BMO-TYPE SEMINORMS FROM ESCHER-TYPE TESSELLATIONS

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Abstract. The paper is about a representation formula introduced by Fusco, Moscariello, and Sbordone in [12]. The formula permits to characterize the gradient norm of a Sobolev function, defined on the whole space $\mathbb{R}^n$, as the limit of non-local energies (BMO-type seminorms) defined on tessellations of $\mathbb{R}^n$ generated by cubic cells. We extend the main result in [12] in two different regards: we analyze the case of a generic open subset $\Omega \subseteq \mathbb{R}^n$ and consider tessellations of Ω inspired by the creative mind of the graphic artist M.C. Escher.

Keywords: BMO-type spaces, Sobolev spaces, tessellations, tilings, cells, M.C. Escher

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1. Introduction

In a lecture given at the Université Paris VI in 2001, entitled *Sobolev spaces revisited*, Professor Haïm Brezis communicated that while working on the limiting behavior of the norms of fractional Sobolev spaces [2], a new characterization of the classical Sobolev spaces $W^{1,p}(\Omega)$ came out: if $f \in L^p(\Omega)$, $1 < p < \infty$, and Ω is a smooth bounded domain in $\mathbb{R}^n$, then, with the understanding that $\|\nabla f\|^p_{L^p(\Omega)} = +\infty$ if $f \notin W^{1,p}(\Omega)$,

$$\lim_{m\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_m(x-y)\, \mathrm{d}x\, \mathrm{d}y = K_{p,n} \|\nabla f\|^p_{L^p(\Omega)} \,. \tag{1}$$

Here, $K_{p,n}$ is a positive constant depending only on p and n, and $(\rho_m)_{m\in\mathbb{N}}$ denotes a sequence of radial mollifiers whose masses tend to concentrate around the origin.

Over the years, the results in [2] induced several researchers to look for variants of (1) that could lead to analogous characterizations of other Sobolev-type spaces. Here, we mention [5,15–17] as general references, [19] for magnetic Sobolev spaces, and [7,10] for the validity of (1) in the *variable* Sobolev space setting where the second author obtained a weak (rougher) form of (1). The interest in this kind of representation formulas is twofold: on the one hand, they explain p-Dirichlet energies as short-range limits of non-local energies; on the other hand [4, Remark 6], since they do not involve the concept of weak derivative, they suggest a definition of Sobolev spaces in the more general setting of metric measure spaces (see, e.g., [13] and the bibliography therein).

Several years later, in [3] the authors introduced the function space $B_0(\Omega)$, defined through a generalization of the classical BMO seminorm. When $p > 1$, the space $B_0(\Omega)$ includes $\mathrm{VMO}(\Omega) + \mathrm{BMO}(\Omega) + W^{1/p,p}(\Omega)$ and provides a common ground for certain regularity results that can now be unified under the statement: any integer-valued function belonging to $B_0(\Omega)$ is necessarily a constant function. Some of the ideas contained in [3] have been later on extended in [1] to give a new characterization of sets of finite perimeter.

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Recently, in [12], in line with the ideas in [3], the authors introduced a new BMO-type seminorm. Given a function $f \in L^p_{\mathrm{loc}}(\Omega)$ and any $\varepsilon > 0$, they define

$$\kappa_p^\varepsilon[f] \quad := \quad \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\mathbb{R}^n)} \sum_{Q_\varepsilon \in \mathcal{G}_\varepsilon} \varepsilon^{n-p} \fint_{Q_\varepsilon} \left| f(x) - \fint_{Q_\varepsilon} f(y)\,\mathrm{d}y \right|^p \mathrm{d}x\,, \tag{2}$$

where $\mathfrak{S}_\varepsilon(\mathbb{R}^n)$ denotes the set of all families of disjoint open cubes $Q_\varepsilon \subseteq \mathbb{R}^n$ of side-length ε and arbitrary orientation, and show that, as in [2], it is possible to give a representation formula for the gradient norm of a Sobolev function that makes no use of distributional derivatives. Precisely, if $1 < p < \infty$ and $\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f] < \infty$, then $|\nabla f| \in L^p(\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \kappa_p^\varepsilon[f] = C_{p,n} \|\nabla f\|_{L^p(\mathbb{R}^n)}\,, \tag{3}$$

where $C_{p,n}$ is a positive constant depending only on p and n. Finally, in [8], a similar derivative-free representation formula is obtained for the total variation of SBV functions.

1.1. Contributions of present work. In this paper, moving beyond [12], we give at the same time a new proof and an extension of the representation formula (3). In particular, we extend the main result in [12] to the case of a generic open set $\Omega \subseteq \mathbb{R}^n$. We show that if $\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] < \infty$ then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and we provide an estimate which controls, for every open set U strictly contained in Ω, the blow-up of the norm $\|\nabla f\|^p_{L^p(U)}$ in terms of the distance of U from $\partial\Omega$. For $\Omega = \mathbb{R}^n$, we recover the main result in [12].

In extending the results, we provide a concise proof which emphasizes the role played by each assumption. In particular, our proof highlights how tessellations by open cubes play no special role in the analysis. Indeed, our result applies to a broader class of tessellations: from pentagonal and hexagonal tilings to space-filling polyhedrons and creative tessellations inspired by the artistic genius of M.C. Escher (Figure 1).

1.2. Outline. The paper is organized as follows. In Section 2 we fix the notation, state the main result (Theorem 1), and prove some preliminary lemmas; the proof of the main result is then given in Sections 3 and 5. In Section 4, we present again the proof given in Section 3 but in a very compact form that highlights the main idea behind our argument; its correctness, a fortiori, is justified by the results of Section 3.

2. Statement of the main result

The intuitive idea of a tessellation of $\mathbb{R}^n$ is that of a collection of objects, called *tiles* or *cells*, that cover the whole space without gaps or overlaps. To formalize the idea, we need to set up the notation and terminology.

