Uniqueness of critical points of the anisotropic isoperimetric problem for finite perimeter sets

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Abstract

Given an elliptic integrand of class $\mathscr{C}^3$, we prove that finite unions of disjoint open Wulff shapes with equal radii are the only volume-constrained critical points of the anisotropic surface energy among all sets with finite perimeter and reduced boundary almost equal to its closure.

1 Introduction

The classical anisotropic isoperimetric problem (or Wulff problem) consists in minimizing the anisotropic boundary energy among all sets of finite perimeter with prescribed volume. For all positive (continuous) integrands the solution is uniquely characterized, up to translation, by the Wulff shape, as proved by Taylor in [36]. Alternative proofs can be found in [18, 27, 6]. This isoperimetric shape was constructed by Wulff in [37] and plays a central role in crystallography.

Instead of considering minima, a more subtle question is to characterize *critical points* of the anisotropic isoperimetric problem. For integrands of class $\mathscr{C}^1$, this is equivalent to characterize sets of finite perimeter whose anisotropic mean curvature in the sense of varifolds is constant. For all convex integrands in $\mathbf{R}^2$, Morgan proved in [29] that Wulff shapes are the only critical points among all planar regions with boundary given by a closed and connected rectifiable curve. To the best of our knowledge, the characterization in every dimension for smooth boundaries has been conjectured for the first time by Giga in [19] and Morgan in [29]. This has been positively answered for smooth elliptic integrands in [20] for dimension 3, and in [21] for every dimension. These works are the anisotropic counterpart of the celebrated Alexandrov's result [1]. Moreover, quantitative stability versions of this rigidity theorem have been showed in [7, 15, 14].

In the non-smooth setting, Maggi has conjectured in [25, Conjecture] the characterization of the Wulff shapes among *sets of finite perimeter*:

Conjecture ([25]). *F-Wulff shapes are the unique sets of finite perimeter and finite volume that are critical points of $\mathcal{F}$ at fixed volume.*

Since F is assumed to be convex, but may fail to be $\mathscr{C}^1$, the notion of first variation and critical points are suitably defined in [25, p. 35-36], using the convexity in time of the functional along any prescribed variational flow. Maggi specifies in [25] the significant interest from the physical viewpoint for crystalline integrands. Moreover he points out that this question is open even for smooth elliptic anisotropic energies and among sets with Lipschitz boundary.

Delgadino and Maggi have settled the conjecture [25, Conjecture] for the special case of the *area functional* in [9], proving that among sets of finite perimeter, finite unions of balls with equal radii are the unique volume-constrained critical points of the isotropic surface area. Their beautiful proof provides a measure-theoretic revisiting of the Montiel-Ros argument [28] by means of the Heintze-Karcher inequality. Recently the third author has obtained with different techniques in [33] a similar Heintze-Karcher inequality for sets of finite perimeter and bounded isotropic mean curvature and has proved that the equality case is uniquely characterized by finite unions of disjoint open balls, thus recovering the characterization of isotropic critical points.

To deal with the lack of regularity of finite perimeter sets, Delgadino and Maggi need to use in [9] the strong maximum principle for integral varifolds of Schätzle [35]. Unfortunately, as they point out, this result is only available in the isotropic setting, preventing the extension of the method of [9] to anisotropic integrands. They can threat in [8] the special case of *local minimizers*, since this allows to apply an anisotropic strong maximum principle proved in [12] and provides the required regularity through the use of suitable competitors. Nevertheless these competition arguments are not applicable to study the general case of *anisotropic critical points*.

In the present paper we address this problem, providing a positive answer to [25, Conjecture] for elliptic integrands of class $\mathscr{C}^3$ among finite perimeter sets with reduced boundary almost equal to its closure, see Corollary 6.8. Our main result, see Theorem 6.4, is actually more general and it consists in the following anisotropic Heintze-Karcher inequality for sets of finite perimeter (we refer to Section 2 for the notation) and in the characterization of finite unions of disjoint open Wulff shapes (of possibly different radii) as the unique configurations realizing the equality case.

Theorem. *Suppose F is an elliptic integrand of class $\mathscr{C}^3$ (see 2.16), $\alpha \in (0,1)$, $c \in (0,\infty)$, $E \subseteq \mathbf{R}^{n+1}$ is a set of finite perimeter such that $\mathscr{H}^n(\operatorname{Clos}(\partial^* E) \sim \partial^* E) = 0$ and the distributional anisotropic mean curvature H of $\partial^* E$ with respect to F in the direction of the interior normal satisfies $0 < H \leq c$ and it is locally of class $\mathscr{C}^{0,\alpha}$ on the $\mathscr{C}^{1,\alpha}$ regular part of $\operatorname{spt} \|V\|$. Then*

$$\mathscr{L}^{n+1}(E) \leq \frac{n}{n+1} \int_{\partial E} \frac{F(\mathbf{n}(E,x))}{H(x)} \, \mathrm{d}\mathscr{H}^n(x) \, .$$

Equality holds if and only if E coincides up to a set of $\mathscr{L}^{n+1}$ measure zero with a finite union of disjoint open Wulff shapes with radii not smaller than n/c.

As mentioned before, for any elliptic integrand of class $\mathscr{C}^3$, we obtain the following characterization of finite unions of Wulff shapes as the only volume-constrained anisotropic critical points among finite perimeter sets with reduced boundary almost equal to its closure, see Corollary 6.8. We denote by $\mathcal{P}_F$ *the F-perimeter functional*, i.e.

$$\mathcal{P}_F(E) = \int_{\partial^* E} F(\mathbf{n}(E,x)) \, \mathrm{d}\mathscr{H}^n(x)$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter.

Corollary. *Suppose $E \subseteq \mathbf{R}^{n+1}$ is a finite perimeter set with finite volume such that*

$$\mathscr{H}^n(\operatorname{Clos}(\partial^* E) \sim \partial^* E) = 0.$$

If E is a volume-constrained critical point of $\mathcal{P}_F$, then E is equivalent to a finite union of disjoint open Wulff shapes.

We describe now the structure of the paper. In Section 2, after having recalled some background material, we provide some classical facts on Wulff shapes and we study some basic properties of the anisotropic nearest point projection onto an arbitrary closed set. In Section 3 we prove that the only totally umbilical closed and connected hypersurface of class $\mathscr{C}^{1,1}$ is the Wulff shape. In Section 4 we recall the notion of anisotropic (n,h)-sets introduced in [11] and we prove that their generalized normal bundle satisfies a Lusin (N) condition with respect to the n dimensional Hausdorff measure $\mathscr{H}^n$, thus extending an analogous result for isotropic (n,h) sets obtained in [34, 3.7]. This is the key to obtain the main result of the paper. In Section 5 we introduce the anisotropic normal bundle and we study its relation with the isotropic one and with the anisotropic nearest point projection; moreover we consider the anisotropic Steiner formula for closed sets and we prove that every closed set satisfying such a formula has positive reach. To conclude, in Section 6 we combine all these tools to prove Theorem 6.4 and Corollary 6.8.

2 Preliminaries

Notation

The natural number $n \geq 1$ shall be fixed for the whole paper.

In principle, but with some exceptions explained below, we shall follow the notation of Federer (see [17, pp. 669 – 671]). Whenever $A \subseteq \mathbf{R}^{n+1}$ we denote by $\operatorname{Clos} A$ the closure of A in $\mathbf{R}^{n+1}$. Following Almgren (e.g. [4]) if $T \in \mathbf{G}(n+1,k)$, then we write $T_\natural$ for the linear orthogonal projection of $\mathbf{R}^{n+1}$ onto T. The symbol $\mathbb{N}$ stands for the set of non-negative integers. We use standard abbreviations for intervals $(a,b) = \mathbf{R} \cap \{t : a < t < b\}$ and $[a,b] = \mathbf{R} \cap \{t : a \leq t \leq b\}$. We also employ the terminology introduced in [17, 3.2.14] when dealing with rectifiable sets. Moreover, given a measure ϕ and a positive integer m the notions of (ϕ, m) approximate tangent cone $\operatorname{Tan}^m(\phi, \cdot)$, (ϕ, m) approximate differentiability and (ϕ, m) approximate differential are used in agreement with [17, 3.2.16]. We also introduce the symbol $\mathbf{S}^n$ for the unit n-dimensional sphere in $\mathbf{R}^{n+1}$.

Concerning varifolds and submanifolds of $\mathbf{R}^{n+1}$ we use the notation introduced in [3]. If M is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^1$, we write $\mathscr{X}(M)$ for compactly supported tangent vectorfields on M of class $\mathscr{C}^1$; cf. [3, 2.5]. We say that M is a *closed submanifold* of $\mathbf{R}^{n+1}$ if it is a submanifold of $\mathbf{R}^{n+1}$ and a closed (but not necessarily compact) subset of $\mathbf{R}^{n+1}$; in particular, $\partial M \sim M = \varnothing$.

We also use the following convention. Whenever X, Y are vectorspaces, $A \subseteq X$, and $f : A \to Y$ we write $\mathrm{D}f$ for the derivative of f that is a $\operatorname{Hom}(X, Y)$ valued function whose domain is the set of points of differentiability of f. If $Y = \mathbf{R}$ and X is equipped with a scalar product, then we write $\operatorname{grad} f$ for the X valued function characterised by

$$\langle u, \mathrm{D}f(x) \rangle = \operatorname{grad} f(x) \bullet u \quad \text{for } x \in \operatorname{dmn} \mathrm{D}f \text{ and } u \in X\,.$$

Pointwise differentiability

2.1 Definition (cf. [26, §2.7])**.** Let $k \in \mathbb{N}$, X, Y be normed vectorspaces, $A \subseteq X$, $f : A \to Y$, and $a \in X$. Then f is called *pointwise differentiable of order k at a* if there exists an open set $U \subseteq X$ and a function $g : U \to Y$ of class k such that

$$a \in U \subseteq A\,, \quad f(a) = g(a)\,, \quad \text{and} \quad \lim_{x \to a} \frac{|f(x) - g(x)|}{|x - a|^k} = 0\,.$$

Whenever this is satisfied one defines also the *pointwise differential of order i of f at a* by

$$\operatorname{pt} \mathrm{D}^i f(a) = \mathrm{D}^i g(a) \quad \text{for } i \in \{0, 1, \ldots, k\}\,.$$

2.2 Definition (cf. [26, §3.3])**.** Suppose $k, n \in \mathbb{N}$ and $A \subseteq \mathbf{R}^{n+1}$. Then A is called *pointwise differentiable of order k at a* if there exists a submanifold B of $\mathbf{R}^{n+1}$ of class k such that $a \in B$,

$$\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, A) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a, r)] = 0\,, \tag{1}$$

$$\text{and} \quad \lim_{r \downarrow 0} r^{-k} \sup \operatorname{distance}(\cdot, B)\, [A \cap \mathbf{B}(a, r)] = 0\,. \tag{2}$$

2.3 Definition (cf. [26, §3.12])**.** Suppose $n, k \in \mathbb{N}$ and $A \subseteq \mathbf{R}^{n+1}$. Then $\operatorname{pt} \mathrm{D}^k A$ is the function whose domain consists of pairs (a, S) such that $a \in \operatorname{Clos} A$, A is pointwise differentiable of order k at a, $S \in \mathbf{G}(n+1, \dim \operatorname{Tan}(A, a))$, and $S^\perp \cap \operatorname{Tan}(A, a) = \{0\}$ and whose value at (a, S) equals the unique $\phi \in \bigodot^k(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ such that whenever $f : S \to S^\perp$ is of class k and satisfies

$$\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, A) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a, r)] = 0\,, \tag{3}$$

$$\text{and} \quad \lim_{r \downarrow 0} r^{-k} \sup \operatorname{distance}(\cdot, B)\, [A \cap \mathbf{B}(a, r)] = 0\,, \tag{4}$$

where $B = \{x + f(x) : x \in S\}$, then $\phi = \mathrm{D}^k(f \circ S_\natural)(a)$.

2.4 Remark (cf. [26, §§3.14, 3.15])*.* Assume $n, d, k \in \mathbb{N}$, $S \in \mathbf{G}(n+1, d)$, $U \subseteq S$ is open, $f : U \to S^\perp$ is continuous, $x \in U$, $A = \{\chi + f(\chi) : \chi \in S\}$. Then A is pointwise differentiable of order k at $a = x + f(x)$ if and only if f is pointwise differentiable of order k at x. Moreover, $\operatorname{pt} \mathrm{D}^i A(a, S) = \operatorname{pt} \mathrm{D}^i(f \circ S_\natural)(x)$ for $i \in \{0, 1, \ldots, k\}$.

The unit normal bundle of a closed set

Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set.

2.5 Definition. Given $A \subseteq \mathbf{R}^{n+1}$ we define *the distance function to A* as

$$\boldsymbol{\delta}_A(x) = \inf\{|x - a| : a \in A\} \quad \text{for every } x \in \mathbf{R}^{n+1}.$$

Moreover,

$$S(A, r) = \{x : \boldsymbol{\delta}_A(x) = r\} \quad \text{for } r > 0.$$

2.6 Remark (cf. [31, 2.13]). If $r > 0$ then $\mathscr{H}^n(S(A,r) \cap K) < \infty$ whenever $K \subseteq \mathbf{R}^n$ is compact and $S(A,r)$ is countably $(\mathscr{H}^n, n)$ rectifiable of class 2.

2.7 Definition (cf. [31, 3.1]). If U is the set of all $x \in \mathbf{R}^{n+1}$ such that there exists a unique $a \in A$ with $|x-a| = \boldsymbol{\delta}_A(x)$, we define the *nearest point projection onto* A as the map $\boldsymbol{\xi}_A$ characterised by the requirement

$$|x - \boldsymbol{\xi}_A(x)| = \boldsymbol{\delta}_A(x) \quad \text{for } x \in U.$$

We set $U(A) = \operatorname{dmn} \boldsymbol{\xi}_A \sim A$. The functions $\boldsymbol{\nu}_A$ and $\boldsymbol{\psi}_A$ are defined by

$$\boldsymbol{\nu}_A(z) = \boldsymbol{\delta}_A(z)^{-1}(z - \boldsymbol{\xi}_A(z)) \quad \text{and} \quad \boldsymbol{\psi}_A(z) = (\boldsymbol{\xi}_A(z), \boldsymbol{\nu}_A(z)),$$

whenever $z \in U(A)$.

2.8 Definition (cf. [31, 3.6, 3.8, 3.13]). We define the function $\rho(A, \cdot)$ setting

$$\rho(A,x) = \sup\bigl\{t : \boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))) = t\boldsymbol{\delta}_A(x)\bigr\} \quad \text{for } x \in U(A)\,,$$

and we say that $x \in U(A)$ is a *regular point of* $\boldsymbol{\xi}_A$ if and only if $\boldsymbol{\xi}_A$ is approximately differentiable at x with symmetric approximate differential and $\operatorname{ap}\lim_{y \to x} \rho(A,y) \geq \rho(A,x) > 1$. The set of regular points of $\boldsymbol{\xi}_A$ is denoted by $R(A)$.

For $\tau \geq 1$ we define

$$A_\tau = U(A) \cap \{x : \rho(A,x) \geq \tau\}\,.$$

2.9 Remark (cf. [31, 3.7]). The function $\rho(A, \cdot)$ is upper semicontinuous and its image is contained $[1, \infty]$.

2.10 Definition (cf. [31, 4.9]). Suppose $x \in R(A)$. Then $\chi_{A,1}(x) \leq \ldots \leq \chi_{A,n}(x)$ denote the eigenvalues of the symmetric linear map $\operatorname{ap} \mathrm{D}\boldsymbol{\nu}_A(x)|\{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$.

2.11 Remark. Notice that $\mathscr{H}^n(S(A,r) \sim R(A)) = 0$ for $\mathscr{L}^1$ a.e. $r > 0$ (cf. [31, 3.16]) and

$$\operatorname{Tan}^n(\mathscr{H}^n \llcorner S(A,r), x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$$

for $\mathscr{H}^n$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$, cf. [31, 3.12].

The functions $\chi_{A,i}$ are the approximate principal curvatures of $S(A,r)$ in the direction of $\boldsymbol{\nu}_A(x)$. In fact, as proved in [31, 3.12], they coincide with the eigenvalues the approximate second-order differential $\operatorname{ap} \mathrm{D}^2 S(A,r)$ of $S(A,r)$; cf. [32] for the general theory of higher order approximate differentiability for sets.

