HOMOGENIZATION OF FIBER REINFORCED BRITTLE MATERIALS: THE INTERMEDIATE CASE

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Abstract. We derive a cohesive fracture model by homogenizing a periodic composite material whose microstructure is characterized by the presence of brittle inclusions in a reticulated unbreakable elastic structure.

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1. Introduction

Since the seminal papers by Barrenblatt [6, 7] and Dugdale [14], cohesive zone models have been widely used in fracture mechanics. Compared with the models of brittle cracks, based on the original Griffith's criterion [15], they provide a more accurate description of the process of crack opening. Moreover, they permit to avoid the singularity of the stress near the crack tip.

In this paper we provide a possible justification of a macroscopic cohesive zone model in terms of microscopic brittle cracks. More precisely, at a microscopic level we consider an antiplane problem for linearized elasticity with purely brittle cracks, assuming that the material under examination is reinforced by an orthogonal periodic grid of unbreakable elastic fibers (see Figure 1).

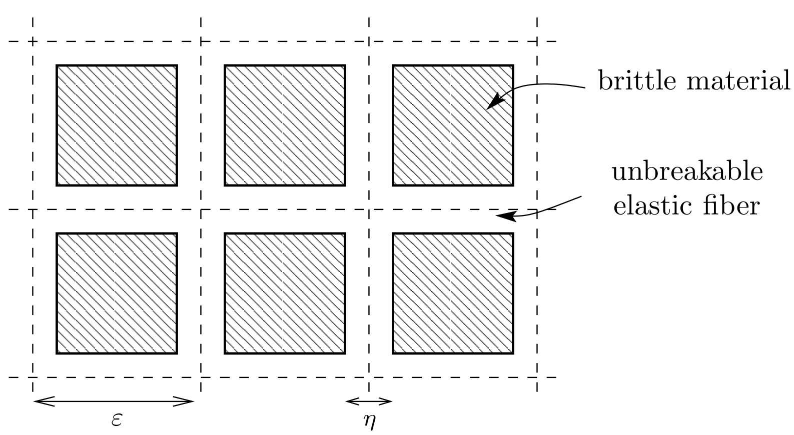

Figure 1. Schematic of the composite material.

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Two scales play a crucial role in the problem: the period ε of the grid and the thickness η of the fibers. We study the asymptotic behavior of this material as $\varepsilon \to 0$ and $\eta \to 0$, when $\eta = 2\varepsilon^2$. When $\eta \to 0$ with a different rate of convergence, it has been proved in [5] that the limit is purely elastic with no cracks when $\eta \gg \varepsilon^2$, while the limit is elastic with purely brittle cracks if $\eta \ll \varepsilon^2$.

According to Griffith's theory, in our problem, with $\eta = 2\varepsilon^2$, the energy to be considered is given by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^1(S_u) & \text{if } S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is the reference configuration, u denotes the displacement, S_u is the jump set of u and represents the crack, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, and Ω_ε is the brittle part of the material, defined as the disconnected part of Ω not contained in the fibers.

The effective behavior of the macroscopic material is obtained through a homogenization procedure. The effective energy $\mathcal{F}_{\rm hom}$ is the Γ-limit of $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$ along a suitable sequence. We prove that $\mathcal{F}_{\rm hom}$ is given by the functional

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega |\nabla u|^2\,dx + \int_{S_u} g([u],\nu_u)d\mathcal{H}^1, \tag{1.1}$$

where $[u]$ is the amplitude of the jump of u and represents the crack opening, ν_u is the normal to the jump set and describes the orientation of the crack, while g is a surface energy density satisfying the estimates

$$\max\{\alpha|t|, 1\} \leq g(t,\nu) \leq \beta(1+|t|) \quad \text{for every pair } (t,\nu) \in \mathbb{R}\times S^1, \tag{1.2}$$

for suitable $\alpha, \beta > 0$. Hence, the effective energy $\mathcal{F}_{\rm hom}$ actually describes a cohesive zone model.

The plan of the paper is as follows. In Section 2, after recalling the basic notions on the space SBV of special functions of bounded variation, we make precise the variational setting of the problem and state the main result (Theorem 2.2).

In Section 3 we prove a compactness result (Theorem 3.4) for the sequence $(\mathcal{F}_\varepsilon)$ with respect to Γ-convergence. The main step in the proof (see Propositions 3.3 and 3.5) is an estimate from above of the Γ-limit, which also leads to the second inequality in (1.2).

Finally, in Section 4 we represent the Γ-limit as an integral functional of the form (1.1).

2. Setting of the problem and statement of the main result

Let U be a bounded open subset of $\mathbb{R}^2$ and let $1 \leq p \leq +\infty$, we use standard notation for the Sobolev and Lebesgue spaces $W^{1,p}(U)$ and $L^p(U)$.

The scalar product of $x, y \in \mathbb{R}^2$ is denoted by $x \cdot y$. For $\rho > 0$, $B_\rho(x)$ is the open ball centered in x with radius ρ, while for $\nu \in S^1 := \{x \in \mathbb{R}^2 \colon |x| = 1\}$, $Q^\nu_\rho(x)$ denotes the open square of center x, side length ρ and one face orthogonal to ν.

The Lebesgue measure and the one-dimensional Hausdorff measure in $\mathbb{R}^2$ are denoted by $\mathcal{L}^2$ and $\mathcal{H}^1$, respectively.

For the general theory of *special functions of bounded variation* we refer to [4]; here we just recall some notation and definitions.

The space of special functions of bounded variation on U is denoted by $SBV(U)$. For every $u \in SBV(U)$, ∇u denotes the *approximate gradient* of u, S_u the *approximate discontinuity set* of u, and ν_u the *generalized normal* to S_u, which is defined up to the sign. If u^+ and u^- are the traces of u on the sides of S_u determined by ν_u and $-\nu_u$, respectively, the difference $u^+ - u^-$ is called the *jump* of u, and is denoted by $[u]$. Note that, with our convention, if we reverese the orientation of ν_u, we change the sign of $[u]$. It turns out that $[u] \in L^1(S_u; \mathcal{H}^1)$.

We consider the vector subspace of $SBV(U)$

$$SBV^2(U) := \{u \in SBV(U) \colon \nabla u \in L^2(U; \mathbb{R}^2) \ \text{ and } \ \mathcal{H}^1(S_u) < +\infty\}.$$

We consider also the larger space of the *generalized special functions of bounded variation* on U, $GSBV(U)$, which is made of all measurable functions $u \colon U \to \mathbb{R}$ whose truncations $u^n := (u \wedge n) \vee (-n)$ belong to $SBV(U')$ for every $n \in \mathbb{N}$ and for every open set $U' \subset\subset U$; *i.e.,* with $\overline{U'}$ compact and contained in U. Notice that

$$\begin{gathered} |\nabla u^n(x)| \leq |\nabla u(x)| \quad \text{a.e.} \quad \text{in} \ \ U, \\ \nabla u^n(x) \to \nabla u(x) \quad \text{a.e.} \quad \text{in} \ \ U \ \ \text{as} \ \ n \to +\infty, \\ S_{u^n} \subseteq S_u, \quad (u^n)^\pm = (u^\pm)^n, \\ \mathcal{H}^1(S_{u^n}) \to \mathcal{H}^1(S_u) \quad \text{as} \quad n \to +\infty. \end{gathered}$$

Moreover, if $u \in GSBV(U)$ in general it is no longer true that u^+ and u^- are finite $\mathcal{H}^1$-a.e. on S_u; however, it is still possible to define the jump $[u]$ on $\mathcal{H}^1$-almost all S_u because the points where the traces are both $+\infty$ or $-\infty$ do not belong to S_u. We remark that $[u]$ is now an extended real-valued function, and it may happen that $[u] \notin L^1(S_u; \mathcal{H}^1)$.

In analogy with the case of SBV functions, we say that $u \in GSBV^2(U)$ if $u \in GSBV(U)$, $\nabla u \in L^2(U, \mathbb{R}^2)$ and $\mathcal{H}^1(S_u) < +\infty$.

It can be proved that

$$GSBV^2(U) \cap L^\infty(U) = SBV^2(U) \cap L^\infty(U).$$

Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary. Let $A_\varepsilon := [-\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon]^2$; for $i \in \mathbb{Z}^2$ we set $\Omega_\varepsilon^i := i\varepsilon + \varepsilon A_\varepsilon$ (see Figure 2). Moreover we define

$$\Omega_\varepsilon := \Omega \cap \bigcup_{i \in \mathbb{Z}^2} \Omega_\varepsilon^i.$$

Let $\mathcal{F}_\varepsilon \colon L^1(\Omega) \longrightarrow [0, +\infty]$ be the functionals defined as

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) & \text{if } u \in SBV^2(\Omega),\ S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

We also consider the Mumford-Shah functional $MS \colon L^1(\Omega) \longrightarrow [0, +\infty]$

$$MS(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) & \text{if } \ u \in SBV^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

In what follows the Γ-convergence of $(\mathcal{F}_\varepsilon)$ is understood with respect to the strong $L^1(\Omega)$-topology (see [13]).

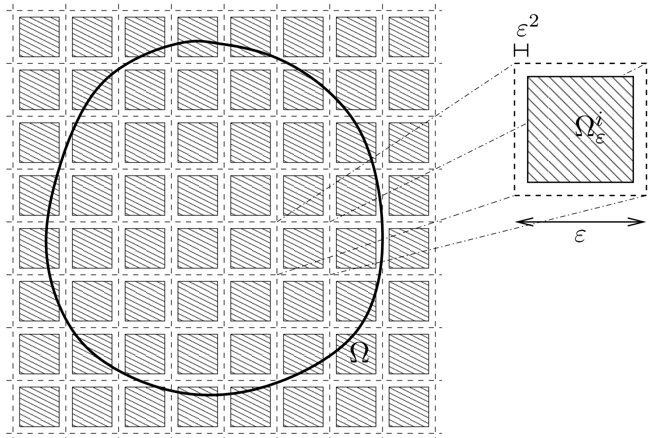

FIGURE 2. A portion of the fiber reinforced brittle material.