2.1. The concept of tessellation. We denote by $E(n)$ the Euclidean group of $\mathbb{R}^n$, i.e., the group of all distance-preserving maps of $\mathbb{R}^n$ onto $\mathbb{R}^n$, and by $O(n) := \{R \in \mathbb{R}^{n\times n} : RR^\top = I\}$ the orthogonal group of $\mathbb{R}^n$, i.e., the subgroup of $E(n)$ that leaves the origin $0 \in \mathbb{R}^n$ fixed. We recall that for every $h \in E(n)$ there exists a unique $(z_h, R_h) \in \mathbb{R}^n \times O(n)$ such that $h(x) = z_h + R_h x$ for every $x \in \mathbb{R}^n$. Given a set Ω, we denote by $\wp(\Omega)$ the family of all subsets of Ω. For Lebesgue measurable sets $A \subseteq \mathbb{R}^n$, we denote by $|A|$ the Lebesgue measure of A and by $\langle f\rangle_A$ the average of f on A. We give the following definition.

Definition 1. *Let $\varepsilon > 0$. An ε-tessellation of $\mathbb{R}^n$ is a set $\mathcal{T}_\varepsilon(\mathbb{R}^n) \subseteq \wp(\mathbb{R}^n)$, whose elements are called cells, satisfying the following properties* (cf. Figure 1)*:*

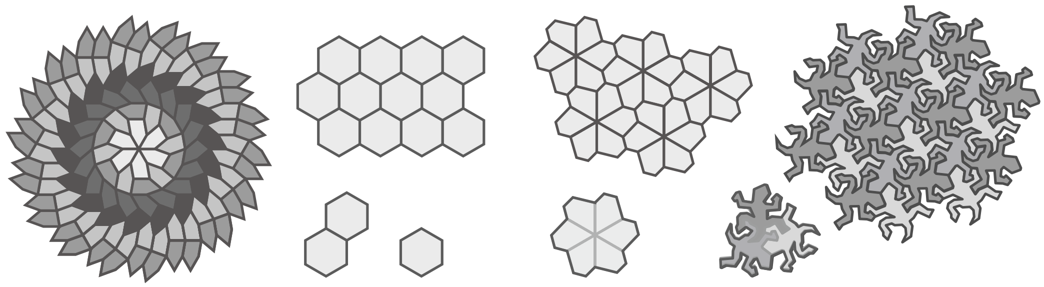

FIGURE 1. Pentagonal and hexagonal tessellations of the plane. On the right, a tessellation inspired by the lithograph *Reptiles* of M.C. Escher. Note that, in general, the choice of a reference cell is not unique. The leftmost tessellation is not rigid. Instead, the rightmost one is rigid because one can take the union three reptiles as a reference cell.

i. (*non-overlapping condition*) *The set* $\mathcal{T}_\varepsilon(\mathbb{R}^n)$ *is a* countable *family of* disjoint *open subsets of* $\mathbb{R}^n$.

ii. (*no-gaps condition*) *The family* $\mathcal{T}_\varepsilon(\mathbb{R}^n)$ *covers the space in a measure-theoretic sense:*

$$|\mathbb{R}^n \setminus \cup\{T_\varepsilon : T_\varepsilon \in \mathcal{T}_\varepsilon(\mathbb{R}^n)\}| = 0.$$

iii. (*homothetic condition*) *There exists an open* reference cell $\hat{Q} \subseteq \mathbb{R}^n$, *i.e., an open set such that*

$$\int_{\hat{Q}} y \,\mathrm{d}y = 0, \qquad \operatorname{diam}(\hat{Q}) = 1, \tag{4}$$

so that for every $T_\varepsilon \in \mathcal{T}_\varepsilon(\mathbb{R}^n)$, $T_\varepsilon = h(\varepsilon\hat{Q})$ *for some* $h \in E(n)$.

We say that the ε*-tessellation* $\mathcal{T}_\varepsilon(\mathbb{R}^n)$ *is* rigid *whenever all the cells in* $\mathcal{T}_\varepsilon(\mathbb{R}^n)$ *have the same orientation, i.e., for every* $T_\varepsilon \in \mathcal{T}_\varepsilon(\mathbb{R}^n)$, $T_\varepsilon = z \pm \varepsilon\hat{Q}$ *for some* $z \in \mathbb{R}^n$.

Clearly, $\operatorname{diam}(T_\varepsilon) = \varepsilon$ *for every* $T_\varepsilon \in \mathcal{T}_\varepsilon(\mathbb{R}^n)$ *and, therefore, we will refer to* ε *as the* resolution *of the tessellation.*

REMARK 2.1. The condition (4) imposed on the reference cell means that the origin of $\mathbb{R}^n$ is the geometric center (centroid) of $\hat{Q}$. Such a condition is not essential, but it is convenient: indeed, in this way, if one denotes by $T_\varepsilon(z)$ the generic cell of the form $z+R(\hat{Q}_\varepsilon)$ with $R \in O(n)$ and $\hat{Q}_\varepsilon := \varepsilon\hat{Q}$, then z is the geometric center of $T_\varepsilon(z)$ and

$$\int_{T_\varepsilon(z)} (y-z)\,\mathrm{d}y = 0\,. \tag{5}$$

The definition of ε-tessellation, as given in Definition 1, is intuitive and sounds familiar. Indeed, it is based on the natural way one thinks about space tessellations whose reference cell is a cube, or, in the plane, tessellations whose reference cell is, for example, an equilateral triangle. The notion is common in recreational mathematics where it is well-known that the multitude of possible plane tessellations is limited only by one's own imagination (see, e.g., [6, p. 40]). The concept of tessellation is also well known in art, and in particular in the production of M.C. Escher, whose artworks have been the primary source of inspiration for our refined notion of tessellation as introduced below (see, e.g., [18, p. 14 III.19, p. 17 III.24a, p. 21 III.35, p. 26 III.48, p. 30 III.52]).

However, the definition of ε-tessellation is unsuited to tile a generic open subset $\Omega \subseteq \mathbb{R}^n$. Indeed, when $\partial\Omega \neq \emptyset$, the presence of the boundary may create obstructions to the fulfillment of the no-gaps condition by cells with the same diameter. The question can be overcome in two different ways: either one allows for countable unions of cells with different diameters (as in the *Smaller and smaller* artwork by M.C. Escher) or, as we shall do, one considers *families* of non-overlapping cells, sharing the same shape and whose resolution shrinks to zero. This leads to the following generalization.

