UNTANGLING OF TRAJECTORIES FOR NON-SMOOTH VECTOR FIELDS AND BRESSAN'S COMPACTNESS CONJECTURE

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(Communicated by the associate editor name)

Abstract. Given $d \geq 1$, $T > 0$ and a vector field $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we study the problem of uniqueness of weak solutions to the associated transport equation $\partial_t u + \boldsymbol{b} \cdot \nabla u = 0$ where $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a scalar function. In the classical setting, the method of characteristics provides an explicit formula for the solution of the PDE, in terms of the flow of $\boldsymbol{b}$. However, when we drop regularity assumptions on the velocity field, uniqueness is in general lost. We present an approach to the problem of uniqueness based on the concept of Lagrangian representation. This tool allows to represent a suitable class of vector fields as superposition of trajectories: we then give local conditions to ensure that this representation induces a partition of the space-time made up of disjoint trajectories, along which the PDE can be disintegrated into a family of 1-dimensional equations. We finally show that, if $\boldsymbol{b}$ is locally of class BV in the space variable, the decomposition satisfies this structural assumption, yielding a positive answer to the (weak) Bressan's Compactness Conjecture.

1. **Introduction.** We present some recent advances (obtained in [12]) in the study of two partial differential equations of the first order, namely the *continuity equation*

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0, & \text{in } [0,T] \times \mathbb{R}^d \\ u(0,\cdot) = \overline{u}(\cdot) \end{cases} \tag{1}$$

and the *transport equation*

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, & \text{in } [0,T] \times \mathbb{R}^d \\ u(0,\cdot) = \overline{u}(\cdot) \end{cases} \tag{2}$$

where $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field, $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\overline{u}\colon \mathbb{R}^d \to \mathbb{R}$ is the initial datum.

The continuity and the transport equations are among the *cardinal equations* of Mathematical Physics: for instance, the conservation of mass in Euler's equations

2000 *Mathematics Subject Classification.* 35F10, 35L03, 28A50, 35D30.

Key words and phrases. Transport equation, continuity equation, renormalization, uniqueness, Superposition Principle.

The second author is supported by ERC Starting Grant 676675 FLIRT.

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of fluid-mechanics has the form of (1). In that case, a solution u to (1) can be thought as the density of a continuous distribution of particles moving according to the velocity field $\boldsymbol{b}$; in other terms, the quantity $u(t,x)$ represents the number of particles per unit volume at time $t \in [0,T]$ and position $x \in \mathbb{R}^d$. Notice, moreover, that (1) and (2) are equivalent when $\operatorname{div} \boldsymbol{b} = 0$.

When $\boldsymbol{b}$ is sufficiently regular, existence and uniqueness results for (classical) solutions to Problems (1) and (2) are well known. They rely on the so called *method of characteristics* which establishes a deep connection between the "Eulerian" problems (1), (2) and their "Lagrangian" counterpart, given by the ordinary differential equation driven by $\boldsymbol{b}$:

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t, \boldsymbol{X}(t,x)), & (t,x) \in [0,T] \times \mathbb{R}^d \\ \boldsymbol{X}(0,x) = x. \end{cases} \tag{3}$$

Under suitable regularity assumptions on $\boldsymbol{b}$, it is well known (and goes under the name of *Cauchy-Lipschitz theory*) that a *flow* exists, i.e. there is a smooth map $\boldsymbol{X}$ solving (3). A simple observation yields that, if u is a solution to (2), then the function $t \mapsto u(t, \boldsymbol{X}(t,x))$ has to be constant: this allows to conclude that the unique solution u of (2) is the *transport* of the initial data $\overline{u}$ along the *characteristics* of (3), i.e. along the curves $[0,T] \ni t \mapsto \boldsymbol{X}(t,x)$. Thus we end up with an explicit formula for the solution u to (2):

$$u(t,x) = \overline{u}\left(\boldsymbol{X}(t,\cdot)^{-1}(x)\right).$$

Similarly one can obtain an explicit formula for solutions to (1).

However, in view of the applications to fluid-mechanics, one would like to deal with velocity fields or densities which are not necessarily smooth. For instance, continuity equation and transport equation with non-smooth vector fields are related to Boltzmann [23, 25] and Vlasov-Poisson equations [22], and also to *hyperbolic conservation laws*. In particular the *Keyfitz and Kranzer system* (introduced in [27]) is a system of conservation laws that reads as

$$\partial_t u + \operatorname{div}\left(\boldsymbol{f}(|u|)u\right) = 0 \qquad \text{in } [0,T] \times \mathbb{R}^d, \tag{4}$$

where the map $\boldsymbol{f} \colon \mathbb{R}^+ \to \mathbb{R}^d$ is assumed to be smooth. It has been shown in [5] that (4) can be formally decoupled in a scalar conservation law for the modulus $r = |u|$ and a transport equation (with field $\boldsymbol{f}(r)$) for the angular part $\vartheta = u/|u|$:

$$\begin{cases} \partial_t r + \operatorname{div}(\boldsymbol{f}(r) r) = 0, \\ \partial_t \vartheta + \boldsymbol{f}(r) \cdot \nabla \vartheta = 0. \end{cases}$$

As it is well known, solutions to systems of conservation laws are in general non-smooth, hence the vector field $\boldsymbol{f}(r)$ appearing in the transport equation is not regular enough to apply the method of characteristics: we thus have to go beyond the Cauchy-Lipschitz setting.

1.1. **The classical approach: renormalized solutions.** The exploration of the non-smooth framework started with the paper of DiPerna and Lions [24]. They realized that an interplay between Eulerian and Lagrangian coordinates could be exploited to deduce well-posedness results for the ODE (3) from analogous results on PDEs (1) and (2).

On the one hand, due to the linearity of the PDEs, the *existence* of weak solutions to (1), (2) is always guaranteed under reasonable summability assumptions on the vector field $\boldsymbol{b}$ and its spatial divergence; on the other hand, the problem of uniqueness turns out to be much more delicate. A possible strategy, introduced by [24], to recover uniqueness, is based on the notions of *renormalized solution* and of *renormalization property.*

Roughly speaking, a bounded function $u \in L^\infty([0,T]\times \mathbb{R}^d)$ is said to be a *renormalized solution* to (2) if for all $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ is a solution to the corresponding Cauchy problem:

$$\begin{cases}\partial_t u + \boldsymbol{b}\cdot\nabla u = 0,\\ u(0,\cdot)=\bar u\end{cases} \Longrightarrow \begin{cases}\partial_t(\beta(u)) + \boldsymbol{b}\cdot\nabla(\beta(u)) = 0\\ \beta(u(0,\cdot)) = \beta(\bar u(\cdot))\end{cases} \quad \text{for every } \beta\in C^1(\mathbb{R}).$$

This can be interpreted as a sort of weak "Chain Rule" for the function u, saying that u is differentiable along the flow generated by $\boldsymbol{b}$. In [24] it is shown that the validity of this property for every $\beta \in C^1(\mathbb{R})$ implies, under general assumptions, uniqueness of weak solutions for (2). Moreover, when this property is satisfied by all solutions, this can be transferred into a property of the vector field itself, which will be said to have the *renormalization property.*

The problem of uniqueness of solutions is thus shifted to prove the renormalization property for $\boldsymbol{b}$: this seems to require some regularity of vector field (tipically in terms of spatial weak differentiability), as counterexamples by Depauw [21] and Bressan [17] show. With an approximation scheme, in [24] the authors proved that renormalization property holds under Sobolev regularity assumptions on the vector field; some years later, Ambrosio [4] improved upon this result, showing that renormalization holds for vector fields which are of class BV (locally in space) with absolutely continuous divergence.

