LOSS OF DOUBLE-INTEGRAL CHARACTER DURING RELAXATION

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Abstract. We provide explicit examples to show that the relaxation of functionals

$$L^p(\Omega) \ni u \mapsto \int_\Omega \int_\Omega W(u(x), u(y))\, dx\, dy,$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set, $1 < p < \infty$ and $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a suitable integrand, is in general not of double-integral form. This proves an up to now open statement in [Pedregal, *Rev. Mat. Complut.* **29** (2016)] and [Bellido & Mora-Corral, *SIAM J. Math. Anal.* **50** (2018)]. The arguments are inspired by recent results regarding the structure of (approximate) nonlocal inclusions, in particular, their invariance under diagonalization of the constraining set. For a complementary viewpoint, we also discuss a class of double-integral functionals for which relaxation is in fact structure preserving and the relaxed integrands arise from separate convexification.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a non-empty, open and bounded set and $1 < p < \infty$. Moreover, let $W : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function satisfying p-growth, i.e., $W(\xi, \zeta) \le C(|\xi|^p + |\zeta|^p + 1)$ for all $(\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m$ with a constant $C > 0$. For any such W, we define a double-integral functional

$$I_W(u) = \int_\Omega \int_\Omega W(u(x), u(y))\, dx\, dy \tag{1.1}$$

for $u \in L^p(\Omega; \mathbb{R}^m)$. Without loss of generality (see e.g. [18]), one may assume W to be symmetric, that is, $W(\xi, \zeta) = W(\zeta, \xi)$, for every $(\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m$.

Nonlocal functionals of this type and their inhomogeneous versions with explicit dependence of W on $x, y \in \Omega$ have recently become of increasing interest in the literature. Besides their nonlocal character, which gives rise to interesting mathematical questions that require the development of new techniques [2, 4, 17, 19], this can also be attributed to their relevance in various modern modeling approaches, e.g. in image processing [5, 10, 12], in machine learning [1, 22, 24], in the theory of phase transitions [8, 21], or in continuum mechanics through the theory of peridynamics [3, 9, 14, 16, 23] and crystal plasticity [15].

Under the additional assumption that W is p-coercive, i.e., there are constants $c, C > 0$ such that

$$W(\xi, \zeta) \ge c(|\xi|^p + |\zeta|^p) - C \qquad \text{for all } (\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m,$$

the existence of minimizers of I_W is guaranteed by the direct method in the calculus of variations, if I_W is L^p-weakly lower semicontinuous, or equivalently, if W is separately convex [4, 17, 19]. In situations when W fails to have this property, minimizers of I_W do in general not exist due to oscillation effects. A common strategy to capture the asymptotic behavior of minimizing sequences of I_W is resorting to a related variational problem, called the relaxed problem, which involves the L^p-weak lower semicontinuous envelope of I_W, i.e., for $u \in L^p(\Omega; \mathbb{R}^m)$,

$$I_W^{\mathrm{rlx}}(u) := \inf\{\liminf_{j\to\infty} I_W(u_j) : u_j \rightharpoonup u \text{ in } L^p(\Omega; \mathbb{R}^m)\}. \tag{1.2}$$

The major challenge in relaxation theory lies in finding alternative representations of $I_W^{\rm rlx}$, ideally via closed formulas. Contrary to the single-integral case, where a body of works has emerged over the last decades, see e.g. [6, 7] and the references therein, relaxation in the nonlocal setting is still largely unsolved. In the following, we give some background and outline briefly the latest developments related to this problem.

The first paper to present a characterization of L^p-weak lower semicontinuity of I_W in the scalar case $m = 1$ goes back to Pedregal [18] in the late 1990s. Separate convexity of W as a necessary and sufficient condition was identified almost ten years later in [4], and generalized to the case of vector-valued fields, meaning for $m \geq 1$, in [17]. More recent results, in particular on the inhomogeneous setting, can be found in [2, 19].

Motivated by these findings, a natural first guess for the relaxed functional associated with I_W seems a double-integral with the separately convex hull $W^{\rm sc}$ of W as integrand. However, there are one-dimensional counterexamples to disprove this conjecture, see e.g. [4, Example 3.1] or [2, Example 7.2] for integral functionals involving suitably chosen integrands with eight or six wells, respectively. Here, as a consequence of Corollary 4.7 and Corollary 4.8, which both provide different necessary conditions for the relaxation via separate convexification of W, we can generate a whole class of counterexamples. The probably simplest one for $m = 1$ is when W is a four-well integrand with minima in the corners of a square, cf. (5.1).

In [19], Pedregal claims even more, namely that $I_W^{\rm rlx}$ may not be representable as a double-integral at all. His reasoning is based on a monotonicity argument along the lines of a basic observation for single-integrals. However, as Bellido & Mora-Coral point out in [2, Section 7], this argument is in general not valid in the nonlocal context, see Section 4.1 for more details.

In this paper, we give proofs based on two different approaches to confirm that Pedregal's statement is indeed correct, see Propositions 5.1 an 5.4. Both counterexamples in the proofs feature double-integrands $W : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ of distance-type (see (5.1) and (5.9)), and take inspiration from recent insights on the properties of nonlocal supremal functionals in [13]; cf. also (1.3) below. Especially the operation of diagonalization of sets in the sense of Definition 2.3, applied here to the zero sublevel sets of W, and its interplay with approximate nonlocal inclusions plays a central role. Indeed, the latter are invariant under diagonalization as a consequence of Theorem 3.1.

In the related nonlocal supremal setting, which was mentioned briefly above already and corresponds formally to the limiting case $p \to \infty$, one considers in place of I_W as in (1.1), functionals

$$L^\infty(\Omega; \mathbb{R}^m) \ni u \mapsto \operatorname{esssup}_{(x,y)\in\Omega\times\Omega} Z(u(x), u(y)) \tag{1.3}$$

with a suitable symmetric supremand $Z : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. The problem of relaxing (1.3) has been settled recently; it is shown in [13] that the relaxation of (1.3) is structure preserving, meaning that it is again of supremal form, and that the relaxed supremand corresponds to the separately level convexification of the diagonalization of Z, cf. (4.1). Here, in contrast, the challenging open question remains: What kind of representation for $I_W^{\rm rlx}$ in (1.2) can be expected if double-integrals are out of the picture? For first steps towards a better understanding, we refer to the Young measure relaxation result in [2, Theorem 6.1], as well as to Proposition 6.1, where we contribute a partial result by giving a closed formula for the relaxation of a specific class of double-integrals.

This article is organized as follows. After introducing notation and collecting some auxiliary results in Section 2, Section 3 is concerned with the asymptotic behavior of approximate nonlocal inclusions; in particular, we provide a characterization of Young measures generated by sequences of nonlocal fields of the form $(u(x), u(y))$ for $(x, y) \in \Omega \times \Omega$ subject to approximate pointwise constraints, see Theorem 3.3. Even though these are technical tools for the remaining paper, the results are also interesting in their own right. In Section 4, we address the issue of order relations and comparison arguments for double-integrals as in (1.1), and deduce conditions on W that are necessary for the identity $I_W^{\rm rlx} = I_{W^{\rm sc}}$. The heart piece of this paper, namely the two counterexamples to structure preservation during relaxation, are presented, along with their proofs, in Section 5. For a complementary viewpoint, we close in Section 6 by discussing functionals with double-integrands in the form of distances to Cartesian sets and extended-valued indicators; the relaxations in both cases give rise to the intuitively expected double-integrals with separately convexified integrands.

2. Notation and preliminaries

To make the paper self-contained, we fix notations and collect some well-known results that will be used later on.

We denote the Euclidean norm of a vector $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d$ by $|\eta| = (\sum_{i=1}^m \eta_i^2)^{\frac{1}{2}}$, and use the notation $\|\cdot\|$ for a generic norm on $\mathbb{R}^m \times \mathbb{R}^m$ (without explicit mention, we often idenitfy $\mathbb{R}^m \times \mathbb{R}^m$ with $\mathbb{R}^{2m}$); specific choices of norms in the following include the 1-norm $\|(\xi,\zeta)\|_1 := |\xi| + |\zeta|$, the Euclidean norm $\|(\xi,\zeta)\|_2 = \sqrt{|\xi|^2 + |\zeta|^2}$ or the maximum norm $\|(\xi,\zeta)\|_\infty = \max\{|\xi|, |\zeta|\}$ for $(\xi,\zeta) \in \mathbb{R}^m \times \mathbb{R}^m$. Further, $B_r(\xi,\zeta)$ represents the closed ball in $\mathbb{R}^m \times \mathbb{R}^m$ centered at (ξ,ζ) of radius $r > 0$, and for the distance of a point $(\xi,\zeta) \in \mathbb{R}^m \times \mathbb{R}^m$ to a non-empty, closed set $E \subset \mathbb{R}^m \times \mathbb{R}^m$, we write

$$\operatorname{dist}((\xi,\zeta), E) = \min_{(\alpha,\beta)\in E} \|(\xi,\zeta) - (\alpha,\beta)\|; \tag{2.1}$$

if relevant, the use of a specific norm is indicated by super- and subscript indices, e.g. $B_3^1(0,0) = \{(\xi,\zeta) \in \mathbb{R}^m \times \mathbb{R}^m : \|(\xi,\zeta)\|_1 \leq 3\}$ or $\operatorname{dist}_\infty(\cdot, E) = \min_{(\alpha,\beta)\in E} \|\cdot - (\alpha,\beta)\|_\infty$. Besides, the generalized closed interval $[\xi,\zeta]$ for $\xi,\zeta \in \mathbb{R}^m$ is the set $\{\lambda\xi + (1-\lambda)\zeta \in \mathbb{R}^m : \lambda \in [0,1]\}$.

For the complement of a set $A \subset \mathbb{R}^d$, we write $A^c = \mathbb{R}^d \setminus A$, and let $\mathbb{1}_A$ be the indicator function of A, i.e.

$$\mathbb{1}_A(\eta) := \begin{cases} 1 & \text{if } \eta \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta \in \mathbb{R}^d.$$

To refer to the minimum of a function $f : \mathbb{R}^d \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ (if existent), we usually use the short-hand notation $\min f$ rather than $\min_{\eta\in\mathbb{R}^d} f(\eta)$.

For any probability measure $\mu \in \mathcal{P}r(\mathbb{R}^d)$,

$$[\mu] := \langle \mu, \mathrm{id} \rangle = \int_{\mathbb{R}^d} \eta \, d\mu(\eta)$$

stands for its barycenter. The product measure of $\nu, \mu \in \mathcal{P}r(\mathbb{R}^d)$ is denoted by $\nu \otimes \mu$, and for the Lebesgue measure of a Lebesgue measurable set $U \subset \mathbb{R}^l$, we write $\mathcal{L}^l(U)$, or simply $|U|$. We employ standard notation for L^p-spaces with $p \in [1,\infty]$; particularly, our way to symbolize weak and weak* convergence of a sequence $(u_j)_j \subset L^p(U;\mathbb{R}^d)$ with $U \subset \mathbb{R}^l$ bounded and open to a function $u \in L^p(U;\mathbb{R}^d)$ is $u_j \rightharpoonup u$ in $L^p(U;\mathbb{R}^d)$ if $p \in [1,\infty)$ and $u_j \rightharpoonup^* u$ in $L^\infty(U;\mathbb{R}^d)$ if $p = \infty$. Moreover, $S^\infty(U;\mathbb{R}^d)$ refers to the set of simple functions on U with values in $\mathbb{R}^d$.