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A family $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$, $I = (0,1)$, with $\mathcal{Q}_\varepsilon(\Omega) \subseteq \wp(\Omega)$ for any $\varepsilon \in I$, is called a* regular tessellation family *for Ω, if the following properties are satisfied:*

i. (*non-overlapping condition*) *For any $\varepsilon \in I$ the set $\mathcal{Q}_\varepsilon(\Omega)$, is a* countable *family of* pairwise disjoint *open subsets of Ω, called cells.*

ii. (*no-gaps condition*)
$$\liminf_{\varepsilon\to 0} \chi_{\mathcal{Q}^{\cup}_\varepsilon(\Omega)} = \chi_\Omega. \tag{6}$$
Here, for $A \subseteq \Omega$, χ_A denotes the characteristic function of A, $\mathcal{Q}^{\cup}_\varepsilon(\Omega) \subseteq \Omega$ denotes the union of all the disjoint open sets of the family $\mathcal{Q}_\varepsilon(\Omega)$, and the $\liminf$ *has to be intended a.e. in Ω.*

iii. (*uniform homothetic condition*) *There exists a* reference open cell $\hat{Q} \subseteq \Omega$, *i.e., an open set such that*
$$\int_{\hat{Q}} y \, \mathrm{d}y = 0, \qquad \operatorname{diam}(\hat{Q}) = 1,$$
so that for every $\varepsilon \in I$, $Q_\varepsilon \in \mathcal{Q}_\varepsilon(\Omega)$, one has $Q_\varepsilon = h(\varepsilon\hat{Q})$ for some $h \in E(n)$.

We say that the regular tessellation family $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$ is rigid *whenever all the cells have the same orientation, i.e., for every $\varepsilon \in I$, $Q_\varepsilon \in \mathcal{Q}_\varepsilon(\Omega)$, one has $Q_\varepsilon = z \pm (\varepsilon\hat{Q})$ for some $z \in \mathbb{R}^n$.*

Clearly, $\operatorname{diam}(Q_\varepsilon) = \varepsilon$ *for every $Q_\varepsilon \in \mathcal{Q}_\varepsilon(\Omega)$ and, therefore, we will refer to $\mathcal{Q}_\varepsilon(\Omega)$ as the element of the family at* resolution ε, *and to $\hat{Q}$ as an open reference cell for the regular tessellation family.*

Remark 2.2. In giving our definition of regular tessellation family, we implicitly assumed Ω sufficiently large so that $h(\hat{Q}) \subset \Omega$ for some $h \in E(n)$. This assumption has been made only for the sake of clarity; indeed, one can always replace I with a smaller interval.

Notation 2.1. For any $\varepsilon > 0$ we set $\hat{Q}_\varepsilon := \varepsilon\hat{Q}$. We denote by $Q_\varepsilon(z)$ a generic open cell of the form $z + R(\hat{Q}_\varepsilon)$, for some $R \in O(n)$. In particular, denote by Q_ε a cell of the type $Q_\varepsilon(0) = R(\hat{Q}_\varepsilon)$ with $R \in O(n)$. Finally, we say that two cells $Q_\varepsilon(z'), Q_\varepsilon(z'')$ have the *same orientation* if $Q_\varepsilon(z'') = (z'' - z') + Q_\varepsilon(z')$ or $Q_\varepsilon(z'') = (z'' - z') - Q_\varepsilon(z')$.

Remark 2.3. Note that, due to the non-overlapping condition, the no-gaps condition (6) can be equivalently stated as
$$\liminf_{\varepsilon\to 0} \left(\sum_{Q_\varepsilon(z)\in\mathcal{Q}_\varepsilon(\Omega)} \chi_{Q_\varepsilon(z)} \right) = \chi_\Omega \, .$$

Regular tessellation families do exist: indeed, any open set $\Omega \subseteq \mathbb{R}^n$ admits a regular tessellation family having the open unit cube as a reference cell [12, (2.13)]. We note that the key property of the cube which allows for the existence of regular tessellation families for any open subset of $\mathbb{R}^n$, is that it is a reference cell for some ε-tessellation of $\mathbb{R}^n$. This can be formalized as follows:

Proposition 1. *Let $\varepsilon_0 > 0$. Any ε_0-tessellation $\mathcal{T}_{\varepsilon_0}(\mathbb{R}^n)$ of $\mathbb{R}^n$ induces a regular tessellation family $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$ for every open subset $\Omega \subseteq \mathbb{R}^n$. If $\mathcal{T}_{\varepsilon_0}(\mathbb{R}^n)$ is rigid, so is $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$.*

Proof. Let $\varepsilon_0 > 0$, and let $\hat{Q}$ be a reference cell for an ε_0-tessellation $\mathcal{T}_{\varepsilon_0}(\mathbb{R}^n)$ of $\mathbb{R}^n$. Note that, for any $\varepsilon > 0$, the set $\hat{Q}$ is also a reference cell for a ε-tessellation of $\mathbb{R}^n$. In fact, the family

$$\mathcal{T}_\varepsilon(\mathbb{R}^n) := \{W(Q_{\varepsilon_0})\}_{Q_{\varepsilon_0}\in\mathcal{T}_{\varepsilon_0}(\mathbb{R}^n)}\,,$$

with $W : x \in \mathbb{R}^n \mapsto (\varepsilon/\varepsilon_0)x \in \mathbb{R}^n$, is a ε-tessellation of $\mathbb{R}^n$. Next, we set

$$\Omega_\varepsilon := \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Clearly, we have $\lim_{\varepsilon\to 0} \chi_{\Omega_\varepsilon} = \chi_\Omega$ in Ω. Also, the family $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$ with $\mathcal{Q}_\varepsilon(\Omega)$ given by the elements of $\mathcal{T}_\varepsilon(\mathbb{R}^n)$ having nonempty intersection with Ω_ε, satisfies the relation

$$\chi_{\Omega_\varepsilon} \leqslant \chi_{\mathcal{Q}_\varepsilon^\cup(\Omega)} \leqslant \chi_\Omega \qquad \text{in } \Omega\,.$$

Hence, $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$ is a regular tessellation family for Ω. Finally, by construction, it is clear that if $\mathcal{T}_{\varepsilon_0}(\mathbb{R}^n)$ is rigid, so is $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$. □

REMARK 2.4. Proposition 1 shows that the *rigidity* of a tessellation is a hereditary property: if a tessellation has cells with the same orientation, also the induced regular tessellation family will consist of cells sharing the same orientation. As an example, in the *Pessimist-Optimist* (No. 63) artwork by M.C. Escher, the union of one optimist and one pessimist (not necessarily adjacent, but thought scaled so that its diameter is 1) can be considered as a reference cell for a tessellation of $\mathbb{R}^n$. Note that there are both optimists looking at the right, and optimists looking at the left: they correspond to cells that, although reflected, have (by definition) the same orientation. A similar conclusion holds, for instance, for the 1946 India ink, colored pencil, watercolor *Horseman* (No.67).

