EXISTENCE AND REGULARITY OF OPTIMAL CONVEX SHAPES FOR FUNCTIONALS INVOLVING THE ROBIN EIGENVALUES

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Abstract. The aim of this paper is to study the existence and the regularity of optimal convex domains for a large class of shape optimization problems involving functions of the eigenvalues of the Robin-Laplacian on convex sets. We will prove that convex solutions exist and that, under some additional hypotheses on the functional, these optimal sets have C^1 boundary.

1. Introduction

Let $\Omega \subset \mathbf{R}^d$ be a bounded Lipschitz domain and $\beta > 0$ be a fixed positive real number. A number $\lambda \in \mathbf{R}$ is told an *eigenvalue of the Robin problem for the Laplace operator with boundary parameter* β if there exists a non-zero function $u \in H^1(\Omega)$ solving, in the weak sense, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

(here n is the outer normal on $\partial\Omega$), i.e., following the approach in Chapter 6 of [3],

$$\forall\ v \in H^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} uv \, d\mathcal{H}^{d-1} = \lambda \int_\Omega uv \, dx.$$

For every $k \in \mathbf{N}$, the k-th eigenvalue, that we will denote by the symbol $\lambda_{k,\beta}(\Omega)$, is given by the usual Rayleigh min-max formula

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx}, \tag{1.2}$$

where $\mathcal{S}_k$ denotes the family of all k-dimensional subspaces of $H^1(\Omega)$.
We are interested in solving in $\mathbf{R}^d$ the following problem

$$\min \left\{ F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) + \Lambda P(\Omega) : \Omega \subset \mathbf{R}^d \text{ bounded and convex} \right\}, \tag{1.3}$$

where $\Lambda > 0$, $F : \mathbf{R}^k \to \mathbf{R}$ is non decreasing and lower semicontinuous in each variable and $P(\Omega)$ is the perimeter in the sense of De Giorgi (for Lipschitz set it holds $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$, see [1] for details). The prototypic problem of this family is

$$\min\left\{\lambda_{1,\beta}(\Omega) + \ldots + \lambda_{k,\beta}(\Omega) + \Lambda P(\Omega) : \Omega \subset \mathbf{R}^d \text{ bounded and convex}\right\}.$$

We will prove an existence result for problem (1.3) and, under slightly stronger hypotheses on F, we will be able to gain also regularity of the optimal sets. The main result of the paper is the following theorem.

Theorem 1.1. *Let $F : \mathbf{R}^k \to \mathbf{R}$ be non decreasing and lower semicontinuous in each variable. Then, problem* (1.3) *admits at least a solution.*
Moreover, if F is differentiable and its partial derivative with respect to the first variable is strictly positive, then every optimal solution has C^1 boundary.

The result presented in the paper, up to the authors' knowledge, is new in literature, although similar problems have already been studied in different settings (e.g. with Dirichlet boundary conditions) and using different techniques. The proof of existence of optimal convex shapes (Theorem 3.2) follows a standard approach used in shape optimization problems with geometrical constraints: the idea is to choose a suitable topology on open (or closed) sets that preserves convexity, then to show that minimizing sequences enjoy some properties that assure compactness in the chosen topology. The proof of the regularity of the convex solutions (Theorem 5.3) is based on a cutting argument that allowed us to show that, in order to minimize (1.3), it is more convenient to remove singularity, as for instance corners in two dimensions.
The structure of the paper is the following. In Section 2 we recall the notation, some tools and well known facts which are necessary to understand and obtain our results; in particular, we will present a short survey on Hausdorff distances (subsection 2.1), their good behaviour in the convex setting and their links with the convergence in measure and of the perimeters and some properties of $\lambda_{k,\beta}(\Omega)$ if Ω is a bounded Lipschitz domain (subsection 2.2). In Section 3 we obtain the existence of optimal convex shapes minimizing (1.3) via direct methods of calculus of variations, proving that $\lambda_{k,\beta}$ is lower semicontinuous and that minimizing sequences are compact and do not degenerate or stretch indefinitely in any direction. Then, in Section 4, we will estimate the gap between $\lambda_{k,\beta}(\Omega)$ and $\lambda_{k,\beta}(\Omega_\varepsilon)$, where Ω is an admissible set with a singularity point on the boundary and Ω_ε is another convex competitor obtained by a suitable cut of Ω around the singularity point. Finally, in Section 5, we will introduce the family of of convex energy subsolutions for (1.3), that generalizes the definition of solutions, then we will complete the proof of Theorem 1.1 showing the regularity of the boundaries of the energy subsolutions.

2. Notation and preliminaries

In this section we will fix the notation used throughout the paper and recall some definitions and tools which are applied in the following sections.
For $x \in \mathbf{R}^d$ and and $r > 0$, $B_r(x)$ will denote the ball of radius r centered in x; when x is omitted, we consider the ball centered in the origin. For every measurable set $E \subseteq \mathbf{R}^d$, we will use the symbols χ_E for the characteristic function of E and E^c for its complement and tE for the rescaled set $\{tx : x \in E\}$. As

usual, $|E|$ and $\mathcal{H}^s(E)$ ($s > 0$) stand respectively for the Lebesgue measure and the Hausdorff s-dimensional measure of E; if E is a piecewise regular hypersurface, $\mathcal{H}^{d-1}(E)$ coincides with its area measure. We will denote by $\mathcal{H} - dim(E)$ the Hausdorff dimension of the set E; for sufficiently regular sets, it coincides with the topological dimension of the set E, e.g. if E is an open set of $\mathbf{R}^d$, $\mathcal{H} - dim(E) = d$ (for further details see Chapter 2, Section 8 in [1]). For every open set $\Omega \subset \mathbf{R}^d$, we will denote by $L^p(\Omega)$ the usual Lebesgue space of (classes of) p-summable functions, by $W^{k,p}(\Omega)$ the Sobolev space of functions whose (weak) derivatives are p-summable up to order k and by $H^k(\Omega)$ the (Hilbert) space $W^{k,2}(\Omega)$. For brevity's sake, we will often use the symbol $R_\Omega^\beta(u)$ to denote the Rayleigh quotient in (1.2) computed for the admissible function u on the set Ω with boundary parameter β; when the boundary parameter does not vary throughout the proofs, β is omitted.
In order to minimize (1.3) using the direct methods of the Calculus of Variation, we need lower semicontinuity of the Robin eigenvalues with respect to the some compact topology on the class of convex sets of $\mathbf{R}^d$; as we will see in Subsection 2.1, a good choice for our purposes is the Haudorff topology, that coincides with the L^1-topology on suitable classes of convex sets.
The following proposition is a particular case of Proposition 2.3 in [4] and gives us a first result that will be useful in Section 3 to obtain the lower semicontinuity of $\lambda_{k,\beta}$ and then to gain an existence result in any dimension d.

Proposition 2.1. *Let $(E_n)_{n\in\mathbf{N}}$ be a sequence of bounded convex sets of $\mathbf{R}^d$ and let $E \subset \mathbf{R}^d$ convex such that*

$$\limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial E_n) < +\infty \qquad \text{and} \qquad \chi_{E_n} \overset{L^1(\mathbf{R}^d)}{\longrightarrow} \chi_E.$$

Let $(u_n)_{n\in\mathbf{N}} \subset H^1(\mathbf{R}^d)$ and $u \in H^1(\mathbf{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbf{R}^d)$. Then

$$\mathcal{H}^{d-1}(\partial E) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(\partial E_n) \quad \text{and} \quad \int_{\partial E} u^2 \, d\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{\partial E_n} u_n^2 \, d\mathcal{H}^{d-1}.$$

The next notion of convergence of functional spaces is very useful in shape optimization problems and will be used to obtain ecistence of minimizers for Problem (1.3).