2.12 Definition (cf. [31, 4.1], [23, §2.1]). *The generalized unit normal bundle of* A is defined as

$$N(A) = (A \times \mathbf{S}^n) \cap \{(a,u) : \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}$$

and $N(A,a) = \{v : (a,v) \in N(A)\}$ for $a \in A$.

2.13 Remark (cf. [31, 4.3]). The set $N(A)$ is a countably n rectifiable subsets of $\mathbf{R}^{n+1} \times \mathbf{S}^n$.

Anisotropic integrands and mean curvature

2.14 Definition. Let $k \in \mathbb{N}$, $\alpha \in [0,1]$. By an *integrand of class* $\mathscr{C}^{k,\alpha}$ we mean a non-negative function $F : \mathbf{R}^{n+1} \to \mathbf{R}$ such that $F|\mathbf{R}^{n+1} \sim \{0\}$ is of class $\mathscr{C}^{k,\alpha}$ and

$$F(\lambda\nu) = |\lambda| F(\nu) \quad \text{for } \nu \in \mathbf{R}^{n+1} \text{ and } \lambda \in \mathbf{R} \tag{5}$$

By an *integrand* we mean an integrand of class $\mathscr{C}^0$.

2.15 Remark. If F is convex, then it is a norm on $\mathbf{R}^{n+1}$. We say that F is a *strictly convex norm* if it is an integrand satisfying

$$F(x+y) < F(x) + F(y) \quad \text{for all linearly independent } x, y \in \mathbf{R}^{n+1}\,.$$

2.16 Definition (cf. [17, 5.1.2] and [2, 3.1(4)]). We say that an integrand F is *elliptic* if there exists a number $\gamma > 0$ such that the map $\mathbf{R}^{n+1} \ni u \mapsto F(u) - \gamma|u|$ is convex. We call γ the *ellipticity constant of* F.

2.17 Remark (cf. [17, 5.1.3]). Assume F is an integrand of class $\mathscr{C}^{1,1}$. Then ellipticity of F with ellipticity constant $\gamma > 0$ is equivalent to the condition

$$\big\langle (v,v),\, \mathrm{D}^2F(u) \big\rangle \geq \gamma \frac{|u \wedge v|^2}{|u|^3} = \gamma \frac{|v|^2 - (v \bullet u/|u|)^2}{|u|} \quad \text{for } u \in \operatorname{dmn} \mathrm{D}^2F,\ u \neq 0,\ v \in \mathbf{R}^{n+1}\,. \tag{6}$$

In particular, if F is elliptic, $u \in \operatorname{dmn} \mathrm{D}^2F$, $|u| = 1$, and $v \in \operatorname{span}\{u\}^{\perp}$, then

$$\big\langle (v,v),\, \mathrm{D}^2F(u) \big\rangle \geq \gamma |v|^2\,,$$

which shows that F is uniformly elliptic in the sense of [11, §2].

2.18 Definition. Assume F is an elliptic integrand with ellipticity constant $\gamma > 0$. We define

$$C(F) = \sup\big(\{\gamma^{-1},\, \sup F[\mathbf{S}^n]/\inf F[\mathbf{S}^n]\} \cup \{\|\mathrm{D}^2F(\nu)\| : \nu \in \mathbf{S}^n \cap \operatorname{dmn} \mathrm{D}^2F\}\big)\,.$$

2.19 Remark. Let $U \subseteq \mathbf{R}^{n+1}$ be open. For any $T \in \mathbf{G}(n+1,n)$ we choose arbitrarily $\nu(T) \in T^{\perp}$ such that $|\nu(T)| = 1$. In the sequel we shall tacitly identify any $V \in \mathbf{V}_n(U)$ with a Radon measure $\bar{V}$ over $U \times \mathbf{R}^{n+1}$ such that

$$\bar{V}(\alpha) = \frac{1}{2} \int \alpha(x, \nu(T)) + \alpha(x, -\nu(T))\, \mathrm{d}V(x,T) \quad \text{for } \alpha \in C^0_c(U, \mathbf{R})\,.$$

Clearly, this definition does not depend on the choice of $\nu(T)$.

2.20 Definition. Let $U \subseteq \mathbf{R}^{n+1}$ be open, F be an integrand of class $\mathscr{C}^1$, $V \in \mathbf{V}_n(U)$. We define the *first variation of V with respect to F* by the formula

$$\delta_F V(g) = \int \mathrm{D}g(x) \bullet B_F(\nu)\, \mathrm{d}V(x,\nu) \quad \text{for } g \in \mathscr{X}(U)\,,$$

where $B_F(\nu) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ is given by

$$B_F(\nu)u = F(\nu)u - \nu \cdot \langle u,\, \mathrm{D}F(\nu)\rangle \quad \text{for } \nu, u \in \mathbf{R}^{n+1},\ \nu \neq 0\,.$$

2.21 Remark (cf. [2],[10, Appendix A],[13]). If $\varphi : \mathbf{R} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is smooth, $\varphi(0,x) = x$ for $x \in \mathbf{R}^{n+1}$, and $g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi(t,\cdot) \in \mathscr{X}(\mathbf{R}^{n+1})$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_F(\varphi_{t\#}V) = \delta_F V(g)\,,$$

where the functional $\Phi_F : \mathbf{V}_n(U) \to [0,\infty]$ is defined as

$$\Phi_F(V) = \int F(\nu)\, \mathrm{d}V(x,\nu)\,.$$

2.22 Definition (cf. [11, §2])**.** Let $\Omega \subseteq \mathbf{R}^{n+1}$ be open, $V \in \mathbf{V}_n(\Omega)$, $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be an integrand of class $\mathscr{C}^1$. Assume that $\|\delta_F V\|$ is a Radon measure. Then

$$\delta_F V(g) = -\int \overline{\mathbf{h}}_F(V,x) \bullet g(x)\, \mathrm{d}\|V\|(x) + \int \boldsymbol{\eta}_F(V,x) \bullet g(x)\, \mathrm{d}\|\delta_F V\|_{\mathrm{sing}}(x) \quad \text{for } g \in \mathscr{X}(\Omega)\,,$$

where $\|\delta_F V\|_{\mathrm{sing}}$ is the singular part of $\|\delta_F V\|$ with respect to $\|V\|$, $\overline{\mathbf{h}}_F(V,\cdot)$ is an $\mathbf{R}^{n+1}$ valued $\|V\|$-integrable function, and $\boldsymbol{\eta}_F(V,\cdot)$ is an $\mathbf{S}^n$ valued $\|\delta_F V\|$-integrable function.

For $\|V\|$-a.e. x we define the *F-mean curvature vector of V at x*, denoted $\mathbf{h}_F(V,x)$, by the formula

$$\mathbf{h}_F(V,x) = \frac{\overline{\mathbf{h}}_F(V,x)}{\int F(\nu)\, \mathrm{d}V^{(x)}(\nu)}\,,$$

where $V^{(x)}$ is the probability measure on $\mathbf{S}^n$ coming from disintegration of V; see [3, §3.3].

2.23 Definition. Define $\boldsymbol{\Xi} : \bigodot^2 \mathbf{R}^{n+1} \to \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ to be the linear map characterised by

$$\langle u,\, \boldsymbol{\Xi}(A)\rangle \bullet v = A(u,v) \quad \text{for } A \in {\textstyle\bigodot^2} \mathbf{R}^{n+1} \text{ and } u, v \in \mathbf{R}^{n+1}\,.$$

2.24 Remark. In particular, if $f : \mathbf{R}^{n+1} \to \mathbf{R}$ is twice differentiable at $x \in \mathbf{R}^{n+1}$, then

$$\boldsymbol{\Xi}(\mathrm{D}^2 f(x)) = \mathrm{D}(\operatorname{grad} f)(x) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1}) .$$

2.25 Remark. Let $G \subseteq \mathbf{R}^{n+1}$ be open, $v_1, \ldots, v_{n+1}$ be an orthonormal basis of $\mathbf{R}^{n+1}$, M be a submanifold of G of dimension n of class $\mathscr{C}^2$, $V = \mathbf{v}_n(M) \in \mathbf{V}_n(G)$, $x \in M$, $\nu : G \to \mathbf{R}^{n+1}$ be of class $\mathscr{C}^1$ and satisfy

$$(7) \qquad |\nu(y)| = 1 , \quad \nu(y) \in \operatorname{Nor}(M, y) , \quad \text{and} \quad \langle \nu(y), \mathrm{D}\nu(y) \rangle = 0 \quad \text{for } y \in M .$$

In [11, Proposition 2.1] the authors show that if F is an elliptic integrand of class $\mathscr{C}^2$ then

$$-F(\nu(x))\mathbf{h}_F(V, x) = \nu(x) \operatorname{tr}\big(\mathrm{D}(\operatorname{grad} F \circ \nu)(x)\big) = \nu(x) \sum_{j=1}^{n+1} \big\langle (\mathrm{D}\nu(x) v_j, v_j),\, \mathrm{D}^2 F(\nu(x)) \big\rangle .$$

2.26 Definition (cf. [17, 4.5.5])**.** Let $A \subseteq \mathbf{R}^{n+1}$ and $b \in \mathbf{R}^{n+1}$. We say that u is an *exterior normal of A at b* if $u \in \mathbf{R}^{n+1}$, $|u| = 1$,

$$(8) \qquad \boldsymbol{\Theta}^{n+1}(\mathscr{L}^{n+1} \llcorner \{x : (x - b) \bullet u > 0\} \cap A, b) = 0 ,$$

$$(9) \qquad \text{and} \quad \boldsymbol{\Theta}^{n+1}(\mathscr{L}^{n+1} \llcorner \{x : (x - b) \bullet u < 0\} \sim A, b) = 0 .$$

We also set $\mathbf{n}(A, b) = u$ if u is the exterior normal of A at b and $\mathbf{n}(A, b) = 0$ if there exists no exterior normal of A at b.

2.27 Definition. Let $E \subseteq \mathbf{R}^{n+1}$ and $x \in \mathbf{R}^{n+1}$. We define

$$\mathbf{n}^F(E, x) = \operatorname{grad} F(\mathbf{n}(E, x)) \quad \text{if } \mathbf{n}(E, x) \neq 0 \quad \text{and} \quad \mathbf{n}^F(E, x) = 0 \quad \text{if } \mathbf{n}(E, x) = 0 .$$

2.28 Remark. Assume X is a Hilbert space, $\dim X = k \in \mathbb{N}$, $A, B \in \operatorname{Hom}(X, X)$ are self-adjoint automorphisms of X, and A is positive definite. With the help of the (tiny) spectral theorem [24, Chap. VIII, Thm. 4.3] we find a self-adjoint and positive definite map $C \in \operatorname{Hom}(X, X)$ such that $A = C \circ C$. Next, we observe that $E = C^{-1} \circ A \circ B \circ C = C \circ B \circ C$ is self-adjoint. Employing again the (tiny) spectral theorem we find an orthonormal basis $v_1, \ldots, v_k \in X$ and real numbers $\lambda_1, \ldots, \lambda_k$ such that $E v_i = \lambda_i v_i$ for $i \in \{1, 2, \ldots, k\}$. We obtain

$$A \circ B(C v_i) = C \circ E v_i = \lambda_i C v_i \quad \text{for } i \in \{1, 2, \ldots, k\}$$

and we see that $C v_1, \ldots, C v_k$ is a basis of eigenvectors of $A \circ B$ with eigenvalues $\lambda_1, \ldots, \lambda_k$.

In particular, if G, M, x, and ν are as in 2.25, F is an elliptic integrand, $u = \nu(x) \in \operatorname{dmn} \mathrm{D}^2 F$, and $X = \operatorname{Tan}(M, x)$, then the maps $A = \boldsymbol{\Xi}(\mathrm{D}^2 F(\nu(x)) | X \times X)$ and $B = \mathrm{D}\nu(x) | X \in \operatorname{Hom}(X, X)$ are self-adjoint and A is positive definite; hence, $A \circ B$ has exactly n real eigenvalues.

Observe also that since F is positively 1-homogeneous, $\operatorname{grad} F$ is positively 0-homogeneous, i.e., $\operatorname{grad} F(\lambda v) = \operatorname{grad} F(v)$ for $\lambda \in (0, \infty)$ and $v \in \operatorname{dmn} \operatorname{grad} F$; hence,

$$(10) \qquad v \in \ker \mathrm{D}(\operatorname{grad} F)(v) \quad \text{for } v \in \operatorname{dmn} \mathrm{D}^2 F .$$

Since $\mathrm{D}^2 F(\nu(x)) \in \bigodot^2 \mathbf{R}^{n+1}$ is symmetric it follows that $\mathrm{D}(\operatorname{grad} F)(\nu(x)) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ is self-adjoint and we have

$$\operatorname{im} \mathrm{D}(\operatorname{grad} F)(\nu(x)) = \big(\ker \mathrm{D}(\operatorname{grad} F)(\nu(x))\big)^{\perp}$$

so that $\mathrm{D}(\operatorname{grad} F)(\nu(x)) | X \in \operatorname{Hom}(X, X)$ by (10). Seeing that also $\mathrm{D}\nu(x) | X \in \operatorname{Hom}(X, X)$ we conclude

$$\mathrm{D}(\operatorname{grad} F \circ \nu)(x) | X \in \operatorname{Hom}(X, X) .$$

2.29 Definition. Let F be an elliptic integrand of class $\mathscr{C}^{1,1}$, $G \subseteq \mathbf{R}^{n+1}$ be open, M be a submanifold of G of dimension n of class $\mathscr{C}^{1,1}$, $\nu : G \to \mathbf{R}^{n+1}$ be Lipschitz continuous and such that $|\nu(z)| = 1$ and $\nu(z) \in \operatorname{Nor}(M, z)$ for $z \in M$, $x \in \operatorname{dmn} \mathrm{D}\nu$, and $u = \operatorname{grad} F(\nu(x))$. We define the *$F$-principal curvatures of M at (x, u)*

$$\kappa^F_{M,1}(x, u) \leq \ldots \leq \kappa^F_{M,n}(x, u)$$

to be the eigenvalues of the map $\mathrm{D}(\operatorname{grad} F \circ \nu)(x) | \operatorname{Tan}(M, x) \in \operatorname{Hom}(\operatorname{Tan}(M, x), \operatorname{Tan}(M, x))$; cf. 2.28.

2.30 Remark. Clearly if $V = \mathbf{v}_n(M) \in \mathbf{V}_n(G)$, then

$$\overline{\mathbf{h}}_F(V,x) = -\nu(x) \textstyle\sum_{i=1}^n \kappa_{M,i}^F(x,u) \,.$$

2.31 Definition. Assume $M \subseteq \mathbf{R}^{n+1}$ is pointwise differentiable of order 2 at $a \in \operatorname{Clos} M$, $T \in \mathbf{G}(n+1,n)$, $f : T \to T^\perp$ is pointwise differentiable of order 2 at 0, $f(0) = 0$, $\operatorname{pt} \mathrm{D} f(0) = 0$, $B = \mathbf{R}^{n+1} \cap \{a + x + f(x) : x \in T\}$, $\nu \in T^\perp$, $|\nu| = 1$, and

$$\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, M) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a,r)] = 0 \,, \tag{11}$$

$$\text{and} \quad \lim_{r \downarrow 0} r^{-2} \sup \operatorname{distance}(\cdot, B) \, [M \cap \mathbf{B}(a,r)] = 0 \,. \tag{12}$$

We define the *pointwise F-mean curvature vector of M at a*, denoted $\operatorname{pt} \mathbf{h}_F(M,a)$, by the formula

$$-F(\nu) \operatorname{pt} \mathbf{h}_F(M,a) = \operatorname{tr}\big(\mathbf{\Xi}(\mathrm{D}^2 F(\nu)) \circ \mathbf{\Xi}(\operatorname{pt} \mathrm{D}^2 (f \circ T_\natural)(0) \bullet \nu)\big) \,.$$

2.32 Remark. Note that the above definition does not depend on the choice of ν and f. In particular, if $\bar\nu = -\nu$, then recalling (5) we obtain

$$F(\nu) = \langle \nu, \mathrm{D}F(\nu) \rangle = \langle \bar\nu, \mathrm{D}F(\bar\nu) \rangle = F(\bar\nu) \,; \tag{13}$$

$$\text{hence,} \quad \nu \big\langle (\nu,\nu), \mathrm{D}^2 F(\nu) \big\rangle = \nu \big\langle (\nu,\bar\nu), \mathrm{D}^2 F(\bar\nu) \big\rangle = \bar\nu \big\langle (\bar\nu,\bar\nu), \mathrm{D}^2 F(\bar\nu) \big\rangle \,. \tag{14}$$

Anisotropic nearest point projection and related objects

2.33 Definition. Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a norm, $x \in \mathbf{R}^{n+1}$ and $r > 0$. We define

$$\mathbf{U}^F(x,r) = \{y : F(y-x) < r\} \quad \text{and} \quad \mathbf{B}^F(x,r) = \{y : F(y-x) \le r\} \,.$$

2.34 Definition. Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a norm. Define the *conjugate norm* F^* on $\mathbf{R}^{n+1}$ by setting

$$F^*(w) = \sup\{w \bullet u : u \in \mathbf{R}^{n+1},\ F(u) \le 1\} \,.$$

By a *Wulff shape (of F)* we mean any open ball with respect to the F^* norm.