The following simple observation motivates our notion of a regular tessellation family.

Lemma 1. *If $(\mathcal{Q}_\varepsilon(\Omega))_{\varepsilon\in I}$ is a regular tessellation family for Ω, then, for any measurable function $f : \Omega \to \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^\cup(\Omega)} |f(x)|\mathrm{d}x \;=\; \int_\Omega |f(x)|\mathrm{d}x.$$

In particular, $f \in L^1(\Omega)$ if and only if

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^\cup(\Omega)} |f(x)|\mathrm{d}x < \infty.$$

Proof. Assume that the no-gaps condition (6) holds. By Fatou's lemma

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^\cup(\Omega)} |f(x)|\mathrm{d}x \geqslant \int_\Omega \liminf_{\varepsilon\to 0} \chi_{\mathcal{Q}_\varepsilon^\cup(\Omega)}(x)|f(x)|\mathrm{d}x = \int_\Omega |f(x)|\mathrm{d}x.$$

On the other hand, since $\mathcal{Q}_\varepsilon^\cup(\Omega) \subseteq \Omega$, we have

$$\int_\Omega |f(x)|\mathrm{d}x \geqslant \liminf_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^\cup(\Omega)} |f(x)|\mathrm{d}x.$$

This completes the proof. □

2.2. Statement of the main result. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $\hat{Q} \subset \mathbb{R}^n$ be an open reference cell for a *rigid* tessellation of $\mathbb{R}^n$. For any $\varepsilon > 0$ we denote by $\mathfrak{S}_\varepsilon(\Omega) = \mathfrak{S}_\varepsilon^{\hat{Q}}(\Omega) = \{\mathcal{G}_\varepsilon\}$ the set of all families $\mathcal{G}_\varepsilon$ consisting of (necessarily countable) pairwise disjoint open cells of the type $h(\hat{Q}_\varepsilon)$ for some $h \in E(n)$. Given a function $f \in L^p_{\mathrm{loc}}(\Omega)$, $p > 1$, we are interested in the limiting behavior, as $\varepsilon \to 0$, of the following family of seminorms:

$$\kappa_p^\varepsilon[f,\Omega] \;:=\; \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} \left| f(x) - \fint_{Q_\varepsilon(z)} f(y)\,\mathrm{d}y \right|^p \mathrm{d}x. \tag{7}$$

We stress that even if in literature the expression $\kappa_p^\varepsilon[f,\Omega]$ is often referred to as a *seminorm*, in fact it is not, because it is not homogeneous of degree 1. It would turn into a seminorm if the outer integral would be raised to the power $1/p$. However, the study of such expression is not of interest for this paper.

Remark 2.5. Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be nonempty open sets. If $\Omega_1 \subseteq \Omega_2$ then $\mathfrak{S}_\varepsilon^{\hat{Q}}(\Omega_1) \subseteq \mathfrak{S}_\varepsilon^{\hat{Q}}(\Omega_2)$. Therefore, as an immediate consequence of the definition of $\kappa_p^\varepsilon[f,\cdot]$, we get that

$$\kappa_p^\varepsilon[f,\Omega_1] \leqslant \kappa_p^\varepsilon[f,\Omega_2]. \tag{8}$$

Moreover, it is clear that if $(\mathcal{Q}_\varepsilon(\Omega_1))_{\varepsilon\in I}$ is a regular tessellation family (see Definition 2) for Ω_1 having $\hat{Q}$ as a reference cell, then $\mathcal{Q}_\varepsilon(\Omega_1) \in \mathfrak{S}_\varepsilon^{\hat{Q}}(\Omega_2)$ for every $\varepsilon \in I$.

Our main result is stated in the next Theorem 1. Note that, in particular, when $\Omega := \mathbb{R}^n$ and $\hat{Q}$ is the open unit cube, we recover [12, Theorem 2.2] in the case $p > 1$ (see next Remark 2.6 for details).

Theorem 1. *For $f \in L^p_{\mathrm{loc}}(\Omega)$, the following assertions hold:*

i. If $\nabla f \in L^p(\Omega)$ then

$$\lim_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] = \gamma \int_\Omega |\nabla f(x)|^p \,\mathrm{d}x \tag{9}$$

with $\gamma := \max_{\sigma\in\mathbb{S}^{n-1}} \int_{\hat{Q}} |\sigma\cdot x|^p \mathrm{d}x.$

ii. (limit inferior condition) If

$$\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] < \infty,$$

then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Moreover, for every open set U compactly contained in Ω (i.e., $\emptyset \neq \bar{U} \subseteq \Omega$) and for every $\tau > 0$, the following estimate holds

$$\lim_{\varepsilon\to 0} \kappa_p^\varepsilon[f,U] = \gamma \|\nabla f\|^p_{L^p(U)} \leqslant \tau_1 \liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] + \tau_2 \frac{c_\Omega^p}{d(U,\partial\Omega)^p} \|f\|^p_{L^p(U)}, \tag{10}$$

where $\tau_1 := (1+\tau)^{p-1}$ and $\tau_2 := \tau_1/\tau^{p-1}$.

Remark 2.6. Let $p > 1$. When $\Omega = \mathbb{R}^n$ one has $\partial\Omega = \emptyset$ and $d(U,\partial\Omega)^{-p} = 0$ for any open set U compactly contained in $\mathbb{R}^n$. Therefore, if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\mathbb{R}^n] < \infty$, from (10) one gets that $|\nabla f| \in L^p(\mathbb{R}^n)$. In particular, if $\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\mathbb{R}^n] < \infty$ and $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ then, by *i.*, we get that $|\nabla f| \in L^p(\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\mathbb{R}^n] = \gamma \int_{\mathbb{R}^n} |\nabla f(x)|^p \,\mathrm{d}x\,. \tag{11}$$

On the other hand, if $\liminf_{\varepsilon\to 0}\ \kappa_p^\varepsilon[f,\mathbb{R}^n]=\infty$, then $|\nabla f|\notin L^p(\mathbb{R}^n)$: in fact, on the contrary, by $i.$ we would get a contradiction. As a conclusion, (11) holds in any case.