Remark 2.1. We notice that for every $\varepsilon > 0$ we have the trivial bound $MS \leq \mathcal{F}_\varepsilon$. Then, if $u, u_\varepsilon \in L^1(\Omega)$ and $u_\varepsilon \to u$ in $L^1(\Omega)$, with $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$, Ambrosio's compactness theorem (see [1, 2]) yields $u \in GSBV^2(\Omega) \cap L^1(\Omega)$. Hence, the domain of the Γ-limit (if it exists) is a subset of $GSBV^2(\Omega) \cap L^1(\Omega)$.

The main result of this paper is the following.

Theorem 2.2 (Γ-convergence)**.** *Let $(\mathcal{F}_\varepsilon)$ be the family of functionals defined in* (2.1)*. Then, for every sequence of positive numbers converging to* 0 *there exists a subsequence (ε_k) such that $(\mathcal{F}_{\varepsilon_k})$ Γ-converges to a functional $\mathcal{F}_{\text{hom}} \colon L^1(\Omega) \longrightarrow [0, +\infty]$ of the form*

$$\mathcal{F}_{\text{hom}}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + \int_{S_u} g([u], \nu_u) d\mathcal{H}^1 & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

for some Borel function $g : \mathbb{R} \times S^1 \longrightarrow [0, +\infty)$ satisfying the following properties:

(i) for any fixed $\nu \in S^1$, $g(\cdot, \nu)$ is nondecreasing on $(0, +\infty)$ and satisfies the symmetry condition $g(-t, -\nu) = g(t, \nu)$;

(ii) there exist $\alpha, \beta > 0$, with $\alpha \leq \beta$, such that

$$\max\{\alpha|t|, 1\} \leq g(t, \nu) \leq \beta(1 + |t|), \tag{2.4}$$

for every pair $(t, \nu) \in \mathbb{R} \times S^1$.

Remark 2.3. By the properties of Γ-convergence we know that $\mathcal{F}_{\text{hom}}$ is lower semicontinuous in $L^1(\Omega)$. As a consequence, we deduce in particular that the functional

$$u \mapsto \int_{S_u} g([u], \nu_u)\, d\mathcal{H}^1 \tag{2.5}$$

is lower semicontinuous on finite partitions; *i.e.*, on the subspace of $SBV^2(\Omega) \cap L^\infty(\Omega)$ made of those BV-functions which take only a finite number of values. Then, it is well-known (see [3]) that two necessary conditions on g for the lower semicontinuity of (2.5) are the following:

1. (subadditivity in the first variable) for any $\nu \in S^1$

$$g(t_1 + t_2, \nu) \leq g(t_1, \nu) + g(t_2, \nu),$$

for every $t_1, t_2 \in \mathbb{R}$;

2. (convexity in the second variable) for any $t \in \mathbb{R}$, the 1-homogeneous extension of $g(t, \cdot)\colon S^1 \longrightarrow [0, +\infty)$ to $\mathbb{R}^2$ is convex. This condition can be equivalently expressed in terms of g as

$$g(t, \nu) \leq \lambda_1 g(t, \nu_1) + \lambda_2 g(t, \nu_2),$$

for every $\nu, \nu_1, \nu_2 \in S^1$, $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 \nu_1 + \lambda_2 \nu_2 = \nu$.

Remark 2.4. We may also consider the functionals $\widetilde{\mathcal{F}}_\varepsilon : L^1(\Omega) \longrightarrow [0, +\infty]$ defined as

$$\widetilde{\mathcal{F}}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^1(S_u) & \text{if } u \in GSBV^2(\Omega) \cap L^1(\Omega),\ S_u \subseteq \Omega_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

(see *e.g.* [5]). Notice that Theorem 2.2 ensures the same Γ-convergence result for $\widetilde{\mathcal{F}}_\varepsilon$. Indeed, Theorem 2.2 immediately yields

$$\mathcal{F}_{\text{hom}}(u) = \Gamma\text{-}\limsup_{k \to +\infty} \mathcal{F}_{\varepsilon_k}(u) \geq \Gamma\text{-}\limsup_{k \to +\infty} \widetilde{\mathcal{F}}_{\varepsilon_k}(u),$$

hence the upper bound.

To achieve the lower bound we need to prove that for every $u, u_k \in GSBV^2(\Omega) \cap L^1(\Omega)$ with $S_{u_k} \subseteq \Omega_{\varepsilon_k}$ and $u_k \to u$ in $L^1(\Omega)$ we have $\mathcal{F}_{\text{hom}}(u) \leq \liminf \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k)$.

To this end, for any fixed $n \in \mathbb{N}$ we let u_k^n be the truncation of u_k at level n. Then, $(u_k^n) \subset SBV^2(\Omega)$, $S_{u_k^n} \subseteq \Omega_{\varepsilon_k}$ and $u_k^n \to u^n$ in $L^1(\Omega)$, as $k \to +\infty$. Moreover, since $\widetilde{\mathcal{F}}_{\varepsilon_k}(u_k^n) \leq \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k)$, by Theorem 2.2 we get

$$\mathcal{F}_{\text{hom}}(u^n) \leq \liminf_{k \to +\infty} \mathcal{F}_{\varepsilon_k}(u_k^n) = \liminf_{k \to +\infty} \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k^n) \leq \liminf_{k \to +\infty} \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k).$$

Finally, since $u^n \to u$ in $L^1(\Omega)$, letting n go to infinity and invoking the lower semicontinuity of the Γ-limit $\mathcal{F}_{\text{hom}}$, we obtain

$$\mathcal{F}_{\text{hom}}(u) \leq \liminf_{n \to +\infty} \mathcal{F}_{\text{hom}}(u^n) \leq \Gamma\text{-}\liminf_{k \to +\infty} \widetilde{\mathcal{F}}_{\varepsilon_k}(u),$$

thus, the lower bound.

3. A compactness result on $SBV^2(\Omega)$

We prove Theorem 2.2 in a nonconstructive way, following the so-called localization method of Γ-convergence, for which we refer the reader to [13, Chapters 14-20].

Loosely speaking, this method consists of two main steps. In the first one, based on compactness arguments, we prove the existence of Γ-converging (sub)sequences. While in the second one we recover enough information on the structure of the Γ-limit as to obtain a representation in an integral form.

We localize the family $(\mathcal{F}_\varepsilon)$ by introducing an explicit dependence on the set of integration.

Let $\mathcal{A}(\Omega)$ be the family of all open subset of Ω. For every pair $(u,U) \in L^1(\Omega)\times\mathcal{A}(\Omega)$ we define

$$\mathcal{F}_\varepsilon(u,U) := \begin{cases} \displaystyle\int_U |\nabla u|^2\,dx + \mathcal{H}^1(S_u\cap U) & \text{if } u\in SBV^2(U),\ S_u\cap U\subseteq\Omega_\varepsilon,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.1}$$

For any $U\in\mathcal{A}(\Omega)$ and any $u\in SBV^2(U)\cap L^1(\Omega)$, we may extend the localized functionals considered above to a measure $\mathcal{F}^*_\varepsilon(u,\cdot)$ defined on the σ-algebra $\mathcal{B}(U)$ of Borel subset of U, by setting

$$\mathcal{F}^*_\varepsilon(u,B) := \int_B |\nabla u|^2\,dx + \mathcal{H}^1(S_u\cap B),\quad \text{for every } B\in\mathcal{B}(U).$$

Given a positive sequence (ε_k) converging to 0, we define $\mathcal{F}',\mathcal{F}'' : L^1(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty]$ as

$$\mathcal{F}'(\cdot,U) = \Gamma\text{-}\liminf_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(\cdot,U),\qquad \mathcal{F}''(\cdot,U) = \Gamma\text{-}\limsup_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(\cdot,U),$$

for every $U\in\mathcal{A}(\Omega)$.

We notice that $\mathcal{F}',\mathcal{F}''$ are lower semicontinuous [13, Propositions 6.8] and that they inherit some of the properties of the functionals $\mathcal{F}_\varepsilon$. Indeed, they are increasing [13, Propositions 6.7], local [13, Propositions 16.15] and it is immediate to show that they decrease by truncation. Notice that in general they are not inner regular. Hence we also consider their inner regular envelope; *i.e.,* the functionals $\mathcal{F}'_-,\mathcal{F}''_-\colon L^1(\Omega)\times\mathcal{A}(\Omega)\longrightarrow[0,+\infty]$ defined as

$$\mathcal{F}'_-(u,U) := \sup\big\{\mathcal{F}'(u,V)\colon V\subset\subset U,\ V\in\mathcal{A}(\Omega)\big\}. \tag{3.2}$$

and

$$\mathcal{F}''_-(u,U) := \sup\big\{\mathcal{F}''(u,V)\colon V\subset\subset U,\ V\in\mathcal{A}(\Omega)\big\}. \tag{3.3}$$

Then, $\mathcal{F}'_-$ and $\mathcal{F}''_-$ are both increasing, lower semicontinuous [13, Remark 15.10], and local [13, Remark 15.25].

By the compactness of Γ-convergence, in Theorem 3.4 we easily show that for every sequence of positive numbers converging to 0 there exists a subsequence (ε_k) such that the corresponding functionals $\mathcal{F}'$ and $\mathcal{F}''$ satisfy $\mathcal{F}'_-=\mathcal{F}''_-$. Moreover, by monotonicity we always have

$$\mathcal{F}''_- = \mathcal{F}'_- \le \mathcal{F}' \le \mathcal{F}''. \tag{3.4}$$

Then, if we show that $\mathcal{F}''$ is inner regular, which is equivalent to the inequality $\mathcal{F}''\le\mathcal{F}''_-$, from (3.4) we deduce the existence of the Γ-limit of $(\mathcal{F}_{\varepsilon_k})$.

A preliminary step towards the proof of the inner regularity of $\mathcal{F}''$, and of other crucial properties of the Γ-limit considered as a set function, is proving that the so-called fundamental estimate holds uniformly for the sequence of functionals $(\mathcal{F}_\varepsilon)$.

The next proposition provides an extension of the fundamental estimate to the SBV-setting.

