RIGIDITY FOR PERIMETER INEQUALITY UNDER SPHERICAL SYMMETRISATION

F. CAGNETTI, M. PERUGINI, AND D. STÖGER

Abstract. Necessary and sufficient conditions for rigidity of the perimeter inequality under spherical symmetrisation are given. That is, a characterisation for the uniqueness (up to orthogonal transformations) of the extremals is provided. This is obtained through a careful analysis of the equality cases, and studying fine properties of the circular symmetrisation, which was firstly introduced by Pólya in 1950.

1. Introduction

In this paper we study the perimeter inequality under spherical symmetrisation, giving necessary and sufficient conditions for the uniqueness, up to orthogonal transformations, of the extremals. Perimeter inequalities under symmetrisation have been studied by many authors, see for instance [20, 21] and the references therein. In general, we say that rigidity holds true for one of these inequalities if the set of extremals is trivial. The study of rigidity can have important applications to show that minimisers of variational problems (or solutions of PDEs) are symmetric.

For instance, a crucial step in the proof of the Isoperimetric Inequality given by Ennio De Giorgi consists in showing rigidity of Steiner's inequality (see, for instance, [22, Theorem 14.4]) for convex sets (see the proof of Theorem I in Section 4 in [16, 17]). After De Giorgi, an important contribution in the understanding of rigidity for Steiner's inequality was given by Chlebík, Cianchi, and Fusco. In the seminal paper [12], the authors give sufficient conditions for rigidity which are much more general than convexity. After that, this result was extended to the case of higher codimensions in [3], where a quantitative version of Steiner's inequality was also given.

Then, necessary and sufficient conditions for rigidity (in codimension 1) were given in [9], in the case where the distribution function is a Special Function of Bounded Variation with locally finite jump set [9, Theorem 1.29]. The anisotropic case has recently been considered in [26], where rigidity for Steiner's inequality in the isotropic and anisotropic setting are shown to be equivalent, under suitable conditions. In the Gaussian setting, where the role of Steiner's inequality is played by Ehrhard's inequality (see [15, Section 4.1]), necessary and sufficient conditions for rigidity are given in [10], by making use of the notion of essential connectedness [10, Theorem 1.3]. Finally, in the smooth case, sufficient conditions for rigidity are given in [24, Proposition 5], for a general class of symmetrisations in warped products.

The main motivation for the study of the spherical symmetrisation is that it can be used to understand the symmetry properties of the solutions of PDEs and variational problems, when the radial symmetry has been ruled out. Moreover, some well established methods (as for instance the moving plane method, see [29, 19]) rely on convexity properties of the domain which fail, for instance, when one deals with annuli.

In particular, in many applications minimisers of variational problems and solutions of PDEs turn out to be *foliated Schwarz symmetric*. Roughly speaking, a function $u : \mathbb{R}^n \to \mathbb{R}$ is foliated Schwarz symmetric if one can find a direction $p \in \mathbb{S}^{n-1}$ such that u only depends on $|x|$ and on the polar angle $\alpha = \arccos(\hat{x} \cdot p)$, and u is non increasing with respect to

α (here $\hat{x} := x/|x|$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$). We direct the interested reader to [4, 5, 6, 31] and the references therein for more information.

1.1. **Spherical Symmetrisation.** To the best of our knowledge, the spherical symmetrisation was first introduced by Pólya in [27], in the case $n = 2$ and in the smooth setting. Let $n \in \mathbb{N}$ with $n \geq 2$. For each $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x, r)$ the open ball of $\mathbb{R}^n$ of radius r centred at x, by ω_n the (n-dimensional) volume of the unit ball, and we write $B(r)$ for $B(0, r)$. Moreover, $e_1, \ldots, e_n$ stand for the vectors of the canonical basis of $\mathbb{R}^n$. Given a set $E \subset \mathbb{R}^n$ and $r > 0$, we define the *spherical slice* E_r of E with respect to $\partial B(r)$ as

$$E_r := E \cap \partial B(r) = \{x \in E : |x| = r\}.$$

Let $v : (0, \infty) \to [0, \infty)$ be a measurable function. We say that E is *spherically v-distributed* if

$$v(r) = \mathcal{H}^{n-1}(E_r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty), \tag{1.1}$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure of $\mathbb{R}^n$, $1 \leq k \leq n$. Note that, in order v to be an admissible distribution, one needs

$$v(r) \leq \mathcal{H}^{n-1}(\partial B(r)) = n\omega_n r^{n-1} \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty). \tag{1.2}$$

In the following, as usual, we set $\mathbb{S}^{n-1} = \partial B(1)$. For every $x, y \in \mathbb{S}^{n-1}$, the *geodesic distance* between x and y is given by

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) := \arccos(x \cdot y).$$

Let $r > 0$, $p \in \mathbb{S}^{n-1}$, and $\beta \in [0, \pi]$ be fixed. The *open geodesic ball* (or *spherical cap*) of centre rp and radius β is the set

$$\mathbf{B}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) < \beta\}.$$

The $(n-1)$-dimensional Hausdorff measure of $\mathbf{B}_\beta(rp)$ can be explicitly calculated, and is given by

$$\mathcal{H}^{n-1}(\mathbf{B}_\beta(rp)) = (n-1)\omega_{n-1} r^{n-1} \int_0^\beta (\sin\tau)^{n-2}\, d\tau.$$

The expression above shows that the function $\beta \mapsto \mathcal{H}^{n-1}(\mathbf{B}_\beta(rp))$ is strictly increasing from $[0, \pi]$ to $[0, n\omega_n r^{n-1}]$. Therefore, if $v : (0, \infty) \to [0, \infty)$ is a measurable function satisfying (1.2), and $E \subset \mathbb{R}^n$ is a spherically v-distributed set, there exists only one (defined up to a subset of zero $\mathcal{H}^1$-measure) measurable function $\alpha_v : (0, \infty) \to [0, \pi]$ satisfying

$$v(r) = \mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v(r)}(re_1)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty). \tag{1.3}$$

Among all the spherically v-distributed sets of $\mathbb{R}^n$, we denote by F_v the one whose spherical slices are open geodesic balls centred at the positive e_1 axis., i.e.

$$F_v := \{x \in \mathbb{R}^n \setminus \{0\} : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) < \alpha_v(|x|)\},$$

see Figure 1.1. Before stating our results, it will be convenient to recall some basic notions about sets of finite perimeter.

1.2. **Basic notions on sets of finite perimeter.** Let $E \subset \mathbb{R}^n$ be a measurable set, and let $t \in [0, 1]$. We denote by $E^{(t)}$ the set of points of density t of E, given by

$$E^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, \rho))}{\omega_n \rho^n} = t \right\}.$$

The essential boundary of E is then defined as

$$\partial^{\mathrm{e}} E := E \setminus (E^{(1)} \cup E^{(0)}).$$

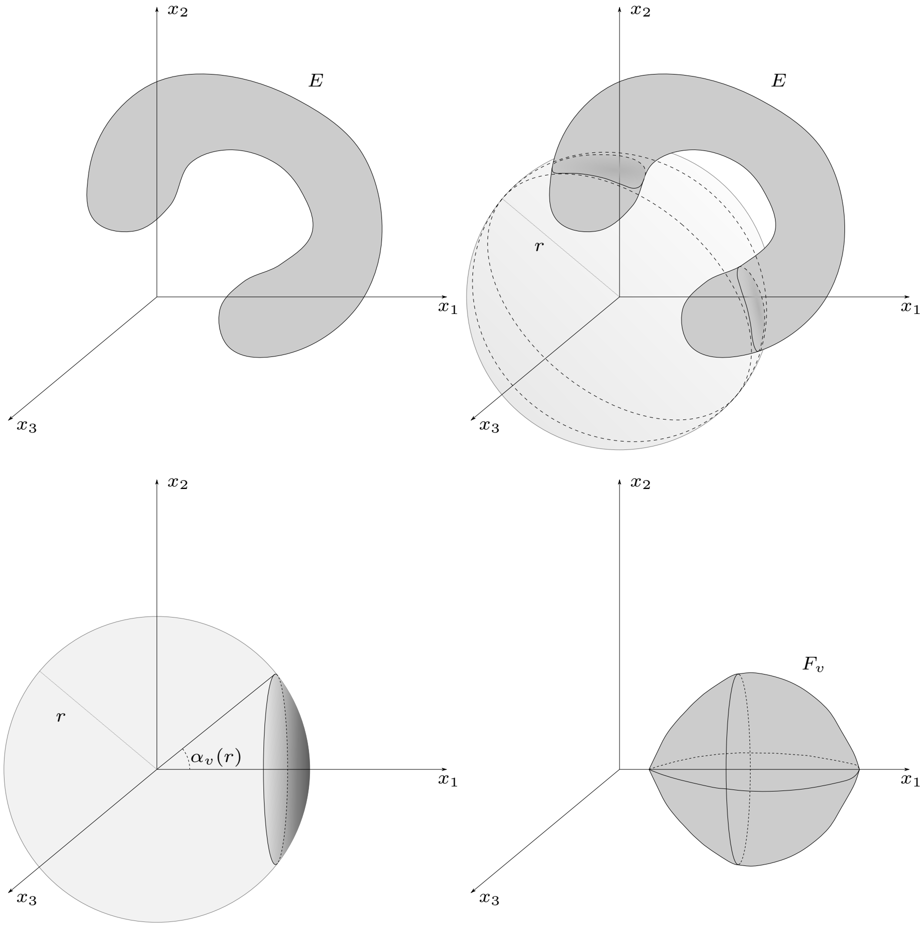

FIGURE 1.1. A pictorial idea of the spherical symmetral F_v of a v-distributed set E, in the case $n = 3$.

Moreover, if $A \subset \mathbb{R}^n$ is any Borel set, we define the perimeter of E relative to A as the extended real number given by

$$P(E; A) := \mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \cap A),$$

and we set $P(E) := P(E; \mathbb{R}^n)$. When E is a set with smooth boundary, it turns out that $\partial^{\mathrm{e}} E = \partial E$, and the perimeter of E agrees with the usual notion of $(n-1)$-dimensional surface measure of ∂E.

If $P(E) < \infty$, it is possible to define the reduced boundary $\partial^* E$ of E. This has the property that $\partial^* E \subset \partial^{\mathrm{e}} E$, $\mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \setminus \partial^* E) = 0$, and is such that for every $x \in \partial^* E$ there exists the *measure theoretic outer unit normal* $\nu^E(x)$ of $\partial^* E$ at x, see Section 2. If $x \in \partial^* E$, it will be convenient to decompose $\nu^E(x)$ as

$$\nu^E(x) = \nu^E_\perp(x) + \nu^E_\parallel(x),$$

where $\nu^E_\perp(x) := (\nu^E(x) \cdot \hat{x})\hat{x}$ and $\nu^E_\parallel(x)$ are the radial and tangential component of $\nu^E(x)$ along $\partial B(|x|)$, respectively. In the following, we will use the diffeomorphism $\Phi : (0, \infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ defined as

$$\Phi(r, \omega) := r\omega \qquad \text{for every } (r, \omega) \in (0, \infty) \times \mathbb{S}^{n-1}.$$

1.3. **Perimeter Inequality under spherical symmetrisation.** Our first result shows that the spherical symmetrisation does not increase the perimeter, and gives some necessary conditions for equality cases. In our analysis we require the set F_v (or, equivalently, any spherically v-distributed set) to have finite volume. This is not restrictive. Indeed, if F_v has finite perimeter but infinite volume, we can consider the complement $\mathbb{R}^n \setminus F_v$ which, by the relative isoperimetric inequality, has finite volume. This change corresponds to considering the complementary distribution function $r \mapsto n\omega_n r^{n-1} - v(r)$, and the spherical symmetrisation with respect to the axis $-e_1$.

Theorem 1.1. *Let $v : (0, \infty) \to [0, \infty)$ be a measurable function satisfying* (1.2)*, and let $E \subset \mathbb{R}^n$ be a spherically v-distributed set of finite perimeter and finite volume. Then, $v \in BV(0, \infty)$. Moreover, F_v is a set of finite perimeter and*

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) \leq P(E; \Phi(B \times \mathbb{S}^{n-1})), \tag{1.4}$$

for every Borel set $B \subset (0, \infty)$.

Finally, if $P(E) = P(F_v)$, then for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$:

(a) *E_r is $\mathcal{H}^{n-1}$-equivalent to a spherical cap and $\mathcal{H}^{n-2}(\partial^*(E_r) \Delta (\partial^* E)_r) = 0$;*

(b) *the functions $x \mapsto \nu^E(x) \cdot \hat{x}$ and $x \mapsto |\nu_\parallel^E|(x)$ are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$.*

The result above shows that the perimeter inequality holds on a local level, provided one considers sets of the type $\Phi(B \times \mathbb{S}^{n-1})$, with $B \subset (0, \infty)$ Borel. Inequality (1.4) is very well known in the literature. In the special case $n = 2$, a short proof was given by Pólya in [27]. In the general n-dimensional case with $B = (0, \infty)$ the result is stated in [25, Theorem 6.2]), but the proof is only sketched (see also [23] and [24, Proposition 3 and Remark 4]). As mentioned by Morgan and Pratelli in [25], certain parts of the proof of (1.4) follow the general lines of analogous results in the context of Steiner symmetrisation (see, for instance, [12, Lemma 3.4] and [3, Theorem 1.1]). There are, however, non trivial technical difficulties that arise when one deals with the spherical symmetrisation. For this reason, we give a detailed proof of Theorem 1.1.

We start by introducing radial and tangential components of a Radon measure, see Section 3.1. These turn out to be useful tools which allow to prove several preliminary results. Moreover, since we are dealing with a symmetrisation of codimension $n - 1$, we need to pay attention to some delicate effects that are not usually observed when the codimension is 1 (as, for instance, in [12]). Indeed, a crucial role is played by the measure λ_E given by:

$$\lambda_E(B) := \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} \hat{x} \cdot \nu^E(x) \, d\mathcal{H}^{n-1}(x), \tag{1.5}$$

for every Borel set $B \subset (0, \infty)$. When $n = 2$, it turns out that λ_E is singular with respect to the Lebesgue measure in $(0, \infty)$. However, for $n > 2$ it may happen that λ_E contains a non trivial absolutely continuous part, see Remark 3.9. This requires some extra care while proving inequality (1.4). A similar phenomenon has already been observed in [3], in the study of the Steiner symmetrisation of codimension higher than 1. Higher codimension effects play an important role also in the study of rigidity, as explained below.

1.4. **Rigidity of the Perimeter Inequality.** Given $v : (0, \infty) \to [0, \infty)$ measurable, satisfying (1.2), and such that F_v is a set of finite perimeter and finite volume, we define $\mathcal{N}(v)$ as the class of extremals of (1.4):

$$\mathcal{N}(v) := \{E \subset \mathbb{R}^n : E \text{ is spherically } v\text{-distributed and } P(E) = P(F_v)\}.$$

Note that, by definition of F_v, and by the invariance of the perimeter under rigid transformations, every time we apply an orthogonal transformation to F_v we obtain a set that belongs to $\mathcal{N}(v)$, i.e.:

$$\mathcal{N}(v) \supset \{E \subset \mathbb{R}^n : \mathcal{H}^n(E\Delta(R\,F_v)) = 0 \text{ for some } R \in O(n)\},$$

where Δ denotes the symmetric difference of sets and $O(n)$ is the set of orthogonal transformations in $\mathbb{R}^n$. We would like to understand when also the opposite inclusion is satisfied, that is, when the class of extremals of (1.4) is just given by rotated copies of F_v. We will say that *rigidity* holds true for inequality (1.4) if

$$\mathcal{N}(v) = \{E \subset \mathbb{R}^n : \mathcal{H}^n(E\Delta(R\,F_v)) = 0 \text{ for some } R \in O(n)\}. \tag{$\mathcal{R}$}$$

In order to explain which conditions we should expect in order ($\mathcal{R}$) to be true, let us first give some examples.

Figure 1.2 shows a set $E \in \mathcal{N}(v)$ that cannot be obtained by applying a single orthogonal transformation to F_v. This is due to the fact that the set $\{0 < \alpha_v < \pi\}$ is disconnected

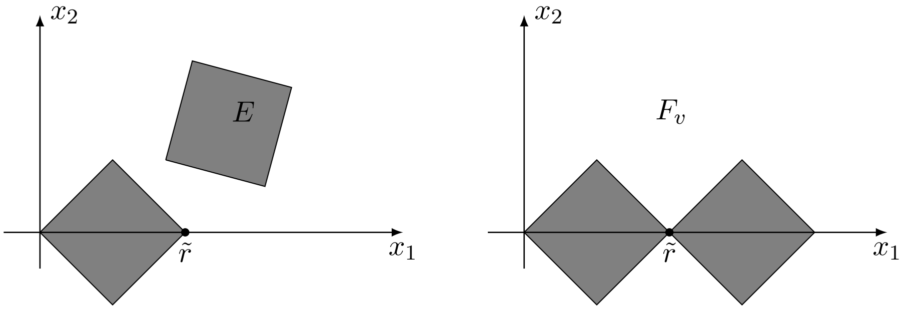

FIGURE 1.2. Rigidity ($\mathcal{R}$) fails, since the set $\{0 < \alpha_v < \pi\}$ is disconnected by a point $\tilde{r} \in (0,\infty)$ such that $\alpha_v(\tilde{r}) = 0$.

by a point $\tilde{r}$ satisfying $\alpha_v(\tilde{r}) = 0$. A similar situation happens when $\{0 < \alpha_v < \pi\}$ is disconnected by points belonging to the set $\{\alpha_v = \pi\}$, see Figure 1.3.

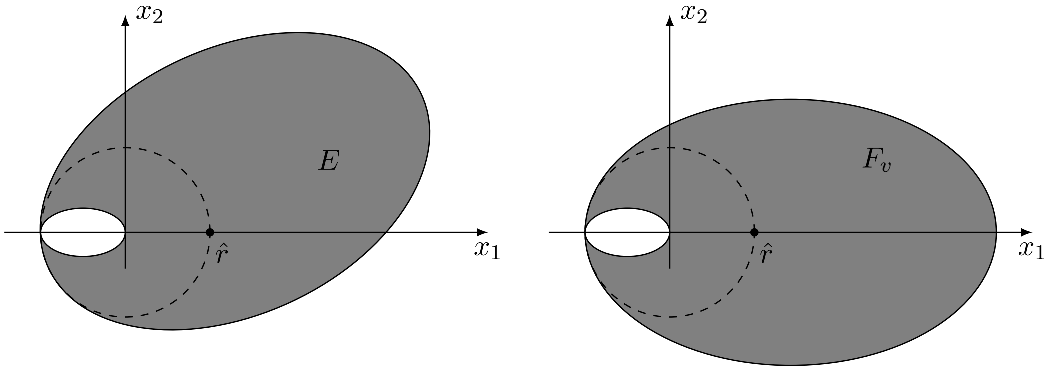

FIGURE 1.3. The set E above cannot be obtained by applying an orthogonal transformation around the origin to the set F_v shown in the right, therefore rigidity ($\mathcal{R}$) fails. This happens because the set $\{0 < \alpha_v < \pi\}$ is disconnected by a point $\hat{r} \in (0,\infty)$ such that $\alpha_v(\hat{r}) = \pi$.

One possibility to avoid such a situation could be to request the set $\{0 < \alpha_v < \pi\}$ to be an interval. However, this condition depends on the representative chosen for α_v, while the perimeters of the sets E and F_v don't. Indeed, in Figure 1.2 one could modify α_v just at the point $\tilde{r}$, in such a way that $\{0 < \alpha_v < \pi\}$ becomes an interval. Nevertheless, rigidity still fails, see Figure 1.4.

To formulate a condition which is independent on the chosen representative, we consider the approximate liminf and the approximate limsup of α_v, which we denote by $\alpha_v^\wedge$ and $\alpha_v^\vee$, respectively (see Section 2). These two functions are defined *at every point* $r \in (0,\infty)$ and satisfy $\alpha_v^\wedge \le \alpha_v^\vee$. In addition, they do not depend on the representative chosen for α_v, and $\alpha_v^\wedge = \alpha_v^\vee = \alpha_v$ $\mathcal{H}^1$-a.e. in $(0,\infty)$. The condition that we will impose is then the following:

$$\{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\} \text{ is a (possibly unbounded) interval.} \tag{1.6}$$

One can check that, in the example given in Figure 1.4 this condition fails, since $\alpha_v^\wedge(\tilde r) = \alpha_v^\vee(\tilde r) = 0$.

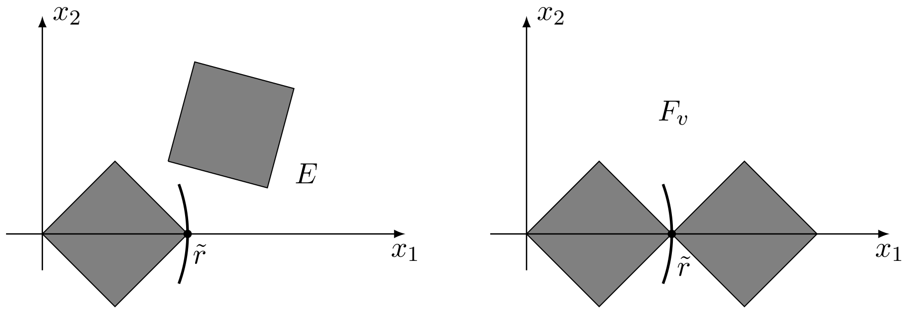

FIGURE 1.4. Modifying the function α_v given in Figure 1.2 at the point $\tilde r$, we can make sure that $\{0 < \alpha_v < \pi\}$ is an open connected interval. However, rigitidy still fails.

Let us show that, even imposing (1.6), rigidity can still be violated. In the example given in Figure 1.5, there is some radius $\overline{r} \in \{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\}$ such that the boundary of F_v contains a non trivial subset of $\partial B(\overline{r})$. In this way, it is possible to rotate a proper subset of F_v around the origin, without affecting the perimeter. Note that at each point of the set $\partial^* F_v \cap \partial B(\overline{r})$ the exterior normal ν^{F_v} is parallel to the radial direction. To rule out the situation described in Figure 1.5, we will impose the following condition:

$$\mathcal{H}^{n-1}(\{x \in \partial^* F_v : \nu_\parallel^{F_v}(x) = 0 \text{ and } |x| \in \{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\}) = 0. \tag{1.7}$$

Note that, from Theorem 1.1 and identity (1.3), it follows that in general we only have $\alpha_v \in BV_{\text{loc}}(0,\infty)$. However, it turns out that (1.7) is equivalent to ask that α_v is $W^{1,1}_{\text{loc}}$ in the interior of $\{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\}$, see Proposition 5.3.

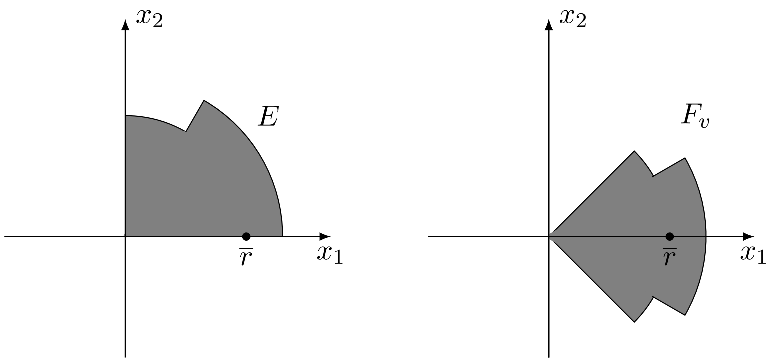

FIGURE 1.5. An example in which rigidity fails. In this case, the tangential part of $\partial^* F_v$ gives a non trivial contribution to $P(F_v)$. This allows to slide a proper subset of F_v around the origin, without modifying the perimeter.

