OPTIMAL REGULARITY OF SOLUTIONS TO NO-SIGN OBSTACLE-TYPE PROBLEMS FOR THE SUB-LAPLACIAN

VALENTINO MAGNANI AND ANDREAS MINNE

Abstract. We establish the optimal $C_H^{1,1}$ interior regularity of solutions to

$$\Delta_H u = f\chi_{\{u\neq 0\}},$$

where Δ_H denotes the sub-Laplacian operator in a stratified group. We assume the weakest regularity condition on f, namely the group convolution $f*\Gamma$ is $C_H^{1,1}$, where Γ is the fundamental solution of Δ_H. The $C_H^{1,1}$ regularity is understood in the sense of Folland and Stein. In the classical Euclidean setting, the first seeds of the above problem are already present in the 1991 paper of Sakai and are also related to quadrature domains. As a special instance of our results, when u is nonnegative and satisfies the above equation we recover the $C_H^{1,1}$ regularity of solutions to the obstacle problem in stratified groups, that was previously established by Danielli, Garofalo and Salsa. Our regularity result is sharp: it can be seen as the subelliptic counterpart of the $C^{1,1}$ regularity result due to Andersson, Lindgren and Shahgholian.

1. Introduction

The main question we consider in this paper is the optimal interior regularity of distributional solutions to the *no-sign obstacle-type problem*

$$\Delta_H u = f\chi_{\{u\neq 0\}} \tag{1.1}$$

on some domain of a stratified group $\mathbb{G}$, see Section 2 for notation and terminology. In the Euclidean setting, the obstacle problem is among the most studied topics in the field of Free Boundary Problems, see for instance the monographs by Rodrigues [Rod87], Friedman [Fri88], and Petrosyan et al. [PSU12]. It asks which properties can be deduced about a function with given boundary values and that minimizes the Dirichlet energy, under the constraint of lying above a given function. This is the classical obstacle problem, that can be studied through the theory of variational inequalities, using the Dirichlet energy, see for instance [KS00, Fre72]. The variational approach, after subtracting the

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obstacle from the solution, leads to the following PDE formulation of the problem

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} & \text{in } B_1, \\ u \geq 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1, \end{cases} \tag{1.2}$$

where B_1 denotes the metric unit ball with respect to the Carnot–Carathéodory distance (Definition 2.1). Our problem is a non-variational counterpart of (1.2), that is

$$\begin{cases} \Delta u = f\chi_{\{u\neq 0\}} & \text{in } B_1, \\ u = g & \text{on } \partial B_1. \end{cases} \tag{1.3}$$

We point out that (1.3) — which is called a no-sign obstacle-type problem — naturally appears also when considering the so-called quadrature domains [Sak91, GS05].

Two important questions on this problem concern the regularity of solutions to (1.3), and the regularity of the free boundary. In Euclidean space, the analysis of both questions is essentially complete [Sak91, CKS00, PS07, ALS13]. In particular, in relation to the regularity of solutions, Andersson et al. [ALS13] show that u has the optimal $C^{1,1}$ regularity if the linear problem $\Delta v = f$ has a $C^{1,1}$ solution. This is the minimal regularity assumption on f in order to establish the $C^{1,1}$ regularity of solutions.

The main result of this paper is the sharp regularity of solutions to (1.1) also in the subelliptic setting of stratified groups.

Theorem 1.1 ($C^{1,1}_H$ regularity)**.** *Let $u \in L^\infty(B_1)$ be a distributional solution to* (1.1) *in the unit ball B_1. Let $f : B_1 \to \mathbb{R}$ be locally summable such that $f * \Gamma \in C^{1,1}_H(B_1)$. Then there exists a universal constant $C > 0$ such that, after a modification on a negligible set, we have $u \in C^{1,1}_H(B_{1/4})$ and*

$$\|D^2_h u\|_{L^\infty(B_{1/4})} \leq C \left(\|D^2_h (f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right). \tag{1.4}$$

In our setting, the natural counterpart of the Euclidean $C^{1,1}$ regularity is the $C^{1,1}_H$ regularity, where the horizontal derivatives are required to be Lipschitz continuous (Definition 2.2). The function Γ denotes the fundamental solution of Δ_H (Definition 2.3). For further notation and terminology, we address the reader to Section 2.

We wish to emphasize that u satisfies (1.1) also in the strong sense. Indeed, the distributional equality $\Delta_H(f * \Gamma) = -f$ joined with the assumed $C^{1,1}_H$ regularity of $f * \Gamma$ show that $f \in L^\infty(B_1)$. Therefore $f\chi_{\{u\neq 0\}} \in L^\infty(B_1)$ and by the regularity result of Folland, [Fol75, Theorem 6.1], we get $u \in W^{2,p}_{H,\mathrm{loc}}(B_1)$ for every $1 \leq p < \infty$. The $C^{1,1}_H$ regularity of solutions to the obstacle problem in stratified groups was obtained by Danielli et al. [DGS03], using the variational formulation of the problem. The regularity of the free boundary was subsequently established in step two groups [DGP07]. Further results in this area have been obtained for Kolmogorov operators and parabolic non-divergence form operators of Hörmander type [DFPP08, FNPP10, FGN12, Fre13]. The no-sign obstacle-type problem in terms of the equation (1.1) does not seem to have been considered before in the subelliptic setting.

Our arguments are remarkably different from the ones used for the obstacle problem. For instance, in this problem without a forcing term the solution is automatically superharmonic with respect to Δ_H, while in our setting we have no such sign condition that would yield a superharmonic solution. We initiate our analysis observing that second order horizontal derivatives of solutions to (1.1) satisfy certain BMO estimates, that have been established by Bramanti et al. [BB05, BF13]. The subsequent step is to construct suitable approximating polynomials, starting from the second order horizontal derivatives of the solution. Indeed these polynomials yield a subquadratic growth estimate (4.10) at small scales. We point out that this estimate is valid for any bounded and $W^{2,p}$ regular function, with bounded sub-Laplacian, so it might be of independent interest. As a consequence, we perform a suitable rescaling of the equation and then infer the crucial decay estimate of the measure of the coincidence set (Proposition 4.6), when the horizontal Hessian of the approximating polynomial is sufficiently large. More details on this procedure can be found at the beginning of Section 4.

Although our ideas mainly follow the path set up by Andersson et al. [ALS13], and Figalli and Shahgholian [FS12], there are several difficulties related to the subelliptic setting. The basic one is concerned with the fact that the sub-Laplacian Δ_H is degenerate elliptic. In addition, since the operator Δ_H is written in terms of Hömander vector fields, we can only consider the *horizontal Hessian* (2.5) of the solution, that is a nonsymmetric matrix. Then the construction of the approximating polynomials starting from the average of the second order noncommuting derivatives X_iX_ju becomes more delicate and requires some preliminary algebraic work, see Section 3. Notwithstanding the technical complications, the proof has become more streamlined: we can stay clear of the projection operator used in [ALS13], and this simplifies several technical points. A suitable quantitative decay estimate of the zero-level set (4.14) can be obtained also in our setting. Finally, we adapt Caffarelli's polynomial iteration technique of [Caf89] to find explicit estimates of the second order horizontal derivatives, see (1.4).

We finish the introduction by giving an overview of the paper. In Section 2 we introduce some basic notions on stratified groups and the related function spaces. In Section 3 we construct suitable second order homogeneous polynomials (Definition 3.2), that have an assigned horizontal Hessian (Corollary 3.3). Then some important $W^{2,p}$ and BMO estimates are presented. Finally, we provide the crucial scaling estimates of Lemma 3.8. In Section 4 we prove a subquadratic growth estimate of the difference between a solution and its approximating polynomial. Then we apply the subquadratic growth estimate to get a suitable decay of the measure of the zero-level set. Finally, we establish the $C^{1,1}_H$ regularity in quantitative terms, according to (1.4).

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2. Basic facts and notation

A *stratified group* is a simply connected, real nilpotent Lie group $\mathbb{G}$, whose Lie algebra $\mathcal{G}$ has a special stratification. We denote by V_i the subspaces of $\mathcal{G}$, having the properties:

$$\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_\iota \quad \text{and} \quad [V_1, V_j] = V_{j+1}$$

for $j = 1, \ldots, \iota$ and $V_{\iota+1} = 0$. Let us denote by n the topological dimension of $\mathbb{G}$ and by m the dimension of V_1. We choose a *graded basis* $X_1, X_2, \ldots, X_n$ of $\mathcal{G}$, that is characterized by the property that

$$X_{m_{j-1}+1}, \ldots, X_{m_j}$$

is a basis of V_j for all $j = 1, \ldots, \iota$, where we have set $m_0 = 0$, $m_1 = m$ and $m_j = \sum_{i=1}^{j} \dim V_i$. We notice that $m_\iota = n$ and with these definitions, if $m_{k-1} < j \le m_k$, then $k \in \mathbb{N}$ is uniquely determined and we define the positive integer

$$d_j := k. \tag{2.1}$$

Through the exponential mapping of $\mathbb{G}$, one can construct a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{G}$. Hence we have defined a *graded basis* $e_1, e_2, \ldots, e_n$ of $\mathbb{R}^n$ and *graded coordinates* $x_1, x_2, \ldots, x_n$ that define the point $x = (x_1, x_2, \ldots, x_n)$ of $\mathbb{G}$. This allows us to identify $\mathbb{G}$ with $\mathbb{R}^n$, as it will be understood in the sequel.

In addition, one may also verify that the Lebesgue measure of $\mathbb{R}^n$ through the graded coordinates yields the Haar measure of the group $\mathbb{G}$. The notation $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$.

The diffeomorphism associated to graded coordinates has also the property that the group operation on $\mathbb{G}$, when read in $\mathbb{R}^n$, is given by a special polynomial group operation

$$xy = x + y + BCH(x, y), \tag{2.2}$$

where the precise form of the vector polynomial $BCH : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by the important *Baker–Campbell–Hausdorff formula*, in short BCH formula, see for instance [Var84]. The *degree* of x_j is the integer d_j defined in (2.1) and we define *intrinsic dilations* as follows

$$\delta_r x = (r x_1, \ldots, r x_m, r^2 x_{m+1}, \ldots, r^2 x_{m_2}, \ldots, r^\iota x_{m_{\iota-1}+1}, \ldots, r^\iota x_n) = \sum_{j=1}^{n} r^{d_j} x_j e_j$$

for any $r > 0$. The notion of degree fits the algebraic properties of dilations, since

$$\delta_r(xy) = (\delta_r x)(\delta_r y) \tag{2.3}$$

for all $x, y \in \mathbb{R}^n$. By the form of dilations, for every measurable set $A \subset \mathbb{R}^n$ we have

$$|\delta_r(A)| = r^Q |A|$$

for all $r > 0$, where $Q = \sum_{j=1}^{n} d_j$ can be proved to be the Hausdorff dimension of $\mathbb{G}$.

The metric structure of $\mathbb{R}^n$ is given by a control distance. We say that $\gamma : [0,T] \to \mathbb{R}^n$, an absolutely continuous curve, is *admissible* if for a.e. $t \in [0,1]$ there holds

$$\dot\gamma(t) = \sum_{i=1}^m b_i(t) X_i(\gamma(t))$$

and $\sum_{i=1}^m b_i(t)^2 \le 1$. We denote by $\mathcal{H}(x,y)$ the family of all admissible curves whose image contains x, y. By Chow's theorem, $\mathcal{H}(x,y)$ is nonempty for every $x, y \in \mathbb{R}^n$, hence the following "control distance"

$$d(x,y) = \inf\left\{T > 0 \middle| \gamma : [0,T] \to \mathbb{R}^n,\ \gamma \in \mathcal{H}(x,y)\right\}$$

is well defined. It is also possible to check that d is actually a distance, corresponding to the well known *Carnot–Carathéodory distance*.