Proposition 3.1 (Fundamental estimate in SBV^2)**.** *For every $\eta>0$ and for every $U',U'',V\in\mathcal{A}(\Omega)$, with $U'\subset\subset U''$, there exists a constant $M(\eta)>0$ satisfying the following property: for every $\varepsilon>0$, for every $u\in L^1(\Omega)$ with $u\in SBV^2(U'')$ and $S_u\cap U''\subseteq\Omega_\varepsilon$, and for every $v\in L^1(\Omega)$ with $v\in SBV^2(V)$ and $S_v\cap V\subseteq\Omega_\varepsilon$, there exists a function $\varphi\in C^\infty_0(\Omega)$ with $\varphi=1$ in a neighborhood of U', $\mathrm{spt}\varphi\subset U''$ and $0\le\varphi\le1$ such that*

$$\mathcal{F}_\varepsilon(\varphi u+(1-\varphi)v, U'\cup V) \le (1+\eta)\Big(\mathcal{F}_\varepsilon(u,U'')+\mathcal{F}_\varepsilon(u,V)\Big) + M(\eta)\|u-v\|^2_{L^2(S)},$$

with $S := (U'' \setminus U') \cap V$.

Proof. This result can be obtained, as a particular case, from [10, Proposition 3.1]. For the reader's convenience we prefer to give here a simplified proof.

Let $\eta > 0$, $U', U'', V \in \mathcal{A}(\Omega)$ be fixed as in the statement and let φ be a function in $C_0^\infty(\Omega)$ with $0 \le \varphi \le 1$, $\mathrm{spt}\varphi \subset U''$ and $\varphi = 1$ in a neighborhood of U'.

Let u and v be two functions as in the statement and let $w := \varphi u + (1-\varphi)v$. Notice that w belongs to $SBV^2(U' \cup V)$ and satisfies the constraint $S_w \cap (U' \cup V) \subseteq \Omega_\varepsilon$.

We have

$$\mathcal{F}_\varepsilon(w, U' \cup V) = \mathcal{F}_\varepsilon(u, U') + \mathcal{F}^*_\varepsilon(v, V \setminus U'') + \mathcal{F}^*_\varepsilon(w, (U'' \setminus U') \cap V). \tag{3.5}$$

We now estimate the last term in the right hand side of (3.5). Let $S := (U'' \setminus U') \cap V$; for any fixed $\eta \in (0,1)$ we have

$$\begin{aligned}
\mathcal{F}^*_\varepsilon(w, S) &\le \int_S \Big| (1-\eta)\,\frac{\varphi \nabla u + (1-\varphi)\nabla v}{1-\eta} + \eta\,\frac{\nabla\varphi(u-v)}{\eta}\Big|^2 dx + \mathcal{H}^1(S_u \cap S) + \mathcal{H}^1(S_v \cap S) \\
&\le \frac{1}{1-\eta}\Big(\int_S |\nabla u|^2\,dx + \int_S |\nabla v|^2\,dx \Big) + \frac{1}{\eta}\int_S |\nabla\varphi|^2 |u-v|^2\,dx + \mathcal{H}^1(S_u \cap S) + \mathcal{H}^1(S_v \cap S) \\
&\le \frac{1}{1-\eta}\Big(\mathcal{F}^*_\varepsilon(u,S) + \mathcal{F}^*_\varepsilon(v,S)\Big) + \frac{1}{\eta}\int_S |\nabla\varphi|^2 |u-v|^2\,dx.
\end{aligned} \tag{3.6}$$

Finally, setting $M := ||\nabla\varphi||_{L^\infty(\Omega)}$ and combining (3.5) and (3.6), we find

$$\mathcal{F}_\varepsilon(w, U' \cup V) \le \frac{1}{1-\eta}\Big(\mathcal{F}_\varepsilon(u, U'') + \mathcal{F}_\varepsilon(v, V)\Big) + \frac{M}{\eta}\|u-v\|^2_{L^2(S)},$$

and thus the thesis. □

Now we show that the restriction of $\mathcal{F}''$ to $SBV^2(\Omega) \cap L^\infty(\Omega)$ and to all open Lipschitz subsets of Ω satisfies a bound from above (see (3.8)).

Before stating this result, it is convenient to introduce some notation.

Let us fix an open rectangle R containing $\overline{\Omega}$ and let $\mathcal{W}(R)$ be the space of all functions $w \in SBV^2(R) \cap L^\infty(R)$ enjoying the following properties:

- $S_w \subseteq L$, with L finite union of pairwise disjoint closed segments contained in R;
- $w \in W^{2,\infty}(R \setminus L)$.

Moreover, we denote by $\mathcal{A}_L(\Omega)$ the class of all open subsets of Ω with Lipschitz boundary.

To obtain the desired estimate we need the following approximation lemma.

Lemma 3.2. *Let* $U \in \mathcal{A}_L(\Omega)$ *and let* $u \in SBV^2(U) \cap L^\infty(U)$. *Then* u *has an extension* $v \in SBV^2(R) \cap L^\infty(R)$ *with compact support in* R*, such that*

$$\mathcal{H}^1(S_v \cap \partial U) = 0, \tag{3.7}$$

and $\|v\|_{L^\infty(R)} = \|u\|_{L^\infty(U)}$. *Moreover, there exist a sequence* $(w_j) \subset \mathcal{W}(R)$ *converging to* v *in* $L^1(R)$*, and a sequence* (L_j) *of finite unions of pairwise disjoint closed segments contained in* R

and such that $S_{w_j} \subseteq L_j$*, with the following properties:*

$$\|w_j\|_{L^\infty(R)} \leq \|v\|_{L^\infty(R)} = \|u\|_{L^\infty(U)},$$

$$\nabla w_j \to \nabla v \quad \text{strongly in } L^2(R;\mathbb{R}^2), \text{ hence} \quad \nabla w_j \to \nabla u \quad \text{strongly in } L^2(U;\mathbb{R}^2),$$

$$\limsup_{j\to+\infty} \int_{L_j\cap U} \psi([w_j],\nu_{w_j})\, d\mathcal{H}^1 \leq \int_{S_v\cap U} \psi([v],\nu_v)\, d\mathcal{H}^1 = \int_{S_u\cap U} \psi([u],\nu_u)\, d\mathcal{H}^1$$

for every upper semicontinuous function $\psi\colon \mathbb{R}\times S^1 \to [0,+\infty)$ *such that* $\psi(t,\nu) = \psi(-t,-\nu)$ *for every* $t\in\mathbb{R}$ *and* $\nu \in S^1$*.*

Proof. To prove the first assertion we can use locally a reflection argument in a curvilinear coordinate system for which the boundary is flat. The global extension can be obtained, as usual, through a partion of unity. Then, by (3.7), the existence of the approximating sequence (w_j) is a consequence of the density result [12, Theorem 3.1] (see also [12, Remark 3.5]). □

We are in a position to prove the following proposition.

Proposition 3.3. *There exists* $\beta > 0$ *such that*

$$\mathcal{F}''(u,U) \leq \int_U |\nabla u|^2\, dx + \beta \int_{S_u\cap U} (1+[u]^2) d\mathcal{H}^1, \tag{3.8}$$

for every $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$ *and for every* $U \in \mathcal{A}_L(\Omega)$*.*

Proof. We fix $U\in\mathcal{A}_L(\Omega)$; in view of Lemma 3.2 and of the locality of $\mathcal{F}''$ it is enough to prove

$$\mathcal{F}''(u,U) \leq \int_U |\nabla u|^2\, dx + \beta \int_{L\cap U} (1+[u]^2) d\mathcal{H}^1, \tag{3.9}$$

for $u \in \mathcal{W}(R)$.

We want to construct a sequence (u_k) converging to u in $L^1(\Omega)$ such that $u_k \in SBV^2(U)$, $S_{u_k}\cap U \subseteq \Omega_{\varepsilon_k}$, and

$$\limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,U) \leq \int_U |\nabla u|^2\, dx + \beta \int_{L\cap U} (1+[u]^2) d\mathcal{H}^1. \tag{3.10}$$

Since U has Lipschitz boundary, we can slightly modify u near each connected component of $\overline{S}_u$ to find a L that intersects ∂U in a finite number of points. This can be done, for instance, by slightly shifting these connected components taking into account the area formula for ∂U.

Now we explicitly construct u_k when L is a single closed segment; then, the general case follows easily.

We divide the proof into two steps.

Step 1. We prove (3.10) for a target function u such that L is parallel to the x_1-axis; more precisely,

$$L = \{x\in\mathbb{R}^2:\ a\leq x_1\leq b,\ x_2 = c\} \subseteq R,$$

for some $a,b,c\in\mathbb{R}$ with $a<b$.

To fulfill the constraints on the jump sets for the recovery sequence u_k, the idea is to regularize the target function u on the portion of the unbreakable fibers intersecting L.

Let R' be an open rectangle such that $\overline{\Omega}\cap L \subseteq R' \subset\subset R$ and consider a horizontal strip of fragile squares $\Omega^i_{\varepsilon_k}$ centered on a line "close" to $x_2 = c$. To fix the ideas, we consider those

$\Omega^i_{\varepsilon_k}$ corresponding to indices $i = (i_1, i_2) \in \mathbb{Z}^2$, with $i_1 \in I_k := \{\lfloor \frac{a}{\varepsilon_k} \rfloor \varepsilon_k, \ldots, \lfloor \frac{b}{\varepsilon_k} \rfloor \varepsilon_k + \varepsilon_k\}$ and $i_2 = \lfloor \frac{c}{\varepsilon_k} \rfloor \varepsilon_k$ (being $\lfloor r \rfloor$ be the integer part of any $r \in \mathbb{R}$). Then, we translate u to prevent the possibility that L entirely falls in an unbreakable horizontal fiber (see Figure 3). More precisely,

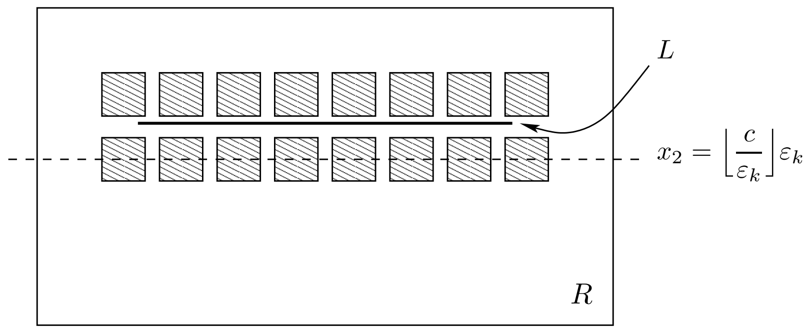

FIGURE 3. A set L falling in an unbreakable fiber.

for k large enough and for $x \in R'$ we set

$$v_k(x) := u\Big(x + \Big(c - \Big\lfloor \frac{c}{\varepsilon_k} \Big\rfloor \varepsilon_k\Big)\, e_2\Big),$$

so that S_{v_k} is contained in the horizontal line $x_2 = \left\lfloor \frac{c}{\varepsilon_k} \right\rfloor \varepsilon_k$.