From the Lagrangian point of view, the uniqueness of the solution to the transport equation (2) translates into well-posedness results of the so-called *Regular Lagrangian Flow* of $\boldsymbol{b}$, which is the by-now standard notion of flow in the non-smooth setting. This concept was introduced by Ambrosio in [4]: in a sense, among all possible integral curves of $\boldsymbol{b}$ passing through a point, the Regular Lagrangian Flow selects the ones that do not allow for concentration, in a quantitative way with respect to some reference measure (usually the Lebesgue measure $\mathscr{L}^d$ in $\mathbb{R}^d$). It is worth pointing out that a number of recent papers are devoted to the study of its properties, in particular we mention [6] where a purely *local* theory of Regular Lagrangian Flows has been proposed, thus establishing a complete analogy with the Cauchy-Lipschitz theory.

1.2. **Bressan's Compactness Conjecture.** As we have seen, the theory developed by DiPerna-Lions-Ambrosio settles the Sobolev and the BV case, when the divergence of $\boldsymbol{b}$ does not contain singular terms (with respect to $\mathscr{L}^d$). However, in connections with applications to conservation laws, it would be interesting to cover also the case in which $\boldsymbol{b}$ is of bounded variation in the space, but its divergence may contain non-trivial singular terms: indeed the natural assumption at the level of the divergence of $\boldsymbol{b}$ seems to be not really absolute continuity with bounded density, as considered in Ambrosio [4], but rather the existence of a nonnegative density ρ transported by $\boldsymbol{b}$, with ρ uniformly bounded from above and from below away from zero. Such vector fields are called *nearly incompressible*, according to the following definition.

Definition 1.1. A locally integrable vector field $\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho\colon (0,T)\times\mathbb{R}^d\to\mathbb{R}$ (called *density* of $\boldsymbol{b}$) and a constant $C>0$ such that $0<C^{-1}\le\rho(t,x)\le C$ for Lebesgue almost every $(t,x)\in(0,T)\times\mathbb{R}^d$ and

$$\partial_t\rho+\operatorname{div}_x(\rho\boldsymbol{b})=0 \qquad \text{in the sense of distributions on } (0,T)\times\mathbb{R}^d.$$

Notice that no assumption is made on the divergence of $\boldsymbol{b}$; on the other hand, it is rather easy to see (for instance, by mollifications) that if $\operatorname{div}\boldsymbol{b}$ is bounded then $\boldsymbol{b}$ is nearly incompressible.

Nearly incompressible vector fields are strictly related to a conjecture, raised by A. Bressan (studying the well-posedness of the Keyfitz and Kranzer system (4)), predicting the strong compactness of a family of flows associated to smooth vector fields:

Conjecture 1 (Bressan's Compactness Conjecture - Lagrangian formulation)**.** *Let $\boldsymbol{b}_k\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $k\in\mathbb{N}$, be a sequence of smooth vector fields and denote by $\boldsymbol{X}_k$ the associated flows, i.e. the solutions of*

$$\begin{cases}\partial_t\boldsymbol{X}_k(t,x)=\boldsymbol{b}_k(t,\boldsymbol{X}_k(t,x))\\ \boldsymbol{X}_k(0,x)=x.\end{cases}$$

Assume that the quantity $\|\boldsymbol{b}_k\|_\infty+\|\nabla\boldsymbol{b}_k\|_{L^1}$ is uniformly bounded and assume furthermore that there exists $C>0$ such that for every $k\in\mathbb{N}$ it holds

$$\frac{1}{C}\le\det\left(\nabla_x\boldsymbol{X}_k(t,x)\right)\le C, \qquad \forall(t,x)\in[0,T]\times\mathbb{R}^d.$$

Then the sequence $\{\boldsymbol{X}_k\}_{k\in\mathbb{N}}$ is strongly precompact in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$.

By standard compactness arguments, it is readily seen that Conjecture 1 deals essentially with an ordinary differential equation, driven by a nearly incompressible, BV vector field. From the Eulerian point of view, one can thus expect that Conjecture 1 is proved as soon as one can show well posedness at the PDE level for a vector field of class BV and nearly incompressible, extending the well-posedness result of Ambrosio [4]. This is indeed the case: as it has been proved in [5], Conjecture 1 would follow from the following one:

Conjecture 2 (Bressan's Compactness Conjecture - Eulerian formulation)**.** *Any nearly incompressible vector field $\boldsymbol{b}\in L^1([0,T];\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ has the renormalization property.*

The main result is the following Theorem, which answers affirmatively to the conjectures above.

Main Theorem. *Bressan's Compactness Conjecture holds true.*

More precisely, we prove Conjecture 2. It is important to mention various approaches that have been tried in the recent years, also at a purely Lagrangian level: for instance, explicit compactness estimates have been proposed in [10, 19] (and further developed in [16]; see also [26, 18]).

Before presenting the techniques we use to prove the Main Theorem we briefly discuss a particular setting, namely the two-dimensional one, where finer results are availble in view of the Hamiltonian structure.