Since left translations preserve the "horizontal velocity", d is also *left invariant*, namely $d(x,y) = d(zx,zy)$ for all $x, y, z \in \mathbb{R}^n$. Furthermore, dilations are Lie group homomorphisms, hence the Carnot–Carathéodory distance is homogeneous in the sense that $d(\delta_r x, \delta_r y) = r d(x,y)$ for every $x, y \in \mathbb{R}^n$ and $r > 0$.

Definition 2.1 (Metric balls)**.** For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x)$ the open ball of center x and radius $r > 0$ with respect to d. Precisely, this is the set $\{y \in \mathbb{R}^n : d(x,y) < r\}$. When $x = 0$, we use the notation $B_r := B_r(0)$.

From the properties of d and δ_r, it is easy to observe that

$$B_r(x) = x\delta_r(B_1).$$

Dilations also allow us to introduce a natural notion of homogeneity, so we may say that a polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is *k-homogeneous* if

$$p(\delta_r x) = r^k p(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0.$$

The number $k \in \mathbb{N}$ is the *degree* of p. Moreover, any vector field X_j of the fixed graded basis can be identified with a first order differential operator of the form

$$X_j = \partial_{x_j} + \sum_{i=m_{d_j}+1}^{n} a_{ji}\partial_{x_i} \tag{2.4}$$

for every $j = 1, \dots, n$. The functions $a_{ji} : \mathbb{R}^n \to \mathbb{R}$ are homogeneous polynomials of degree $d_i - d_j \ge 1$ and in particular $X_j(0) = e_j$ for all $j = 1, \dots, n$. In the sequel Ω will be understood as an open bounded subset of $\mathbb{G}$, that can be also identified with an open subset of $\mathbb{R}^n$, if not otherwise stated.

Given a function $u : \Omega \to \mathbb{R}$ and considering the vector fields X_j as differential operators, we may introduce the *horizontal gradient* and the *horizontal Hessian*

$$\nabla_h u = (X_1 u, \dots, X_m u) \quad \text{and} \quad D_h^2 u = \begin{pmatrix} X_1X_1u & X_1X_2u & \cdots & X_1X_mu \\ X_2X_1u & X_2X_2u & \cdots & X_2X_mu \\ \vdots & \vdots & \ddots & \vdots \\ X_mX_1u & \cdots & \cdots & X_mX_mu \end{pmatrix}, \tag{2.5}$$

respectively, whenever they are pointwise defined. More generally, we can define higher order differential operators considering for $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ the function

$$X^I u := X_1^{i_n} \cdots X_n^{i_1} u.$$

Definition 2.2 (Folland–Stein spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $C^1_H(\Omega)$ the space of all functions $u : \Omega \to \mathbb{R}$ such that the horizontal derivatives $X_j u$ exist on Ω for all $j = 1, \dots, m$ and are continuous. If $0 < \alpha \leq 1$, then $C^{1,\alpha}_H(\Omega)$ is the space of functions u in $C^1_H(\Omega)$ such that there exists $C > 0$ with the property that

$$|X_j f(x) - X_j f(y)| \leq C\, d(x,y)^\alpha$$

for every $x, y \in \Omega$ and $j = 1, \dots, m$.

Notice that $D^2_h u$ is not symmetric, since the vector fields X_j do not commute in general. We say that $X_j u$ are the *horizontal derivatives* and $X_i X_j u$ are the *second order horizontal derivatives.* The *symmetrized horizontal Hessian* is defined as

$$D^{2,s}_h u = \frac{1}{2}\left(D^2_h u + D^2_h u^T\right).$$

The sub-Laplacian is defined as

$$\Delta_H u = \sum_{j=1}^m X_j^2 u.$$

Functions satisfying $\Delta_H u = 0$ are called as usual *harmonic functions*.

Definition 2.3 (Fundamental solution)**.** We say that $\Gamma \in C^\infty(\mathbb{G} \setminus \{0\})$ is a *fundamental solution* for Δ_H if it is locally summable, it vanishes at infinity and satisfies $\Delta_H \Gamma = -\delta_0$, where δ_0 denotes the Dirac distribution centered at the origin.

The fundamental solution Γ defines a gauge $d_G = \Gamma^{1/(2-Q)}$, that is 1-homogeneous with respect to dilations and continuous on $\mathbb{R}^n$. We can readily check that there exists a constant $c_0 > 1$ such that

$$c_0^{-1} d_G(x) \leq d(x,0) \leq c_0 d_G(x) \tag{2.6}$$

for all $x \in \mathbb{R}^n$. Defining $d_G(x,y) := d_G(x^{-1}y)$ we also introduce the *gauge ball*

$$B^G_r(x) = \{y \in \mathbb{R}^n : d_G(x,y) < r\}. \tag{2.7}$$

The previous estimates clearly imply that

$$B^G_r(x) \subset B_{c_0 r}(x) \tag{2.8}$$

for every $r > 0$ and $x \in \mathbb{R}^n$.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let ϑ be harmonic in Ω. We consider an open set $\Omega' \subset \Omega$ and $h > 0$ such that*

$$\mathrm{dist}_G(\Omega', \Omega^c) := \inf\{d_G(x,y) : x \in \Omega',\, y \in \Omega^c\} > h.$$

Then $\vartheta \in C^\infty(\Omega)$ and for every multiindex I there exists a constant $C_{I,h} > 0$ such that

$$|X^I \vartheta(x)| \leq C_{I,h} \|\vartheta\|_{L^1(\Omega)} \tag{2.9}$$

Proof. We consider the function ϕ defined in [BLU07, (5.50e)], where we choose φ appearing in the definition of ϕ, such that $\varphi \in C_c^\infty(]3^{-1}, 1[)$, $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$. It follows that ϕ is smooth and bounded on $\mathbb{R}^n$, along with all of its derivatives, and it is compactly supported in B_1^G, see the definition (2.7). We also consider $\phi_r(z) := r^{-Q}\phi(\delta_{1/r}z)$, that is compactly supported on B_r^G. We finally set $\hat{\phi}_r(z) := \phi_r(z^{-1})$ for all $z \in \mathbb{R}^n$ and $r > 0$. Thus, using [BLU07, (5.50a),(5.50d)], for every $x \in B_\lambda$, we get

$$\vartheta(x) = \int_{B_h^G(x)} \phi_h(x^{-1}y)\vartheta(y)dy = \int_\Omega \hat{\phi}_h(y^{-1}x)\vartheta(y)dy.$$

We can differentiate the last integral an arbitrary number of times, due to the smoothness of $\hat{\phi}_h$, getting the smoothness of ϑ and the following estimate

$$|X^I\vartheta(x)| = \left|\int_\Omega X^I\hat{\phi}_h(y^{-1}x)\vartheta(y)dy\right| \leq \|X^I\hat{\phi}_h\|_{L^\infty(\mathbb{R}^n)}\|\vartheta\|_{L^1(\Omega)}$$

for all $x \in \Omega'$. This concludes the proof. □

In our setting, we need the notion of Sobolev function adapted to the horizontal vector fields $X_1, \dots, X_m$, see [Fol75]. The horizontal Sobolev space $W_H^{k,p}(\Omega)$ consists of those functions $u \in L^p(\Omega)$ for which, for all $j_s \in \{1, \dots, m\}$ and $s \in \{1, \dots, k\}$, there exists a function $v_{j_1,\dots,j_k} \in L^p(\Omega)$ such that

$$\int_\Omega u(y)(X_{j_1}\cdots X_{j_k}\phi)(y)\, dy = (-1)^k \int_\Omega v_{j_1,\dots,j_k}(y)\phi(y)\, dy$$

for any function $\phi \in C_c^\infty(\Omega)$. Also in the more general setting of Hörmander vector fields some Sobolev embedding theorems hold, see [GN96, Theorem 1.11 and (3.19)], or [Lu96, Theorem 1.1]. The next theorem specializes these embedding results for stratified groups.

Theorem 2.5. *Let $p > Q$, where Q is the Hausdorff dimension of $\mathbb{G}$ and let $\Omega' \Subset \Omega$ be any open and relatively compact subset. Then there exists $C > 0$, depending on Ω', such that for every $u \in W_H^{1,p}(\Omega)$, up to a modification of u on a negligible set, we have*

$$|u(x) - u(y)| \leq C\, \|u\|_{W_H^{1,p}(\Omega)}\, d(x,y)^{1-\frac{Q}{p}},$$

for every $x, y \in \Omega'$.

The following (1,1)-Poincaré inequality holds,

$$\int_{B_r(x)} |u(y) - u_{B_r(x)}|dy \leq cr \int_{B_r(x)} |\nabla_h u(y)|dy \tag{2.10}$$

for every $u \in C^1(\overline{B_r(x)})$. This inequality follows from [Jer86], see also [LM00].

For any measurable function u that is summable on a measurable set $A \subset \Omega$, we use the notation

$$u_A := \fint_A u(y)\, dy = \frac{1}{|A|}\int_A u(y)\, dy.$$

Definition 2.6. For $u \in L^1(\Omega)$, we define the BMO seminorms

$$[u]_{BMO(\Omega)} := \sup_{x_0\in\Omega, r>0} \fint_{B_r(x_0)\cap\Omega} |u(y) - u_{B_r(x_0)}|\, dy,$$
$$[u]_{BMO_{\mathrm{loc}}(\Omega)} := \sup_{B_r(x_0)\subset\Omega} \fint_{B_r(x_0)} |u(y) - u_{B_r(x_0)}|\, dy$$

and for $1 \le p < \infty$ the corresponding BMO^p norms

$$\|u\|_{BMO^p(\Omega)} := [u]_{BMO(\Omega)} + \|u\|_{L^p(\Omega)},$$
$$\|u\|_{BMO^p_{\mathrm{loc}}(\Omega)} := [u]_{BMO_{\mathrm{loc}}(\Omega)} + \|u\|_{L^p(\Omega)}.$$

The spaces $BMO^p(\Omega)$ and $BMO^p_{\mathrm{loc}}(\Omega)$ consist of all L^p functions on Ω with finite BMO^p and BMO^p_{loc} norm, respectively. See [BF13] for more information on BMO functions in the subelliptic setting.

3. Preparatory results

We first study the relationship between the coefficients of a 2-homogeneous polynomial and its second order horizontal derivatives. Then, by some $W^{2,p}_H$ and BMO estimates, we show how to control the horizontal Hessian of a Sobolev function by the horizontal Hessian of a suitable 2-homogeneous harmonic polynomial (Corollary 3.7). Finally, in Lemma 3.8 we establish a quantitative control on the growth of these polynomials at small scales.

We need first to find 2-homogeneous polynomials with assigned second order horizontal derivatives. To do this, we first observe that (2.2), combined with (2.3) and (2.1), setting

$$BCH(x,y) = \sum_{j=m+1}^{n} q_j(x,y)e_j,$$

imply that any q_j is a homogeneous polynomial of degree d_j. Due to the BCH formula, one can also prove that q_l is a 2-homogeneous polynomial with respect to the variables $x_1, \dots, x_m, y_1, \dots, y_m$ for all $l = m+1, \dots, m_2$ and

$$q_l(x,y) = -q_l(y,x).$$

From the definition of left invariant vector field, we get

$$a_{jl}(x) = \frac{\partial q_l}{\partial y_j}(x,0),$$

for all $j = 1, \dots, m$ and $l = m+1, \dots, m_2$. As a consequence, we get

$$\frac{\partial a_{jl}}{\partial x_i} = \frac{\partial^2 q_l}{\partial x_i \partial y_j} = -\frac{\partial^2 q_l}{\partial x_j \partial y_i} = -\frac{\partial a_{il}}{\partial x_j} \tag{3.1}$$

for all $i, j = 1, \dots, m$ and $l = m+1, \dots, m_2$. Notice that the partial derivatives in the previous equalities are all constant functions. Equalities (3.1) will be important in the proof of Proposition 3.1.