We also define the numbers

$$a_k^+ := \Big\lfloor \frac{c}{\varepsilon_k} \Big\rfloor \varepsilon_k + \frac{\varepsilon_k}{2} - \varepsilon_k^2 \quad \text{and} \quad a_k^- := \Big\lfloor \frac{c}{\varepsilon_k} \Big\rfloor \varepsilon_k - \frac{\varepsilon_k}{2} + \varepsilon_k^2,$$

corresponding to the upper and lower sides of the squares $\Omega^i_{\varepsilon_k}$, respectively (see Figure 4).

Since the index i_2 will be fixed throughout this proof, to not overburden notation, with a little abuse, we denote by i an index varying in $\mathbb{Z}$. For every $i \in \mathbb{Z}$ we consider the unbreakable rectangles

$$R_k^i := S_k^i \times (a_k^-, a_k^+), \quad \text{where} \quad S_k^i := \Big(\frac{\varepsilon_k}{2} - \varepsilon_k^2, \frac{\varepsilon_k}{2} + \varepsilon_k^2\Big) + i\varepsilon_k.$$

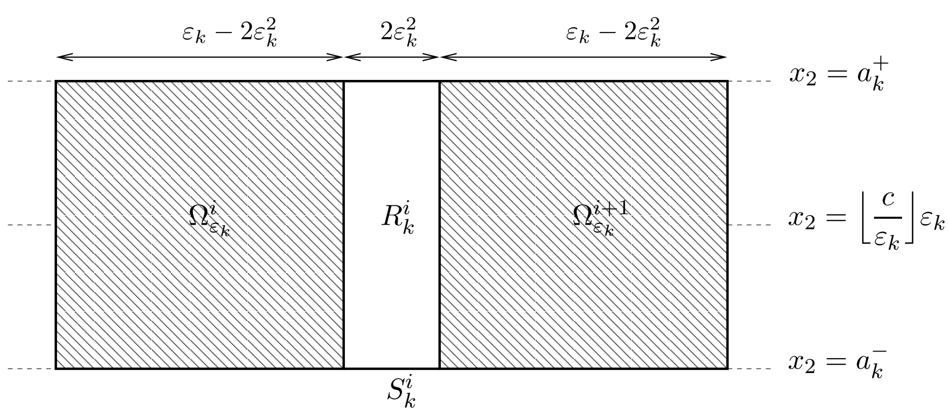

FIGURE 4. An unbreakable rectangle R_k^i between two consecutive brittle squares.

For every $i \in I_k$ we define u_k in each rectangle R_k^i in the following way: for fixed $x_1 \in S_k^i$, $u_k(x_1,\cdot)$ is the affine function connecting the two values $v_k(x_1,a_k^+)$ and $v_k(x_1,a_k^-)$ in $[a_k^-,a_k^+]$; *i.e.*,

$$u_k(x_1,x_2) := \frac{v_k(x_1,a_k^+)-v_k(x_1,a_k^-)}{a_k^+-a_k^-}(x_2-a_k^-)+v_k(x_1,a_k^-),$$

while we set $u_k := v_k$ elsewhere in R'.

By construction, $(u_k) \subset SBV^2(R')$, $S_{u_k}\cap\Omega \subseteq \Omega_{\varepsilon_k}$, and u_k is bounded in $L^\infty(R')$. Then, since u_k differs from v_k only on a set whose Lebesgue measure tends to 0 as $k\to+\infty$, we have $u_k \to u$ in $L^1(\Omega)$. Since in each square $\Omega^i_{\varepsilon_k}$ the jump set of u_k is contained in the horizontal line $x_2 = \lfloor\frac{c}{\varepsilon_k}\rfloor\varepsilon_k$ and in the vertical sides of the square, its lengh is bounded from above by $3(\varepsilon_k-\varepsilon_k^2)$. Therefore, we have

$$\limsup_{k\to+\infty}\mathcal{F}(u_k,U) \le \limsup_{k\to+\infty}\int_U |\nabla v_k|^2\,dx \\ +\limsup_{k\to+\infty}\sum_{i\in J_k}\int_{R_k^i}|\nabla u_k|^2\,dx + \limsup_{k\to+\infty}3N_k\,(\varepsilon_k-\varepsilon_k^2), \tag{3.11}$$

where

$$J_k := \{i\in I_k\colon R_k^i\cap U\neq\emptyset\}\cup\{i\in I_k\colon \Omega^i_{\varepsilon_k}\cap U\neq\emptyset\}$$

and N_k is the number of elements of J_k. Notice that since L intersects ∂U in a finite number of points, we have

$$\varepsilon_k N_k \le \mathcal{H}^1(L\cap U)+o(1) \quad \text{as} \quad k\to+\infty. \tag{3.12}$$

The definition of v_k immediately yields

$$\lim_{k\to+\infty}\int_U|\nabla v_k|^2\,dx = \int_U |\nabla u|^2\,dx. \tag{3.13}$$

Moreover, when we come to estimate the other terms in (3.11), we easily find

$$\int_{R_k^i}|\nabla u_k|^2\,dx \le 2\int_{R_k^i}\Big|\frac{\partial_{x_1}v_k(x_1,a_k^+)-\partial_{x_1}v_k(x_1,a_k^-)}{a_k^+-a_k^-}\Big|^2(x_2-a_k^-)^2\,dx \\ +2\int_{R_k^i}|\partial_{x_1}v_k(x_1,a_k^-)|^2\,dx + \int_{R_k^i}\Big|\frac{v_k(x_1,a_k^+)-v_k(x_1,a_k^-)}{a_k^+-a_k^-}\Big|^2\,dx;$$

hence

$$\sum_{i\in J_k}\int_{R_k^i}|\nabla u_k|^2\,dx \le \frac{16}{3}N_k\varepsilon_k^3\|\nabla u\|^2_{L^\infty(R)} + 4N_k\varepsilon_k^3\|\nabla u\|^2_{L^\infty(R)} \\ +\sum_{i\in J_k}\varepsilon_k\int_{S_k^i}\Big|\frac{v_k(x_1,a_k^+)-v_k(x_1,a_k^-)}{a_k^+-a_k^-}\Big|^2\,dx_1.$$

Now let $v_k^\pm$ be the traces of v_k on both sides of the line $x_2=\lfloor\frac{c}{\varepsilon_k}\rfloor\varepsilon_k$, and for every $i\in J_k$ let $x_k^i := (i\varepsilon_k+\frac{\varepsilon_k}{2},\lfloor\frac{c}{\varepsilon_k}\rfloor\varepsilon_k)$ be the center of the rectangle R_k^i. Since v_k is Lipschtz continuous on each side of the line $x_2 = \lfloor \frac{c}{\varepsilon_k} \rfloor \varepsilon_k$ with constant $\|\nabla u\|_{L^\infty(R)}$, we obtain

$$\begin{aligned} &|v_k(x_1, a_k^+) - v_k(x_1, a_k^-)|^2 \\ &\le 2|[v_k]|(x_k^i)|^2 + 2|v_k(x_1, a_k^+) - v_k^+(x_k^i) - (v_k(x_1, a_k^-) - v_k^-(x_k^i))|^2 \\ &\le 2|[v_k](x_k^i)|^2 + 8\varepsilon_k^2 \|\nabla u\|^2_{L^\infty(R)}, \end{aligned}$$

for every $i \in J_k$ and every $x_1 \in S_k^i$. This implies

$$\begin{aligned} &\limsup_{k\to+\infty} \sum_{i\in J_k} \varepsilon_k \int_{S_k^i} \Big| \frac{v_k(x_1, a_k^+) - v_k(x_1, a_k^-)}{a_k^+ - a_k^-} \Big|^2 dx_1 \\ &\le 2 \limsup_{k\to+\infty} \sum_{i\in J_k} \varepsilon_k |[v_k](\bar{x}_k^i)|^2 + 16 \limsup_{k\to+\infty} N_k \varepsilon_k^3 \|\nabla u\|^2_{L^\infty(R)} = 2 \int_{L\cap U} [u]^2 \, d\mathcal{H}^1, \end{aligned} \tag{3.14}$$

where the last equality is a consequence of (3.12) and of the Lipschitz regularity of $[u]$. Hence, gathering (3.11), (3.12), (3.13), and (3.14), by definition of Γ-limsup we have

$$\mathcal{F}''(u, U) \le \int_U |\nabla u|^2 \, dx + 3 \int_{L\cap U} (1 + [u]^2) d\mathcal{H}^1.$$

Clearly, an analogous construction holds true if L is parallel to the x_2-axis.

Step 2. Let L be neither horizontal nor vertical.

We construct a sequence $(u_j) \subset \mathcal{W}(R)$ such that $S_{u_j} \subseteq L_j$, with L_j finite union of pairwise disjoint horizontal and vertical closed segments, and satisfying the following properties:

$$\begin{aligned} u_j \to u \quad &\text{strongly in } L^1(R), \\ \nabla u_j \to \nabla u \quad &\text{strongly in } L^2(R; \mathbb{R}^2), \\ \limsup_{j\to+\infty} \int_{L_j} (1 + [u_j]^2) \, d\mathcal{H}^1 &\le \sqrt{2} \int_L (1 + [u]^2) \, d\mathcal{H}^1. \end{aligned} \tag{3.15}$$

Then, we deduce (3.9) by Step 1 and by the lower semicontinuity of $\mathcal{F}''$.