Remark 2.7. Note that γ depends only on n, p, and the *shape* of $\hat{Q}$: it does not depend on the *orientation* of $\hat{Q}$, i.e.,

$$\gamma \equiv \max_{\sigma\in\mathbb{S}^{n-1}} \int_{R(\hat{Q})} |\sigma\cdot x|^p \mathrm{d}x \quad \forall R\in O(n). \tag{12}$$

Indeed, for $R\in O(n)$ let $\sigma_{R(\hat{Q})}\in \operatorname{argmax}_{\sigma\in\mathbb{S}^{n-1}}\int_{R(\hat{Q})}|\sigma\cdot x|^p\,\mathrm{d}x$. We have

$$\begin{aligned}\gamma &= \int_{\hat{Q}} |\sigma_{\hat{Q}}\cdot x|^p \mathrm{d}x = \int_{R(\hat{Q})} |\sigma_{\hat{Q}}\cdot R^\top x|^p\,\mathrm{d}x = \int_{R(\hat{Q})} |R(\sigma_{\hat{Q}})\cdot x|^p\,\mathrm{d}x\\ &\leqslant \max_{\sigma\in\mathbb{S}^{n-1}}\int_{R(\hat{Q})}|\sigma\cdot x|^p\,\mathrm{d}x = \int_{R(\hat{Q})}|\sigma_{R(\hat{Q})}\cdot x|^p\,\mathrm{d}x = \int_{\hat{Q}} |R^\top\sigma_{R(\hat{Q})}\cdot x|^p\,\mathrm{d}x \leqslant \gamma.\end{aligned}$$

The proof of Theorem 1 is the object of the next section. One of the main ingredients is the following simple observation:

Lemma 2. *Let $a,b\in\mathbb{R}$. For every $\tau>0$ and every $p\geqslant 1$ the following inequalities hold:*

$$-\frac{1}{\tau^{p-1}}|b|^p+\frac{1}{(1+\tau)^{p-1}}|a|^p \ \leqslant\ |a+b|^p \ \leqslant\ (1+\tau)^{p-1}|a|^p+\left(\frac{1+\tau}{\tau}\right)^{p-1}|b|^p. \tag{13}$$

Setting $\tau_1:=(1+\tau)^{p-1}$, $\tau_3:=\tau^{p-1}$, and $\tau_2:=\tau_1/\tau_3$, (13) can be written

$$-\frac{1}{\tau_3}|b|^p+\frac{1}{\tau_1}|a|^p \ \leqslant\ |a+b|^p \ \leqslant\ \tau_1|a|^p+\tau_2|b|^p \tag{14}$$

with τ_1,τ_2,τ_3 depending on p and τ. Note that $\tau_1\to 1$ for $\tau\to 0^+$.

Proof. The assertion follows from the convexity of the function $t\in\mathbb{R}\to|t|^p$. Indeed, we can write

$$\begin{aligned}|a+b|^p &= \left|\frac{1}{1+\tau}((1+\tau)a)+\frac{\tau}{1+\tau}\left(\frac{1+\tau}{\tau}b\right)\right|^p\\ &\leqslant (1+\tau)^{p-1}|a|^p+\left(\frac{1+\tau}{\tau}\right)^{p-1}|b|^p\end{aligned}$$

After that, we have $|a|^p=|(a+b)-b|^p\leqslant(1+\tau)^{p-1}|a+b|^p+((1+\tau)/\tau)^{p-1}|b|^p$ from which also the left-hand side of (13) follows. □

We close this section with the following instrumental

Lemma 3. *Let $\mathcal{M}$ be a compact Lipschitz hypersurface of $\mathbb{R}^n$, $G\subseteq\mathbb{R}^n$ a set with finite measure, and $g: G\subseteq\mathbb{R}^n\to\mathcal{M}$ a measurable map. For any $\eta>0$ there exists a finite family $(\Gamma_1,\dots,\Gamma_{k(\eta)})$ of pairwise disjoint open subsets in $\mathcal{M}$, with $k(\eta)$ depending on η, such that*

$$\operatorname{diam}(\Gamma_j)<\eta \quad \forall j\in\mathbb{N}_{k(\eta)}:=\{1,\dots,k(\eta)\}\subseteq\mathbb{N}$$

and

$$|\{x\in G: g(x)\in\mathcal{M}_0\}|=0,\quad \mathcal{M}_0:=\mathcal{M}\setminus\cup_{j\in\mathbb{N}_{k(\eta)}}\Gamma_j.$$

Remark 2.8. The Lemma can be generalized to the setting of measurable functions defined on a measure space. The reader may compare its proof with the argument in the proof of Proposition 3.6 in [11].

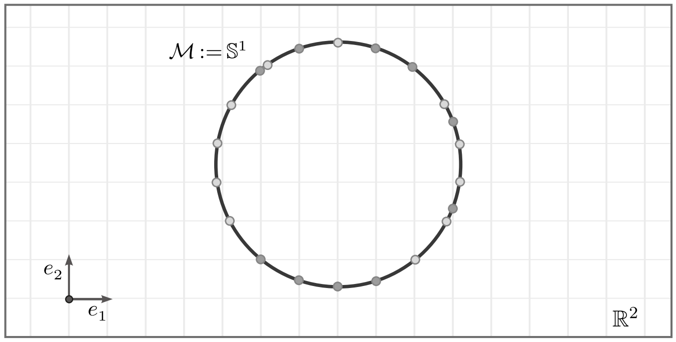

FIGURE 2. A schematic representation of the geometric idea behind the proof of Lemma 3.

Proof. We prove the result for compact Lipschitz hypersurface of $\mathbb{R}^2$, but the argument readily generalizes to higher dimensions.