Every polynomial on $\mathbb{R}^n$, thought of as equipped with dilations δ_r, is the sum of homogeneous polynomials and the maximum among these degrees is the *degree of the polynomial*. Polynomials of degree one are just affine functions ℓ of the form

$$\ell(x) = \alpha + \langle \beta, \pi(x) \rangle$$

with $\beta = (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and we have used the projection

$$\pi : \mathbb{R}^n \to \mathbb{R}^m, \quad \pi(x) = (x_1, \dots, x_m). \tag{3.2}$$

A homogeneous polynomial of degree two must have the form

$$p(x) = \frac{1}{2} \sum_{i,j=1}^{m} c_{ij} x_i x_j + \sum_{l=m+1}^{m_2} c_l x_l, \tag{3.3}$$

where c_{ij} and c_l are real numbers, with $c_{ij} = c_{ji}$ for all $i, j = 1, \dots, m$.

Proposition 3.1. *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a 2-homogeneous polynomial of the form* (3.3) *and let us consider the basis $X_{m+1}, \dots, X_{m_2}$ of V_2. Then we have*

$$c_{ij} = \frac{1}{2}(X_i X_j p + X_j X_i p) \quad \textit{and} \quad X_i X_j p = c_{ij} + \sum_{l=m+1}^{m_2} \gamma_{ij}^l c_l,$$

where γ_{ij}^l are proportional to the structure constants of the Lie algebra, namely

$$[X_i, X_j] = \sum_{l=m+1}^{m_2} 2\gamma_{ij}^l X_l \tag{3.4}$$

and $i, j = 1, \dots, m$.

Proof. We first define the symmetrized second order derivative

$$(X_i X_j)^s := \frac{X_i X_j + X_j X_i}{2},$$

so that we can write

$$X_i X_j = (X_i X_j)^s + \frac{1}{2}[X_i, X_j], \tag{3.5}$$

for every $i, j = 1, \dots, m$. Since $X_i X_j$ and X_l are homogeneous differential operators of order -2 and p has degree 2, the horizontal derivatives $X_i X_j p$ and $X_l p$ are constants.

By (3.3) and (2.4), we get

$$X_j p = \sum_{i=1}^{m} c_{ji} x_i + \sum_{i=m+1}^{n} a_{ji} \partial_{x_i} \left(\sum_{l=m+1}^{m_2} c_l x_l \right) = \sum_{i=1}^{m} c_{ji} x_i + \sum_{i=m+1}^{m_2} a_{ji} c_i,$$

for $j = 1, \dots, m$. As a consequence, taking into account the form of the vector fields (2.4) and of the polynomial (3.3), for any $i, j = 1, \dots, m$ and $l = m+1, \dots, m_2$, we get

$$X_i X_j p = c_{ij} + \sum_{s=m+1}^{m_2} \partial_{x_i} a_{js} c_s. \tag{3.6}$$

To establish the previous equality, we have also observed that the polynomials a_{ji} are homogeneous of degree $d_i - d_j = 1$, therefore they are only depending on their first m variables. In particular, all the partial derivatives $\partial_{x_l} a_{ji}$ are vanishing whenever the integers l and i take values from $m+1$ to m_2 and $j = 1, \dots, m$. Combining (3.6) and (3.1), we also obtain the first of the following equalities

$$X_i X_j p + X_j X_i p = 2c_{ij} \quad \text{and} \quad X_l p = c_l,$$

with $1 \le i, j \le m$ and $m+1 \le l \le m_2$. The latter directly follows from the form of (3.3). In conclusion, by virtue of (3.4), (3.5) and (3.6), we have obtained that

$$X_i X_j p = (X_i X_j)^s p + \sum_{l=m+1}^{m_2} \gamma_{ij}^l X_l p = c_{ij} + \sum_{l=m+1}^{m_2} \gamma_{ij}^l c_l,$$

hence concluding the proof. □

Definition 3.2. For $B_r(x_0) \Subset \Omega$ and $u \in W^{2,1}_{H,\mathrm{loc}}(\Omega)$, we define the matrix

$$P_r^{x_0} := (D_h^2 u)_{B_r(x_0)} - \frac{1}{m} (\Delta_H u)_{B_r(x_0)} I_m \in \mathbb{R}^{n \times n},$$

where I_m stands for the identity matrix and the (i,j) entry of $(D_h^2 u)_{B_r(x_0)}$ is the average $(X_i X_j u)_{B_r(x_0)}$. Associated to the ball $B_r(x_0)$, we also define the coefficients

$$c_{ij}^{r,x_0} := \left(\frac{X_i X_j u + X_j X_i u}{2} \right)_{B_r(x_0)} - \frac{1}{m} \delta_{ij} \, (\Delta_H u)_{B_r(x_0)} \quad \text{and} \quad c_l^{r,x_0} = (X_l u)_{B_r(x_0)}.$$

These numbers define the 2-homogeneous polynomial

$$p_r^{x_0}(x) = \frac{1}{2} \sum_{i,j=1}^{m} c_{ij}^{r,x_0} x_i x_j + \sum_{l=m+1}^{m_2} c_l^{r,x_0} x_l,$$

that we will show to be related to $P_r^{x_0}$.

Corollary 3.3. *In the assumptions of Definition 3.2, the 2-homogeneous polynomial $p_r^{x_0}$ is harmonic and*

$$D_h^2 p_r^{x_0} = P_r^{x_0}.$$

Proof. By Proposition 3.1, we have

$$X_i X_j p_r^{x_0} = c_{ij}^{r,x_0} + \sum_{l=m+1}^{m_2} \gamma_{ij}^l c_l^{r,x_0},$$

where $\gamma_{ii}^l = 0$ and by definition of c_{ii}^{r,x_0} we get

$$\Delta_H p_r^{x_0} = \sum_{i=1}^{m} c_{ii}^{r,x_0} = \sum_{i=1}^{m} \left[(X_i X_i u)_{B_r(x_0)} - \frac{1}{m} \, (\Delta_H u)_{B_r(x_0)} \right] = 0.$$

Finally, we observe that

$$\begin{aligned} X_iX_jp_r^{x_0} &= \Big(\frac{X_iX_ju + X_jX_iu}{2}\Big)_{B_r(x_0)} - \frac{1}{m}\delta_{ij}\left(\Delta_H u\right)_{B_r(x_0)} + \left(\sum_{l=m+1}^{m_2}\gamma_{ij}^l X_l u\right)_{B_r(x_0)} \\ &= \Big(\frac{X_iX_ju + X_jX_iu}{2}\Big)_{B_r(x_0)} - \frac{1}{m}\delta_{ij}\left(\Delta_H u\right)_{B_r(x_0)} + \left(\frac{[X_i,X_j]u}{2}\right)_{B_r(x_0)} \\ &= \left(X_iX_ju\right)_{B_r(x_0)} - \frac{1}{m}\delta_{ij}\left(\Delta_H u\right)_{B_r(x_0)}, \end{aligned}$$

having taken into account that $[X_i, X_j] = \sum_{l=m+1}^{m_2} 2\gamma_{ij}^l X_l$ from Proposition 3.1. □

The following $W_H^{2,p}$ estimates go back to the work of Folland [Fol75], see also the work by Bramanti and Brandolini [BB00] for more general hypoelliptic operators.

Theorem 3.4 ([BB00])**.** *Let $1 < p < \infty$ and consider two bounded open sets Ω and Ω' with $\Omega' \Subset \Omega$. Then there exists a constant $C > 0$ such that for every $u \in W_H^{2,p}(\Omega)$ it holds*

$$\|X_iX_ju\|_{L^p(\Omega')} \leq C\left(\|\Delta_H u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right). \tag{3.7}$$

It is well known that even for the classical Laplacian operator Δ, it is not true that L^∞ bounds on Δu imply the boundedness of second order horizontal derivatives. Indeed our starting point is that bounds on the BMO norm of the sub-Laplacian $\Delta_H u$ show that the BMO norm of the horizontal Hessian of u is bounded, according to the results of Bramanti et al. [BB05], [BF13].

Theorem 3.5 ([BF13, Theorem 2.10])**.** *Let $1 < p < \infty$, $0 < \sigma < 1$, $u \in BMO^p_{loc}(B_1)$ and let $\Delta_H u \in BMO^p_{loc}(B_1)$. Then $X_iX_ju \in BMO^p(B_\sigma)$ for $i, j = 1, \dots, m$ and there exists a universal constant $C(\sigma, p) > 0$ such that*

$$\|X_iX_ju\|_{BMO^p(B_\sigma)} \leq C(\sigma,p)\left(\|\Delta_H u\|_{BMO^p_{loc}(B_1)} + \|u\|_{BMO^p_{loc}(B_1)}\right). \tag{3.8}$$

Remark 3.6. Note that the nonvariational form of the operator in [BF13, Theorem 2.10] needs a priori that the solution u and its horizontal derivatives are BMO. For our purposes, it is very important that the BMO regularity of u is established with no a priori assumptions. This can be obtained for the sub-Laplacian operator, since its distributional form allows us to apply a mollification argument.

In the sequel, we will also use the Frobenius norm $|M|$ for a matrix M of coefficients m_{ij}, setting

$$|M| = \sqrt{\sum_{ij} |m_{ij}|^2}.$$

With this definition we easily notice that $|\Delta_H u| \leq |D_h^2 u|$.

Corollary 3.7. *Let $1 < p < \infty$ and $0 < \sigma < 1$ be fixed. There exists $C(\sigma, p) > 0$ such that for all $u \in BMO^p_{loc}(B_1)$ that satisfy the condition $\Delta_H u \in L^\infty(B_1)$, we have*

$X_iX_ju \in BMO^p(B_\sigma)$ for $i, j = 1, \dots, m$ and whenever $x_0 \in B_\sigma$, $0 < r < 1 - \sigma$, it holds

$$\fint_{B_r(x_0)\cap B_\sigma} |D_h^2 u(y) - P_r^{x_0}|dy \leq C(\sigma, p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{loc}(B_1)}\right), \tag{3.9}$$

where the matrix $P_r^{x_0}$ is introduced in Definition 3.2.

Proof. Theorem 3.5 immediately implies that $X_iX_ju \in BMO^p(B_\sigma)$ and (3.8) holds. Thus, we obtain the following estimates

$$\begin{aligned}
&\fint_{B_r(x_0)\cap B_\sigma} |D_h^2 u(y) - P_r^{x_0}|dy \\
&\leq \fint_{B_r(x_0)\cap B_\sigma} |D_h^2 u(y) - (D_h^2 u)_{B_r(x_0)}|dy + \frac{1}{m}|(\Delta_H u)_{B_r(x_0)} I_m| \\
&\leq [D_h^2 u]_{BMO(B_\sigma)} + \frac{1}{\sqrt{m}}\|\Delta_H u\|_{L^\infty(B_1)} \\
&\leq \widetilde{C}(\sigma, p)\left(\|\Delta_H u\|_{BMO^p_{\text{loc}}(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right) + \|\Delta_H u\|_{L^\infty(B_1)} \\
&\leq C(\sigma, p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right).
\end{aligned}$$

This completes the proof. □

Heuristically, if $D_h^2 u$ is not bounded around x_0, since the difference of $D_h^2 u$ and $P_r^{x_0}$ is controlled, the BMO estimate tells us that also $P_r^{x_0}$ becomes unbounded as $r \to 0^+$. Hence we will turn our attention to $P_r^{x_0}$. In the following lemma, we will derive a general "scaling estimate" for the difference $|P_{r_1}^{x_0} - P_{r_2}^{x_0}|$. In particular, when $r_2 = 2r_1$ we get a uniform bound on the growth of $|P_r^{x_0}|$ on dyadic scales.