2.35 Definition. Given $A \subseteq \mathbf{R}^{n+1}$, we define *the anisotropic distance function to A* as

$$\boldsymbol{\delta}_A^F(x) = \inf\{F^*(a-x) : a \in A\} \quad \text{for every } x \in \mathbf{R}^{n+1}.$$

Moreover,

$$S^F(A,r) = \{x : \boldsymbol{\delta}_A^F(x) = r\} \quad \text{for } r > 0.$$

2.36 Definition. Suppose $A \subseteq \mathbf{R}^n$ is closed and W is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $F^*(x-a) = \boldsymbol{\delta}_A^F(x)$. The *anisotropic nearest point projection onto A* is the map $\boldsymbol{\xi}_A^F : W \to A$ characterised by the requirement

$$F^*(x - \boldsymbol{\xi}_A^F(x)) = \boldsymbol{\delta}_A^F(x) \quad \text{for } x \in W.$$

We also define $\boldsymbol{\nu}_A^F : W \sim A \to \partial \mathbf{B}^{F^*}(0,1)$ and $\boldsymbol{\psi}_A^F : W \sim A \to A \times \partial \mathbf{B}^{F^*}(0,1)$ by the formulas

$$\boldsymbol{\nu}_A^F(z) = \boldsymbol{\delta}_A^F(z)^{-1}(z - \boldsymbol{\xi}_A^F(z)) \quad \text{and} \quad \boldsymbol{\psi}_A^F(z) = (\boldsymbol{\xi}_A^F(z), \boldsymbol{\nu}_A^F(z)) \quad \text{for } z \in W \sim A \,.$$

2.37 Definition (cf. [5, Def. 3.54])**.** Let $A \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and $V = \mathbf{v}_{n+1}(A) \in \mathbf{V}_{n+1}(\mathbf{R}^{n+1})$. Then $\|\delta V\|$ is a Radon measure (cf. [3, 4.7]) and there exists $\|\delta V\|$ measurable function $\boldsymbol{\eta}(V,\cdot)$ with values in $\mathbf{S}^n$ as in [3, 4.3]. We define the *reduced boundary of A*, denoted $\partial^* A$, as the set of points $x \in \operatorname{dmn} \boldsymbol{\eta}(V,\cdot)$ for which

$$\|\delta V\| \, \mathbf{B}(x,r) > 0 \quad \text{for } r > 0 \text{ and} \quad \lim_{r \downarrow 0} \frac{1}{\|\delta V\| \, \mathbf{B}(x,r)} \int_{\mathbf{B}(x,r)} \boldsymbol{\eta}(V,\cdot) \, \mathrm{d}\|\delta V\| = \boldsymbol{\eta}(V,x) \,.$$

In the next lemma we summarize a few facts about relations between F and F^*.

2.38 Lemma. *Let F be a strictly convex norm of class $\mathscr{C}^{1,1}$, F^* its conjugate, $W = \mathbf{U}^F(0,1)$, $W^* = \mathbf{U}^{F^*}(0,1)$, $G, G^* : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be given by $G = \operatorname{grad} F$ and $G^* = \operatorname{grad} F^*$.*

The following hold.

(a) $F^*(G(x)) = 1$ *and* $F(G^*(x)) = 1$ *for any* $x \in \mathbf{R}^{n+1} \sim \{0\}$.

(b) $G|\partial W : \partial W \to \partial W^*$ *is a Lipschitz homeomorphism.*

(c) $F^*(x) = x \bullet G^*(x)$ *and* $F(x) = x \bullet G(x)$ *for* $x \in \mathbf{R}^{n+1} \sim \{0\}$.

(d) $F^{**} = F$.

(e) F^* *is a strictly convex norm.*

(f) $G^*|\partial W^* = (G|\partial W)^{-1}$.

(g) F^* *is of class* $\mathscr{C}^1$.

(h) If F *satisfies* (6)*, then* F^* *is of class* $\mathscr{C}^{1,1}$ *and* $G|\partial W : \partial W \to \partial W^*$ *is bilipschitz.*

(i) $\mathbf{n}(W, x) = G(x)F(\mathbf{n}(W, x))$ *and* $\mathbf{n}(W^*, y) = G^*(y)F(\mathbf{n}(W^*, y))$ *for* $x \in \partial W$ *and* $y \in \partial W^*$. *In particular,* $G(\mathbf{n}(W^*, y)) = y$ *for* $y \in \partial W^*$ *and* $G^*(\mathbf{n}(W, x)) = x$ *for* $x \in \partial W$.

Proof. It is clear from the definition that F^* is a norm; hence, it is Lipschitz and convex. Employ [30, Theorem 25.5] or the Rademacher theorem [17, 3.1.6] to see that F^* is differentiable $\mathscr{L}^{n+1}$ almost everywhere. Observe that $G(x) \bullet x = F(x) > 0$ and $G^*(y) \bullet y = F^*(y) > 0$ for all $x \in \mathbf{R}^{n+1} \sim \{0\}$ and $y \in \operatorname{dmn} G^*$ due to positive 1-homogeneity of F and F^*.

Assume now that $G|\partial W$ is not injective, i.e., that there exist $a, b \in \partial W$ such that $a \neq b$ and $G(a) = G(b)$. Since $F(u) = F(-u)$ we see that $G(u) = -G(-u)$ so $a \neq -b$ and the line segment joining a and b does not pass through the origin. Set $u = b - a$ and define the strictly convex map $f : [0, 1] \to \mathbf{R}^{n+1}$ by the formula $f(t) = F(a + tu)$. Then $f'(0) = G(a) \bullet u$ and $f'(1) = G(b) \bullet u$ so $f'(0) = f'(1)$ which contradicts strict convexity of f. Therefore $G|\partial W$ is injective; hence, since ∂W is compact, $(G|\partial W)^{-1} : G[\partial W] \to \partial W$ is continuous.

For any $w \in \mathbf{R}^{n+1}$ define $g_w : \mathbf{R}^{n+1} \to \mathbf{R}$ by the formula $g_w(u) = u \bullet w$ for $u \in \mathbf{R}^{n+1}$.

Let $w \in \partial W^*$ so that $F^*(w) = 1$. Since the smooth map g_w attains its maximal value on the compact manifold $\partial W = F^{-1}\{1\}$ of class $\mathscr{C}^{1,1}$, using the method of Lagrange multipliers, we deduce that there exists at least one $u \in \partial W$ such that $1 = F^*(w) = w \bullet u$ and $w = \lambda G(u)$ for some $\lambda \in \mathbf{R}$. Then

$$1 = u \bullet w = u \bullet \lambda G(u) = \lambda \langle u, \mathrm{D}F(u) \rangle = \lambda F(u) = \lambda \,.$$

Therefore,

$$F^*(G(u)) = \lambda F^*(w) = \lambda = 1 \,.$$

Since $w \in \partial W$ was arbitrary, it follows that $\partial W^* \subseteq G[\partial W]$. Noting that ∂W and ∂W^* are compact connected submanifolds of $\mathbf{R}^{n+1}$ without boundary of class at least $\mathscr{C}^{0,1}$ and that $G|\partial W$ is a homeomorphism onto its image, employing the invariance of domain theorem, we get $G[\partial W] = \partial W^*$.

For $u \in \partial W$ we have $F^{**}(u) = \sup\{u \bullet w : F^*(w) = 1\} \leq 1$, $u \bullet G(u) = F(u) = 1$, and $F^*(G(u)) = 1$, so $F^{**} = F$. If $F(u) = 1$ and $F(u) = F^{**}(u) = u \bullet w$ for some $w \in \partial W^* \cap \operatorname{dmn} G^*$, then $u = \lambda G^*(w)$ for some $\lambda \in \mathbf{R}$ because the function g_u attains its maximum at w. Hence, the same argument as before shows that $F(G^*(w)) = 1$ for all $w \in \partial W^* \cap \operatorname{dmn} G^*$.

If $F^*(y) = 1$, then there exists exactly one $w = (G|\partial W)^{-1}(y) \in \partial W$ for which $F^*(y) = w \bullet y$. If $y \in \partial W^* \cap \operatorname{dmn} G^*$, then we know also that $F^*(y) = G^*(y) \bullet y$ and $G^*(y) \in \partial W$; hence, $G^*(y) = (G|\partial W)^{-1}(y)$. Employing [16, 4.7] we see that $G^*|\partial W^* = (G|\partial W)^{-1}$ and, since $(G|\partial W)^{-1}$ is continuous, $F^*|\mathbf{R}^{n+1} \sim \{0\}$ is of class $\mathscr{C}^1$.

For $u, v \in \mathbf{R}^{n+1} \sim \{0\}$ we have

$$\begin{aligned}(15) \quad F^*(u + v) = (u + v) \bullet G^*(u + v) &= u \bullet G^*(u + v) + v \bullet G^*(u + v) \\ &< u \bullet G^*(u) + v \bullet G^*(v) = F^*(u) + F^*(v) \,;\end{aligned}$$

because $G^*(u) \neq G^*(v)$, $G^*(u)$ is the unique element of ∂W which realises $\sup\{u \bullet w : F(w) = 1\}$, and $G^*(v)$ is the unique element of ∂W which realises $\sup\{v \bullet w : F(w) = 1\}$; hence, F^* is a strictly convex norm.

Observe that if $x \in \partial W$, $y = G(x)$, $\mathrm{D}G(x)$ exists, and $v, w \in \operatorname{Tan}(\partial W^*, x)$, then

$$\langle (v, w), \mathrm{D}^2 F^*(y) \rangle = \mathrm{D}G^*(y)v \bullet w = (\mathrm{D}G(x)|\operatorname{Tan}(\partial W, x))^{-1} v \bullet w \,.$$

If F satisfies (6), then the right-hand side is bounded and F^* is of class $\mathscr{C}^{1,1}$.

Finally, note that for $x \in \partial W$ we have $G(x) = \operatorname{grad} F(x) \perp \operatorname{Tan}(\partial W, x)$, $G(x) \bullet x = F(x) = 1$, and $F^*(G(x)) = 1$; hence, $G(x) = \mathbf{n}(W,x)/F^*(\mathbf{n}(W,x))$. Similarly, $G^*(y) = \mathbf{n}(W^*,y)/F(\mathbf{n}(W^*,y))$ whenever $F^*(y) = 1$. $\square$

2.39 Corollary. *Assume F is an elliptic integrand of class $\mathscr{C}^{1,1}$, $r \in \mathbf{R}$ is positive, $W = \mathbf{U}^{F^*}(0,r)$, $\eta : \partial W \to \mathbf{R}^{n+1}$ is given by $\eta(z) = \operatorname{grad} F(\mathbf{n}(W,z))$ for $z \in \partial W$. We have $\eta(z) = z/r$ for $z \in \partial W$ so $\mathrm{D}\eta(y)v = v/r$ for $v \in \operatorname{Tan}(\partial W, y)$ and $y \in \operatorname{dmn} \mathrm{D}\eta$; hence, recalling 2.29 and 2.30 we see that*

$$\kappa^F_{\partial W,1}(y,\eta(y)) = \ldots = \kappa^F_{\partial W,n}(y,\eta(y)) = 1/r \quad \text{for } y \in \operatorname{dmn} \mathrm{D}\eta\,.$$

2.40 Lemma. *Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a strictly convex norm of class $\mathscr{C}^{1,1}$, $G = \operatorname{grad} F$, $A \subseteq \mathbf{R}^{n+1}$ be closed. Then*

(a) $|\boldsymbol{\delta}^F_A(y) - \boldsymbol{\delta}^F_A(z)| \le F^*(y-z)$ *for* $y, z \in \mathbf{R}^{n+1}$.

(b) $\boldsymbol{\xi}^F_A$ *is continuous.*

(c) Suppose $x \in \mathbf{R}^{n+1} \sim A$ and $a \in A$ are such that $\boldsymbol{\delta}^F_A(x) = F^*(x-a)$. Then

$$\boldsymbol{\delta}^F_A(a + t(x-a)) = tF^*(x-a) = t\boldsymbol{\delta}^F_A(x) \quad \text{for } 0 < t \le 1\,.$$

(d) Suppose $x \in \mathbf{R}^{n+1} \sim A$ and $a \in A$ are such that $\boldsymbol{\delta}^F_A(x) = F^*(x-a)$ and $\mathrm{D}\boldsymbol{\delta}^F_A(x)$ exists. Then $x \in U$ and $G(\operatorname{grad} \boldsymbol{\delta}^F_A(x))\boldsymbol{\delta}^F_A(x) = x - a$; hence,

$$\boldsymbol{\xi}^F_A(x) = x - G(\operatorname{grad} \boldsymbol{\delta}^F_A(x))\boldsymbol{\delta}^F_A(x) = a\,.$$

(e) The maps $\boldsymbol{\delta}^F_A | \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A \sim A\big)$ and $(\boldsymbol{\delta}^F_A)^2 | \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A\big)$ are continuously differentiable and

$$\big\langle u,\, \mathrm{D}(\boldsymbol{\delta}^F_A)^2(y)\big\rangle = \big\langle u,\, \mathrm{D}(F^*)^2(y - \boldsymbol{\xi}^F_A(y))\big\rangle \quad \text{for } y \in \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A\big) \text{ and } u \in \mathbf{R}^{n+1}\,.$$

(f) $\mathscr{L}^{n+1}(\mathbf{R}^{n+1} \sim \operatorname{dmn} \boldsymbol{\xi}^F_A) = 0$.

(g) Assume $a \in A$, $u \in \partial \mathbf{B}^{F^*}(0,1)$, $t > 0$, and $\boldsymbol{\delta}^F_A(a + tu) = t$. Then $a + su \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ and $\boldsymbol{\xi}^F_A(a + su) = a$ for all $0 < s < t$. In particular,

$$\{s : \boldsymbol{\xi}^F_A(a+su) = a\} \subseteq \{s : \boldsymbol{\delta}^F_A(a+su) = s\} = \operatorname{Clos}\{s : \boldsymbol{\xi}^F_A(a+su) = a\}\,.$$

(h) Assume $a \in A$, $x \in \mathbf{R}^{n+1}$, and $\boldsymbol{\delta}^F_A(x) = F^*(x-a)$. Then

$$x - a \in G(\operatorname{Nor}(A,a))\,.$$

In particular, if $\mathbf{n}(A,a) \ne 0$, then

$$\mathbf{n}^F(A,a) = \boldsymbol{\nu}^F_A(x) = \frac{x-a}{F^*(x-a)}\,.$$

Proof. We mimic parts of the proof of [16, 4.8]. We set $G^* = \operatorname{grad} F^*$.

Let $y, z \in \mathbf{R}^{n+1}$, then

$$\boldsymbol{\delta}^F_A(y) \le \boldsymbol{\delta}^F_A(z) + F^*(y-z) \quad \text{and} \quad \boldsymbol{\delta}^F_A(z) \le \boldsymbol{\delta}^F_A(y) + F^*(y-z)\,;$$

hence, claim (a) follows.