Convexity notions including separate convexity, separate level convexity and the related envelopes are a recurring theme in this paper. We briefly collect here some basics, referring the reader to [13, Sections 2, 3 and 4] for more properties, relations and characterizations of the following definitions.

A set $E \subset \mathbb{R}^m \times \mathbb{R}^m$ is called separately convex (with vectorial components), if for every $t \in (0,1)$ and every $(\xi_1,\zeta_1), (\xi_2,\zeta_2) \in E$ such that $\xi_1 = \xi_2$ or $\zeta_1 = \zeta_2$ it holds that

$$t(\xi_1,\zeta_1) + (1-t)(\xi_2,\zeta_2) \in E.$$

The smallest separately convex set in $\mathbb{R}^m \times \mathbb{R}^m$ that contains E is called the separately convex hull of E and denoted by E^{sc}. Our notation for the convex hull of a convex set $A \subset \mathbb{R}^d$ is A^{co}.

Definition 2.1. *A function $f : \mathbb{R}^d \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is level convex if all level sets of f, that is,*

$$L_c(f) := \{\eta \in \mathbb{R}^d : f(\eta) \leq c\} \quad \text{with } c \in \mathbb{R},$$

are convex sets.

We say that a function $W : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_\infty$ *is separately convex (with vectorial components) if for every* $\xi \in \mathbb{R}^m$*, the functions* $W(\cdot, \xi)$ *and* $W(\xi, \cdot)$ *are convex, and we call* W *separately level convex (with vectorial components) if the sets* $L_c(W) = \{(\xi, \eta) \in \mathbb{R}^m \times \mathbb{R}^m : W(\xi, \eta) \leq c\}$ *are separately convex for all* $c \in \mathbb{R}$.

Moreover, $W^{\rm sc} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_\infty$ ($W^{\rm slc} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_\infty$) stands for the separately (level) convex envelope of W, that is, the largest separately (level) convex function below W. It is well-known that if W is of distance type, meaning that $W(\xi, \zeta) = {\rm dist}^p((\xi, \zeta), E)$ for $(\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m$ with some non-empty and closed $E \subset \mathbb{R}^m \times \mathbb{R}^m$ and $p \geq 1$ (cf. (2.1)), then the (generalized) convex envelopes of W are

$$W^{\rm co}(\xi, \zeta) = {\rm dist}^p((\xi, \zeta), E^{\rm co}), \tag{2.2}$$

and

$$W^{\rm sc}(\xi, \zeta) = W^{\rm slc}(\xi, \zeta) = {\rm dist}^p((\xi, \zeta), E^{\rm sc}) \tag{2.3}$$

for $(\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m$.

The following lemma is a corollary of a classical result in convex analysis, also known as zig-zag lemma (see e.g. [7, Lemma 20.2]). For the readers' convenience, we give here a simple explicit construction.

Lemma 2.2. *Let* $A \subset \mathbb{R}^m$ *and* $v \in S^\infty(\Omega; \mathbb{R}^m)$ *be a simple function with image in* $A^{\rm co}$*. Then there exist a sequence* $(v_i)_i \subset S^\infty(\Omega; \mathbb{R}^m)$ *such that* $v_i \in A$ *a.e. in* Ω *for any* $i \in \mathbb{N}$ *and* $v_i \overset{*}{\rightharpoonup} v$ *in* $L^\infty(\Omega; \mathbb{R}^m)$.

Proof. Let $\xi^{(j)} \in A^{\rm co}$ and $\{\Omega^{(j)}\}_j$ be a decomposition of Ω into measurable subsets such that

$$v = \sum_{j=1}^n \xi^{(j)} \mathbb{1}_{\Omega^{(j)}}.$$

By Caratheodory's theorem, $\xi^{(j)} \in A^{\rm co}$ is the convex combination of $m+1$ elements of A, that is, $\xi^{(j)} = \sum_{l=1}^{m+1} \lambda_l \xi_l^{(j)}$ with $\xi_l^{(j)} \in A$ and $\lambda_l \in [0,1]$ with $\sum_{l=1}^{m+1} \lambda_l = 1$. Let $\Omega_{l,i}^{(j)}$ for $i \in \mathbb{N}$ be measurable subsets of $\Omega^{(j)}$ such that

$$\mathbb{1}_{\Omega_{l,i}^{(j)}} \overset{*}{\rightharpoonup} \lambda_l \mathbb{1}_{\Omega^{(j)}} \quad \text{ in } L^\infty(\Omega) \text{ as } i \to \infty.$$

This can be achieved for instance by choosing $\Omega_{l,i}^{(j)} = \Omega^{(j)} \cap \bigcup_{z \in \mathbb{Z}^m} \frac{1}{i} z + \frac{1}{i}[0, \lambda_l^{\frac{1}{m}}]^m$ for $j = 1, \dots, n$, $l = 1, \dots, m+1$ and $i \in \mathbb{N}$. Then,

$$v_i^{(j)} := \sum_{l=1}^{m+1} \xi_j^{(l)} \mathbb{1}_{\Omega_{l,i}^{(j)}} \overset{*}{\rightharpoonup} v \quad \text{ in } L^\infty(\Omega; \mathbb{R}^m) \text{ as } i \to \infty.$$

With these definitions, the sequence $(v_i)_i$ given by

$$v_i := \sum_{j=1}^n v_i^{(j)} \mathbb{1}_{\Omega^{(j)}} \qquad \text{for } i \in \mathbb{N}$$

has all the desired properties. □

Next, we recall some terminology related to the diagonalization of a symmetric set $E \subset \mathbb{R}^m \times \mathbb{R}^m$ from [13, (4.1)]. Such a set E is symmetric if $(\xi, \zeta) \in E$ if and only if $(\zeta, \xi) \in E$.

Definition 2.3. *Let* $E \subset \mathbb{R}^m \times \mathbb{R}^m$ *be symmetric, then*

$$\widehat{E} = \{(\xi, \zeta) \in E : (\xi, \xi), (\zeta, \zeta) \in E\} \subset \mathbb{R}^m \times \mathbb{R}^m \tag{2.4}$$

is called the diagonalization of E*. We also use the alternative notation* $E^\wedge$.

If $K \subset \mathbb{R}^m \times \mathbb{R}^m$ is symmetric and compact, then also $\widehat{K}$ is compact. Note that for any symmetric $E \subset \mathbb{R}^m \times \mathbb{R}^m$,

$$\widehat{E} = \bigcup_{P \in \mathcal{P}_E} P; \tag{2.5}$$

here $\mathcal{P}_E$ stands for the set of maximal Cartesian subsets of E. A set $P \subset E$ is a maximal Cartesian subset of E if $P = A \times A$ with $A \subset \mathbb{R}^m$ and if for any $B \subset \mathbb{R}^m$ with $A \subset B$ and $B \times B \subset K$ it holds that $B = A$. As a simple consequence of the above definitions, it holds that

$$\mathcal{P}_E = \mathcal{P}_{\widehat{E}} \tag{2.6}$$

Finally, we associate with any suitable $W : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_\infty$ the double-integral functional $I_W : L^p(\Omega; \mathbb{R}^m) \to \mathbb{R}_\infty$ with

$$I_W(u) := \int_\Omega \int_\Omega W(u(x), u(y))\, dx\, dy \tag{2.7}$$

for $u \in L^p(\Omega; \mathbb{R}^m)$. To keep notations light, we dispense with highlighting explicitly the dependence on p and Ω, which will always be clear from the context. Unless mentioned otherwise, Ω is a non-empty, open and bounded subset of $\mathbb{R}^n$ and $p > 1$.

The nonlocal field $v_w \in L^p(\Omega \times \Omega; \mathbb{R}^m \times \mathbb{R}^m)$ corresponding to a function $w \in L^p(\Omega; \mathbb{R}^m)$ with $p \geq 1$ is defined by

$$v_w(x, y) := (w(x), w(y)) \quad \text{for a.e. } (x, y) \in \Omega \times \Omega. \tag{2.8}$$

3. Approximate nonlocal inclusions

Throughout this section, let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $E \subset \mathbb{R}^m \times \mathbb{R}^m$ symmetric.

As in [13], all essentially bounded solutions $u : \Omega \to \mathbb{R}^m$ to the (exact) nonlocal inclusion

$$(u(x), u(y)) \in E \quad \text{for a.e. } (x, y) \in \Omega \times \Omega \tag{3.1}$$

are collected in the set $\mathcal{A}_E$, recalling (2.8), one can write

$$\mathcal{A}_E = \{u \in L^\infty(\Omega; \mathbb{R}^m) : v_u \in E \text{ a.e. in } \Omega \times \Omega\},$$

and we introduce

$$\mathcal{A}_E^\infty := \{u \in L^\infty(\Omega; \mathbb{R}^m) : u_j \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega; \mathbb{R}^m) \text{ with } (u_j)_j \subset \mathcal{A}_E\} \tag{3.2}$$

to describe the limiting behavior of sequences in $\mathcal{A}_E$. Upon relaxing the strict requirement of the exact nonlocal inclusion in (3.1), we obtain an approximate version whose asymptotics is encoded in

$$\begin{aligned} \mathcal{B}_E^\infty := \{u \in L^\infty(\Omega; \mathbb{R}^m) : u_j \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega; \mathbb{R}^m) \text{ with } (u_j)_j \subset L^\infty(\Omega; \mathbb{R}^m) \text{ such that} \\ \operatorname{dist}(v_{u_j}, E) \to 0 \text{ in measure as } j \to \infty\}. \end{aligned} \tag{3.3}$$

Clearly, $\mathcal{A}_E^\infty \subset \mathcal{B}_E^\infty$. Under the additional assumption of compactness, we can show equality of these two sets and provide a new characterization, valid in any dimension.

Theorem 3.1. *Let $K \subset \mathbb{R}^m \times \mathbb{R}^m$ be symmetric and compact. Then,*

$$\mathcal{A}_K^\infty = \mathcal{B}_K^\infty = \{u \in L^\infty(\Omega; \mathbb{R}^m) : u \in A^{\mathrm{co}} \text{ a.e. in } \Omega \text{ with } A \times A \in \mathcal{P}_K\}, \tag{3.4}$$

where $\mathcal{P}_K$ is the set of maximal Cartesian subsets of K.

Remark 3.2. *a) The sets $\mathcal{A}_K^\infty$ and $\mathcal{B}_K^\infty$ remain unchanged under diagonalization of K, that is, $\mathcal{A}_K^\infty = \mathcal{A}_{\widehat{K}}^\infty$ and $\mathcal{B}_K^\infty = \mathcal{B}_{\widehat{K}}^\infty$. Since $\mathcal{P}_K = \mathcal{P}_{\widehat{K}}$ by* (2.6)*, this is apparent from the representation* (3.4)*.*

Even though based on a different argumentation, the diagonalization invariance of $\mathcal{A}_K^\infty$ has been observed before in [13]*. Indeed,* [13, Proposition 5.1] *yields that $\mathcal{A}_K = \mathcal{A}_{\widehat{K}}$, which implies $\mathcal{A}_K^\infty = \mathcal{A}_{\widehat{K}}^\infty$ in view of* (3.2)*.*

b) Note that the assumption of closedness of the set K in Theorem 3.1 cannot be dropped. To see this, we refer to [13, Remark 5.2] *for a simple example of a symmetric, non-closed set $E \subset \mathbb{R}^m \times \mathbb{R}^m$ and a set $\Omega \subset \mathbb{R}^m$ such that $\emptyset = \mathcal{A}_{\widehat{E}} \neq \mathcal{A}_E$. This implies in particular that $\mathcal{A}_E^\infty \neq \mathcal{A}_{\widehat{E}}^\infty$, and hence,* (3.4) *cannot be true.*

c) An equivalent way of writing the characterization formula in (3.4) *is*

$$\mathcal{A}_K^\infty = \mathcal{B}_K^\infty = \bigcup_{A\times A\in\mathcal{P}_K} \mathcal{A}_{A^{\rm co}\times A^{\rm co}} = \mathcal{A}_{\bigcup_{A\times A\in\mathcal{P}_K} A^{\rm co}\times A^{\rm co}},$$

cf. also (6.9)*. Under an additional assumption on K, which is always satisfied for $m=1$, it was shown in* [13, Theorem 1.1] *that $\mathcal{A}_K^\infty = \mathcal{A}_{\widehat{K}^{\rm sc}}$, where $\widehat{K}^{\rm sc}$ is the separately convex hull of $\widehat{K}$.*

We postpone the proof of Theorem 3.1 to the end of the section, since it is a consequence of the characterization of Young measures generated by sequences subject to approximate nonlocal constraints, which we address next.