2. **The two-dimensional case.** The problem of uniqueness of weak solutions to the transport equation (2) in the two dimensional (autonomous) case is addressed in the papers [3], [2] and [15]. In two dimensions and for divergence-free autonomous vector fields, renormalization theorems are available under quite mild assumptions, because of the underlying Hamiltonian structure. Indeed, if $\operatorname{div} \boldsymbol{b} = 0$ in $\mathbb{R}^2$, then there exists a Lipschitz Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\boldsymbol{b} = \nabla^{\perp} H$, where $\nabla^{\perp} = (-\partial_2, \partial_1)$. Heuristically it is readily seens that level sets of H are invariant under the flow of $\boldsymbol{b}$, since

$$\frac{d}{dt} H(\gamma(t)) = \nabla H(\gamma(t)) \cdot \dot{\gamma}(t) = \nabla H(\gamma(t)) \cdot \boldsymbol{b}(\gamma(t)) = 0$$

as $\boldsymbol{b}$ and ∇H are orthogonal. This suggests the possibility of decomposing the two-dimensional transport equation into a family of one-dimensional equations, along the level sets of H. By means of this strategy, and building on a fine description of the structure of level sets of Lipschitz maps (obtained in the paper [2]), in [3], the authors characterize the autonomous, divergence-free vector fields $\boldsymbol{b}$ on the plane for which uniqueness holds, within the class of bounded (or even merely integrable) solutions. The characterization they present relies on the so called *Weak Sard Property*, which is a (weaker) measure theoretic version of Sard's Lemma and is used to separate the dynamic where $\boldsymbol{b} \neq 0$ from the regions in which $\boldsymbol{b} = 0$. An extension of these Hamiltonian techniques to the two-dimensional nearly incompressible case was obtained in [14], whose main result is the following:

Theorem 2.1 ([14])**.** *Every bounded, autonomous, compactly supported, nearly incompressible* BV *vector field on* $\mathbb{R}^2$ *has the renormalization property.*

However, that in the general d-dimensional case, with $d > 2$, the Hamiltonian approach cannot be applied, as there are not enough first integrals of the ODE (which is to say, bounded divergence-free vector fields in $\mathbb{R}^d$ do not admit in general a Lipschitz potential).

3. **The chain rule approach.** We now come back to the general d-dimensional setting and we briefly discuss an approach towards Bressan's Conjecture 2 that has been tried.

In [9], the authors proposed to face the conjecture by establishing a *Chain rule formula* for the divergence operator. Given a bounded, Borel vector field $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$, a bounded, scalar function $\rho\colon \mathbb{R}^d \to \mathbb{R}$, one would like to characterize (compute) the distribution $\operatorname{div}(\beta(\rho)\boldsymbol{b})$, for $\beta \in C^1(\mathbb{R};\mathbb{R})$, in terms of the quantities $\operatorname{div} \boldsymbol{b}$ and $\operatorname{div}(\rho \boldsymbol{b})$. In the smooth setting one can use the standard chain rule formula to get

$$\operatorname{div}(\beta(\rho)\boldsymbol{b}) = \beta'(\rho) \operatorname{div}(\rho \boldsymbol{b}) + (\beta(\rho) - \rho\beta'(\rho)) \operatorname{div} \boldsymbol{b} \tag{5}$$

In the general case, however, the r.h.s. of (5) cannot be written in that form, being only a distribution. In the case the vector field $\boldsymbol{b} \in \mathrm{BV}(\mathbb{R}^d)$, it can be shown that $\operatorname{div}(\beta(\rho)\boldsymbol{b})$ is a measure, controlled by $\operatorname{div} \boldsymbol{b}$ but, as noted in [9], the main problem is to give a meaning to the r.h.s. of (5) when the measure $\operatorname{div} \boldsymbol{b}$ is singular and ρ is only defined almost everywhere with respect to Lebesgue measure. To overcome this difficulty, in the BV setting, the authors split the measure $\operatorname{div} \boldsymbol{b}$ into its absolutely continuous part, jump part and Cantor part and treat the cases separately.

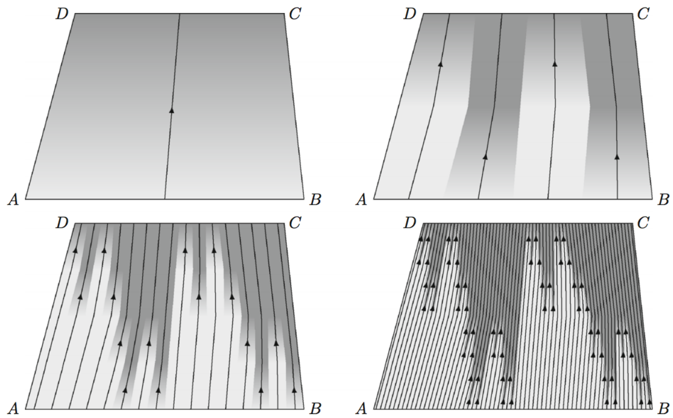


FIGURE 1. Example of [15]: the tangential set of the vector field $\boldsymbol{b}$ (only the integral curves have been drawn here) is a Cantor like set of dimension 3/2. Notice that each trajectory γ meets the tangential set in exactly one point, at time t_γ: the density ρ, computed along the curve, is piecewise constant, having a unique jump of size 1 in t_γ.

The absolutely continuous part. Their first result ([9, Thm. 3]) is that in all Lebesgue points of ρ the formula (5) holds (possibly being $\operatorname{div} \boldsymbol{b}$ singular), where ρ is replaced by its Lebsgue value $\tilde{\rho}$. This is achieved along the same techniques of [4], which are in turn a (non-trivial) extension of the ones employed in [24]: essentially, an approximation argument via convolution is performed (leading to the study of the so called *commutators*). One can control the singular terms by taking suitable convolution kernels which look more elongated in some directions.

The jump part. By exploiting properties of Anzellotti's weak *normal traces* for measure divergence vector fields (see [11]), Ambrosio, De Lellis and Malý managed to settle also the jump part: they obtain an explicit formula (in the spirit of (5)), involving the traces of $\boldsymbol{b}$ and $\rho\boldsymbol{b}$ along a $\mathscr{H}^{d-1}$-rectifiable set (see also [8] for an extension of these results to the BD case).

The Cantor part. In order to tackle the Cantor part, a "transversality condition" between the vector field and its derivative is assumed in [9]: it is shown that, if in a point $(\bar{t}, \bar{x})$ one has $(D\boldsymbol{b} \cdot \boldsymbol{b})(\bar{t}, \bar{x}) \neq 0$ (where $\boldsymbol{b}(\bar{t}, \bar{x})$ is the Lebesgue value of $\boldsymbol{b}$ in $(\bar{t}, \bar{x})$) then the point $(\bar{t}, \bar{x})$ is a Lebesgue point for ρ.

From the analysis of [9], it thus remains open the case of tangential points, i.e. the set of points at which $D\boldsymbol{b} \cdot \boldsymbol{b}$ vanishes, which make up the so called *tangential set*. This is actually relevant, as shown in [15]: answering negatively to one of the questions in [9], in [15] the authors exhibited an example of BV, nearly incompressible vector field with non empty tangential set. Even worse, the tangential set is a Cantor-like set of non integer dimension but, at level of the density ρ, one sees a pure jump. This severe pathology is depicted in Figure 1 and we refer the reader to [15] for a detailed construction.