Lemma 3.8 (Scaling estimates)**.** *Let $1 < p < \infty$ and $0 < \lambda_1 < 1$ be fixed. Then there exists a universal constant $C(\lambda_1, p) > 0$ such that for all $u \in BMO^p_{loc}(B_1)$ that satisfy the condition $\Delta_H u \in L^\infty(B_1)$ the following holds. We have $X_iX_ju \in L^1_{loc}(B_1)$, where $i, j = 1, \dots, m$ and for $x_0 \in B_{\lambda_1/3}$ the matrices of the form*

$$P_r^{x_0} := (D_h^2 u)_{B_r(x_0)} - \frac{1}{m}(\Delta_H u)_{B_r(x_0)} I_m,$$

with $0 < r_1 < \min\{2\lambda_1/3, 1 - \lambda_1\}$ and $r_1 < r_2 < 1 - \lambda_1$, satisfy the following inequality

$$|P_{r_1}^{x_0} - P_{r_2}^{x_0}| \leq \left(\frac{r_2}{r_1}\right)^Q C(\lambda_1, p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{loc}(B_1)}\right).$$

Proof. Due to the BMO estimate (3.9) with $\sigma = \lambda_1$, we can estimate $|P_{r_1}^{x_0} - P_{r_2}^{x_0}|$ as follows

$$\begin{aligned}
&\fint_{B_{r_1}(x_0)\cap B_{\lambda_1}} |P_{r_1}^{x_0} - P_{r_2}^{x_0}|dx \\
&\le \fint_{B_{r_1}(x_0)\cap B_{\lambda_1}} |D_h^2 u(y) - P_{r_1}^{x_0}|dy + \fint_{B_{r_1}(x_0)\cap B_{\lambda_1}} |D_h^2 u(y) - P_{r_2}^{x_0}|dy \\
&\le \fint_{B_{r_1}(x_0)\cap B_{\lambda_1}} |D_h^2 u(y) - P_{r_1}^{x_0}|dy + \frac{|B_{r_2}(x_0)\cap B_{\lambda_1}|}{|B_{r_1}(x_0)\cap B_{\lambda_1}|} \fint_{B_{r_2}(x_0)\cap B_{\lambda_1}} |D_h^2 u(y) - P_{r_2}^{x_0}|dy \\
&\le C(\lambda_1, p)\left(1 + \left(\frac{r_2}{r_1}\right)^Q\right)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\mathrm{loc}}(B_1)}\right).
\end{aligned}$$

The last inequality follows by taking into account our conditions on the radii r_1 and r_2. Indeed, we have

$$\frac{|B_{r_2}(x_0)\cap B_{\lambda_1}|}{|B_{r_1}(x_0)\cap B_{\lambda_1}|} \le \frac{|B_{r_2}(x_0)|}{|B_{r_1}(x_0)|} = \left(\frac{r_2}{r_1}\right)^Q.$$

Finally, with a slight abuse of notation, we denote the constant $2C(\lambda_1, p)$ again by $C(\lambda_1, p)$ in the inequality of the lemma, concluding the proof. □

4. Proof of $C_H^{1,1}$ regularity

This section represents the core of the paper. We establish the sub-quadratic growth of the difference

$$u(y) - u(x_0) - \langle \nabla_h u(x_0), \pi(x_0^{-1}y)\rangle - p_r^{x_0}(x_0^{-1}y)$$

on the ball $B_r(x_0)$, where $p_r^{x_0}$ is the harmonic polynomial introduced in Definition 3.2. We show that when the norm of $D_h^2 p_r^{x_0}$ is sufficiently large, then the measure of the coincidence set $\{u = 0\}$ decays in a quantitative way. This is one of the central facts, that leads us to the dichotomy argument of [ALS13] to reach the $C_H^{1,1}$ regularity. There are indeed two cases: (i) when $|D_h^2 p_r^{x_0}|$ is uniformly bounded as $r \to 0^+$, we immediately infer the regularity from the subquadratic growth, (ii) if otherwise $|D_h^2 p_r^{x_0}|$ grows without bound as $r \to 0^+$, then the coincidence set is "small" and we show that a suitable adaptation of Caffarelli's polynomial iteration technique can lead us to the $C_H^{1,1}$ regularity.

In the sequel, whenever we consider a function u with essentially bounded sub-Laplacian $\Delta_H u$, then it is understood that u is chosen to be of class C_H^1. The following remark rigorously justifies this convention.

Remark 4.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u : \Omega \to \mathbb{R}$ be a locally summable function such that $\Delta_H u \in L^\infty(\Omega)$. From [Fol75, Theorem 6.1], we have $u \in W^{2,p'}_{H,\mathrm{loc}}(\Omega)$ for every $p' > 1$. In view of Theorem 2.5, by standard arguments, we can modify u on a negligible set such that $u \in C_H^{1,\alpha}(\Omega')$ for any relatively compact open set $\Omega' \Subset \Omega$, where we have fixed some $p' > Q$ and $\alpha = 1 - \frac{Q}{p'}$. In particular, we have shown that, after the modification, $u \in C_H^1(\Omega)$.

Lemma 4.2 (Sub-quadratic growth). *Assume $u \in BMO^p_{loc}(B_1)$ such that $\Delta_H u \in L^\infty(B_1)$. Let $\lambda, \sigma \in (0,1)$ and fix $p > 1$. Then there exist $r_0 > 0$ and a universal constant $C(\lambda,\sigma,p) > 0$, such that for any $x_0 \in B_\lambda$ and $0 < r \le r_0$, assuming that*

$$u(x_0) = X_i u(x_0) = 0, \quad 1 \le i \le m$$

and considering $p_r^{x_0}$, as given in Definition 3.2, the following estimate holds

$$\sup_{y \in B_{\sigma r}(x_0)} |u(y) - p_r^{x_0}(x_0^{-1}y)| \le C(\lambda,\sigma,p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{loc}(B_1)}\right) r^2.$$

Proof. We fix $x_0 \in B_\lambda$ and $\lambda' = (1+\lambda)/2$, so that for $0 < r \le \lambda' - \lambda$, we have the inclusion

$$B_r(x_0) \subset B_{\lambda'}. \tag{4.1}$$

Let us introduce the translated and rescaled function

$$u_{r,x_0}(x) := \frac{u(x_0 \delta_r x) - p_r^{x_0}(\delta_r x)}{r^2},$$

observing that it is well defined in B_1. Taking into account that $u \in W^{2,p}_{H,\mathrm{loc}}(B_1)$ and $\overline{B_r(x_0)} \subset \overline{B_{\lambda'}} \subset B_1$, then $u_{r,x_0} \in W^{2,p}_H(B_1)$. We are in the position to apply Corollary 3.7 to u with $\sigma = \lambda'$. As a consequence of both Corollary 3.3 and (3.9), taking into account (4.1), it follows that

$$\begin{aligned}\|D_h^2 u_{r,x_0}\|_{L^1(B_1)} &= |B_1| \fint_{B_1} |D_h^2 u_{r,x_0}(x)|\,dx \\ &= |B_1| \fint_{B_r(x_0)} |D_h^2 u(y) - P_r^{x_0}|dy \\ &\le C(\lambda,p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\mathrm{loc}}(B_1)}\right).\end{aligned} \tag{4.2}$$

Now we wish to apply the Poincaré inequality (2.10) to $u_{r,x_0} - \ell_{r,x_0}$, where ℓ_{r,x_0} is an affine function to be properly defined. If we let

$$\ell_{r,x_0}(x) := (u_{r,x_0})_{B_1} + \langle (\nabla_h u_{r,x_0})_{B_1}, \pi(x)\rangle,$$

where $\pi(x) = (x_1, \dots, x_m)$, it follows that

$$\|u_{r,x_0} - \ell_{r,x_0}\|_{L^1(B_1)} \le c \fint_{B_1} \left|\nabla_h u_{r,x_0} - (\nabla_h u_{r,x_0})_{B_1}\right| dx,$$

since the average over B_1 of the linear part of ℓ_{r,x_0} is zero. Again, from the Poincaré inequality, using (4.2), we get

$$\begin{aligned}\|u_{r,x_0} - \ell_{r,x_0}\|_{L^1(B_1)} &\le C\|D_h^2 u_{r,x_0}\|_{L^1(B_1)} \\ &\le C(\lambda,p)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\mathrm{loc}}(B_1)}\right).\end{aligned} \tag{4.3}$$

For the sequel, we set $\hat{u}_{r,x_0} := u_{r,x_0} - \ell_{r,x_0}$. Since both $p_r^{x_0}$ and ℓ_{r,x_0} are harmonic, we observe that

$$\Delta_H \hat{u}_{r,x_0}(x) = (\Delta_H u)(x_0 \delta_r x) = f(x_0 \delta_r x)$$

for a.e. $x \in B_1$, where we have set $f := \Delta_H u \in L^\infty(B_1)$. We set $g_{r,x_0}(x) = f(x_0 \delta_r x)\chi_{B_1}$ and we consider the decomposition $\hat{u}_{r,x_0} = \hat{v}_{r,x_0} + \hat{w}_{r,x_0}$, where

$$\hat{v}_{r,x_0} = -g_{r,x_0} * \Gamma \quad \text{and} \quad \hat{w}_{r,x_0} = \hat{u}_{r,x_0} + g_{r,x_0} * \Gamma$$

and Γ is the fundamental solution for Δ_H, introduced in Definition 2.3. The explicit form of $\hat{v}_{r,x_0}$ allows us to get the estimate

$$|\hat{v}_{r,x_0}(x)| = \left| \int_{\mathbb{R}^n} \Gamma(z^{-1}x) g_{r,x_0}(z) dz \right| = \left| \int_{B_1} \Gamma(z^{-1}x) g_{r,x_0}(z) dz \right| \leq C \|g_{r,x_0}\|_{L^{Q_0}(B_1)}$$

for every $x \in B_1$, where $Q_0 = Q + 1$ and $C > 0$ can be seen as a universal constant. The previous estimate follows from the Hölder inequality, setting $q = Q_0/Q$ and taking into account the $(2 - Q)$-homogeneity of Γ. Indeed, it holds

$$\left| \int_{B_1} \Gamma(z^{-1}x) g_{r,x_0}(z) dz \right| \leq \left(\int_{B_2} |\Gamma|^q \right)^{1/q} \|g_{r,x_0}\|_{L^{Q_0}(B_1)} \tag{4.4}$$

for every $x \in B_1$. As a consequence, we have proved that

$$\|\hat{v}_{r,x_0}\|_{L^\infty(B_1)} \leq C \|\Delta_H \hat{u}_{r,x_0}\|_{L^{Q_0}(B_1)}. \tag{4.5}$$

Since $\hat{w}_{r,x_0}$ is harmonic, from [BLU07, (5.52)] we have the mean value type formula

$$\hat{w}_{r,x_0}(x) = \fint_{B^G_{(1-\sigma)/c_0}(x)} \Psi(x^{-1}z) \hat{w}_{r,x_0}(z)\, dz$$

for any $x \in B_\sigma$, whenever $0 < \sigma < 1$ and with $c_0 > 1$ defined in (2.6). We point out that the function Ψ is 0-homogeneous with respect to dilations and smooth on $\mathbb{R}^n \setminus \{0\}$, see [BLU07, Definition 5.5.1] for more information. Notice that with our assumptions we have the inclusion $B^G_{(1-\sigma)/c_0}(x) \subset B_1$. For every $x \in B_\sigma$, it holds

$$\begin{aligned} |\hat{w}_{r,x_0}(x)| &= \left| \fint_{B^G_{(1-\sigma)/c_0}(x)} \Psi(x^{-1}z) \hat{w}_{r,x_0}(z)\, dz \right| \\ &\leq \|\Psi\|_{L^\infty(B_1)} \fint_{B^G_{(1-\sigma)/c_0}(x)} |\hat{w}_{r,x_0}(z)|\, dz \\ &\leq \|\Psi\|_{L^\infty(B_1)} \frac{\|\hat{w}_{r,x_0}\|_{L^1(B_1)}}{|B^G_{(1-\sigma)/c_0}(x)|} \leq C(\sigma) \|\hat{w}_{r,x_0}\|_{L^1(B_1)}. \end{aligned}$$