Definition 2.2 (convergence in the sense of Mosco)**.** *Let X be a Banach space and $(G_n)n$ a sequence of closed subsets of X. We define weak upper and strong lower limits in the sense of Kuratowski the spaces*

$$w - \limsup_{n\to+\infty} G_n := \{u \in X : \exists\, (n_k)_k,\ \exists\, u_{n_k} \in G_{n_k} \ s.t.\ u_{n_k} \rightharpoonup u \ weakly\ in\ X\},$$

$$s - \liminf_{n\to+\infty} G_n := \{u \in X : \exists\, u_{n_k} \in G_{n_k} \ s.t.\ u_n \to u \ strongly\ in\ X\}.$$

We say that G_n converges to the closed subspace G in the sense of Mosco if

$$w - \limsup_{n\to+\infty} G_n \subseteq G$$

and

$$G \subseteq s - \liminf_{n\to+\infty} G_n,$$

i.e. if

$$w - \limsup_{n\to+\infty} G_n = G = s - \liminf_{n\to+\infty} G_n.$$

2.1. **Hausdorff convergences.** Theorem 2.1 is a key result in order to obtain lower semicontinuity of the Robin eigenvalues with respect to the L^1-topology. The only disadvantage is that no topological properties of converging sequences can be deduced for the limit set in this framework (we imposed in the hypotheses the convexity of the limit set). This difficulty may be overcome choosing a suitable (compact) topology on the class of convex sets that, under suitable hypotheses, coincides with the L^1-topology for converging sequences. To this aim, we introduce the Hausdorff distance between closed sets.

Definition 2.3 (Hausdorff topology on closed sets)**.** *Let $A, B \subseteq \mathbf{R}^d$ be closed. We define the* Hausdorff distance between A and B *by*

$$d_H(A,B) := \max\left\{\sup_{x\in A} dist(x,B), \sup_{x\in B} dist(x,A)\right\}.$$

The topology induced by this distance is called Hausdorff topology (or simply H-topology) on closed sets*.*

The counterpart for open sets of the Hausdorff topology is defined below.

Definition 2.4 (Hausdorff topology on open sets)**.** *Let $A, B \subseteq \mathbf{R}^d$ be open. We define the* Hausdorff-complementary distance between A and B *by*

$$d_{H^c}(A,B) := d_H(A^c, B^c).$$

The topology induced by this distance is called Hausdorff-complementary topology (or simply H^c-topology) on open sets*.*

These topologies on turn out to be suited for our purposes as, under not so restrictive hypotheses on the functional, they guarantee compactness of sequence of convex sets. The following proposition contains some results proved in [6], Section 2.4, and shows us that Hausdorff convergences preserve convexity and assure continuity for Lebesgue measure and perimeters of convex sets.

Proposition 2.5. *The following results hold for convex sets:*

(i) *If $A \subseteq B$, then $\mathcal{H}^{d-1}(\partial A) \le \mathcal{H}^{d-1}(\partial B)$;*
(ii) *If A_n, A are closed (respectively open) and convex $A_n \to A$ with respect to the H-topology (respectively H^c-topology), then $\chi_{A_n} \to \chi_A$ in L^1; moreover, if $\mathcal{H} - dim(A) = \mathcal{H} - dim(A_n)$, then $\mathcal{H}^{d-1}(\partial A_n) \to \mathcal{H}^{d-1}(\partial A)$.*
(iii) *$|A| \le \rho\mathcal{H}^{d-1}(\partial A)$, where ρ is the radius of the biggest ball contained in A.*
(iv) *If a sequence $(A_n)_n$ of closed convex sets H-converges to a closed set A, then A is a closed convex set; if a sequence $(B_n)_n$ of open convex sets H^c-converges to an open set B, then B is an open convex set.*
(v) *Let $B \subset \mathbf{R}^d$ a fixed compact set. Then, the class of the closed convex sets contained in B is compact in the Hausdorff topology.*

Remark 2.6. *Let $A_n \subset \mathbf{R}^d$ be convex and uniformly bounded with $|A_n| \geq m > 0$ for every $n \in \mathbf{N}$; it is easy to show that*

$$\overline{A_n} \stackrel{H}{\longrightarrow} \overline{A} \qquad \Leftrightarrow \qquad \mathring{A}_n \stackrel{H^c}{\longrightarrow} \mathring{A}.$$

The implication "$\Leftarrow$" follows by Proposition 2.4.10 in [11] *(applied to a sequence of convex sets). The converse implication follows by the definition of uniform convergence of sets*[1] *and its equivalence with the H-topology on compact sets (see Remark 2.4.2 in* [6]*). Indeed, since $\overline{A_n}, \overline{A}$ are contained in a compact set $B \subset \mathbf{R}^d$, we have that there exists $n_\varepsilon \in \mathbf{N}$ such that, for every $n \geq n_\varepsilon$*

$$\overline{A_n} \subset \overline{A} + B_\varepsilon, \quad \overline{A} \subset \overline{A_n} + B_\varepsilon.$$

Since A_n, A are convex, we deduce that

$$D \setminus \mathring{A}_n \subset (D \setminus \mathring{A}) + B_\varepsilon, \quad D \setminus \mathring{A} \subset (D \setminus \mathring{A}_n) + B_\varepsilon.$$

Then $D \setminus \mathring{A}_n$ H-converges to $D \setminus \mathring{A}$ and so $\mathring{A}_n$ H^c-converges to $\mathring{A}$.
In view of the previous equivalence, in the following we will speak only of Hausdorff convergence*, specifying if the involved convex sets are open or closed only where necessary.*

Hausdorff convergence of convex sets implies the convergence in the sense of Mosco of the H^1 spaces under weak hypotheses.

Theorem 2.7 (see Theorem 7.2.7 in [6])**.** *Let $B \subset \mathbf{R}^d$ a compact set and let $\Omega_n, \Omega \subset B$ be open and convex. If Ω_n H^c-converges to Ω, then $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco.*

Remark 2.8. *Let us take Ω_n, Ω as in Theorem 2.7 and $u_n \in H^1(\Omega_n)$ such that $\|u_n\|_{H^1(\Omega_n)} < C$, with $C > 0$ independent on n. Using Definition 2.2, it is possible to prove that there exists $u \in H^1(\Omega)$ such that, up to subsequences, $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\mathbf{R}^d)$ and $\tilde{\nabla u_n} \to \tilde{\nabla u}$ weakly in $L^2(\mathbf{R}^d; \mathbf{R}^d)$, where we denoted by $\tilde{f}$ the zero extension of the function f outside its domain (Ω_n and Ω for $u_n, \nabla u_n$ and $u, \nabla u$, respectively).*

2.2. **The Robin eigenvalues.** In this section we will recall some properties of $\lambda_{k,\beta}(\Omega)$ and remark some well known facts about it. Many proofs and details of the results of this section could be found in [7].
By means of classical spectral theory, we can observe that $(\lambda_{n,\beta}(\Omega))_n$ is an increasing, diverging sequence of strictly positive values for every $\beta > 0$ and $\Omega \subset \mathbf{R}^d$ open, bounded and Lipschitz (as one deduce also by the formula (1.2)). Moreover, the function $\beta \mapsto \lambda_{k,\beta}(\Omega)$ is strictly increasing.