To construct this sequence, we consider the intersections $R^\ominus$ and $R^\oplus$ of R with the open half-planes determined by the line containing L. As $u \in \mathcal{W}(R)$, there exist two functions $u^\ominus, u^\oplus \in W^{2,\infty}(R)$ such that

$$u = \begin{cases} u^\ominus & \text{in } R^\ominus \\ u^\oplus & \text{in } R^\oplus. \end{cases}$$

For every integer $j \ge 1$, let P_j be a polygonal line consisting only of horizontal and vertical segments within a distance proportional to $1/j$ from L, as in Figure 5. We assume that P_j and L have the same end points. By completing P_j with the half-lines prolonging L, R is divided into two sets $R_j^\ominus$, $R_j^\oplus$, such that

$$\mathcal{L}^2(R^\oplus \triangle R_j^\oplus) \longrightarrow 0 \quad \text{as} \quad j \to +\infty.$$

Then, we define

$$v_j := u^\ominus \chi_{R_j^\ominus} + u^\oplus \chi_{R_j^\oplus}.$$

Clearly $v_j \in SBV^2(R) \cap L^\infty(R)$, $v_j \in W^{2,\infty}(R \setminus P_j)$. Moreover, v_j satisfies (3.15) with $L_j = P_j$.

Finally, using a capacitary argument as in [11, Corollary 3.11], we may modify (v_j) near the vertices of P_j, obtaining a new sequence (u_j) still satisfying (3.15), with $u_j \in SBV^2(R) \cap L^\infty(R)$

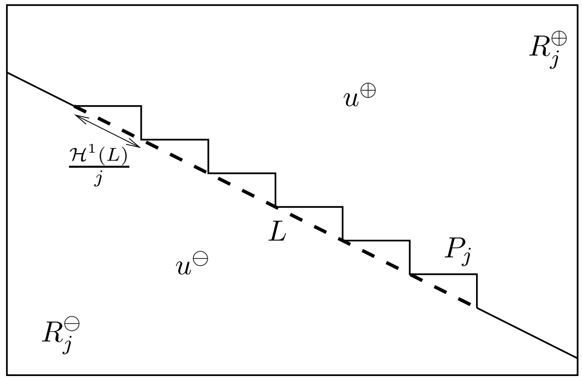

FIGURE 5. Construction of the polygonal line P_j.

and $u_j \in W^{2,\infty}(R\setminus\widetilde{P}_j)$, where $\widetilde{P}_j$ is obtained from P_j by removing small balls around its vertices and is therefore the union of a finite number of pairwise disjoint horizontal and vertical closed segments. □

A first consequence of the fundamental estimate Proposition 3.1 and of the upper bound stated in Proposition 3.3 is given by the following compactness result.

Theorem 3.4 (Compactness by Γ-convergence). *Let $\mathcal{F}_\varepsilon$ be as in* (3.1). *Then, for every sequence of positive numbers converging to* 0 *there exist a subsequence* (ε_k) *and a functional* $\mathcal{F}: SBV^2(\Omega)\times\mathcal{A}(\Omega)\longrightarrow[0,+\infty]$ *such that for every* $U\in\mathcal{A}(\Omega)$

$$\mathcal{F}(\cdot,U)=\mathcal{F}'_-(\cdot,U)=\mathcal{F}''_-(\cdot,U),$$

with $\mathcal{F}'_-,\mathcal{F}''_-$ *as in* (3.2) *and* (3.3), *respectively.*

Moreover, $\mathcal{F}$ *satisfies the following properties:*

- *for every* $U\in\mathcal{A}(\Omega)$, *the functional* $\mathcal{F}(\cdot,U)$ *is local and lower semicontinuous with respect to the strong* $L^1(\Omega)$*-topology;*
- *for every* $u\in SBV^2(\Omega)$, *the set function* $\mathcal{F}(u,\cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω.
- *for every* $U\in\mathcal{A}_L(\Omega)$

$$\mathcal{F}(\cdot,U)=\mathcal{F}'(\cdot,U)=\mathcal{F}''(\cdot,U)\quad \textit{on}\ \ SBV^2(\Omega).$$

Proof. Let $\mathcal{R}$ be the class of all finite unions of open rectangles contained in Ω, whose vertices have rational coordinates. By the compactness of Γ-convergence [13, Theorem 8.5], a diagonal argument yields the existence of a subsequence (ε_k) such that $\mathcal{F}_{\varepsilon_k}(\cdot,R)$ Γ-converges to a functional $\mathcal{F}_0(\cdot,R)$, for all $R\in\mathcal{R}$; *i.e.,*

$$\mathcal{F}_0(u,R)=\mathcal{F}'(u,R)=\mathcal{F}''(u,R)\quad\text{for every } u\in L^1(\Omega),\ R\in\mathcal{R}. \tag{3.16}$$

For $u\in SBV^2(\Omega)$ and for all $U\in\mathcal{A}(\Omega)$, set

$$\mathcal{F}(u,U):=\sup\{\mathcal{F}_0(u,R)\colon R\subset\subset U,\ R\in\mathcal{R}\}.$$

For every $U, U' \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$, there exists $R \in \mathcal{R}$ such that $U' \subset\subset R \subset\subset U$; hence by (3.16) we get

$$\begin{aligned}\mathcal{F}(u,U) &= \sup\{\mathcal{F}'(u,U') \colon U' \subset\subset U,\, U' \in \mathcal{A}(\Omega)\}\\ &= \sup\{\mathcal{F}''(u,U') \colon U' \subset\subset U,\, U' \in \mathcal{A}(\Omega)\},\end{aligned} \tag{3.17}$$

for all $U \in \mathcal{A}(\Omega)$; that is $\mathcal{F}$ is the inner regular envelope of both $\mathcal{F}'$ and $\mathcal{F}''$. Hence the set function $\mathcal{F}(u,\cdot)$ is inner regular (see [13, Remark 15.10]) and superadditive (see [13, Proposition 16.12]).

Let us prove now that $\mathcal{F}(u,\cdot)$ is also subadditive for every $u \in SBV^2(\Omega)$. The main difference from the general treatment developed in [13, Chapter 18] is that in our case the rest in the fundamental estimate vanishes only for recovery sequences converging in $L^2(\Omega)$ (see Proposition 3.1), while we are studying the Γ-convergence with respect to the $L^1(\Omega)$-topology.

We start by observing that on $SBV^2(\Omega) \cap L^\infty(\Omega)$, (3.17) is equivalent to the two following conditions:

i) for every $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, for every $U \in \mathcal{A}(\Omega)$ and for every sequence $(u_k) \subset SBV^2(U) \cap L^1(\Omega)$ with $S_{u_k} \cap U \subseteq \Omega_{\varepsilon_k}$, such that $u_k \to u$ in $L^1(\Omega)$ it is

$$\mathcal{F}(u,U) \leq \liminf_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,U);$$

ii) for every $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ and for every $U, U' \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$, there exists a sequence $(u_k) \subset SBV^2(U') \cap L^1(\Omega)$ with $S_{u_k} \cap U' \subseteq \Omega_{\varepsilon_k}$, $u_k \to u$ in $L^1(\Omega)$ such that

$$\mathcal{F}(u,U) \geq \limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,U')$$

(see also [13, Proposition 16.4 and Remark 16.5]).

Now let $U, V \in \mathcal{A}(\Omega)$ and let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Fix any $U' \subset\subset U$, $V' \subset\subset V$, $U', V' \in \mathcal{A}(\Omega)$. Choose an open set U'' such that $U' \subset\subset U'' \subset\subset U$ and two sequences $(u_k) \subset SBV^2(U'') \cap L^1(\Omega)$, $S_{u_k} \cap U'' \subseteq \Omega_{\varepsilon_k}$, $u_k \to u$ in $L^1(\Omega)$ and $(v_k) \subset SBV^2(V') \cap L^1(\Omega)$, $S_{v_k} \cap V' \subseteq \Omega_{\varepsilon_k}$, $v_k \to u$ in $L^1(\Omega)$ such that

$$\limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,U'') \leq \mathcal{F}(u,U), \quad \limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(v_k,V') \leq \mathcal{F}(u,V).$$

Since the functionals $\mathcal{F}_{\varepsilon_k}$ decrease by truncation, we can additionally assume that $\|u_k\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$, $\|v_k\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$; hence $u_k \to u$ in $L^2(\Omega)$ and $v_k \to u$ in $L^2(\Omega)$.

Let us fix $\eta > 0$. Then, the fundamental estimate Proposition 3.1 gives a constant $M(\eta) > 0$ and a sequence (φ_k) of cut-off functions between U' and U'' such that

$$\begin{aligned}&\mathcal{F}_{\varepsilon_k}(\varphi_k u_k + (1-\varphi_k)v_k, U' \cup V')\\ &\leq (1+\eta)\Big(\mathcal{F}_{\varepsilon}(u_k,U'') + \mathcal{F}_{\varepsilon_k}(v_k,V')\Big) + M(\eta)\|u_k - v_k\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence, taking the limit as $k \to +\infty$, and noticing that $\varphi_k u_k + (1-\varphi_k)v_k \to u$ in $L^1(\Omega)$, we get

$$\mathcal{F}(u, U' \cup V') \leq (1+\eta)\Big(\mathcal{F}(u,U) + \mathcal{F}(u,V)\Big).$$

Now let $\eta \to 0$, and then $U' \nearrow U$, $V' \nearrow V$; it turns out that

$$\mathcal{F}(u, U \cup V) \leq \mathcal{F}(u,U) + \mathcal{F}(u,V),$$

so we have proved the subadditivity of $\mathcal{F}$ at least for $SBV^2(\Omega) \cap L^\infty(\Omega)$ functions.

Now let $u \in SBV^2(\Omega)$ and, for every $n \in \mathbb{N}$, set $u^n := (u \wedge n) \vee (-n)$. Since $u^n \in L^\infty(\Omega)$, and moreover since the values of $\mathcal{F}$ decrease by truncation, we have

$$\mathcal{F}(u^n, U \cup V) \leq \mathcal{F}(u^n, U) + \mathcal{F}(u^n, V) \leq \mathcal{F}(u, U) + \mathcal{F}(u, V).$$

On the other hand, as $u^n \to u$ in $L^1(\Omega)$, the lower semicontinuity of $\mathcal{F}$ yields

$$\mathcal{F}(u, U \cup V) \leq \liminf_{n \to +\infty} \mathcal{F}(u^n, U \cup V) \leq \mathcal{F}(u, U) + \mathcal{F}(u, V),$$

and hence the subadditivity of $\mathcal{F}$. Therefore, by the measure-property criterion of De Giorgi and Letta, $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω (see [13, Theorem 14.23]).