We consider an orthogonal basis (e_1, e_2) of $\mathbb{R}^2$ with $|e_i| = \eta/\sqrt{2}$. We denote by E_1 the subset of $\mathbb{R}^2$ consisting of the union of all lines perpendicular to e_1 and passing through the points of the form me_1 for some $m \in \mathbb{Z}$. The family $\{(te_1 + E_1) \cap \mathcal{M}\}_{t\in[0,1)}$ forms a partition of $\mathcal{M}$. We then consider, for any $t \in [0, 1)$ the set

$$C_1(t) := \{x \in G : g(x) \in (te_1 + E_1) \cap \mathcal{M}\}\,.$$

We note that the set of $t \in [0, 1)$ such that $|C_1(t)| > 0$ can be at most countable because $|G| < +\infty$. Therefore, there exists $t_1 \in [0, 1)$ such that $|C_1(t_1)| = 0$. Repeating the argument for the subset E_2 of $\mathbb{R}^2$ consisting of the union of all lines perpendicular to e_2 and passing through points of the form me_2 for some $m \in \mathbb{Z}$, we get, for the set

$$C_2(t) := \{x \in G : g(x) \in (te_2 + E_2) \cap \mathcal{M}\}\,,$$

the existence of a $t_2 \in [0, 1)$ such that $|C_2(t_2)| = 0$.

Therefore, the grid $(t_1e_1 + E_1) \cup (t_2e_2 + E_2)$ determines a tessellation $\mathcal{T}_\eta = \{T_\eta\}$ in $\mathbb{R}^2$ consisting of open cells of diameter less than η. The finite family $\{T_\eta \cap \mathcal{M} : T_\eta \in \mathcal{T}_\eta,\ T_\eta \cap \mathcal{M} \neq \emptyset\}$ induces the desired decomposition of $\mathcal{M}$. $\square$

3. PROOF OF THEOREM 1.*i*

We split the proof of Theorem 1 in four steps. In what follows, to shorten notation, we denote by $\langle f\rangle_{Q_\varepsilon(z)}$ the average of f on the cell $Q_\varepsilon(z)$.

Step 1 (density argument). We show that it is sufficient to prove (9) for every $f \in C^2_c(\mathbb{R}^n)$. Indeed, assume that (9) holds for any function in $C^2_c(\mathbb{R}^n)$, and consider $f \in L^p_{\mathrm{loc}}(\Omega)$ such that $\nabla f \in L^p(\Omega)$. We denote by $(\Omega_m)_{m\in\mathbb{N}}$ an exhaustion of Ω by compact and smooth domains (see, e.g., [20, Lemma 1]). By our assumption on f, we have $f \in W^{1,p}(\Omega_m)$. Therefore, for each $m \in \mathbb{N}$ there exists a sequence of functions $f^{(m)}_k \in C^2_c(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} f^{(m)}_k = f \quad \text{in } W^{1,p}(\Omega_m), \qquad \|\nabla g^{(m)}_k\|^p_{L^p(\Omega_m)} < \frac{1}{k} \qquad \forall m \in \mathbb{N}, \tag{15}$$

with $g_k^{(m)} := f_k^{(m)} - f$. For any fixed $m \in \mathbb{N}$, by Lemma 2 and Poincaré inequality, we have

$$\begin{aligned}
\kappa_p^\varepsilon[f,\Omega] &\geqslant \kappa_p^\varepsilon[f_k^{(m)} - g_k^{(m)}, \Omega_m] \\
&\geqslant \frac{1}{\tau_1} \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega_m)} \frac{1}{\varepsilon^p} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q_\varepsilon(z)} |f_k^{(m)}(x) - \langle f_k^{(m)}\rangle_{Q_\varepsilon(z)}|^p \,\mathrm{d}x \\
&\qquad - \frac{1}{\tau_3} \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega_m)} \frac{1}{\varepsilon^p} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q_\varepsilon(z)} |g_k^{(m)}(x) - \langle g_k^{(m)}\rangle_{Q_\varepsilon(z)}|^p \,\mathrm{d}x \\
&\geqslant \frac{1}{\tau_1} \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega_m)} \frac{1}{\varepsilon^p} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q_\varepsilon(z)} |f_k^{(m)}(x) - \langle f_k^{(m)}\rangle_{Q_\varepsilon(z)}|^p \mathrm{d}x \\
&\qquad - \frac{c_P}{\tau_3} \int_\Omega |\nabla g_k^{(m)}(x)|^p \,\mathrm{d}x,
\end{aligned}$$

where τ_1, τ_3 are the constants of Lemma 2 and c_P the Poincaré constant related to $\hat{Q}$. Because of (9), taking the limit inferior of both sides as $\varepsilon \to 0$, we get,

$$\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] \geqslant \frac{\gamma}{\tau_1} \int_{\Omega_m} |\nabla f_k^{(m)}(x)|^p \mathrm{d}x - \frac{c_P}{\tau_3} \|\nabla g_k^{(m)}\|^p_{L^p(\Omega_m)}.$$

Next, by *first* taking the limit for $k \to \infty$ and *then* the limit for $\tau \to 0^+$, we infer that

$$\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] \geqslant \gamma \int_{\Omega_m} |\nabla f(x)|^p \,\mathrm{d}x\,.$$

Eventually, taking the limit for $m \to \infty$ we conclude that:

$$\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] \geqslant \gamma \int_{\Omega} |\nabla f(x)|^p \,\mathrm{d}x. \tag{16}$$

It remains to prove that

$$\limsup_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] \leqslant \gamma \int_{\Omega} |\nabla f(x)|^p \,\mathrm{d}x. \tag{17}$$

Now, let $0 < \varepsilon < 1$. For every $\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega)$ and every $m \in \mathbb{N}$, we set

$$\mathcal{G}_\varepsilon^{(m)} := \{Q_\varepsilon(z) \in \mathcal{G}_\varepsilon \,:\, Q_\varepsilon(z) \subseteq \Omega_m\}\,.$$