Remark 2.9 (Scaling property and monotonicity under dilatation)**.** *For every $t > 0$, we have*

$$\lambda_{k,\beta}(t\Omega) = \frac{1}{t^2} \lambda_{k,t\beta}(\Omega).$$

[1] A sequence of closed sets $(K_n)_n$ converges uniformly to a closed set K if, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbf{N}$ such that, for every $n \geq n_\varepsilon$

$$K_n \subset K + B_\varepsilon, \quad K \subset K_n + B_\varepsilon.$$

Then, the Robin eigenvalues are not scale invariant. We only have a decreasing monotonicity under dilatation; more precisely, for every $t > 1$, by the scaling property we have

$$\lambda_{k,\beta}(t\Omega) \leq \frac{1}{t}\lambda_{k,\beta}(\Omega) < \lambda_{k,\beta}(\Omega). \tag{2.1}$$

By several examples (see [9] *for details), one can notice that there is no general monotonicity under inclusions, unlike it happens for other problems (e.g. Dirichlet, Neumann, Steklov).*

Remark 2.10 (Faber-Krahn inequality and global estimates on $\lambda_{1,\beta}(B_r)$)**.** *For every admissible set Ω, if B is a ball with the same measure of Ω, it holds*

$$\lambda_{k,\beta}(B) \leq \lambda_{k,\beta}(\Omega); \tag{2.2}$$

the equality holds if and only if also Ω is a ball (for details see [5],[8]*). Together with this property, we will use the following estimate on the first eigenvalue on a ball of radius r (see e.g.* [10]*, Theorem 4.5):*

$$\frac{\beta}{4r(1+\beta r)} \leq \lambda_{1,\beta}(B_r) \leq \frac{C_d \beta}{r(1+\beta r)}, \tag{2.3}$$

where $C_d > 0$ is a dimensional constant. This implies that $\lambda_{1,\beta}(B_r)$ is infinitesimal as the radius r explodes and explodes as the radius r tends to zero.

We conclude this section giving two results regarding some properties of the Robin-Laplacian eigenfunctions. The first proposition describes their regularity.

Proposition 2.11 (Regularity of the eigenfunctions)**.** *Let $\Omega \subset \mathbf{R}^d$ bounded and Lipschitz; then every solution of* (1.1) *is analytic in Ω and belongs to $H^1(\Omega) \cap C(\overline{\Omega})$.*

The next theorem will play a key role in proving the regularity of optimal convex shapes. It provides a strictly positive lower bound for an eigenfunction for the first Robin eigenvalue of a Lipschitz domain. A proof of this result is contained in [2], Theorem 6.11(j), where the authors use a technique based on C_0-semigroups.

Theorem 2.12 (Strictly positive first eigenfunctions)**.** *Let Ω be a connected Lipschitz domain. Then there exist $\alpha > 0$ and a first Robin eigenfunction $u \in C(\overline{\Omega})$ such that $u \geq \alpha$.*

3. Existence of convex minimizers

In this section we will prove the existence of bounded convex minimizers for problem (1.3) using the direct methods of calculus of variations. In view of this strategy, we need lower semicontinuity of the functional in (1.3) with respect a topology that ensures compactness of minimizing sequences of convex sets. A first step is the following proposition.

Proposition 3.1 (Lower semicontinuity of $\lambda_{k,\beta}$)**.** *Let $(\Omega_n)_n$ be a sequence of open convex sets converging to an open, non empty, convex set Ω in the*

Hausdorff topology and let Ω_n, Ω be contained in a compact set $B \subset \mathbf{R}^d$. Then, for every $k \in \mathbf{N}$,

$$\lambda_{k,\beta}(\Omega) \leq \liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n).$$

Proof. Without loss of generality, we can suppose $\liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n) < +\infty$. Let V_n be an admissible space for the computation of $\lambda_{k,\beta}(\Omega_n)$ such that

$$\lambda_{k,\beta}(\Omega_n) = \max_{V_n} R_{\Omega_n}.$$

Let $\{u_1^n, \dots, u_k^n\} \subset H^1(\Omega_n)$ an $L^2(\Omega_n)$-orthonormal basis for V_n. Without loss of generality we can suppose the sequence $(\lambda_{k,\beta}(\Omega_n))_n$ bounded. So, for every $i = 1, \dots, k$,

$$\int_{\Omega_n} |\nabla u_i^n|^2 \, dx + \beta \int_{\partial\Omega_n} (u_i^n)^2 \, d\sigma < C.$$

Then, $\sup_n \|u_n^i\|_{H^1(\Omega_n)} < +\infty$ for every $i = 1, \dots, k$. Moreover, by Theorem 2.7, $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco; then, by Remark 2.8, there exist $u_1, \dots, u_k \in H^1(\Omega)$ such that, up to subsequences, $\tilde{u}_n^i \to u^i$ strongly in $L^2(\mathbf{R}^d)$ and $\tilde{\nabla u_i^n} \rightharpoonup \tilde{\nabla} u_i$ weakly in $L^2(\mathbf{R}^d; \mathbf{R}^d)$. Notice that $u_1, \dots, u_k$ are linearly independent in $H^1(\Omega)$, since Ω_n converges to Ω also in measure; hence, the linear space $V := \operatorname{span}\{u_1, \dots, u_k\}$ is a competitor for the computation of $\lambda_{k,\beta}(\Omega)$. Let $w = \sum \alpha_i u_i$ realizing the maximum of the Rayleigh quotient $R_\Omega(\cdot)$ on V and let $w_n := \sum \alpha_i u_i^n \in V_n$. Observe that, up to subsequences, $w_n \to w$ strongly in $L^2(\mathbf{R}^d)$ and $\chi_{\Omega_n} \nabla w^n \rightharpoonup \chi_\Omega \nabla w$ weakly in $L^2(\mathbf{R}^d; \mathbf{R}^d)$. Since Ω, Ω_n are convex, there exists a bounded operator that extends w_n and w to the whole of $\mathbf{R}^d$ in such a way that $w_n \rightharpoonup w$ weakly in $H^1(\mathbf{R}^d)$. We can thus apply Proposition 2.1 to w_n, w and $\partial\Omega_n, \partial\Omega$ to have the lower semicontinuity of the boundary integrals. Finally, we have the convergence of the volume integrals at the denominator and the lower semicontinuity of the L^2-norms of the gradients in the Rayleigh quotient. Using the fact that $w_n \in V_n$, we conclude that

$$\begin{aligned}\lambda_{k,\beta}(\Omega) &\leq \max_V R_\Omega = R_\Omega(w) \leq \liminf_{n\to+\infty} R_{\Omega_n}(w_n) \leq \liminf_{n\to+\infty} \max_{V_n} R_{\Omega_n} \\ &= \liminf_{n\to+\infty} \lambda_{k,\beta}(\Omega_n),\end{aligned}$$

obtaining the required lower semicontinuity of the eigenvalues. □

We are now in a position to prove the existence of solutions of (1.3). The key point of the following theorem is to proof that the diameters of the sets of a minimizing sequence are uniformly bounded and that the limit set does not degenerate in any direction.