Assume that (b) does not hold. Then there are $y_i \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ for $i \in \mathbb{N}$ and $\varepsilon > 0$ such that $\lim_{i\to\infty} y_i = y \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ but $F^*(\boldsymbol{\xi}^F_A(y_i) - \boldsymbol{\xi}^F_A(y)) > \varepsilon$. Using (a) we get

$$F^*(\boldsymbol{\xi}^F_A(y_i) - y) \le \boldsymbol{\delta}^F_A(y) + 2F^*(y_i - y) \quad \text{for } i \in \mathbb{N}\,;$$

hence, the set $\{\boldsymbol{\xi}^F_A(y_i) : i \in \mathbb{N}\}$ is a bounded subset of the closed set A and we may assume that $\lim_{i\to\infty} \boldsymbol{\xi}^F_A(y_i) = z \in A$. Then

$$\boldsymbol{\delta}^F_A(y) = \lim_{i\to\infty} \boldsymbol{\delta}^F_A(y_i) = \lim_{i\to\infty} F^*(\boldsymbol{\xi}^F_A(y_i) - y_i) = F^*(z-y)\,;$$

hence, $\boldsymbol{\xi}_A^F(y) = z$ which is incompatible with

$$F^*(z - \boldsymbol{\xi}_A^F(y)) = \lim_{i \to \infty} F^*(\boldsymbol{\xi}_A^F(y_i) - \boldsymbol{\xi}_A^F(y)) \geq \varepsilon\,.$$

Assume (c) does not hold. Then there are $0 < t < 1$ and $b \in A$ such that setting $y = a + t(x - a)$ we get $F^*(y - b) < F^*(y - a)$ and

$$F^*(x-a) \leq F^*(x-b) \leq F^*(x-y) + F^*(y-b) < F^*(x-y) + F^*(y-a) = F^*(x-a)$$

a contradiction.

Now we prove (d). We have

$$\boldsymbol{\delta}_A^F(x + t(a - x)) = \boldsymbol{\delta}_A^F(x) - t\boldsymbol{\delta}_A^F(x) \quad \text{for } 0 < t < 1\,,$$

which implies

$$\operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet \frac{x-a}{\boldsymbol{\delta}_A^F(x)} = \frac{\mathrm{D}\boldsymbol{\delta}_A^F(x)(a-x)}{-\boldsymbol{\delta}_A^F(x)} = 1\,. \tag{16}$$

From (16) and (a) we conclude using 2.38(d)(c)

$$\begin{aligned}(17)\quad 1 = \sup\{\mathrm{D}\boldsymbol{\delta}_A^F(x)u : u \in \mathbf{R}^{n+1},\ F^*(u) \leq 1\}& \\ = \sup\{\operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet u : u \in \mathbf{R}^{n+1},\ F^*(u) \leq 1\}& \\ = F^{**}(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = F(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = \operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet G(\operatorname{grad} \boldsymbol{\delta}_A^F(x))\,.&\end{aligned}$$

However, due to 2.38(b) there is exactly one $w \in \mathbf{R}^{n+1}$ with $F^*(w) = 1$ such that $\operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet w = F(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = 1$; thus,

$$G(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = \frac{x-a}{\boldsymbol{\delta}_A^F(x)}\,.$$

The formula for $\mathrm{D}(\boldsymbol{\delta}_A^F)^2$ postulated in (e) is proven exactly as in [16, 4.8(5)] noting

$$F^*(\tfrac{x-a}{\boldsymbol{\delta}_A^F(x)}) = 1\,, \quad \operatorname{grad} \boldsymbol{\delta}_A^F(x) = G^*(\tfrac{x-a}{\boldsymbol{\delta}_A^F(x)})\,, \quad \langle u,\, \mathrm{D}(F^*)^2(y)\rangle = 2F^*(y)G^*(y) \bullet u\,.$$

Continuity of the derivatives of $\boldsymbol{\delta}_A^F | \mathbf{R}^{n+1} \sim A$ and $(\boldsymbol{\delta}_A^F)^2$ follows from the formulas and a reasoning completely analogous to the proof of [16, 4.8(5)].

Item (f) is now a consequence of the Rademacher theorem [17, 3.1.6].

For the proof of (g) recall (c) and assume to the contrary, that there exist $0 < s < t$ and $b \in A$, $b \neq a$ such that $s = F^*(a + su - a) = F^*(a + su - b) = \boldsymbol{\delta}_A^F(a + su)$. Set $p = a + su$ and $q = a + tu$. Clearly $b \neq p + su$ since otherwise $t = \boldsymbol{\delta}_A^F(q) \leq F^*(q - b) = F^*(a + tu - (a + 2su)) = t - 2s < t$ which is impossible. Therefore, $q - a$ and $q - b$ are linearly independent and, using 2.38(e), we obtain the contradictory estimate

$$t \leq F^*(q-b) < F*(q-p) + F^*(p-b) = t - s + s = t\,.$$

To prove (h) we observe that

$$\mathbf{U}^{F^*}(x, F^*(x-a)) \cap A = \varnothing\,; \quad \text{hence,} \quad -\mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a) \in \operatorname{Nor}(A, a)\,.$$

Indeed, otherwise there would exist $v \in \operatorname{Tan}(A, a)$ such that $v \bullet \mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a) < 0$ so there would be points $y_i \in A$ such that $|y_i - a| \to 0$ and $(y_i - a)/|y_i - a| \to v$ as $i \to \infty$ and then, since F^* is of class $\mathscr{C}^1$, we could find $i \in \mathbb{N}$ for which $y_i \in \mathbf{U}^{F^*}(x, F^*(x-a)) \cap A$ and this cannot happen. Employing 2.38(i) we see that

$$G(-\mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a)) = \frac{x-a}{F^*(x-a)} \in G(\operatorname{Nor}(A, a))\,. \qquad \square$$

3 Totally umbilical hypersurfaces

In 2.39 we proved that $\partial \mathbf{B}^{F^*}(0,r)$ has all F-principal curvatures equal to $1/r$. In this section we show that this condition actually characterises the manifold $\partial \mathbf{B}^{F^*}(0,r)$.

3.1 Lemma. *Suppose M is a connected submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ of dimensions n, $\eta : M \to \mathbf{R}^{n+1}$ is Lipschitz, and $\kappa : M \to \mathbf{R}$ is such that*

$$\mathrm{D}\eta(z)(u) = \kappa(z)u \quad \text{for } \mathscr{H}^n \text{ almost all } z \in M \text{ and all } u \in \operatorname{Tan}(M,z)\,.$$

Then κ is a constant function.

Proof. Since M is connected it suffices to show the claim only locally. Let $a \in M$. We represent M near a as the graph of some $\mathscr{C}^{1,1}$ function f, i.e., we find $p \in \mathbf{O}^*(n+1,n)$, $q \in \mathbf{O}^*(n+1,1)$, $U \subseteq \mathbf{R}^n$ an open ball centred at $p(a)$, and $f : U \to \mathbf{R}$ of class $\mathscr{C}^{1,1}$ such that, setting $L = p^* + q^* \circ f$, there holds

$$a \in L[U] \subseteq M \quad \text{and} \quad q \circ p^* = 0\,.$$

For each $v \in \mathbf{R}^n$ we define

$$\gamma_v : U \to \mathbf{R} \quad \text{by} \quad \gamma_v(x) = \eta(L(x)) \bullet v\,.$$

Then

$$\begin{aligned}(18)\quad \mathrm{D}\gamma_v(x)u &= \mathrm{D}\eta(L(x))(\mathrm{D}L(x)u) \bullet v = \kappa(L(x))(\mathrm{D}L(x)u) \bullet v \\ &= \kappa(L(x))(p^*(u) + q^*(\mathrm{D}f(x)u)) \bullet v = \kappa(L(x))(u \bullet p(v) + \mathrm{D}f(x)u \bullet q(v)) \\ &\qquad \text{for } \mathscr{L}^n \text{ almost all } x \in U,\ u \in \mathbf{R}^n,\ v \in \mathbf{R}^{n+1}\,.\end{aligned}$$

Now, choose an orthonormal basis $e_1, \dots, e_n$ of $\mathbf{R}^n$ and set $\gamma_i = \gamma_{p^*(e_i)}$ for $i = 1, 2, \dots, n$. Since $q \circ p^* = 0$ and $p \circ p^* = \mathbf{1}_{\mathbf{R}^n}$, we obtain

$$\begin{aligned}(19)\quad \mathrm{D}\gamma_i(x)e_j = \kappa(F(x))(e_i \bullet e_j) = 0 \quad &\text{and} \quad \mathrm{D}\gamma_i(x)e_i = \kappa(F(x)) \\ &\text{for } \mathscr{L}^n \text{ almost all } x \in U,\ i,j \in \{1,2,\dots,n\}, \text{ and } i \neq j\,.\end{aligned}$$

Recall that U is an open ball centred at $p(a)$. Define $J = \{(x - p(a)) \bullet e_1 : x \in U\}$. Since η is Lipschitz we see that $\gamma_1, \dots, \gamma_n$ are absolutely continuous and deduce from (19) that there exist Lipschitz functions $a_1, \dots, a_n : J \to \mathbf{R}$ such that

$$\begin{aligned}(20)\quad \gamma_i(x) &= a_i((x - p(a)) \bullet e_i) \\ &\text{and} \quad a_i'((x - p(a)) \bullet e_i) = a_j'((x - p(a)) \bullet e_j) = \kappa(F(x)) \\ &\qquad \text{for } \mathscr{L}^n \text{ almost all } x \in U,\ i,j \in \{1,2,\dots,n\}\,.\end{aligned}$$

It follows that a_i' is a constant function for $i = 1, 2, \dots, n$; hence, κ is also constant. □

3.2 Lemma. *Suppose F is an elliptic integrand of class $\mathscr{C}^{1,1}$, M is a connected n-dimensional submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ satisfying $\operatorname{Clos} M \sim M = \varnothing$, $\nu : M \to \mathbf{R}^{n+1}$ is Lipschitz and such that $\nu(z) \in \operatorname{Nor}(M,z)$ and $|\nu(z)| = 1$, $\eta : M \to \mathbf{R}^{n+1}$ is defined by $\eta(y) = \operatorname{grad} F(\nu(y))$, and there exists a scalar function $\kappa : M \to \mathbf{R}$ such that*

$$\mathrm{D}\eta(y)u = \kappa(y)u \quad \text{for } \mathscr{H}^n \text{ almost all } y \in M \text{ and all } u \in \operatorname{Tan}(M,y)\,.$$

Then there exists $\lambda \in \mathbf{R}$ such that $\kappa(y) = \lambda$ for $y \in M$ and either $\lambda = 0$ and M is a hyperplane in $\mathbf{R}^{n+1}$ or $\lambda \neq 0$ and $M = \partial \mathbf{B}^F(a, |\lambda|^{-1})$ for some $a \in \mathbf{R}^{n+1}$.

Proof. In view of 3.1 we obtain $\lambda \in \mathbf{R}$ such that

$$\mathrm{D}\eta(z)u = \lambda u \quad \text{for all } \mathscr{H}^n \text{ almost all } z \in M \text{ and } u \in \operatorname{Tan}(M,z)\,.$$

Therefore, $\mathrm{D}(\eta - \lambda \mathrm{id}_{\mathbf{R}^n}) = 0$ and we obtain $c \in \mathbf{R}^n$ such that

$$\eta(z) - \lambda z = c \quad \text{for all } z \in M\,.$$

If $\lambda = 0$, then η is constant and M must be a hyperplane because $\operatorname{Clos} M \sim M = \varnothing$. In case $\lambda \neq 0$ we set $a = -c\lambda^{-1}$ and $\rho = |\lambda|^{-1}$. Then

$$F^*(z - a) = \rho F^*(\eta(z)) = \rho F^*(\operatorname{grad} F(\nu(z))) = \rho \quad \text{for all } z \in M\,,$$

by 2.38(a). Hence, $M = \partial \mathbf{B}^F(a, \rho)$ because $\operatorname{Clos} M \sim M = \varnothing$. □

4 The Lusin property for anisotropic (n,h)-sets

In this section F is an elliptic integrand of class $\mathscr{C}^2$ and $\Omega \subseteq \mathbf{R}^{n+1}$ is open.

4.1 Definition (cf. [11, Definition 3.1])**.** We say that $Z \subseteq \Omega$ is an *(n,h)-set with respect to F* if Z is relatively closed in Ω and for any open set $N \subseteq \Omega$ such that $\partial N \cap \Omega$ is smooth and $Z \subseteq \operatorname{Clos} N$ there holds

$$F(\mathbf{n}(N,p))\mathbf{h}_F(\mathbf{v}_n(\partial N),p) \bullet \mathbf{n}(N,p) \geq -h \quad \text{for } p \in Z \cap \partial N \cap \Omega\,.$$

4.2 Lemma. *Suppose $T \in \mathbf{G}(n+1,n)$, $\eta \in T^\perp$, $|\eta| = 1$, $f : T \to T^\perp$ is pointwise differentiable of order 2 at 0 and satisfies $f(0) = 0$ and $\operatorname{pt} \mathrm{D}f(0) = 0$, $\Sigma = \{x + f(x) : x \in T\}$, $h \geq 0$, and Γ is an (n,h) subset of Ω with respect to F such that $0 \in \Gamma$ and*

$$\Gamma \cap V \subseteq \big\{ z : z \bullet \eta \leq f \circ T_\natural(z) \bullet \eta \big\}$$

for some open neighbourhood V of 0. Then

$$F(\eta) \operatorname{pt} \mathbf{h}_F(\Sigma,0) \bullet \eta \geq -h.$$

Proof. We mimic the proof of [34, 3.4]. Fix $\varepsilon > 0$, define $P, \psi : T \to T^\perp$ by

$$P(x) = \tfrac{1}{2}\langle (x,x), \operatorname{pt} \mathrm{D}^2 f(0)\rangle \quad \text{for } x \in T\,, \tag{21}$$

$$\psi(x) = \big(P(x) \bullet \eta + \varepsilon |x|^2\big)\eta \quad \text{for } x \in T\,, \tag{22}$$

$$\text{and set} \quad M = \mathbf{R}^n \cap \big\{ x + \psi(x) : x \in T \big\}\,. \tag{23}$$

Note that since f is pointwise differentiable of order 2 at 0, it follows that

$$\lim_{x \to 0} \frac{|f(x) - P(x)|}{|x|^2} = 0\,.$$

Hence, we choose $r > 0$ such that $f(x) \bullet \eta \leq \psi(x) \bullet \eta$ for $x \in \mathbf{U}(0,r) \cap T$. Since Γ is an (n,h) subset of Ω, M is smooth and touches Γ at 0, and $\Gamma \cap \mathbf{U}(0,r) \subseteq \mathbf{R}^{n+1}\{x : x \bullet \eta \leq \psi(x) \bullet \eta\}$, we may use the barrier principle [11, Proposition 3.1(iii)] to derive the estimate

$$F(\eta)\mathbf{h}_F(M,0) \bullet \eta \geq -h\,.$$

Recall 2.31 to see that

$$-F(\eta) \operatorname{pt} \mathbf{h}_F(M,0) = \eta \operatorname{tr}\big(\boldsymbol{\Xi}(\mathrm{D}^2 F(\eta)) \circ \boldsymbol{\Xi}(\mathrm{D}^2(\psi \circ T_\natural)(0) \bullet \eta)\big)\,.$$

Since

$$\mathrm{D}^2(\psi \circ T_\natural)(0)(u,v) \bullet \eta = \operatorname{pt} \mathrm{D}^2(f \circ T_\natural)(0)(u,v) \bullet \eta + 2\varepsilon u \bullet T_\natural v \quad \text{for } u, v \in \mathbf{R}^n$$

we see that

$$-F(0,\eta) \operatorname{pt} \mathbf{h}_F(\Sigma,0) = -F(\eta) \operatorname{pt} \mathbf{h}_F(M,0) - 2\varepsilon\eta \operatorname{tr}\big(\boldsymbol{\Xi}(\mathrm{D}^2 F(\eta))\big)\,.$$

Passing to the limit $\varepsilon \downarrow 0$ we obtain the claim. □

4.3 Definition. Suppose $A \subseteq \mathbf{R}^{n+1}$ is a closed set. We say that $N(A)$ satisfies the *n dimensional Lusin (N) condition in* Ω if and only if

$$S \subseteq A \cap \Omega \quad \text{and} \quad \mathscr{H}^n(S) = 0 \quad \text{implies that} \quad \mathscr{H}^n(N(A)|S) = 0\,.$$

4.4 Theorem. *Suppose $0 \leq h < \infty$, A is an (n,h) subset of Ω with respect to F that is a countable union of sets with finite $\mathscr{H}^n$ measure.*

Then $N(A)$ satisfies the n dimensional Lusin (N) condition in Ω.

Proof. We modify the proof of [34, 3.7]. Let $\tau > \lambda = 2C(F)^2(n-1) + 1$, where $C(F) > 0$ is defined in (2.18).