Following the notation of [13, Section 2.2], we consider the sets of parameterized measures

$$\begin{aligned}\mathcal{Y}_E := \{\Lambda \in L^\infty_w(\Omega\times\Omega;\mathcal{P}r(\mathbb{R}^m\times\mathbb{R}^m)) : \Lambda_{(x,y)} = \nu_x\otimes\nu_y \text{ with } \nu\in L^\infty_w(\Omega;\mathcal{P}r(\mathbb{R}^m))\\ \text{and } \operatorname{supp}\Lambda_{(x,y)}\subset E \text{ for a.e. } (x,y)\in\Omega\times\Omega\},\end{aligned} \tag{3.5}$$

$$\mathcal{Y}_E^\infty := \{\Lambda \in L^\infty_w(\Omega\times\Omega;\mathcal{P}r(\mathbb{R}^m\times\mathbb{R}^m)) : v_{u_j} \stackrel{YM}{\longrightarrow} \Lambda \text{ with } (u_j)_j\subset\mathcal{A}_E\},$$

and

$$\begin{aligned}\widetilde{\mathcal{Y}}_E^\infty := \{\Lambda \in L^\infty_w(\Omega\times\Omega;\mathcal{P}r(\mathbb{R}^m\times\mathbb{R}^m)) : v_{u_j} \stackrel{YM}{\longrightarrow} \Lambda \text{ with } (u_j)_j\subset L^\infty(\Omega;\mathbb{R}^m) \text{ such that}\\ \operatorname{dist}(v_{u_j},E)\to 0 \text{ in measure as } j\to\infty\};\end{aligned} \tag{3.6}$$

here, $L^\infty_w(U;\mathcal{P}r(\mathbb{R}^d))$ denotes the space of weakly measurable functions defined on an open set $U\subset\mathbb{R}^l$ with values in the space of probability measures on $\mathbb{R}^d$. By $v_j \stackrel{YM}{\longrightarrow} \mu$, we mean that a sequence $(v_j)_j\subset L^\infty(U;\mathbb{R}^d)$ generates the Young measure $\mu\in L^\infty_w(U;\mathcal{P}r(\mathbb{R}^d))$, see e.g. [11, 18, 20] for more details.

It was shown in [13, (5.21) and Theorem 5.11] that for symmetric and compact $K\subset\mathbb{R}^m\times\mathbb{R}^m$,

$$\bigcup_{P\in\mathcal{P}_K}\mathcal{Y}_P = \mathcal{Y}_K^\infty \subset \widetilde{\mathcal{Y}}_K^\infty = \mathcal{Y}_K. \tag{3.7}$$

In light of Proposition 3.5, all four sets in (3.7) have to coincide, which gives rise to the following theorem.

Theorem 3.3. *Let $K\subset\mathbb{R}^m\times\mathbb{R}^m$ be compact and symmetric. Then*

$$\widetilde{\mathcal{Y}}_K^\infty = \mathcal{Y}_K^\infty = \mathcal{Y}_K = \bigcup_{P\in\mathcal{P}_K}\mathcal{Y}_P. \tag{3.8}$$

Due to (2.6), all the sets in (3.8) are invariant under diagonalization of K. In particular, $\mathcal{Y}_K = \mathcal{Y}_{\widehat{K}}$.

The next lemma serves as the main tool for the proof of Proposition 3.5.

Lemma 3.4. *Let $\nu,\mu\in\mathcal{P}r(\mathbb{R}^m)$ and $\Lambda=\nu\otimes\mu\in\mathcal{P}r(\mathbb{R}^m\times\mathbb{R}^m)$. If $(\xi,\zeta),(\zeta,\xi),(\alpha,\beta),(\beta,\alpha)\in\operatorname{supp}\Lambda$, then*

$$\{\xi,\zeta,\alpha,\beta\}\times\{\xi,\zeta,\alpha,\beta\}\subset\operatorname{supp}\Lambda.$$

Proof. It suffices to prove that one element of $\{\xi,\zeta,\alpha,\beta\}\times\{\xi,\zeta,\alpha,\beta\}$ different from (ξ,ζ), (ζ,ξ), (α,β) and (β,α) is contained in $\operatorname{supp}\Lambda$, say (ξ,α). For the other elements, the argumentation is analogous.

Recalling the definition of the support of Λ, that is,

$$\operatorname{supp}\Lambda = \{(\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m : \Lambda(U)>0 \text{ for any open neighborhood } U \text{ of } (\xi,\zeta)\},$$

let U be an open neighborhood of (ξ,α). Within U one can find another neighborhood of the form $A\times B\subset U$ with $A,B\subset\mathbb{R}^m$ open such that $\xi\in A$ and $\alpha\in B$. From

$$\Lambda(U)\geq\Lambda(A\times B)=(\nu\otimes\mu)(A\times B)=\nu(A)\mu(B)>0, \tag{3.9}$$

we conclude that $(\xi,\alpha)\in\operatorname{supp}\Lambda$, as desired. For the last estimate in (3.9), we have used that $A\times(B-\alpha+\zeta)$ is an open set containing (ξ,ζ), and thus,

$$0<\Lambda(A\times(B-\alpha+\zeta))=\nu(A)\mu(B-\alpha+\zeta)$$

by the assumption that $(\xi,\zeta)\in\operatorname{supp}\Lambda$. This implies in particular that $\nu(A)>0$. Similarly, we show that $\mu(B)>0$. □

Proposition 3.5. *Let $E\subset\mathbb{R}^m\times\mathbb{R}^m$ be symmetric. Then,*

$$\mathcal{Y}_E=\bigcup_{P\in\mathcal{P}_E}\mathcal{Y}_P.$$

Proof. We prove the first two identities of

$$\mathcal{Y}_E=\mathcal{Y}_{\widehat{E}}=\bigcup_{P\in\mathcal{P}_{\widehat{E}}}\mathcal{Y}_P=\bigcup_{P\in\mathcal{P}_E}\mathcal{Y}_P. \tag{3.10}$$

in separate steps; the last one is immediate, since $\mathcal{P}_E=\mathcal{P}_{\widehat{E}}$ by (2.6).

Step 1: Invariance under diagonalization. Since $\widehat{E}\subset E$, we only need to prove that $\mathcal{Y}_E\subset\mathcal{Y}_{\widehat{E}}$. Let $\Lambda\in\mathcal{Y}_E$, and assume to the contrary that there exists a measurable set $N\subset\Omega\times\Omega$ with positive $\mathcal{L}^{2n}$-measure such that

$$\operatorname{supp}\Lambda_{(x,y)}\cap E\setminus\widehat{E}\neq\emptyset$$

for all $(x,y)\in N$. Due to the symmetry of E and $\widehat{E}$, we may take N to be symmetric.

Now fix $(x,y)\in N$ and let $(\xi,\zeta)\in\operatorname{supp}\Lambda_{(x,y)}$ with $(\xi,\zeta)\notin\widehat{E}$. Then also $(\zeta,\xi)\in\operatorname{supp}\Lambda_{(x,y)}$, and we infer from Lemma 3.4 that

$$\{\xi,\zeta\}\times\{\xi,\zeta\}\subset\operatorname{supp}\Lambda_{(x,y)}\subset E.$$

Hence, $(\xi,\zeta)\in\widehat{E}$ in view of Definition 2.3, which is a contradiction.

Step 2: Alternative representation of $\mathcal{Y}_{\widehat{E}}$. By definition, any $P\in\mathcal{P}_{\widehat{E}}$ is contained in $\widehat{E}$; hence, $\bigcup_{P\in\mathcal{P}_E}\mathcal{Y}_P\subset\mathcal{Y}_{\widehat{E}}$ is immediate. For the reverse inclusion, let $\Lambda\in\mathcal{Y}_{\widehat{E}}$. To show that $\Lambda\in\mathcal{Y}_P$ for some $P\in\mathcal{P}_{\widehat{E}}$, we argue again by contradiction, assuming that there is a measurable set $N\subset\Omega\times\Omega$ with $\mathcal{L}^{2n}(N)>0$, as well as a maximal Cartesian set $\overline{P}\in\mathcal{P}_{\widehat{E}}$ such that for all $(x,y)\in N$,

$$\operatorname{supp}\Lambda_{(x,y)}\cap\overline{Q}\neq\emptyset\quad\text{and}\quad\operatorname{supp}\Lambda_{(x,y)}\cap\underline{Q}\neq\emptyset,$$

with

$$\overline{Q}:=\overline{P}\setminus\bigcup_{P\in\mathcal{P}_{\widehat{E}},P\neq\overline{P}}P\quad\text{and}\quad\underline{Q}:=\bigcup_{P\in\mathcal{P}_{\widehat{E}},P\neq\overline{P}}P\setminus\overline{P}. \tag{3.11}$$

Since $\overline{Q}$ and $\underline{Q}$ are both symmetric, N can be chosen to be symmetric, too.