The constant $C(\sigma)$ only depends on σ and it blows up as $\sigma \to 1^-$. By the triangle inequality and (4.5) we obtain that

$$\begin{aligned} \|\hat{w}_{r,x_0}\|_{L^\infty(B_\sigma)} &\leq C(\sigma) \|\hat{w}_{r,x_0}\|_{L^1(B_1)} \\ &\leq C(\sigma) \left(\|\hat{u}_{r,x_0}\|_{L^1(B_1)} + \|\hat{v}_{r,x_0}\|_{L^1(B_1)} \right) \\ &\leq C_1(\sigma) \left(\|\hat{u}_{r,x_0}\|_{L^1(B_1)} + \|\Delta_H \hat{u}_{r,x_0}\|_{L^{Q_0}(B_1)} \right). \end{aligned} \tag{4.6}$$

We conclude from both (4.5) and (4.6) that

$$\begin{aligned}\|\hat{u}_{r,x_0}\|_{L^\infty(B_\sigma)} &\leq \|\hat{v}_{r,x_0}\|_{L^\infty(B_\sigma)} + \|\hat{w}_{r,x_0}\|_{L^\infty(B_\sigma)} \\ &\leq C_2(\sigma)\left(\|\hat{u}_{r,x_0}\|_{L^1(B_1)} + \|\Delta_H \hat{u}_{r,x_0}\|_{L^{Q_0}(B_1)}\right).\end{aligned} \tag{4.7}$$

Differentiating $\hat{v}_{r,x_0}$, seen as an integral, it turns out that $\hat{v}_{r,x_0} \in C^1_H(B_1)$. Again arguing as in the proof of (4.4), from the Hölder inequality and the $(1-Q)$-homogeneity of $X_j\Gamma$, we get the estimate

$$|X_j \hat{v}_{r,x_0}(x)| = \left| \int_{B_1} X_j \Gamma(y^{-1}x)\, g_{r,x_0}(y) dy \right| \leq \overline{C} \|g_{r,x_0}\|_{L^{Q_0}(B_1)} \tag{4.8}$$

for every $j = 1, \ldots, m$, $x \in B_1$ and a fixed geometric constant $\overline{C} > 0$. By Proposition 2.4, we get a constant $C_3(\sigma) > 0$, such that

$$\|\nabla_h \hat{w}_{r,x_0}\|_{L^\infty(B_\sigma)} \leq C_3(\sigma) \|\hat{w}_{r,x_0}\|_{L^1(B_1)}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), along with the third inequality of (4.6), we establish the first of the following inequalities:

$$\begin{aligned}\|\hat{u}_{r,x_0}\|_{C^1_H(B_\sigma)} &\leq C_4(\sigma)\left(\|\hat{u}_{r,x_0}\|_{L^1(B_1)} + \|\Delta_H \hat{u}_{r,x_0}\|_{L^{Q_0}(B_1)}\right) \\ &\leq C_5(\sigma, p, \lambda)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right).\end{aligned}$$

The second inequality is a consequence of (4.3). Since $u_{r,x_0}(0) = X_j u_{r,x_0}(0) = 0$, by our assumptions on u, and taking into account that $p_r^{x_0}(0) = X_j p_r^{x_0}(0) = 0$, we immediately infer from the C^1_H estimate above that

$$\begin{aligned}&|\ell_{r,x_0}(0)| + \sum_{i=1}^m |X_i \ell_{r,x_0}(0)| \\ &= |\ell_{r,x_0}(0) - u_{r,x_0}(0)| + \sum_{i=1}^m |X_i \ell_{r,x_0}(0) - X_i u_{r,x_0}(0)| \\ &\leq \|\hat{u}_{r,x_0}\|_{C^1(B_\sigma)} \leq C_5(\sigma, p, \lambda)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right).\end{aligned}$$

It follows that

$$\|\ell_{r,x_0}\|_{L^\infty(B_\sigma)} \leq C_6(\sigma, p, \lambda)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right).$$

As a consequence, it follows that

$$\begin{aligned}\frac{1}{r^2} \sup_{y \in B_{\sigma r}(x_0)} |u(y) - p_r^{x_0}(x_0^{-1}y)| &= \sup_{x \in B_\sigma} \left| \frac{u(x_0 \delta_r x) - p_r^{x_0}(\delta_r x)}{r^2} \right| \\ &= \sup_{x \in B_\sigma} |u_{r,x_0}(x)| \\ &\leq \sup_{x \in B_\sigma} |\hat{u}_{r,x_0}(x)| + \sup_{x \in B_\sigma} |\ell_{r,x_0}(x)| \\ &\leq C(\sigma, p, \lambda)\left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{BMO^p_{\text{loc}}(B_1)}\right).\end{aligned}$$

This finishes the proof. □

Corollary 4.3. *Assume $u \in L^\infty(B_1)$ such that $\Delta_H u \in L^\infty(B_1)$ and fix $0 < \lambda, \sigma < 1$. If we consider π as in (3.2), then there exists $r_0 > 0$ such that the affine function*

$$\ell^{x_0}(z) := u(x_0) + \langle \nabla_h u(x_0), \pi(z) \rangle, \quad x_0 \in B_\lambda$$

satisfies the following properties. There exists a universal constant $C(\lambda, \sigma) > 0$ such that for every $r \in (0, r_0]$ the following estimate holds

$$\sup_{y \in B_{\sigma r}(x_0)} |u(y) - \ell^{x_0}(x_0^{-1}y) - p_r^{x_0}(x_0^{-1}y)| \leq C(\lambda, \sigma) \left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right) r^2, \tag{4.10}$$

where $p_r^{x_0}$ is as in Definition 3.2.

Proof. Our assumptions allow us to apply Lemma 4.2 to $y \to u(y) - \ell^{x_0}(x_0^{-1}y)$ with $p = 2$. Then there exist $r_0, C(\lambda, \sigma) > 0$ such that

$$\sup_{y \in B_{\sigma r}(x_0)} |u(y) - \ell^{x_0}(x_0^{-1}y) - p_r^{x_0}(x_0^{-1}y)| \leq C(\lambda, \sigma) \left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u - \ell^{x_0}\|_{BMO^2_{\mathrm{loc}}(B_1)} \right) r^2$$

for every $r \in (0, r_0]$. In addition, we have

$$\|u - \ell^{x_0}\|_{BMO^2_{\mathrm{loc}}(B_1)} \leq C \|u - \ell^{x_0}\|_{L^\infty(B_1)} \leq C'(\|u\|_{L^\infty(B_1)} + |\nabla_h u(x_0)|).$$

We set $f = \Delta_H u \in L^\infty(B_1)$ and write $v = f * \Gamma$, getting

$$|\nabla_h u(x_0)| \leq |\nabla_h (u + v)(x_0)| + |\nabla_h (f * \Gamma)(x_0)|.$$

Arguing as in [GT01, Lemma 4.1], we establish

$$|\nabla_h v(x_0)| = |\nabla_h (f * \Gamma)(x_0)| \leq \|\Delta_H u\|_{L^\infty(B_1)} \|\nabla_h \Gamma\|_{L^1(B_2)},$$

therefore we have

$$|\nabla_h u(x_0)| \leq |\nabla_h (u + v)(x_0)| + \|\Delta_H u\|_{L^\infty(B_1)} \|\nabla_h \Gamma\|_{L^1(B_2)}.$$

Since $u + v$ is harmonic in B_1, by (2.9), it follows that

$$\begin{aligned} |\nabla_h (u + v)(x_0)| &\leq C_0(\|u\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)}) \\ &\leq C_0(\|u\|_{L^\infty(B_1)} + \|\Delta_H u\|_{L^\infty(B_1)} \|\Gamma\|_{L^1(B_2)}). \end{aligned}$$

This immediately leads us to our claim. □

Remark 4.4. Notice that under the same assumptions of Corollary 4.3, we can assume that for every $\lambda, \sigma \in (0, 1)$ and any $x_0 \in B_\lambda$, there exist $\tilde{r}_0 > 0$ and $C > 0$, only depending on λ and σ, such that for all $r \in (0, \tilde{r}_0]$ the following estimate holds

$$\sup_{y \in B^G_{\sigma_0 r}(x_0)} |u(y) - \ell^{x_0}(x_0^{-1}y) - p_r^{x_0}(x_0^{-1}y)| \leq C(\lambda, \sigma) \left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right) r^2,$$

for $\sigma_0 = \sigma / c_0 \in (0, 1/c_0)$ and additionally we have the inclusion $\overline{B^G_{r_0}(x_0)} \subset B_1$. This is a consequence of the definition of c_0 in (2.8). If we set $r_0 := \tilde{r}_0 / c_0$, replacing r by $c_0 r$, we may rephrase the previous estimate as follows

$$\sup_{y \in B^G_{\sigma r}(x_0)} |u(y) - \ell^{x_0}(x_0^{-1}y) - p_{c_0 r}^{x_0}(x_0^{-1}y)| \leq C(\lambda, \sigma) c_0^2 \left(\|\Delta_H u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right) r^2 \tag{4.11}$$

for every $0 < r \leq r_0$.

We introduce now the important definition of *coincidence set*:

$$\Lambda := \{x \in B_1 : u(x) = 0\}.$$

We will perform a blow-up of Λ around a fixed point $x_0 \in B_{1/2}$, considering the rescaled and translated coincidence sets

$$\Lambda_r(x_0) := \{x \in B_1^G : u(x_0\delta_r x) = 0\},$$

for $0 < r \leq r_0$ and some $r_0 > 0$ such that $B_r^G(x_0) \subset B_1$. Notice that in the previous definition the gauge distance is used for technical reasons, related to the existence of solutions to the Dirichlet problem with respect to the sub-Laplacian.

The next result is a technical lemma, that will be used both to get the decay estimates in Proposition 4.6 and to establish the regularity in Theorem 4.8.