Claim 1: *Assume $r \in \mathbf{R}$ satisfies $0 \leq h < \frac{1}{2C(F)r}$, and $x \in S(A,r) \cap R(A) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ (see 2.8) is such that $\boldsymbol{\Theta}^n(\mathscr{H}^n \llcorner S(A,r) \sim A_\tau, x) = 0$, and the conclusions of [34, Lemma 2.8] are* *satisfied. Consider an orthonormal basis $v_1, \ldots, v_{n+1}$ in which the matrix of* $\operatorname{ap} \mathrm{D}\boldsymbol{\nu}_A(x)$ *is diagonal and* $v_{n+1} = \boldsymbol{\nu}_A(x)$*. We introduce abbreviations*

$$\partial_{ij} F(\nu) = \langle (v_i, v_j),\, \mathrm{D}^2 F(\nu) \rangle \quad \text{for } i, j \in \{1, 2, \ldots, n+1\}\,.$$

Then we have

$$\sum_{i=1}^{n} \partial_{ii} F(\boldsymbol{\nu}_A(x)) \chi_{A,i}(x) \le h \quad \text{and} \quad \|{\textstyle\bigwedge_n}\big((\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x)\big)\| > 0\,.$$

Noting that $\boldsymbol{\xi}_A | A_\lambda$ is approximately differentiable at x (since $x \in R(A)$), we employ [31, 3.8, 3.10(3)(6)] and [17, 3.2.16] to conclude that

$$\chi_{A,j}(x) \ge -(\lambda - 1)^{-1} r^{-1} \quad \text{for } j = 1, \ldots, n\,, \tag{24}$$

$$\operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x) | \operatorname{Tan}(\mathscr{H}^n \llcorner S(A,r), x) = (\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x). \tag{25}$$

We choose f, V and T as in [34, Lemma 2.8] and $0 < s < r/2$ such that $\mathbf{U}(x,s) \subseteq V$. We assume $\boldsymbol{\xi}_A(x) = 0 \in \Gamma$ and we notice that $T_\natural(x) = 0$ and $\boldsymbol{\nu}_A(x) = r^{-1}x$. Then we define $g(\zeta) = f(\zeta) - x$ for $\zeta \in T$,

$$U = T_\natural\big(\mathbf{U}(x,s) \cap \{\chi + f(\chi) : \chi \in T\}\big), \quad W = \{y - x : y \in T_\natural^{-1}(U) \cap \mathbf{U}(x,s)\}\,.$$

It follows that W is an open neighbourhood of 0 and

$$W \cap A \subseteq \{z : z \bullet \boldsymbol{\nu}_A(x) \le g(T_\natural(z)) \bullet \boldsymbol{\nu}_A(x)\}\,. \tag{26}$$

Indeed, if (26) did not hold, then there would be $y \in \mathbf{U}(x,s) \cap T_\natural^{-1}[U]$ such that $y - x \in A$ and $y \bullet \boldsymbol{\nu}_A(x) > f(T_\natural(y)) \bullet \boldsymbol{\nu}_A(x)$; noting that

$$T_\natural(y) + f(T_\natural(y)) \in \mathbf{U}(x,s) \cap S(A,r) \quad \text{and} \quad |T_\natural(y) + f(T_\natural(y)) - y| < r\,,$$

we would conclude

$$|T_\natural(y) + f(T_\natural(y)) - (y - x)| = r - (y - f(T_\natural(y))) \bullet \boldsymbol{\nu}_A(x) < r = \boldsymbol{\delta}_A(T_\natural(y) + f(T_\natural(y)))$$

which is a contradiction.

Since $-\chi_{A,1}(x), \ldots, -\chi_{A,n}(x)$ are the eigenvalues of $\operatorname{pt} \mathrm{D}^2 g(0) \bullet \boldsymbol{\nu}_A(x)$ and $0 \in A$, we may apply 4.2 to infer that

$$\partial_{11} F(\boldsymbol{\nu}_A(x)) \chi_{A,1}(x) + \ldots + \partial_{nn} F(\boldsymbol{\nu}_A(x)) \chi_{A,n}(x) \le h \tag{27}$$

and combining (6), 2.18, (24), and (27) we get that for every $j = 1, \ldots, n$

$$\begin{aligned} \chi_{A,j}(x) \le C(F) \partial_{jj} F(\boldsymbol{\nu}_A(x)) \chi_{A,j}(x) &\le C(F)h - C(F) \sum_{k \ne j, k=1}^{n} \partial_{kk} F(\boldsymbol{\nu}_A(x)) \chi_{A,k}(x) \\ &\le C(F)h - \frac{C(F)^2(n-1)}{(\lambda - 1)r} < \frac{1}{r}\,. \end{aligned} \tag{28}$$

From (25) and [31, 3.5] follows that $1 - r\chi_{A,j}(x)$ are the eigenvalues of $(\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x)$ for $j = 1, \ldots, n$; hence, we obtain

$$\|{\textstyle\bigwedge_n}\big((\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x)\big)\| \ge \prod_{i=1}^{n} \big(1 - \chi_{A,i}(x) r\big) > 0\,.$$

Claim 2: *For $\mathscr{H}^n$ a.e. $x \in S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ and for $\mathscr{L}^1$ a.e. $0 < r < \frac{1}{2C(F)h}$ the conclusion of Claim 1 holds.*

This is immediate since

$$\boldsymbol{\Theta}^n(\mathscr{H}^n \llcorner S(A,r) \sim A_\tau, x) = 0$$

for $\mathscr{H}^n$ a.e. $x \in S(A,r) \cap A_\tau$ and for every $r > 0$ by [31, 2.13(1)] and [17, 2.10.19(4)], and $\mathscr{H}^n(S(A,r) \sim R(A)) = 0$ for $\mathscr{L}^1$ a.e. $r > 0$ by [31, 3.16].

Claim 3: *$N(A)$ satisfies the n dimensional Lusin (N) condition in Ω.*

Let $R \subseteq A$ be such that $\mathscr{H}^n(R) = 0$. For $r > 0$ it follows from [31, 3.17, 3.18(1), 4.3] that $\boldsymbol{\psi}_A | A_\tau \cap S(A,r)$ is a bilipschitz homeomorphism and

$$\boldsymbol{\psi}_A(\boldsymbol{\xi}_A^{-1}\{x\} \cap A_\tau \cap S(A,r)) \subseteq N(A,x) \quad \text{for } x \in A\,.$$

Noting Claim 2 and [31, 3.10(1)], we can apply [34, Lemma 3.5] with W and f replaced by $S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ and $\boldsymbol{\xi}_A | S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ to infer that

$$\mathscr{H}^n(\boldsymbol{\xi}_A^{-1}(R) \cap S(A,r) \cap A_\tau) = 0 \quad \text{for } \mathscr{L}^1 \text{ a.e. } 0 < r < \frac{1}{2C(F)} h^{-1}.$$

We notice that $N(A)|R = \bigcup_{r>0} \boldsymbol{\psi}_A(S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(R))$ by [31, 4.3] and $\boldsymbol{\psi}_A(S(A,r) \cap A_\tau) \subseteq \boldsymbol{\psi}_A(S(A,s) \cap A_\tau)$ if $s < r$ by [31, 3.18(2)]. Henceforth, it follows that

$$\mathscr{H}^n(N(A)|R) = 0\,. \qquad \square$$

The following weak maximum principle is a simple consequence of [11, Theorem 3.4].

4.5 Lemma. *Assume*

$$V \in \mathbf{V}_n(\Omega)\,, \quad F(\overline{\mathbf{h}}_F(V,x)) \le h \quad \textit{for } \|V\| \textit{ almost all } x\,, \quad \|\delta_F V\|_{\mathrm{sing}} = 0\,.$$

Then $\operatorname{spt}\|V\|$ *is an (n,h) subset of Ω with respect to F.*

Proof. For every $k \in \mathbb{N}$ let $V_k = k \cdot V$. Note that

$$u \bullet v = \frac{u}{F(u)} \bullet \frac{v}{F^*(v)} F(u)F^*(v) \le F(u)F^*(v) \quad \text{whenever } u, v \in \mathbf{R}^{n+1}\,;$$

thus,

$$\delta_F V_k(g) = -\int \overline{\mathbf{h}}_F(V,x) \bullet g(x)\, \mathrm{d}\|V_k\|(x) \le h \int F^*(g(x))\, \mathrm{d}\|V_k\|(x) \quad \text{for } k \in \mathbb{N} \text{ and } g \in \mathscr{X}(\Omega)\,.$$

Moreover, the area blowup set

$$Z = \big\{x \in \operatorname{Clos}\Omega : \limsup_{k\to\infty} \|V_k\|(\mathbf{B}(x,r)) = +\infty \text{ for every } r > 0\big\}$$

coincides with $\operatorname{spt}\|V\|$; hence, [11, Theorem 3.4] yields that $\operatorname{spt}\|V\| = Z$ is an (n,h) set. $\square$

5 The anisotropic unit normal bundle

In this section we will need to work with a suitable anisotropic variant of the normal bundle for closed sets. Let us introduce some definitions.

5.1 Definition. Suppose F is an elliptic integrand and $A \subseteq \mathbf{R}^{n+1}$ is closed. The *generalized anisotropic unit normal bundle of A* is defined as

$$N^F(A) = (A \times \partial \mathbf{B}^{F^*}(0,1)) \cap \{(a,u) : \boldsymbol{\delta}_A^F(a+su) = s \text{ for some } s > 0\}\,.$$

5.2 Lemma. *Suppose F is an elliptic integrand of class $\mathscr{C}^{1,1}$ and $A \subseteq \mathbf{R}^{n+1}$ is closed. Then*

$$N^F(A) = (\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)[N(A)] = \big\{(a, \operatorname{grad} F(u)) : (a,u) \in N(A)\big\}\,.$$

In particular, $N^F(A)$ is a countably n rectifiable Borel subset of $\mathbf{R}^{n+1} \times \partial\mathbf{B}^{F^*}(0,1)$.

Proof. Given $(a,u) \in N^F(A)$, there exists $s > 0$ such that

$$a \in A \cap \partial \mathbf{U}^{F^*}(a+su,s) \quad \text{and} \quad \mathbf{U}^{F^*}(a+su,s) \cap A = \varnothing\,.$$

Since $\partial\mathbf{U}^{F^*}(a+su,s)$ is submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ (see 2.38(h)), there exists $r > 0$ and $x \in \mathbf{R}^{n+1}$ such that $\mathbf{U}(x,r) \subseteq \mathbf{U}^{F^*}(a+su,s)$ and $a \in \partial\mathbf{U}(x,r)$. It follows that

$$\mathbf{n}(\mathbf{U}(x,r),a) = \mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a) \quad \text{and} \quad (a, -\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)) \in N(A)\,.$$

Since $\operatorname{grad} F(\mathbf{n}(\mathbf{U}^{F^*}(0,1),z)) = z$ for every $z \in \partial \mathbf{U}^{F^*}(0,1)$ (see 2.38(i)), it follows that

$$\operatorname{grad} F\big(-\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)\big) = -\operatorname{grad} F\big(\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)\big) = -\frac{a-(a+su)}{s} = u\,,$$

i.e. $(a,u) \in (\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A))$.

The proof of the reverse inclusion $(\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A)) \subseteq N^F(A)$ is completely analogous and the postscript follows from [31, 4.3]. $\square$

5.3 Definition. Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is open, F is an elliptic integrand, and $A \subseteq \mathbf{R}^{n+1}$ is closed. We say that $N^F(A)$ satisfies the *n dimensional Lusin (N) condition in Ω* if and only if the following implication holds,

$$S \subseteq A \cap \Omega, \quad \mathscr{H}^n(S) = 0 \quad \implies \quad \mathscr{H}^n(N^F(A)|S) = 0\,.$$

5.4 Lemma. *Assume F is an elliptic integrand of class $\mathscr{C}^{1,1}$, $\Omega \subseteq \mathbf{R}^{n+1}$ is open, and $A \subseteq \mathbf{R}^{n+1}$ is closed. Then $N(A)$ satisfies the n dimensional Lusin (N) condition in Ω if and only if $N^F(A)$ satisfies the n dimensional Lusin (N) condition in Ω.*

Proof. Let $S \subseteq A \cap \Omega$ be such that $\mathscr{H}^n(S) = 0$. Assume that either $\mathscr{H}^n(N^F(A)|S) = 0$ or $\mathscr{H}^n(N(A)|S) = 0$. Since the map $\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F$ is a bilipschitz homeomorphism (see 2.38(h)), we deduce that $\mathscr{H}^n(N^F(A)|S) = \mathscr{H}^n((\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A)|S)) = \mathscr{H}^n(N(A)|S) = 0$ as desired. $\square$

5.5 Definition. Let F be an elliptic integrand and $A \subseteq \mathbf{R}^{n+1}$ be closed. The *anisotropic reach function* $r^F_A : N^F(A) \to [0,\infty]$ is defined by

$$r^F_A(a,u) = \sup\{s : \boldsymbol{\delta}^F_A(a+su) = s\} \quad \text{for } (a,u) \in N^F(A)\,.$$

The *anisotropic reach of A* is defined by

$$\operatorname{reach}^F(A) = \inf\big\{\sup\{r : \mathbf{U}^{F^*}(a,r) \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A\} : a \in A\big\} = \sup\big\{r : \{x : \boldsymbol{\delta}^F_A(x) < r\} \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A\big\}\,.$$

5.6 Remark. Since $\boldsymbol{\delta}^F_A$ is Lipschitz continuous (see 2.38(a)), the function $f_s : N^F(A) \to \mathbf{R}$ given by $f_s(a,u) = \boldsymbol{\delta}^F_A(a+su) - s$ is also Lipschitz for any $s \in \mathbf{R}$. Therefore r^F_A is lower-semicontinuous. In particular, r^F_A is a Borel function.

5.7 Lemma. *Suppose F is an elliptic integrand of class $\mathscr{C}^{1,1}$ and A is a closed submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^1$ such that* $\operatorname{reach}^F A > 0$. *Then* $\operatorname{reach} A > 0$ *and A is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$.*

Proof. Set $W = \mathbf{B}^{F^*}(0,1)$. First observe that ∂W is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ by 2.38(h). Therefore, there exists $\rho \in (0,1)$ such that for each $x \in \partial W$ we have

$$\mathbf{B}(x + \rho \mathbf{n}(W,x), \rho) \subseteq W\,.$$

Assume $\operatorname{reach}^F A = s > 0$. Let $z \in \mathbf{R}^{n+1}$ be such that $\boldsymbol{\delta}_A(z) = r < \rho s$ and find $x \in A$ with $|z-x| = \boldsymbol{\delta}_A(z)$. Set $B = \mathbf{B}(z,r)$, $u = -\mathbf{n}(B,x)$, and $w = x + r \operatorname{grad} F(u)/\rho$. Note that $u \in \operatorname{Tan}(A,x)^\perp$. We have $\boldsymbol{\delta}^F_A(w) = r/\rho < s$ so $w \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ and $\mathbf{B}^{F^*}(w, r/\rho) \cap A = \{x\}$ and $\mathbf{B}(z,r) \subseteq \mathbf{B}^{F^*}(w, r/\rho)$; hence, $z \in \operatorname{dmn} \boldsymbol{\xi}_A$.

Since z was arbitrary we see that $\{x : \boldsymbol{\delta}_A(x) < \rho s\} \subseteq \operatorname{dmn} \boldsymbol{\xi}_A$ which shows that $\operatorname{reach} A \geq \rho s$. The second part of the conclusion readily follows from [16, 4.20]. $\square$

5.8 Corollary. *Suppose $A \subseteq \mathbf{R}^{n+1}$ is closed and* $\operatorname{reach}^F A > 0$. *Then $S^F(A,r)$ is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ of dimension n for every* $0 < r < \operatorname{reach}^F A$.

Proof. Since $R = \operatorname{reach}^F A > 0$, we have that $\mathbf{R}^{n+1} \cap \{y : \boldsymbol{\delta}^F_A(y) < R\} \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A$. Therefore, from 2.40(e)(d) and 2.38(f) it follows that $\boldsymbol{\delta}^F_A | \mathbf{R}^{n+1} \cap \{y : 0 < \boldsymbol{\delta}^F_A(y) < R\}$ is of class $\mathscr{C}^1$ and

$$\operatorname{grad} \boldsymbol{\delta}^F_A(y) = \operatorname{grad} F^*\Big(\frac{x - \boldsymbol{\xi}^F_A(y)}{\boldsymbol{\delta}^F_A(y)}\Big) \neq 0 \quad \text{for } y \in \mathbf{R}^{n+1} \text{ with } 0 < \boldsymbol{\delta}^F_A(y) < R\,.$$

Consequently, for every $0 < r < R$ we see that $S^F(A,r) = (\boldsymbol{\delta}^F_A)^{-1}\{r\}$ is a closed submanifold of Ω of class $\mathscr{C}^1$ of dimension n. Moreover, we have $\operatorname{reach}^F S^F(A,r) \geq \min\{R-r, r\} > 0$ so the conclusion follows from 5.7. $\square$

We prove now the anisotropic version of [22, Theorem 3], whose proof is essentially along the same lines.