4. **A new approach.** We now want to present in more details our main contribution, discussing briefly the theorems we obtained in [12] and the strategy leading to their proofs. The starting point of our approach is the notion of *Lagrangian representation* η of the $\mathbb{R}^{d+1}$-valued vector field $\rho(1,\boldsymbol{b})$, defined in the subsequent paragraph.

4.1. **Lagrangian representations.** In the general non-smooth setting, one could recover a link between the continuity equation (1) and the ODE (3) thanks to the so called *Superposition Principle*, which has been established by Ambrosio in [4] (see also [28]). Roughly speaking, it asserts that, if the vector field is globally bounded, every non-negative (possibly measure-valued) solution to the PDE (1) can be written as a superposition of solutions obtained via propagation along integral curves of $\boldsymbol{b}$, i.e. solutions to the ODE (3).
More generally, let us consider a locally integrable vector field $\boldsymbol{b} \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ and let ρ be a non-negative solution to the balance law

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{b}) = \mu, \qquad \mu \in \mathscr{M}((0,T)\times\mathbb{R}^d). \tag{6}$$

with $\rho \in L^1_{\mathrm{loc}}((1+|\boldsymbol{b}|)\mathscr{L}^{d+1})$ (so that a distributional meaning can be given). For simplicity, we will often write (6) in the shorter form

$$\operatorname{div}_{t,x}\big(\rho(1,\boldsymbol{b})\big) = \mu. \tag{7}$$

Let us denote the space of continuous curves by

$$\Upsilon := \big\{(t_1,t_2,\gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C(\mathbb{R}^+,\mathbb{R}^d),\, t_1 < t_2\big\}$$

and let us tacitly identify the triplet $(t^-_\gamma, t^+_\gamma, \gamma) \in \Upsilon$ with γ, so that we will simply write $\gamma \in \Gamma$. We say that a finite, non negative measure η over the set Υ is a *Lagrangian representation* of the vector field $\rho(1,\boldsymbol{b})$ if the following conditions hold:

1. η is concentrated on the set of characteristics Γ, defined as

$$\Gamma := \{(t_1,t_2,\gamma) \in \Upsilon : \gamma \text{ characteristic of } \boldsymbol{b} \text{ in } (t_1,t_2)\};$$

   we explicitly recall that a curve γ is said to be a characteristic of the vector field $\boldsymbol{b}$ in the interval I_γ if it is an absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \boldsymbol{b}(t,\gamma(t)),$$

   in I_γ, which means that for every $(s,t) \subset I_\gamma$ we have

$$\int_\Gamma \left| \gamma(t) - \gamma(s) - \int_s^t \boldsymbol{b}(\tau,\gamma(\tau))\,d\tau \right| \eta(d\gamma) = 0.$$

2. The solution ρ can be seen as a superposition of the curves selected by η, i.e. if $(\mathbb{I},\gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t,\gamma(t))$, we ask that

$$\rho \mathscr{L}^{d+1} = \int_\Gamma (\mathbb{I},\gamma)_\sharp \mathscr{L}^1\, \eta(d\gamma);$$

3. we can decompose μ, the divergence of $\rho(1,\boldsymbol{b})$, as a local superposition of Dirac masses without cancellation, i.e.

$$\mu = \int_\Gamma \Big[\delta_{t^-_\gamma,\gamma(t^-_\gamma)}(dt\,dx) - \delta_{t^+_\gamma,\gamma(t^+_\gamma)}(dt\,dx)\Big]\, \eta(d\gamma),$$
$$|\mu| = \int_\Gamma \Big[\delta_{t^-_\gamma,\gamma(t^-_\gamma)}(dt\,dx) + \delta_{t^+_\gamma,\gamma(t^+_\gamma)}(dt\,dx)\Big]\, \eta(d\gamma).$$

The existence of such a decomposition into curves is a consequence of general structural results of 1-dimensional normal currents (see [28] and, for the case $\mu = 0$, [7, Thm. 12]). The non-negativity assumption on $\rho \geq 0$ (i.e. the *a-cyclicity* of $\rho(1, \boldsymbol{b})$ in the language of currents) plays here a role, allowing to reparametrize the curves in such a way they become characteristic of $\boldsymbol{b}$, i.e. they satisfy Point (1).

4.2. **Restriction of Lagrangian representations and proper sets.** One problem we face immediately lies in the fact that η is a *global* object, thus it is not immediate to relate suitable *local estimates* with η: in other words, in general, η cannot be restricted to a set, without losing the property of being a Lagrangian representation. If we are given an open set $\Omega \subset \mathbb{R}^{d+1}$ and a curve γ, we can write

$$\gamma^{-1}(\Omega) = \bigcup_{i=1}^{\infty} (t_\gamma^{i,-}, t_\gamma^{i,+})$$

and then consider the family of curves

$$\mathtt{R}_\Omega^i \gamma := \gamma \llcorner_{(t_\gamma^{i,-}, t_\gamma^{i,+})}.$$

We can now define

$$\eta_\Omega := \sum_{i=1}^{\infty} (\mathtt{R}_\Omega^i)_\sharp \eta. \tag{8}$$

In general, the series in (8) does not converge. Moreover, even if the quantity in (8) is well defined as a measure, since η satisfies Points (1) and (2) of the definition of Lagrangian representation given above, it certainly holds

$$\rho(1, \boldsymbol{b}) \, \mathscr{L}^{d+1} \llcorner_\Omega = \int_\Gamma (\mathbb{I}, \gamma)_\sharp \big((1, \dot\gamma) \mathscr{L}^1 \big) \, \eta_\Omega(d\gamma).$$

but, in general, Point (3) is not satisfied by η_Ω (more precisely the second formula): in other words, η_Ω might not be a Lagrangian representation of $\rho(1, \boldsymbol{b}) \mathscr{L}^{d+1} \llcorner_\Omega$: the key point is that the sets of γ which are exiting from or entering in Ω are not disjoint.