Lemma 4.5. *Let f be such that $f * \Gamma \in C_H^{1,1}(B_1)$ and let u solve* (1.1) *in B_1. Then for every $0 < \lambda, \sigma < 1$, there exists $r_0 > 0$ such that for every $x_0 \in B_\lambda$ we have $\overline{B_{r_0}^G(x_0)} \subset B_1$ and the following holds. Let us consider the translated and rescaled function*

$$u_{r,x_0}(x) := \frac{u(x_0\delta_r x) - \ell^{x_0}(\delta_r x) - p_{c_0 r}^{x_0}(\delta_r x)}{r^2}, \tag{4.12}$$

where $p_{c_0 r}^{x_0}$ is introduced in Definition 3.2 and $\ell^{x_0}(z) = u(x_0) + \langle \nabla_h u(x_0), \pi(z)\rangle$. For each $r \in (0, r_0]$ we also define v_{r,x_0} as the solution to

$$\begin{cases} \Delta_H v_{r,x_0} = f_{r,x_0}, & \text{in } B_\sigma^G, \\ v_{r,x_0} = u_{r,x_0}, & \text{on } \partial B_\sigma^G, \end{cases} \tag{4.13}$$

where $f_{r,x_0}(x) = f(x_0\delta_r x)\chi_{B_\sigma^G}$. Then there exists a universal constant $C(\lambda, \sigma) > 0$, depending on λ and σ, such that

$$\|D_h^2 v_{r,x_0}\|_{L^\infty(B_{\sigma^2}^G)} \leq C(\lambda, \sigma)(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}).$$

Proof. Due to Remark 4.4, there exists $r_0 > 0$ such that for every $x_0 \in B_\lambda$ we have $\overline{B_{r_0}^G(x_0)} \subset B_1$ and (4.11) holds for every $r \in (0, r_0]$. We write the solution to the Dirichlet problem (4.13) in the form

$$v_{r,x_0} = \eta_{r,x_0} + \zeta_{r,x_0},$$

where ζ_{r,x_0} solves

$$\begin{cases} \Delta_H \zeta_{r,x_0} = 0, & \text{in } B_\sigma^G, \\ \zeta_{r,x_0} = u_{r,x_0} - \eta_{r,x_0}, & \text{on } \partial B_\sigma^G \end{cases}$$

and we have defined

$$\eta_{r,x_0} = -f_{r,x_0} * \Gamma.$$

Indeed, the open set B_σ^G is regular with respect to Δ_H, see [BLU07, Proposition 7.2.8]. From the identity

$$D_h^2 v_{r,x_0} = -D_h^2(f_{r,x_0} * \Gamma) + D_h^2 \zeta_{r,x_0},$$

taking into account the equality $D_h^2(f*\Gamma)(x_0\delta_r x) = D_h^2(f_{r,x_0}*\Gamma)(x)$ for a.e. $x \in B^G_{\sigma^2}$ and the estimate (2.9) we obtain that

$$\begin{aligned}\|D_h^2 v_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})} &\le \|D_h^2(f_{r,x_0}*\Gamma)\|_{L^\infty(B^G_{\sigma^2})} + \|D_h^2\zeta_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})}\\ &\le \|D_h^2(f*\Gamma)\|_{L^\infty(B^G_{\sigma^2 r}(x_0))} + C(\sigma)\|\zeta_{r,x_0}\|_{L^\infty(B^G_\sigma)}.\end{aligned}$$

Now we combine the maximum principle and the Dirichlet problem (4.13) to get

$$\|D_h^2 v_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})} \le \|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + C(\sigma)\|u_{r,x_0} + f_{r,x_0}*\Gamma\|_{L^\infty(\partial B^G_\sigma)}.$$

Due to the version of the sub-quadratic growth in (4.11), taking into account the definition (4.12) and the immediate estimate

$$\|f_{r,x_0}*\Gamma\|_{L^\infty(B^G_\sigma)} \le C\|f\|_{L^\infty(B_1)},$$

where $C > 0$ only depends on Γ, it follows that

$$\|D_h^2 v_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})} \le \|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + C(\lambda,\sigma)(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

In conclusion, we have established the following estimate

$$\|D_h^2 v_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})} \le C(\lambda,\sigma)(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}),$$

concluding the proof. □

Proposition 4.6 (Decay of the coincidence set)**.** *Let f be such that $f*\Gamma \in C^{1,1}_H(B_1)$ and let u solve* (1.1)*. Then for every $\beta > 0$, there exist $C_\beta > 0$ and $r_0 > 0$ so that if $0 < r \le r_0$, $x_0 \in B_{1/2}$, and the estimate*

$$|P_r^{x_0}| \ge C_\beta\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)$$

holds, we have

$$|\Lambda_{r/2}(x_0)| \le \frac{|\Lambda_r(x_0)|}{2^{\beta Q}}. \tag{4.14}$$

Proof. Our assumptions allow us to apply Lemma 4.5 with $\lambda = 1/2$, where we choose $\sigma \in [1/\sqrt{2}, 1)$. Let $r_0 > 0$, v_{r,x_0} and u_{r,x_0} be as in the same lemma and define

$$w_{r,x_0} := v_{r,x_0} - u_{r,x_0}.$$

Lemma 4.5 yields a constant $C(\sigma) > 0$ such that

$$\|D_h^2 v_{r,x_0}\|_{L^\infty(B^G_{\sigma^2})} \le C(\sigma)(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)})$$

for every $r \in (0, r_0]$. In addition, from the definition of w_{r,x_0} we observe that

$$\begin{cases}\Delta_H w_{r,x_0} = f_{r,x_0}\chi_{\Lambda_r(x_0)} & \text{in } B^G_\sigma,\\ w_{r,x_0} = 0 & \text{on } \partial B^G_\sigma.\end{cases}$$

By uniqueness, it follows that

$$w_{r,x_0} = -\Big(f_{r,x_0}\chi_{\Lambda_r(x_0)}\Big) * G_{B^G_\sigma}$$

where $G_{B^G_\sigma}$ is the Green function of B^G_σ, according to [BLU07, Definition 9.2.1]. From the definition of Green function, we have $G_{B^G_\sigma} \geq 0$. In addition, taking into account [BLU07, Proposition 9.2.12(iv)], we also notice that the maximum principle gives

$$G_{B^G_\sigma}(x,y) \leq \Gamma(x^{-1}y)$$

for every $x, y \in B^G_\sigma$ with $x \neq y$. Then a standard convolution estimate yields

$$\|w_{r,x_0}\|_{L^\infty(B^G_\sigma)} \leq C\|f\|_{L^\infty(B^G_{r\sigma}(x_0))}\|\chi_{\Lambda_r(x_0)}\|_{L^Q(B_1)} \leq C\|f\|_{L^\infty(B_1)}|\Lambda_r(x_0)|^{1/Q} \tag{4.15}$$

for some geometric constant $C > 0$. The $W^{2,p}_H$ estimates (3.7) give a universal constant C_1, depending on σ, such that

$$\begin{aligned}\int_{B^G_{\sigma^2}} |D^2_h w_{r,x_0}(x)|^{2Q}\,dx &\leq C_1\left(\|f_{r,x_0}\chi_{\Lambda_r(x_0)}\|_{L^{2Q}(B^G_\sigma)} + \|w_{r,x_0}\|_{L^{2Q}(B^G_\sigma)}\right)^{2Q}\\ &\leq C_2\|f\|^{2Q}_{L^\infty(B_1)}(|\Lambda_r(x_0)| + |\Lambda_r(x_0)|^2)\\ &\leq C_3\|f\|^{2Q}_{L^\infty(B_1)}|\Lambda_r(x_0)|.\end{aligned}$$

The second inequality is again a consequence of a convolution estimate, joined with (4.15). Since $|\Lambda_r(x_0)| \leq |B^G_1|$, the third inequality is also established. Furthermore, taking the second order horizontal derivatives in the definition (4.12), we get the equality

$$P^{x_0}_{c_0 r} = (D^2_h u)(x_0\delta_r x) + D^2_h w_{r,x_0}(x) - D^2_h v_{r,x_0}(x).$$

and also $\Lambda_{r\sigma^2}(x_0) = \delta_{\sigma^{-2}}(\Lambda_r(x_0) \cap B^G_{\sigma^2})$. In addition, arguing as in [GT01, Lemma 7.7], we can establish that $(D^2_h u)(x_0\delta_r x) = 0$ a.e. on the coincidence set $\Lambda_r(x_0)$. Taking into account all previous facts, we get

$$\begin{aligned}&\sigma^{2Q}|\Lambda_{r\sigma^2}(x_0)||P^{x_0}_{c_0 r}|^{2Q}\\ &= |\Lambda_r(x_0) \cap B^G_{\sigma^2}||P^{x_0}_{c_0 r}|^{2Q}\\ &= \int_{\Lambda_r(x_0)\cap B^G_{\sigma^2}} |P^{x_0}_{c_0 r}|^{2Q}\,dx\\ &= \int_{\Lambda_r(x_0)\cap B^G_{\sigma^2}} |(D^2_h u)(x_0\delta_r x) + D^2_h w_{r,x_0}(x) - D^2_h v_{r,x_0}(x)|^{2Q}\,dx\\ &= \int_{\Lambda_r(x_0)\cap B^G_{\sigma^2}} |D^2_h w_{r,x_0}(x) - D^2_h v_{r,x_0}(x)|^{2Q}\,dx\\ &\leq 4^Q\int_{\Lambda_r(x_0)\cap B^G_{\sigma^2}} |D^2_h w_{r,x_0}(x)|^{2Q} + |D^2_h v_{r,x_0}(x)|^{2Q}\,dx\\ &\leq C_2(\sigma)\left(\|f\|^{2Q}_{L^\infty(B_1)}|\Lambda_r(x_0)| + |\Lambda_{r\sigma^2}(x_0)|\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)^{2Q}\right).\end{aligned}$$

Consequently,

$$\frac{\sigma^{2Q}|P^{x_0}_{c_0 r}|^{2Q} - C_2(\sigma)\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)^{2Q}}{C_2(\sigma)\|f\|^{2Q}_{L^\infty(B_1)}}|\Lambda_{r\sigma^2}(x_0)| \le |\Lambda_r(x_0)|.$$

We see that the coefficient in front of $|\Lambda_{r\sigma^2}(x_0)|$ is bigger than $2^{\beta Q}$ if

$$\sigma^{2Q}|P^{x_0}_{c_0 r}|^{2Q} \ge C_2(\sigma)2^{\beta Q}\|f\|^{2Q}_{L^\infty(B_1)} + C_2(\sigma)\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)^{2Q}. \tag{4.16}$$

By the simple inequality $\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} \ge \|f\|_{L^\infty(B_1)}$, a few more computations lead us to the following sufficient condition

$$|P^{x_0}_{c_0 r}| \ge \left[C_2(\sigma)\sigma^{-2Q}(2^{\beta Q} + 2^{2Q-1})\right]^{1/2Q}\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right),$$

to get (4.16) to hold. Finally, we choose $\sigma = 1/\sqrt{2}$, then the proof follows by choosing the constant C_β in our statement equal to $\sqrt[2Q]{C_2(1/\sqrt{2})2^Q(2^{\beta Q} + 2^{2Q-1})}$ and replacing $c_0 r$ by r. □

To carry out the proof of the $C^{1,1}_H$ regularity, we need a Calderón type second order differentiability, according to the next definition.

Definition 4.7. We say that $u \in L^1_{\text{loc}}(\Omega)$ is *twice L^1 differentiable at x_0* if there exists a polynomial t of degree less than or equal to two, such that

$$\frac{1}{r^2}\fint_{B_r(x_0)} |u(z) - t(z)|dz \to 0 \quad \text{as} \quad r \to 0^+.$$

The polynomial t has the following form

$$t(x) = c_0 + \sum_{l=1}^{m} v_l(x_l - x_{0l}) + \frac{1}{2}\sum_{i,j=1}^{m} c_{ij}(x_i - x_{0i})(x_j - x_{0j}) + \sum_{l=m+1}^{m_2} c_l(x_l - x_{0l}),$$

$x = (x_1, \dots, x_n)$, $x_0 = (x_{01}, \dots, x_{0n})$ and $c_0, c_{ij}, c_l \in \mathbb{R}$.

It is possible to show that any $u \in W^{2,1}_{H,\text{loc}}(\Omega)$ is twice L^1 differentiable a.e. in Ω. Furthermore, if the function is twice L^1 differentiable at a Lebesgue point $x_0 \in \Omega$ of all functions X_iX_ju, X_ju and u, then the corresponding polynomial is unique and it has the following form

$$\begin{aligned} u(x_0) + \sum_{j=1}^{m} X_ju(x_0)\,(x_j - x_{0j}) + \frac{1}{2}\sum_{i,j=1}^{m}((X_iX_j + X_jX_i)u)(x_0)\,(x_i - x_{0i})(x_j - x_{0j}) \\ + \sum_{l=m+1}^{m_2} X_lu(x_0)\,(x_l - x_{0l}), \end{aligned}$$

see [Mag05] for more information. We are now in the position to prove the optimal interior regularity of solutions to the no-sign obstacle-type problem (1.1).