Next, we fix $(x,y)\in N$ and take $(\xi,\zeta),(\alpha,\beta)\in\operatorname{supp}\Lambda_{(x,y)}$ such that

$$(\xi,\zeta)\in\overline{Q}\quad\text{and }(\alpha,\beta)\in\underline{Q}. \tag{3.12}$$

By symmetry, also $(\zeta,\xi),(\beta,\alpha)\in\operatorname{supp}\Lambda_{(x,y)}$, and we infer from Lemma 3.4 that

$$M:=\{\xi,\zeta,\alpha,\beta\}\times\{\xi,\zeta,\alpha,\beta\}\subset\operatorname{supp}\Lambda_{(x,y)}\subset\widehat{E}.$$

Since M is a Cartesian product, it is contained in some maximal Cartesian subset P of $\widehat{E}$. However, in view of (3.12) and (3.11), P cannot coincide with any element of $\mathcal{P}_{\widehat{E}}$; indeed, $(\xi,\zeta)\in P$, but (ξ,ζ) does not lie in any maximal Cartesian subset of $\widehat{E}$ other than $\overline{P}$, and $(\alpha,\beta)\in P$, but $(\alpha,\beta)\notin\overline{P}$. This is the sought contradiction. □

Proof of Theorem 3.1. The equality of $\mathcal{A}_K^\infty$ and $\mathcal{B}_K^\infty$ is an immediate consequence of Theorem 3.3 after retrieving to barycenters. Indeed, it suffices to use (3.8) along with the observation that

$$\mathcal{A}_K^\infty = \{u \in L^\infty(\Omega;\mathbb{R}^m) : v_u = [\Lambda],\ \Lambda \in \mathcal{Y}_K^\infty\} \text{ and } B_K^\infty = \{u \in L^\infty(\Omega;\mathbb{R}^m) : v_u = [\Lambda],\ \Lambda \in \widetilde{\mathcal{Y}}_K^\infty\}.$$

For the desired representation formula, we invoke again (3.8) to deduce that

$$\begin{aligned}\mathcal{A}_K^\infty = \mathcal{B}_K^\infty &= \{u \in L^\infty(\Omega;\mathbb{R}^m) : v_u = [\Lambda], \Lambda \in \bigcup_{P\in\mathcal{P}_K} \mathcal{Y}_P\}\\ &= \bigcup_{A\times A\in\mathcal{P}_K} \{u \in L^\infty(\Omega;\mathbb{R}^m) : v_u = [\Lambda], \Lambda \in \mathcal{Y}_{A\times A}\}\\ &= \bigcup_{A\times A\in\mathcal{P}_K} \{u \in L^\infty(\Omega;\mathbb{R}^m) : u = [\nu], \nu \in L_w^\infty(\Omega;\mathcal{P}r(\mathbb{R}^m)), \operatorname{supp}\nu_x \subset A \text{ for a.e. } x\in\Omega\}\\ &= \bigcup_{A\times A\in\mathcal{P}_K} \{u \in L^\infty(\Omega;\mathbb{R}^m) : u \in A^{\mathrm{co}} \text{ a.e. in } \Omega\},\end{aligned}$$

which was the claim. □

Remark 3.6. *Let $p \geq 1$ and $E \subset \mathbb{R}^m \times \mathbb{R}^m$ symmetric. Replacing in the above definitions of $\widetilde{\mathcal{Y}}_E^\infty$ and $\mathcal{B}_E^\infty$ (see* (3.6) *and* (3.3)*) the weakly* converging L^∞-sequences by weakly converging L^p-sequences results in new sets of functions and parametrized measures, which we want to call $\widetilde{\mathcal{Y}}_{E,p}^\infty$ and $\mathcal{B}_{E,p}^\infty$, respectively.*

If $K \subset \mathbb{R}^m \times \mathbb{R}^m$ is symmetric and compact, then

$$\mathcal{B}_{K,p}^\infty = \mathcal{B}_K^\infty \qquad \text{and} \qquad \widetilde{\mathcal{Y}}_{K,p}^\infty = \widetilde{\mathcal{Y}}_K^\infty,$$

which is a consequence of [18, Proposition 2.2] *and the fundamental theorem on Young measures, see e.g.* [11, Theorem 8.6 (iii)].

4. Necessary conditions for relaxation via separate convexification

As pointed out in the introduction, the papers [2, 4] present each a specific example of a double-integral functional of multi-well form for which separate convexification of the integrand fails in providing a correct relaxation formula. Here, we generalize these findings and generate a whole class of such examples (see Corollary 4.7), motivated by recent insights from the study of nonlocal supremal functionals and nonlocal inclusions [13]. A key ingredient is the following notion of diagonalization for functions introduced in [13, (7.1)].

Definition 4.1. *The diagonalization of a symmetric function $W : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is defined as*

$$\widehat{W} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, \quad \widehat{W}(\xi,\zeta) = \inf\{c \in \mathbb{R} : (\xi,\zeta) \in \widehat{L_c(W)}\},$$

where $\widehat{L_c(W)}$ is the diagonalization of the sublevel set $L_c(W)$ with $c \in \mathbb{R}$ in the sense of Definition 2.3.

Notice in particular that the previous definition implies

$$L_c(\widehat{W}) = \widehat{L_c(W)} = L_c(W)^\wedge \tag{4.1}$$

for all $c \in \mathbb{R}$, cf. [13, (7.2)].

4.1. **Double-integrals and order relations.**

An important difference between the theory of single- and double-integral functionals, which has substantial conceptual and technical ramifications, lies in the order relations for the functionals and their integrands.

Whereas it holds for any suitable $f : \mathbb{R}^m \to \mathbb{R}$ that

$$\inf_{u\in L^p(\Omega;\mathbb{R}^m)} \int_\Omega f(u)\,dx \geq 0 \quad \Rightarrow \quad f \geq 0,$$

the analogy of this implication is in general not true in the context of double-integral functionals. In fact, even if

$$\inf_{u\in L^p(\Omega;\mathbb{R}^m)} \int_\Omega\int_\Omega W(u(x),u(y))\,dx\,dy \geq 0,$$

for a suitable $W:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$, the integrand W may take both positive and negative values, which is owed to nonlocal effects. This observation was pointed out first by Bellido & Mora-Corral in [2, Section 7] and illustrated with an explicit scalar example of the form $W(\xi,\zeta)=w(\xi-\zeta)$ for $(\xi,\zeta)\in\mathbb{R}$, where $w:\mathbb{R}\to\mathbb{R}$ is a fourth order even polynomial; see [2, Example 7.2] for the details. In the next proposition, we investigate a more general class of related integrands, cf. Remark 4.3 c) below.

Proposition 4.2. *Let $W:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be a symmetric, lower semicontinuous function with p-growth and p-coercivity. If*

$$\min W < \min \widehat{W}, \tag{4.2}$$

then,

$$|\Omega|^2\min\widehat{W} \geq \inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u) > |\Omega|^2 \min W.$$

Remark 4.3. *a) It is clear that W attains its infimum on $\mathbb{R}^m\times\mathbb{R}^m$ due to its coercivity and lower semicontinuity of W. Since these two properties carry over to $\widehat{W}$ (cf. comment right after Definition 2.3), also $\min\widehat{W}$ is well defined.*

b) As a consequence of the previous result and the properties of $\widehat{W}$, one finds that $\min W = \min\widehat{W}$ if and only if

$$\inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u) = \min_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u) = |\Omega|^2\min W.$$

c) The above result is valid also for double-integrands $W:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ of the form $W(\xi,\zeta)=w(\xi-\zeta)$ for $(\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m$, where $w:\mathbb{R}^m\to\mathbb{R}$ is a lower semicontinuous with p-growth and p-coercivity, provided we consider I_W only on the smaller space of $L^p(\Omega;\mathbb{R}^m)$-functions with vanishing mean.

Proof of Proposition 4.2. For simplicity of notation, we write $\inf I_W := \inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u)$ in what follows, and we assume without loss of generality that $\min W=0$; otherwise, W can be translated suitably.

First, we show the estimate

$$\min\widehat{W} \geq \inf I_W. \tag{4.3}$$

Let $(\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m$ be a minimizer of $\widehat{W}$. Then, $(\xi,\zeta)\in L_{\min\widehat{W}}(\widehat{W}) = L_{\min\widehat{W}}(W)^\wedge$ (cf. (4.1)), so that

$$\widehat{W}(\xi,\zeta)=\widehat{W}(\zeta,\xi)=\widehat{W}(\xi,\xi)=\widehat{W}(\zeta,\zeta)=\min\widehat{W}, \tag{4.4}$$

by the symmetry of $\widehat{W}$ and the definition of diagonalization of sets in (2.4). Considering the piecewise constant function $v:\Omega\to\mathbb{R}^m$ given by

$$v=\xi\mathbb{1}_{\Omega_\xi}+\zeta\mathbb{1}_{\Omega\setminus\Omega_\xi}$$

with $\Omega_\xi\subset\Omega$ measurable such that $0<|\Omega_\xi|<|\Omega|$, we conclude in view of (4.4) and $W\leq\widehat{W}$ that

$$\begin{aligned}\inf I_W \leq I_W(v) &= |\Omega_\xi|^2 W(\xi,\xi) + |\Omega\setminus\Omega_\xi|^2 W(\zeta,\zeta) + 2|\Omega_\xi|\,|\Omega\setminus\Omega_\xi| W(\xi,\zeta)\\ &\leq (|\Omega_\xi|+|\Omega\setminus\Omega_\xi|)^2\min\widehat{W} = |\Omega|^2\min\widehat{W}.\end{aligned}$$

This implies (4.3).

To prove the strict inequality $\inf I_W > \min W = 0$, we assume to the contrary that $\inf I_W = 0$, meaning that there exists a sequence $(u_j)_j \subset L^p(\Omega;\mathbb{R}^m)$ such that

$$\lim_{j\to\infty} I_W(u_j) = \lim_{j\to\infty} \int_\Omega \int_\Omega W(v_{u_j}(x,y))\,dx\,dy = 0, \tag{4.5}$$

cf. (2.8). As a consequence of the p-coercivity of W, $(v_{u_j})_j$ is uniformly bounded in $L^p(\Omega \times \Omega;\mathbb{R}^m \times \mathbb{R}^m)$.

If $\Lambda = \{\Lambda_{(x,y)}\}_{(x,y)} = \{\nu_x \otimes \nu_y\}_{(x,y)}$ with $\nu \in L^\infty_w(\Omega;\mathcal{P}r(\mathbb{R}^m))$ is the Young measure generated by a (non-relabeled) subsequence of $(v_{u_j})_j \subset L^p(\Omega\times\Omega;\mathbb{R}^m\times\mathbb{R}^m)$ according to [18, Proposition 2.3], the fundamental theorem on Young measures (see e.g. [11, Theorem 8.6 (i)]) yields that

$$\lim_{j\to\infty} \int_\Omega \int_\Omega W(v_{u_j}(x,y))\,dx\,dy \geq \int_\Omega \int_\Omega \langle \Lambda_{(x,y)}, W\rangle\,dx\,dy,$$

where $\langle \Lambda_{(x,y)}, W\rangle := \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} W(\xi,\zeta)\,d\Lambda_{(x,y)}(\xi,\zeta)$.

In light of (4.5) and the non-negativity of W, it follows that $\langle \Lambda_{(x,y)}, W\rangle = 0$ for a.e. $(x,y) \in \Omega\times\Omega$, and hence, $\operatorname{supp}\Lambda_{(x,y)} \subset L_0(W)$ for a.e. $(x,y)\in\Omega\times\Omega$, or equivalently by (3.5),

$$\Lambda \in \mathcal{Y}_{L_0(W)}. \tag{4.6}$$

On the other hand, (3.10) in the proof of Lemma 3.5 together with (4.2) results in

$$\mathcal{Y}_{L_0(W)} = \mathcal{Y}_{\widehat{L_0(W)}} = \emptyset. \tag{4.7}$$

Combining (4.6) with (4.7) produces the desired contradiction. □

We continue our discussion of order relations for double-integrals with the following basic, yet useful, observation.

Lemma 4.4. *Let $V, W : \mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ be symmetric, lower semicontinuous integrands with p-growth such that $I_V \leq I_W$. Then $V(\xi,\xi) \leq W(\xi,\xi)$ for all $\xi\in\mathbb{R}^m$.*

Moreover, if $V(\xi,\xi) = W(\xi,\xi)$ for all $\xi \in A \subset \mathbb{R}^m$, then $V \leq W$ on $A\times A$.