Our main result shows that the two conditions above give a complete characterisation of rigidity for inequality (1.4) (below, $\mathring{\mathcal{I}}$ stands for the interior of the set $\mathcal{I}$).

Theorem 1.2. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume, and let α_v be defined by* (1.3)*. Then, the following two statements are equivalent:*

(i) $(\mathcal{R})$ *holds true;*

(ii) $\{0 < \alpha_v^\wedge \leq \alpha_v^\vee < \pi\}$ *is a (possibly unbounded) interval $\mathcal{I}$, and $\alpha_v \in W^{1,1}_{\mathrm{loc}}(\mathring{\mathcal{I}})$.*

Let us point out that, although similar results in the context of Steiner and Ehrhard's inequalities already appeared in [9, 10], the proof of Theorem 1.2 cannot simply use previous ideas, especially in the implication (i) $\Longrightarrow$ (ii). We cannot rely, as in [9], on a general formula for the perimeter of sets E satisfying equality in (1.4). Instead, we exhibit explicit counterexamples to rigidity, whenever one of the assumptions in (ii) fails. This requires a careful analysis of the transformations that one can apply to the set F_v, without modifying its perimeter. This turns out to be non trivial, especially if one assumes α_v to have a non zero Cantor part (see Proposition 8.4).

Also the proof of the implication (ii) $\Longrightarrow$ (i) presents some difficulties. In the context of Steiner symmetrisation, this has been proved in [12, Theorem 1.3] and [3, Theorem 1.2], for codimension 1 and for every codimension, respectively. In the smooth case, a proof is given in [24, Proposition 5], for the general class of symmetrisations in warped products. For the spherical setting without any smoothness assumption, this implication has already been stated in [25, Theorem 6.2], but the proof is only sketched. A rigorous proof of this fact turns out to be more delicate than one would expect, and relies on the following result.

Lemma 1.3. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume. Let $E \subset \mathbb{R}^n$ be a spherically v-distributed set, and let $I \subset (0,+\infty)$ be a Borel set. Assume that*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{E}(x) = 0\right\}\right) = 0. \tag{1.8}$$

Then,

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{F_v}(x) = 0\right\}\right) = 0. \tag{1.9}$$

Viceversa, let (1.9) *be satisfied, and suppose that $P(E; \Phi(I\times\mathbb{S}^{n-1})) = P(F_v; \Phi(I\times\mathbb{S}^{n-1}))$. Then,* (1.8) *holds true.*

A direct proof of Lemma 1.3 does not seem to be obvious, due to the fact that, as pointed out above, the measure λ_E defined in (1.5) can have an absolutely continuous part when $n > 2$. In the context of Steiner symmetrisation of higher codimension, a result playing the role of Lemma 1.3 (see [3, Proposition 3.6]) is proved using the fact that the statement holds true in codimension 1, see [12, Proposition 4.2]. For this reason, we are led to consider the *circular symmetrisation*, which is the codimension 1 version of the spherical symmetrisation, and was originally introduced by Pólya in the case $n = 3$ (see [27]). Note that, when $n = 2$, spherical and circular symmetrisation coincide.

1.5. **Circular Symmetrisation.** In order to introduce the circular symmetrisation, let us first observe how the spherical symmetrisation operates on a given set E, in the special case $n = 2$. In this situation, for each $r > 0$ one intersects E with the circle $\partial B(r)$ of radius r centred at the origin. Then, the symmetric set F_v is obtained by centring, for each $r > 0$, an open circumference arc of length $\mathcal{H}^1(E \cap \partial B(r))$ at the point re_1. When $n > 2$ one can proceed in a similar way, by first slicing the set E with parallel planes, and then by symmetrising it (in each plane) with the procedure just described. Note that, in this case, one needs to specify both the direction along which the open arcs are centred, and the direction along which the slicing through planes is performed.

Let us then choose an ordered pair of orthogonal directions in $\mathbb{R}^n$, which we will assume to be (e_1, e_2) (we will be centring open circumference arcs along e_1, while we will be slicing the set E with parallel planes that are orthogonal to e_2). In the following, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will write $x = (x_{12}, x')$, where $x_{12} = (x_1, x_2) \in \mathbb{R}^2$ and $x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$. When $x_{12} \neq 0$, we set $\hat{x}_{12} := x_{12}/|x_{12}|$. For each given $z' \in \mathbb{R}^{n-2}$, we denote by $\Pi_{z'}$ the two-dimensional plane defined by

$$\Pi_{z'} := \{x = (x_{12}, x') \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : x' = z'\}.$$

Given a set $E \subset \mathbb{R}^n$ and $(r, z') \in (0, \infty) \times \mathbb{R}^{n-2}$, we define the *circular slice* $E_{(r,z')}$ of E with respect to $\partial B((0, z'), r) \cap \Pi_{z'}$ as

$$E_{(r,z')} := E \cap \partial B((0, z'), r) \cap \Pi_{z'} = \{x = (x_{12}, x') \in E : x' = z' \text{ and } |x_{12}| = r\}.$$

Let $\ell : (0, \infty) \times \mathbb{R}^{n-2} \to [0, \infty)$ be a measurable function. We say that E is *circularly ℓ-distributed* if

$$\ell(r, x') = \mathcal{H}^1(E_{(r,x')}), \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (r, x') \in (0, \infty) \times \mathbb{R}^{n-2}.$$

If ℓ is a circular distribution, then we have

$$\ell(r, x') \leq \mathcal{H}^1(\partial B((0, x'), r) \cap \Pi_{x'}) = 2\pi r \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (r, x') \in (0, \infty) \times \mathbb{R}^{n-2}. \tag{1.10}$$

Among all the sets in $\mathbb{R}^n$ that are circularly ℓ-distributed, we denote by F^ℓ the one whose circular slices are open circumference arcs centred at the positive e_1 axis. That is, we set

$$F^\ell := \left\{(x_{12}, x') \in \mathbb{R}^n \setminus \{x_{12} = 0\} : \operatorname{dist}_{\mathbb{S}^1}(\hat{x}_{12}, e_1) < \frac{1}{2r}\ell(r, x')\right\}.$$

In the following, we introduce the diffeomorphism $\Phi_{12} : (0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{S}^1 \to \mathbb{R}^n \setminus \{\hat{x}_{12} = 0\}$ given by

$$\Phi_{12}(r, x', \omega) := (r\omega, x') \qquad \text{for every } (r, x', \omega) \in (0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{S}^1.$$

Moreover, for every $x \in \partial^* E$ we write $\nu^E(x) = (\nu^E_{12}(x), \nu^E_{x'}(x))$, where $\nu^E_{12}(x) = (\nu^E_1(x), \nu^E_2(x))$ and $\nu^E_{x'}(x) = (\nu^E_3(x), \ldots, \nu^E_n(x))$. Then, we further decompose $\nu^E_{12}(x)$ as

$$\nu^E_{12}(x) = \nu^E_{12\perp}(x) + \nu^E_{12\parallel}(x),$$

where $\nu^E_{12\perp}(x) := (\nu^E(x) \cdot \hat{x}_{12})\hat{x}_{12}$ and $\nu^E_{12\parallel}(x) := \nu^E_{12}(x) - \nu^E_{12\perp}(x)$. We can now state a result that plays the role of Theorem 1.1 for the circular symmetrisation.

Theorem 1.4. *Let $\ell : (0, \infty) \times \mathbb{R}^{n-2} \to [0, \infty)$ be a measurable function satisfying* (1.10)*, and let $E \subset \mathbb{R}^n$ be a circularly ℓ-distributed set of finite perimeter and finite volume. Then, $\ell \in BV_{\text{loc}}((0, \infty) \times \mathbb{R}^{n-2})$. Moreover, F^ℓ is a set of finite perimeter and*

$$P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1)) \leq P(E; \Phi_{12}(B \times \mathbb{S}^1)), \tag{1.11}$$

for every Borel set $B \subset (0, \infty) \times \mathbb{R}^{n-2}$.

Finally, if $P(E) = P(F^\ell)$, then for $\mathcal{H}^{n-1}$-a.e. $(r, x') \in (0, \infty) \times \mathbb{R}^{n-2}$:

(a) *$E_{(r,x')}$ is $\mathcal{H}^1$-equivalent to a circular arc and $\partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$;*

(b) *the three functions*

$$x \longmapsto \nu^E(x) \cdot \hat{x}_{12}, \qquad x \longmapsto |\nu^E_{12\parallel}|(x), \qquad x \longmapsto \nu^E_{x'}(x),$$

are constant in $(\partial^* E)_{(r,x')}$.

In the smooth setting and in the case $n = 3$, inequality (1.11) was proved by Pólya. The following result is the counterpart of Lemma 1.3 in the context of circular symmetrisation.

Lemma 1.5. *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10) *such that F^ℓ is a set of finite perimeter and finite volume. Let $E \subset \mathbb{R}^n$ be a circularly ℓ-distributed set, and let $I \subset (0,\infty) \times \mathbb{R}^{n-2}$ be a Borel set. Assume that*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \cap \Phi(I \times \mathbb{S}^1) : \nu^E_{12\|}(x) = 0\right\}\right) = 0. \tag{1.12}$$

Then,

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F^\ell \cap \Phi(I \times \mathbb{S}^1) : \nu^{F^\ell}_{12\|}(x) = 0\right\}\right) = 0. \tag{1.13}$$

Viceversa, let (1.13) *be satisfied, and suppose that $P(E; \Phi(I \times \mathbb{S}^1)) = P(F^\ell; \Phi(I \times \mathbb{S}^1))$. Then,* (1.12) *holds true.*

Once Lemma 1.5 is established, we can show Lemma 1.3 through a slicing argument. Finally, the proof of (ii) $\Longrightarrow$ (i) is concluded by showing that, if E satisfies equality in (1.4), the function associating to every $r \in (0,\infty)$ the center of E_r (see (7.1)) is $W^{1,1}_{\mathrm{loc}}$ and, ultimately, constant (see Section 7).

The paper is divided as follows. Section 2 contains basic results of Geometric Measure Theory that are extensively used in the following. In Section 3 we give the setting of the problem and introduce useful tools to deal with the spherical framework. Section 4 is devoted to the study of the properties of the functions v and ξ_v, while Theorem 1.1 is proven in Section 5. Important properties of the circular symmetrisation are discussed in Section 6, where we also give the proof of Lemma 1.3. The implications (ii) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (ii) of Theorem 1.2 are proven in Section 7 and Section 8, respectively.

2. Basic notions of Geometric Measure Theory

In this section we introduce some tools from Geometric Measure Theory. The interested reader can find more details in the monographs [2, 18, 22, 30]. For $n \in \mathbb{N}$, we denote with $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, i.e.

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\},$$

where $|\cdot|$ stands for the Euclidean norm, and we set $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$. For every $x \in \mathbb{R}^n_0$, we write $\hat{x} := x/|x|$ for the radial versor of x. We denote by $e_1, \dots, e_n$ the canonical basis in $\mathbb{R}^n$, and for every $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the standard scalar product in $\mathbb{R}^n$ between x and y. For every $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x,r)$ the open ball of $\mathbb{R}^n$ with radius r centred at x. In the special case $x = 0$, we set $B(r) := B(0,r)$. In the following, we will often make use of the diffeomorphism $\Phi : (0,\infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n_0$ defined as

$$\Phi(r,\omega) := r\omega \qquad \text{for every } (r,\omega) \in (0,\infty) \times \mathbb{S}^{n-1}.$$

For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, we will denote by $H^+_{x,\nu}$ and $H^-_{x,\nu}$ the closed half-spaces whose boundaries are orthogonal to ν:

$$\begin{aligned} H^+_{x,\nu} &:= \left\{y \in \mathbb{R}^n : (y - x) \cdot \nu \geq 0\right\}, \\ H^-_{x,\nu} &:= \left\{y \in \mathbb{R}^n : (y - x) \cdot \nu \leq 0\right\}. \end{aligned} \tag{2.1}$$

If $1 \leq k \leq n$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ with finite volume, and $E \subset \mathbb{R}^n$ is also measurable with finite volume, we say that $\{E_h\}_{h\in\mathbb{N}}$ converges to E as $h \to \infty$, and write $E_h \to E$, if $\mathcal{H}^n(E_h \Delta E) \to 0$ as $h \to \infty$. In the following, we will denote by χ_E the characteristic function of a measurable set $E \subset \mathbb{R}^n$.

2.1. **Density points.** Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $x \in \mathbb{R}^n$. The upper and lower n-dimensional densities of E at x are defined as

$$\theta^*(E,x) := \limsup_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,, \qquad \theta_*(E,x) := \liminf_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,,$$

respectively. It turns out that $x \mapsto \theta^*(E,x)$ and $x \mapsto \theta_*(E,x)$ are Borel functions that agree $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$. Therefore, the n-dimensional density of E at x

$$\theta(E,x) := \lim_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,,$$

is defined for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^n$, and $x \mapsto \theta(E,x)$ is a Borel function on $\mathbb{R}^n$. Given $t \in [0,1]$, we set

$$E^{(t)} := \{x \in \mathbb{R}^n : \theta(E,x) = t\}.$$

By the Lebesgue differentiation theorem, the pair $\{E^{(0)}, E^{(1)}\}$ is a partition of $\mathbb{R}^n$, up to a $\mathcal{H}^n$-negligible set. The set $\partial^{\mathrm{e}} E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$ is called the *essential boundary* of E.

2.2. **Rectifiable sets.** Let $1 \le k \le n$, $k \in \mathbb{N}$. If $A, B \subset \mathbb{R}^n$ are Borel sets we say that $A \subset_{\mathcal{H}^k} B$ if $\mathcal{H}^k(B \backslash A) = 0$, and $A =_{\mathcal{H}^k} B$ if $\mathcal{H}^k(A \Delta B) = 0$, where Δ denotes the symmetric difference of sets. Let $M \subset \mathbb{R}^n$ be a Borel set. We say that M is *countably $\mathcal{H}^k$-rectifiable* if there exist Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$ ($h \in \mathbb{N}$) such that $M \subset_{\mathcal{H}^k} \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)$. Moreover, we say that M is *locally $\mathcal{H}^k$-rectifiable* if $\mathcal{H}^k(M \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^n$, or, equivalently, if $\mathcal{H}^k \llcorner M$ is a Radon measure on $\mathbb{R}^n$.

A Lebesgue measurable set $E \subset \mathbb{R}^n$ is said of *locally finite perimeter* in $\mathbb{R}^n$ if there exists a $\mathbb{R}^n$-valued Radon measure μ_E, called the *Gauss–Green measure* of E, such that

$$\int_E \nabla\varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, d\mu_E(x)\,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n)\,,$$

where $C^1_c(\mathbb{R}^n)$ denotes the class of C^1 functions in $\mathbb{R}^n$ with compact support. The relative perimeter of E in $A \subset \mathbb{R}^n$ is then defined by setting $P(E;A) := |\mu_E|(A)$ for any Borel set $A \subset \mathbb{R}^n$. The perimeter of E is then defined as $P(E) := P(E;\mathbb{R}^n)$. If $P(E) < \infty$, we say that E is a set of *finite perimeter* in $\mathbb{R}^n$. The *reduced boundary* of E is the set $\partial^* E$ of those $x \in \mathbb{R}^n$ such that

$$\nu^E(x) = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \qquad \text{exists and belongs to } \mathbb{S}^{n-1}\,.$$

The Borel function $\nu^E : \partial^* E \to \mathbb{S}^{n-1}$ is called the *measure-theoretic outer unit normal* to E. If E is a set of locally finite perimeter, it is possible to show that $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ [22, Corollary 16.1], with $\mu_E = \nu^E\, \mathcal{H}^{n-1} \llcorner \partial^* E$, and

$$\int_E \nabla\varphi(x)\, dx = \int_{\partial^* E} \varphi(x)\, \nu^E(x)\, d\mathcal{H}^{n-1}(x)\,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n)\,.$$

Thus, $P(E;A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$ for every Borel set $A \subset \mathbb{R}^n$. If E is a set of locally finite perimeter, it turns out that

$$\partial^* E \subset\subset E^{(1/2)} \subset \partial^{\mathrm{e}} E\,.$$

Moreover, *Federer's theorem* holds true (see [2, Theorem 3.61] and [22, Theorem 16.2]):

$$\mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \setminus \partial^* E) = 0\,,$$

thus implying that the essential boundary $\partial^{\mathrm{e}} E$ of E is locally $\mathcal{H}^{n-1}$-rectifiable in $\mathbb{R}^n$.

2.3. **General facts about measurable functions.** Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Lebesgue measurable function. We define the *approximate upper limit* $f^\vee(x)$ and the *approximate lower limit* $f^\wedge(x)$ of f at $x\in\mathbb{R}^n$ as

$$f^\vee(x)=\inf\Big\{t\in\mathbb{R}:x\in\{f>t\}^{(0)}\Big\}\,, \tag{2.2}$$

$$f^\wedge(x)=\sup\Big\{t\in\mathbb{R}:x\in\{f<t\}^{(0)}\Big\}\,. \tag{2.3}$$

We observe that $f^\vee$ and $f^\wedge$ are Borel functions that are defined at *every* point of $\mathbb{R}^n$, with values in $\mathbb{R}\cup\{\pm\infty\}$. Moreover, if $f_1:\mathbb{R}^n\to\mathbb{R}$ and $f_2:\mathbb{R}^n\to\mathbb{R}$ are measurable functions satisfying $f_1=f_2$ $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$, then $f_1^\vee=f_2^\vee$ and $f_1^\wedge=f_2^\wedge$ *everywhere* on $\mathbb{R}^n$. We define the *approximate discontinuity* set S_f of f as

$$S_f:=\{f^\wedge<f^\vee\}.$$

Note that, by the above considerations, it follows that $\mathcal{H}^n(S_f)=0$. Although $f^\wedge$ and $f^\vee$ may take infinite values on S_f, the difference $f^\vee(x)-f^\wedge(x)$ is well defined in $\mathbb{R}\cup\{\pm\infty\}$ for every $x\in S_f$. Then, we can define the *approximate jump* $[f]$ of f as the Borel function $[f]:\mathbb{R}^n\to[0,\infty]$ given by

$$[f](x):=\begin{cases} f^\vee(x)-f^\wedge(x)\,, & \text{if } x\in S_f\,,\\ 0\,, & \text{if } x\in\mathbb{R}^n\setminus S_f\,.\end{cases}$$

Let $A\subset\mathbb{R}^n$ be a Lebesgue measurable set. We say that $t\in\mathbb{R}\cup\{\pm\infty\}$ is the approximate limit of f at x with respect to A, and write $t=\operatorname{ap\,lim}(f,A,x)$, if

$$\begin{aligned} &\theta\Big(\{|f-t|>\varepsilon\}\cap A;x\Big)=0\,, && \forall\varepsilon>0\,, && (t\in\mathbb{R})\,,\\ &\theta\Big(\{f<M\}\cap A;x\Big)=0\,, && \forall M>0\,, && (t=+\infty)\,,\\ &\theta\Big(\{f>-M\}\cap A;x\Big)=0\,, && \forall M>0\,, && (t=-\infty)\,.\end{aligned}$$

We say that $x\in S_f$ is a *jump point* of f if there exists $\nu\in\mathbb{S}^{n-1}$ such that

$$f^\vee(x)=\operatorname{ap\,lim}(f,H^+_{x,\nu},x)\,,\qquad f^\wedge(x)=\operatorname{ap\,lim}(f,H^-_{x,\nu},x)\,.$$

If this is the case, we say that $\nu_f(x):=\nu$ is the approximate jump direction of f at x. If we denote by J_f the set of approximate jump points of f, we have that $J_f\subset S_f$ and $\nu_f:J_f\to\mathbb{S}^{n-1}$ is a Borel function.

2.4. **Functions of bounded variation.** Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Lebesgue measurable function, and let $\Omega\subset\mathbb{R}^n$ be open. We define the *total variation of f in Ω* as

$$|Df|(\Omega)=\sup\Big\{\int_\Omega f(x)\operatorname{div}T(x)\,dx:T\in C^1_c(\Omega;\mathbb{R}^n)\,,|T|\le1\Big\}\,,$$

where $C^1_c(\Omega;\mathbb{R}^n)$ is the set of C^1 functions from Ω to $\mathbb{R}^n$ with compact support. We also denote by $C_c(\Omega;\mathbb{R}^n)$ the class of all continuous functions from Ω to $\mathbb{R}^n$. Analogously, for any $k\in\mathbb{N}$, the class of k times continuously differentiable functions from Ω to $\mathbb{R}^n$ is denoted by $C^k_c(\Omega;\mathbb{R}^n)$. We say that f belongs to the space of functions of bounded variations, $f\in BV(\Omega)$, if $|Df|(\Omega)<\infty$ and $f\in L^1(\Omega)$. Moreover, we say that $f\in BV_{\rm loc}(\Omega)$ if $f\in BV(\Omega')$ for every open set Ω' compactly contained in Ω. Therefore, if $f\in BV_{\rm loc}(\mathbb{R}^n)$ the distributional derivative Df of f is an $\mathbb{R}^n$-valued Radon measure. In particular, E is a set of locally finite perimeter if and only if $\chi_E\in BV_{\rm loc}(\mathbb{R}^n)$. If $f\in BV_{loc}(\mathbb{R}^n)$, one can write the Radon–Nykodim decomposition of Df with respect to $\mathcal{H}^n$ as $Df=D^af+D^sf$, where D^sf and $\mathcal{H}^n$ are mutually singular, and where $D^af\ll\mathcal{H}^n$. We denote the density of D^af with respect to $\mathcal{H}^n$ by ∇f, so that $\nabla f\in L^1(\Omega;\mathbb{R}^n)$ with $D^af=\nabla f\,d\mathcal{H}^n$. Moreover, for $\mathcal{H}^n$-a.e. $x\in\mathbb{R}^n$, $\nabla f(x)$ is the approximate differential of f at x. If $f\in BV_{\rm loc}(\mathbb{R}^n)$, then S_f

is countably $\mathcal{H}^{n-1}$-rectifiable. Moreover, we have $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$, $[f] \in L^1_{loc}(\mathcal{H}^{n-1} \llcorner J_f)$, and the $\mathbb{R}^n$-valued Radon measure $D^j f$ defined as

$$D^j f = [f] \, \nu_f \, d\mathcal{H}^{n-1} \llcorner J_f \,,$$

is called the *jump part of* Df. If we set $D^c f = D^s f - D^j f$, we have that $Df = D^a f + D^j f + D^c f$. The $\mathbb{R}^n$-valued Radon measure $D^c f$ is called the *Cantorian part* of Df, and it is such that $|D^c f|(M) = 0$ for every $M \subset \mathbb{R}^n$ which is σ-finite with respect to $\mathcal{H}^{n-1}$.