Appealing to Propositions 3.1 and 3.3, now we prove that $\mathcal{F}''$ is inner regular on the class of all open subsets of Ω with Lipschitz boundary. Indeed, let $\mathcal{G}\colon SBV^2(\Omega) \cap L^\infty(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0, +\infty)$ be the functional defined as

$$\mathcal{G}(u, U) := \int_U |\nabla u|^2\, dx + \int_{S_u \cap U} (1 + [u]^2) d\mathcal{H}^1,$$

and fix $W \in \mathcal{A}_L(\Omega)$. Since $\mathcal{G}$ is a measure, for every $\eta > 0$ there exists a compact set $K \subset W$ such that $W \setminus K \in \mathcal{A}_L(\Omega)$ and $\mathcal{G}(u, W \setminus K) < \eta$.

Choose $U, U' \in \mathcal{A}(\Omega)$ satisfying

$$K \subset U' \subset\subset U \subset\subset W$$

and set $V := W \setminus K$.

Recalling that $\mathcal{F}''$ is increasing, Proposition 3.1 easily yields

$$\mathcal{F}''(u, W) \leq \mathcal{F}''(u, U' \cup V) \leq \mathcal{F}''(u, U) + \mathcal{F}''(u, V) = \mathcal{F}''(u, U) + \mathcal{F}''(u, W \setminus K).$$

Moreover, by the definition of $\mathcal{F}''_-$ and Proposition 3.3 we have

$$\mathcal{F}''(u, W) \leq \mathcal{F}''_-(u, W) + C\,\mathcal{G}(u, W \setminus K) \leq \mathcal{F}''_-(u, W) + C\eta,$$

for some $C > 0$. Hence by the arbitrariness of η we get

$$\mathcal{F}''(u, W) \leq \mathcal{F}''_-(u, W) \quad \text{for every } W \in \mathcal{A}_L(\Omega),\ u \in SBV^2(\Omega) \cap L^\infty(\Omega).$$

Invoking the $L^1(\Omega)$ lower semicontinuity of $\mathcal{F}''$, the above inequality can be recovered on the whole $SBV^2(\Omega)$ with the usual truncation argument. Therefore, as the opposite inequality is trivial, we may deduce that $\mathcal{F}''(u, \cdot)$ is inner regular on the class of all open subsets of Ω with Lipschitz boundary, hence by (3.17)

$$\mathcal{F}(u, U) = \mathcal{F}'(u, U) = \mathcal{F}''(u, U),$$

for all $U \in \mathcal{A}_L(\Omega)$ and for every $u \in SBV^2(\Omega)$. Thus, the complete proof is achieved. □

In the next proposition we relax estimate (3.8) and we show that the restriction of $\mathcal{F}''$ to $SBV^2(\Omega) \times \mathcal{A}_L(\Omega)$ satisfies a bound from above similar to (4.1).

Proposition 3.5 (Upper bound)**.** *There exists $\beta > 0$ such that*

$$\mathcal{F}''(u, U) \leq \int_U |\nabla u|^2\, dx + \beta \int_{S_u \cap U} (1 + |[u]|) d\mathcal{H}^1, \tag{3.18}$$

for every $U \in \mathcal{A}_L(\Omega)$ and for every $u \in SBV^2(\Omega)$.

Proof. We fix $U \in \mathcal{A}_L(\Omega)$; since $\mathcal{F}''(\cdot, U)$ is lower semicontinuous in $L^1(\Omega)$, by a truncation argument it is enough to prove (3.18) when $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, in view of Lemma 3.2 and of the locality of $\mathcal{F}''$, we may restrict ourselves to showing that

$$\mathcal{F}''(u, U) \leq \int_U |\nabla u|^2 \, dx + \beta \int_{L \cap U} (1 + |[u]|) d\mathcal{H}^1,$$

for $u \in \mathcal{W}(R)$.

To simplify the exposition, we additionally assume that L is a single segment; the general case follows easily. Then, using the same notation employed in Proposition 3.3, Step 2 we have

$$u = \begin{cases} u^\ominus & \text{in } R^\ominus \\ u^\oplus & \text{in } R^\oplus. \end{cases}$$

Let $n \geq 1$, we define a sequence (u_j) as

$$u_j := \begin{cases} u^\ominus & \text{in } R^\ominus \\ u^\ominus + \dfrac{i}{n}(u^\oplus - u^\ominus) & \text{in } T_j^i, \ i = 1, \dots, n \\ u^\oplus & \text{in } R^\oplus \setminus \bigcup_{i=1}^n T_j^i, \end{cases}$$

where T_j^i $(i = 1, \dots, n)$ are the "thin" open rectangles in Figure 6.

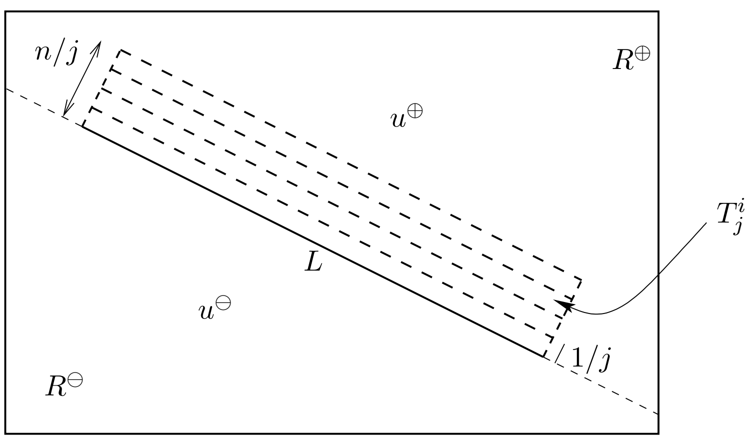

FIGURE 6. Construction of the function u_j.

Clearly, $u_j \in SBV^2(R) \cap L^\infty(R)$, $S_{u_j} \subseteq L_j$, where L_j is given by the union of L with the sides of the rectangles T_j^i, for $i = 1, \dots, n$, and $u_j \in W^{2,\infty}(R \setminus L_j)$. Moreover, by definition of u_j and by virtue of its regularity, it is easy to check that the following conditions are satisfied:

$$\begin{gathered} u_j \to u \quad \text{strongly in } L^1(R), \\ \nabla u_j \to \nabla u \quad \text{strongly in } L^2(R; \mathbb{R}^2), \\ \lim_{j \to +\infty} \int_{L_j} (1 + [u_j]^2) \, d\mathcal{H}^1 = \int_L n \Big(1 + \frac{[u]^2}{n^2} \Big) \, d\mathcal{H}^1. \end{gathered} \tag{3.19}$$

On the other hand, Proposition 3.3 and Theorem 3.4 ensure that

$$\begin{aligned}\mathcal{F}(u_j,V) &\le \int_V |\nabla u_j|^2\,dx + \beta\int_{S_{u_j}\cap V}(1+[u_j]^2)d\mathcal{H}^1\\ &\le \int_V |\nabla u_j|^2\,dx + \beta\int_{L_j\cap V}(1+[u_j]^2)d\mathcal{H}^1,\end{aligned}$$

for every $V\in\mathcal{A}(\Omega)$. Therefore, the $L^1(\Omega)$ lower semicontinuity of $\mathcal{F}$ and (3.19) directly yield

$$\mathcal{F}(u,V) \le \int_V |\nabla u|^2\,dx + \beta\int_{L\cap V} n\Big(1+\frac{[u]^2}{n^2}\Big)d\mathcal{H}^1, \tag{3.20}$$

for every $V\in\mathcal{A}(\Omega)$ and for every $n\ge 1$.

In view of Theorem 3.4 we know that we may extend $\mathcal{F}(u,\cdot)$ to a measure on $\mathcal{B}(\Omega)$ as follows

$$\mathcal{F}^*(u,B) := \inf\{\mathcal{F}(u,V)\colon B\subseteq V,\ V\in\mathcal{A}(\Omega)\},\quad \text{for every } B\in\mathcal{B}(\Omega).$$

Then, setting

$$f_n(x) := n\Big(1+\frac{[u]^2(x)}{n^2}\Big)\quad \text{for } x\in L,\ n\ge 1$$

by (3.20) we deduce

$$\mathcal{F}^*(u,B) \le \int_B |\nabla u|^2\,dx + \beta\int_{L\cap B} f_n\,d\mathcal{H}^1, \tag{3.21}$$

for every $B\in\mathcal{B}(\Omega)$ and for every $n\ge 1$.

Let $N\ge 1$ be an integer; there exists a Borel partition $\{S^n\}_{n=1,\dots,N}$, of L such that

$$f_n(x) = \min_{1\le i\le N} f_i(x),\quad \text{for every } x\in S^n,\quad n=1,\dots,N.$$

Hence, for every $B\in\mathcal{B}(\Omega)$ we have

$$\begin{aligned}\mathcal{F}^*(u,B) = \mathcal{F}^*\Big(u,(B\setminus L)\cup\bigcup_{i=1}^N (S^i\cap B)\Big) &= \mathcal{F}^*(u,B\setminus L) + \sum_{i=1}^N \mathcal{F}^*(u,S^i\cap B)\\ &\le \int_B |\nabla u|^2\,dx + \sum_{i=1}^N \beta\int_{S^i\cap B} f_i\,d\mathcal{H}^1,\end{aligned}$$

the last inequality being a consequence of (3.21). Then, by the definition of S^i and f_i we get

$$\begin{aligned}\mathcal{F}^*(u,B) &\le \int_B |\nabla u|^2\,dx + \beta\int_{L\cap B}\min_{1\le n\le N} f_n\,d\mathcal{H}^1\\ &= \int_B |\nabla u|^2\,dx + \beta\int_{L\cap B}\min_{1\le n\le N} n\Big(1+\frac{[u]^2}{n^2}\Big)\,d\mathcal{H}^1,\end{aligned} \tag{3.22}$$

for every $B\in\mathcal{B}(\Omega)$ and for every $N\ge 1$.