Proof. Trivially, the first statement follows by evaluating I_V and I_W for constant functions. To show the second statement, let $(\xi,\zeta)\in A\times A$, and consider a piecewise constant function $v = \xi\mathbb{1}_{\Omega_\xi} + \zeta\mathbb{1}_{\Omega\setminus\Omega_\xi} \in S^\infty(\Omega;\mathbb{R}^m)$ with a measurable set $\Omega_\xi \subset\Omega$ such that $|\Omega_\xi| = \frac{1}{2}|\Omega|$. Then,

$$\begin{aligned}\frac{|\Omega|^2}{4}V(\xi,\xi) + \frac{|\Omega|^2}{4}V(\zeta,\zeta) + \frac{|\Omega|^2}{2}V(\xi,\zeta) &= I_V(u)\\ \leq I_W(u) = \frac{|\Omega|^2}{4}W(\xi,\xi) + &\frac{|\Omega|^2}{4}W(\zeta,\zeta) + \frac{|\Omega|^2}{2}W(\xi,\zeta).\end{aligned}$$

Since V and W coincide on the diagonal elements in $A\times A$, this implies $V(\xi,\zeta)\leq W(\xi,\zeta)$, concluding the proof. □

Remark 4.5. *The previous lemma shows in particular that a double-integral I_W as in* (2.7) *determines its integrand W uniquely. We point out that this is in contrast to the supremal setting, where according to* [13, (7.3)]*), all supremands with the same diagonalization in the sense of Definition 4.1 generate the same supremal functional.*

With the help of the previous lemma, we can derive the following bounds for certain relaxed double-integrands.

Proposition 4.6. *Let $W, G : \mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be symmetric, lower semicontinuous functions with p-growth. Suppose that $I_W^{\mathrm{rlx}} = I_G$ and that there exists $A\subset\mathbb{R}^m$ open such that $W(\xi,\xi) = W^{\mathrm{sc}}(\xi,\xi)$ for every $\xi\in A^c$. Then,*

$$W^{\mathrm{sc}} \leq G \leq W \quad \text{on } (A\times A)^{\mathrm{c}}. \tag{4.8}$$

If $A=\emptyset$, it holds that $G = W^{\mathrm{sc}}$.

Proof. From

$$I_{W^{\mathrm{sc}}} \leq I_W^{\mathrm{rlx}} = I_G \leq I_W,$$

we conclude with Lemma 4.4 that $W^{\mathrm{sc}}(\xi,\xi) \leq G(\xi,\xi) \leq W(\xi,\xi)$ for all $\xi \in A^c$, which, in view of our hypothesis, yields

$$W^{\mathrm{sc}}(\xi,\xi) = G(\xi,\xi) = W(\xi,\xi) \quad \text{for } \xi \in A^c. \tag{4.9}$$

Let $(\xi,\zeta) \in (A\times A)^c$, and assume without loss of generality that $\xi \notin A$; otherwise interchange the roles of ξ and ζ. Moreover, suppose for simplicity that $|\Omega| = 1$. We define

$$v = \xi \mathbb{1}_{\Omega_\xi} + \zeta \mathbb{1}_{\Omega\setminus\Omega_\xi} \in S^\infty(\Omega;\mathbb{R}^m),$$

where $\Omega_\xi \subset \Omega$ is measurable such that $|\Omega_\xi| = \lambda$ with $\lambda \in (0,1)$. Then,

$$\begin{aligned}\lambda^2 G(\xi,\xi) + (1-\lambda)^2 G(\zeta,\zeta) + 2\lambda(1-\lambda)G(\xi,\zeta) &= I_G(v)\\ \leq I_W(v) = \lambda^2 W(\xi,\xi) + (1-\lambda)^2 W(\zeta,\zeta) &+ 2\lambda(1-\lambda)W(\xi,\zeta).\end{aligned}$$

Due to (4.9), this can be rewritten as

$$G(\xi,\zeta) \leq \frac{1-\lambda}{2\lambda}\big(W(\zeta,\zeta) - G(\zeta,\zeta)\big) + W(\xi,\zeta).$$

Letting λ tend to 1, allows us to conclude that $G \leq W$ on the complement of $A\times A$.

If we replace G in the argument above with W^{sc}, and W with G, the exact same reasoning provides that $W^{\mathrm{sc}} \leq G$ outside of $A \times A$. Overall, this proves (4.8).

For the statement on the special case $A = \emptyset$, we infer from (4.8) that $W^{\mathrm{sc}} \leq G \leq W$. Since G has to be separately convex due to the L^p-weakly lower semicontinuity of I_G (see e.g. [4, Theorem 1.1]), it follows that even $G \leq W^{\mathrm{sc}}$, which entails the claim. □

4.2. **Implications for relaxation formulas.** Based on the results of Section 4.1, one can derive necessary conditions for the relaxation of I_W as in (2.7) via separate convexification of the double-integrand W. We distinguish in the following between the two cases when $\min W = \min \widehat{W}$ and $\min W \neq \min \widehat{W}$, which we address in Corollary 4.7 and Corollary 4.8, respectively.

Corollary 4.7. *Let $W : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ as in Proposition 4.2 with $\min \widehat{W} > \min W$. If $I_W^{\mathrm{rlx}} = I_{W^{\mathrm{sc}}}$, then*

$$\min \widehat{W^{\mathrm{sc}}} \neq \min W. \tag{4.10}$$

Proof. By contrapositive, we show that if $\min \widehat{W^{\mathrm{sc}}} = \min W$, then $I_W^{\mathrm{rlx}} \neq I_{W^{\mathrm{sc}}}$. Observe first that $\widehat{W^{\mathrm{sc}}} \geq W^{\mathrm{sc}}$ and $W^{\mathrm{sc}} \geq \min W$ implies

$$\min \widehat{W^{\mathrm{sc}}} = \min W^{\mathrm{sc}} = \min W. \tag{4.11}$$

Thus, by Proposition 4.2,

$$\inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u) > |\Omega|^2 \min W = |\Omega|^2 \min W^{\mathrm{sc}}. \tag{4.12}$$

On the other hand,

$$|\Omega|^2 \min \widehat{W^{\mathrm{sc}}} = \inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_{\widehat{W^{\mathrm{sc}}}}(u) \geq \inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_{W^{\mathrm{sc}}}(u) \geq |\Omega|^2 \min W^{\mathrm{sc}}; \tag{4.13}$$

for the first identity, it suffices to consider any piecewise constant function with values $\xi,\zeta \in \mathbb{R}^m$ such that (ξ,ζ) minimizes the diagonalized $\widehat{W^{\mathrm{sc}}}$. Along with (4.11), all inequalities in (4.13) turn into equalities, and the infima are in particular attained.

Combining (4.12) and (4.13) finally proves that

$$\min_{u\in L^p(\Omega;\mathbb{R}^m)} I_W^{\mathrm{rlx}}(u) = \inf_{u\in L^p(\Omega;\mathbb{R}^m)} I_W(u) > \min_{u\in L^p(\Omega;\mathbb{R}^m)} I_{W^{\mathrm{sc}}}(u),$$

which implies the statement. □

For a simple, one-dimensional example of a double-integrand that fails to satisfy the necessary conditions (4.10), see Section 5.1, especially (5.5).

Next, we formulate a corresponding result in the case when the minima of the double-integrand and its diagonalization coincide.

Corollary 4.8. *Let $W:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be symmetric and lower semicontinuous with p-growth and p-coercivity such that* $\min W=\min\widehat{W}=0$. *If* $I_W^{\rm rlx}=I_{W^{\rm sc}}$, *then,*

$$\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}} A^{\rm co}\times A^{\rm co}=(L_0(W)^{\rm sc})^{\wedge}. \tag{4.14}$$

For $m=1$, (4.14) *reduces to*

$$(L_0(W)^{\wedge})^{\rm sc}=(L_0(W)^{\rm sc})^{\wedge}. \tag{4.15}$$

Proof. According the classical abstract theory of relaxation (see e.g. [6, Section 9] and [7, Section 3] and the references therein), the set of L^p-weak limits of sequences of almost minimizers for I_W coincides with the set of minimizers of the relaxed functional $I_W^{\rm rlx}=I_{W^{\rm sc}}$.

Translated into the language of nonlocal inclusions in the spirit of Section 3, this means

$$\mathcal{B}^\infty_{L_0(W),p}=\mathcal{A}_{L_0(W^{\rm sc})}.$$

Because $L_0(W)$ is symmetric and compact as the sublevel set of a lower semicontinuous and coercive symmetric function, we can infer from Theorem 3.1 in conjunction with Remark 3.2 c) that

$$\mathcal{A}_{\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}}A^{\rm co}\times A^{\rm co}}=\mathcal{A}^\infty_{L_0(W)}=\mathcal{A}_{L_0(W^{\rm sc})}.$$

Upon exploiting [13, Proposition 5.1], this is equivalent to saying that

$$\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}} A^{\rm co}\times A^{\rm co}=\Big(\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}} A^{\rm co}\times A^{\rm co}\Big)^{\wedge}=L_0(W^{\rm sc})^{\wedge}; \tag{4.16}$$

notice that also $L_0(W^{\rm sc})$ is compact, as the symmetry, growth and continuity properties of W carry over to $W^{\rm sc}$, and that the left-hand side of (4.16) is the union of Cartesian products and thus, already diagonal.

The simplification in the case $m=1$ follows from

$$\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}} A^{\rm co}\times A^{\rm co}=\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}}(A\times A)^{\rm sc}=\Big(\bigcup_{A\times A\in\mathcal{P}_{L_0(W)}} A\times A\Big)^{\rm sc}=(L_0(W)^{\wedge})^{\rm sc}, \tag{4.17}$$

where we have used in particular (2.5) and the fact that the diagonalization of a separately convex set in $\mathbb{R}\times\mathbb{R}$ is again separately convex, see [13, Lemma 4.5]. Let us point out that in higher dimensions, the latter is not true and the second identity in (4.17) fails in general, cf. [13, Remark 4.6 b)]. □

Given that the operations of diagonalization and separate convexification do not commute, (4.14) and (4.15) impose in general non-trivial restrictions on W, as the following example for $m=1$ illustrates.

Example 4.9. *Let* $K=\{(\pm1,0),(0,\pm1),(2,2)\}\subset\mathbb{R}\times\mathbb{R}$ *and consider*

$$W(\xi,\zeta)=\operatorname{dist}^2((\xi,\zeta),K)\qquad \text{for } (\xi,\zeta)\in\mathbb{R}\times\mathbb{R},$$

with respect to any norm on $\mathbb{R}\times\mathbb{R}$. *Then* $L_0(\widehat{W})=\widehat{L_0(W)}=\widehat{K}=\{(2,2)\}$, *which is already separately convex, whereas* $L_0(W)^{\rm sc}=K^{\rm sc}=\{0\}\times[-1,1]\cup[-1,1]\times\{0\}\cup\{(2,2)\}$ *turns after diagonalization into* $\widehat{K^{\rm sc}}=\{(0,0),(2,2)\}$.

5. Counterexamples for preservation of double-integral character

In this section, we present and analyze two examples to disprove that the relaxation of a double-integral yields again a double-integral. In both cases, the integrand W of I_W as in (1.1) with $m = 1$ is given by a function measuring the distance to a compact set K in the 1-norm. This choice of norm turns out to make calculations particularly simple. The conceptual difference between the two densities is that for the first, $\min W \neq \min \widehat{W}$, while the second satisfies $\min W = \min \widehat{W}$.

5.1. First counterexample.

Let $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$W(\xi,\zeta) = \mathrm{dist}_1^2((\xi,\zeta), K) \tag{5.1}$$

for $(\xi,\zeta) \in \mathbb{R}\times\mathbb{R}$ with $K = \{(\pm 1, 0), (0, \pm 1)\}$ and the underlying norm $\|\cdot\|_1$.