In the special case $n = 1$, if $(a, b) \subset \mathbb{R}$ is an open (possibly unbounded) interval, *every* $f \in BV((a, b))$ can be written as

$$f = f^a + f^j + f^c, \tag{2.4}$$

where $f \in W^{1,1}((a, b))$, f^j is a jump function (i.e. $Df = D^j f$) and f^c is a Cantor function (i.e. $Df = D^c f$), see [2, Corollary 3.33]. Moreover, if $f^j = 0$ (or, more in general, if f is a *good representative*, see [2, Theorem 3.28]), the total variation of Df can be obtained as

$$|Df|(a, b) = \sup \left\{ \sum_{i=1}^{N} |f(x_{i+1}) - f(x_i)| : a < x_1 < x_2 < \ldots < x_N < b \right\}, \tag{2.5}$$

where the supremum runs over all $N \in \mathbb{N}$, and over all the possible partitions of (a, b) with $a < x_1 < x_2 < \ldots < x_N < b$. When $n = 1$, we will often write f' instead of ∇f.

3. Setting of the problem and preliminary results

In this section we give the notation for the chapter, and we introduce some results that will be extensively used later. For every $x, y \in \mathbb{S}^{n-1}$, the *geodesic distance* between x and y is given by

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) := \arccos(x \cdot y).$$

We recall that the geodesic distance satisfies the triangle inequality:

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(x, z) + \mathrm{dist}_{\mathbb{S}^{n-1}}(z, y) \qquad \text{for every } x, y, z \in \mathbb{S}^{n-1}.$$

Let $r > 0$, $p \in \mathbb{S}^{n-1}$ and $\beta \in [0, \pi]$ be fixed. The *open geodesic ball* (or *spherical cap*) of centre rp and radius β is the set

$$\mathbf{B}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) < \beta\}.$$

Note in the extreme cases $\beta = 0$ and $\beta = \pi$ we have $\mathbf{B}_0(rp) = \emptyset$ and $\mathbf{B}_\pi(rp) = \partial B(r) \setminus \{-rp\}$, respectively. Accordingly, the *geodesic sphere* of centre rp and radius β is the boundary of $\mathbf{B}_\beta(rp)$, which is given by

$$\mathbf{S}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) = \beta\}.$$

The $(n-1)$-dimensional Hausdorff measure of a geodesic ball and the $(n-2)$-dimensional Hausdorff measure of a geodesic sphere are given by

$$\mathcal{H}^{n-1}(\mathbf{B}_\beta(rp)) = (n-1)\omega_{n-1} r^{n-1} \int_0^\beta (\sin \tau)^{n-2} \, d\tau, \tag{3.1}$$

$$\mathcal{H}^{n-2}(\mathbf{S}_\beta(rp)) = (n-1)\omega_{n-1} r^{n-2} (\sin \beta)^{n-2}. \tag{3.2}$$

Let $E \subset \mathbb{R}^n$ be a measurable set. For every $r > 0$, we define the *spherical slice of radius* r *of* E as the set

$$E_r := E \cap \partial B(r) = \{x \in \partial B(r) : x \in E\}.$$

Let $v : (0, \infty) \to [0, \infty)$ be a Lebesgue measurable function, and let $E \subset \mathbb{R}^n$ be a measurable set in $\mathbb{R}^n$. We say that E is spherically v-distributed if

$$v(r) = \mathcal{H}^{n-1}(E_r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty).$$

If E is spherically v-distributed, we can define the function

$$\xi_v(r) := \frac{v(r)}{r^{n-1}} = \frac{\mathcal{H}^{n-1}(E_r)}{r^{n-1}}, \qquad \text{for every } r \in (0,\infty). \tag{3.3}$$

Note that $\mathcal{H}^{n-1}(\mathbf{B}_\pi) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$, so that

$$0 \le \xi_v(r) \le n\omega_n, \qquad \text{for every } r \in (0,\infty). \tag{3.4}$$

From (3.1), it follows that the function $\mathcal{F} : [0,\pi] \to [0, n\omega_n]$ given by

$$\mathcal{F}(\beta) := \mathcal{H}^{n-1}(\mathbf{B}_\beta(e_1)) \text{ is strictly increasing and smoothly invertible in } (0, n\omega_n). \tag{3.5}$$

Therefore, if $v : (0,\infty) \to [0,\infty)$ is measurable, thanks to (3.4), there exists a unique function $\alpha_v : (0,\infty) \to [0,\pi]$ such that

$$\xi_v(r) = \mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v(r)}(e_1)) \qquad \text{for every } r \in (0,\infty). \tag{3.6}$$

Among all the spherically v-distributed sets of $\mathbb{R}^n$, we denote by F_v the one whose spherical slices are open geodesic balls centred at the positive e_1 axis., i.e.

$$F_v := \{x \in \mathbb{R}^n_0 : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) < \alpha_v(|x|)\}, \tag{3.7}$$

where α_v is defined by (3.3) and (3.6). The next result (see [2, Lemma 2.35]) will be used in the proof of Theorem 1.1.

Lemma 3.1. *Let $B \subset \mathbb{R}^n$ be a Borel set and let $\varphi_h, \varphi : B \to \mathbb{R}$, $h \in \mathbb{N}$ be summable Borel functions such that $|\varphi_h| \le |\varphi|$ for every h. Then*

$$\int_B \sup_h \varphi_h dx = \sup_H \left\{ \sum_{h\in H} \int_{A_h} \varphi_h dx \right\},$$

where the supremum ranges over all finite sets $H \subset \mathbb{N}$ and all finite partitions A_h, $h \in H$ of B in Borel sets.

3.1. **Normal and tangential components of functions and measures.** For every $\varphi \in C_c(\mathbb{R}^n_0;\mathbb{R}^n)$, we decompose φ as $\varphi = \varphi_\perp + \varphi_\parallel$, where

$$\varphi_\perp(x) := (\varphi(x)\cdot\hat{x})\,\hat{x} \qquad \text{and} \qquad \varphi_\parallel(x) := \varphi(x) - \varphi_\perp(x)$$

are the radial and tangential components of φ, respectively. If $\varphi \in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$, $\mathrm{div}_\parallel\varphi(x)$ stands for the tangential divergence of φ at x along the sphere $\partial B(|x|)$:

$$\mathrm{div}_\parallel\varphi(x) := \mathrm{div}\varphi(x) - (\nabla\varphi(x)\hat{x})\cdot\hat{x}. \tag{3.8}$$

The following lemma gives some useful identities that will be needed later.

Lemma 3.2. *Let $\varphi \in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n_0$ one has*

$$\mathrm{div}\varphi_\perp(x) = (\nabla\varphi(x)\hat{x})\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}, \tag{3.9}$$

$$\mathrm{div}\varphi_\parallel(x) = \mathrm{div}_\parallel\varphi_\parallel(x). \tag{3.10}$$

Remark 3.3. *Let $\varphi \in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$. Recalling that $\varphi = \varphi_\perp + \varphi_\parallel$, combining* (3.9) *and* (3.10) *it follows that*

$$\mathrm{div}\varphi(x) = (\nabla\varphi(x)\hat{x})\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|} + \mathrm{div}_\parallel\varphi_\parallel(x) \qquad \forall\, x \in \mathbb{R}^n_0.$$

Proof. First of all, note that

$$\nabla\left(\varphi(x)\cdot\hat{x}\right)=(\nabla\varphi(x))^T\hat{x}+\frac{1}{|x|}\varphi_{\|}(x). \tag{3.11}$$

Indeed,

$$\nabla\left(\varphi(x)\cdot\hat{x}\right)=(\nabla\varphi(x))^T\hat{x}+\frac{I-\hat{x}\otimes\hat{x}}{|x|}\varphi(x)=(\nabla\varphi(x))^T\hat{x}+\frac{1}{|x|}\varphi_{\|}(x),$$

where I represents the identity map in $\mathbb{R}^n$, and $\hat{x}\otimes\hat{x}$ is the usual tensor product of $\hat{x}$ with itself (so that $I-\hat{x}\otimes\hat{x}$ is the orthogonal projection on the tangent plane to $\mathbb{S}^{n-1}$ at $\hat{x}$). Thanks to (3.11), we have

$$\begin{aligned}\mathrm{div}\varphi_{\perp}(x)&=\mathrm{div}\left((\varphi(x)\cdot\hat{x})\hat{x}\right)=\nabla\left(\varphi(x)\cdot\hat{x}\right)\cdot\hat{x}+(\varphi(x)\cdot\hat{x})\,\mathrm{div}\hat{x}\\&=\left[(\nabla\varphi(x))^T\hat{x}+\frac{1}{|x|}\varphi_{\|}(x)\right]\cdot\hat{x}+(\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}\\&=(\nabla\varphi(x)\hat{x})\cdot\hat{x}+(\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|},\end{aligned}$$

which proves (3.9). Note now that, by definition (3.8), it follows that

$$\mathrm{div}\varphi(x)=\mathrm{div}_{\|}\varphi(x)+(\nabla\varphi(x)\hat{x})\cdot\hat{x}. \tag{3.12}$$

On the other hand, from (3.9)

$$\begin{aligned}\mathrm{div}\varphi(x)&=\mathrm{div}\varphi_{\|}(x)+\mathrm{div}\varphi_{\perp}(x)\\&=\mathrm{div}\varphi_{\|}(x)+(\nabla\varphi(x)\hat{x})\cdot\hat{x}+(\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}.\end{aligned}$$

Comparing last identity with (3.12) we obtain that for every $\varphi\in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$

$$\mathrm{div}_{\|}\varphi(x)=\mathrm{div}\varphi_{\|}(x)+(\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}.$$

Applying the last identity to the function $\varphi_{\|}$ we obtain (3.10). □

If μ is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n_0$, we will write $\mu=\mu_{\perp}+\mu_{\|}$, where $\mu_{\perp}$ and $\mu_{\|}$ are the $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^n_0$ such that

$$\int_{\mathbb{R}^n_0}\varphi\cdot d\mu_{\perp}=\int_{\mathbb{R}^n_0}\varphi_{\perp}\cdot d\mu,\qquad\text{and}\qquad\int_{\mathbb{R}^n_0}\varphi\cdot d\mu_{\|}=\int_{\mathbb{R}^n_0}\varphi_{\|}\cdot d\mu,$$

for every $\varphi\in C_c(\mathbb{R}^n_0;\mathbb{R}^n)$. Note that $\mu_{\perp}$ and $\mu_{\|}$ are well defined by Riesz Theorem (see, for instance, [2, Theorem 1.54]). In the special case $\mu=Df$, with $f\in BV_{\mathrm{loc}}(\mathbb{R}^n_0)$, we will shorten the notation writing $D_{\|}f$ and $D_{\perp}f$ in place of $(Df)_{\|}$ and $(Df)_{\perp}$, respectively. In particular, if $f=\chi_E$ and $E\subset\mathbb{R}^n$ is a set of finite perimeter, by De Giorgi structure theorem we have

$$D_{\perp}\chi_E=\nu^E_{\perp}d\mathcal{H}^{n-1}\llcorner\partial^*E\qquad\text{and}\qquad D_{\|}\chi_E=\nu^E_{\|}d\mathcal{H}^{n-1}\llcorner\partial^*E. \tag{3.13}$$

Next lemma gives some useful identities concerning the radial and tangential components of the gradient of a BV_{loc} function.

Lemma 3.4. *Let $f\in BV_{\mathrm{loc}}(\mathbb{R}^n_0)$. Then,*

$$\int_{\mathbb{R}^n_0}\varphi(x)\cdot dD_{\|}f=-\int_{\mathbb{R}^n_0}f(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)\,dx, \tag{3.14}$$

$$\int_{\mathbb{R}^n_0}\varphi(x)\cdot dD_{\perp}f=-\int_{\mathbb{R}^n_0}f(x)\,(\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\,dx-\int_{\mathbb{R}^n_0}f(x)\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\;dx, \tag{3.15}$$

for every $\varphi\in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$.

Proof. Let $\varphi \in C_c^1(\mathbb{R}_0^n;\mathbb{R}^n)$. By definition of $D_\| f$ and thanks to (3.10) we have

$$\begin{aligned}\int_{\mathbb{R}_0^n} \varphi(x)\cdot dD_\| f &= \int_{\mathbb{R}_0^n} \varphi_\|(x)\cdot dDf \\ &= -\int_{\mathbb{R}_0^n} \mathrm{div}\varphi_\|(x) f(x)\,dx = -\int_{\mathbb{R}_0^n} \mathrm{div}_\|\varphi_\|(x) f(x)\,dx,\end{aligned}$$

and this shows (3.14). Similarly, by definition of $D_\perp f$

$$\int_{\mathbb{R}_0^n} \varphi(x)\cdot dD_\perp f = \int_{\mathbb{R}_0^n} \varphi_\perp(x)\cdot dDf = -\int_{\mathbb{R}_0^n} \mathrm{div}\varphi_\perp(x) f(x)\,dx.$$

Thanks to (3.9), identity (3.15) follows. □

An immediate consequence of identity (3.14) is the following.

Corollary 3.5. *Let $f \in BV_{\mathrm{loc}}(\mathbb{R}_0^n)$ and let $\Omega \subset\subset \mathbb{R}_0^n$ be open and bounded. Then,*

$$\left|D_\| f\right|(\Omega) = \sup\left\{\int_{\mathbb{R}^n} f(x)\,\mathrm{div}_\|\varphi_\|(x)dx:\ \varphi \in C_c^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1\right\}.$$

We conclude this subsection with an important proposition, that is a special case of the Coarea Formula (see [2, Theorem 2.93]).

Proposition 3.6. *Let E be a set of finite perimeter in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to [0,\infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} g(x)|\nu_\|^E(x)|d\mathcal{H}^{n-1}(x) = \int_0^\infty dr \int_{(\partial^* E)_r} g(x)\,d\mathcal{H}^{n-2}(x).$$

Proof. The result follows by applying [2, Remark 2.94] with $N = n-1$, $M = n$, $k = 1$, and $f(x) = |x|$. □

In the next subsection we show how the notion of set of finite perimeter can be given in a natural way also for subsets of the sphere $\mathbb{S}^{n-1}$ (and, more in general, of $\partial B(r)$, for any $r > 0$).

3.2. **Sets of finite perimeter on $\mathbb{S}^{n-1}$.** We now give a very brief introduction to sets of finite perimeter on $\mathbb{S}^{n-1}$, by using the notion of integer multiplicity rectifiable currents, see [30, Chapter 6] for more details (see also [7]). Let $k \in \mathbb{N}$ with $1 \le k \le n-1$. We denote by $\Lambda_k(\mathbb{R}^n)$ and $\Lambda^k(\mathbb{R}^n)$ the linear spaces of k-vectors and k-covectors in $\mathbb{R}^n$, respectively, while $\mathcal{D}^k(\mathbb{R}^n)$ stands for the set of smooth k-forms with compact support in $\mathbb{R}^n$.

A *k-dimensional current* in $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}^k(\mathbb{R}^n)$. The family of k-dimensional currents in $\mathbb{R}^n$ is denoted by $\mathcal{D}_k(\mathbb{R}^n)$. We say that $T \in \mathcal{D}_k(\mathbb{R}^n)$ is an *integer multiplicity rectifiable k-current* if it can be represented as

$$T(\omega) = \int_M \langle \omega(x), \eta(x)\rangle\,\theta(x)\,d\mathcal{H}^k(x) \quad \text{for every } \omega \in \mathcal{D}^k(\mathbb{R}^n),$$

where M is an $\mathcal{H}^k$-measurable countably k-rectifiable subset of $\mathbb{R}^n$, θ is an $\mathcal{H}^k$-measurable positive integer-valued function, and $\eta : M \to \Lambda_k(\mathbb{R}^n)$ is an $\mathcal{H}^k$-measurable function such that for $\mathcal{H}^k$-a.e. $x \in M$ one has $\eta(x) = \tau_1(x) \wedge \ldots \wedge \tau_k(x)$, with $\tau_1(x),\ldots,\tau_k(x)$ an orthonormal basis for the approximate tangent space of M at x, and $\langle\cdot,\cdot\rangle$ denotes the usual pairing between $\Lambda^k(\mathbb{R}^n)$ and $\Lambda_k(\mathbb{R}^n)$. In the special case when

$$T(\omega) = \int_M \langle \omega(x), \eta(x)\rangle\,d\mathcal{H}^k(x) \quad \text{for every } \omega \in \mathcal{D}^k(\mathbb{R}^n),$$

we write $T = [\![M]\!]$. The boundary ∂T of T is then defined as the element of $\mathcal{D}_{k-1}(\mathbb{R}^n)$ such that

$$\partial T(\omega) = T(d\omega) \quad \text{for every } \omega \in \mathcal{D}^k(\mathbb{R}^n),$$

while the mass $\mathbf{M}(T)$ of T is given by

$$\mathbf{M}(T) := \sup\Big\{T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n),\ |\omega| \le 1\Big\}.$$

More in general, for any open set $U \subset \mathbb{R}^n$, we set

$$\mathbf{M}_U(T) := \sup\Big\{T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n),\ |\omega| \le 1, \operatorname{supp}\omega \in U\Big\}.$$

Let $A \subset \mathbb{S}^{n-1}$ be an $\mathcal{H}^{n-1}$-measurable set. We will say that A is a set of finite perimeter on $\mathbb{S}^{n-1}$ if there exists $Q \in \mathcal{D}_{n-2}(\mathbb{R}^n)$ with $\operatorname{supp} Q \subset \mathbb{S}^{n-1}$ and

$$Q = \partial[\![A]\!],$$

with the property that $\mathbf{M}_U(Q) < \infty$ for every $U \subset\subset \mathbb{R}^n$. By the Riesz representation theorem it follows that there exists a Radon measure μ_Q and a μ_Q-measurable function $\nu : \mathbb{S}^{n-1} \to T_x\mathbb{S}^{n-1}$ such that $|\nu(x)| = 1$ for μ_T-a.e. x and

$$\int_A \operatorname{div}_{\|}\varphi(x)\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} \varphi(x)\cdot\nu(x)\, d\mu_Q(x),$$

for every smooth vector field with $\varphi = \varphi_{\|}$. If $A \subset \mathbb{S}^{n-1}$ is a set of finite perimeter on the sphere, the reduced boundary $\partial^* A$ is the set of points $x \in \mathbb{S}^{n-1}$ such that the limit

$$\nu^A(x) := \lim_{\rho\to 0} \frac{1}{\mu_Q(B(x,\rho))} \int_{B(x,\rho)} \nu(y)\, d\mu_Q(y)$$

exists, $\nu^A(x) \in T_x\mathbb{S}^{n-1}$, and $\nu^A(x) = 1$. The De Giorgi structure theorem holds true also for sets of finite perimeter on the sphere. In particular, $\partial^* A$ is countably $(n-2)$-rectifiable, $\mu_Q = \mathcal{H}^{n-2} \llcorner \partial^* A$, and

$$\int_A \operatorname{div}_{\|}\varphi(x)\, d\mathcal{H}^{n-1}(x) = \int_{\partial^* A} \varphi(x)\cdot\nu^A(x)\, d\mathcal{H}^{n-2}(x), \tag{3.16}$$

for every smooth vector field with $\varphi = \varphi_{\|}$. The isoperimetric inequality on the sphere states that, if $\beta \in (0,\pi)$ and $A \subset \mathbb{S}^{n-1}$ is a set of finite perimeter on $\mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(A) = \mathcal{H}^{n-1}(\mathbf{B}_\beta(e_1))$, then (see [28])

$$\mathcal{H}^{n-2}(\partial^*\mathbf{B}_\beta(e_1)) \le \mathcal{H}^{n-2}(\partial^* A). \tag{3.17}$$

The next theorem is a version of a result by Vol'pert (see [32]).

Theorem 3.7. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2)*, and let $E \subset \mathbb{R}^n$ be a spherically v-distributed set of finite perimeter and finite volume. Then, there exists a Borel set $G_E \subset \{\alpha_v > 0\}$ with $\mathcal{H}^1(\{\alpha_v > 0\} \setminus G_E) = 0$, such that*

- (i) *for every $r \in G_E$:*
  - (ia) *E_r is a set of finite perimeter in $\partial B(r)$;*
  - (ib) *$\mathcal{H}^{n-2}(\partial^*(E_r)\Delta(\partial^* E)_r) = 0$;*
- (ii) *for every $r \in G_E \cap \{0 < \alpha_v < \pi\}$:*
  - (iia) *$|\nu^E_{\|}(r\omega)| > 0$,*
  - (iib) *$\nu^E_{\|}(r\omega) = \nu^{E_r}(r\omega)|\nu^E_{\|}(r\omega)|$,*

  for $\mathcal{H}^{n-2}$-a.e. $\omega \in \mathbb{S}^{n-1}$ such that $r\omega \in \partial^*(E_r) \cap (\partial^* E)_r$.

Proof. The result follows applying [30, Theorem 28.5] with $f(x) = |x|$, and recalling the definition of slicing of a current (see [30, Definition 28.4]). □

We now make some important remarks about Theorem 3.7.

Remark 3.8. *Thanks to property (ib), we have*

$$\partial^*(E_r) =_{\mathcal{H}^{n-2}} (\partial^* E)_r \qquad \text{for every } r \in G_E.$$

Therefore, whenever $r \in G_E$ we will often write $\partial^* E_r$ instead of $\partial^*(E_r)$ or $(\partial^* E)_r$, without any risk of ambiguity. Moreover, for every $r \in G_E$ we will also use the notation

$$p_E(r) := \mathcal{H}^{n-2}(\partial^* E_r).$$

Remark 3.9. *In dimension $n = 2$, the theorem above implies that, if $r \in G_E \cap \{0 < \theta < \pi\}$, then $\partial^*(E_r) = (\partial^* E)_r$ and*

$$|\nu_{\parallel}^E(r\omega)| > 0 \quad \textbf{for every } \omega \in \mathbb{S}^1 \text{ such that } r\omega \in (\partial^* E)_r. \tag{3.18}$$

Let now λ_E be the measure defined in (1.5):

$$\lambda_E(B) = \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^1) \cap \{\nu_{\parallel}^E = 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^1(x) \quad \text{for every Borel set } B \subset (0, \infty).$$

If $B \subset G_E$, then by (3.18)

$$|\lambda_E(B)| \le \mathcal{H}^1(\partial^* E \cap \Phi(G_E \times \mathbb{S}^1) \cap \{\nu_{\parallel}^E = 0\}) = 0,$$

so that $\lambda_E(B) = 0$. As a consequence, λ_E is singular with respect to the Lebesgue measure in $(0, \infty)$. If $n > 2$ this conclusion is in general false (unless one chooses $E = F_v$, see Remark 3.10 below), and it may happen that λ_E has a non trivial absolutely continuous part.