Obviously $\mathcal{G}_\varepsilon^{(m)} \in \mathfrak{S}_\varepsilon(\Omega_m)$. Also, with no loss of generality, we can assume that $\Omega_m = \Omega_m \cap \{|x| \leqslant m\}$ for every $m \in \mathbb{N}$. In this way, since $0 < \varepsilon < 1$, we have

$$\begin{aligned}
\kappa_p^\varepsilon[f,\Omega] &= \sup_{\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega)} \left(\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon^{(m)}} \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} \left| f(x) - \fint_{Q_\varepsilon(z)} f(y)\mathrm{d}y \right|^p \mathrm{d}x \right. \\
&\qquad\qquad \left. + \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon\setminus\mathcal{G}_\varepsilon^{(m)}} \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} \left| f(x) - \fint_{Q_\varepsilon(z)} f(y)\mathrm{d}y \right|^p \mathrm{d}x \right) \\
&\leqslant \kappa_p^\varepsilon[f,\Omega_m] + c_P \int_{\Omega\setminus\Omega_{m-1}} |\nabla f(x)|^p \mathrm{d}x\,,
\end{aligned}$$

where for the last estimate we used Poincaré's inequality. Summarizing, we obtained that

$$\kappa_p^\varepsilon[f,\Omega] \leqslant \kappa_p^\varepsilon[f,\Omega_m] + c_P \int_{\Omega\setminus\Omega_{m-1}} |\nabla f(x)|^p \mathrm{d}x\,. \tag{18}$$

Again, using Lemma 2, Poincaré's inequality, and arguing as before we obtain

$$\begin{aligned}
\kappa_p^\varepsilon[f,\Omega_m] &= \kappa_p^\varepsilon[f_k^{(m)}-g_k^{(m)},\Omega_m] \\
&\leqslant \tau_1 \sup_{\mathcal{G}_\varepsilon\in\mathfrak{G}_\varepsilon(\Omega_m)} \frac{1}{\varepsilon^p}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\int_{Q_\varepsilon(z)} |f_k^{(m)}(x)-\langle f_k^{(m)}\rangle_{Q_\varepsilon(z)}|^p \mathrm{d}x \\
&\qquad + c_P\tau_2\int_{\Omega_m}|\nabla g_k^{(m)}(x)|^p\mathrm{d}x, \\
&\overset{(15)}{\leqslant} \tau_1 \sup_{\mathcal{G}_\varepsilon\in\mathfrak{G}_\varepsilon(\Omega_m)} \frac{1}{\varepsilon^p}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\int_{Q_\varepsilon(z)} |f_k^{(m)}(x)-\langle f_k^{(m)}\rangle_{Q_\varepsilon(z)}|^p \mathrm{d}x + \frac{c_P\tau_2}{k}\,.
\end{aligned}$$

By (9), taking the limit superior of both sides of (18) as $\varepsilon\to 0$, we get

$$\limsup_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega]\leqslant \tau_1\gamma\int_{\Omega_m}|\nabla f_k^{(m)}(x)|^p\mathrm{d}x+\frac{c_P\tau_2}{k}+c_P\int_{\Omega\setminus\Omega_{m-1}}|\nabla f(x)|^p\mathrm{d}x$$

and again, by *first* taking the limit for $k\to\infty$ and *then* the limit for $\tau\to 0^+$, we infer that

$$\limsup_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega]\leqslant \gamma\int_{\Omega_m}|\nabla f(x)|^p\,\mathrm{d}x+c_P\int_{\Omega\setminus\Omega_{m-1}}|\nabla f(x)|^p\mathrm{d}x\,.$$

Finally, the inequality (17) follows by taking the limit for $m\to\infty$ and using that $\nabla f\in L^p(\Omega)$. This concludes the proof of Step 1.

Step 2 (sharp upper and lower bounds). In what follows, to shorten notation, it will be convenient to define

$$M_p^\varepsilon[f,Q_\varepsilon(z)]:=\varepsilon^{n-p}\fint_{Q_\varepsilon(z)}|F(x,Q_\varepsilon(z))|^p\mathrm{d}x, \tag{19}$$

with $F(x,Q_\varepsilon(z)):=f(x)-\langle f\rangle_{Q_\varepsilon(z)}$. Note that, our BMO-type seminorm (7) reads as

$$\kappa_p^\varepsilon[f,\Omega]:=\sup_{\mathcal{G}_\varepsilon\in\mathfrak{G}_\varepsilon(\Omega)}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}M_p^\varepsilon[f,Q_\varepsilon(z)]. \tag{20}$$

We shall need the following key observation, where the constants τ_1,τ_2, and τ_3, are the ones of Lemma 2.

Lemma 4. *Let $f\in C_c^2(\mathbb{R}^n)$. For any $Q_\varepsilon(z)$ the following estimates hold:*

$$\begin{aligned}
M_p^\varepsilon[f,Q_\varepsilon(z)] &\leqslant \tau_1^2\gamma_f(z,Q)\left(\int_{Q_\varepsilon(z)}|\nabla f(x)|^p\,\mathrm{d}x\right) \\
&\qquad +\tau_2(1+\tau_1\gamma_f(z,Q))|\mathcal{O}(\varepsilon^{n+p})|, &(21)\\
M_p^\varepsilon[f,Q_\varepsilon(z)] &\geqslant \tau_1^{-2}\gamma_f(z,Q)\left(\int_{Q_\varepsilon(z)}|\nabla f(x)|^p\,\mathrm{d}x\right) \\
&\qquad -\tau_3^{-1}(1+\tau_1^{-1}\gamma_f(z,Q))|\mathcal{O}(\varepsilon^{n+p})|, &(22)
\end{aligned}$$

with $Q:=Q_1(0)=\varepsilon^{-1}(-z+Q_\varepsilon(z))$ and

$$\gamma_f(z,Q):=\begin{cases}\displaystyle\int_Q\left|\frac{\nabla f(z)}{|\nabla f(z)|}\cdot x\right|^p\mathrm{d}x & \textit{if } \nabla f(z)\neq 0,\\ 0 & \textit{otherwise.}\end{cases} \tag{23}$$

Proof. Since $f \in C_c^2(\mathbb{R}^n)$, the Hessian of f is bounded on $\mathbb{R}^n$. We denote by $c_{H(f)}$ the maximum modulus of the Hessian of f on $\mathbb{R}^n$. We then have, by classical Taylor expansion, that for every $x, z \in \mathbb{R}^n$

$$f(x) - f(z) = \nabla f(z) \cdot (x-z) + \mathcal{O}(|x-z|^2), \tag{24}$$

and

$$|\nabla f(x) - \nabla f(z)| \leqslant \mathcal{O}(|x-z|) \tag{25}$$

where, to be precise, $\mathcal{O}(|x-z|) \leqslant c_{H(f)}|x-z|$.

