyperbolic Differ. Equ.*, 15(4):623–691, 2018.

[Ole63] O. A. Oleĭnik. Discontinuous solutions of non-linear differential equations. *Amer. Math. Soc. Transl. (2)*, 26:95–172, 1963.

[Sil19] L. Silvestre. Oscillation properties of scalar conservation laws. *Comm. Pure Appl. Math.*, 72(6):1321–1348, 2019.

[TT07] E. Tadmor and T. Tao. Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs. *Comm. Pure Appl. Math.*, 60(10):1488–1521, 2007.

[Vil09] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

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