5.9 Theorem. *Assume F is an elliptic integrand of class $\mathscr{C}^{1,1}$ and $A \subseteq \mathbf{R}^{n+1}$ is closed. Let $r > 0$ and suppose that for every $\mathscr{H}^n$ measurable bounded function $f : \mathbf{R}^{n+1} \times \partial \mathbf{U}^{F^*}(0,1) \to \mathbf{R}$ with compact support there are numbers $c_1(f), \ldots, c_{n+1}(f) \in \mathbf{R}$ such that*

$$\int_{\mathbf{R}^{n+1} \sim A} f \circ \boldsymbol{\psi}_A^F \cdot \mathbf{1}_{\{x : \boldsymbol{\delta}_A^F(x) \le t\}} \,\mathrm{d}\mathscr{L}^{n+1} = \sum_{j=1}^{n+1} c_j(f) t^j \quad \text{for } 0 < t < r\,. \tag{29}$$

Then $\operatorname{reach}^F(A) \ge r$.

Proof. Let $S = \{(x,u,t) : (x,u) \in N^F(A),\ r_A^F(x,u) > t\}$ and define $\phi : N^F(A) \times (0,\infty) \to \mathbf{R}^{n+1}$

$$\phi(x,u,t) = x + tu \quad \text{for } (x,u,t) \in N^F(A) \times (0,\infty).$$

Claim 1: $\mathscr{L}^{n+1}(\operatorname{dmn} \boldsymbol{\xi}_A^F \sim (A \cup \phi(S))) = 0$*; hence,*

$$\mathscr{L}^{n+1}(\mathbf{R}^{n+1} \sim (A \cup \phi(S))) = 0\,.$$

Recalling 2.38(g) we see that

$$\operatorname{dmn} \boldsymbol{\xi}_A^F \sim (A \cup \phi(S)) = \phi(\{(x,u,t) : (x,u) \in N^F(A),\ t = r_A^F(x,u) > 0\})\,.$$

Since ϕ is a locally Lipschitz map, it suffices to prove that

$$\mathscr{H}^{n+1}(\{(x,u,t) : (x,u) \in K,\ M > t = r_A^F(x,u) > 0\}) = 0 \tag{30}$$

for all $M \in \mathbb{N}$ and $K \subseteq N^F(A)$ bounded. By 5.2 and [17, 3.2.29] we know that $N^F(A)$ is countably n rectifiable. Hence, it suffices to prove (30) for all $M \in \mathbb{N}$ and $K \subseteq A$ being n rectifiable. Assume K and M are such. Employing [17, 3.2.23] we get

$$\mathscr{H}^{n+1}(K \times (0, M+1)) = (M+1)\mathscr{H}^n(K) < \infty\,. \tag{31}$$

Recall 5.6. For $q \in \mathbf{R}$ define the Borel set

$$V_q = \{(x,u,t+q) : (x,u) \in K,\ M > t = r_A^F(x,u) > 0\}$$

and observe that

$$V_q \cap V_p = \varnothing \quad \text{whenever } p \ne q\,, \quad V_q \subseteq K \times (0, M+1) \quad \text{for } 0 < q < 1\,, \tag{32}$$

$$\text{and} \quad \mathscr{H}^{n+1}(V_q) = \mathscr{H}^{n+1}(V_0) \quad \text{for any } q \in \mathbf{R}\,. \tag{33}$$

Therefore, if $\mathscr{H}^{n+1}(V_0) > 0$, then $\mathscr{H}^{n+1}(\bigcup\{V_q : 0 < q < 1,\ q \text{ rational}\}) = \infty$ which contradicts (31).

Claim 2:

$$\mathscr{L}^{n+1}\big(\{z : 0 < \boldsymbol{\delta}_A^F(z) \le r,\ r_A^F(\boldsymbol{\psi}_A^F(z)) < r\}\big) = 0\,. \tag{34}$$

In the following sequence of estimates we have to deal with the problem that $N^F(A)$ might not have locally finite measure so $\mu = \mathscr{H}^n \llcorner N^F(A)$ might not be Radon and (μ, n) approximate derivative of ϕ might not be well defined.

Recalling 2.38(g) one readily infers that $\phi|S$ is injective. Since $N^F(A)$ is Borel and countably n rectifiable (see 5.2) we may find a partition

$$N^F(A) = \textstyle\bigcup_{i=1}^\infty N_i$$

such that each N_i is a Borel n rectifiable set (in particular, $\mathscr{H}^n(N_i) < \infty$) and the family $\{N_i : i \in \mathbb{N}\}$ is disjointed; cf. [17, 2.1.6]. For $i \in \mathbb{N}$ w define

$$\mu_i = \mathscr{H}^n \llcorner N_i\,, \quad S_i = S \cap \big(N_i \times (0,\infty)\big)\,, \quad \text{and} \quad J = \sum_{i=1}^\infty \big\| \textstyle\bigwedge_n [(\mu_i, n) \operatorname{ap} \mathrm{D}\phi] \big\| \mathbf{1}_{S_i}\,.$$

We apply Claim 1 and the coarea formula [17, 3.2.22] to find that

$$
\begin{aligned}
(35)\quad \int_{\mathbf{R}^{n+1}\sim A} g\,\mathrm{d}\mathscr{L}^{n+1} &= \int_{\phi(S)} g\,\mathrm{d}\mathscr{L}^{n+1} = \sum_{i=1}^{\infty}\int_{\phi(S_i)} g\,\mathrm{d}\mathscr{L}^{n+1} \\
&= \int_0^\infty \sum_{i=1}^{\infty}\int_{N_i} \big\|\textstyle\bigwedge_n[(\mu_i,n)\,\mathrm{ap}\,\mathrm{D}\phi(x,u,t)]\big\| g(x+tu)\mathbf{1}_{\{(w,v):r^F_A(w,v)>t\}}(x,u)\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}t \\
&= \int_0^\infty J(x,u,t)g(x+tu)\mathbf{1}_{\{(w,v):r^F_A(w,v)>t\}}(x,u)\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}t
\end{aligned}
$$

whenever $g:\mathbf{R}^{n+1}\to\mathbf{R}$ is a non-negative Borel function with compact support.

Let $B\subseteq\mathbf{R}^{n+1}$ be compact, $0<\tau<r$ and $\tau<t<r$. We define

$$N_{\tau,B} = N^F(A)\cap\{(x,u): r^F_A(x,u)\le\tau,\ x\in B\}\,,$$

and we apply (29) to the function $\mathbf{1}_{N_{\tau,B}}$ and (35) to the function $g=(\mathbf{1}_{N_{\tau,B}}\circ\boldsymbol{\psi}^F_A)\cdot\mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\le t\}}$ to compute

$$
\begin{aligned}
(36)\quad \sum_{j=1}^{n+1} c_j(f)t^j &\overset{(29)}{=} \int_{\mathbf{R}^{n+1}\sim A}\mathbf{1}_{N_{\tau,B}}(\boldsymbol{\psi}^F_A(z))\mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\le t\}}(z)\,\mathrm{d}\mathscr{L}^{n+1}z \\
&\overset{(35)}{=} \int_0^\infty\int_{N^F(A)} J(x,u,s)\mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\le t\}}(x+su)\mathbf{1}_{\{(w,v):r^F_A(w,v)>s\}}(x,u)\mathbf{1}_{N_{\tau,B}}(\boldsymbol{\psi}^F_A(x+su))\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}s \\
&= \int_0^\infty\int_{N^F(A)} J(x,u,s)\mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\le t\}}(x+su)\mathbf{1}_{\{(w,v):r^F_A(w,v)>s\}}(x,u)\mathbf{1}_{N_{\tau,B}}(x,u)\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}s \\
&= \int_0^\infty\int_{N^F(A)} J(x,u,s)\mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\le t\}}(x+su)\mathbf{1}_{\{(w,v):s<r^F_A(w,v)\le\tau\}}(x,u)\mathbf{1}_B(x)\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}s \\
&= \int_0^\infty\int_{N^F(A)} J(x,u,s)\mathbf{1}_{\{(w,v):s<r^F_A(w,v)\le\tau\}}(x,u)\mathbf{1}_B(x)\,\mathrm{d}\mathscr{H}^n(x,u)\,\mathrm{d}s\,,
\end{aligned}
$$

where the last equality follows because $\boldsymbol{\delta}^F_A(x+su)=s<r^F_A(x,u)\le\tau<t$, for every $\tau<t<r$. Whence, we deduce that $\sum_{j=1}^{n+1}c_j(f)t^j$ is independent of t, for every $\tau<t<r$. Therefore, this polynomial is identically zero, a condition that implies, by the first equality in (36),

$$\mathscr{L}^{n+1}\big(\{z: 0<\boldsymbol{\delta}^F_A(z)\le r,\ \boldsymbol{\psi}^F_A(z)\in N_{\tau,B}\}\big)=0\,.$$

Since the last equation holds for every $0<\tau<r$ and for every compact set $B\subseteq\mathbf{R}^{n+1}$, we conclude that (34) holds.

Claim 3: $\operatorname{reach}^F(A)\ge r$.

Let $z\in\mathbf{R}^{n+1}\sim A$ satisfy $0<\boldsymbol{\delta}^F_A(z)<r$. Then there exists a sequence $\{z_i : i\in\mathbb{N}\}\subseteq\operatorname{dmn}\boldsymbol{\xi}^F_A$ which converges to z and such that

$$0<\boldsymbol{\delta}^F_A(z_i)\le r \quad\text{and}\quad r^F_A(\boldsymbol{\psi}^F_A(z_i))\ge r\,.$$

Noting that $(\boldsymbol{\xi}^F_A(z_i))$ is a bounded sequence, and passing to a subsequence if necessary, we find $p\in A$ and $u\in\partial\mathbf{U}^{F^*}(0,1)$ such that

$$\boldsymbol{\xi}^F_A(z_i)\to p,\qquad \boldsymbol{\nu}^F_A(z_i)\to u\,.$$

In particular, $z=p+\boldsymbol{\delta}^F_A(z)u$. We find $t\in\mathbf{R}$ such that $\boldsymbol{\delta}^F_A(z)<t<r$, and notice that

$$(37)\qquad \mathbf{U}^{F^*}(\boldsymbol{\xi}^F_A(z_i)+t\boldsymbol{\nu}^F_A(z_i),t)\cap A=\varnothing\quad\text{for } i\ge 1;\quad\text{hence,}\quad \mathbf{U}^{F^*}(p+tu,t)\cap A=\varnothing\,.$$

This shows that $\boldsymbol{\delta}^F_A(p+tu)=t>\boldsymbol{\delta}^F_A(z)$; hence, 2.38(g) yields $z\in\operatorname{dmn}\boldsymbol{\xi}^F_A$ and $\boldsymbol{\xi}^F_A(z)=p$. $\square$

6 Heintze Karcher inequality

Here we prove our main theorem 6.4.

6.1 Remark. Let F be an elliptic integrand. Recalling [17, 5.1.1] we define $\Phi : \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1} \to \mathbf{R}$, a parametric integrand of degree n on $\mathbf{R}^{n+1}$, by setting

$$\Phi(z,\xi) = F(*\xi) \quad \text{for } z \in \mathbf{R}^{n+1} \text{ and } \xi \in \textstyle\bigwedge_n \mathbf{R}^{n+1} ,$$

where $*$ denotes the Hodge star operator associated with the standard scalar product and orientation on $\mathbf{R}^{n+1}$; see [17, 1.7.8]. By 2.16 and [17, 5.1.2] we see that Φ is elliptic in the sense of [17, 5.1.2]. Moreover, if $\Phi^{\S}$ is the nonparametric integrand associated with Φ (see [17, 5.1.9]) and $\Phi^{\S}_z(\xi) = \Phi^{\S}(z,\xi)$ for $(z,\xi) \in \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1}$, then $\mathrm{D}^2\Phi^{\S}_z(\xi)$ is strongly elliptic in the sense of [17, 5.2.3] for all $(z,\xi) \in \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1}$ by [17, 5.2.17].

Let $W \subseteq \mathbf{R}^n$ be open and bounded, $V \in \mathbf{V}_n(W \times \mathbf{R})$, $\mathbf{p} : \mathbf{R}^{n+1} \to \mathbf{R}^n$ and $\mathbf{q} : \mathbf{R}^{n+1} \to \mathbf{R}$ be given by $\mathbf{p}(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_n)$ and $\mathbf{q}(z_1, \ldots, z_{n+1}) = z_{n+1}$ for $(z_1, \ldots, z_{n+1}) \in \mathbf{R}^{n+1}$. Assume $f : \mathbf{R}^n \to \mathbf{R}$ is of class $\mathscr{C}^1$, and V is the unit density varifold associated to the graph of f, i.e., $V = \mathbf{v}_n(\operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f))$. Recalling [17, 5.1.9] we see that for any $\theta : W \to \mathbf{R}$ of class $\mathscr{C}^1$ with compact support there holds

$$\delta_F V(\mathbf{q}^* \circ \theta \circ \mathbf{p}) = \int \big\langle (0, \theta(x), \mathrm{D}\theta(x)),\, \mathrm{D}\Phi^{\S}(x, f(x), \mathrm{D}f(x)) \big\rangle \,\mathrm{d}\mathscr{L}^{n+1}(x) .$$

Suppose F is of class $\mathscr{C}^3$, $\alpha \in (0,1)$, f is of class $\mathscr{C}^{1,\alpha}$, $\|\delta_F V\|$ is a Radon measure, $\|\delta_F V\|_{\mathrm{sing}} = 0$, and $\mathbf{h}_F(V,\cdot) : \operatorname{spt}\|V\| \to \mathbf{R}^{n+1}$ is of class $\mathscr{C}^{0,\alpha}$. Define $\eta : W \to \mathbf{R}^{n+1}$ and $H : W \to \mathbf{R}$ by the formulas

$$\eta(x) = (\mathbf{q}^*(1) - \mathbf{p}^*(\operatorname{grad} f(x))) \cdot (1 + |\operatorname{grad} f(x)|^2)^{-1/2} \tag{38}$$

$$\text{and} \quad H(x) = -F(\eta(x)) \cdot \mathbf{q} \circ \mathbf{h}_F(V, (\mathbf{p}^* + \mathbf{q}^* \circ f)(x)) \cdot \sqrt{1 + |\operatorname{grad} f|^2} \tag{39}$$

for $x \in W$. Note that $\eta(x)$ is the unit normal vector to the graph of f at $(\mathbf{p}^* + \mathbf{q}^* \circ f)(x)$ for $x \in W$. Employing the area formula [17, 3.2.3] we get

$$\delta_F V(\mathbf{q}^* \circ \theta \circ \mathbf{p}) = -\int_{\operatorname{spt}\|V\|} \theta(\mathbf{p}(z)) \cdot \mathbf{q}(\mathbf{h}(V,z)) \cdot F(\eta(\mathbf{p}(z)) \,\mathrm{d}\mathscr{H}^n(z) = \int_W \theta(x) \cdot H(x) \,\mathrm{d}\mathscr{L}^n(x)$$

so that

$$\int_W \big\langle (0, \theta(x), \mathrm{D}\theta(x)),\, \mathrm{D}\Phi^{\S}(x, f(x), \mathrm{D}f(x)) \big\rangle \,\mathrm{d}\mathscr{L}^{n+1}(x) = \int_W \theta(x) \cdot H(x) \,\mathrm{d}\mathscr{L}^n(x) \quad \text{for any } \theta \in \mathscr{D}(W, \mathbf{R}) . \tag{40}$$

Since H is of class $\mathscr{C}^{0,\alpha}$ a slight modification of the proof of [17, 5.2.15] shows that f is actually of class $\mathscr{C}^{2,\alpha}$.