Now let $N\ge \|u\|_{L^\infty(R)}+1$; for any $t\in\mathbb{R}$ such that $|t|\le\|u\|_{L^\infty(R)}$, using as a test $n=\lfloor|t|\rfloor+1$ we find

$$\min_{1\le n\le N} n\Big(1+\frac{t^2}{n^2}\Big) \le 1+2|t|,$$

this combined with (3.22) yields

$$\mathcal{F}^*(u,B) \le \int_B |\nabla u|^2\,dx + \beta \int_{L\cap B} (1+2|[u]|)d\mathcal{H}^1.$$

Finally, choosing $B = U \in \mathcal{A}_L(\Omega)$ and appealing to Theorem 3.4 we obtain the thesis. □

Remark 3.6. Proposition 3.5, combined with the trivial bound $MS \le \mathcal{F}'$ and the inner regularity of $MS(u,\cdot)$, easily yields

$$MS(u,U) \le \mathcal{F}(u,U) \le \int_U |\nabla u|^2\,dx + \beta \int_{S_u\cap U} (1+|[u]|)\,d\mathcal{H}^1$$

for every $u \in SBV^2(\Omega)$, $U \in \mathcal{A}(\Omega)$. Hence, from this estimate and from Theorem 3.4 we may deduce that for $u \in SBV^2(\Omega)$, $\mathcal{F}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on Ω.

4. Integral representation on $SBV^2(\Omega)$

On account of Theorem 3.4, we now complete the proof of Theorem 2.2, that is we identify the functional $\mathcal{F}$. Therefore, we assume that a sequence (ε_k) of positive numbers converging to 0 is given, such that for every $U \in \mathcal{A}(\Omega)$

$$\mathcal{F}(\cdot,U) = \mathcal{F}'_-(\cdot,U) = \mathcal{F}''_-(\cdot,U) \quad \text{on} \quad SBV^2(\Omega).$$

For the reader's convenience we recall here the statement of the representation theorem we are going to employ.

The following theorem is a particular case of the representation result [8, Theorem 1] by Bouchitté, Fonseca, Leoni, and Mascarenhas (see also the earlier work by Braides and Chiadò Piat [9]).

Theorem 4.1. *Let $\mathcal{E} : SBV^2(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0,+\infty]$ be a functional satisfying the following conditions:*

(i) (locality) *$\mathcal{E}(u,U) = \mathcal{E}(v,U)$ whenever $u = v$ $\mathcal{L}^2$-a.e. on $U \in \mathcal{A}(\Omega)$;*
(ii) (measure property) *for every $u \in SBV^2(\Omega)$ the set function $\mathcal{E}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;*
(iii) (lower semicontinuity) *for all $U \in \mathcal{A}(\Omega)$ the functional $\mathcal{E}(\cdot,U)$ is lower semicontinuous on $SBV^2(\Omega)$ with respect to the strong $L^1(\Omega)$-topology;*
(iv) (growth condition) *there exists $C > 0$ such that for every $(u,U) \in SBV^2(\Omega) \times \mathcal{A}(\Omega)$*

$$\begin{aligned}\frac{1}{C}\Big(\int_U |\nabla u|^2\,dx + \int_{S_u\cap U} (1+|[u]|)d\mathcal{H}^1\Big) &\le \mathcal{E}(u,U)\\ &\le C\Big(\int_U (1+|\nabla u|^2)\,dx + \int_{S_u\cap U}(1+|[u]|)d\mathcal{H}^1\Big).\end{aligned} \tag{4.1}$$

Then, there exist Borel functions $f_0 : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow [0,+\infty)$ and $g_0 : \Omega\times\mathbb{R}\times\mathbb{R}\times S^1 \longrightarrow [0,+\infty)$ such that

$$\mathcal{E}(u,U) = \int_U f_0(x,u,\nabla u)\,dx + \int_{S_u\cap U} g_0(x,u^+,u^-,\nu_u)d\mathcal{H}^1$$

for every pair $(u,U) \in SBV^2(\Omega)\times\mathcal{A}(\Omega)$. Moreover, the following derivation formulas hold true

$$f_0(y,s,\xi) = \limsup_{\rho\to 0^+} \frac{\mathbf{m}(s+\xi(\cdot-y); Q^\nu_\rho(y))}{\rho^2} \tag{4.2}$$

$$g_0(y,a,b,\nu) = \limsup_{\rho\to 0^+} \frac{\mathbf{m}(u^{\nu,y}_{a,b}; Q^\nu_\rho(y))}{\rho}, \tag{4.3}$$

for every $y \in \Omega$, $s, a, b \in \mathbb{R}$, $\xi \in \mathbb{R}^2, \nu \in S^1$, where for every $u \in SBV^2(\Omega)$ and $U \in \mathcal{A}(\Omega)$

$$\mathbf{m}(u; U) := \inf\{\mathcal{E}(v, U)\colon v \in SBV^2(U),\ v = u \ \textit{in a neighborhood of}\ \partial U\}, \tag{4.4}$$

and

$$u^{\nu,y}_{a,b}(x) := \begin{cases} a & \textit{if}\,(x-y)\cdot\nu > 0 \\ b & \textit{if}\,(x-y)\cdot\nu \le 0. \end{cases}$$

As $u^{\nu,y}_{a,b} = u^{-\nu,y}_{b,a}$ $\mathcal{L}^2$-a.e. in $Q^\nu_\rho(y) = Q^{-\nu}_\rho(y)$, we have $g_0(y,a,b,\nu) = g_0(y,b,a,-\nu)$ for every $y \in \Omega$, $a, b \in \mathbb{R}$, $\nu \in S^1$.

Let $U \in \mathcal{A}(\Omega)$ and $y \in \mathbb{R}^2$; we define the set $\tau_y U := U + y$ and for $u \in L^1(U)$ we define the function $\tau_y u \in L^1(\tau_y U)$ as $\tau_y u(x) := u(x-y)$.

Now we are ready to state the following lemma.

Lemma 4.2 (Translational invariance of the Γ-limit)**.** *Let (ε_k) be a sequence of positive numbers converging to 0 such that for every $U \in \mathcal{A}_L(\Omega)$*

$$\mathcal{F}(\cdot, U) = \mathcal{F}'(\cdot, U) = \mathcal{F}''(\cdot, U) \quad \textit{on} \quad SBV^2(\Omega).$$

Then, for every $u \in SBV^2(\Omega)$

1. (translation invariance in u) *$\mathcal{F}(u+s, U) = \mathcal{F}(u, U)$, for all $s \in \mathbb{R}$, $U \in \mathcal{A}_L(\Omega)$;*
2. (translation invariance in x) *$\mathcal{F}(v, \tau_y U) = \mathcal{F}(u, U)$, for every $y \in \mathbb{R}^2$, for every $U \in \mathcal{A}_L(\Omega)$ such that $\tau_y U \subset\subset \Omega$, and for every $v \in SBV^2(\Omega)$ such that $v = \tau_y u$ a.e. on $\tau_y U$.*

Proof. We start proving (1). For fixed $U \in \mathcal{A}_L(\Omega)$ and $u \in SBV^2(\Omega)$, let $(u_k) \subset L^1(\Omega)$ with $u_k \in SBV^2(U)$, $S_{u_k} \cap U \subseteq \Omega_{\varepsilon_k}$ be a sequence converging to u in $L^1(\Omega)$, and such that $\lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k, U) = \mathcal{F}(u, U)$. Since for $s \in \mathbb{R}$, $(u_k + s)$ converges to $u + s$ in $L^1(\Omega)$, we get

$$\mathcal{F}(u+s, U) \le \liminf_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k + s, U) = \lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k, U) = \mathcal{F}(u, U).$$

On the other hand, $\mathcal{F}(u, U) = \mathcal{F}((u+s) + (-s), U) \le \mathcal{F}((u+s), U)$.

Now we prove (2). For every fixed $y \in \mathbb{R}^2$ and $U \in \mathcal{A}_L(\Omega)$ such that $\tau_y U \subset\subset \Omega$, let $y_k := \lfloor \frac{y}{\varepsilon_k} \rfloor \varepsilon_k$ (here the integer part is meant component-wise) then $\tau_{y_k} U \subset\subset \Omega$, for k large enough.

Let u_k be as in the proof of (1) and let $v_k \in L^1(\Omega)$ be such that $v_k := \tau_{y_k} u_k$ in $\tau_{y_k} U$. Then, $v_k \in SBV^2(\tau_{y_k} U) \cap L^1(\Omega)$ and $S_{v_k} \cap \tau_{y_k} U \subseteq \Omega_{\varepsilon_k}$.

Taking into account the definition of $\mathcal{F}_{\varepsilon_k}$, a change of variable directly yields

$$\mathcal{F}_{\varepsilon_k}(u_k, U) = \mathcal{F}_{\varepsilon_k}(v_k, \tau_{y_k} U).$$

Let $V \subset\subset U$; for k sufficiently large we may assume that $\tau_y V \subseteq \tau_{y_k} U$, hence in view of the nondeceasing character of $\mathcal{F}_\varepsilon$ we have

$$\mathcal{F}_{\varepsilon_k}(u_k, U) \ge \mathcal{F}_{\varepsilon_k}(v_k, \tau_y V). \tag{4.5}$$

Setting

$$\tilde{v}_k := \begin{cases} v_k & \text{in } \tau_y V, \\ \tau_y u & \text{elsewhere in } \Omega, \end{cases}$$

since $\tilde{v}_k \to \tau_y u$ in $L^1(\Omega)$, taking the liminf of both sides of (4.5), in view of the locality of $\mathcal{F}_{\varepsilon_k}$, we get

$$\mathcal{F}(u,U) = \lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,U) \geq \liminf_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(v_k,\tau_y V) \geq \mathcal{F}(\tau_y u,\tau_y V).$$

Then, by the arbitrariness of $V \subset\subset U$ and the inner regularity of $\mathcal{F}$ we finally obtain

$$\mathcal{F}(u,U) \geq \mathcal{F}(\tau_y u,\tau_y U).$$

We conclude the proof of (ii) by noticing that $\mathcal{F}(\tau_y u,\tau_y U) \geq \mathcal{F}(\tau_{-y}(\tau_y u),\tau_{-y}(\tau_y U)) = \mathcal{F}(u,U)$. $\square$

Now we are in a position to prove the main result Theorem 2.2.

Proof of Theorem 2.2. To get the proof, we first apply the integral representation result Theorem 4.1, on $SBV^2(\Omega)$, and then we recover the Γ-convergence result on the whole $L^1(\Omega)$ by using a truncation argument.

We start noticing that in view of Theorem 3.4 and Remark 3.6, $\mathcal{F}$ satisfies hypotheses (i)-(iii) of Theorem 4.1, as well as the upper bound in hypothesys (iv).