Theorem 4.8 ($C^{1,1}$ regularity)**.** *Let* $u \in L^\infty(B_1)$ *be a distributional solution to* (1.1) *in the unit ball* B_1*. Let* $f : B_1 \to \mathbb{R}$ *be locally summable such that* $f * \Gamma \in C_H^{1,1}(B_1)$*. Then there exists a universal constant* $C > 0$ *such that, after a modification on a negligible set, we have* $u \in C_H^{1,1}(B_{1/4})$ *and*

$$\|D_h^2 u\|_{L^\infty(B_{1/4})} \leq C \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

Proof. We consider C_β as in Proposition 4.6 and fix $\beta = 4$. We consider a priori the following constant

$$K = C_4 \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

Combining the Hölder inequality and Theorem 3.4, taking into account that the constant C_β in Proposition 4.6 is bounded from below by a universal positive constant independent of β, we can find a universal constant $C_1' > 0$ such that

$$\|D_h^2 u\|_{L^1(B_{1/2})} \leq C_1' K. \tag{4.17}$$

Let $r_0 > 0$ be the minimum among the r_0's of Remark 4.4, Lemma 4.5 with $\lambda = 1/4$ and Proposition 4.6 with $\beta = 4$. We fix an integer i_0 such that

$$i_0 \geq 3 + \log_2 c_0, \tag{4.18}$$

such that $2^{-i_0} \leq r_0$, where c_0 is the geometric constant appearing in (2.6). Then (4.17) provides us with a universal constant $\overline{C}_1 \geq 1$ such that

$$|P_{2^{-i_0}}^y| \leq \overline{C}_1 K \tag{4.19}$$

for all $y \in B_{1/4}$. Notice that $\overline{C}_1$ actually depends on i_0. However, this integer is fixed throughout the proof. We have chosen i_0 to satisfy also (4.18) in view of the subsequent application of Lemma 3.8 with $\lambda_1 = 3/4$. We can fix $x_0 \in B_{1/4}$ such that u is twice L^1 differentiable at x_0. Using [Mag05, Theorem 3.8] for $p = 1$ and $k = 2$, the set of these differentiability points has full measure in B_1. We can further write $u = v - w$, such that

$$\Delta_H v = f \quad \text{and} \quad \Delta_H w = f\chi_\Lambda$$

on B_1, where $v = -f * \Gamma$. By assumption $v \in C_H^{1,1}(B_1)$, hence it is also a.e. twice L^1 differentiable, therefore we can further assume v is twice L^1 differentiable at x_0, having the set of these points full measure in B_1. Now, only two cases may occur.

Case 1: $\boldsymbol{\liminf_{k\to\infty}|P^{x_0}_{2^{-k}}| \le \overline{C}_1 K}$. At our point x_0, we have

$$\begin{aligned}
|D^2_h u(x_0)| &= \left|\lim_{k\to\infty} \fint_{B_{2^{-k}}(x_0)} D^2_h u(y)\,dy\right| \\
&= \lim_{k\to\infty}\left|\left(P^{x_0}_{2^{-k}} + \frac{(\Delta_H u)_{B_{2^{-k}}(x_0)}}{m} I_m\right)\right| \\
&\le \liminf_{k\to\infty}\left(|P^{x_0}_{2^{-k}}| + \frac{1}{\sqrt{m}}|(\Delta_H u)_{B_{2^{-k}}(x_0)}|\right) \\
&= \frac{1}{\sqrt{m}}|\Delta_H u(x_0)| + \liminf_{k\to\infty}|P^{x_0}_{2^{-k}}| \\
&\le \frac{1}{\sqrt{m}}\|f\|_{L^\infty(B_1)} + \overline{C}_1 K \\
&\le \|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \overline{C}_1 K.
\end{aligned}$$

Therefore

$$|D^2_h u(x_0)| \le (\overline{C}_1 C_4 + 1)(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}).$$

Case 2: $\boldsymbol{\liminf_{k\to\infty}|P^{x_0}_{2^{-k}}| > \overline{C}_1 K}$. Then the following integer is well defined

$$k_0 := \min\{k\in\mathbb{N} : k \ge i_0,\ |P^{x_0}_{2^{-j}}| > \overline{C}_1 K,\ \text{for all } j \ge k\}.$$

The positive integer k_0 possibly depends on x_0. We notice that from the definition of k_0, we have $|P^{x_0}_{2^{-k_0+1}}| \le \overline{C}_1 K$. The strict inequality $k_0 > i_0$ follows by (4.19). In view of our choice of i_0, that satisfies (4.18) and then $i_0 > 3$, we can apply Lemma 3.8 with $\lambda_1 = 3/4$. Indeed, we have $B_{1/4} = B_{\lambda_1/3}$ and

$$2^{-k_0+1} < \min\left\{\frac{2}{3}\lambda_1, 1-\lambda_1\right\} = 2^{-2},$$

so Lemma 3.8 with $r_1 = 2^{-k_0}$ and $r_2 = 2^{-k_0+1}$ yields

$$\begin{aligned}
|P^{x_0}_{2^{-k_0}}| &\le |P^{x_0}_{2^{-k_0+1}}| + C\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right), \\
&\le (\overline{C}_1 C_4 + C)\left(\|D^2_h(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right).
\end{aligned} \tag{4.20}$$

We consider the "rescaled function" defined in Lemma 4.5:

$$u_0(x) := \frac{u(x_0\delta_{2^{-k_0}}x) - \ell^{x_0}(\delta_{2^{-k_0}}x) - p^{x_0}_{c_0 2^{-k_0}}(\delta_{2^{-k_0}}x)}{4^{-k_0}}. \tag{4.21}$$

This function coincides with $u_{2^{-k_0},x_0}$ of the same lemma. Now we set $f_0(x) := f(x_0\delta_{2^{-k_0}}x)$, that is also defined on B^G_1. We can find a harmonic function h_0 such that

$$v_0 = 2^{2k_0} v(x_0\delta_{2^{-k_0}}x) + h_0$$

and v_0 satisfies the Dirichlet problem

$$\begin{cases}
\Delta_H v_0 = f_0 & \text{in } B^G_\sigma, \\
v_0 = u_0, & \text{on } \partial B^G_\sigma,
\end{cases} \tag{4.22}$$

with $0 < \sigma < 1$. Notice that v_0 is also twice L^1 differentiable at 0, being a consequence of the twice L^1 differentiability of v at x_0. For the same reason, the twice L^1 differentiability of u at x_0 gives the twice L^1 differentiability of u_0 at 0. From Lemma 4.5 with $\lambda = 1/4$, there exists $C_\sigma > 0$ such that

$$\|D_h^2 v_0\|_{L^\infty(B^G_{\sigma^2})} \leq C_\sigma \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right). \tag{4.23}$$

For the sequel, it is now important to remark that the difference

$$w_0 := v_0 - u_0 \tag{4.24}$$

is twice L^1 differentiable at the origin. Then we know the existence of a polynomial

$$R(x) = w_0(0) + \sum_{j=1}^{m} X_j w_0(0)\, x_j + \frac{1}{2} \sum_{i,j=1}^{m} \Big((X_i X_j + X_j X_i) w_0 \Big)(0)\, x_i x_j + \sum_{l=m+1}^{m_2} X_l w_0(0)\, x_l$$

such that we get

$$\frac{1}{r^2} \fint_{B_{\kappa r}} |w_0(z) - R(z)| dz \to 0 \tag{4.25}$$

as $r \to 0^+$ and for an arbitrary $\kappa > 0$. The definition of w_0 immediately gives

$$\begin{cases} \Delta_H w_0 = f_0 \chi_{\Lambda_{2^{-k_0}}(x_0)} & \text{in } B^G_\sigma, \\ w_0 = 0 & \text{on } \partial B^G_\sigma. \end{cases}$$

Claim: for a fixed $0 < \alpha < 1$, there exist $l_0 \geq 1$ and $C > 0$, depending on α and on universal constants, such that for $\tau = 2^{-l_0}$ and for every $k \in \mathbb{N} \setminus \{0\}$, there exist harmonic polynomials q_k with the property that

$$\|w_0 - q_k\|_{L^\infty(B^G_{\tau^k})} \leq C \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right) \tau^{(2+\alpha)(k-1)} \tag{4.26}$$

where the constants are independent of x_0.

To prove (4.26) by induction, we need first to establish the case $k = 1$. Here we choose the null harmonic polynomial $q_1 = 0$. We consider the decomposition (4.24) and observe that standard L^∞ estimates for v_0 are available, since it solves (4.22). Indeed, we may further decompose v_0 into the sum of $z_0 = -f_0 * \Gamma$ and of a harmonic function h_0 such that $h_0|_{\partial B^G_\sigma} = u_0 - z_0$. Then we apply the sub-quadratic growth estimate (4.11) of Remark 4.4 where we fix $\lambda = 1/4$. This leads us to the following estimate

$$\|v_0\|_{L^\infty(B_\sigma)} \leq C_{1\sigma} \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

Then using again estimate (4.11), we obtain

$$\begin{aligned} \|w_0\|_{L^\infty(B^G_\sigma)} &\leq \|v_0\|_{L^\infty(B^G_{r_1})} + \|u_0\|_{L^\infty(B^G_{r_1})} \\ &\leq C_{2\sigma} \left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right). \end{aligned}$$

Taking $\sigma = \tau = 2^{-l_0}$, the estimate (4.26) is established for $k = 1$. We may take $l_0 \in \mathbb{N}$ possibly larger, such that

$$\tau = 2^{-l_0} \leq \frac{1}{2|B_1^G|^{1/Q}}. \tag{4.27}$$

In view of Proposition 2.4, there exists a universal constant $c > 0$ such that

$$\|D_h^3 H\|_{L^\infty(B_{1/2}^G)} \leq c\|H\|_{L^\infty(B_1^G)} \tag{4.28}$$

for any harmonic function H on B_1^G. Now we assume the statement (4.26) is true for any fixed $k \geq 1$ and define

$$w_k(x) := \frac{w_0(\delta_{\tau^k} x) - q_k(\delta_{\tau^k} x)}{\tau^{(2+\alpha)(k-1)}}$$

on $\overline{B_1^G}$. We choose the harmonic function h_k such that

$$\begin{cases} \Delta_H h_k = 0 & \text{in } B_1^G, \\ h_k = w_k & \text{on } \partial B_1^G. \end{cases}$$

From the definition of w_k, we get

$$\Delta_H w_k = \tau^{2-\alpha(k-1)} f(x_0 \delta_{2^{-k_0}\tau^k} \cdot)\chi_{\Lambda_{2^{-k_0}\tau^k}(x_0)}$$

on B_1^G, and the induction assumption yields

$$\|w_k\|_{L^\infty(B_1^G)} \leq C\left(\|D_h^2(f * \Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right). \tag{4.29}$$