Theorem 3.2. *Problem* (1.3) *admits at least a bounded convex minimizer.*

Proof. Let $(\Omega_n)_n$ be a minimizing sequence of admissible sets. From the optimality of $(\Omega_n)_n$, we have that $\sup_n \mathcal{H}^{d-1}(\partial\Omega_n) < +\infty$. Then, via isoperimetric inequality, we also have $\sup_n |\Omega_n| < +\infty$. Without loss of generality, up to translations and rotations we can suppose that

$$\operatorname{diam}(\Omega_n) = \mathcal{H}^1(\Omega_n \cap \{x_2 = \ldots = x_d = 0\})$$

and that

$$\min_{i=2,\dots,d}\left(\max_{\Omega_n} x_i - \min_{\Omega_n} x_i\right) = \max_{\Omega_n} x_d - \min_{\Omega_n} x_d$$

i.e. the width of Ω_n is minimal on the direction of the axe x_d. We claim that $\sup_n \operatorname{diam}(\Omega_n) < +\infty$ and that, up to subsequences,

$$\lim_n \left(\max_{\Omega_n} x_d - \min_{\Omega_n} x_d\right) > 0. \tag{3.1}$$

We start proving (3.1) arguing by contradiction. Let us suppose that the limit in (3.1) is zero; define

$$\Omega_n^0 := \Omega_n \cap \{x_d = 0\}$$

and, for every $x' \in \Omega_n^0$, the segment

$$l_n(x') := \{(x', x_d) \in \Omega_n\}.$$

Let us consider an admissible function $u \in H^1(\Omega_n)$ for the computation of the Robin eigenvalues of Ω_n and observe that, for every $x' \in \Omega_n^0$, the function $x_d \mapsto u(x', x_d)$ is admissible for the computation of the Robin eigenvalues of $l_n(x')$. Then we have

(3.2)

$$\begin{aligned}
R(u) &= \frac{\displaystyle\int_{\Omega_n} |\nabla u|^2 \, dx + \beta \int_{\partial\Omega_n} u^2 \, d\sigma}{\displaystyle\int_{\Omega_n} u^2 \, dx} \\
&= \frac{\displaystyle\int_{\Omega_{n_0}} dx' \int_{l_n(x')} \left[|\nabla_{x'} u|^2 + \left(\frac{\partial u}{\partial x_d}\right)^2\right] dx_d + \beta \int_{\Omega_{n_0}} dx' \int_{\partial l_n(x')} u^2(x', x_d) d\mathcal{H}^0(x_d)}{\displaystyle\int_{\Omega_{n_0}} dx' \int_{l_n(x')} u^2 \, dx_d} \\
&\geq \frac{\displaystyle\int_{\Omega_{n_0}} \left(\int_{l_n(x')} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n(x')} u^2(x', x_d) \, d\mathcal{H}^0(x_d)\right) dx'}{\displaystyle\int_{\Omega_{n_0}} \left(\int_{l_n(x')} u^2 \, dx_d\right) dx'} \\
&\geq \min_{x' \in \Omega_n^0} \frac{\displaystyle\int_{l_n(x')} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n(x')} u^2(x', x_d) \, d\mathcal{H}^0(x_d)}{\displaystyle\int_{l_n(x')} u^2 \, dx_d}.
\end{aligned}$$

Now, the term on the last side is a minimum computed among one dimensional Rayleigh quotients on segments. Thanks to the monotonicity under homotheties (2.1) and to the fact that all the $l_n(x')$ are homothetical, we can conclude that the required minimum is achieved on the longest segment $l_n^{max} := l_n(x_{max})$, so

$$R(u) \geq \frac{\displaystyle\int_{l_n^{max}} \left(\frac{\partial u}{\partial x_d}\right)^2 dx_d + \beta \int_{\partial l_n^{max}} u^2(x_{max}, x_d)\, d\mathcal{H}^0(x_d)}{\displaystyle\int_{l_n(x')} u^2\, dx_d} \geq \lambda_{1,\beta}(l_n^{max})$$

as $u(x_{max}, \cdot)$ is a competitor in the computation of $\lambda_{1,\beta}(l_n^{max})$. Let us observe that, as we are contradicting (3.1), the length of l_n^{max} tends to zero as n goes to infinity; then, by estimates (2.3), $\lambda_{1,\beta}(l_n^{max})$ tends to $+\infty$, so $R(u) = +\infty$ for every admissible function $u \in H^1(\Omega_n)$, which is impossible.

To prove that the diameters of the Ω_n sets are uniformly bounded, we argue straightforwardly by contradiction. If the sequence of the diameters was unbounded, as the Ω_n are convex and uniformly bounded in measure, the product

$$\prod_{j=1}^{d} \left(\max_{\Omega_n} x_j - \min_{\Omega_n} x_j\right)$$

has to be uniformly bounded in measure. In view of our assumptions, as the diameter of Ω_n tends to infinity, necessarily the first term of the product diverges and so at least the smallest among the remaining $d-1$ terms has to vanish; in other words, we would have

$$\lim_n \left(\max_{\Omega_n} x_d - \min_{\Omega_n} x_d\right) = 0,$$

in contradiction with (3.1).

Then $(\Omega_n)_n$ is an equibounded sequence of convex sets which converge (up to subsequences) to a bounded convex set Ω in the Hausdorff topology; moreover, by Proposition 2.5, the convergence is also in measure. In addition, thanks to (3.1), the limit set Ω is not degenerate (i.e. it has positive measure) and

$$P(\Omega) \leq \liminf_{n\to+\infty} P(\Omega_n).$$

Finally, thanks to the lower semicontinuity of the Robin eigenvalues (Proposition 3.1) and to the monotonicity and lower semicontinuity of the function F in each variable, we obtain

$$F(\lambda_{1,\beta}(\Omega), \ldots, \lambda_{k,\beta}(\Omega)) \leq \liminf_{n\to+\infty} F(\lambda_{1,\beta}(\Omega_n), \ldots, \lambda_{k,\beta}(\Omega_n)),$$

so we can conclude that Ω is a minimizer of (1.3). □

Remark 3.3. *The previous existence theorem is still valid if, instead of penalizing the perimeter as in Problem* (1.3)*, we impose an uniform constraint on the measures, on the perimeters or on the diameters of the admissible convex sets and minimize only the functional* $F(\lambda_{1,\beta}(\Omega), \ldots, \lambda_{k,\beta}(\Omega))$.

4. Estimates on the cut set

The aim of the following part of the work is to show that, under some additional hypotheses, the optimal convex shapes have C^1 boundary. We will prove this regularity result for a larger class of sets, the so-called *energy subsolutions* for Problem (1.3) (see Definition 5.1); we will see that optimal sets for Problem

(1.3) are also energy subsolutions. The technique to prove the regularity of the boundary is rather intuitive: supposing, by contradiction, that an energy subsolution Ω has a singularity point x_0 for its boundary, we can cut a suitable "ε-neighbourhood" of x_0, obtaining a convex competitor Ω_ε; then, comparing the values of (1.3) for Ω and Ω_ε, we can observe that there exists a cut set strictly better than the optimal set Ω, obtaining a contradiction. The key point of this approach is to estimate the gap between $\lambda_{h,\beta}(\Omega)$ and $\lambda_{h,\beta}(\Omega_\varepsilon)$, for every $h \in \mathbf{N}$.
We will distinguish between the case $d = 2$ and $d > 2$, since the arguments used are based on 2-dimensional sections and on a lower bound on the ratio between two surface areas; in dimension larger than two this is not immediate as in $\mathbf{R}^2$.