Remark 3.10. *If $n \ge 2$, but we consider the special case $E = F_v$, Theorem 3.7 gives much more information than the one we can obtain for a generic set of finite perimeter. Indeed, let $R \in O(n)$ be any orthogonal transformation that keeps fixed the e_1 axis. By definition of F_v, and thanks to* [22, Exercise 15.10], *we have that if $x \in \partial^* F_v$, then $Rx \in \partial^* F_v$ and*

$$\nu_{\parallel}^{F_v}(Rx) = R\,\nu_{\parallel}^{F_v}(x) \qquad \text{and} \qquad \nu_{\perp}^{F_v}(Rx) = R\,\nu_{\perp}^{F_v}(x).$$

Therefore, applying Theorem 3.7 to F_v we infer that

- (j) *for every $r \in G_{F_v}$:*
  - (ja) *$(F_v)_r$ is a spherical cap;*
  - (jb) *$\partial^*(F_v)_r = (\partial^* F_v)_r$;*
- (jj) *for every $r \in G_{F_v} \cap \{0 < \alpha_v < \pi\}$:*
  - (jja) *$|\nu_{\parallel}^{F_v}(r\omega)| > 0$,*
  - (jjb) *$\nu_{\parallel}^{F_v}(r\omega) = \nu^{(F_v)_r}(r\omega)|\nu_{\parallel}^{F_v}(r\omega)|$,*

  for every *$\omega \in \mathbb{S}^{n-1}$ such that $r\omega \in (\partial^* F_v)_r \cap \partial^*(F_v)_r$.*

Therefore,

$$\mathcal{H}^1(B_0) = 0, \tag{3.19}$$

where

$$B_0 := \left\{ r \in (0, +\infty) : \exists\, \omega \in \mathbb{S}^{n-1} \text{ such that } r\omega \in \partial^* F_v \text{ and } \nu_{\parallel}^{F_v}(r\omega) = 0 \right\}.$$

Moreover, repeating the argument used in Remark 3.9 one obtains that

$$\mathcal{H}^{n-1}(\partial^* F_v \cap \Phi(G_{F_v} \times \mathbb{S}^{n-1}) \cap \{\nu_{\parallel}^{F_v} = 0\}) = 0.$$

Thus, the measure λ_{F_v} defined in (1.5) is purely singular with respect to the Lebesgue measure in $(0, \infty)$.

4. Properties of v and ξ_v

In this section we discuss several properties of the functions v and ξ_v. These are the natural counterpart in the spherical setting of analogous results proven in [12] and [3]. We start by showing that, if $E \subset \mathbb{R}^n$ is a set of finite perimeter and volume, then $v \in BV(0, \infty)$.

Lemma 4.1. *Let v be as in Theorem 1.1, and let $E \subset \mathbb{R}^n$ be a spherically v-distributed set of finite perimeter and finite volume. Then, $v \in BV(0, \infty)$. Moreover, $\xi_v \in BV_{\text{loc}}(0, \infty)$ and*

$$\int_0^\infty \psi(r) r^{n-1} dD\xi_v(r) = \int_{\mathbb{R}^n_0} \psi(|x|)\, \hat{x} \cdot dD_\perp \chi_E(x), \tag{4.1}$$

for every bounded Borel function $\psi : (0, \infty) \to \mathbb{R}$. As a consequence,

$$|r^{n-1} D\xi_v|(B) \leq |D_\perp \chi_E|(\Phi(B \times \mathbb{S}^{n-1})), \tag{4.2}$$

for every Borel set $B \subset (0, \infty)$. In particular, $r^{n-1} D\xi_v$ is a bounded Radon measure on $(0, \infty)$.

Proof. We divide the proof into steps.

Step 1: We show that $v \in BV(0, \infty)$. First of all, note that $v \in L^1(0, \infty)$, since

$$\|v\|_{L^1(0,\infty)} = \int_0^\infty v(r)\, dr = \int_0^\infty dr \int_{\partial B(r)} \chi_E(x)\, d\mathcal{H}^{n-1}(x) = \mathcal{H}^n(E) < \infty.$$

Let now $\psi \in C^1_c(0, \infty)$ with $|\psi| \leq 1$. Applying formula (3.9) to the radial function $\psi(|x|)\hat{x}$, we obtain that for every $x \in \mathbb{R}^n_0$

$$\begin{aligned}
&\operatorname{div}(\psi(|x|)\hat{x}) = [\nabla(\psi(|x|)\hat{x})\,\hat{x}] \cdot \hat{x} + [\psi(|x|)\hat{x} \cdot \hat{x}]\,\frac{n-1}{|x|} \\
&= \left[\left(\psi'(|x|)\hat{x} \otimes \hat{x} + \psi(|x|)\frac{I - \hat{x} \otimes \hat{x}}{|x|}\right)\hat{x}\right] \cdot \hat{x} + \psi(|x|)\frac{n-1}{|x|} \\
&= \psi'(|x|) + \psi(|x|)\frac{n-1}{|x|}.
\end{aligned} \tag{4.3}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left[\psi'(|x|) + \psi(|x|)\frac{n-1}{|x|}\right] \chi_E(x)\, dx = \int_{\mathbb{R}^n} \operatorname{div}(\psi(|x|)\,\hat{x})\, \chi_E(x)\, dx \\
&= -\int_{\mathbb{R}^n} \psi(|x|)\,\hat{x} \cdot dD\chi_E(x) = -\int_{\mathbb{R}^n} \psi(|x|)\,\hat{x} \cdot dD_\perp \chi_E(x),
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \psi'(|x|)\chi_E(x)\, dx \\
&= -\int_{\mathbb{R}^n} \psi(|x|)\frac{n-1}{|x|}\chi_E(x)\, dx - \int_{\mathbb{R}^n} \psi(|x|)\,\hat{x} \cdot dD_\perp \chi_E(x).
\end{aligned} \tag{4.4}$$

By Coarea formula, the integral in the left hand side can be written as

$$\int_{\mathbb{R}^n} \psi'(|x|)\chi_E(x)\, dx = \int_0^\infty dr\, \psi'(r) \int_{\partial B(r)} \chi_E(x)\, d\mathcal{H}^{n-1}(x) = \int_0^\infty \psi'(r) v(r)\, dr. \tag{4.5}$$

Combining (4.4) and (4.5) we find that

$$\begin{aligned}
&\int_0^\infty \psi(r)\,dDv(r) \\
&= \int_{\mathbb{R}^n} \psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx + \int_{\mathbb{R}^n} \psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E(x). \\
&\le \int_{B(1)} \psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx + \int_{\mathbb{R}^n\setminus B(1)} \psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx + P(E) \\
&\le n(n-1)\omega_n\int_0^1 \rho^{n-2}\,d\rho + (n-1)|E| + P(E) \\
&= n\omega_n + (n-1)|E| + P(E) < \infty.
\end{aligned} \tag{4.6}$$

Taking the supremum over ψ we obtain that

$$|Dv|(0,\infty) < \infty,$$

so that $v \in BV(0,\infty)$.

Step 2: We conclude the proof. Since the function $r \mapsto 1/(r^{n-1})$ is smooth and locally bounded in $(0,\infty)$, we also have that $\xi_v(r) \in BV_{\mathrm{loc}}(0,\infty)$. Moreover, recalling that $v(r) = r^{n-1}\xi_v(r)$, by the chain rule in BV (see [2, Example 3.97])

$$Dv = (n-1)r^{n-2}\xi_v(r)\,dr + r^{n-1}D\xi_v = (n-1)\frac{v(r)}{r}dr + r^{n-1}D\xi_v. \tag{4.7}$$

Let now $\psi \in C_c^1(0,\infty)$. From the previous identity it follows that

$$\begin{aligned}
\int_0^\infty \psi(r)\,dDv(r) &= \int_0^\infty \psi(r)\frac{n-1}{r}\,v(r)\,dr + \int_0^\infty \psi(r)r^{n-1}dD\xi_v(r) \\
&= \int_0^\infty \psi(r)\frac{n-1}{r}\mathcal{H}^{n-1}(\partial B(r)\cap E)\,dr + \int_0^\infty \psi(r)r^{n-1}dD\xi_v(r) \\
&= \int_{\mathbb{R}^n} \psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx + \int_0^\infty \psi(r)r^{n-1}dD\xi_v(r).
\end{aligned}$$

Combining the previous identity and (4.6),

$$\int_0^\infty \psi(r)r^{n-1}dD\xi_v(r) = \int_{\mathbb{R}^n} \psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E, \qquad \text{for every } \psi \in C_c^1(0\,\infty).$$

By approximation, the identity above is true also when ψ is a bounded Borel function, and this gives (4.1).

If $B \subset (0,\infty)$ is open, thanks to (4.1) we have that for every $\psi \in C_c(B)$ with $|\psi| \le 1$

$$\int_B \psi(r)r^{n-1}dD\xi_v(r) = \int_{\Phi(B\times\mathbb{S}^{n-1})} \psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E \le |D_\perp\chi_E|(\Phi(B\times\mathbb{S}^{n-1})).$$

Taking the supremum over all such ψ gives

$$|r^{n-1}D\xi_v|(B) \le |D_\perp\chi_E|(\Phi(B\times\mathbb{S}^{n-1})) \qquad \text{for every open set } B \subset (0,\infty).$$

By approximation, the inequality above holds true for every Borel set, and this shows inequality (4.2). □

The next lemma gives an important property of the measure $r^{n-1}D\xi_v$.

Lemma 4.2. *Let v be as in Theorem 1.1, and let $E \subset \mathbb{R}^n$ be a spherically v-distributed set of finite perimeter and finite volume. Then*

$$(r^{n-1}D\xi_v)(B) = \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_\parallel^E=0\}} \hat{x}\cdot\nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_B dr \int_{(\partial^* E)_r\cap\{\nu_\parallel^E\neq 0\}} \frac{\hat{x}\cdot\nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x). \tag{4.8}$$

for every Borel set $B \subset (0,+\infty)$.

Moreover, $r^{n-1}D\xi_v \llcorner G_{F_v} = r^{n-1}\xi_v' dr$ and for $\mathcal{H}^1$-a.e. $r \in G_{F_v}\cap\{0<\alpha_v<\pi\}$

$$r^{n-1}\xi_v'(r) = \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1))\frac{\hat{x}\cdot\nu^{F_v}(x)}{|\nu_\parallel^{F_v}(x)|}, \qquad \textit{for every } x\in\mathbf{S}_{\alpha_v(r)}(re_1).$$

Proof. Let $B \subset (0,+\infty)$ be a Borel set. Then, choosing $\psi = \chi_B$ in (4.1), and recalling (3.13),

$$\begin{aligned}
(r^{n-1}D\xi_v)(B) &= \int_0^{+\infty} \chi_B(r)r^{n-1}dD\xi_v(r)\\
&= \int_{\Phi(B\times\mathbb{S}^{n-1})} \hat{x}\cdot dD_\perp\chi_E(x) = \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})} \hat{x}\cdot\nu^E(x)\, d\mathcal{H}^{n-1}(x)\\
&= \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_\parallel^E=0\}} \hat{x}\cdot\nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_\parallel^E\neq0\}} \hat{x}\cdot\nu^E(x)\, d\mathcal{H}^{n-1}(x)\\
&= \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_\parallel^E=0\}} \hat{x}\cdot\nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_B dr \int_{(\partial^* E)_r\cap\{\nu_\parallel^E\neq 0\}} \frac{\hat{x}\cdot\nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x),
\end{aligned}$$

where in the last equality we have used the Coarea formula.

Let us now prove the second part of the statement. If one chooses $E = F_v$, thanks to Remark 3.10 we have

$$\begin{aligned}
r^{n-1}D\xi_v \llcorner G_{F_v} &= \left(\int_{(\partial^* F_v)_r\cap\{\nu_\parallel^{F_v}\neq 0\}} \frac{\hat{x}\cdot\nu^{F_v}(x)}{|\nu_\parallel^{F_v}(x)|} d\mathcal{H}^{n-2}(x)\right) dr \llcorner G_{F_v}\\
&= \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1))\frac{\hat{x}\cdot\nu^{F_v}(x)}{|\nu_\parallel^{F_v}(x)|}.
\end{aligned}$$

In particular,

$$r^{n-1}D\xi_v \llcorner G_{F_v} = r^{n-1}\xi_v'(r)\, dr \llcorner G_{F_v}.$$

Moreover, since $\xi_v'(r) = 0$ $\mathcal{H}^1$-a.e. in $\{\alpha = 0\}\cup\{\alpha = \pi\}$, we obtain that for $\mathcal{H}^1$-a.e. $r \in (0,\infty)$

$$r^{n-1}\xi'(r) = \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1))\frac{\hat{x}\cdot\nu^{F_v}(x)}{|\nu_\parallel^{F_v}(x)|}, \qquad \text{for every } x\in\mathbf{S}_{\alpha_v(r)}(re_1).$$

□

We now prove an auxiliary inequality that will be useful later.

Proposition 4.3. *Let v be as in Theorem 1.1, and suppose that there exists a spherically v-distributed set $E \subset \mathbb{R}^n$ of finite perimeter and finite volume. Then, F_v is a set of finite perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B \subset (0,+\infty)$*

$$P(F_v;\Phi(B\times\mathbb{S}^{n-1})) \le \left|r^{n-1}D\xi_v\right|(B) + \left|D_\parallel\chi_{F_v}\right|(\Phi(B\times\mathbb{S}^{n-1})). \tag{4.9}$$

Proof. The proof is based on the arguments of [12, Lemma 3.5] and [3, Lemma 3.3]. Thanks to Lemma 4.1, $v \in BV(0,\infty)$. Let $\{v_j\}_{j\in\mathbb{N}} \subset C_c^1(0,\infty)$ be a sequence of non-negative functions such that $v_j \to v$ $\mathcal{H}^1$-a.e. in $(0,\infty)$ and $|Dv_j| \overset{*}{\rightharpoonup} |Dv|$. For every $j \in \mathbb{N}$, we denote by $F_{v_j} \subset \mathbb{R}^n$ the set defined by (3.7), with v_j in place of v. Let now $\Omega \subset (0,\infty)$ be open, and let $\varphi \in C_c^1(\Phi(\Omega \times \mathbb{S}^{n-1});\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \le 1$. Thanks to Remark 3.3, we have

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}\varphi(x)dx = \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)dx \tag{4.10}$$
$$+ \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\left(\nabla\varphi(x)\,\hat{x}\right)\cdot\hat{x}\,dx + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\frac{n-1}{|x|}\left(\varphi(x)\cdot\hat{x}\right)\,dx.$$

In the following, it will be convenient to introduce the function $V_j : (0,\infty) \to \mathbb{R}$ given by

$$V_j(r) := \int_{\mathbf{B}_{\alpha_{v_j}(r)}(re_1)} \varphi(x)\cdot\hat{x}\,d\mathcal{H}^{n-1}(x) = r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega),$$

where $\alpha_{v_j} : (0,r) \to [0,\pi]$ is defined by (3.6), with v_j in place of v. We divide the proof into several steps.

Step 1: We show that V_j is Lipschitz continuous with compact support. Indeed,

$$\mathrm{supp}\,V_j \subset \Lambda(\mathrm{supp}\,\varphi) := \{r \in (0,+\infty) : (\mathrm{supp}\,\varphi)\cap\partial B(r) \neq \emptyset\}\,.$$

Moreover, for every $r_1, r_2 \in (0,\infty)$,

$$\begin{aligned}
|V_j(r_1) - V_j(r_2)| &\le \int_{\mathbf{B}_{\alpha_{v_j}(r_1)}(e_1)} |r_1^{n-1}\varphi(r_1\omega)\cdot\omega - r_2^{n-1}\varphi(r_2\omega)\cdot\omega|\,d\mathcal{H}^{n-1}(\omega)\\
&+ r_2^{n-1}\left|\int_{\mathbf{B}_{\alpha_{v_j}(r_1)}(e_1)} \varphi(r_2\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega) - \int_{\mathbf{B}_{\alpha_{v_j}(r_2)}(e_1)} \varphi(r_2\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega)\right|\\
&\le c|r_1 - r_2| + r_2^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(\widetilde{r}_1)}(e_1)\setminus\mathbf{B}_{\alpha_{v_j}(\widetilde{r}_2)}(e_1)} |\varphi(r_2\omega)\cdot\omega|\,d\mathcal{H}^{n-1}(\omega)\\
&\le c|r_1 - r_2| + r_2^{n-1}|\xi_{v_j}(r_1) - \xi_{v_j}(r_2)| \le c|r_1 - r_2|,
\end{aligned}$$

where we used the fact that ξ_{v_j} is compactly supported in $(0,\infty)$ (since v_j is), and $\widetilde{r}_1$ and $\widetilde{r}_2$ are such that $\alpha_{v_j}(\widetilde{r}_1) = \max\{\alpha_{v_j}(r_1), \alpha_{v_j}(r_2)\}$ and $\alpha_{v_j}(\widetilde{r}_2) := \min\{\alpha_{v_j}(r_1), \alpha_{v_j}(r_2)\}$.

Step 2: We show that α_{v_j} is $\mathcal{H}^1$-a.e. differentiable and that

$$\begin{aligned}
V_j'(r) = (n-1)r^{n-2}&\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega)\\
&+ r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-2}(\omega)\right)\\
&+ r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega),
\end{aligned} \tag{4.11}$$

for $\mathcal{H}^1$-a.e. $r > 0$. Let us set $A_j := \{0 < \alpha_{v_j} < \pi\}$. Since $v_j \in C^1_c(0, \infty)$, from (3.5) it follows that $\alpha_{v_j} \in C^1(A_j)$. Moreover, for every $r \in A_j$

$$
\begin{aligned}
V_j'(r) &= \frac{d}{dr}\left(r^{n-1}\int_0^{\alpha_{v_j}(r)} d\beta \int_{\mathbf{S}_\beta(e_1)} \varphi(r\omega)\cdot\omega \, d\mathcal{H}^{n-2}(\omega)\right)\\
&= (n-1)r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) + r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right)\\
&\quad + r^{n-1}\int_0^{\alpha_{v_j}(r)} d\beta\int_{\mathbf{S}_\beta(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\\
&= (n-1)r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) + r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right)\\
&\quad + r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega).
\end{aligned}
$$

This shows (4.11) whenever $r \in A_j$. Note now that

$$
\begin{aligned}
&V_j(r) = 0 && \text{for every } r \in \mathrm{Int}(\{\alpha_{v_j} = 0\}),\\
&V_j(r) = r^{n-1}\int_{\mathbb{S}^{n-1}} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) && \text{for every } r \in \mathrm{Int}(\{\alpha_{v_j} = \pi\}),
\end{aligned}
$$

where $\mathrm{Int}(\cdot)$ stands for the interior of a set. Since $\alpha'_{v_j}(r) = 0$ for every $r \in \mathrm{Int}(\{\alpha_{v_j} = 0\}) \cup \mathrm{Int}(\{\alpha_{v_j} = \pi\})$, using the identities above one can see that (4.11) holds true for $\mathcal{H}^1$-a.e. $r > 0$.

Step 3: We show that

$$
\begin{aligned}
&\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\ (\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\, dx + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\ dx\\
&= -\int_\Omega dr\, r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right).
\end{aligned}
$$

Integrating (4.11), thanks to the classical divergence theorem applied in Ω, and recalling that V_j has compact support, we obtain

$$
\begin{aligned}
0 &= (n-1)\int_\Omega dr\, r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega)\\
&\quad + \int_\Omega dr\, r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right)\\
&\quad + \int_\Omega dr\, r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega)\\
&= \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\ dx\\
&\quad + \int_\Omega dr\, r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right)\\
&\quad + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\ (\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\, dx,
\end{aligned}
$$

which gives the claim.

Step 4: we prove that

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}\varphi(x)dx \le \left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)) + \int_\Omega \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)})dr, \quad (4.12)$$

where $\Lambda(\operatorname{supp}\varphi) \subset (0,\infty)$ is the compact set defined in Step 1. Thanks to (4.10) and Step 3

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}\varphi(x)\,dx = \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)\,dx$$
$$- \int_\Omega dr\, r^{n-1}\left(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right). \quad (4.13)$$

We now estimate the right hand side of the expression above. Thanks to (3.6) and arguing as in Step 2 we have that

$$\xi'_{v_j}(r) = \alpha'_{v_j}(r)\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j(r)}}(e_1)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r\in(0,\infty).$$

Therefore,

$$- \int_\Omega dr\, r^{n-1}\left(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right)$$
$$\le \int_{\Lambda(\operatorname{supp}\varphi)} r^{n-1}\left|\alpha'_{v_j}(r)\right|\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(e_1))dr \quad (4.14)$$
$$= \int_{\Lambda(\operatorname{supp}\varphi)} r^{n-1}\left|\xi'_{v_j}(r)\right|dr = \left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)).$$

Let us now focus on the second integral in the right hand side of (4.13). Applying the divergence theorem (3.16) with $A = \mathbf{B}_{\alpha_{v_j}(r)}(re_1)$, and denoting by $\nu_*(x)$ the exterior unit normal to $\mathbf{S}_{\alpha_{v_j}(r)}(re_1)$, we have

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)\,dx = \int_\Omega dr \int_{\mathbf{B}_{\alpha_{v_j}(r)}(re_1)} \mathrm{div}_{\|}\varphi_{\|}(x)\,d\mathcal{H}^{n-1}(x)$$
$$= \int_\Omega dr \int_{\mathbf{S}_{\alpha_{v_j}(r)}(re_1)} \varphi_{\|}(x)\cdot\nu_*(x)d\mathcal{H}^{n-2}(x) \le \int_\Omega dr\,\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(re_1)). \quad (4.15)$$

Combining (4.13), (4.14), and (4.15), we obtain (4.12).

Step 5: We show that F_v is a set of finite perimeter. Note that $\chi_{F_{v_j}} \to \chi_{F_v}$ $\mathcal{H}^n$-a.e. in $\mathbb{R}^n$, and $\alpha_{v_j} \to \alpha$ $\mathcal{H}^1$-a.e. in $(0,\infty)$. Note also that, from our choice of the sequence $\{v_j\}_{j\in\mathbb{N}}$ and thanks to (4.7), it follows that

$$|r^{n-1}D\xi_{v_j}| \overset{*}{\rightharpoonup} |r^{n-1}D\xi_v| \qquad \text{as } j\to\infty.$$

Therefore, taking the limsup as $j\to\infty$ in (4.12), and using the fact that $\Lambda(\operatorname{supp}\varphi)$ is compact,

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x)\,\mathrm{div}\,\varphi(x)dx = \limsup_{j\to\infty}\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}\,\varphi(x)dx$$
$$\le \limsup_{j\to\infty}\left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)) + \limsup_{j\to\infty}\int_\Omega \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(re_1))\,dr$$
$$\le \left|r^{n-1}D\xi_v\right|(\Lambda(\operatorname{supp}\varphi)) + \int_\Omega \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1))\,dr \le \left|r^{n-1}D\xi_v\right|(\Omega) + \int_\Omega \mathcal{H}^{n-2}(\partial^* E_r)\,dr$$
$$\le \left|r^{n-1}D\xi_v\right|(\Omega) + P(E;\Phi(\Omega\times\mathbb{S}^{n-1})),$$

where we also used the isoperimetric inequality in the sphere (see (3.17)) and the Coarea formula. Taking the supremum of the above inequality over all functions $\varphi \in C_c^1(\Phi(\Omega \times \mathbb{S}^{n-1}); \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \le 1$, we obtain

$$P(F_v; \Phi(\Omega \times \mathbb{S}^{n-1})) \le \left|r^{n-1} D\xi_v\right|(\Omega) + P(E; \Phi(\Omega \times \mathbb{S}^{n-1})).$$

Thanks to (4.2) we have

$$P(F_v; \Phi(\Omega \times \mathbb{S}^{n-1})) \le 2P(E; P(F_v; \Phi(\Omega \times \mathbb{S}^{n-1}))) < \infty,$$

since E is a set of finite perimeter by assummption. Since Ω was arbitrary, this shows that F_v is a set of locally finite perimeter.