Recalling (2.3), we know that the separately convex envelope of W coincides with the separately level convex one and is given by

$$W^{\mathrm{sc}}(\xi,\zeta) = \mathrm{dist}_1^2((\xi,\zeta), K^{\mathrm{sc}}) \quad \text{for } (\xi,\zeta) \in \mathbb{R}\times\mathbb{R},$$

where $K^{\mathrm{sc}} = \{0\} \times [-1,1] \cup [-1,1] \times \{0\}$.

In terms of sublevel sets, W and W^{sc} can be expressed as follows: For $c \in \mathbb{R}$,

$$L_c(W) = \begin{cases} \bigcup_{(\xi,\zeta)\in K} B^1_{\sqrt{c}}(\xi,\zeta) & \text{for } 0 \le c \le 1, \\ B^1_{1+\sqrt{c}}(0,0) & \text{for } c \ge 1, \\ \emptyset & \text{for } c < 0, \end{cases} \tag{5.2}$$

and

$$L_c(W^{\mathrm{sc}}) = L_c(W)^{\mathrm{sc}} = \begin{cases} L_c(W) \cup [(-\sqrt{c},\sqrt{c}) \times (-1,1)] \cup [(-1,1)\times(-\sqrt{c},\sqrt{c})] & \text{for } 0 \le c \le 1, \\ L_c(W) & \text{for } c \ge 1, \\ \emptyset & \text{for } c < 0, \end{cases} \tag{5.3}$$

see also Figure 5.1. After diagonalization, (5.2) and (5.3) turn into

$$\widehat{L_c(W)} = \begin{cases} [-\sqrt{c},\sqrt{c}]^2 & \text{for } c \ge 1, \\ \emptyset & \text{for } c < 1, \end{cases} \quad \text{and} \quad \widehat{L_c(W^{\mathrm{sc}})} = \begin{cases} [-\sqrt{c},\sqrt{c}]^2 & \text{for } c \ge 0, \\ \emptyset & \text{for } c < 0, \end{cases} \tag{5.4}$$

for $c \in \mathbb{R}$. Observe in particular that $\widehat{L_0(W)} = \widehat{K} = \emptyset$. In view of (4.1) and (5.4), one can deduce explicit expressions for the diagonalizations of W and W^{sc}, that is,

$$\widehat{W}(\xi,\zeta) = \mathrm{dist}^2_\infty((\xi,\zeta), [-1,1]^2) + 1 \quad \text{and} \quad \widehat{W^{\mathrm{sc}}}(\xi,\zeta) = \|(\xi,\zeta)\|^2_\infty$$

for $(\xi,\zeta) \in \mathbb{R}^m$, where $\|\cdot\|_\infty$ stands for the maximum norm.

The above computations allow us to conclude that

$$\min W = 0 < 1 = \min \widehat{W} \quad \text{and} \quad \min \widehat{W^{\mathrm{sc}}} = \min W^{\mathrm{sc}} = \min W = 0. \tag{5.5}$$

Thus, according to Corollary 4.7, separate convexification of the double-integrand W fails to give a representation for I_W^{rlx}. The next result provides even more, namely that the relaxation of I_W is not of double-integral form at all.

Proposition 5.1. *Let $W : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ as in (5.1). There exists no symmetric, lower semicontinuous double-integrand function $G : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with quadratic growth such that $I_W^{\mathrm{rlx}} = I_G$.*

Proof. We argue by contradiction, and suppose therefore that

$$0 \le I_{W^{\mathrm{sc}}} \le I_W^{\mathrm{rlx}} = I_G \le I_W \tag{5.6}$$

for some $G : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as in the statement. A comparision of (5.2) and (5.3) yields that W^{sc} coincides with W outside of $(-1,1)^2$, that is,

$$W = W^{\mathrm{sc}} \text{ on } [(-1,1)^2]^c,$$

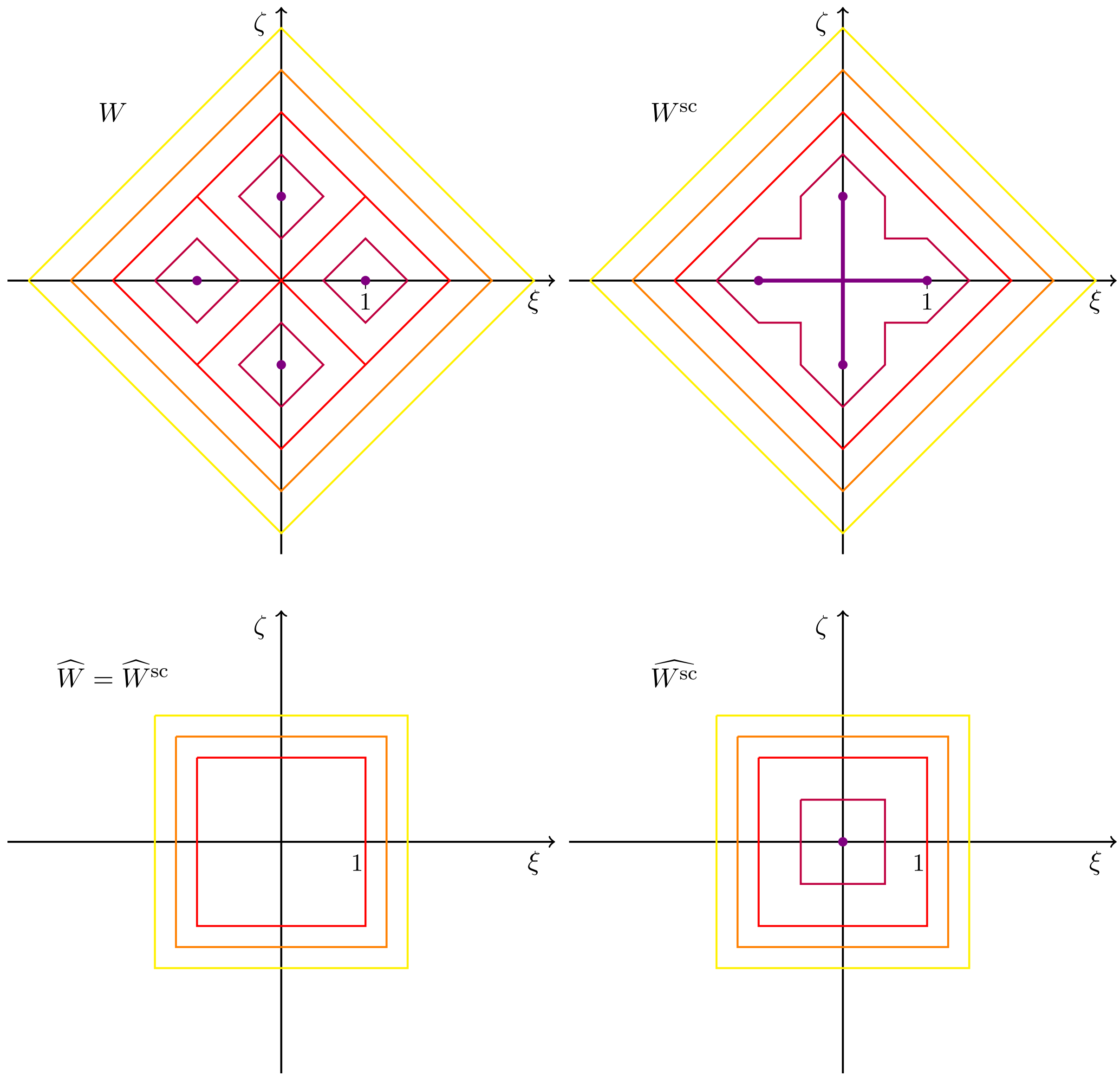

FIGURE 1. Illustration of the sublevel sets of W, $\widehat{W}$, W^{sc}, and $\widehat{W^{\mathrm{sc}}}$ for the levels $c = 0$ (violet), $c = \frac{1}{4}$ (purple), $c = 1$ (red), $c = \frac{5}{4}$ (orange), $c = \frac{3}{2}$ (yellow).

which enables us to invoke Proposition 4.6. Hence,

$$G = W^{\mathrm{sc}} = W \quad \text{ on } [(-1,1)^2]^c,$$

and especially, $G = 0$ on K. On the other hand, we know in light of (5.6) and Lemma 4.4 that $G(0,0) \geq W^{\mathrm{sc}}(0,0) = 0$. Consequently, the L^2-weakly lower semicontinuity of $I_G = I_W^{\mathrm{rlx}}$, which implies the separate convexity of G (see e.g. [4, Theorem 1.1]), leads us to conclude that G vanishes $\{0\} \times [-1,1] \cup [-1,1] \times \{0\} = K^{\mathrm{sc}}$; in particular, $G(0,0) = 0$.

This proves that $\min_{u \in L^2(\Omega)} I_G(u) = I_G(0) = 0$, in view of (5.6). As I_G coincides with the relaxation of I_W by assumption, one has that

$$\inf_{u \in L^2(\Omega)} I_W(u) = \min_{u \in L^2(\Omega)} I_W^{\mathrm{rlx}}(u) = \min_{u \in L^2(\Omega)} I_G(u) = 0. \tag{5.7}$$

However, by Proposition 4.2 in combination with (5.5), $\inf_{u \in L^2(\Omega)} I_W(u) > \min W = 0$, which contradicts (5.7) and concludes the proof. □

Remark 5.2. *Alternatively, there is also a direct and self-contained argument for the last step in the proof above, meaning, one that does not make use of Proposition 4.2. Following along the lines of* [19, Section 3], *we argue by contradiction and let* $\inf_{u\in L^2(\Omega)} I_W(u) = 0$*, so that*

$$I_W(u_j) = \int_\Omega \int_\Omega W(v_{u_j})\,dx\,dy \to 0 \tag{5.8}$$

for some sequence $(u_j)_j \subset L^2(\Omega)$*. Due to* $W \geq 0$*, we know that* $(v_{u_j})_j$ *needs to concentrate around* $L_0(W) = K$*, and since* K *is finite, it has to concentrate partially around at least one point, without loss of generality* $(1, 0)$*. Then there exists a set* $\omega \subset \Omega$ *of positive measure where* $(u_j)_j$ *concentrates around* 1*, which again entails that* $(v_{u_j})_j$ *concentrates around* $(1, 1)$ *on* $\omega\times\omega$*. Finally, we derive a contradiction with* (5.8) *by concluding that*

$$\int_\omega \int_\omega W(v_{u_j})\,dx\,dy \to W(1, 1)|\omega \times \omega| > 0.$$

Indeed, the limit is strictly positive because $(1, 1) \notin K = L_0(W)$ *and* $\omega \times \omega$ *has non-vanishing* $\mathcal{L}^{2n}$*-measure.*

Not only do homogeneous double-integrals fail to provide an explicit representation for the L^2-weak lower semicontinuous envelope of I_W, allowing for inhomogeneous double-integrands does not help in obtaining correct relaxation formulas either.