4.1. **The case $d = 2$.** In the planar setting, many assumptions can be done. First of all, as the boundary of a convex 2-dimensional set is a Lipschitz curve, the only singularity points for the outer normal are sharp corners, in which we can distinguish two different tangent lines. In the following, without loss of generality, we will assume that Ω has only a singularity point coinciding with the origin, that Ω lies in the halfplane $\{x_1 > 0\}$ and that the bisector of the corner between the distinguished tangent lines is $\{x_2 = 0\}$.
In any dimension d the situation is more involved. In this case, the singularity set for the outer normal is at most a $(d-2)$-dimensional locally Lipschitz variety and it can happen that we have in every singularity point more than one couple of distinguished tangent hyperplanes (e.g., in $\mathbf{R}^3$ the vertex of a circular cone has infinite couples of tangent planes). In every case, without loss of generality, we can assume that, chosen a singularity point for the boundary, it coincides with the origin and that Ω lies in the halfspace $\{x_1 > 0\}$; moreover, we can rotate Ω and chose one of the couples of distinguished tangent hyperplanes in such a way that their bisector is the hyperplane $\{x_d = 0\}$.
Under these assumptions, for every dimension d and every $\varepsilon > 0$, we define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{x_1 > \varepsilon\}, \; m_\varepsilon := \Omega \setminus \Omega_\varepsilon, \; \sigma_\varepsilon := \Omega \cap \{x_d = \varepsilon\}, \; s_\varepsilon := \partial\Omega \setminus \partial\Omega_\varepsilon. \tag{4.1}$$

Let us observe that, in view of our choice, the origin turns out to realize the maximum

$$\max_{x \in \overline{m_\varepsilon}} \operatorname{dist}(x, \sigma_\varepsilon) = \varepsilon.$$

In the following lemma we show how the first eigenvalues decreases after a small cut.

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^2$ be an admissible set for* (1.3) *with a singularity point for the outer normal. Then, there exist $\varepsilon_0 > 0$ and $C = C(\Omega) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\lambda_{1,\beta}(\Omega_\varepsilon) \le \lambda_{1,\beta}(\Omega) - C\varepsilon. \tag{4.2}$$

Proof. First of all let us remark that both $\mathcal{H}^1(\sigma_\varepsilon)$ and $\mathcal{H}^1(s_\varepsilon)$ are infinitesimal of the same order of ε and that $|m_\varepsilon|$ is infinitesimal of the same order of ε^2 as ε goes to 0; moreover, there exists a constant $C_1 > 1$, depending only on the set Ω, such that $\mathcal{H}^1(s_\varepsilon) \ge C_1 \mathcal{H}^1(\sigma_\varepsilon)$.

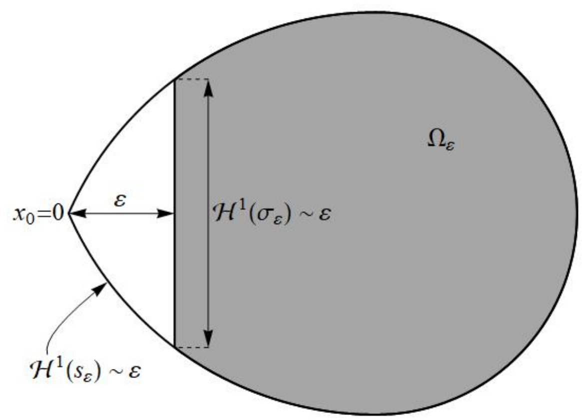

FIGURE 1. Cutting procedure in dimension $d = 2$.

Under the same assumptions of the beginning of the section and using the same notation as in (4.1), let us compare $\lambda_{1,\beta}(\Omega_\varepsilon)$ with $\lambda_{1,\beta}(\Omega)$. Let us consider $u \in H^1(\Omega)$ a $L^2(\Omega)$-normalized eigenfunction for $\lambda_{1,\beta}(\Omega)$ positively bounded from below (it exists in view of 2.12); its restriction on Ω_ε is a test function for $\lambda_{1,\beta}(\Omega_\varepsilon)$ and it holds

$$\begin{aligned}\lambda_{1,\beta}(\Omega_\varepsilon) &\le \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u^2\,d\sigma}{\displaystyle\int_{\Omega_\varepsilon} u^2\,dx}\\ &\le \frac{\displaystyle\int_{\Omega} |\nabla u|^2\,dx + \beta\int_{\partial\Omega} u^2\,d\sigma + \beta\int_{\sigma_\varepsilon} u^2\,d\sigma - \beta\int_{s_\varepsilon} u^2\,d\sigma}{\displaystyle 1-\int_{m_\varepsilon} u^2\,dx} \qquad (4.3)\\ &\le \left[\lambda_{1,\beta}(\Omega) + \beta\int_{\sigma_\varepsilon} u^2\,d\sigma - \beta\int_{s_\varepsilon} u^2\,d\sigma\right]\left(1+2\int_{m_\varepsilon} u^2\,dx\right)\\ &\le \lambda_{1,\beta}(\Omega) + \beta\left(\int_{\sigma_\varepsilon} u^2\,d\sigma - \int_{s_\varepsilon} u^2\,d\sigma\right) + C_2\varepsilon^2\end{aligned}$$

for ε small enough. Fixed $\delta > 0$, with $(u(0)+\delta)^2 \le C_1(u(0)-\delta)^2$, there exists $\varepsilon_0 > 0$ such that

$$0 < u(0) - \delta < u(x) < u(0) + \delta$$

for every $x \in \overline{m_{\varepsilon_0}}$, since $u \in C(\overline{\Omega})$ and $u(0) > 0$. In particular, as m_ε is decreasing in ε with respect to inclusions, we can choose ε_0 small enough to satisfy (4.3). Then we obtain

$$\begin{aligned}\lambda_{1,\beta}(\Omega_\varepsilon) &\le \lambda_{1,\beta}(\Omega) + \beta\left(\mathcal{H}^1(\sigma_\varepsilon)(u(0)+\delta)^2 - \mathcal{H}^1(s_\varepsilon)(u(0)-\delta)^2\right) + C_2\varepsilon^2\\ &\le \lambda_{1,\beta}(\Omega) + \beta\mathcal{H}^1(\sigma_\varepsilon)\left((u(0)+\delta)^2 - C_1(u(0)-\delta)^2\right) + C_2\varepsilon^2\\ &\le \lambda_{1,\beta}(\Omega) - \beta C_3\varepsilon + C_2\varepsilon^2 \le \lambda_{1,\beta}(\Omega) - C\varepsilon,\end{aligned}$$

where the last constant C takes into account all the previous constants and depends only on the domain Ω. $\square$

The behaviour of higher order eigenvalues is studied in the following lemma.