Step 6: We conclude. Let $\Omega \subset (0,\infty)$ be open, and let $\varphi \in C_c^1(\Phi(\Omega \times \mathbb{S}^{n-1}); \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \le 1$. Combining (4.10), Step 3, and (4.14), we have that for every $j \in \mathbb{N}$

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \operatorname{div}\varphi(x)dx \le \left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)) + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \operatorname{div}_\parallel \varphi_\parallel(x)\,dx.$$

Taking the limsup as $j \to \infty$ and thanks to Corollary 3.5,

$$\begin{aligned}\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x) \operatorname{div}\varphi(x)dx &\le \left|r^{n-1}D\xi_v\right|(\Lambda(\operatorname{supp}\varphi)) + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x)\operatorname{div}_\parallel\varphi_\parallel(x)\,dx \\ &\le \left|r^{n-1}D\xi_v\right|(\Lambda(\operatorname{supp}\varphi)) + |D_\parallel \chi_{F_v}|(\Phi(\Omega\times\mathbb{S}^{n-1})),\end{aligned}$$

where we also used the fact that $\Lambda(\operatorname{supp}\varphi)$ is compact.

Taking the supremum over all $\varphi \in C_c^1(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \le 1$,

$$P(F_v; \Phi(\Omega \times \mathbb{S}^{n-1})) \le \left|r^{n-1}D\xi_v\right|(\Omega) + |D_\parallel\chi_{F_v}|(\Phi(\Omega\times\mathbb{S}^{n-1})), \tag{4.16}$$

which shows (4.9) when B is an open set. Let now $B \subset (0,\infty)$ be a Borel set. From (4.16) it follows that

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) \le \left|r^{n-1}D\xi_v\right|(\Omega) + P(E; \Phi(\Omega\times\mathbb{S}^{n-1})),$$

for any open set $\Omega \subset (0,\infty)$ with $B \subset \Omega$. Taking the infimum of the above inequality over all open sets $\Omega \subset (0,\infty)$ with $B \subset \Omega$, we obtain inequality (4.9) when B is a Borel set. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1, and state some important auxiliary results. The proof of Lemma 1.3 is postponed to Section 6, since it requires some results related to the circular symmetrisation. We start by proving Theorem 1.1.

Proof of Theorem 1.1. We will adapt the arguments of the proof of [3, Theorem 1.1]. Let G_{F_v} be the set associated with F_v given by Theorem 3.7. We start by proving (1.4). We will first prove the inequality when $B \subset (0,\infty)\setminus G_{F_v}$, and then in the case $B \subset G_{F_v}$. The case of a general Borel set $B \subset (0,\infty)$ then follows by decomposing B as $B = (B\setminus G_{F_v}) \cup (B \cap G_{F_v})$.

Step 1: We prove inequality (1.4) when $B \subset (0,\infty) \setminus G_{F_v}$. First observe that, thanks to Proposition 3.6 and (3.13),

$$\begin{aligned}\left|D_\parallel\chi_{F_v}\right|(\Phi(B\times\mathbb{S}^{n-1})) &= \int_{\partial^* F_v \cap \Phi(B\times\mathbb{S}^{n-1})} |\nu_\parallel^{F_v}(x)| d\mathcal{H}^{n-1}(x) = \int_B \mathcal{H}^{n-2}((\partial^* F_v)_r)dr \\ &= \int_{B\cap\{0<\alpha_v\}} \mathcal{H}^{n-2}((\partial^*F_v)_r)dr = \int_{B\cap(\{0<\alpha_v\}\setminus G_{F_v})} \mathcal{H}^{n-2}((\partial^*F_v)_r)dr = 0,\end{aligned} \tag{5.1}$$

where we used the fact that $B \subset (0,\infty) \setminus G_{F_v}$ and $\mathcal{H}^1(\{0 < \alpha_v\} \setminus G_{F_v}) = 0$. Therefore, thanks to Proposition 4.3

$$\begin{aligned} P(F_v; \Phi(B \times \mathbb{S}^{n-1})) &\leq r^{n-1} |D\xi_v| (B) + \left|D_{\|}\chi_{F_v}\right| (\Phi(B \times \mathbb{S}^{n-1})) \\ &= r^{n-1} |D\xi_v| (B) \leq P(E; \Phi(B \times \mathbb{S}^{n-1})), \end{aligned} \tag{5.2}$$

where in the last inequality we used (4.2).

Step 2: We prove inequality (1.4) when $B \subset G_{F_v}$. We divide this part of the proof into further substeps.

Step 2a: we prove that

$$P(E; \Phi(B \times \mathbb{S}^{n-1})) \geq P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)} dr, \tag{5.3}$$

where $g : (0,\infty) \to \mathbb{R}$ and $p_E : (0,\infty) \to [0,\infty)$ are defined as

$$g(r) := \int_{\partial^* E \cap \partial B(r)} \frac{\hat{x} \cdot \nu^E(x)}{|\nu_{\|}^E(x)|} d\mathcal{H}^{n-2}(x) \quad \text{ and } \quad p_E(r) := \mathcal{H}^{n-2}(\partial^* E \cap \partial B(r)),$$

for $\mathcal{H}^1$-a.e. $r \in (0,\infty)$, respectively. We have

$$\begin{aligned} &P(E; \Phi(B \times \mathbb{S}^{n-1})) \\ &= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E \neq 0\}) \\ &= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E \neq 0\}} d\mathcal{H}^{n-1}(x) \\ &= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_B dr \int_{\partial^* E \cap \partial B(r)} \frac{1}{|\nu_{\|}^E(x)|} d\mathcal{H}^{n-2}(x) \\ &= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_B dr \int_{\partial^* E \cap \partial B(r)} \sqrt{1 + \left(\frac{\hat{x} \cdot \nu^E(x)}{|\nu_{\|}^E(x)|}\right)^2} d\mathcal{H}^{n-2}(x), \end{aligned}$$

where in the last equality we used the fact that

$$1 = |\nu_{\perp}^E|^2 + |\nu_{\|}^E|^2 = (\hat{x} \cdot \nu^E)^2 + |\nu_{\|}^E|^2.$$

Defining the function $f : \mathbb{R} \to [0,\infty)$ as

$$f(t) := \sqrt{1 + t^2},$$

we obtain

$$\begin{aligned} &P(E; \Phi(B \times \mathbb{S}^{n-1})) \\ &= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_B dr \int_{\partial^* E \cap \partial B(r)} f\left(\frac{\hat{x} \cdot \nu^E(x)}{|\nu_{\|}^E(x)|}\right) d\mathcal{H}^{n-2}(x). \end{aligned}$$

Observing that f is strictly convex, (5.3) follows applying Jensen's inequality.

Step 2b: We show that

$$\begin{aligned} &\int_B \sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2} \, dr \\ &\leq P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\|}^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)} \, dr. \end{aligned} \tag{5.4}$$

Let $H \subset \mathbb{N}$ be a finite set, and let $\{A_h\}_{h\in H}$ be a finite partition of Borel sets of B. Note that, for each $h \in H$, we have $A_h \subset B \subset G_{F_v}$. Therefore, thanks to Lemma 4.2, for every $h \in H$ we have $r^{n-1} D\xi_v \llcorner A_h = r^{n-1}\xi_v' dr \llcorner A_h$ and

$$\begin{aligned}
&\int_{A_h} w_h r^{n-1} \xi_v'(r)\, dr = \int_{A_h} w_h r^{n-1} dD\xi_v(r) \\
&= \int_{\partial^* E \cap \Phi(A_h \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} w_h\, \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) \\
&\quad + \int_{A_h} dr \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} w_h \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x) \\
&= \int_{\partial^* E \cap \Phi(A_h \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} w_h\, \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_{A_h} w_h\, g(r)\, dr. \qquad (5.5)
\end{aligned}$$

We will now use the fact that, by duality, we can write

$$\sqrt{1+t^2} = \sup_{h\in\mathbb{N}} \left\{ w_h t + \sqrt{1-w_h^2} \right\} \qquad \text{for every } t \in \mathbb{R}, \qquad (5.6)$$

where $\{w_h\}_{h\in\mathbb{N}}$ is a countable dense set in $(-1,1)$. Then, thanks to (5.5)

$$\begin{aligned}
&\sum_{h\in H} \int_{A_h} \left(w_h r^{n-1} \xi_v'(r) + p_E(r) \sqrt{1-w_h^2} \right) dr \\
&= \sum_{h\in H} \int_{\partial^* E \cap \Phi(A_h \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} w_h\, \hat{x} \cdot \nu^E(x) d\mathcal{H}^{n-1}(x) \\
&\quad + \sum_{h\in H} \int_{A_h} \left(w_h\, g(r) + p_E(r) \sqrt{1-w_h^2} \right) dr \\
&\leq \sum_{h\in H} \int_{\partial^* E \cap \Phi(A_h \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} |\hat{x} \cdot \nu^E(x)| d\mathcal{H}^{n-1}(x) \\
&\quad + \sum_{h\in H} \int_{A_h} p_E(r) \left(w_h \frac{g(r)}{p_E(r)} + \sqrt{1-w_h^2} \right) dr \\
&\leq \sum_{h\in H} \left(P(E; \Phi(A_h \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}) \right) + \int_{A_h} p_E(r) \sqrt{1 + \frac{g^2(r)}{p_E^2(r)}} dr \\
&= P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)} dr,
\end{aligned}$$

where we applied identity (5.6) with $t = g(r)/p_E(r)$, and we also used the fact that $p_E(r) = 0$ for $\mathcal{H}^1$-a.e. $r \notin \{0 < \alpha_v < \pi\}$, thanks to Volper't theorem. Applying Lemma 3.1 to the functions

$$\varphi_h(r) = p_E(r) \left(w_h \frac{r^{n-1}\xi_v'(r)}{p_E(r)} + \sqrt{1-w_h^2} \right),$$

we obtain (5.4).

Step 2c: We conclude the proof of Step 2. In the special case $E = F_v$, thanks to Vol'pert Theorem and Lemma 4.2 we have

$$
\begin{aligned}
P(F_v; \Phi(B \times \mathbb{S}^{n-1})) &= \mathcal{H}^{n-1}(\partial^* F_v \cap \Phi(B \times \mathbb{S}^{n-1})) \\
&= \int_{B \cap \{0 < \alpha_v < \pi\}} \int_{\partial^*(F_v)_r} \frac{1}{|\nu_{\parallel}^{F_v}(x)|} d\mathcal{H}^{n-2}(x) dr \\
&= \int_{B \cap \{0 < \alpha_v < \pi\}} \int_{\partial^*(F_v)_r} \sqrt{1 + \left(\frac{\nu^{F_v}(x)}{|\nu_{\parallel}^{F_v}(x)|} \right)^2} d\mathcal{H}^{n-2}(x) dr \\
&= \int_{B \cap \{0 < \alpha_v < \pi\}} \sqrt{p_{F_v}^2(r) + (r^{n-1} \xi_v'(r))^2} dr. \qquad (5.7)
\end{aligned}
$$

Using the isoperimetric inequality (3.17) together with (5.4) and (5.3) we then have,

$$
\begin{aligned}
P(F_v; \Phi(B \times \mathbb{S}^{n-1})) &\leq \int_{B \cap \{0 < \alpha_v < \pi\}} \sqrt{p_E^2(r) + (r^{n-1} \xi_v'(r))^2} dr \\
&\leq P(E; \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\parallel}^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)} dr \\
&\leq P(E; \Phi(B \times \mathbb{S}^{n-1})),
\end{aligned}
$$

from which we conclude.

Step 3: We conclude the proof of the theorem. Suppose $P(E) = P(F_v)$. Then, in particular, all the inequalities in Step 2 hold true as equalities. At the end of Step 2c we used the fact that, by the isoperimetric inequality (3.17), we have

$$
p_{F_v}(r) \leq p_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in \{0 < \alpha_v < \pi\}.
$$

If the above becomes an equality, this means that for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$ the slice E_r is a spherical cap. Finally, the fact that for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$ we have

$$
\mathcal{H}^{n-2}(\partial^*(E_r) \Delta (\partial^* E)_r) = 0
$$

follows from Vol'pert Theorem 3.7, and this shows (a).

Let us now prove (b). If $P(E) = P(F_v)$, the Jensen's inequality at the end of Step 2b, for the strictly convex function

$$
f(t) := \sqrt{1 + t^2},
$$

becomes an equality. This implies that for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$ the function

$$
x \longmapsto \frac{\hat{x} \cdot \nu^E(x)}{|\nu_{\parallel}^E(x)|}
$$

is $\mathcal{H}^{n-2}$-a.e. constant in $\partial^* E_r$. Since, for $\mathcal{H}^{n-2}$-a.e. $x \in \partial^* E_r$, we have

$$
1 = |\nu_{\parallel}^E(x)|^2 + (\hat{x} \cdot \nu^E(x))^2,
$$

this implies that

$$
x \longmapsto \frac{(\hat{x} \cdot \nu^E(x))^2}{|\nu_{\parallel}^E(x)|^2} = 1 - \frac{1}{|\nu_{\parallel}^E(x)|^2}
$$

is $\mathcal{H}^{n-2}$-a.e. constant in $\partial^* E_r$. Therefore, the two functions

$$
x \longmapsto \nu^E(x) \cdot \hat{x} \qquad \text{and} \qquad x \longmapsto |\nu_{\parallel}^E|(x)
$$

are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$.

□

The previous result allows us to prove a useful proposition (see also [3, Proposition 3.4]).

Proposition 5.1. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume, let E be a spherically v-distributed set of finite perimeter, and let $f : (0,\infty) \to [0,\infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} f(|x|)\, d\mathcal{H}^{n-1}(x) \geq \int_0^\infty f(r)\sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty f(r) r^{n-1} d|D^s\xi_v|(r). \tag{5.8}$$

Moreover, in the special case $E = F_v$, equality holds true.

Proof. To prove the proposition it is enough to consider the case in which $f = \chi_B$, with $B \subset (0,\infty)$ Borel set.

First, suppose $B \subset (0,\infty) \setminus G_{F_v}$. Thanks to Lemma 4.2, in this case we have $\xi_v' = 0$ in B and $|r^{n-1}D\xi_v|(B) = |r^{n-1}D^s\xi_v|(B)$. Then, from (4.2) it follows that

$$\begin{aligned}
&\int_{\partial^* E} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(E;\Phi(B\times\mathbb{S}^{n-1})) \geq |D_\perp \chi_E|(\Phi(B\times\mathbb{S}^{n-1}))\\
&\geq |r^{n-1}D\xi_v|(B) = |r^{n-1}D^s\xi_v|(B) = \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r)\\
&= \int_0^\infty \chi_B(r)\sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r),
\end{aligned}$$

where we also used the fact that $p_E = 0$ $\mathcal{H}^1$-a.e. in B, since

$$\mathcal{H}^n(E \cap \Phi(B\times\mathbb{S}^{n-1})) \leq \int_{\{v=0\}} dr \int_{E_r} d\mathcal{H}^{n-1}(x) = \int_{\{v=0\}} v(r)\, dr = 0.$$

Let us now assume $B \subset G_{F_v}$. In this case, by Lemma 4.2 we have $|r^{n-1}D^s\xi_v|(B) = 0$. Then, thanks to (5.3) and (5.4) we obtain

$$\begin{aligned}
&\int_{\partial^* E} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(E;\Phi(B\times\mathbb{S}^{n-1}))\\
&\geq P(E;\Phi(B\times\mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)} dr\\
&\geq \int_B \sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr\\
&= \int_0^\infty \chi_B(r)\sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r),
\end{aligned}$$

so that (5.8) follows.

Consider now the case $E = F_v$. If $B \subset G_{F_v}$, recalling again that by Lemma 4.2 we have $|r^{n-1}D^s\xi_v|(B) = 0$, thanks to (5.7) we obtain

$$\begin{aligned}
&\int_{\partial^* F_v} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(F_v;\Phi(B\times\mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr\\
&= \int_0^\infty \chi_B(r)\sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r).
\end{aligned}$$

If, instead, $B \subset (0,\infty) \setminus G_{F_v}$, then $\xi_v' = 0$ in B and $|r^{n-1}D\xi_v|(B) = |r^{n-1}D^s\xi_v|(B)$. Therefore, thanks to (5.2),

$$\begin{aligned}
&\int_{\partial^* F_v} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(F_v;\Phi(B\times\mathbb{S}^{n-1})) \leq r^{n-1}\left|D\xi_v\right|(B) = |r^{n-1}D^s\xi_v|(B)\\
&= \int_0^\infty \chi_B(r)\sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r).
\end{aligned}$$

□

An important consequence of the above proposition is a formula for the perimeter of F_v.

Corollary 5.2. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume. Then*

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_B r^{n-1} d|D^s\xi_v|(r). \tag{5.9}$$

We conclude this section with an important result, that will be used later.

Proposition 5.3. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume, and let $I \subset (0,+\infty)$ be an open set. Then the following three statements are equivalent:*

(i) $\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{F_v}(x) = 0\right\}\right) = 0$*;*

(ii) $\xi_v \in W^{1,1}_{\mathrm{loc}}(I)$*;*

(iii) $P(F_v; \Phi(B \times \mathbb{S}^{n-1})) = 0$ *for every Borel set $B \subset I$, such that $\mathcal{H}^1(B) = 0$.*

Remark 5.4. *Note that the equivalence* (iii) $\iff$ (i) *holds true also if I is a Borel set. To show this, we only need to prove that* (i) $\implies$ (iii)*, since the opposite implication is given by repeating Step 3 of the proof of Proposition 5.3. Suppose* (i) *is satisfied. Then from* (4.8) *we have $r^{n-1}D\xi_v \llcorner I = r^{n-1}\xi_v' \llcorner I$. Therefore, thanks to* (5.9)

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr \qquad \text{for every Borel set } B \subset I,$$

which implies (iii).

Proof. We divide the proof into three steps.

Step 1: (i) $\implies$ (ii). Recall that, by Lemma 4.1, $\xi_v \in BV_{\mathrm{loc}}(I)$. If (i) is satisfied, from (4.8) we have $r^{n-1}D\xi_v \llcorner I = r^{n-1}\xi_v' \llcorner I$, which implies (ii).

Step 2: (ii) $\implies$ (iii). This implication follows from formula (5.9).

Step 3: (iii) $\implies$ (i) (note that we will not use the fact that I is open). Assume (iii) holds true. Then,

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{\partial^* F_v}(x) = 0\right\}\right) \le P(\partial^* F_v; \Phi((B_0 \cap I) \times \mathbb{S}^{n-1})) = 0,$$

where we used the fact that $\mathcal{H}^1(B_0) = 0$, thanks to (3.19). $\square$

6. Circular symmetrisation and proof of Lemma 1.3

In this section we show Theorem 1.4, Lemma 1.5, and finally Lemma 1.3. We will only sketch the proofs, since in most cases the arguments follow the lines of the proofs in Section 3, Section 4, and Section 5.

We start with some notation which, together with that one already given in the Introduction, will be extensively used in this section. Let $(r, x') \in (0,\infty) \times \mathbb{R}^{n-2}$, $\beta \in [0,\pi]$, and let $p \in \mathbb{S}^1$. The circular arc of centre (rp, x') and radius β is the set

$$\mathcal{B}_\beta(rp, x') := \{x \in \partial B((0,x'), r) \cap \Pi_{x'} : \mathrm{dist}_{\mathbb{S}^1}(\hat{x}_{12}, rp) < \beta\},$$

If $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ is a measurable function satisfying (1.10), we define $\alpha^\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\pi]$ and $\xi^\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,2\pi]$ as

$$\alpha^\ell := \frac{1}{2r}\ell(r,x') \qquad \text{and} \qquad \xi^\ell(r,x') = \frac{1}{r}\ell(r,x') = 2\alpha^\ell(r,x').$$

Note that in this case the relation between α^ℓ and ξ^ℓ is linear. If μ is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n \setminus \{x_{12} = 0\}$, we will write $\mu = \mu_{12\perp} + \mu_{12\|}$, where $\mu_{12\perp}$ and $\mu_{12\|}$ are the $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^n \setminus \{x_{12} = 0\}$ such that

$$\int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi \cdot d\mu_{12\perp} = \int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi_{12\perp} \cdot d\mu,$$

and

$$\int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi \cdot d\mu_{12\|} = \int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi_{12\|} \cdot d\mu,$$

for every $\varphi \in C_c(\mathbb{R}^n \setminus \{x_{12} = 0\}; \mathbb{R}^n)$. The next two results play the role of Proposition 3.6 and Vol'pert Theorem 3.7, in the context of circular symmetrisation.

Proposition 6.1. *Let E be a set of finite perimeter in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to [0, \infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} g(x)|\nu^E_{12\|}(x)| d\mathcal{H}^{n-1}(x) = \int_{(0,\infty)\times\mathbb{R}^{n-2}} dr\, dx' \int_{(\partial^* E)_{(r,x')}} g(x)\, d\mathcal{H}^0(x).$$

Proof. In this case, the result follows applying [2, Remark 2.94] with $N = n - 1$, $M = n$, $k = n - 1$, and $f(x) = (|x_{12}|, x')$. □

Theorem 6.2. *Let $\ell : (0, \infty) \times \mathbb{R}^{n-2} \to [0, \infty)$ be a measurable function satisfying* (1.10), *and let $E \subset \mathbb{R}^n$ be an circularly ℓ-distributed set of finite perimeter and finite volume. Then, there exists a Borel set $G^\ell_E \subset \{\alpha^\ell > 0\}$ with $\mathcal{H}^{n-1}(\{\alpha^\ell > 0\} \setminus G^\ell_E) = 0$, such that*

- (i) *for every $(r, x') \in G^\ell_E$:*
  - (ia) *$E_{(r,x')}$ is a set of finite perimeter in $\partial B_r(0, x') \cap \Pi_{x'}$;*
  - (ib) *$\partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$;*
- (ii) *for every $(r, x') \in G^\ell_E \cap \{0 < \alpha^\ell < \pi\}$:*
  - (iia) *$|\nu^E_{12\|}(r\omega, x')| > 0$;*
  - (iib) *$\nu^E_{12\|}(r\omega, x') = \nu^{E_{(r,x')}}(r\omega, x')|\nu^E_{12\|}(r\omega, x')|$,*

    *for **every** $\omega \in \mathbb{S}^1$ such that $(r\omega, x') \in \partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$.*

Proof. The statement follows applying the results of [18, Section 2.5], where the slicing of codimension higher than 1 for currents is defined. □

Remark 6.3. *Note that, if $(r, x') \in G^\ell_E$, conditions (iia) and (iib) are satisfied for **every** $\omega \in \mathbb{S}^1$ such that $(r\omega, x') \in \partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$. This is due to the fact that the circular symmetrisation has codimension* 1. *Such property fails, in general, for the spherical symmetrisation (see Remark 3.9).*

Remark 6.4. *An argument similar to that one used in Remark 3.9 shows that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \Phi_{12}(G^\ell_E \times \mathbb{S}^1) \cap \{\nu^E_{12\|} = 0\}) = 0.$$

As a consequence, the measure λ^ℓ_E defined as:

$$\lambda^\ell_E(B) := \int_{\partial^* E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}} \hat{x}_{12} \cdot \nu^E(x)\, d\mathcal{H}^1(x),$$

for every Borel set $B \subset (0, \infty) \times \mathbb{R}^{n-2}$, is singular with respect to the Lebesgue measure in $(0, \infty) \times \mathbb{R}^{n-2}$.