Thus the first question we have to answer to is to characterize the open sets $\Omega \subset \mathbb{R}^{d+1}$ for which η_Ω is a Lagrangian representation of $\rho(1, \boldsymbol{b}) \mathscr{L}^{d+1} \llcorner_\Omega$. It turns out that there are sufficiently many open sets Ω with this property: apart from having a piecewise C^1-regular boundary and assuming that $\mathscr{H}^d \llcorner_{\partial\Omega}$-a.e. point is a Lebesgue point for $\rho(1, \boldsymbol{b})$, the fundamental fact is that there are two Lipschitz functions $\phi^{\delta,\pm}$ such that

$$\mathbb{1}_\Omega \leq \phi^{\delta,+} \leq \mathbb{1}_{\Omega + B_\delta^{d+1}(0)}, \qquad \mathbb{1}_{\mathbb{R}^{d+1} \setminus \Omega} \leq \phi^{\delta,-} \leq \mathbb{1}_{\mathbb{R}^{d+1} \setminus \Omega + B_\delta^{d+1}(0)}$$

and

$$\lim_{\delta \to 0} \rho |(1, \boldsymbol{b}) \cdot \nabla \phi^{\delta,\pm}| \, \mathscr{L}^{d+1} = \rho |(1, \boldsymbol{b}) \cdot \mathbf{n}| \, \mathscr{H}^d \llcorner_{\partial\Omega} \qquad \text{in the sense of measures on } \mathbb{R}^{d+1},$$

which essentially mean that $\rho(1, \boldsymbol{b}) \mathscr{H}^d \llcorner_{\partial\Omega}$ is measuring the flux of $\rho(1, \boldsymbol{b})$ across $\partial\Omega$. We call these set $\rho(1, \boldsymbol{b})$-*proper* (or just *proper* for shortness) and we study carefully their properties: we show that there are sufficiently many proper sets and that they can be perturbed in order to adapt to the vector field under study.

4.3. **Cylinders of approximate flow.** Once we are able to localize the problem in a proper set, we can start studying which are the pieces of information on the local behavior of the vector field that one needs in order to deduce global uniqueness results.

Given a proper set $\Omega \subset \mathbb{R}^{d+1}$, we assume we can construct locally *cylinders of approximate flow* as follows:

Assumption 4.1. *There are constants* $\mathtt{M}, \varpi > 0$ *and a family of functions* $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ *such that:*

1. *for every* $\gamma \in \Gamma, \ell \in \mathbb{R}^+$*, the function* $\phi^\ell_\gamma \colon [t^-_\gamma, t^+_\gamma] \times \mathbb{R}^d \to [0,1]$ *is Lipschitz, so that it can be used as a test function;*
2. *the* shrinking ratio *of the cylinder* ϕ^ℓ_γ *is controlled in time, preventing it collapses to a point: more precisely, for* $t \in [t^-_\gamma, t^+_\gamma]$ *and* $x \in \mathbb{R}^d$*,*
$$\mathbb{1}_{\gamma(t)+B^d_{\ell/\mathtt{M}}(0)}(x) \le \phi^\ell_\gamma(t,x) \le \mathbb{1}_{\gamma(t)+B^d_{\mathtt{M}\ell}(0)}(x);$$
3. *we control in a quantitative way the flux through the "lateral boundary of the cylinder" (compared to the total amount of curves starting from the "base of the cylinder") with the quantity* ϖ*: more precisely, denoting by*
$$\begin{aligned}\mathsf{Flux}^\ell(\gamma) :=& \begin{array}{c}\textit{flux of the the vector field } \rho(1,\boldsymbol{b}) \\ \textit{across the "boundary of the cylinder" } \phi^\ell_\gamma\end{array} \\ =& \iint_{(t^-_\gamma,t^+_\gamma)\times\mathbb{R}^d} \rho(t,x)\big|(1,\boldsymbol{b})\cdot\nabla\phi^\ell_\gamma(t,x)\big|\,\mathscr{L}^{d+1}(dx\,dt),\end{aligned}$$
$\sigma^\ell(\gamma) :=$ *amount of curves starting from the base of the cylinder* ϕ^ℓ_γ
and
$$\eta^{\rm in}_\Omega := \eta_{\Omega}\llcorner_{\{\textit{curves entering in } \Omega\}}$$
we ask that
$$\int_\Gamma \frac{1}{\sigma^\ell(\gamma)} \mathsf{Flux}^\ell(\gamma)\,\eta^{\rm in}_\Omega(d\gamma) \le \varpi. \tag{9}$$

We decided to call *cylinders of approximate flow* the family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$: indeed, if γ is a characteristic of the vector field $\boldsymbol{b}$, the function ϕ^ℓ_γ can be thought as generalized, smoothed cylinder centered at γ. Notice that the measure $\eta^{\rm in}_\Omega$ makes sense if Ω is a proper set, in view of the above analysis. Thus the ultimate meaning of the assumption is that one controls the ratio between the flux of $\rho(1,\boldsymbol{b})$ across the lateral boundary of the cylinders and the total amount of curves entering through its base in a uniform way (w.r.t. ℓ), as the cylinder shrinks to a trajectory γ. A completely similar computation can be performed backward in time, by considering η_Ω restricted to the exiting trajectories and adopting suitable modifications.

4.4. **Passing to the limit via transport plans.** At this point, one would like to determine what the cylinder estimate (9) yields in the limit $\ell \to 0$. In order to perform this passage to the limit, we borrow some tools from the Optimal Transportation Theory. The language of transference plans is particularly suited for our purposes: we define
$$\Gamma^{\rm cr}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^\pm_\gamma) \in \partial\Omega\big\}, \qquad \Gamma^{\rm in}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^-_\gamma) \in \partial\Omega\big\}$$
and we consider plans between $\eta^{\rm cr}_\Omega := \eta_\Omega\llcorner_{\Gamma^{\rm cr}(\Omega)}$ and the entering trajectory measure $\eta^{\rm in}_\Omega$. Notice that $\eta^{\rm cr}_\Omega$ is concentrated, by definition, on the set of trajectories entering in and exiting from Ω (*crossing* trajectories).

In the correct estimate one has to take into account also of trajectories which end inside the set Ω and this, in view of Point 3 of the definition of Lagrangian representation, is estimated by the negative part μ^- of the divergence μ, defined in (7). Thus one obtains the following

Proposition 1. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and η be a Lagrangian representation of $\rho(1, \boldsymbol{b})$. If Assumption 4.1 holds then there exist $N_1 \subset \mathit{\Gamma}^{\mathrm{cr}}(\Omega), N_2 \subset \mathit{\Gamma}^{\mathrm{in}}(\Omega)$ such that*

$$\eta_\Omega^{\mathrm{cr}}(N_1) + \eta_\Omega^{\mathrm{in}}(N_2) \leq \inf_{C>1} \left\{ 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1} \right\}$$

and for every $(\gamma, \gamma') \in (\mathit{\Gamma}^{\mathrm{cr}} \setminus N_1) \times (\mathit{\Gamma}^{\mathrm{in}} \setminus N_2)$

$$\textit{either } \operatorname{clos}\operatorname{Graph}\gamma' \subset \operatorname{clos}\operatorname{Graph}\gamma \textit{ or } \operatorname{clos}\operatorname{Graph}\gamma, \operatorname{clos}\operatorname{Graph}\gamma' \textit{ are disjoint.} \tag{$\star$}$$

Proposition 1 gives essentially a uniqueness result (from the Lagrangian point of view) at a *local* level, namely inside a proper set Ω: it says that, under Assumption 4.1, up to removing a set of trajectories whose measure is controlled, one gets a family of essentially disjoint trajectories (meaning that are either disjoint or one contained in the other).