To completely fit the assumptions of Theorem 4.1, we now use a perturbation argument which permits to recover the growth condition from below required in (iv).

In this respect, we fix $\sigma > 0$ and we consider the functionals

$$\mathcal{F}^\sigma(u,U) := \mathcal{F}(u,U) + \sigma \int_{S_u\cap U} (1+|[u]|)d\mathcal{H}^1.$$

Then, $\mathcal{F}^\sigma$ satisfies hypotheses (i)-(iv), for every $\sigma > 0$. Indeed, (i) and (ii) are trivial, while (iii) and (iv) follow from Remark 3.6 using Ambrosio's lower semicontinuity Theorem [2, Theorem 3.7]. Hence, Theorem 4.1 ensures the existence of two Borel functions $f_0^\sigma : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow [0,+\infty)$ and $g_0^\sigma : \Omega\times\mathbb{R}\times\mathbb{R}\times S^1 \longrightarrow [0,+\infty)$ such that

$$\mathcal{F}^\sigma(u,U) = \int_U f_0^\sigma(x,u,\nabla u)\,dx + \int_{S_u\cap U} g_0^\sigma(x,u^+,u^-,\nu_u)d\mathcal{H}^1$$

for every $u \in SBV^2(\Omega)$, $U \in \mathcal{A}(\Omega)$.

By virtue of Lemma 4.2, and in view of (4.2)-(4.3), we may conclude that both f_0^σ and g_0^σ are independent of x, that f_0^σ does not depend on u, and that g_0^σ depends on (u^+,u^-) only through their difference $[u]$; *i.e.*, $f_0^\sigma(y,s,\xi) = f^\sigma(\xi)$ and $g_0^\sigma(y,a,b,\nu) = g^\sigma(a-b,\nu)$ for some Borel functions $f^\sigma : \mathbb{R}^2 \longrightarrow [0,+\infty)$, $g^\sigma : \mathbb{R}\times S^1 \longrightarrow [0,+\infty)$. Moreover, setting $u_\xi(x) := \xi\cdot x$, the growth conditions on $\mathcal{F}$ immediately yield

$$|\xi|^2\mathcal{L}^2(\Omega) \leq f^\sigma(\xi)\mathcal{L}^2(\Omega) = \mathcal{F}^\sigma(u_\xi,\Omega) \leq |\xi|^2\mathcal{L}^2(\Omega), \quad \forall\,\xi\in\mathbb{R}^2$$

hence $f^\sigma(\xi) = |\xi|^2$. On the other hand, by construction, the family $(g^\sigma)_{\sigma>0}$ is decreasing as σ decreases, therefore setting $g := \lim_{\sigma\to0^+} g^\sigma$, by the pointwise convergence of $(\mathcal{F}^\sigma)_{\sigma>0}$ to $\mathcal{F}$ and the Monotone Convergence Theorem, we get

$$\mathcal{F}(u,U) = \int_U |\nabla u|^2\,dx + \int_{S_u\cap U} g([u],\nu_u)d\mathcal{H}^1 \tag{4.6}$$

for every $u \in SBV^2(\Omega)$, $U \in \mathcal{A}(\Omega)$.

To get the bounds (2.4) on g, we let $y \in \Omega$, $\nu \in S^1$, choose $\rho > 0$ such that $Q_\rho^\nu(y) \subset \Omega$, and set

$$u_t^{\nu,y}(x) := \begin{cases} t & \text{if} \quad (x-y)\cdot\nu > 0 \\ 0 & \text{if} \quad (x-y)\cdot\nu \leq 0, \end{cases}$$

hence

$$g(t,\nu)\rho = \mathcal{F}(u_t^{\nu,y}, Q_\rho^\nu(y)) = \Gamma\text{-}\lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_t^{\nu,y}, Q_\rho^\nu(y)).$$

Then, the upper bound on g directly follows from Proposition 3.5, while the lower bound is a consequence of [5, Theorem 3.1] (see also [5, Remark 3.3]) combined with the trivial bound $MS \leq \mathcal{F}$.

We now prove that for every fixed $\nu \in S^1$, $g(\cdot,\nu)$ is nondecreasing on $(0,+\infty)$.

Let $0 < t_1 < t_2$, $y \in \Omega$ and $\rho > 0$ be such that $Q_\rho^\nu(y) \subset \Omega$; we have

$$g(t_1,\nu)\rho = \mathcal{F}(u_{t_1}^{\nu,y}, Q_\rho^\nu(y)) \quad \text{and} \quad g(t_2,\nu)\rho = \mathcal{F}(u_{t_2}^{\nu,y}, Q_\rho^\nu(y)) = \mathcal{F}\Big(\frac{t_2}{t_1} u_{t_1}^{\nu,y}, Q_\rho^\nu(y)\Big).$$

Let $(v_k) \subset SBV^2(Q_\rho^\nu(y)) \cap L^1(\Omega)$ be a sequence such that $S_{v_k} \cap Q_\rho^\nu(y) \subseteq \Omega_{\varepsilon_k}$, $v_k \to u_{t_2}^{\nu,y}$ in $L^1(\Omega)$, and $\mathcal{F}_{\varepsilon_k}(v_k, Q_\rho^\nu(y)) \to \mathcal{F}(u_{t_2}^{\nu,y}, Q_\rho^\nu(y))$. Then, if we set $u_k := \frac{t_1}{t_2} v_k$ we get

$$\begin{aligned} \mathcal{F}(u_{t_1}^{\nu,y}, Q_\rho^\nu(y)) &\leq \liminf_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k, Q_\rho^\nu(y)) = \liminf_{k\to+\infty} \mathcal{F}_{\varepsilon_k}\Big(\frac{t_1}{t_2} v_k, Q_\rho^\nu(y)\Big) \\ &\leq \limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(v_k, Q_\rho^\nu(y)) = \mathcal{F}(u_{t_2}^{\nu,y}, Q_\rho^\nu(y)). \end{aligned}$$

Hence, in view of Theorem 3.4, so far we have proved that for every $U \in \mathcal{A}_L(\Omega)$, and for every $u \in SBV^2(\Omega)$

$$\Gamma\text{-}\lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u,U) = \mathcal{F}(u,U) = \int_U |\nabla u|^2\,dx + \int_{S_u\cap U} g([u],\nu_u)\,d\mathcal{H}^1;$$

thus, in particular choosing $U = \Omega$

$$\Gamma\text{-}\lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u) = \mathcal{F}(u,\Omega) = \int_\Omega |\nabla u|^2\,dx + \int_{S_u} g([u],\nu_u)\,d\mathcal{H}^1, \tag{4.7}$$

on $SBV^2(\Omega)$.

To complete the proof it remains to show that for every $u \in L^1(\Omega)$

$$\mathcal{F}'(u,\Omega) < +\infty \quad \Longrightarrow \quad u \in SBV^2(\Omega). \tag{4.8}$$

Let $u \in L^1(\Omega)$ with $\mathcal{F}'(u,\Omega) < +\infty$. By Remark 2.1 we know that u belongs also to $GSBV^2(\Omega)$. Then, if $u^n := (u \wedge n) \vee (-n)$ we have $u^n \in SBV^2(\Omega)$, for every $n \in \mathbb{N}$.

Appealing to (4.7) and recalling that $\mathcal{F}'$ decreases by truncation give

$$\mathcal{F}(u^n,\Omega) = \mathcal{F}'(u^n,\Omega) \leq \mathcal{F}'(u,\Omega) < +\infty.$$

Hence, in view of the lower bound on g we may further deduce that

$$\int_\Omega |\nabla u^n|^2\,dx + \frac{1}{2}\int_{S_{u^n}} (1+\alpha|[u^n]|)\,d\mathcal{H}^1 \leq C < +\infty, \tag{4.9}$$

for every $n \in \mathbb{N}$ and for some $C > 0$.

This ensures that (u_n) is bounded in $BV(\Omega)$. As $u^n \to u$ in $L^1(\Omega)$, we deduce that $u \in BV(\Omega)$ and $u^n \rightharpoonup u$ weakly* in $BV(\Omega)$. Finally, the closure theorem of SBV [4, Theorem 4.7] entails $u \in SBV^2(\Omega)$. This proves (4.8). $\square$

References

[1] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881.

[2] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[3] L. Ambrosio and A. Braides, Functionals defined on partitions of sets of finite perimeter II: semicontinuity, relaxation and homogenization, *J. Math. Pures Appl.* **69** (1990), 307-333.

[4] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] M. Barchiesi and G. Dal Maso, Homogenization of fiber reinforced brittle materials: the extremal cases, *preprint*, downloadable at `http://cvgmt.sns.it`.

[6] G. I. Barenblatt, On some basic ideas of the theory of equilibrium cracks, forming during brittle fracture. 1961 *Problems of continuum mechanics (Muskhelishvili anniversary volume)*, 21-38 *SIAM, Philadelphia, Pa.*

[7] G. I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture. 1962 *Advances in Applied Mechanics*, Vol. **7**, 55-129 Academic Press, New York.

[8] G. Bouchitté, I. Fonseca, G. Leoni and L. Mascarenhas, A global method for relaxation in $W^{1,p}$ and in SBV_p. *Arch. Rational Mech. Anal.* **165** (2002), 187-242.

[9] A. Braides and V. Chiadò Piat, Integral representation results for functionals defined on $SBV(\Omega;\mathbb{R}^m)$. *J. Math. Pures Appl.* **75** (1996), 595-626.

[10] A. Braides, A. Defranceschi and E. Vitali, Homogenization of free discontinuity problems. *Arch. Rational Mech. Anal.* **135** (1996), 297-356.

[11] G. Cortesani, Strong approximation of $GSBV$ functions by piecewise smooth functions. *Ann. Univ. Ferrara Sez. VII (N.S.)* **43** (1998), 27-49.

[12] G. Cortesani and R. Toader, A density result in SBV with respect to non-isotropic energies. *Nonlinear Anal.* **38** (1999), 585-604.

[13] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[14] D. S. Dugdale, Yielding of steel sheets containing sleets. *J. Mech. Phys. Solids* **8** (1960), 100-104

[15] A. A. Griffith, The phenomena of rupture and flow in solids. *Phil. Trans. Roy. Soc. London A* **221** (1921), 163-198.

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