Lemma 4.2. *Let $\Omega \subset \mathbf{R}^2$ be an admissible set for* (1.3) *with a singularity point for the outer normal. Then, for every $h \in \mathbf{N}, h \geq 2$,*

$$\lambda_{h,\beta}(\Omega_\varepsilon) \leq \lambda_{h,\beta}(\Omega) + o(\varepsilon). \tag{4.4}$$

Proof. Let us consider h eigenfunctions for the Laplacian Robin, say $u_1, \dots, u_h$, associated to $\lambda_{1,\beta}(\Omega), \dots, \lambda_{h,\beta}(\Omega)$ such that they form an L^2-orthonormal basis of $S := \operatorname{span}\{u_1, \dots, u_h\}$. Let us consider S as a test space for the computation of $\lambda_{h,\beta}(\Omega_\varepsilon)$; precisely, we can restrict ourselves to the subset of $\operatorname{span}\{u_1|_{\Omega_\varepsilon}, \dots, u_h|_{\Omega_\varepsilon}\}$ of functions of the form $\sum_{i=1}^h \alpha_i^\varepsilon u_i$ with $\sum_{i=1}^h (\alpha_i^\varepsilon)^2 = 1$. This compactness hypothesis on the coefficients ensures us that, up to subsequences, $\alpha_i^\varepsilon \to \alpha_i \in [-1,1]$ and

$$\sum_{i=1}^h \alpha_i^\varepsilon u_i \longrightarrow \sum_{i=1}^h \alpha_i u_i$$

strongly in $H^1(\Omega)$. In the following we will denote by $(\overline{\alpha}_1^\varepsilon, \dots, \overline{\alpha}_h^\varepsilon)$ and $(\overline{\alpha}_1, \dots, \overline{\alpha}_h)$ two h-ple of coefficients that maximizes $R^\beta_{\Omega_\varepsilon}$ and R^β_Ω in S. We claim that $\overline{\alpha}_i^\varepsilon \to 0$ if $\lambda_{i,\beta}(\Omega) < \lambda_{h,\beta}(\Omega)$. To prove this claim, observe first that $\lambda_{h,\beta}(\Omega) = \max_{u \in S} R(u)$, since at least $u_h \in S$ is associated to $\lambda_{h,\beta}(\Omega)$. Then we have

$$\begin{aligned}\lambda_{h,\beta}(\Omega) &= \max_{\substack{\alpha_1,\dots,\alpha_h \in \mathbf{R} \\ \sum_i \alpha_i^2 = 1}} \frac{\displaystyle\int_\Omega \Big| \sum_i \alpha_i \nabla u_i \Big|^2 dx + \beta \int_{\partial\Omega} \Big(\sum_i \alpha_i u_i \Big)^2 d\sigma}{\displaystyle\int_\Omega \Big(\sum_i \alpha_i u_i \Big)^2 dx} \\ &= \sum_{i,j} \overline{\alpha}_i \overline{\alpha}_j \left(\int_\Omega \nabla u_i \cdot \nabla u_j \, dx + \beta \int_{\partial\Omega} u_i u_j \, d\sigma \right)\end{aligned} \tag{4.5}$$

Let us compute the quantity between brackets using the Robin boundary conditions, integrating by parts and recalling that the u_i and u_j belong to an orthonormal basis of eigenfunctions:

$$\begin{aligned}\int_\Omega \nabla u_i \cdot \nabla u_j \, dx + \beta \int_{\partial\Omega} u_i u_j \, d\sigma &= \int_\Omega \nabla u_i \cdot \nabla u_j \, dx - \int_{\partial\Omega} u_i \frac{\partial u_j}{\partial n} \, d\sigma \\ &= - \int_\Omega u_i \Delta u_j \, dx = \lambda_{j,\beta}(\Omega) \int_\Omega u_i u_j \, dx = \lambda_{j,\beta}(\Omega) \delta_{ij}.\end{aligned}$$

So, by (4.5), we obtain

$$\lambda_{h,\beta}(\Omega) = \sum_{i,j} \overline{\alpha}_i \overline{\alpha}_j \lambda_{j,\beta}(\Omega) \delta_{ij} = \sum_i \overline{\alpha}_i^2 \lambda_{i,\beta}(\Omega),$$

that implies that all the coefficients related to $\lambda_{i,\beta}(\Omega) < \lambda_{h,\beta}(\Omega)$ have to be 0.

In view of this remark, for any $\varepsilon > 0$ sufficiently small, we estimate $\lambda_{h,\beta}(\Omega_\varepsilon)$ using S as a test space:

(4.6)

$$\begin{aligned}
\lambda_{h,\beta}(\Omega_\varepsilon) = &\max_{\substack{\alpha_1^\varepsilon,\dots,\alpha_h^\varepsilon\in\mathbf{R}\\ \sum_i(\alpha_i^\varepsilon)^2=1}} \frac{\displaystyle\int_{\Omega_\varepsilon}\Big|\sum_i \alpha_i^\varepsilon \nabla u_i\Big|^2 dx + \beta\int_{\partial\Omega_\varepsilon}\Big(\sum_i \alpha_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle\int_{\Omega_\varepsilon}\Big(\sum_i \alpha_i^\varepsilon u_i\Big)^2 dx}\\
&\le \frac{\displaystyle\int_{\Omega}\Big|\sum_i \overline{\alpha}_i^\varepsilon \nabla u_i\Big|^2 dx + \beta\int_{\partial\Omega}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle 1-\int_{m_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx}\\
&+\frac{\displaystyle\beta\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma - \beta\int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma}{\displaystyle 1-\int_{m_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx}\\
&\le \lambda_{h,\beta}(\Omega) + \beta\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma - \beta\int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\sigma + C|m_\varepsilon|.
\end{aligned}$$

Observe that, for every $i = 1,\dots,h$, $\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i \to 0$; moreover, following the remark at the beginning of Lemma 4.1 about the infinitesimal order of $\mathcal{H}^1(\sigma_\varepsilon)$, $\mathcal{H}^1(s_\varepsilon)$ and $|m_\varepsilon|$, by (4.6) we have

(4.7)

$$\begin{aligned}
\lambda_{h,\beta}(\Omega_\varepsilon) \le \lambda_{h,\beta}(\Omega) + \beta\Bigg[&\int_{\sigma_\varepsilon}\Big(\sum_i(\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i + \overline{\alpha}_i u_i\Big)^2 d\sigma\\
&-\int_{s_\varepsilon}\Big(\sum_i(\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i + \overline{\alpha}_i u_i\Big)^2 d\sigma\Bigg] + C\varepsilon^2\\
\le \lambda_{h,\beta}(\Omega) + \beta&\left(\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma\right) + o(\varepsilon).
\end{aligned}$$

To estimate the boundary integrals in the last term, we have to distinguish two cases. If

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 \neq 0,$$

then, for any sufficiently small ε, we can proceed as in Lemma 4.1 and conclude that

$$\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\sigma \le 0.$$

If

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 = 0,$$

the uniform continuity of the eigenfunctions u_i on m_ε implies that both integrands go to zero as ε goes to zero and so the boundary integrals are infinitesimal of higher order than ε. In both cases, by (4.7) we obtain

$$\lambda_{h,\beta}(\Omega_\varepsilon) \leq \lambda_{h,\beta}(\Omega) + o(\varepsilon).$$

□

Remark 4.3. *Let us compare the results of the previous lemmas. In Lemma 4.1 we proved that, after a small cut, the first eigenvalue decreases by a term of the same order as the perimeter. On the other hand, in Lemma 4.2, we proved that a small cut could at most increase $\lambda_{h,\beta}$ ($h \geq 2$) by a term infinitesimal of higher order than the perimeter. In other words, the possible increase of $\lambda_{h,\beta}$ ($h \geq 2$) is infinitesimal of higher order than the decrease of $\lambda_{1,\beta}$.*

4.2. **The case $d > 2$.** The case of higher dimension is quite different. Recalling the notation introduced in (4.1), the key point is to prove that the ratio $\mathcal{H}^{d-1}(s_\varepsilon)/\mathcal{H}^{d-1}(\sigma_\varepsilon)$ has a lower bound strictly greater than 1, as in the planar case. It is not trivial at a first sight, but fortunately this obstacle can be overcome taking into account suitable 2-dimensional sections of Ω around the singularity point of the boundary. We will get the required lower estimate in the following lemmas, the first holding in dimension 3, the second holding in any dimension. We chose to expose separately the cases of dimension $d = 3$ and of higher dimension for a better clarity for the reader, although the proofs are quite similar.