The following result plays the role of Lemma 4.1 in the context of circular symmetrisation.

Lemma 6.5. *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ be a measurable function satisfying* (1.10), *and let $E\subset\mathbb{R}^n$ be an circularly ℓ-distributed set of finite perimeter and finite volume. Then, $\ell\in BV_{\rm loc}((0,\infty)\times\mathbb{R}^{n-2})$. Moreover, $\xi^\ell\in BV_{\rm loc}((0,\infty)\times\mathbb{R}^{n-2})$ and*

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,x')\,r\,dD_r\xi^\ell(r,x')=\int_{\mathbb{R}^n\setminus\{x_{12}=0\}}\psi(|x_{12}|,x')\,\hat{x}_{12}\cdot dD_{12\perp}\chi_E(x),$$

for every bounded Borel function $\psi:(0,\infty)\times\mathbb{R}^{n-2}\to\mathbb{R}$, where $D_r\xi^\ell$ denotes the r-component of the $\mathbb{R}^{n-1}$-valued Radon measure $D\xi^\ell$. As a consequence,

$$|rD_r\xi^\ell|(B)\le|D_{12\perp}\chi_E|(\Phi_{12}(B\times\mathbb{S}^1)),$$

for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$. In particular, $rD_r\xi^\ell$ is a bounded Radon measure on $(0,\infty)\times\mathbb{R}^{n-2}$. Finally,

$$D_{x'}\ell(B)=\int_{\partial^*E\cap\Phi_{12}(B\times\mathbb{S}^1)}\nu^E_{x'}(x)\,d\mathcal{H}^{n-1}(x),$$

for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$.

Remark 6.6. *Unlike what happened when we were considering the spherical symmetrisation, now the function ℓ might fail to be in $BV((0,\infty)\times\mathbb{R}^{n-2})$. Indeed, in Step 1 of the proof of Lemma 4.1 we used the fact that for r bounded we are in a bounded set. This is not true in the context of circular symmetrisation.*

The next lemma, which is related to Lemma 4.2, will show the advantage of considering a symmetrisation of codimension 1.

Lemma 6.7. *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ be a measurable function satisfying* (1.10), *and let $E\subset\mathbb{R}^n$ be an circularly ℓ-distributed set of finite perimeter and finite volume. Then*

$$\begin{aligned}(r\,dD_r\xi^\ell)(B)=&\int_{\partial^*E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}}\hat{x}_{12}\cdot\nu^E(x)\,d\mathcal{H}^{n-1}(x)\\&+\int_B dr\,dx'\int_{(\partial^*E)_{(r,x')}\cap\{\nu^E_{12\|}\neq0\}}\frac{\hat{x}_{12}\cdot\nu^E(x)}{|\nu^E_{12\|}(x)|}d\mathcal{H}^0(x).\end{aligned}$$

for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$. Moreover,

$$r(\xi^\ell)'(r,x')=\int_{(\partial^*E)_{(r,x')}\cap\{\nu^E_{12\|}\neq0\}}\frac{\hat{x}_{12}\cdot\nu^E(x)}{|\nu^E_{12\|}(x)|}d\mathcal{H}^0(x),$$

for $\mathcal{H}^{n-1}$-a.e. $(r,x')\in(0,\infty)\times\mathbb{R}^{n-2}$, where $(\xi^\ell)'$ denotes the approximate differential of ξ^ℓ with respect to r. Similarly,

$$\begin{aligned}D_{x'}\ell(B)=&\int_{\partial^*E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}}\nu^E_{x'}(x)\,d\mathcal{H}^{n-1}(x)\\&+\int_B dr\,dx'\int_{(\partial^*E)_{(r,x')}\cap\{\nu^E_{12\|}\neq0\}}\frac{\nu^E_{x'}(x)}{|\nu^E_{12\|}(x)|}d\mathcal{H}^0(x).\end{aligned}$$

for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$, and

$$\nabla_{x'}\ell(r,x')=\int_{(\partial^*E)_{(r,x')}\cap\{\nu^E_{12\|}\neq0\}}\frac{\nu^E_{x'}(x)}{|\nu^E_{12\|}(x)|}d\mathcal{H}^0(x),$$

for $\mathcal{H}^{n-1}$-a.e. $(r,x')\in(0,\infty)\times\mathbb{R}^{n-2}$, where $\nabla_{x'}\ell$ denotes the approximate gradient of ℓ with respect to x'.

The next result should be compared to Proposition 4.3.

Proposition 6.8. *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10)*, and suppose that there exists an circularly ℓ-distributed set $E \subset \mathbb{R}^n$ be of finite perimeter and finite volume. Then, F^ℓ is a set of finite perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B \subset (0,+\infty) \times \mathbb{R}^{n-2}$*

$$P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1)) \le |D_{x'}\ell|(B) + |rD_r\xi^\ell|(B) + \left|D_{12\|}\chi_{F_v}\right|(\Phi_{12}(B \times \mathbb{S}^1)).$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Using the results shown above, Theorem 1.4 can be proved by following the lines of the proof of Theorem 1.1. □

We will now state the results that are need to prove Lemma 1.5. The next proposition should be compared to Proposition 5.1.

Proposition 6.9. *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10) *such that F^ℓ is a set of finite perimeter and finite volume, let $E \subset \mathbb{R}^n$ be an circularly ℓ-distributed set of finite perimeter, and let $f : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty]$ be a Borel function. Then,*

$$\begin{aligned}
&\int_{\partial^* E} f(|x_{12}|, x')\, d\mathcal{H}^{n-1}(x)\\
&\ge \int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x')\sqrt{p_E^2(r,x') + (r(\xi^\ell)'(r,x'))^2 + |\nabla_{x'}\ell(r,x')|^2}\, dr\, dx'\\
&+ \int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x')\, r\, d|D_r^s\xi^\ell|(r,x') + \int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x') d|D_{x'}^s\ell|(r,x').
\end{aligned}$$

Moreover, in the special case $E = F^\ell$, equality holds true.

A straightforward consequence of the previous result is the following formula for the perimeter of F^ℓ.

Corollary 6.10. *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10) *such that F^ℓ is a set of finite perimeter and finite volume. Then*

$$\begin{aligned}
&P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1))\\
&= \int_B \sqrt{p_E^2(r,x') + (r(\xi^\ell)'(r,x'))^2 + |\nabla_{x'}\ell(r,x')|^2}\, dr\, dx' + |rD_r^s\xi^\ell|(B) + |D_{x'}^s\ell|(B).
\end{aligned}$$

Next lemma relies on the fact that the circular symmetrisation has codimension 1. The proof can be obtained by repeating the arguments used in the proof of [12, Lemma 4.1].

Lemma 6.11. *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10)*, let $E \subset \mathbb{R}^n$ be an circularly ℓ-distributed set of finite perimeter and finite volume, and let $A \subset (0,+\infty) \times \mathbb{R}^{n-2}$ be a Borel set. Then,*

$$\mathcal{H}^{n-1}\Big(\{x \in \partial^* E : \nu_{12\|}^E(x) = 0\} \cap \Phi_{12}(A \times \mathbb{S}^1)\Big) = 0.$$

if and only if

$$P(E; \Phi_{12}(B \times \mathbb{S}^1)) = 0 \quad \text{for every Borel set } B \subset A \text{ with } \mathcal{H}^{n-1}(B) = 0.$$

The next proposition can be proved with the same arguments used to show Proposition 5.3.

Proposition 6.12. *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying* (1.10) *such that F^ℓ is a set of finite perimeter and finite volume, and let $\Omega \subset (0,+\infty) \times \mathbb{R}^{n-2}$ be an open set. Then the following three statements are equivalent:*

(i) $\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F^\ell \cap \Phi_{12}(\Omega \times \mathbb{S}^1) : \nu_{12\|}^{F^\ell}(x) = 0\right\}\right) = 0;$

(ii) $\xi^\ell \in W^{1,1}_{\rm loc}(\Omega)$ *and* $\ell \in W^{1,1}_{\rm loc}(\Omega)$*;*

(iii) $P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1)) = 0$ *for every Borel set* $B \subset \Omega$*, such that* $\mathcal{H}^{n-1}(B) = 0$.

Proof of Lemma 1.5. Once all the results above are established, Lemma 1.5 can be shown by adapting the arguments used in the proof of [12, Proposition 4.2]. □

We can now prove Lemma 1.3. As already mentioned in the Introduction, the proof relies on Theorem 1.4 and Lemma 1.5.

Proof of Lemma 1.3. We divide the proof into steps.

Step 1: We show that (1.8) $\Longrightarrow$ (1.9). Suppose (1.8) is satisfied. Then, from (4.8) we have $r^{n-1} D\xi_v \llcorner I = r^{n-1}\xi_v' \llcorner I$. Thanks to (5.9), this implies that

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr. \qquad \text{for every Borel set } B \subset I.$$

In particular, condition (iii) of Proposition 5.3 is satisfied. Then, (1.9) follows from Remark 5.4.

Step 2: We show that if $P(E; \Phi(I \times \mathbb{S}^{n-1})) = P(F_v; \Phi(I \times \mathbb{S}^{n-1}))$, then (1.9) implies (1.8). To this aim, we first prove an auxiliary result.

Step 2a: We show that if $\overline{F} \subset \mathbb{R}^n$ is a set of finite perimeter such that $(\overline{F})_r$ is a spherical cap for $\mathcal{H}^1$-a.e. $r > 0$, and

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* \overline{F} \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_\|^{\overline{F}}(x) = 0\right\}\right) = 0, \tag{6.1}$$

then $\mathcal{H}^{n-1}(B^j) = 0$ for every $j = 2, \dots, n$, where

$$B^j := \left\{x \in \partial^* \overline{F} \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{1j\|}^{\overline{F}}(x) = 0\right\}.$$

Here, the vector $\nu_{1j\|}^{\overline{F}}$ is defined in the following way. Let $j \in \{2, \dots, n\}$, and let $\nu_{1j}^{\overline{F}}$ be the orthogonal projection of $\nu^{\overline{F}}$ on the bi-dimensional plane generated by e_1 and e_j. In this plane, we consider the following orthonormal basis $\{\widehat{x}_{1j}, \widetilde{x}_{1j}\}$:

$$\widehat{x}_{1j} = \frac{1}{\sqrt{x_1^2 + x_j^2}}(x_1, \overbrace{0, \dots, 0}^{j-2 \text{ times}}, x_j, \overbrace{0, \dots, 0}^{n-j \text{ times}}),$$

and

$$\widetilde{x}_{1j} = \frac{1}{\sqrt{x_1^2 + x_j^2}}(-x_j, \overbrace{0, \dots, 0}^{j-2 \text{ times}}, x_1, \overbrace{0, \dots, 0}^{n-j \text{ times}}),$$

where $\widehat{x}_{1j}$ is directed along the radial direction, and $\widetilde{x}_{1j}$ is parallel to the tangential direction. To show the claim, first of all note that, by Vol'pert Theorem 3.7, for $\mathcal{H}^1$-a.e. $r > 0$ we have

$$(B^j)_r = \left\{x \in \partial^* \overline{F}_r \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_\|^{\overline{F}_r}(x) \cdot \widetilde{x}_{1j} = 0\right\}.$$

up to an $\mathcal{H}^{n-2}$-negligible set. Since $(B^j)_r$ is a spherical cap, we have $\mathcal{H}^{n-2}((B^j)_r) = 0$. Then, thanks to (6.1),

$$\begin{aligned}\mathcal{H}^{n-1}(B^j) &= \mathcal{H}^{n-1}\left(B^j \cap \left\{x \in \partial^* \overline{F} \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_\|^{\overline{F}}(x) \neq 0\right\}\right) \\ &= \int_I dr \int_{\partial^* \overline{F}_r \cap (B^j)_r} \chi_{\{\nu_\|^{\overline{F}} \neq 0\}}(x) \frac{1}{|\nu_\|^{\overline{F}}(x)|}\, d\mathcal{H}^{n-2}(x) = 0.\end{aligned}$$

Step 2b: We conclude. Let $E^1 := E$, and let E^2 be set obtained by applying to E the circular symmetrisation with respect to (e_1, e_2). Then, for $j = 3, \dots, n$, we define iteratively the set E^j as the circular symmetral of E^{j-1} with respect to (e_1, e_j). Note that, since $\mathcal{H}^1$-a.e. spherical section of E is a spherical cap, we have $E^n = F_v$. Therefore, thanks to the perimeter inequality (1.11) under circular symmetrisation (see Theorem 1.4), we have

$$P(F_v; \Phi(I \times \mathbb{S}^{n-1})) = P(E^{n-1}; \Phi(I \times \mathbb{S}^{n-1})) = \dots = P(E; \Phi(I \times \mathbb{S}^{n-1})).$$

Moreover, for $j = 3, \dots, n$, we define $I_j := \Phi(I \times \mathbb{S}^{n-1}) \cap \{x_j = 0\} \cap \{x_1 > 0\}$. It is not difficult to check that

$$\Phi(I \times \mathbb{S}^{n-1}) = \Phi_{1j}(I_j \times \mathbb{S}^1) \quad \textit{for } j = 3, \dots, n.$$

Then, applying Lemma 1.5 to F_v and E^{n-1}, we obtain that

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E^{n-1} \cap \Phi_{1\,n-1}(I_{n-1} \times \mathbb{S}^1) : \nu^{E^{n-1}}_{1(n-1)\parallel}(x) = 0\right\}\right) = 0,$$

which, in turns, implies

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E^{n-1} \cap \Phi_{1\,n-1}(I_{n-1} \times \mathbb{S}^1) : \nu^{E^{n-1}}_{\parallel}(x) = 0\right\}\right) = 0.$$

Applying iteratively this argument to $E^{n-2}, \dots, E$, we conclude. □

7. Proof of Theorem 1.2: (ii) $\Longrightarrow$ (i)

Before giving the proof of the implication (ii) $\Longrightarrow$ (i) of Theorem 1.2, it will be convenient to introduce some useful notation. Let v and $\mathcal{I} = \{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\}$ be as in the statement of Theorem 1.2. By assumption, $\mathcal{I}$ is an interval and $\alpha_v \in W^{1,1}_{\mathrm{loc}}(I)$ where, to ease the notation, we set $I := \mathring{\mathcal{I}}$. Let now E be a spherically v-distributed set of finite perimeter. We define the *average direction of* E as the map $d_E : I \to \mathbb{S}^{n-1}$ given by

$$d_E(r) := \begin{cases} \dfrac{1}{\omega_{n-1}(\sin\alpha_v(r))^{n-1} r^{n-1}} \displaystyle\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x), & \text{if } r \in I \cap G_E, \\ e_1 & \text{otherwise in } I, \end{cases} \tag{7.1}$$

where $G_E \subset (0, \infty)$ is the set given by Theorem 3.7. To ease our calculations, it will also be convenient to introduce the *barycentre function* $b_E : I \to \mathbb{R}^n$ *of* E as

$$b_E(r) := \begin{cases} \dfrac{1}{r^{n-1}} \displaystyle\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x), & \text{if } r \in I \cap G_E, \\ e_1 & \text{otherwise in } I. \end{cases}$$

The importance of the functions d_E and b_E is given by the following lemma.

Lemma 7.1. *Let v be as in Theorem 1.2, let $I \subset (0, \infty)$ be an open interval, and let E be a spherically v-distributed set of finite perimeter such that E_r is $\mathcal{H}^{n-1}$-equivalent to a spherical cap for $\mathcal{H}^1$-a.e. $r \in I$. Then,*

$$E \cap \Phi(I \times \mathbb{S}^{n-1}) =_{\mathcal{H}^n} \{x \in \Phi(I \times \mathbb{S}^{n-1}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, d_E(|x|)) < \alpha_v(|x|)\}.$$

Moreover,

$$b_E(r) = \omega_{n-1}(\sin\alpha_v(r))^{n-1} d_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{7.2}$$

Proof. Let us immediately observe that (7.2) follows by the definitions of d_E and b_E. By assumption, for $\mathcal{H}^1$-a.e. $r \in I$, there exists $\omega(r) \in \mathbb{S}^{n-1}$ such that $E_r = \mathbf{B}_{\alpha_v(r)}(r\omega(r))$. We are left to show that

$$\omega(r) = d_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{7.3}$$

Note that for $\mathcal{H}^1$-a.e. $r \in I$ we have $E_r = \mathbf{B}_{\alpha_v(r)}(r\omega(r))$ and $\partial^* E_r = \mathbf{S}_{\alpha_v(r)}(r\omega(r))$. Therefore, for $\mathcal{H}^1$-a.e. $r \in I$

$$\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x) = \int_0^{\alpha_v(r)} d\beta \int_{\mathbf{S}_\beta(r\omega(r))} x\, d\mathcal{H}^{n-2}(x). \tag{7.4}$$

Observe now that, thanks to the symmetry of the geodesic sphere and recalling (3.2), for every $\beta \in (0, \alpha_v(r))$ we have

$$\begin{aligned}\int_{\mathbf{S}_\beta(r\omega(r))} x\, d\mathcal{H}^{n-2}(x) &= \left(\int_{\mathbf{S}_\beta(r\omega(r))} (x \cdot \omega(r))\, d\mathcal{H}^{n-2}(x) \right) \omega(r) \\ &= r\cos\beta\, \mathcal{H}^{n-2}(\mathbf{S}_\beta(r\omega(r)))\, \omega(r) = (n-1)\, \omega_{n-1} r^{n-1} \cos\beta\, (\sin\beta)^{n-2}\, \omega(r).\end{aligned} \tag{7.5}$$

Combining (7.4) and (7.5) we obtain that for $\mathcal{H}^1$-a.e. $r \in I$

$$\begin{aligned}\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x) &= (n-1)\, \omega_{n-1} r^{n-1} \left(\int_0^{\alpha_v(r)} \cos\beta\, (\sin\beta)^{n-2}\, d\beta \right) \omega(r) \\ &= \omega_{n-1} r^{n-1} (\sin\alpha_v(r))^{n-1} \omega(r).\end{aligned}$$

Recalling the definition of d_E, identity (7.3) follows.

□

Remark 7.2. *Let us point out that here we are using the term barycentre in a slightly imprecise way. Indeed, for a given $r \in I \cap G_E$, the geometric barycentre of E_r is given by*

$$\begin{aligned}&\frac{1}{\mathcal{H}^{n-1}(E_r)} \int_{E_r} x\, d\mathcal{H}^{n-1}(x) = \frac{1}{\xi_v(r) r^{n-1}} \int_{E_r} x\, d\mathcal{H}^{n-1}(x) \\ &= \frac{r}{\xi_v(r)} \frac{1}{r^{n-1}} \int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x) = \frac{r}{\xi_v(r)} b_E(r).\end{aligned}$$

Nevertheless, we will still keep this terminology, since b_E turns out to be very useful for our analysis.

We are now ready to prove the implication (ii) $\Longrightarrow$ (i) of Theorem 1.2.

Proof of Theorem 1.2: (ii) $\Longrightarrow$ (i). Suppose (ii) is satisfied, and let $E \in \mathcal{N}(v)$. We are going to show that there exists an orthogonal transformation $R \in SO(n)$ such that $\mathcal{H}^n(E \Delta (RF_v)) = 0$. We now divide the proof into steps.

Step 1: First of all, we observe that

$$\mathcal{H}^{n-1}\left(\left\{ x \in \partial^* E \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_\parallel^E(x) = 0 \right\} \right) = 0.$$

Indeed, since $\alpha_v \in W^{1,1}_{\mathrm{loc}}(I)$, thanks to Proposition 5.3 we have

$$\mathcal{H}^{n-1}\left(\left\{ x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_\parallel^{F_v}(x) = 0 \right\} \right) = 0.$$

Since $E \in \mathcal{N}(v)$, applying Lemma 1.3 the claim follows.

Step 2: We show that $b_E \in W^{1,1}_{\mathrm{loc}}(I; \mathbb{R}^n)$ and

$$b_E'(r) = \frac{1}{r^n} \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} x \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|}\, d\mathcal{H}^{n-2}(x). \tag{7.6}$$

Indeed, let $\psi \in C^1_c(I)$ be arbitrary, and let $i \in \{1, \dots, n\}$. By definition of b_E

$$\begin{aligned}\int_I (b_E)_i(r)\psi'(r)dr &= \int_I \int_{E\cap\partial B(r)} \frac{1}{r^{n-1}} \frac{x_i}{|x|} d\mathcal{H}^{n-1}(x)\psi'(r)dr \\ &= \int_{\Phi(I\times\mathbb{S}^{n-1})} \frac{x_i}{|x|^n}\psi'(|x|)\chi_E(x)\,dx.\end{aligned}$$

Note now that

$$\operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right) = \frac{x_i}{|x|^n}\psi'(|x|).$$

Indeed, recalling (4.3),

$$\begin{aligned}&\operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right) = \psi(|x|)\nabla\left(\frac{x_i}{|x|^n}\right)\cdot\hat{x} + \frac{x_i}{|x|^n}\operatorname{div}(\psi(|x|)\hat{x}) \\ &= \psi(|x|)\left(\frac{e_i}{|x|^n} - \frac{n\,x_i}{|x|^{n+1}}\hat{x}\right)\cdot\hat{x} + \frac{x_i}{|x|^n}\left(\psi'(|x|) + \psi(|x|)\frac{n-1}{|x|}\right) = \frac{x_i}{|x|^n}\psi'(|x|).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_I (b_E)_i(r)\psi'(r)dr = \int_{\Phi(I\times\mathbb{S}^{n-1})} \operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right)\chi_E(x)\,dx \\ &= -\int_{\Phi(I\times\mathbb{S}^{n-1})} \frac{x_i}{|x|^n}\psi(|x|)\hat{x}\cdot dD\chi_E(x) \\ &= \int_{\partial^* E\cap\Phi(I\times\mathbb{S}^{n-1})} \frac{x_i}{|x|^n}\psi(|x|)\,\hat{x}\cdot\nu^E(x)d\mathcal{H}^{n-1}(x).\end{aligned}$$

Thanks to Step 1 we then obtain

$$\begin{aligned}&\int_I (b_E)_i(r)\psi'(r)dr = \int_{\partial^* E\cap\{\nu^E_\parallel\neq 0\}\cap\Phi(I\times\mathbb{S}^{n-1})} \frac{x_i}{|x|^n}\psi(|x|)\,\hat{x}\cdot\nu^E(x)d\mathcal{H}^{n-1}(x) \\ &= \int_I \psi(r)\frac{1}{r^n}\left[\int_{(\partial^* E)_r\cap\{\nu^E_\parallel\neq 0\}} x_i\frac{\hat{x}\cdot\nu^E(x)}{|\nu^E_\parallel(x)|}\,d\mathcal{H}^{n-2}(x)\right]dr,\end{aligned}$$

so that (7.6) follows.