Remark 5.3. *In generalization of Proposition 5.1, one can show that it is not possible to express* $I_W^{\rm rlx}$ *with* W *as in* (5.1) *in terms of*

$$L^2(\Omega) \ni u \mapsto \int_\Omega \int_\Omega G(x, y, u(x), u(y))\,dx\,dy,$$

where $G : \Omega \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a symmetric and normal function with quadratic growth, i.e.,* G *satisfies*

(*i*) $G(x, y, \xi, \zeta) = G(y, x, \xi, \zeta)$ *and* $G(x, y, \xi, \zeta) = G(x, y, \zeta, \xi)$ *for all* $x, y \in \Omega$ *and* $\xi, \zeta \in \mathbb{R}$*;*

(*ii*) $G(x, y, \cdot, \cdot)$ *is lower semicontinuous for a.e.* $(x, y) \in \Omega \times \Omega$ *and* $G(\cdot, \cdot, \xi, \zeta)$ *is measurable for all* $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}$*;*

(*iii*) $|G(x, y, \xi, \zeta)| \leq C(a(x, y) + |\xi|^2 + |\zeta|^2)$ *for all* $(x, y) \in \Omega \times \Omega$ *and* $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}$ *with a constant* $C > 0$ *and* $a \in L^1(\Omega \times \Omega)$*.*

To see this, it suffices to substitute G *in the proof of Proposition* (5.1) *by*

$$\overline{G}(\xi, \zeta) := \int_\Omega \int_\Omega G(x, y, \xi, \zeta)\,dx\,dy \quad \textit{for } (\xi, \zeta) \in \mathbb{R} \times \mathbb{R},$$

and to use [19, Theorem 2.5] *in place of* [4, Theorem 1.1].

5.2. **Second counterexample.** The double-integrand for our second example is qualitatively different from the first, in the sense that its minimum does not change under diagonalization.

Proposition 5.4. *Let* $K = \partial B_1^1(0, 0) = \{(\xi, \zeta) \in \mathbb{R} \times \mathbb{R} : |\xi| + |\zeta| = 1\}$ *and* $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$W(\xi, \zeta) = \mathrm{dist}_1^2((\xi, \zeta), K) \quad \textit{for } (\xi, \zeta) \in \mathbb{R} \times \mathbb{R}. \tag{5.9}$$

There exists no symmetric, lower semicontinuous double-integrand $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *with quadratic growth such that* $I_W^{\rm rlx} = I_G$*.*

Proof. Arguing by contradiction, we suppose that $I_W^{\rm rlx} = I_G$ with G as in the statement, and split the proof into two steps. First, Step 1 shows that necessarily $G = W^{\rm sc}$, and then, we conclude in Step 2 that $I_W^{\rm rlx} \neq I_{W^{\rm sc}}$, which yields the desired contradiction.

Step 1: $G = W^{\rm sc}$. Since $W^{\rm sc}(\xi, \zeta) = \mathrm{dist}_1^2((\xi, \zeta), K^{\rm sc})$ for $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}$ with $K^{\rm sc} = K^{\rm co} = B_1^1(0, 0)$, the functions W and $W^{\rm sc}$ coincide outside of $K^{\rm sc}$. A comparison between G with W and $W^{\rm sc}$ along the diagonal therefore gives

$$W^{\rm sc}(\xi, \xi) = G(\xi, \xi) = W(\xi, \xi) \quad \text{for } \xi \in \mathbb{R} \text{ with } |\xi| \geq \tfrac{1}{2}.$$

In view of Proposition 4.6, $W^{\rm sc} \le G \le W$ on $[(-\frac{1}{2}, \frac{1}{2})]^c$, and thus in particular,

$$G = W^{\rm sc} \quad \text{on } B_1^1(0,0)^c. \tag{5.10}$$

Next, we prove that

$$G = W^{\rm sc} \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}]^2. \tag{5.11}$$

The argument is based on the observation that $\widehat{K} = \{-\frac{1}{2}, \frac{1}{2}\}^2$, and therefore $\widehat{K}^{\rm sc} = \widehat{K}^{\rm co} = [-\frac{1}{2}, \frac{1}{2}]^2$.

For $(\xi, \zeta) \in [-\frac{1}{2}, \frac{1}{2}]^2$, let $v = \xi \mathbb{1}_{\Omega_\xi} + \zeta \mathbb{1}_{\Omega \setminus \Omega_\xi}$ with a measurable set $\Omega_\xi \subset \Omega$ of measure $\lambda|\Omega|$ with $\lambda \in (0,1)$. By Lemma 2.2, one can find a sequence $(v_j)_j \in S^\infty(\Omega)$ oscillating suitably between the values $\pm\frac{1}{2}$ such that $v_j \rightharpoonup^* v$ in $L^\infty(\Omega)$. As

$$0 \le I_{W^{\rm sc}}(v) = I_W^{\rm rlx}(v) = I_G(v) \le \limsup_{j\to\infty} I_W(v_j) = 0$$

due to $W = 0$ on $\widehat{K} \subset K$, it follows that

$$\lambda^2 G(\xi,\xi) + (1-\lambda)^2 G(\zeta,\zeta) + 2\lambda(1-\lambda) G(\xi,\zeta) = 0 \quad \text{for all } \lambda \in (0,1).$$

Letting $\lambda \to 0$ and $\lambda \to 1$ yields first that $G(\xi,\xi) = G(\zeta,\zeta) = 0$, and eventually, also $G(\xi,\zeta) = 0$. This finishes the proof of (5.11).

We can now infer from (5.10) and (5.11) that G coincides with $W^{\rm sc}$ on the diagonal, and since $I_{W^{\rm sc}} \le I_W^{\rm rlx} = I_G$, Lemma 4.4 implies that $G \ge W^{\rm sc} \ge 0$. Since G has to be separately convex (see e.g. [4, Theorem 1.1]),

$$G = 0 = W^{\rm sc} \quad \text{on } K^{\rm sc} = B_1^1(0,0).$$

In combination with (5.10), this shows $G = W^{\rm sc}$.

Step 2: $I_W^{\rm rlx} \neq I_{W^{\rm sc}}$. Considering the simple function $v = \mathbb{1}_{\Omega_1}$, where $\Omega_1 \subset \Omega$ is a set of positive Lebesgue measure, we aim to show that

$$I_W^{\rm rlx}(v) \neq I_{W^{\rm sc}}(v).$$

Assume to the contrary that $I_W^{\rm rlx}(v) = I_{W^{\rm sc}}(v)$. Then, by the definition of the relaxed functional, there exists a sequence $(u_j)_j \subset L^2(\Omega)$ such that $u_j \rightharpoonup v$ in $L^2(\Omega)$ and

$$\lim_{j\to\infty} I_W(u_j) = I_{W^{\rm sc}}(v) = |\Omega_1|^2 W^{\rm sc}(1,1) + |\Omega \setminus \Omega_1|^2 W^{\rm sc}(0,0) = |\Omega_1|^2 W(1,1) = |\Omega_1|^2; \tag{5.12}$$

here, we have used that $W^{\rm sc}(1,1) = W(1,1) = 1$ and $W^{\rm sc}(0,0) = 0$. On the other hand, along with for the weak convergence of the restrictions of $(u_j)_j$ to Ω_1 and $\Omega \setminus \Omega_1$, i.e. $u_j|_{\Omega_1} \rightharpoonup 1$ in $L^2(\Omega_1)$ and $u_j|_{\Omega\setminus\Omega_1} \rightharpoonup 0$ in $L^2(\Omega \setminus \Omega_1)$, as well as the symmetry and non-negativity of W,

$$\begin{aligned}
\lim_{j\to\infty} I_W(u_j) &\ge \lim_{j\to\infty} \Big(\int_{\Omega_1}\int_{\Omega_1} W(u_j(x), u_j(y))\,dx\,dy + \int_{\Omega\setminus\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x), u_j(y))\,dx\,dy \\
&\qquad + 2\int_{\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x), u_j(y))\,dx\,dy \Big) \\
&\ge W^{\rm sc}(1,1)|\Omega_1|^2 + W^{\rm sc}(0,0)|\Omega\setminus\Omega_1|^2 + 2\liminf_{j\to\infty} \int_{\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x), u_j(y))\,dx\,dy \qquad (5.13) \\
&= |\Omega_1|^2 + 2\liminf_{j\to\infty} \int_{\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x), u_j(y))\,dx\,dy \ge |\Omega_1|^2.
\end{aligned}$$

Comining (5.13) with (5.12) turns all equalities in (5.13) into equalities. Hence, after passing to a suitable (not relabelled) subsequence,

$$\lim_{j\to\infty} \int_{\Omega_1}\int_{\Omega_1} W(u_j(x), u_j(y))\,dx\,dy = |\Omega_1|^2, \tag{5.14}$$

$$\lim_{j\to\infty}\int_{\Omega\setminus\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x),u_j(y))\,dx\,dy = 0, \tag{5.15}$$

and

$$\lim_{j\to\infty}\int_{\Omega_1}\int_{\Omega\setminus\Omega_1} W(u_j(x),u_j(y))\,dx\,dy = 0. \tag{5.16}$$

We conclude from (5.15) that the sequence $(u_j|_{\Omega\setminus\Omega_1})_j \subset L^2(\Omega\setminus\Omega_1)$ concentrates around $\pm\frac{1}{2}$, while (5.14) shows that this is not the case for $(u_j|_{\Omega_1})_j \subset L^2(\Omega_1)$. Now, let $\omega\subset\Omega\setminus\Omega_1$ and $\omega_1\subset\Omega_1$ be sets of positive $\mathcal{L}^n$-measure and $\varepsilon>0$ such that $(u_j|_\omega)_j$ concentrates around $\frac{1}{2}$, and $(u_j|_{\omega_1})_j$ concentrates on the complement of $(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon)\cup(-\frac{1}{2}-\varepsilon,-\frac{1}{2}+\varepsilon)$. Such sets exist without loss of generality, otherwise replace $\frac{1}{2}$ by $-\frac{1}{2}$ for the set of concentrations of $(u_j|_\omega)_j$. With $\nu\in L^\infty_w(\omega_1;\mathcal{P}r(\mathbb{R}))$ the Young measure generated by a suitable (non-relabelled) subsequence of $(u_j|_{\omega_1})_j$, it follows that

$$\lim_{j\to\infty}\int_{\omega}\int_{\omega_1} W(u_j(x),u_j(y))\,dx\,dy = |\omega|\int_{\omega_1}\int_{\mathbb{R}} W(\xi,\tfrac{1}{2})\,d\nu_x(\xi)\,dx > 0, \tag{5.17}$$

which contradicts (5.16). The estimate in (5.17) makes use of the fact that by the choice of ω_1, $\pm\frac{1}{2}\notin\operatorname{supp}\nu_x$ for a.e. $x\in\omega_1$, along with the observation that $W(\xi,\frac{1}{2})=0$ if and only if $\xi\in\{-\frac{1}{2},\frac{1}{2}\}$. □

Remark 5.5. *We notice that condition* (4.15) *from Proposition 4.8, which is necessary for structure-preserving relaxation of double-integrals via separate convexification in the scalar setting, is in general not sufficient.*

Indeed, for the double-integrand W *from* (5.9)*, one has that* $L_0(W)=K$ *with* $\widehat{K}=\{\frac{1}{2},\frac{1}{2}\}^2$ *and* $K^{\rm sc}=B^1_1(0,0)$*, and therefore,*

$$(L_0(W)^\wedge)^{\rm sc} = \widehat{K}^{\rm sc} = [-\tfrac{1}{2},\tfrac{1}{2}]^2 = B^1_1(0,0)^\wedge = \widehat{K^{\rm sc}} = (L_0(W)^{\rm sc})^\wedge,$$

which verifies (4.15)*. However, we have just proven in Proposition 5.4 that* $I_W^{\rm rlx}\neq I_{W^{\rm sc}}$*.*

6. Examples of structure-preserving relaxation

In this last section, we provide examples of non-trivial relaxation where the double-integral structure is preserved and the integrands result from taking the separately convex envelope.