Lemma 4.4. *Let $\Omega \subset \mathbf{R}^3$ be an admissible set for* (1.3) *with a singularity point at the origin for the outer normal and let us consider s_ε and σ_ε as in* (4.1)*. Then there exists $C > 1$ such that $\mathcal{H}^2(s_\varepsilon)/\mathcal{H}^2(\sigma_\varepsilon) \geq C$ for every $\varepsilon > 0$.*

Proof. Let us use the same convention as in (4.1), so that the origin is a singularity point for $\partial\Omega$, and let us assume that the outer normal to $\partial\Omega$ is discontinuous in the origin with respect to the direction of the x_2 axe. Let us consider two distinguished tangent hyperplanes at the singularity point; without loss of generality we can assume that the bisector of the two planes is the plane $\{x_2 = 0\}$ their intersection V is the line $\{x_1 = x_2 = 0, \}$. Under these assumptions, the orthogonal projection V_ε of V onto σ_ε is a segment on the line $\{x_1 = \varepsilon, x_2 = 0\}$, and it can be expressed by

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ a_\varepsilon \leq x_3 \leq b_\varepsilon \end{cases}$$

with $a_\varepsilon \leq 0 \leq b_\varepsilon$. Moreover, the orthogonal space $V^\perp$ is a 2-dimensional plane and the section $c_\varepsilon := s_\varepsilon \cap V^\perp$ is given by a Lipschitz curve with a corner point in the origin. Notice that

$$V^\perp = \{x_3 = 0\}$$

and that, denoting by l_ε the segment $\sigma_\varepsilon \cap V^\perp$, the curve c_ε is the graph of a concave function defined on l_ε. So, as in the planar setting (see 4.1), there exists $\alpha > 0$ such that

$$\frac{\mathcal{H}^1(c_\varepsilon)}{\mathcal{H}^1(l_\varepsilon)} \geq 1+\alpha \tag{4.8}$$

for every $\varepsilon > 0$ sufficiently small.
The idea to estimate $\mathcal{H}^2(\sigma_\varepsilon)$ in terms of $\mathcal{H}^2(s_\varepsilon)$ is based on the Fubini's theorem: we will split the 2-dimensional surface integral in two 1-dimensional integrals in the variables x_2, x_3 and we will estimate uniformly from above the 1-dimensional sections of s_ε with the 1-dimensional sections of σ_ε. Let us define

$$l_\varepsilon(x_3) := \sigma_\varepsilon \cap (V^\perp + x_3)$$

the 1-dimensional slice of σ_ε passing through $(0,0,x_3)$ and parallel to $l_\varepsilon = l_\varepsilon(0)$. If we denote by

$$c_\varepsilon(x_3) := \sigma_\varepsilon \cap (V^\perp + x_3),$$

then $c_\varepsilon(0) = c_\varepsilon$. Moreover, by continuity and (4.8), the above constant $\alpha > 0$ can be chosen in such a way that

$$\frac{\mathcal{H}^1(c_\varepsilon(x_3))}{\mathcal{H}^1(l_\varepsilon(x_3))} \geq 1+\alpha \tag{4.9}$$

for every $a_\varepsilon/2 \leq x_3 \leq b_\varepsilon/2$. Let us remark that, in every case, the ratio above is greater or equal than 1 for every $x_3 \in]a_\varepsilon, b_\varepsilon[$.
Recalling that s_ε is the graph of a concave function $\phi = \phi(x_2, x_3)$ on σ_ε, let us estimate from below the area of $s_\varepsilon \cap \{x_3 \geq 0\}$:

$$\begin{aligned}
&\mathcal{H}^2(s_\varepsilon \cap \{x_3 \geq 0\}) \\
&\quad = \int_{\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}} \sqrt{1+|\nabla\phi|^2}\, dx_2\, dx_3 + \mathcal{H}^2(s_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} dx_3 \int_{l_\varepsilon(x_3)} \sqrt{1+|\nabla\phi|^2}\, dx_2 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} dx_3 \int_{l_\varepsilon(x_3)} \sqrt{1+(\partial_{x_2}\phi)^2}\, dx_2 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq \int_0^{b_\varepsilon/2} \mathcal{H}^1(c_\varepsilon(x_3))\, dx_3 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq (1+\alpha) \int_0^{b_\varepsilon/2} \mathcal{H}^1(l_\varepsilon(x_3))\, dx_3 + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad \geq (1+\alpha)\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}) + \mathcal{H}^2(\sigma_\varepsilon \cap \{b_\varepsilon/2 \leq x_3 \leq b_\varepsilon\}) \\
&\quad = \mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}) + \alpha\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}).
\end{aligned} \tag{4.10}$$

Let us notice that, as $\sigma_\varepsilon \cap \{x_3 \geq 0\}$ is convex, there exists a positive constant $\gamma < 1$ depending only on Ω such that

$$\mathcal{H}^2(\sigma_\varepsilon \cap \{0 \leq x_3 \leq b_\varepsilon/2\}) \geq \gamma\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}).$$

Then, replacing the estimate in (4.10), we obtain

$$\mathcal{H}^2(s_\varepsilon \cap \{x_3 \geq 0\}) \geq (1+\alpha\gamma)\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \geq 0\}). \tag{4.11}$$

Reasoning for $x_3 \leq 0$ as above we obtain

$$\mathcal{H}^2(s_\varepsilon \cap \{x_3 \leq 0\}) \geq (1+\alpha\gamma)\mathcal{H}^2(\sigma_\varepsilon \cap \{x_3 \leq 0\}),$$

that combined with (4.11) gives us the thesis (with $C = 1+\alpha\gamma$). □

In higher dimension we obtain the same result, after noticing that we can reason similarly to the previous Lemma on each dimension that is orthogonal to a suitable 2-dimensional section.

Lemma 4.5. *Let $\Omega \subset \mathbf{R}^d$ $(d > 3)$ be an admissible set for* (1.3) *with a singularity point at the origin and let us consider s_ε and σ_ε as in* (4.1)*. Then, there exists $C > 1$ such that $\mathcal{H}^{d-1}(s_\varepsilon)/\mathcal{H}^{d-1}(\sigma_\varepsilon) \geq C$ for every $\varepsilon > 0$.*

Proof. Let us start remarking that, in view of the assumptions below (4.1), the intersection V between two distinguished tangent $(d-1)$-dimensional hyperplanes at the singularity point is a $(d-2)$-dimensional hyperplane whose orthogonal projection onto σ_ε, say V_ε, is a convex, $(d-2)$-dimensional set. Let us observe also that $c_\varepsilon := s_\varepsilon \cap V^\perp$ is given by a Lipschitz curve with a corner point in the origin and that, told $l_\varepsilon := \sigma_\varepsilon \cap V^\perp$, there exists a constant $\alpha > 0$ such that the same estimate as in (4.8). To achieve the thesis, it is enough to repeat the same argument in the second part of 4.4 on each of $d-2$ (orthogonal) segments passing by the orthogonal projection of the origin onto σ_ε and parallel to the first $d-2$ Cartesian axes. □

Now we are in a position to state the analogous of 4.1 and 4.2; we omit the proof as it is straightforward, replacing ε by the surface area $\mathcal{H}^{d-1}(\sigma_\varepsilon)$.