Step 3: We show that

$$b'_E(r) = (n-1)\alpha'_v(r)\,\frac{\cos\alpha_v(r)}{\sin\alpha_v(r)}\,b_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{7.7}$$

Since $E \in \mathcal{N}(v)$, from Theorem 1.1 we know that for $\mathcal{H}^1$-a.e. $r \in I$ the spherical slice E_r is a spherical cap. Then, thanks to Lemma 7.1

$$E_r = \mathbf{B}_{\alpha_v(r)}(rd_E(r)) \quad \text{and} \quad (\partial^* E)_r = \mathbf{S}_{\alpha_v(r)}(rd_E(r)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I.$$

Still thanks to Theorem 1.1, we know that for $\mathcal{H}^1$-a.e. $r \in I$ the functions $x \mapsto \nu^E(x)\cdot\hat{x}$ and $x \mapsto |\nu^E_\parallel|(x)$ are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$, say

$$\nu^E(x)\cdot\hat{x} = a(r) \quad \text{and} \quad |\nu^E_\parallel|(x) = c(r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I,$$

for some measurable functions $a : I \to (-1,1)$ and $c : I \to (0,1]$. Therefore, recalling the definition of d_E together with (7.4)-(7.5) we obtain

$$\begin{aligned} b_E'(r) &= \frac{1}{r^n}\int_{(\partial^* E)_r\cap\{\nu_\|^E\neq 0\}} x\frac{\hat{x}\cdot\nu^E(x)}{|\nu_\|^E(x)|}\, d\mathcal{H}^{n-2}(x) \\ &= \frac{1}{r^n}\frac{a(r)}{c(r)}\int_{\mathbf{S}_{\alpha_v(r)}(rd_E(r))} x\, d\mathcal{H}^{n-2}(x) \\ &= \frac{1}{r^n}\frac{a(r)}{c(r)} r\cos(\alpha_v(r))\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(rd_E(r)))d_E(r) \\ &= \frac{1}{r^{n-1}}\frac{a(r)}{c(r)}\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(rd_E(r)))\cos(\alpha_v(r))d_E(r). \end{aligned} \tag{7.8}$$

Note now that from Step 1 and (4.8) it follows that for $\mathcal{H}^1$-a.e. $r \in I$

$$\begin{aligned} r^{n-1}\xi_v'(r) &= \int_{(\partial^* E)_r\cap\{\nu_\|^E\neq 0\}} \frac{\hat{x}\cdot\nu^E(x)}{|\nu_\|^E(x)|} d\mathcal{H}^{n-2}(x) \\ &= \frac{a(r)}{c(r)}\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(rd_E(r))). \end{aligned}$$

Plugging last identity into (7.8) and using (7.2), we obtain

$$\begin{aligned} b_E'(r) &= \xi_v'(r)\cos(\alpha_v(r))d_E(r) = \xi_v'(r)\cos(\alpha_v(r))\frac{b_E(r)}{\omega_{n-1}(\sin\alpha_v(r))^{n-1}} \\ &= (n-1)\alpha_v'(r)\,\frac{\cos\alpha_v(r)}{\sin\alpha_v(r)}\, b_E(r), \end{aligned}$$

where we used the fact that, thanks to (3.1) and (3.3),

$$\xi_v'(r) = (n-1)\omega_{n-1}(\sin\alpha_v(r))^{n-2}\alpha_v'(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I.$$

Step 4: We conclude. First of all, note that from (7.2) and Step 2 it follows that $d_E \in W^{1,1}_{\mathrm{loc}}(I;\mathbb{S}^{n-1})$. Then, thanks to Step 3, for $\mathcal{H}^1$-a.e. $r \in I$

$$\begin{aligned} \omega_{n-1}d_E'(r) &= \frac{d}{dr}\left[\frac{b_E(r)}{(\sin\alpha_v(r))^{n-1}}\right] = \frac{b_E'(r)}{(\sin\alpha_v(r))^{n-1}} + b_E(r)\frac{d}{dr}\left[\frac{1}{(\sin\alpha_v(r))^{n-1}}\right] \\ &= (n-1)\alpha_v'(r)\,\frac{\cos\alpha_v(r)}{(\sin\alpha_v(r))^n}\, b_E(r) + b_E(r)\left[-\frac{n-1}{(\sin\alpha_v(r))^n}(\cos\alpha_v(r))\alpha_v'(r)\right] = 0, \end{aligned}$$

for $\mathcal{H}^1$-a.e. $r \in I$. This shows that d_E is $\mathcal{H}^1$-a.e. constant in I. Therefore, $E\cap\Phi(I\times\mathbb{S}^{n-1})$ can be obtained by applying an orthogonal transformation to $F_v\cap\Phi(I\times\mathbb{S}^{n-1})$. □

8. Proof of Theorem 1.2: (i) $\Longrightarrow$ (ii)

We start by showing that the fact that $\{0 < \alpha^\wedge \leq \alpha^\vee < \pi\}$ is an interval is a necessary condition for rigidity.

Proposition 8.1. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2)*, such that F_v is a set of finite perimeter and finite volume, and let α_v be defined by* (1.3)*. Suppose that the set $\{0 < \alpha^\wedge \leq \alpha^\vee < \pi\}$ is not an interval. That is, suppose that there exists $\overline{r} \in \{\alpha^\wedge = 0\}\cup\{\alpha^\vee = \pi\}$ such that*

$$(0,\overline{r})\cap\{0 < \alpha^\wedge \leq \alpha^\vee < \pi\} \neq \emptyset \qquad \text{and} \qquad (\overline{r},\infty)\cap\{0 < \alpha^\wedge \leq \alpha^\vee < \pi\} \neq \emptyset.$$

Then, rigidity fails. More precisely, setting $E_1 := F_v\cap B(\overline{r})$ and $E_2 := F_v\setminus B(\overline{r})$, we have

$$E_1\cup(RE_2) \in \mathcal{N}(v) \qquad \text{for every } R \in O(n).$$

Before giving the proof of Proposition 8.1 we need the following lemma.

Lemma 8.2. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2)*, such that F_v is a set of finite perimeter and finite volume. Let α_v be defined by* (1.3)*, and let $\overline{r} > 0$. Then,*

$$(\partial^* F_v)_{\overline{r}} =_{\mathcal{H}^{n-1}} \mathbf{B}_{\alpha_v^{\vee}(\overline{r})}(\overline{r}e_1) \setminus \mathbf{B}_{\alpha_v^{\wedge}(\overline{r})}(\overline{r}e_1).$$

Proof. We divide the proof in two steps.

Step 1: We show that

$$(\partial^* F_v)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^{\vee}(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^{\wedge}(\overline{r})}(\overline{r}e_1).$$

To this aim, it will be enough to show that

$$\alpha_v^{\wedge}(\overline{r}) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \leq \alpha_v^{\vee}(\overline{r}) \qquad \text{for every } x \in (\partial^* F_v)_{\overline{r}}. \tag{8.1}$$

Let us first prove that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \leq \alpha_v^{\vee}(\overline{r}) \qquad \text{for every } x \in (\partial^* F_v)_{\overline{r}} \tag{8.2}$$

Note that (8.2) is trivial if $\alpha_v^{\vee}(\overline{r}) = \pi$. For this reason, we will assume $\alpha_v^{\vee}(\overline{r}) < \pi$. Note now that (8.2) follows if we prove that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^{\vee}(\overline{r}) \quad \Longrightarrow \quad x \in F_v^{(0)}. \tag{8.3}$$

Let now $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^{\vee}(\overline{r}) + \delta.$$

Let now $\overline{\rho} > 0$ be so small that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \hat{x}) < \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

By triangle inequality for the geodesic distance we have, in particular, that

$$\alpha_v^{\vee}(\overline{r}) + \delta = \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, \hat{y}) + \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \frac{\delta}{2} + \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1),$$

so that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^{\vee}(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}). \tag{8.4}$$

Thanks to the inequality above, by definition of F_v we have

$$F_v \cap B(x, \overline{\rho}) \subset \left\{ y \in \mathbb{R}^n : \alpha_v^{\vee}(\overline{r}) + \frac{\delta}{2} < \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \alpha_v(|y|) \right\} \cap B(x, \overline{\rho}).$$

Therefore, for every $\rho \in (0, \overline{\rho})$

$$\begin{aligned}
\mathcal{H}^n(F_v \cap B(x,\rho)) &= \int_{\overline{r}-\rho}^{\overline{r}+\rho} \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr \\
&\leq \int_{\overline{r}-\rho}^{\overline{r}+\rho} \chi_{\{\alpha_v > \alpha_v^{\vee}(\overline{r}) + \delta/2\}}(r) \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr \\
&= \int_{(\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^{\vee}(\overline{r}) + \delta/2\}} \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr.
\end{aligned}$$

Note now that, for ρ small enough, there exists $C = C(\overline{r}) > 0$ such that

$$B(x,\rho) \cap \partial B(r) \subset \mathbf{B}_{C\rho}(r\hat{x}) \qquad \text{for every } r \in (\overline{r} - \rho, \overline{r} + \rho).$$

Therefore,

$$\begin{aligned}\mathcal{H}^n(F_v\cap B(x,\rho)) &\le \int_{(\overline{r}-\rho,\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\vee(\overline{r})+\delta/2\}} \mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\,dr\\ &= (n-1)\omega_{n-1}\int_{(\overline{r}-\rho,\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\vee(\overline{r})+\delta/2\}} r^{n-1}\int_0^{C\rho}(\sin\tau)^{n-2}\,d\tau\,dr\\ &\le (n-1)\omega_{n-1}\int_{(\overline{r}-\rho,\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\vee(\overline{r})+\delta/2\}} r^{n-1}\int_0^{C\rho}\tau^{n-2}\,d\tau\,dr\\ &= \omega_{n-1}C^{n-1}(\overline{r}+\overline{\rho})^{n-1}\rho^{n-1}\mathcal{H}^1((\overline{r}-\rho,\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\vee(\overline{r})+\delta/2\}).\end{aligned}$$

Thus, recalling the definition of $\alpha_v^\vee(\overline{r})$,

$$\begin{aligned}&\lim_{\rho\to 0^+}\frac{\mathcal{H}^n(F_v\cap B(x,\rho))}{\omega_n\rho^n}\\ &\le \frac{\omega_{n-1}C^{n-1}}{\omega_n}(\overline{r}+\overline{\rho})^{n-1}\lim_{\rho\to0^+}\frac{\mathcal{H}^1((\overline{r}-\rho,\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\vee(\overline{r})+\delta/2\})}{\rho}=0,\end{aligned}$$

which gives (8.3) and, in turn, (8.2). By similar arguments, one can prove that

$$x\in\partial B(\overline{r})\quad\text{and}\quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)<\alpha_v^\wedge(\overline{r})\quad\Longrightarrow\quad x\in F_v^{(1)},$$

which implies that

$$\alpha_v^\wedge(\overline{r})\le \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)\qquad\text{for every } x\in(\partial^*F_v)_{\overline{r}}.$$

The above inequality, together with (8.2), shows (8.1).

Step 2: We conclude. Thanks to Corollary 5.2,

$$\begin{aligned}&\mathcal{H}^{n-1}((\partial^*F_v)_{\overline{r}}) = \mathcal{H}^{n-1}(\partial^*F_v\cap\partial B(\overline{r})) = P(F_v;\partial B(\overline{r})) = \overline{r}^{n-1}(\xi_v^\vee(\overline{r})-\xi_v^\wedge(\overline{r}))\\ &= v^\vee(\overline{r})-v^\wedge(\overline{r}) = \mathcal{H}^{n-1}(\overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)})-\mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1))\\ &= \mathcal{H}^{n-1}\left(\overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)}\setminus\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1)\right)\end{aligned}$$

Since, by Step 1,

$$(\partial^*F_v)_{\overline{r}}\subset\overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)}\setminus\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1),$$

we have

$$(\partial^*F_v)_{\overline{r}} =_{\mathcal{H}^{n-1}} \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)}\setminus\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1) =_{\mathcal{H}^{n-1}} \mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)\setminus\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1).$$

□

We can now give the proof of Proposition 8.1.

Proof of Proposition 8.1. Note that, since $B(\overline{r})$ is open and $E\cap B(\overline{r}) = F_v\cap B(\overline{r})$, we have

$$E^{(t)}\cap B(\overline{r}) = (E\cap B(\overline{r}))^{(t)} = (F_v\cap B(\overline{r}))^{(t)} = F_v^{(t)}\cap B(\overline{r})\qquad\text{for every } t\in[0,1].$$

From this, it follows that

$$\partial^*E\cap B(\overline{r}) = \partial^*F_v\cap B(\overline{r}). \tag{8.5}$$

Similarly, we obtain

$$\partial^*E\setminus\overline{B(\overline{r})} = \partial^*(RF_v)\setminus\overline{B(\overline{r})} = (R\,\partial^*F_v)\setminus(R\overline{B(\overline{r})}) = R(\partial^*F_v\setminus\overline{B(\overline{r})}). \tag{8.6}$$

Thus, thanks to (8.5) and (8.6)

$$\begin{aligned}P(E) &= \mathcal{H}^{n-1}(\partial^*E\cap B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*E\setminus\overline{B(\overline{r})})\\ &= \mathcal{H}^{n-1}(\partial^*F_v\cap B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))+\mathcal{H}^{n-1}\left(R(\partial^*F_v\setminus\overline{B(\overline{r})})\right)\\ &= \mathcal{H}^{n-1}(\partial^*F_v\cap B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*F_v\setminus\overline{B(\overline{r})}).\end{aligned}$$

Therefore, in order to conclude the proof we only need to show that

$$\mathcal{H}^{n-1}(\partial^* E \cap B(\overline{r})) = \mathcal{H}^{n-1}(\partial^* F_v \cap B(\overline{r})). \tag{8.7}$$

Without any loss of generality, we will assume that

$$\alpha_v^\vee(\overline{r}) = \operatorname{ap}\lim(f,(0,\overline{r}),\overline{r}) \,, \qquad 0 = \alpha_v^\wedge(\overline{r}) = \operatorname{ap}\lim(f,(\overline{r},\infty),\overline{r}) \,. \tag{8.8}$$

Let now E_1, E_2, and R be as in the statement. We divide the proof of (8.7) into steps.

Step 1: We show that

$$(\partial^* E)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \cup \{R(\overline{r}e_1)\}.$$

To this aim, it will be enough to prove that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \le \alpha_v^\vee(\overline{r}) \qquad \text{for every } x \in (\partial^* E)_{\overline{r}}. \tag{8.9}$$

If $\alpha_v^\vee(\overline{r}) = \pi$ inequality (8.9) is obvious, so we will assume that $\alpha_v^\vee(\overline{r}) < \pi$.

Step 1a: We show that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r}) \quad \Longrightarrow \quad x \in E_1^{(0)}.$$

Indeed, let $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^\vee(\overline{r}) + \delta.$$

By repeating the argument used to show (8.4), we can choose $\overline{\rho} > 0$ so small that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

By definition of E_1, we then have

$$\begin{aligned} &E_1 \cap B(x,\overline{\rho}) = F_v \cap B(\overline{r}) \cap B(x,\overline{\rho}) \\ &\subset \left\{ y \in \mathbb{R}^n : |y| < \overline{r} \text{ and } \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} < \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \alpha_v(|y|) \right\} \cap B(x,\overline{\rho}). \end{aligned}$$

Therefore, for every $\rho \in (0, \overline{\rho})$, by repeating the calculations done in Step 1 of Lemma 8.2, we obtain

$$\begin{aligned} &\lim_{\rho \to 0^+} \frac{1}{\omega_n \rho^n} \mathcal{H}^n(E_1 \cap B(x,\rho)) \\ &= \lim_{\rho \to 0^+} \frac{1}{\omega_n \rho^n} \int_{\overline{r}-\rho}^{\overline{r}} \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr \\ &\le \frac{\omega_{n-1} C^{n-1}}{\omega_n} (\overline{r} + \overline{\rho})^{n-1} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1((\overline{r}-\rho, \overline{r}) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\})}{\rho} = 0, \end{aligned}$$

where we used (8.8).

Step 1b: We show that

$$\partial B(\overline{r}) \setminus \{R(\overline{r}e_1)\} \subset (RE_2)^{(0)}.$$

Indeed, let $x \in \partial B(\overline{r})$, and suppose that $\eta := \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, Re_1) > 0$. We are going to prove that $x \in (RE_2)^{(0)}$. By repeating the argument used to show (8.4), we can choose $\overline{\rho} > 0$ so small that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, Re_1) > \frac{\eta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

Then,

$$\begin{aligned} &(RE_2) \cap B(x,\overline{\rho}) = \Big(R(F_v \setminus \overline{B(\overline{r})}) \Big) \cap B(x,\overline{\rho}) \\ &\subset_{\mathcal{H}^n} \left\{ y \in \mathbb{R}^n : |y| > \overline{r} \text{ and } \frac{\eta}{2} < \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, Re_1) < \alpha_v(|y|) \right\} \cap B(x,\overline{\rho}). \end{aligned}$$

For ρ small enough, there exists $C = C(\overline{r}) > 0$ such that

$$B(x,\rho) \cap \partial B(r) \subset \mathbf{B}_{C\rho}(r\hat{x}) \qquad \text{for every } r \in (\overline{r}-\rho, \overline{r}+\rho).$$

Therefore, for every $\rho \in (0, \overline{\rho})$,

$$\begin{aligned}
\mathcal{H}^n((RE_2) \cap B(x,\rho)) &\leq \int_{(\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\}} \mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\, dr \\
&= (n-1)\omega_{n-1} \int_{(\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\}} r^{n-1} \int_0^{C\rho} (\sin\tau)^{n-2}\, d\tau\, dr \\
&= \omega_{n-1} C^{n-1} (\overline{r}+\overline{\rho})^{n-1} \rho^{n-1} \mathcal{H}^1((\overline{r},\overline{r}+\rho)\cap\{\alpha_v > \eta/2\}).
\end{aligned}$$

From this, thanks to (8.8), we obtain

$$\begin{aligned}
&\lim_{\rho\to 0^+} \frac{\mathcal{H}^n((RE_2)\cap B(x,\rho))}{\omega_n \rho^n} \\
&\leq \frac{\omega_{n-1}C^{n-1}}{\omega_n} (\overline{r}+\overline{\rho})^{n-1} \lim_{\rho\to 0^+} \frac{\mathcal{H}^1((\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\})}{\rho} = 0.
\end{aligned}$$

Step 1c: We conclude the proof of Step 1. By definition of E, from Step 1a and Step 1b it follows that

$$\{x \in \partial B(\overline{r}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r})\} \setminus \{Re_1\} \subset E_1^{(0)} \cap (RE_2)^{(0)} = E^{(0)}.$$

Therefore,

$$\begin{aligned}
(\partial^* E)_r &\subset \partial B(\overline{r}) \setminus \big(\{x \in \partial B(\overline{r}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r})\} \setminus \{Re_1\}\big) \\
&= \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \cup \{Re_1\}.
\end{aligned}$$

Step 2: We show (8.7), concluding the proof. Thanks to Step 1 and Lemma 8.2 we have

$$\begin{aligned}
P(E; \partial B(\overline{r})) &= \mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r})) = \mathcal{H}^{n-1}((\partial^* E)_{\overline{r}}) \leq \mathcal{H}^{n-1}\left(\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)\right) \\
&= \mathcal{H}^{n-1}(\partial^* F_v \cap \partial B(\overline{r})) = P(F_v; \partial B(\overline{r})) \leq P(E; \partial B(\overline{r})),
\end{aligned}$$

where we also used (1.4) with $B = \{\overline{r}\}$. ☐

We now show that, if the jump part $D^j\alpha_v$ of $D\alpha_v$ is non zero, rigidity fails.

Proposition 8.3. *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume, and let α_v be defined by* (1.3). *Suppose that α_v has a jump at some point $\overline{r} > 0$. Then, rigidity fails. More precisely, setting $E_1 := F_v \cap B(\overline{r})$ and $E_2 := F_v \setminus B(\overline{r})$, we have*

$$E_1 \cup (RE_2) \in \mathcal{N}(v),$$

for every $R \in O(n)$ such that

$$0 < \mathrm{dist}_{\mathbb{S}^{n-1}}(Re_1, e_1) < \lambda(\alpha_v^\vee(\overline{r}) - \alpha_v^\wedge(\overline{r})) \quad \textit{for some } \lambda \in (0,1). \tag{8.10}$$

Proof. Let $R \in O(n)$, $\lambda \in (0,1)$, and $E \in \mathbb{R}^n$ be as in the statement, and set $\omega := Re_1$. Arguing as in the proof of Proposition 8.1 we have:

$$P(E) = \mathcal{H}^{n-1}(\partial^* F_v \cap B(\overline{r})) + \mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r})) + \mathcal{H}^{n-1}(\partial^* F_v \setminus \overline{B(\overline{r})}).$$

Therefore, in order to conclude the proof we only need to show that

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r})) = \mathcal{H}^{n-1}(\partial^* F_v \cap \partial B(\overline{r})). \tag{8.11}$$

Without any loss of generality, we will assume that

$$\alpha_v^\vee(\overline{r}) = \mathrm{ap}\lim(f, (0,\overline{r}), \overline{r}), \qquad \alpha_v^\wedge(\overline{r}) = \mathrm{ap}\lim(f, (\overline{r},\infty), \overline{r}). \tag{8.12}$$

We now proceed by steps.

Step 1: We show that

$$(\partial^* E)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}\omega). \tag{8.13}$$

To show (8.13), it is enough to prove that for every $x \in (\partial^* E)_{\overline{r}}$ we have

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \leq \alpha_v^\vee(\overline{r}) \qquad \text{for every } x \in (\partial^* E)_{\overline{r}}, \tag{8.14}$$

and

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, \omega) \geq \alpha_v^\wedge(\overline{r}) \qquad \text{for every } x \in (\partial^* E)_{\overline{r}}. \tag{8.15}$$

We will only show (8.14), since (8.15) can be obtained in a similar way. Note that (8.14) is automatically satisfied if $\alpha_v^\vee(\overline{r}) = \pi$, so we will assume $\alpha_v^\vee(\overline{r}) < \pi$.