Clearly $w_k - h_k$ vanishes on ∂B_1^G. Taking into account our choice of i_0 such that $2^{-i_0} \leq r_0$, the decay estimate of the coincidence set (4.14) applies in particular for $\beta = 4$ and for every $r \in (0, 2^{-k_0}]$, that is

$$|\Lambda_{r/2}(x_0)| \leq \frac{|\Lambda_r(x_0)|}{2^{4Q}}. \tag{4.30}$$

Arguing as before for v_0, we can decompose $w_k - h_k$ into the sum of a harmonic function and a convolution with the fundamental solution, therefore standard convolution estimates yield the first of the following inequalities

$$\begin{aligned}
\|w_k - h_k\|_{L^\infty(B_1^G)} &\leq C\|\tau^{2-\alpha(k-1)} f(x_0\delta_{2^{-k_0-l_0k}})\chi_{\Lambda_{2^{-k_0-l_0k}}(x_0)}\|_{L^Q(B_1^G)} \\
&\leq C\tau^{-\alpha(k-1)}\|f\|_{L^\infty(B_1)}|\Lambda_{2^{-k_0-l_0k}}(x_0)|^{1/Q} \\
&\leq C2^{\alpha l_0(k-1)}\|f\|_{L^\infty(B_1)}2^{-4l_0k}|\Lambda_{2^{-k_0}}(x_0)|^{1/Q} \\
&\leq C|B_1^G|^{1/Q}\|f\|_{L^\infty(B_1)}\tau^{k(4-\alpha)+\alpha} \\
&\leq \frac{C}{2}\|f\|_{L^\infty(B_1)}\tau^{2+\alpha},
\end{aligned}$$

where the third inequality is a consequence of (4.30) and the last inequality follows from (4.27). Combining the estimate (4.28) for harmonic functions, the maximum principle and our induction assumption as stated in (4.29), we get

$$\begin{aligned}\|D_h^3 h_k\|_{L^\infty(B^G_{1/2})} &\le c\|h_k\|_{L^\infty(B^G_1)} \le c\|w_k\|_{L^\infty(B^G_1)}\\ &\le cC\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right).\end{aligned}$$

Define $\overline{q}_k(x)$ as the second order Taylor polynomial of h_k at the origin. In particular, $\overline{q}_k$ is harmonic. The previous estimates joined with the application of the stratified Taylor inequality stated in [FS82, Corollary 1.44 with $k = 2$ and $x = 0$] give

$$\begin{aligned}\|h_k - \overline{q}_k\|_{L^\infty(B^G_\tau)} &\le C_2' c\, C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^3\\ &\le \frac{C}{2}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{2+\alpha},\end{aligned}$$

where we have chosen l_0 possibly larger, such that the following conditions

$$b^3\tau = b^3 2^{-l_0} < 1/2 \quad \text{and} \quad \tau = 2^{-l_0} \le \left(\frac{1}{2cC_2'}\right)^{1/(1-\alpha)}$$

also hold. The constants b and C_2' are from [FS82, Corollary 1.44 with $k = 2$ and $x = 0$] and this corollary is applied with the gauge distance d_G. We stress that l_0 does not depend on either k_0 or x_0. This is very important for the final estimate of $D_h^2 u(x_0)$. As a consequence, we obtain

$$\begin{aligned}\|w_k - \overline{q}_k\|_{L^\infty(B^G_\tau)} &\le \|w_k - h_k\|_{L^\infty(B^G_\tau)} + \|h_k - \overline{q}_k\|_{L^\infty(B^G_\tau)}\\ &\le \frac{C}{2}\|f\|_{L^\infty(B_1)}\tau^{2+\alpha} + \frac{C}{2}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{2+\alpha}\\ &\le C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{2+\alpha}.\end{aligned}$$

Taking into account the definition of w_k, we have proved that

$$\left\|\frac{w_0(\delta_{\tau^k}\cdot) - q_k(\delta_{\tau^k}\cdot)}{\tau^{(2+\alpha)(k-1)}} - \overline{q}_k\right\|_{L^\infty(B^G_\tau)} \le C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{2+\alpha},$$

from which we infer that

$$\|w_0 - q_k - \tau^{(2+\alpha)(k-1)}\overline{q}_k(\delta_{\tau^{-k}}\cdot)\|_{L^\infty(B^G_{\tau^{k+1}})} \le C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{(2+\alpha)k}.$$

If we define the new polynomial

$$q_{k+1}(x) := q_k(x) + \tau^{(2+\alpha)(k-1)}\overline{q}_k(\delta_{\tau^{-k}}x),$$

then the induction step is proved and this concludes the proof of our claim. By the same previous argument, we have another universal constant $c' > 0$ such that

$$\begin{aligned}&\max\left\{\|h_k\|_{L^\infty(B^G_{1/2})}, \|\nabla_h h_k\|_{L^\infty(B^G_{1/2})}, \|D_h^2 h_k\|_{L^\infty(B^G_{1/2})}\right\}\\ &\le c'\|w_k\|_{L^\infty(B^G_1)}\\ &\le c'C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right).\end{aligned} \tag{4.31}$$

We introduce the following notation:

$$q_k(x) = a^k + \sum_{i=1}^{m} b_i^k x_i + \frac{1}{2}\sum_{i,j=1}^{m} c_{ij}^k x_i x_j + \sum_{l=m+1}^{m_2} c_l^k x_l,$$
$$\overline{q}_k(x) = \overline{a}^k + \sum_{i=1}^{m} \overline{b}_i^k x_i + \frac{1}{2}\sum_{i,j=1}^{m} \overline{c}_{ij}^k x_i x_j + \sum_{l=m+1}^{m_2} \overline{c}_l^k x_l.$$

From the definition of $\bar{q}_k$ and taking into account the estimates (4.31), we get

$$\max_{i,j,l}\{\overline{a}^k, \overline{b}_i^k, \overline{c}_{ij}^k, \overline{c}_l^k\} \le c'C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right). \tag{4.32}$$

Consequently, differentiating the equality

$$q_{k+1}(x) - q_k(x) = \tau^{(2+\alpha)(k-1)}\overline{q}_k(\delta_{\tau^{-k}}x),$$

with the differential operators X_i, X_iX_j for $i,j = 1,\dots,m$ and X_l for $l = m+1,\dots,m_2$, and evaluating all the equalities at the origin, we get

$$|a^{k+1} - a^k| \le c'C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{(2+\alpha)(k-1)},$$
$$\max_{1\le i\le m}|b_i^{k+1} - b_i^k| \le c'C\tau^{-1}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{(1+\alpha)(k-1)},$$
$$\max_{1\le i,j\le m}|c_{ij}^{k+1} - c_{ij}^k| \le c'C\tau^{-2}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{\alpha(k-1)},$$
$$\max_{m+1\le l\le m_2}|c_l^{k+1} - c_l^k| \le c'C\tau^{-2}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{\alpha(k-1)}.$$

Representing any of these coefficients as γ^k, we notice that they are Cauchy sequences converging to some γ. In addition, for any of them we have

$$|\gamma^k - \gamma| \le \frac{c'C\tau^{-2}}{1-\tau^\alpha}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{(\alpha+l)(k-1)}. \tag{4.33}$$

In these estimates, we have set $l = 0$ when $\gamma^k = c_l^k, c_{ij}^k$, $l = 1$ for $\gamma^k = b_i^k$ and $l = 2$ for $\gamma^k = a_i^k$. As a consequence, the polynomials q_k uniformly converge on compact sets to a polynomial $\tilde{q}$, that has the form

$$\tilde{q}(x) = \tilde{a} + \sum_{i=1}^{m} \tilde{b}_i x_i + \frac{1}{2}\sum_{i,j=1}^{m} \tilde{c}_{ij} x_i x_j + \sum_{l=m+1}^{m_2} \tilde{c}_l x_l.$$

Any coefficient γ of $\tilde{q}$, can be written for instance as $\gamma^2 + \gamma - \gamma^2$. By (4.33), we can find a universal estimate for $\gamma - \gamma^2$, that depends on α. We also observe that the coefficients of q_2 are given by the formula $q_2 = \overline{q}_1 \circ \delta_{\tau^{-1}}$. The estimate (4.32) for $k = 2$ and the fact that $\tau = 2^{-l_0}$ can be universally fixed, independently of x_0, finally lead us to the estimate

$$|D_h^2\tilde{q}| \le C_{\tau,\alpha}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right) \tag{4.34}$$

for a suitable geometric constant $C_{\tau,\alpha} > 0$ depending on the constants of (4.33) and (4.32). From (4.33), defining

$$\overline{C}_{\tau,\alpha,f,u} = \frac{c'C\tau^{-2}\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)}{1-\tau^\alpha},$$

we can establish the following quantitative estimate on small balls:

$$\begin{aligned}\|q_k - \tilde{q}\|_{L^\infty(B^G_{\tau^{k+1}})} \le & \overline{C}_{\tau,\alpha,f,u}\Bigg(\tau^{(\alpha+2)(k-1)} + \tau^{(\alpha+1)(k-1)}\sum_{i=1}^m \|x_i\|_{L^\infty(B^G_{\tau^{k+1}})} \\ & + \tau^{\alpha(k-1)}\sum_{i,j=1}^m \|x_i x_j\|_{L^\infty(B^G_{\tau^{k+1}})} + \tau^{\alpha(k-1)}\sum_{l=m+1}^{m_2}\|x_l\|_{L^\infty(B^G_{\tau^{k+1}})}\Bigg) \\ \le & \widetilde{C}\,\overline{C}_{\tau,\alpha,f,u}\,\tau^{(2+\alpha)(k-1)},\end{aligned}$$

where we have used the proper intrinsic homogeneity of all the monomials x_i, x_ix_j and x_l, with $i,j = 1,\dots,m$ and $l = m+1,\dots,m_2$. We would like to prove that

$$\lim_{r\to 0}\frac{1}{r^2}\fint_{B^G_{\tau r}} |w_0(z) - \tilde{q}(z)|dz = 0.$$

Let us consider $r = \tau^k$ and then

$$\begin{aligned}\frac{1}{r^2}\fint_{B^G_{\tau r}} |w_0(z) - \tilde{q}(z)|dz \le & \frac{1}{r^2}\fint_{B^G_{\tau r}} |w_0(z) - q_k(z)|dz + \frac{1}{r^2}\fint_{B^G_{\tau r}} |\tilde{q}(z) - q_k(z)|dz \\ = & \tau^{-2k}\fint_{B^G_{\tau^{k+1}}} |w_0(z) - q_k(z)|dz + \tau^{-2k}\fint_{B^G_{\tau^{k+1}}} |\tilde{q}(z) - q_k(z)|dz \\ \le & C\left(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right)\tau^{\alpha k} + \tau^{-2k}\|\tilde{q} - q_k\|_{L^\infty(B^G_{\tau^k})}\end{aligned}$$

goes to zero as $k \to \infty$. If we choose $\kappa_0 > 0$ sufficiently small, then we have proved that

$$\frac{1}{\tau^{2k}}\fint_{B_{\kappa_0\tau^{2k}}} |w_0(z) - R(z)|dz \le \frac{1}{\tau^{2k}}\fint_{B^G_{\tau^k}} |w_0(z) - R(z)|dz \to 0.$$

By the uniqueness of the second order polynomial R satisfying (4.25) with $\kappa = \kappa_0$, we get $\tilde{q} = R$. Taking into account (4.23) with $\sigma = \tau$ and (4.34) with $\alpha = 1/2$, we obtain a new constant $C > 0$, such that

$$|D_h^2 u_0(0)| \le |D_h^2 v_0(0)| + |D_h^2 w_0(0)| \le C(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}).$$

As a consequence, since $k_0 > i_0$ and i_0 satisfies (4.18) we can apply Lemma 3.8 with $r_1 = 2^{-k_0}$ and $r_2 = c_0 2^{-k_0}$, so that taking into account the estimate (4.20) and the definition (4.21) of u_0, we finally obtain a possibly larger constant, that we still denote by $C > 0$, such that

$$|D_h^2 u(x_0)| \le |D_h^2 u_0(0)| + |P^{x_0}_{c_0 2^{-k_0}}| \le C(\|D_h^2(f*\Gamma)\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}),$$

concluding the proof. □

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Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127, Pisa, Italy

Email address: `valentino.magnani@unipi.it`

KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

Email address: `minne@kth.se`