To support the last claim recall the proof of [17, 5.2.15] with 2, $n+1$, n, α, W, $\Phi^{\S}$ in place of q, n, m, δ, U, G. Using all the symbols defined therein, for any integer ν such that $\nu > 1/d$, define $R_\nu : \mathbf{B}(b, \rho - d) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$ so that

$$\sigma \bullet R_\nu(x) = \int_0^1 \sigma(e_i) \cdot H(x + te_i/\nu) \,\mathrm{d}\mathscr{L}^1(t) \quad \text{for } \sigma \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}) \text{ and } x \in \mathbf{B}(b, \rho - d) .$$

Since, in our case, f satisfies (40) rather than [17, 5.2.15(4)] the displayed equation in the middle of page 556 of [17], i.e.,

$$\int_{\mathbf{U}(b,\rho-d)} \big\langle \mathrm{D}f_\nu(x) \odot \mathrm{D}\theta(x),\, A_\nu(x) \big\rangle \,\mathrm{d}\mathscr{L}^n(x) = (P_\nu - Q_\nu,\, \mathrm{D}\theta)_{b,\rho-d}$$

turns into

$$\int_{\mathbf{U}(b,\rho-d)} \big\langle \mathrm{D}f_\nu(x) \odot \mathrm{D}\theta(x),\, A_\nu(x) \big\rangle \,\mathrm{d}\mathscr{L}^n(x) = (P_\nu - Q_\nu - R_\nu,\, \mathrm{D}\theta)_{b,\rho-d} .$$

Clearly R_ν is α-Hölder continuous with Hölder constant independent of ν so all the estimates from the upper half of page 557 of [17] hold in the modified case with an additional term coming from R_ν. Thus, one can still use [17, 5.2.2] to conclude that $\mathrm{D}_i f$ is of class $\mathscr{C}^{1,\alpha}$; hence, f is of class $\mathscr{C}^{2,\alpha}$.

6.2 Remark. Suppose $E \subseteq \mathbf{R}^{n+1}$ is of finite perimeter. We recall that the reduced boundary (see 2.37) and the essential boundary (cf. [17, 4.5.12] and [5, Def. 3.60]) of E are $\mathscr{H}^n$ almost the same (see [5, Thm. 3.61]). Recalling [3, 4.7] we deduce that $\mathbf{n}(E,\cdot)|\partial^* E : \partial^* E \to \mathbf{R}^{n+1}$ equals the negative of the *generalised inner normal to* E defined in [5, Def. 3.54].

6.3 Definition. Let $A \subseteq \mathbf{R}^{n+1}$, $k \in \mathbb{N}$, $\alpha \in [0,1]$. We say that $x \in A$ is a $\mathscr{C}^{k,\alpha}$*-regular point of* A if there exists an open set $W \subseteq \mathbf{R}^{n+1}$ such that $x \in W$ and $A \cap W$ is an n-dimensional submanifold of class $\mathscr{C}^{k,\alpha}$ of $\mathbf{R}^{n+1}$. The set of all $\mathscr{C}^{k,\alpha}$ regular points of A shall be called the $\mathscr{C}^{k,\alpha}$ *regular part of* A.

The strategy for the proof of our main theorem can be summarised in the following way. First we replace the set E with an open set Ω with the same essential boundary using [33, 2.2]. Using standard regularity theory for codimension one varifolds with bounded anisotropic mean curvature [2] and 6.1 we deduce that $\mathscr{H}^n$ almost all of $\partial^*\Omega$ is $\mathscr{C}^{2,\alpha}$ regular. On the $\mathscr{C}^{2,\alpha}$ regular part we can express the, variationally defined, anisotropic mean curvature vector $\mathbf{h}_F(V,\cdot)$ as the trace of the anisotropic second fundamental form as in 2.25. However, this does not reduce the problem to the smooth case because we have no control of the singular set and we do not know how different parts of the regular set are arranged in space. Therefore, we look at level-sets $S^F(C,r)$ of the anisotropic distance function from $C = \mathbf{R}^{n+1} \sim \Omega$. These sets are easily seen to be $\mathscr{C}^{1,1}$ submanifolds of Ω of dimension n so we gain a priori regularity. Nonetheless, we need to transfer the information we have from $\mathscr{H}^n$ almost all of $\partial^*\Omega$ onto $S^F(C,r)$ and then back to $\partial^*\Omega$. To this end we need the Lusin (N) condition for $\partial^*\Omega$ which follows from the weak maximum principle 4.5 and 4.4. The Lusin (N) property of $\partial\Omega$ allows to represent $\mathscr{L}^{n+1}$ almost all of Ω as the image of the map $\zeta(x,t) = x + t\mathbf{n}^F(C,x)$, where x belongs to the *regular part of* ∂C and $t > 0$ is bounded by the first eigenvalue of the anisotropic second fundamental form of $\partial\Omega$ at x. At this point we apply the Montiel-Ros argument to estimate the measure of Ω and derive the Heintze-Karcher inequality.

Next, we deal with the equality case. First we note that the principal curvatures of $\partial\Omega$ must all equal $-n/H(z)$ for z in the regular part of $\partial\Omega$. We use the Steiner formula 5.9 to deduce that $\operatorname{reach}^F C > n/c$. Then we let $0 < r < n/c$ and we compute the principal curvatures of the level-set $S^F(C,r)$ using the information we have on the regular part of $\partial\Omega$. This and the Lusin (N) property show that $S^F(C,r)$ is totally umbilical at $\mathscr{H}^n$ almost all points. Since we know that $S^F(C,r)$ is of class $\mathscr{C}^{1,1}$, the $\mathscr{H}^n$ almost everywhere information is enough to apply 3.2 to see that $S^F(C,r)$ is a finite union of boundaries of Wulff shapes of radii $n/c - r$. After that, it is rather easy to see that each connected component of Ω must be a Wulff shape of radius at least n/c. Since the perimeter of Ω is finite we see also that there may be at most finitely many connected components of Ω.

6.4 Theorem. *Suppose*

$$F \text{ is an elliptic integrand of class } \mathscr{C}^3\,, \quad n \geq 2\,, \quad c \in (0,\infty)\,, \tag{41}$$

$$E \subseteq \mathbf{R}^{n+1} \text{ is a set of finite perimeter}\,, \quad \mathscr{H}^n\big(\operatorname{Clos}(\partial^* E) \sim \partial^* E\big) = 0\,, \tag{42}$$

$$V = \mathbf{v}_n(\partial^* E) \in \mathbf{RV}_n(\mathbf{R}^{n+1})\,, \quad \|\delta_F V\|_{\mathrm{sing}} = 0\,, \tag{43}$$

$$\mathbf{h}_F(V,\cdot)|K \text{ is of class } \mathscr{C}^{0,\alpha} \text{ for each compact subset } K \text{ of the } \mathscr{C}^{1,\alpha} \text{ regular part of } \operatorname{spt}\|V\|\,, \tag{44}$$

$$0 < -\overline{\mathbf{h}}_F(V,x) \bullet \mathbf{n}(E,x) \leq c \quad \text{for } \|V\| \text{ almost all } x\,. \tag{45}$$

Then

$$\mathscr{L}^{n+1}(E) \leq \frac{n}{n+1} \int_{\partial E} \frac{1}{|\mathbf{h}_F(V,x)|} \,\mathrm{d}\mathscr{H}^n(x) \tag{46}$$

and equality holds if and only if here there exists a finite union Ω *of disjoint open Wulff shapes with radii not smaller than* n/c *such that* $\mathscr{L}^{n+1}\big((\Omega \sim E) \cup (E \sim \Omega)\big) = 0$.

Proof. First we employ [33, 2.2] to obtain an open set $\Omega \subseteq \mathbf{R}^{n+1}$ such that

$$\mathscr{L}^{n+1}\big((\Omega \sim E) \cup (E \sim \Omega)\big) = 0 \quad \text{and} \quad \mathscr{H}^n(\partial\Omega \sim \partial^*\Omega) = 0\,.$$

Directly from the definition (see [17, 4.5.12, 4.5.11]) it follows that the essential boundaries of Ω and E coincide; hence, recalling 6.2, we obtain $V = \mathbf{v}_n(\partial^*\Omega)$. We shall consider Ω instead of E in the sequel. Let us define

$$H : \operatorname{spt}\|V\| \to [0,c] \quad \text{so that} \quad H(x) = -\overline{\mathbf{h}}_F(V,x) \bullet \mathbf{n}(E,x) \quad \text{for } \|V\| \text{ almost all } x\,, \tag{47}$$

$$C = \mathbf{R}^{n+1} \sim \Omega\,, \quad Q = \partial C \cap \big\{x : x \text{ is a } \mathscr{C}^{2,\alpha}\text{-regular point of } \partial C\big\}\,. \tag{48}$$

Note that $\partial^* C = \partial^* \Omega$, $\mathbf{n}^F(C,\cdot) = -\mathbf{n}^F(\Omega,\cdot)$, and $H(x) = F(\mathbf{n}(E,x))|\mathbf{h}_F(V,x)|$ for $\|V\|$ almost all x.

Claim 1: *If $x \in Q$, $y \in \Omega$, and $\boldsymbol{\xi}^F_C(y) = x$ (in other words: $y \in \Omega \cap (\boldsymbol{\xi}^F_C)^{-1}(Q)$), then*

$$0 \le \frac{1}{n} H(x) \le -\kappa^F_{Q,1}(\boldsymbol{\psi}^F_C(y)) \le \boldsymbol{\delta}^F_C(y)^{-1}\,.$$

We clearly have

$$\mathbf{U}^{F^*}(y, \boldsymbol{\delta}^F_C(y)) \cap C = \varnothing \quad \text{and} \quad \partial \mathbf{U}^{F^*}(y, \boldsymbol{\delta}^F_C(y)) \cap C = \{x\}\,;$$

hence, recalling 2.39, 2.30, and that x is a $\mathscr{C}^{2,\alpha}$-regular point of ∂C, wee see that

$$\frac{1}{n} H(x) \le -\kappa^F_{Q,1}(\boldsymbol{\psi}^F_C(y)) \le -\kappa^F_{\partial \mathbf{U}^{F^*}(y,\boldsymbol{\delta}^F_C(y)),1}(\boldsymbol{\psi}^F_C(y)) = \boldsymbol{\delta}^F_C(y)^{-1}$$

and the claim is proven.

Claim 2: *$\mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)) = 0$.*

Note that $F(\overline{\mathbf{h}}_F(V,x)) = H(x)F(\mathbf{n}(\Omega,x))$ for $\|V\|$ almost all x so applying Lemma 4.5 we conclude that $\partial\Omega$ is an $(n, c\,C(F))$ subset of $\mathbf{R}^{n+1}$. It follows by Theorem 4.4 that $\mathscr{H}^n(N(\partial\Omega)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. Combining this with Lemma 5.4, we deduce that $\mathscr{H}^n(N^F(\partial\Omega)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. Since $N^F(C) \subseteq N^F(\partial\Omega)$, one readily infers that $\mathscr{H}^n(N^F(C)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. We also observe that for $\|V\|$ almost all z there exists a radius $r > 0$ such that V satisfies all the assumption of [2, The Regularity Theorem, pp. 27-28] inside $\mathbf{U}(z,r)$. This implies that there exists $\alpha \in (0,1)$ such that for $\mathscr{H}^n$ almost all $z \in \partial C$ there exists an open set $G \subset \mathbb{R}^{n+1}$ with $z \in G$ and such that $\partial C \cap G$ coincides with a rotated graph of some function $f : \mathbf{R}^n \to \mathbf{R}$ of class $\mathscr{C}^{1,\alpha}$. However, employing 6.1, we see that f is actually of class $\mathscr{C}^{2,\alpha}$. Therefore,

$$\mathscr{H}^n(\partial C \sim Q) = 0 \quad \text{and} \quad \mathscr{H}^n(N^F(C)|(\partial C \sim Q)) = 0\,. \tag{49}$$

Since $\boldsymbol{\psi}^{\boldsymbol{F}}_C\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}^F_C) \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)\big) \subseteq N(C)|(\partial C \sim Q)$ for every $r > 0$, we get

$$\mathscr{H}^n\big(\boldsymbol{\psi}^{\boldsymbol{F}}_C\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}^F_C) \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)\big)\big) = 0 \quad \text{for every } r > 0\,.$$

Moreover, we have $\big(\boldsymbol{\psi}^{\boldsymbol{F}}_C|(S^F(C,r) \cap \operatorname{dmn} \boldsymbol{\xi}^F_C \sim C)\big)^{-1} \in \mathscr{C}^1$ and we deduce that

$$\mathscr{H}^n\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}^F_C) \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)\big) = 0 \quad \text{for every } r > 0.$$

Combining 2.40(f)(a)(d) with the coarea formula [17, 3.2.22], we get

$$\mathscr{H}^n(S^F(C,r) \sim \operatorname{dmn} \boldsymbol{\xi}^F_C) = 0 \quad \text{for } \mathscr{L}^1 \text{ almost all } r > 0\,.$$

From 2.40(d) it follows that $F(\operatorname{grad} \boldsymbol{\delta}^F_C(x)) = 1$; hence, recalling 2.18, we obtain $|\operatorname{grad} \boldsymbol{\delta}^F_C(x)| \ge \frac{1}{C(F)}$. Using the coarea formula, we compute

$$\begin{aligned} &\frac{1}{C(F)} \mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)) \\ &\qquad \le \int_{\Omega \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)} |\operatorname{grad} \boldsymbol{\delta}^F_C(x)| dx = \int_0^\infty \mathscr{H}^n(S^F(C,r) \sim (\boldsymbol{\xi}^F_C)^{-1}(Q))\, dr = 0\,. \end{aligned} \tag{50}$$

In particular we get that $\mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}^F_C)^{-1}(Q)) = 0$, which settles Claim 2.

We define

$$Z = (Q \times \mathbf{R}) \cap \big\{(x,t) : 0 < t \le -\kappa^F_{Q,1}(x, \mathbf{n}^F(C,x))^{-1}\big\}\,, \tag{51}$$

$$\zeta : Z \to \mathbf{R}^{n+1}\,, \quad \zeta(x,t) = x + t\mathbf{n}^F(C,x)\,. \tag{52}$$

For brevity of the notation we also set

$$J_{n+1}\zeta(x,t) = \|{\textstyle\bigwedge_{n+1}}(\mathscr{H}^{n+1} \llcorner Z, n+1) \operatorname{ap} \mathrm{D}\zeta(x,t)\| \quad \text{whenever } (x,t) \in Z\,.$$

Claim 3: *There holds*

$$J_{n+1}\zeta(x,t) = F(\mathbf{n}(C,x)) \prod_{i=1}^{n} \big(1 + t\,\kappa^F_{Q,i}(x, \mathbf{n}^F(C,x))\big) \quad \textit{for } (x,t) \in Z\,. \tag{53}$$

Let $(x,t) \in Z$ and $u = \mathbf{n}^F(C,x)$. Recalling 2.28 we find a basis $\tau_1(x), \ldots, \tau_n(x)$ of $\mathrm{Tan}(Q,x)$ consisting of eigenvectors of $\mathrm{D}(\mathbf{n}^F(C,\cdot))(x)$ and such that

$$\big\langle \tau_i(x),\, \mathrm{D}\mathbf{n}^F(C,\cdot)(x) \big\rangle = \kappa^F_{Q,i}(x,u)\,\tau_i(x) \quad \text{for } i \in \{1,2,\ldots,n\}\,, \tag{54}$$

$$|\tau_1(x) \wedge \cdots \wedge \tau_n(x)| = 1\,. \tag{55}$$

Noting that $\mathrm{Tan}(Z,(x,t)) = \mathrm{Tan}(Q,x) \times \mathbf{R}$,

$$\big\langle (0,1),\, \mathrm{D}\zeta(x,t) \big\rangle = \mathbf{n}^F(C,x) = \operatorname{grad} F(\mathbf{n}(C,x))\,, \tag{56}$$

$$\big\langle (\tau_i(x),0),\, \mathrm{D}\zeta(x,t) \big\rangle = (1 + t\kappa^F_{Q,i}(x,u))\,\tau_i(x) \quad \text{for } i \in \{1,\ldots,n\}\,, \tag{57}$$

we compute

$$\begin{aligned} J_{n+1}\zeta(x,t) &= \prod_{i=1}^{n}(1 + t\kappa^F_{Q,i}(x,u))\,|\mathbf{n}^F(C,x) \wedge \tau_1(x) \wedge \cdots \wedge \tau_n(x)| \\ &= \operatorname{grad} F(\mathbf{n}(C,x)) \bullet \mathbf{n}(C,x) \prod_{i=1}^{n}(1 + t\kappa^F_{Q,i}(x,u))\,|\mathbf{n}(C,x) \wedge \tau_1(x) \wedge \cdots \wedge \tau_n(x)| \end{aligned} \tag{58}$$

and Claim 3 follows from 2.38(c) and [17, 1.7.5].