By arguing as in Step 1a of the proof of Proposition 8.1 we obtain

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r}) \quad \Longrightarrow \quad x \in E_1^{(0)}. \tag{8.16}$$

Let us now prove that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r}) \quad \Longrightarrow \quad x \in (R\, E_2)^{(0)}. \tag{8.17}$$

Let $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^\vee(\overline{r}) + \delta.$$

Thanks to the argument we used to show (8.4), we can choose $\overline{\rho} > 0$ so small that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

Therefore, for every $y \in B(x, \overline{\rho})$ we have

$$\begin{aligned}\alpha_v^\vee(\overline{r}) + \frac{\delta}{2} &< \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) + \mathrm{dist}_{\mathbb{S}^{n-1}}(\omega, e_1)\\ &< \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) + \lambda(\alpha_v^\vee(\overline{r}) - \alpha_v^\wedge(\overline{r})).\end{aligned}$$

Since $\overline{r}$ is a jump point for α_v, we have $\alpha_v^\vee(\overline{r}) > \alpha_v^\wedge(\overline{r})$, and the above inequality implies that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) > (1-\lambda)\alpha_v^\vee(\overline{r}) + \lambda\alpha_v^\wedge(\overline{r}) + \frac{\delta}{2} > (1-\lambda)\alpha_v^\wedge(\overline{r}) + \lambda\alpha_v^\wedge(\overline{r}) + \frac{\delta}{2} = \alpha_v^\wedge(\overline{r}) + \frac{\delta}{2},$$

for every $y \in B(x, \overline{\rho})$. Then, by definition of E_2,

$$\begin{aligned}(RE_2) \cap B(x, \overline{\rho}) &= \Big(R(F_v \setminus \overline{B(\overline{r})})\Big) \cap B(x, \overline{\rho})\\ &\subset_{\mathcal{H}^n} \left\{ y \in \mathbb{R}^n : |y| > \overline{r} \text{ and } \alpha_v^\wedge(\overline{r}) + \frac{\delta}{2} < \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) < \alpha_v(|y|) \right\} \cap B(x, \overline{\rho}).\end{aligned}$$

As already observed in the previous proofs, for ρ small enough there exists $C = C(\overline{r}) > 0$ such that

$$B(x, \rho) \cap \partial B(r) \subset \mathbf{B}_{C\rho}(r\hat{x}) \qquad \text{for every } r \in (\overline{r} - \rho, \overline{r} + \rho).$$

Therefore, for every $\rho \in (0, \overline{\rho})$ sufficiently small

$$\begin{aligned}\mathcal{H}^n((RE_2) \cap B(x, \rho)) &\leq \int_{(\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\wedge(\overline{r}) + \delta/2\}} \mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\, dr\\ &= (n-1)\omega_{n-1} \int_{(\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\wedge(\overline{r}) + \delta/2\}} r^{n-1} \int_0^{C\rho} (\sin\tau)^{n-2}\, d\tau\, dr\\ &= \omega_{n-1} C^{n-1} (\overline{r} + \overline{\rho})^{n-1} \rho^{n-1} \mathcal{H}^1((\overline{r}, \overline{r} + \rho) \cap \{\alpha_v > \alpha_v^\wedge(\overline{r}) + \delta/2\}).\end{aligned}$$

From this, thanks to (8.12), we obtain

$$\begin{aligned}&\lim_{\rho\to 0^+}\frac{\mathcal{H}^n((RE_2)\cap B(x,\rho))}{\omega_n\rho^n}\\&\le\frac{\omega_{n-1}C^{n-1}}{\omega_n}(\overline{r}+\overline{\rho})^{n-1}\lim_{\rho\to 0^+}\frac{\mathcal{H}^1((\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\alpha_v^\wedge(\overline{r})+\delta/2\})}{\rho}=0,\end{aligned}$$

which shows (8.17). This, together with (8.16), implies (8.14). As already mentioned, (8.15) can be proved in a similar way, and therefore (8.13) follows.

Step 2: We conclude. From (8.10) it follows that

$$\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}\omega)\subset\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1).$$

Therefore, thanks to (8.13) and Lemma 8.2

$$\begin{aligned}&P(E;\partial B(\overline{r}))=\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))=\mathcal{H}^{n-1}((\partial^*E)_{\overline{r}})\le\mathcal{H}^{n-1}\left(\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)\setminus\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}\omega)\right)\\&=v^\vee(\overline{r})-v^\wedge(\overline{r})=P(F_v;\partial B(\overline{r}))\le P(E;\partial B(\overline{r})),\end{aligned}$$

where we also used (1.4) with $B=\{\overline{r}\}$. Then, (8.11) follows from the last chain of inequalities. □

We conclude this section showing that, if $D^c\alpha_v\neq 0$, rigidity fails.

Proposition 8.4. *Let $v:(0,\infty)\to[0,\infty)$ be a measurable function satisfying* (1.2) *such that F_v is a set of finite perimeter and finite volume, and let α_v be defined by* (1.3)*. Suppose that $D^c\alpha_v\neq 0$. Then, rigidity fails.*

Proof. We are going to construct a spherically v-distributed set $E\in\mathcal{N}(v)$ that cannot be obtained by applying a single orthogonal transformation to F_v (see (8.20) below).

First of all, let us note that it is not restrictive to assume that α_v is purely Cantorian. Indeed, by (2.4) one can decompose α_v into

$$\alpha_v=\alpha_v^a+\alpha_v^j+\alpha_v^c, \tag{8.18}$$

where $\alpha_v^a\in W^{1,1}_{\mathrm{loc}}(0,\infty)$, α_v^j is a purely jump function, and α_v^c is purely Cantorian. Thanks to (8.18), in the general case when $\alpha_v\neq\alpha_v^c$, the proof can be repeated by applying our argument just to the Cantorian part α_v^c of α_v. Therefore, from now on we will assume that

$$D\alpha_v=D^c\alpha_v.$$

Thanks to Proposition 8.1, we can also assume that $\{0<\alpha_v^\wedge\le\alpha_v^\vee<\pi\}$ is an interval (otherwise there is nothing to prove, since rigidity fails). Moreover, since α_v is continuous, there exist $a,b>0$, with $a<b$, such that $I:=(a,b)\subset\subset\{0<\alpha_v^\wedge\le\alpha_v^\vee<\pi\}$ and

$$0<\alpha_v(r)<\pi\qquad\text{for every } r\in I. \tag{8.19}$$

Since $D^c\alpha_v\neq 0$, it is not restrictive to assume $|D^c\alpha_v|(I)>0$. For each $\gamma\in(-\pi,\pi)$, we define $R_\gamma\in O(n)$ in the following way:

$$R_\gamma\begin{pmatrix}x_1\\x_2\\x_3\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}x_1\cos\gamma-x_2\sin\gamma\\x_1\sin\gamma+x_2\cos\gamma\\x_3\\\vdots\\x_n\end{pmatrix}.$$

That is, R_γ is a counterclockwise rotation of the angle γ in the plane (x_1, x_2). Let now fix $\lambda \in (0,1)$, and define $\beta : (0,\infty) \to (-\pi, \pi)$ as

$$\beta(r) := \begin{cases} 0 & \text{if } r \in (0,a), \\ \lambda(\alpha_v(r) - \alpha_v(a)) & \text{if } r \in [a,b], \\ \lambda(\alpha_v(b) - \alpha_v(a)) & \text{if } r \in (b,\infty). \end{cases}$$

We set

$$E := \{x \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, R_{\beta(|x|)} e_1) < \alpha_v^\vee(|x|)\}. \tag{8.20}$$

Clearly, E cannot be obtained by applying a single orthogonal transformation to F_v. Let us show that $E \in \mathcal{N}(v)$, so that rigidity fails. We proceed by steps.

Step 1: We construct a sequence of functions $v^k : I \to [0,\infty)$ satisfying the following properties:

(a) $\lim_{k\to\infty} \alpha_{v^k}(r) = \alpha_v(r)$ for $\mathcal{H}^1$-a.e. $r \in I$;

(b) $D\xi_{v^k} = D^j \xi_{v^k}$ for every $k \in \mathbb{N}$;

(c) $\lim_{k\to\infty} P(F_{v^k}; \Phi(I \times \mathbb{S}^{n-1})) = P(F_v; \Phi(I \times \mathbb{S}^{n-1}))$.

First of all note that, by (3.5) and by the chain rule in BV (see, [2, Theorem 3.96]), it follows that ξ_v is purely Cantorian, where ξ_v is given by (3.3). Moreover, from (2.5) and from the fact that ξ_v is continuous, we have

$$|D\xi_v|(I) = \sup\left\{\sum_{i=1}^{N-1} |\xi_v(r_{i+1}) - \xi_v(r_i)| : a < r_1 < r_2 < \ldots < r_N < b\right\},$$

where the supremum runs over $N \in \mathbb{N}$ and over all $r_1, \ldots, r_N$ with $a < r_1 < r_2 < \ldots < r_N < b$. Therefore, for every $k \in \mathbb{N}$ there exist $N_k \in \mathbb{N}$ and $r_1^k, \ldots, r_N^k$ with $a < r_1^k < r_2^k < \ldots < r_N^k < b$ such that

$$|D\xi_v|(I) \le \sum_{i=1}^{N_k - 1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| + \frac{1}{k}$$

and

$$|r_{i+1}^k - r_i^k| < \frac{1}{k} \qquad \text{for every } i = 1, \ldots, N_k - 1.$$

Without any loss of generality, we can assume that the partitions are increasing in k. That is, we will assume that

$$\{r_1^k, \ldots, r_{N_k}^k\} \subset \{r_1^{k+1}, \ldots, r_{N_{k+1}}^{k+1}\} \quad \text{for every } k \in \mathbb{N}.$$

Define now, for every $k \in \mathbb{N}$,

$$\xi_v^k(r) := \sum_{i=0}^{N_k} \xi_v(r_i^k) \chi_{[r_i^k, r_{i+1}^k)}(r), \tag{8.21}$$

where we set $r_0^k := a$ and $r_{N_k+1}^k := b$. Let us now set

$$v^k(r) := \xi_v^k(r)/r^{n-1} \qquad \text{for every } r \in I \text{ and for every } k \in \mathbb{N},$$

and note that, by definition, $\xi_v^k = \xi_{v^k}$. Since ξ_v is continuous, we have that

$$\lim_{k\to\infty} \xi_v^k(r) = \xi_v(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{8.22}$$

Recalling (3.5) and (3.6), last relation implies property (a). Moreover, from (8.21) we have (b).

Let us now show (c). Thanks to (8.19) and (8.22), we have

$$\lim_{k\to\infty} p_{F_v^k}(r) = p_{F_v}(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{8.23}$$

Moreover,

$$\begin{aligned} |D\xi_v^k|(I) &= \sum_{i=0}^{N_k} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| \\ &= |\xi_v(r_1^k) - \xi_v(a)| + |\xi_v(b) - \xi_v(r_{N_k}^k)| + \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)|. \end{aligned} \tag{8.24}$$

Since

$$|D\xi_v|(I) - \frac{1}{k} \le \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| \le |D\xi_v|(I),$$

using (8.24) and the fact that ξ_v is continuous we obtain

$$|D\xi_v|(I) = \lim_{k\to\infty} \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| = \lim_{k\to\infty} |D\xi_v^k|(I). \tag{8.25}$$

Thanks to [2, Theorem 3.23], up to subsequences ξ_v^k weakly* converges in $BV(I)$ to ξ_v. Since, in addition, (8.25) holds true, we can apply [2, Proposition 1.80] to the sequence of measures $\{|D\xi_v^k|\}_{k\in\mathbb{N}}$. Therefore, recalling that $D\xi_v^k = D^s\xi_v^k$ and $D\xi_v = D^s\xi_v$, we have

$$\lim_{k\to\infty} \int_I r^n d|D^s\xi_v^k|(r) = \lim_{k\to\infty} \int_I r^n d|D\xi_v^k|(r) = \int_I r^n d|D\xi_v|(r) = \int_I r^n d|D^s\xi_v|(r).$$

Then, from Corollary 5.2

$$\begin{aligned} \lim_{k\to\infty} P(F_{v^k}; \Phi(I\times\mathbb{S}^{n-1})) &= \lim_{k\to\infty} \left(\int_I p_{F_{v^k}}(r)\,dr + \int_I r^{n-1} d|D^s\xi_v^k|(r) \right) \\ &= \left(\int_I p_{F_v}(r)\,dr + \int_I r^{n-1} d|D^s\xi_v|(r) \right) = P(F_v; \Phi(I\times\mathbb{S}^{n-1})), \end{aligned}$$

where we also used (8.23).

Step 2: For each $k \in \mathbb{N}$, we construct a spherically v^k-distributed set E^k such that

$$P(E^k; \Phi(I\times\mathbb{S}^{n-1})) = P(F_{v^k}; \Phi(I\times\mathbb{S}^{n-1})).$$

From (3.5) and (3.6) it follows that $\alpha_{v^k} = \mathcal{F}^{-1}(\xi_v^k) \in BV(I)$, and

$$\alpha_{v^k}(r) = \sum_{i=0}^{N_k} \alpha_v(r_i^k)\chi_{[r_i^k, r_{i+1}^k)}(r). \tag{8.26}$$

Therefore, for each $k \in \mathbb{N}$ we have that $D\alpha_{v^k} = D^j\alpha_{v^k}$, and the jump set of α_{v^k} is a finite set. More precisely,

$$D\alpha_{v^k} = \sum_{i=1}^{N_k} (\alpha_v(r_i^k) - \alpha_v(r_{i-1}^k))\delta_{r_i^k},$$

where δ_r denotes the Dirac delta measure concentrated at r. Let $\lambda \in (0,1)$ be fixed, and define the set $E_1^k \subset \Phi(I\times\mathbb{S}^{n-1})$ as

$$E_1^k := \left[F_{v^k} \cap (B(r_1^k) \setminus \overline{B(a)}) \right] \cup \left[R_{\lambda(\alpha_v(r_1^k)-\alpha_v(a))}(F_{v^k} \cap (B(b) \setminus B(r_1^k))) \right].$$

Thanks to Proposition 8.3, we have that

$$P(E_1^k; \Phi(I\times\mathbb{S}^{n-1})) = P(F_{v^k}; \Phi(I\times\mathbb{S}^{n-1})).$$

Define now $E_2^k \subset \Phi(I \times \mathbb{S}^{n-1})$ as

$$E_2^k := (E_1^k \cap B(r_2^k)) \cup \left[R_{\lambda(\alpha_v(r_2^k)-\alpha_v(r_1^k))}(E_1^k \setminus B(r_2^k)) \right].$$

Applying again Proposition 8.3, we have

$$P(E_2^k; \Phi(I \times \mathbb{S}^{n-1})) = P(E_1^k; \Phi(I \times \mathbb{S}^{n-1})) = P(F_{v^k}; \Phi(I \times \mathbb{S}^{n-1})).$$

Note that, since R_γ is associative with respect to γ (that is, we have $R_{\gamma_1} R_{\gamma_2} = R_{\gamma_1+\gamma_1}$), we can write E_2^k as

$$\begin{aligned} E_2^k = &\left[F_{v^k} \cap (B(r_1^k) \setminus \overline{B(a)}) \right] \cup \left[R_{\lambda(\alpha_v(r_1^k)-\alpha_v(a))}(F_{v^k} \cap (B(r_2^k) \setminus B(r_1^k))) \right] \\ &\cup \left[R_{\lambda(\alpha_v(r_2^k)-\alpha_v(a))}(F_{v^k} \cap (B(b) \setminus B(r_2^k))) \right]. \end{aligned}$$

Iterating this procedure N_k times, we obtain that

$$P(E^k; \Phi(I \times \mathbb{S}^{n-1})) = P(F_{v^k}; \Phi(I \times \mathbb{S}^{n-1})),$$

where

$$E_k := E_{N_k}^k = \{x \in \Phi(I \times \mathbb{S}^{n-1}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, R_{\lambda(\alpha_{v^k}(|x|)-\alpha_{v^k}(a))}e_1) < \alpha_{v^k}(|x|)\}. \tag{8.27}$$

Step 3: We show that $E^k \longrightarrow \widehat{E}$ in $\Phi(I \times \mathbb{S}^{n-1})$, for some spherically v-distributed set $\widehat{E}$ such that

$$P(\widehat{E}; \Phi(I \times \mathbb{S}^{n-1})) = P(F_v; \Phi(I \times \mathbb{S}^{n-1})).$$

From (8.26) and (8.22) it follows that

$$\lim_{k\to\infty} \alpha_{v^k}(r) = \alpha_v(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I.$$

Therefore, from (8.27) we have $E^k \longrightarrow \widehat{E}$ (in $(\Phi(I \times \mathbb{S}^{n-1}))$), where $\widehat{E}$ is the spherically v-distributed set in $\Phi(I \times \mathbb{S}^{n-1})$ given by

$$\widehat{E} := \{x \in \Phi(I \times \mathbb{S}^{n-1}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, R_{\lambda(\alpha_v(|x|)-\alpha_v(a))}e_1) < \alpha_v(|x|)\}. \tag{8.28}$$

Then, by the lower semicontinuity of the perimeter with respect to the L^1 convergence (see, for instance, [22, Proposition 12.15]):

$$\begin{aligned} &P(\widehat{E}; \Phi(I \times \mathbb{S}^{n-1})) \leq \lim_{k\to\infty} P(E^k; \Phi(I \times \mathbb{S}^{n-1})) \\ &\lim_{k\to\infty} P(F_{v^k}; \Phi(I \times \mathbb{S}^{n-1})) = P(F_v; \Phi(I \times \mathbb{S}^{n-1})) \\ &\leq P(\widehat{E}; \Phi(I \times \mathbb{S}^{n-1})), \end{aligned}$$

where we also used (1.4).

Step 4: We conclude. Let E be given by (8.20). Then, E is spherically v-distributed and satisfies

$$E =_{\mathcal{H}^n} (F_v \cap (B(a))) \cup \left[\widehat{E} \cap (B(b) \setminus B(a)) \right] \cup \left[R_{\lambda(\alpha_v(b)-\alpha_v(a))}(F_v \setminus (B(b))) \right],$$

where $\widehat{E}$ is defined in (8.28). By repeating the arguments used in the proof of Proposition 8.1, and using the fact that $\Phi(I \times \mathbb{S}^{n-1}) = B(b) \setminus \overline{B(a)}$, one can see that

$$\begin{aligned} P(E) &= P(E; B(a)) + P(E; \partial B(a)) + P(E; B(b) \setminus \overline{B(a)}) \\ &+ P(E; \partial B(b)) + P(E; \mathbb{R}^n \setminus \overline{B(b)}) \\ &= P(F_v; B(a)) + P(E; \partial B(a)) + P(\widehat{E}; B(b) \setminus \overline{B(a)}) \\ &+ P(E; \partial B(b)) + P(F_v; \mathbb{R}^n \setminus \overline{B(b)}) \\ &= P(F_v; B(a)) + P(E; \partial B(a)) + P(F_v; B(b) \setminus \overline{B(a)}) \\ &+ P(E; \partial B(b)) + P(F_v; \mathbb{R}^n \setminus \overline{B(b)}), \end{aligned}$$

where we also used Step 3 and the invariance of the perimeter under orthogonal transformations. Since α_v is continuous, an argument similar to the one used to prove (8.13) shows that

$$P(E; \partial B(a)) = P(E; \partial B(b)) = 0.$$

Therefore,

$$P(E) = P(F_v; B(a)) + P(F_v; B(b) \setminus \overline{B(a)}) + P(F_v; \mathbb{R}^n \setminus \overline{B(b)}) = P(F_v).$$

□

We can now give the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 1.2.

Proof of Theorem 1.2: (i) $\Longrightarrow$ (ii). To show the implication, it suffices to combine Proposition 8.1, Proposition 8.3, and Proposition 8.4. □

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References

[1] L. Ambrosio, V. Caselles, S. Masnou, and J. Morel, *Connected components of sets of finite perimeter and applications to image processing*, J. Eur. Math. Soc. (JEMS), 3 (2001), pp. 39–92.
[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[3] M. Barchiesi, F. Cagnetti, and N. Fusco, *Stability of the Steiner symmetrization of convex sets*, J. Eur. Math. Soc., 15 (2013), pp. 1245–1278.
[4] T. Bartsch, T. Weth, M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math., 96 (2005), pp. 1–18.
[5] A. Baernstein, *A unified approach to symmetrization*, Symposia Matematica, 35 (1995), pp. 47–91.
[6] I , Birindelli, F Leoni, F Pacella, *Symmetry and spectral properties for viscosity solutions of fully nonlinear equations*, J. Math. Pures Appl. 107(9), (2017), pp. 409–428.
[7] V. Bögelein, F. Duzaar, N. Fusco, *A quantitative isoperimetric inequality on the sphere*, Adv. Calc. Var. (3), 10 (2017), pp. 223–265.
[8] J. E. Brothers and W. P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math., 384 (1988), pp. 153–179.
[9] F. Cagnetti, M. Colombo, G. De Philippis, and F. Maggi, *Rigidity of equality cases in Steiner's perimeter inequality*, Anal. PDE, 7 (2014), pp. 1535–1593.
[10] ———, *Essential connectedness and the rigidity problem for Gaussian symmetrization*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 395–439.
[11] G. M. Capriani, *The Steiner rearrangement in any codimension*, Calc. Var. Partial Differential Equations, 49 (2014), pp. 517–548.

[12] M. Chlebík, A. Cianchi, and N. Fusco, *The perimeter inequality under Steiner symmetrization: cases of equality*, Ann. of Math. (2), 162 (2005), pp. 525–555.
[13] A. Cianchi and N. Fusco, *Functions of bounded variation and rearrangements*, Arch. Ration. Mech. Anal., 165 (2002), pp. 1–40.
[14] ———, *Steiner symmetric extremals in Pólya-Szegö type inequalities*, Adv. Math., 203 (2006), pp. 673–728.
[15] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli, *On the isoperimetric deficit in Gauss space*, Amer. J. Math., 133 (2011), pp. 131–186.
[16] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8), 5 (1958), pp. 33–44.
[17] ———, *Selected papers*, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013. [Author name on title page: Ennio Giorgi], Edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo, Reprint of the 2006 edition.
[18] M. Giaquinta, G. Modica, and J. Soucek, *Cartesian currents in the Calculus of Variations. I. Cartesian currents*, vol. 37 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998.
[19] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68 (1979), 209–243.
[20] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, vol. 1150 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985.
[21] ———, *On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems*, Arch. Rational Mech. Anal., 94 (1986), pp. 227–243.
[22] F. Maggi, *Sets of finite perimeter and geometric variational problems*, vol. 135 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2012. An introduction to Geometric Measure Theory.
[23] I. McGillivray, *An isoperimetric inequality in the plane with a log-convex density*, Ric. Mat., 67 (2018), No. 2, pp. 817–874.
[24] F. Morgan, S. Howe, N. Harman, *Steiner and Schwarz symmetrization in warped products and fiber bundles with density*, Rev. Mat. Iberoam. 27 (2011), no. 3, 909–918.
[25] F. Morgan, A. Pratelli, *Existence of isoperimetric regions in* $\mathbb{R}^n$ *with density* Ann. Global Anal. Geom. 43 (2013), no. 4, 331–365.
[26] M. Perugini, *Rigidity of Steiner's inequality for the anisotropic perimeter* (preprint).
[27] G. Pólya, *Sur la symétrisation circulaire.* (French) C. R. Acad. Sci. Paris 230, (1950), pp 25–27.
[28] E Schmidt, *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionszahl*, Math. Z. 49 (1943/44), pp 1–109.
[29] J. Serrin, *A symmetry property in potential theory*, Arch. Rational Mech. Anal., 43 (1971), pp 304–318.
[30] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 3, Centre for Mathematical Analysis, Canberra, 1983.
[31] D. Smets, M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. Partial Differ. Equ., 18 (2003), pp 57–75.
[32] A. I. Vol′ pert, *Spaces* BV *and quasilinear equations*, Mat. Sb. (N.S.), 73 (115) (1967), pp. 255–302.

Department of Mathematics, University of Sussex, Pevensey 2, BN1 9QH, Brighton, UK
Email address: f.cagnetti@sussex.ac.uk

Department of Mathematics, University of Sussex, Pevensey 2, BN1 9QH, Brighton, UK
Email address: m.perugini@sussex.ac.uk

Technische Universität München, Zentrum Mathematik - M15, Boltzmannstrasse 3, 85747 Garching, Germany
Email address: dominik.stoeger@ma.tum.de