6.1. **Integrands of distance type.** Let $K=A\times A$ with a compact set $A\subset\mathbb{R}^m$ and $p>1$. We consider functions $W:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ defined via

$$W(\xi,\zeta)=\operatorname{dist}^p((\xi,\zeta),K)=\operatorname{dist}^p((\xi,\zeta),A\times A)\quad\text{for } (\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m, \tag{6.1}$$

cf. (2.1).

Owing to $K^{\rm sc}=K^{\rm co}=A^{\rm co}\times A^{\rm co}$, the separately convex envelope of W is identical with the convex one, that is,

$$W^{\rm sc}(\xi,\zeta)=W^{\rm co}(\xi,\zeta)=\operatorname{dist}^p((\xi,\zeta),A^{\rm co}\times A^{\rm co})=(\operatorname{dist}^2(\xi,A^{\rm co})+\operatorname{dist}^2(\zeta,A^{\rm co}))^{\frac{p}{2}}$$

for $(\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m$, see (2.3) and (2.2). Under the additional hypothesis on A that

$$\operatorname{dist}^2(\xi,A^{\rm co})=\operatorname{dist}^2(\xi,A)\qquad\text{for all }\xi\notin A^{\rm co}, \tag{6.2}$$

one can express the (separate) convexification of W in the following way: With any $\alpha,\beta\in A$,

$$W^{\rm sc}(\xi,\zeta)=\begin{cases}0 & \text{if }\xi\in A^{\rm co}\text{ and }\zeta\in A^{\rm co},\\ W(\xi,\zeta) & \text{if }\xi\notin A^{\rm co}\text{ and }\zeta\notin A^{\rm co},\\ W(\alpha,\zeta) & \text{if }\xi\in A^{\rm co}\text{ and }\zeta\notin A^{\rm co},\\ W(\xi,\beta) & \text{if }\xi\notin A^{\rm co}\text{ and }\zeta\in A^{\rm co},\end{cases} \tag{6.3}$$

for $(\xi,\zeta)\in\mathbb{R}^m\times\mathbb{R}^m$. Notice that the condition (6.2) is always satisfied for A if $m=1$; a sufficient condition for (6.2) in the vectorial case $m>1$ is for instance that $A\subset\mathbb{R}^m$ is of the form $A=B\setminus C$ with $B\subset\mathbb{R}^m$ compact and convex and $C\subset B$ open.

Due to the use of the Euclidean distance in the definition of W, clearly $W\neq\widehat{W}$ and $W^{\rm sc}\neq\widehat{W}^{\rm sc}$; yet the necessary condition for $I_W^{\rm rlx}=I_{W^{\rm sc}}$ in (4.15) holds. One can even prove the following statement.

Proposition 6.1. *Let W as in* (6.1) *and suppose that A satisfies the condition in* (6.2)*. Then, $I_W^{\rm rlx}=I_{W^{\rm sc}}$.*

Proof. Since $W^{\rm sc}$ is separately convex, and hence $I_{W^{\rm sc}}$ L^p-weakly lower semicontinuous (see e.g. [17, Theorem 1.1], [19, Theorem 2.6]), it is clear that $I_W^{\rm rlx}\geq I_{W^{\rm sc}}$. To prove the reverse inequality, let $u\in L^p(\Omega;\mathbb{R}^m)$, which we approximate by a sequence of simple functions $(u_k)_k$ such that $u_k\to u$ in $L^p(\Omega;\mathbb{R}^m)$. In view of the continuity and p-growth of $W^{\rm sc}$, the Vitali-Lebesgue convergence theorem yields that $I_{W^{\rm sc}}(u)=\lim_{k\to\infty}I_{W^{\rm sc}}(u_k)$.

It remains to find for any simple function

$$v=\sum_{j=1}^n\xi^{(j)}\mathbb{1}_{\Omega^{(j)}} \tag{6.4}$$

with $\xi^{(j)}\in\mathbb{R}^m$ and $\{\Omega^{(j)}\}_j$ a decomposition of Ω into measurable subsets, a sequence $(v_i)_i\subset L^p(\Omega;\mathbb{R}^m)$ such that $v_i\rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$, and

$$\limsup_{i\to\infty}I_W(v_i)\leq I_{W^{\rm sc}}(v); \tag{6.5}$$

the claim follows then from a diagonalization argument.

In the following, we take v as in (6.4) and detail the construction of $(v_i)_i$ with the desired properties. If $\xi^{(j)}\in A^{\rm co}$, let $(v_i^{(j)})_i\subset L^p(\Omega^{(j)};\mathbb{R}^m)$ be a sequence that converges weakly to v in $L^p(\Omega^{(j)};\mathbb{R}^m)$ and takes values only in A, meaning,

$$v_i^{(j)}\in A\qquad\text{a.e. in }\Omega^{(j)}\text{ for all }i\in\mathbb{N}, \tag{6.6}$$

cf. Lemma 2.2. If $\xi^{(j)}\notin A^{\rm co}$, let $v_i^{(j)}$ for any $i\in\mathbb{N}$ be the constant function on $\Omega^{(j)}$ with value $\xi^{(j)}$. With these definitions, consider the sequence $(v_i)_i\subset L^p(\Omega;\mathbb{R}^m)$ given by

$$v_i:=\sum_{j=1}^n v_i^{(j)}\mathbb{1}_{\Omega^{(j)}}\qquad\text{for }i\in\mathbb{N}.$$

By construction, $v_i\rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$, and

$$\begin{aligned}I_W(v_i)&=\int_\Omega\int_\Omega W(v_i(x),v_i(y))\,dx\,dy=\sum_{j,k=1}^n\int_{\Omega^{(j)}}\int_{\Omega^{(k)}}W(v_i^{(j)}(x),v_i^{(k)}(y))\,dx\,dy\\&=\sum_{j,k=1}^n W^{\rm sc}(\xi^{(j)},\xi^{(k)})|\Omega^{(j)}|\,|\Omega^{(k)}|=I_{W^{\rm sc}}(v)\end{aligned}$$

for all $i\in\mathbb{N}$, which implies (6.5) and concludes the proof. The third identity follows from the observation that for any $i\in\mathbb{N}$ and $l,k\in\{1,\dots,n\}$,

$$W(v_i^{(j)}(x),v_i^{(k)}(y))=W^{\rm sc}(\xi^{(j)},\xi^{(k)})\qquad\text{for a.e. }(x,y)\in\Omega^{(j)}\times\Omega^{(k)}. \tag{6.7}$$

To see the latter, we distinguish three different cases. If $\xi^{(j)},\xi^{(k)}\notin A^{\rm co}$, the functions $v_i^{(j)}$ and $v_i^{(k)}$ are constant, and $W^{\rm sc}(\xi^{(j)},\xi^{(k)})=W(\xi^{(j)},\xi^{(k)})$ by (6.3). For $\xi^{(j)},\xi^{(k)}\in A^{\rm co}$, both expressions in (6.7) are zero (almost everywhere) according to (6.6) and (6.3). In the case $\xi^{(j)}\in A^{\rm co}$ and $\xi^{(k)}\notin A^{\rm co}$, we invoke again (6.3) to obtain that $W^{\rm sc}(\xi^{(j)},\xi^{(k)})=W(\alpha,\xi^{(k)})$ for any $\alpha\in A$; hence, (6.7) holds in light of (6.6). For $\xi^{(j)}\notin A^{\rm co}$ and $\xi^{(k)}\in A^{\rm co}$, the reasoning is analogous. ☐

6.2. **Indicator functionals.** For a symmetric, diagonal and compact set $K \subset \mathbb{R}^m \times \mathbb{R}^m$, we define the associated indicator functional J_K via

$$L^\infty(\Omega;\mathbb{R}^m) \ni u \mapsto J_K(u) := I_{\chi_K}(u) = \int_\Omega\int_\Omega \chi_K(u(x),u(y))\,dx\,dy, \tag{6.8}$$

where χ_K is the characteristic function of K in the sense of convex analysis, i.e., $\chi_K(\xi,\zeta)=0$ if $(\xi,\zeta)\in K$ and $\chi_K=\infty$ otherwise in $\mathbb{R}^m\times\mathbb{R}^m$. Note that J_K can be expressed in terms of nonlocal inclusions as

$$J_K(u) = \begin{cases} 0 & \text{if } u \in \mathcal{A}_K, \\ \infty & \text{otherwise}, \end{cases}$$

for $u\in L^\infty(\Omega;\mathbb{R}^m)$; recall the definition of $\mathcal{A}_K$ in Section 3.

According to [13, Corollary 6.2] and Theorem 3.3, the Young measure relaxation of J_K is

$$\begin{aligned} J_K^{\mathcal{Y}}(\nu) &= \begin{cases} 0 & \operatorname{supp}\nu\otimes\nu \subset \widehat{K} \text{ a.e. in } \Omega\times\Omega, \\ \infty & \text{otherwise}, \end{cases} \\ &= \begin{cases} 0 & \operatorname{supp}\nu\otimes\nu \subset P \text{ a.e. in } \Omega\times\Omega \text{ with } P\in\mathcal{P}_K, \\ \infty & \text{otherwise}, \end{cases} \\ &= \begin{cases} 0 & \operatorname{supp}\nu \subset A \text{ a.e. in } \Omega \text{ with } A\times A\in\mathcal{P}_K, \\ \infty & \text{otherwise}, \end{cases} \end{aligned}$$

for $\nu\in L^\infty_w(\Omega;\mathcal{P}r(\mathbb{R}^m))$, and (3.4) provides a representation formula for the relaxation of J_K with respect to the L^∞-weak* topology; precisely, for $u\in L^\infty(\Omega;\mathbb{R}^m)$,

$$\begin{aligned} J_K^{\mathrm{rlx}}(u) &:= \inf\{\liminf_{j\to\infty} J_K(u_j) : u_j \rightharpoonup^* u \text{ in } L^\infty(\Omega;\mathbb{R}^m)\} \\ &= \begin{cases} 0 & \text{if } u\in\mathcal{A}_K^\infty, \\ \infty & \text{otherwise}, \end{cases} = \begin{cases} 0 & \text{if } u\in A^{\mathrm{co}} \text{ a.e. in } \Omega \text{ with } A\times A\in\mathcal{P}_K, \\ \infty & \text{otherwise}, \end{cases} \\ &= \int_\Omega\int_\Omega \chi_{\left[\bigcup_{A\times A\in\mathcal{P}_K} A^{\mathrm{co}}\times A^{\mathrm{co}}\right]}(u(x),u(y))\,dx\,dy = J_{K^{\mathrm{rlx}}}, \end{aligned} \tag{6.9}$$

with $K^{\mathrm{rlx}} := \bigcup_{A\times A\in\mathcal{P}_K} A^{\mathrm{co}}\times A^{\mathrm{co}}$.

It is generally not true that K^{rlx} coincides with the separately convex hull of K (see [13, Example 4.6 b)]); yet, under the additional assumption that

$$\widehat{K^{\mathrm{sc}}} = \bigcup_{(\alpha,\beta)\in K} [\alpha,\beta]\times[\alpha,\beta],$$

which is for instance satisfied for $m=1$ (see [13, Lemma 4.7]), it was shown in [13, Corollary 6.1] that $J_K^{\mathrm{rlx}} = J_{K^{\mathrm{sc}}}$. Whether this identity holds in general, or equivalently, if $K^{\mathrm{rlx}} = \widehat{K^{\mathrm{sc}}}$ without further assumptions on K, remains unknown.

In conclusion, we have seen that the relaxation of indicator functionals of the type (6.8) is always structure preserving.

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