Claim 4: *Inequality* (46) *holds.*

Employing Claim 1 and Claim 2 we see that $\mathscr{L}^{n+1}(\Omega \sim \zeta(Z)) = 0$. Hence, using the area formula and then Claim 3, we get

$$\begin{aligned} \mathscr{L}^{n+1}(\Omega) \le \mathscr{L}^{n+1}(\zeta(Z)) &\le \int_{\zeta(Z)} \mathscr{H}^0(\zeta^{-1}(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) = \int_Z J_{n+1}\zeta\,\mathrm{d}\mathscr{H}^{n+1} \\ &= \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa^F_{Q,1}(x,\mathbf{n}^F(C,x))} \prod_{i=1}^{n}\big(1 + t\kappa^F_{Q,i}(x,\mathbf{n}^F(C,x))\big)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x)\,. \end{aligned} \tag{59}$$

Using again Claim 1, then the standard inequality between the arithmetic and the geometric mean, and finally 2.30, we obtain

$$\begin{aligned} \mathscr{L}^{n+1}(\Omega) &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa^F_{Q,1}(x,\mathbf{n}^F(C,x))} \Big(\frac{1}{n}\sum_{i=1}^{n}\big(1 + t\kappa^F_{Q,i}(x,\mathbf{n}^F(C,x))\big)\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x) \\ &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{n/H(x)} \Big(1 - t\frac{H(x)}{n}\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x) \\ &= \frac{n}{n+1} \int_{\partial\Omega} \frac{F(\mathbf{n}(C,x))}{H(x)}\,\mathrm{d}\mathscr{H}^n(x)\,, \end{aligned} \tag{60}$$

which implies (46) by 2.22.

We assume now that equality holds in (46). Since the chains of inequalities (59) and (60) become chains of equalities, we deduce that

$$\mathscr{L}^{n+1}(\zeta(Z) \sim \Omega) = 0\,, \tag{61}$$

$$\mathscr{H}^0(\zeta^{-1}(y)) = 1 \quad \text{for } \mathscr{L}^{n+1} \text{ almost all } y \in \zeta(Z)\,, \tag{62}$$

$$-\kappa^F_{Q,j}(z, \mathbf{n}^F(C,z))^{-1} = \frac{n}{H(z)} \quad \text{for } \mathscr{H}^n \text{ almost all } z \in Q \text{ and all } j = 1,\ldots,n\,. \tag{63}$$

Our goal is to prove that Ω is a finite union of disjoint open Wulff shapes. We need two preliminary claims, whence the conclusion will be easily deduced.

Claim 5: $\operatorname{reach}^F C \geq n/c$.

Recall that $H(z) \leq c$ for $\mathscr{H}^n$ almost all $z \in \partial C$. Let $0 < \rho < n/c$ and

$$Q_\rho = Q \cap \{z : \rho < -\kappa^F_{Q,1}(z, \mathbf{n}^F(C,z))^{-1}\}.$$

It follows from (49), (63), and the fact that ∂C is an $(n, c\,C(F))$ subset of $\mathbf{R}^{n+1}$, that

$$\mathscr{H}^n(\partial C \sim Q_\rho) = 0 \quad \text{and} \quad \mathscr{H}^n(N(C)|\partial C \sim Q_\rho) = 0\,;$$

hence, we argue as in Claim 2 to conclude that $\mathscr{L}^{n+1}(\Omega \sim \boldsymbol{\xi}_C^{-1}(Q_\rho)) = 0$. We define

$$C^F_\rho = \{z : \boldsymbol{\delta}^F_C(z) \leq \rho\} \quad \text{and} \quad Z_\rho = Q_\rho \times \{t : 0 < t \leq \rho\}$$

and we notice that

$$\boldsymbol{\xi}_C^{-1}(Q_\rho) \cap \Omega \cap C^F_\rho \subseteq \zeta(Z_\rho) \subseteq C^F_\rho\,, \quad \mathscr{L}^{n+1}(\Omega \cap C^F_\rho \sim \zeta(Z_\rho)) = 0\,.$$

Let $f : \mathbf{R}^{n+1} \times \mathbf{S}^n \to \mathbf{R}$ be a Borel measurable function with compact support. Then we use Claim 1, (61), (62), (63), and [31, 5.4] to compute

$$\int_{\Omega \cap C^F_\rho} f(\boldsymbol{\psi}^F_C(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) = \int_{\Omega \cap \zeta(Z_\rho)} f(\boldsymbol{\psi}^F_C(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) \tag{64}$$

$$= \int_{\Omega \cap \zeta(Z_\rho)} \int_{\zeta^{-1}(y)} f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^0(z)\,\mathrm{d}\mathscr{L}^{n+1}(y) \tag{65}$$

$$= \int_{\zeta(Z_\rho)} \int_{\zeta^{-1}(y)} f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^0(z)\,\mathrm{d}\mathscr{L}^{n+1}(y) \tag{66}$$

$$= \int_{Z_\rho} J_{n+1}\zeta(z,t)\, f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^{n+1}(z,t) \tag{67}$$

$$= \int_{Q_\rho} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) \int_0^\rho \Big(1 - t\frac{H(z)}{n}\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(z) \tag{68}$$

$$= \int_{\partial C} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) \int_0^\rho \Big(1 - t\frac{H(z)}{n}\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(z) \tag{69}$$

$$= \sum_{i=1}^{n+1} c_i(f)\rho^i, \tag{70}$$

where, for $i = 1, \ldots, n+1$,

$$c_i(f) = \Big(-\frac{1}{n}\Big)^{i-1} \frac{n!}{i!(n-i+1)!} \int_{\partial C} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) H(z)^{i-1}\,\mathrm{d}\mathscr{H}^n(z)\,.$$

Therefore, $\operatorname{reach}^F C \geq n/c$ by Theorem 5.9.

Claim 6: *Let $0 < r < n/c \leq \operatorname{reach}^F C$. Then $S^F(C,r)$ is a finite union of Wulff shapes of radii not smaller than $c^{-1}(n - rc)$.*

Since $\operatorname{reach}^F C \geq n/c$ we employ 5.8 to find that $S^F(C,r)$ is a submanifold of $\mathbf{R}^{n+1}$ of dimension n of class $\mathscr{C}^{1,1}$. We define

$$C_r = \mathbf{R}^{n+1} \cap \{z : \boldsymbol{\delta}^F_C(z) < r\}\,.$$

Noting that $\mathbf{n}^F(C_r, \cdot)|S^F(C,r) = \operatorname{grad} F \circ \mathbf{n}(C_r, \cdot)|S^F(C,r)$ and $\operatorname{grad} F$ is a $\mathscr{C}^1$ function, we deduce that $\mathbf{n}^F(C_r, \cdot)|S^F(C,r)$ is a Lipschitzian vector field. We define

$$T = Q \cap \big\{z : \kappa^F_{Q,j}(z) = -H(z)/n \ \text{ for } j = 1, \ldots, n\big\}\,,$$

and we notice that $\mathscr{H}^n(\partial C \sim T) = 0$ by (49) and (63); then the Lusin (N) condition implies

$$\mathscr{H}^n(S^F(C,r) \sim (\boldsymbol{\xi}^F_C)^{-1}(T)) = 0\,. \tag{71}$$

Recalling 2.40(h) we see that

$$\mathbf{n}^F(C_r, z) = \frac{z - \boldsymbol{\xi}^F_C(z)}{r} = \operatorname{grad} F\big(\mathbf{n}(C, \xi(z))\big) = \mathbf{n}^F(C, \cdot) \circ \boldsymbol{\xi}^F_C(z) \quad \text{whenever } z \in S^F(C,r)\,. \tag{72}$$

Let us set

$$\sigma = \boldsymbol{\xi}_C^F | S^F(C,r) \cap (\boldsymbol{\xi}_C^F)^{-1}(T) \quad \text{and} \quad \varphi = \zeta | T \times \{r\}\,.$$

Observe that if $x \in T$, then $z = x + r\mathbf{n}^F(C,x) \in S^F(C,r)$, $\boldsymbol{\xi}_C^F(z) = x$, and $\operatorname{Tan}(S^F(C,r),z) = \operatorname{Tan}(T,x)$; hence, $\sigma = \varphi^{-1}$ and we get

$$\langle u,\, \mathrm{D}\varphi(x)\rangle = (1 - rH(x)/n)u \quad \text{for } x \in T \text{ and } u \in \operatorname{Tan}(T,x)\,, \tag{73}$$

$$\langle u,\, \mathrm{D}\sigma(z)\rangle = (1 - rH(\boldsymbol{\xi}_C^F(z))/n)^{-1}u \quad \text{for } z \in \operatorname{dmn}\sigma \text{ and } u \in \operatorname{Tan}(T,\boldsymbol{\xi}_C^F(z))\,, \tag{74}$$

$$\mathrm{D}\mathbf{n}^F(C_r,\cdot)(z)u = \frac{-H(\boldsymbol{\xi}_C^F(z))}{n - rH(\boldsymbol{\xi}_C^F(z))}u \quad \text{for } \mathscr{H}^n \text{ a.a. } z \in S^F(C,r) \text{ and } u \in \operatorname{Tan}(T,\boldsymbol{\xi}_C^F(z))\,. \tag{75}$$

Employing 3.2 we conclude that $S^F(C,r)$ is a union of at most countably many boundaries of Wulff shapes with radii not smaller than $c^{-1}(n-rc)$. Since E has finite perimeter we have $\mathscr{H}^n(\partial\Omega) < \infty$ so using (73) and (71) we conclude that $\mathscr{H}^n(S^F(C,r)) < \mathscr{H}^n(\partial^*\Omega) < \infty$ and Claim 6 follows.

We are now ready to conclude the proof. We notice from [16, 4.20] that

$$\partial C = \{x : \dim \operatorname{Nor}(C,x) \geq 1\}$$

and by Lemma 5.2, we also get that

$$\partial C = \{x : \dim \operatorname{Nor}^F(C,x) \geq 1\}\,.$$

We claim that

$$\boldsymbol{\xi}_C^F(S^F(C,r)) = \partial C \quad \text{for } 0 < r < n/c\,. \tag{76}$$

Indeed, since $0 < r < \operatorname{reach}^F C$, for every $x \in \partial C$ there exists $\nu \in \operatorname{Nor}^F(C,x)$ such that $x + r\nu \in S^F(C,r) \cap \operatorname{dmn}\boldsymbol{\xi}_C^F$ and consequently $\boldsymbol{\xi}_C^F(x+r\nu) = x$. We deduce that $\partial C \subseteq \boldsymbol{\xi}_C^F(S^F(C,r))$. The reverse inclusion is trivial.

Consider a connected component S_1 of $S^F(C,r)$. By Claim 6 we obtain $s \geq n/c - r$ and $z \in \mathbf{R}^{n+1}$ such that $S_1 = \partial\mathbf{B}^{F^*}(z,s)$. Observe that

$$S^F(\mathbf{R}^{n+1} \sim \mathbf{B}^{F^*}(z,s+r),r) = S_1\,;$$

hence,

$$\partial\mathbf{B}^{F^*}(z,s+r) = \boldsymbol{\xi}_C^F(S_1) \subseteq \partial C$$

and, using, e.g., the constancy theorem [17, 4.1.7], we deduce that $\mathbf{U}^{F^*}(z,s+r)$ is a connected component of Ω. Since S_1 was chosen arbitrarily we see that Ω must be a finite union of open disjoint Wulff shapes of radii at least n/c. $\square$

6.5 Remark. This theorem extends to sets of finite perimeter the analogous result for smooth boundaries in [21, Theorem 4].

We use now Theorem 6.4 to study the critical points of the anisotropic surface area for a given volume.

6.6 Definition (cf. [3, 4.1])**.** A smooth function $h : (-\epsilon,\epsilon) \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is called *local variation* if and only if

(a) $h(0,x) = x$ for every $x \in \mathbf{R}^{n+1}$,

(b) $h(t,\cdot) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a diffeomorphism for every $t \in (-\epsilon,\epsilon)$,

(c) the set $\{x : h(t,x) \neq x \text{ for some } t \in (-\epsilon,\epsilon)\}$ has compact closure in $\mathbf{R}^{n+1}$.

We set $h_t = h(t,\cdot)$ and $\dot{h}_t(x) = \lim_{u\to 0} u^{-1}(h_{t+u}(x) - h_t(x))$ for every $(t,x) \in (-\epsilon,\epsilon) \times \mathbf{R}^{n+1}$.

Given an integrand F we define *the F-perimeter functional* as

$$\mathcal{P}_F(E) = \int_{\partial^* E} F(\mathbf{n}(E,x))\, \mathrm{d}\mathscr{H}^n x \tag{77}$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter, and *the F-isoperimetric functional* as

$$\mathcal{I}_F(E) = \frac{\mathcal{P}_F(E)^{n+1}}{\mathscr{L}^{n+1}(E)^n}$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter and finite volume.

6.7 Corollary. *Let $E \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and finite volume such that*

$$\mathscr{H}^n(\mathrm{Clos}(\partial^* E) \sim \partial^* E) = 0.$$

If F is an elliptic integrand of class $\mathscr{C}^3$ and for every local variation h it holds that

$$\frac{d}{dt}\mathcal{I}_F(h_t(E))\Big|_{t=0} = 0, \tag{78}$$

then there exists a finite union Ω of disjoint open Wulff shapes with equal radii such that

$$\mathscr{L}^{n+1}((\Omega \sim E) \cup (E \sim \Omega)) = 0.$$

Proof. Let h be a local variation and $V = \mathbf{v}(\partial^* E)$. Define $p(t) = \mathcal{P}_F(h_t(E))$ and $v(t) = \mathscr{L}^{n+1}(h_t(E))$ for $-\epsilon < t < \epsilon$. We observe that

$$a'(0) = \delta_F V(\dot{h}_0)$$

$$v'(0) = \int_E \operatorname{div} \dot{h}_0 \, d\mathscr{L}^{n+1} = \int_{\partial^* E} \dot{h}_0(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)$$

Noting that the derivative in t the function $\frac{a^{n+1}}{v^n}$ equals

$$\Big(\frac{p(t)}{v(t)}\Big)^n \Big[(n+1)p'(t) - n\frac{p(t)}{v(t)}v'(t)\Big],$$

it follows that

$$(n+1)p'(0) - n\frac{p(0)}{v(0)}v'(0) = 0$$

and the arbitrariness of h implies that

$$\|\delta_F V\|_{\mathrm{sing}} = 0 \quad \text{and} \quad \overline{\mathbf{h}}_F(V,x) = -\frac{n}{n+1}\frac{\mathcal{P}_F(E)}{\mathscr{L}^{n+1}(E)}\mathbf{n}(E,x).$$

It follows that the hypothesis of Theorem 6.4 and the equality is realized in (46). Henceforth, the conclusion follows from Theorem 6.4. ☐

6.8 Corollary. *Let $E \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and finite volume such that*

$$\mathscr{H}^n(\mathrm{Clos}(\partial^* E) \sim \partial^* E) = 0.$$

If for every local variation h such that $\mathscr{L}^{n+1}(h_t(E)) = \mathscr{L}^{n+1}(E)$ for every $t \in (-\epsilon, \epsilon)$ it holds that

$$\frac{d}{dt}\mathcal{P}_F(h_t(E))\Big|_{t=0} = 0, \tag{79}$$

then there exists a finite union Ω of disjoint open Wulff shapes with equal radii such that

$$\mathscr{L}^{n+1}((\Omega \sim E) \cup (E \sim \Omega)) = 0.$$

Proof. Thanks to Corollary 6.7, we just need to prove that such a set E satisfies (78) for every local variation h. To this aim we define the variation

$$f_t(x) = \left(\frac{\mathscr{L}^{n+1}(E)}{\mathscr{L}^{n+1}(h_t(E))}\right)^{\frac{1}{n+1}} h_t(x)$$

and we observe that for every $t \in (-\epsilon, \epsilon)$ it holds

$$\mathscr{L}^{n+1}(f_t(E)) = \left(\frac{\mathscr{L}^{n+1}(E)}{\mathscr{L}^{n+1}(h_t(E))}\right)^{\frac{n+1}{n+1}} \mathscr{L}^{n+1}(h_t(E)) = \mathscr{L}^{n+1}(E).$$

We deduce from (79) that

$$0 = \frac{d}{dt}\mathcal{P}_F(f_t(E))\Big|_{t=0} = \mathscr{L}^{n+1}(E)^{\frac{n}{n+1}} \frac{d}{dt}\frac{\mathcal{P}_F(h_t(E))}{\mathscr{L}^{n+1}(h_t(E))^{\frac{n}{n+1}}}\Big|_{t=0},$$

which implies (78), as desired. ☐

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