4.5. ***Untangling* of trajectories.** It seems at this point natural to try to perform some "local-to-global" argument, seeking a global analog of Proposition 1. In order to do this, we introduce the following *untangling functional for* η^{in}, defined on the class of proper sets as

$$\boldsymbol{f}^{\,\mathrm{in}}(\Omega) := \inf \left\{ \eta_\Omega^{\mathrm{cr}}(N_1) + \eta_\Omega^{\mathrm{in}}(N_2) : \forall (\gamma, \gamma') \in (\mathit{\Gamma} \setminus N_1) \times (\mathit{\Gamma} \setminus N_2) \text{ condition } (\star) \text{ holds } \right\}$$

and, in a similar fashion, one can define an untangling functional for the trajectories that are exiting from the domain Ω. In a sense, these functionals are measuring the minimum amount of curves one has to remove so that the remaning ones are essentially disjoint, i.e. they satisfy condition ($\star$). The main property of these functionals is that they are subadditive with respect to the domain Ω, meaning that

$$\boldsymbol{f}^{\,\mathrm{in}}(\Omega) \leq \boldsymbol{f}^{\,\mathrm{in}}(\mathrm{U}) + \boldsymbol{f}^{\,\mathrm{in}}(\mathrm{V}),$$

whenever $\mathrm{U}, \mathrm{V} \subset \mathbb{R}^{d+1}$ are proper sets whose union $\Omega := \mathrm{U} \cup \mathrm{V}$ is proper. The subadditivity suggests the possibility of having a local control in terms of a measure ϖ^τ, whose mass is $\tau > 0$, replacing the constant ϖ in Proposition 1 with $\varpi^\tau(\Omega)$. In view of Proposition 1 one has to combine ϖ^τ with the divergence and this can be done by introducing a suitable measure $\zeta_C^\tau \approx C\varpi^\tau + \frac{|\mu|}{C}$ on $\mathbb{R}^{d+1}$. If Assumption 4.1 is satisfied locally by a suitable family of balls, then one can show, by means of a non-trivial covering argument, the following fundamental proposition, which is the global analog of Proposition 1.

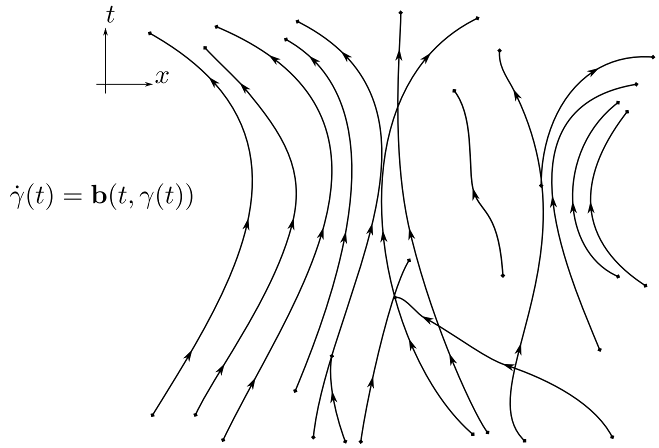
Proposition 2. *There exists a set of trajectories $N \subset \mathit{\Gamma}$ such that*

$$\eta(N) \leq C_d \zeta_C^\tau(\mathbb{R}^{d+1})$$

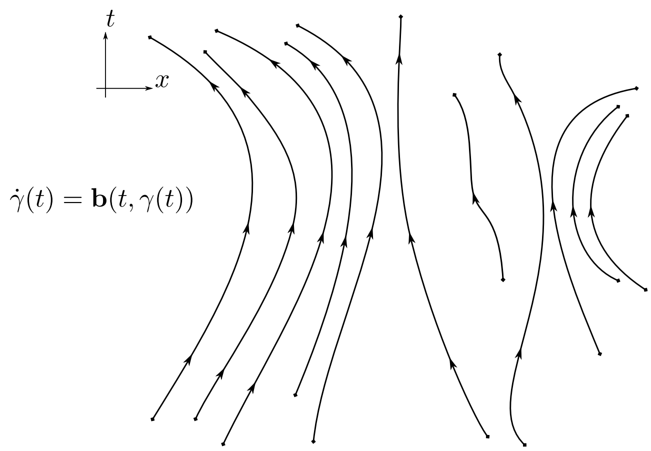
and for every $(\gamma, \gamma') \in (\mathit{\Gamma} \setminus N)^2$ it holds

$$\begin{gathered} \textit{either } \operatorname{Graph}\gamma \subset \operatorname{Graph}\gamma' \textit{ or } \operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma \\ \textit{or } \operatorname{Graph}\gamma, \operatorname{Graph}\gamma' \textit{ are disjoint (up to the end points).} \end{gathered} \tag{$\star\star$}$$

The interesting situation is when the measure ζ^C_τ can be taken arbitrarily small, i.e. when $\tau \to 0$: in that case η is said to be *untangled*, i.e. it is concentrated on a set Δ such that for every $(\gamma, \gamma') \in \Delta \times \Delta$ the condition $(\star\star)$ holds (see also Figure 2).



(A) Initial configuration: the curves ay intersect several times, overlap and bifurcate.



(B) Final configuration: after the untangling, the curves are disjoint, thus forming a partition $\{\wp_\mathfrak{a}\}_\mathfrak{a}$ of $\mathbb{R}^{d+1}$ up to a $\rho\mathcal{L}^{d+1}$-negligible set.

FIGURE 2. Visual effect of the *untangling* of trajectories: we start by removing locally a set of curves, whose η measure is controlled, in such a way that the curves are disjoint in a small ball. Iterating this step - thanks to subadditivity - we end up with a family of disjoint, untangled trajectories.

4.6. **Partition via characteristics and Lagrangian uniqueness.** The *untangling* of trajectories is the core of our approach and it encodes, in our language, the uniqueness issues and the computation of the chain rule. Indeed, once the untangled set $\varDelta$ is selected, we can construct an equivalence relation on it, identifying trajectories whenever they coincide in some time interval: this gives a partition of $\varDelta$ into equivalence classes $E_{\mathfrak{a}} := \{\wp_{\mathfrak{a}}\}_{\mathfrak{a}}$, being $\mathfrak{A}$ a suitable set of indexes. This, in turn, induces a partition of $\mathbb{R}^{d+1}$ (up to a set $\rho\mathscr{L}^{d+1}$-negligible) into disjoint trajectories (that we still denote by $\wp_{\mathfrak{a}}$): both partitions admit a Borel section (i.e. there exist Borel functions $\mathtt{f}\colon \mathbb{R}^{d+1} \to \mathfrak{A}$ and $\hat{\mathtt{f}}\colon \varDelta \to \mathfrak{A}$ such that $\wp_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})$ and $\hat{\mathtt{f}}^{-1}(\mathfrak{a}) = E_{\mathfrak{a}}$ for every $\mathfrak{a} \in \mathfrak{A}$): hence a disintegration approach can be adopted, like in the two-dimensional setting. One reduces the PDE (7) into a family of one-dimensional ODEs along the trajectories $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$: we are thus recovering a sort of method of the characteristic in the weak setting.