Lemma 4.6. *Let $\Omega \subset \mathbf{R}^d$ be an admissible set for* (1.3) *with a singularity point for the outer normal. Then, there exists $\varepsilon_0 > 0$ ad $C = C(\Omega) > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\lambda_{1,\beta}(\Omega_\varepsilon) \leq \lambda_{1,\beta}(\Omega) - C\mathcal{H}^{d-1}(\sigma_\varepsilon).$$

Moreover, for every $h \in \mathbf{N}, h \geq 2$

$$\lambda_{h,\beta}(\Omega_\varepsilon) \leq \lambda_{h,\beta}(\Omega) + o(\mathcal{H}^{d-1}(\sigma_\varepsilon)).$$

5. Regularity of optimal convex shapes

The aim of this section is to prove a regularity result for the optimal shape whose existence has been proved in 3, to complete the proof of Theorem 1.1. We will prove a more general result, i.e. the C^1-regularity of the boundary for a larger class of sets.

Definition 5.1 (Energy subsolutions)**.** *Let $\Omega \subset \mathbf{R}^d$ be a convex bounded set. Ω is said an energy subsolution for problem* (1.3) *if, for every convex set $\tilde{\Omega} \subseteq \Omega$, it holds*

$$F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \leq F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})).$$

Remark 5.2. *Intuitively, a convex set Ω is an energy subsolution when its "energy" $F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega))$ is minimal compared to his convex subsets. Roughly speaking, thanks to the monotonicity of F and of $\lambda_{h,\beta}$ under dilatations, if $\tilde{\Omega} \subset \Omega$ is an admissible set and we focus only on the energy term, it* *is convenient to rescale $\tilde{\Omega}$ to obtain a wider convex set with lower energy; on the other hand, this increases the perimeter term in* (1.3)*, as the two terms seems to be in competition. This behaviour suggests us that a convex solution should balance the two competing terms with the lowest energy possible. In view of this, let us remark that every minimizer Ω of* (1.3) *is also an energy subsolution; in fact, for every $\tilde{\Omega} \subset \Omega$, using the monotonicity of the perimeter of convex sets under inclusions, we have*

$$\begin{aligned} &F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) + \Lambda P(\Omega) \\ &\qquad \le F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})) + \Lambda P(\tilde{\Omega}) \le F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega})) + \Lambda P(\Omega), \end{aligned}$$

that implies $F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \le F(\lambda_{1,\beta}(\tilde{\Omega}), \dots, \lambda_{k,\beta}(\tilde{\Omega}))$.

The following theorem will give us the required regularity for energy subsolutions. To prove it, we will argue by contradiction, supposing that an energy subsolution Ω has at least a singularity point for the outer normal and cutting a piece of Ω around this point; the obtained cut subset will give a strictly smaller energy than Ω, in contradiction with the definition of energy subsolution.

Theorem 5.3 (regularity of the energy subsolutions)**.** *Let $F : \mathbf{R}^k \to R$ satisfy the same hypotheses as in* (1.3) *and, in addition, let it be differentiable in each variable each variable with strictly positive derivative with respect to the first variable. Then, every energy subsolution for problem* (1.3) *has C^1 boundary.*

Proof. Let Ω be an energy subsolution for (1.3) and consider Ω_ε and σ_ε as in (4.1). Considering a Taylor expansion of F and the results in Lemma 4.6, we obtain for sufficiently small ε

$$\begin{aligned} &F(\lambda_{1,\beta}(\Omega_\varepsilon), \dots, \lambda_{k,\beta}(\Omega_\varepsilon)) \\ &= F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) + \sum_{h=1}^{k} \frac{\partial F}{\partial x_h}(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \cdot (\lambda_{h,\beta}(\Omega_\varepsilon) - \lambda_{h,\beta}(\Omega)) \\ &+ o(\mathcal{H}^{d-1}(\sigma_\varepsilon)) \\ &\le F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) - \frac{\partial F}{\partial x_1}(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \cdot (C\mathcal{H}^{d-1}(\sigma_\varepsilon)) + o(\mathcal{H}^{d-1}(\sigma_\varepsilon)) \\ &< F(\lambda_{1,\beta}(\Omega), \dots, \lambda_{k,\beta}(\Omega)) \end{aligned}$$

in contradiction with the fact that Ω is an energy subsolution. □

Proof of Theorem 1.1. Problem (1.3) admits a convex bounded solution Ω thanks to 3.2; by Remark 5.2, this solution is also an energy subsolution, then, under the additional hypotheses of F, by Theorem 5.3, Ω has C^1 boundary. □

5.1. **Further remarks.** Using analogous techniques, it is possible to prove an existence and regularity result for the problem

$$\min\left\{\lambda_{1,\beta}(\Omega) : \Omega \subseteq D, \Omega \text{ bounded and convex}, |\Omega| = m\right\},$$

where D is a fixed bounded open set and $m > 0$. In this case, if the ball of measure m is not contained in D, the problem is not trivially solved: the existence is due to a standard compactness argument for *a priori* uniformly bounded convex sets of fixed measure and lower semicontinuity of the Rayleigh quotient, the regularity arises from the same argument used in Theorem 4.1, which remains valid in every dimension. In a similar way, the problem

$$\min\left\{\lambda_{k,\beta}(\Omega)+\Lambda P(\Omega):\Omega \text{ bounded and convex}\right\},$$

with $\Lambda>0$, has a C^1 solution. The existence is gained with the same arguments in 3.2. The regularity is obtained by contradiction as a consequence of 4.6: the gap between $P(\Omega)$ and $P(\Omega_\varepsilon)$ decreases to zero more slowly than the difference $\lambda_{k,\beta}(\Omega)-\lambda_{k,\beta}(\Omega_\varepsilon)$.

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References

[1] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press (2000)
[2] W. Arendt, A.F.M. ter Elst: *The Laplacian-to-Neumann operator on $C(\partial\Omega)$*, Preprint (2017)
[3] H. Attouch, G. Buttazzo, G. Michaille: *Variational Analysis in Sobolev and BV Spaces*, MPS-SIAM (2006)
[4] B. Bogosel, D. Bucur, A. Giacomini: *Optimal shapes maximizing the Steklov eigenvalues* (preprint), SIAM J. Math. Analysis (to appear, 2017)
[5] M.-H. Bossel: *Membranes élastiquement liées: exstension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger*, C. R. Acad. Sci. Paris Sér. I Math., 302(1):47-50 (1986)
[6] D. Bucur, G. Buttazzo: *Variational Methods in Shape Optimization Problems*, Birkhauser (2005)
[7] D. Bucur, P. Freitas, J. Kennedy: *The Robin Problem*, in A. Henrot (Ed.): *Shape optimization and spectral theory*, Pagg. 78-119, De Gruyter Open (2017)
[8] D. Daners: *A Faber-Krahn inequality for Robin problems in any space dimension*, Math. Ann., 335(4):767-785 (2017)
[9] T. Giorgi, R. G. Smits: *Monotonicity results for the principal eigenvalue of the generalized Robin problem*, Illinois J. Math., 49(4):1133-1143 (electronic) (2005)
[10] H. Kovarik: *On the lowest eigenvalue of Laplace operators with mixed boundary conditions*, J. Geom. Anal., 24(3):1509-1525 (2014)
[11] A. Henrot, M. Pierre: *Variation et optimisation de formes. Une analyse gomtrique*, Springer-Verlag Berlin Heidelberg (2005)

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