To formalize this disintegration issue, we propose to call a Borel map $\mathtt{g}\colon \mathbb{R}^{d+1} \to \mathfrak{A}$ a *partition via characteristics* of the vector field $\rho(1,\boldsymbol{b})$ if:

- for every $\mathfrak{a} \in \mathfrak{A}$, $\mathtt{g}^{-1}(\mathfrak{a})$ coincides with $\operatorname{Graph}\gamma_{\mathfrak{a}}$, where $\gamma_{\mathfrak{a}}\colon I_{\mathfrak{a}} \to \mathbb{R}^{d+1}$ is a characteristic of $\boldsymbol{b}$ in some open domain $I_{\mathfrak{a}} \subset \mathbb{R}$;
- if $\hat{\mathtt{g}}$ denotes the corresponding map $\hat{\mathtt{g}}\colon \varDelta \to \mathfrak{A}$, $\hat{\mathtt{g}}(\gamma) := \mathtt{g}(\operatorname{Graph}\gamma)$, setting $m := \hat{\mathtt{g}}_\sharp\eta$ and letting $w_{\mathfrak{a}}$ be the disintegration
$$\rho\,\mathscr{L}^{d+1} = \int_{\mathfrak{A}} (\mathbb{I},\gamma_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}}\mathscr{L}^1)\, m(d\mathfrak{a})$$
then
$$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}} \in \mathscr{M}(\mathbb{R}), \tag{10}$$
where $w_{\mathfrak{a}}$ is considered extended to 0 outside the domain of $\gamma_{\mathfrak{a}}$;
- it holds
$$\mu = \int (\mathbb{I},\gamma_{\mathfrak{a}})_\sharp \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \qquad \text{and} \qquad |\mu| = \int (\mathbb{I},\gamma_{\mathfrak{a}})_\sharp |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}).$$

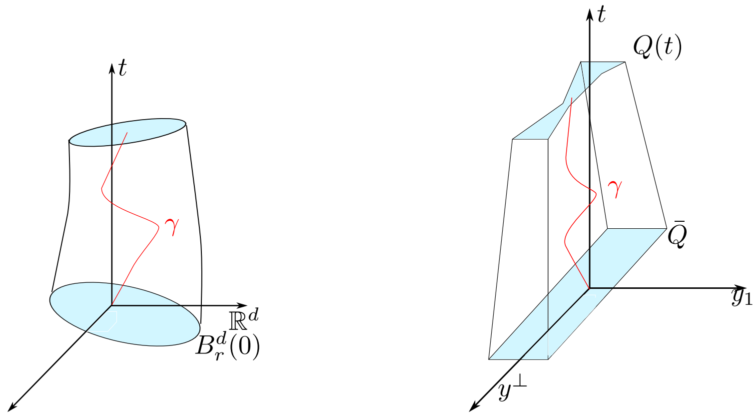
We will say the partition is *minimal* if moreover
$$\lim_{t\to\bar{t}\pm} w_{\mathfrak{a}}(t) > 0 \qquad \forall \bar{t} \in I_{\mathfrak{a}}.$$

In view of the discussion above, the family of equivalence classes $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$ arising from the untangled set $\varDelta$ constitutes a partition via characteristics. Since the function $w_{\mathfrak{a}}$ is a BV function on $\mathbb{R}$, in view of (10), we can further split the equivalence classes so that it becomes a minimal partition via characteristics of $\rho(1,\boldsymbol{b})$. Furthermore, if we take $u \in L^\infty((0,T)\times\mathbb{R}^d)$ such that $\operatorname{div}(u\rho(1,\boldsymbol{b})) = \mu'$ is a measure, we can repeat the computations for the vector field $(2\|u\|_\infty + u)\rho(1,\boldsymbol{b})$ obtaining that the same partition via characteristics works also for $u\rho(1,\boldsymbol{b})$. This yields the following uniqueness result, which is the core of our work:

Theorem 4.1 ([12])**.** *If η is untangled, then there exists a minimal partition via characteristics $\mathtt{f}$ of $\rho(1,\boldsymbol{b})$. Furthermore, if $u \in L^\infty((0,T)\times\mathbb{R}^d)$ is a solution to $\operatorname{div}(u\rho(1,\boldsymbol{b})) = \mu'$, then map $\mathtt{f}$ is a partition via characteristics of $u\rho(1,\boldsymbol{b})$ as well.*

In particular, by disintegrating the PDE $\operatorname{div}(u\rho(1,\boldsymbol{b})) = \mu'$ along the characteristics $\wp_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})$, we obtain the one-dimensional equation
$$\frac{d}{dt}\Big(u\big(t,\wp_{\mathfrak{a}}(t)\big)w_{\mathfrak{a}}(t)\Big) = \mu'_{\mathfrak{a}}.$$

At this point, an application of Volpert's formula for one-dimensional BV functions allows an explicit computation of $\frac{d}{dt}(\beta(u\circ\wp_a)w_{\mathfrak{a}})$, i.e. of $\operatorname{div}(\beta(u)\rho(1,\boldsymbol{b}))$ thus establishing the Chain rule in the general setting.



(A) In the absolutely continuous part of $D\boldsymbol{b}$ the cylinders evolve under a constant matrix A, which will be taken close to $D^a\boldsymbol{b}$.

(B) The singular case: the cylinders shrink (if $\operatorname{div}\boldsymbol{b} < 0$) in a controlled way, their sides being graph of monotone Lipschitz functions which solve suitable differential equations.

FIGURE 3. Approximate cylinders of flow in the BV (nearly incompressible) case.

4.7. **The BV nearly incompressible case and Bressan's Compactness Conjecture.** To conclude the proof of the Main Theorem, establishing Bressan's Compactness Conjecture, it remains to show how we can construct cylinders of approximate flow satisfying Assumption 4.1, for a vector field of the form $\rho(1, \boldsymbol{b})$, with $\rho \in (C^{-1}, C)$ and $\boldsymbol{b} \in L^1((0,T); \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$. In view of Theorem 4.1, without loss of generality, we can assume $\rho = 1$ so that the vector field under consideration is exactly $(1, \boldsymbol{b})$: as usual, we denote by $D\boldsymbol{b}$ the derivative of $\boldsymbol{b}$ and we split it into the absolutely continuous part and the singular part.

In a Lebesgue point $(\bar{t}, \bar{x})$ of the absolutely continuous part, the construction of the cylinders is rather easy: essentially, one replaces the real evolution under the flow of $\boldsymbol{b}$ of a ball $B_\ell^d(0)$ with an ellipsoid, obtained by letting everything evolve under a fixed matrix A (compare with Figure 3a). Some standard computations show that the difference between the two evolutions can be made arbitrarily small, when compared to the volume of $B_\ell^d(0)$, by taking A to be the Lebesgue value of $D\boldsymbol{b}$ in the point $(\bar{t}, \bar{x})$.

The estimates for the singular part are more delicate and depend heavily on the shape of the approximate cylinders of flow. Here the geometric structure of BV functions (Alberti's Rank-One Theorem [1, 20]) plays a role, as in the original proof of [4]. The main idea is to choose properly the (non-transversal) sides' lenghts of the cylinders, in such a way to cancel the effect of the divergence. Indeed, by Rank One Theorem, we can find a suitable (local) coordinate system $\boldsymbol{y} = (y_1, y^\perp) \in \mathbb{R}^d$ in which the derivative $D\boldsymbol{b}$ is essentially directed toward a fixed direction (without loss of generality, the one given by $\boldsymbol{e}_1$). Accordingly, we define the (section at time

t of the) cylinder

$$Q = Q_{\ell^{\pm}_{1,\gamma},\ell}(t) := \gamma(t) + \Big\{ \boldsymbol{y} = (y_1, y^\perp) : -\ell_1^-(t, y^\perp) \le y_1 \le \ell_1^+(t, y^\perp),\, |y^\perp| \le \ell \Big\}, \tag{11}$$

where $\ell > 0$ is a real number and $\ell^{\pm}_{1,\gamma}$ are suitable functions to be chosen, Lipschitz in $y^\perp$ and monotone in t. This is indeed a crucial step: we show it is possible to adapt locally the cylinders of approximate flows, by imposing that the sides' lengths $\ell^{\pm}_{1,\gamma}(t)$ are monotone functions satisfying suitable differential equations (see Figure 3b). In a simplified setting, i.e. if the level set of $b_1(t)$ were exactly of the form $y_1 =$ constant, then we would impose

$$\frac{d}{dt}\ell^+_{1,\gamma}(t) = (Db_1)\big(\gamma(t), \gamma(t) + \ell^+_{1,\gamma}(t)\big) \tag{12}$$

(and an analogous relation for $\ell^-_{1,\gamma}$). Plugging the solution of (12) into the definition of the cylinder (11), we can show that the flux of $\boldsymbol{b}$ through the lateral boundary of Q is under control. Actually, a technical variation of this is needed in order to take into account the fact that the level sets are not of the form $y_1 =$ constant: to do this we exploit Coarea Formula and a classical decomposition of finite perimeter sets into rectifiable parts (relying ultimately on De Giorgi's Rectifiability Theorem). We show that, up to a $|D^{\mathrm{sing}}\boldsymbol{b}|$-small set, one can find Lipschitz functions $y_1 = L_{t,h}(y^\perp)$ in a fixed set of coordinates $(y_1, y^\perp) \in \mathbb{R} \times \mathbb{R}^{d+1}$, whose graphs cover a large fraction of the singular part $D^{\mathrm{sing}}\boldsymbol{b}\llcorner_{B^{d+1}_r(\bar{t},\bar{x})}$. We can at this point reverse the procedure, i.e. we construct a vector field starting from the level sets: this yields a BV vector field $\mathcal{U}(t)$ whose component $\mathcal{U}_1$ can be put into the right hand side of (12) and we can now perform the precise estimate of the flux of $\boldsymbol{b}$ through the lateral boundary of Q.

By an application of the Radon-Nikodym Theorem, it follows that on large compact set it holds that the flow integral (9) is controlled by $\tau|D^{\mathrm{sing}}\boldsymbol{b}|(B^{d+1}_r(\bar{t},\bar{x}))$. Finally a covering argument implies that the measure ζ^C_τ can be taken, in the BV case, to be $\tau|D\boldsymbol{b}|$: in view of the discussion above this is enough to conclude finally the proof of the Main Theorem.

5. **Further developments of the *untangling*.** In a work in progress (that will appear in a forthcoming paper [13]) we study some possible refinements of the concept of *untangling*. In particular, by imposing a control on the intersection of the curves only *forward in time* some estimates and propositions of the approach presented above simplify. More precisely, we define a Lagrangian representation η of $\rho(1,\boldsymbol{b})$, with $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu \in \mathscr{M}([0,T]\times\mathbb{R}^d)$, to be *forward untangled* when it is concentrated on a set $\Delta^{\mathrm{forward}}$ of curves which may intersect, but if they do then they remain the same curve in the future. In a sense, this means that trajectories can bifurcate only in the past.

This formulation arises naturally when one translates well-posedness of the ODEs in terms of Lagrangian representations: restricting for simplicity to the case in which $\mu = 0$ one would like to replace Assumption 4.1 with the following one:

Assumption 5.1. *Let η be a Lagrangian representation of $\rho(1,\boldsymbol{b})$ in $(0,T)\times\mathbb{R}^d$. Let $\varpi > 0$ and assume that for all $R > 0$ there exists $r = r(R) > 0$ such that*

$$\int_\Gamma \frac{1}{\sigma^r(\gamma)}\eta\Big(\Big\{\gamma' \in \Gamma :\, |\gamma(0) - \gamma'(0)| \le r, |\gamma(T) - \gamma'(T)| \ge R\Big\}\Big)\eta(d\gamma) \le \varpi.$$

where now

$$\sigma^r(\gamma) := \text{amount of curves starting from the ball or radius } r > 0 \text{ around } \gamma(0).$$

Assumption 5.1 has the advantage of making more transparent and easier some of the proofs used in the approach presented above. One can repeat the general scheme presented above: first one formulates Assumption 5.1 locally, in a proper set and shows that - up to a set of curves whose measure is controlled - the (restricted) Lagrangian representation η is forward untangled. In this way, one obtains a simpler proof of Theorem 1, avoiding the introduction of the crossing trajectories. Then one introduces the *forward untangling functional*, which turns out to be subadditive as well, exactly as in the setting above, allowing the usual local-to-globalargument. Using this formulation of the untangling, we are able to recover in our setting the results of [16], where the authors considered vector fields whose derivative can be written as convolution between a singular kernel and a L^1 function and we also derive a quantitative stability estimate for a class of vector fields satifying a suitable weak L^p bound on the gradient.

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Received xxxx 20xx; revised xxxx 20xx.

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