Next, let $Q_\varepsilon(z)$ be an open cell centered at $z \in \Omega$, and note that by (5)

$$\int_{Q_\varepsilon(z)} \nabla f(z) \cdot (y-z) \mathrm{d}y = 0.$$

Therefore, for every $x \in Q_\varepsilon(z)$,

$$\begin{aligned} F(x, Q_\varepsilon(z)) &= [f(x) - f(z)] - \fint_{Q_\varepsilon(z)} [f(y) - f(z)] \mathrm{d}y \\ &\overset{(24)}{=} \nabla f(z) \cdot (x-z) + \mathcal{O}(|x-z|^2) + \fint_{Q_\varepsilon(z)} \mathcal{O}(|y-z|^2) \mathrm{d}y \\ &= \nabla f(z) \cdot (x-z) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Here, to be precise, the $\mathcal{O}(\varepsilon^2)$ notation means that

$$\sup_{x \in Q_\varepsilon(z)} |F(x, Q_\varepsilon(z)) - \nabla f(z) \cdot (x-z)| \leqslant c_{H(f)}^2 \varepsilon^2.$$

Applying Lemma 2, we get

$$|F(x, Q_\varepsilon(z))|^p \leqslant \tau_1 |\nabla f(z) \cdot (x-z)|^p + \tau_2 |\mathcal{O}(\varepsilon^{2p})|, \tag{26}$$

$$|F(x, Q_\varepsilon(z))|^p \geqslant \tau_1^{-1} |\nabla f(z) \cdot (x-z)|^p - \tau_3^{-1} |\mathcal{O}(\varepsilon^{2p})|. \tag{27}$$

Now, define the open set $A(f) := \{x \in \mathbb{R}^n : |\nabla f(x)| \neq 0\} \subseteq \mathrm{supp}_{\mathbb{R}^n} f$ and observe that if $z \notin A(f)$ then (26) and (27) imply $-\tau_3^{-1}|\mathcal{O}(\varepsilon^{n+p})| \leqslant M_p^\varepsilon[f, Q_\varepsilon(z)] \leqslant \tau_2 |\mathcal{O}(\varepsilon^{n+p})|$ and, a fortiori, (21) and (22). Therefore, we can focus on the case $z \in A(f)$. We have

$$\varepsilon^{n-p} \fint_{Q_\varepsilon(z)} |\nabla f(z) \cdot (x-z)|^p \,\mathrm{d}x = \varepsilon^n \int_Q |\nabla f(z) \cdot x|^p \,\mathrm{d}x = \varepsilon^n \gamma_f(z, Q) \, |\nabla f(z)|^p \,, \tag{28}$$

with $Q := \varepsilon^{-1}(-z + Q_\varepsilon(z))$ and $\gamma_f(z, Q)$ given by (23). Again by Lemma 2 and Taylor expansion (25), we infer that for any $z \in A(f)$

$$\begin{aligned} |\nabla f(z)|^p &\leqslant \fint_{Q_\varepsilon(z)} \tau_1 |\nabla f(x)|^p + \tau_2 |\nabla f(x) - \nabla f(z)|^p \,\mathrm{d}x \\ &\leqslant \tau_1 \fint_{Q_\varepsilon(z)} |\nabla f(x)|^p \,\mathrm{d}x + \tau_2 |\mathcal{O}(\varepsilon^p)|, \end{aligned} \tag{29}$$

and, similarly,

$$|\nabla f(z)|^p \geqslant \frac{1}{\tau_1} \fint_{Q_\varepsilon(z)} |\nabla f(x)|^p \,\mathrm{d}x - \frac{1}{\tau_3} |\mathcal{O}(\varepsilon^p)|. \tag{30}$$

Overall, combining the estimates (26), (28), and (29), we conclude that for any $z \in A(f)$

$$\begin{aligned} M_p^\varepsilon[f, Q_\varepsilon(z)] &\overset{(26)}{\leqslant} \tau_1 \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} |\nabla f(z) \cdot (x-z)|^p \, \mathrm{d}x + \tau_2 |\mathcal{O}(\varepsilon^{n+p})| \\ &\overset{(28)}{=} \varepsilon^n \tau_1 \gamma_f(z, Q)\, |\nabla f(z)|^p + \tau_2 |\mathcal{O}(\varepsilon^{n+p})| \\ &\overset{(29)}{\leqslant} \varepsilon^n \tau_1 \gamma_f(z, Q) \left(\tau_1 \fint_{Q_\varepsilon(z)} |\nabla f(x)|^p \, \mathrm{d}x + \tau_2 |\mathcal{O}(\varepsilon^p)| \right) + \tau_2 |\mathcal{O}(\varepsilon^{n+p})| \\ &= \tau_1 \gamma_f(z, Q) \left(\tau_1 \int_{Q_\varepsilon(z)} |\nabla f(x)|^p \, \mathrm{d}x + \tau_2 |\mathcal{O}(\varepsilon^{n+p})| \right) + \tau_2 |\mathcal{O}(\varepsilon^{n+p})| \\ &= \tau_1^2 \gamma_f(z, Q) \left(\int_{Q_\varepsilon(z)} |\nabla f(x)|^p \, \mathrm{d}x \right) + \tau_2 (1 + \tau_1 \gamma_f(z, Q)) |\mathcal{O}(\varepsilon^{n+p})|. \end{aligned}$$

This proves (21). Likewise, from (27) and (30), we get the lower bound (22). $\square$

Step 3 (upper bound). By Step 1, we may consider f compactly supported and, taking into account that cells outside the support of f do not contribute to the supremum in $\kappa_p^\varepsilon[f, \Omega]$, without loss of generality, we may restrict ourselves to the case of f with compact support in an open set Ω of finite measure. By definition, we have $\gamma_f(z, Q) \leqslant \gamma$ for every $z \in \mathbb{R}^n$. Moreover, for every $\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(\Omega)$ we have

$$|\Omega| \geqslant \sum_{Q_\varepsilon(z) \in \mathcal{G}_\varepsilon} |Q_\varepsilon(z)| = \#\mathcal{G}_\varepsilon |\hat{Q}| \varepsilon^n \,, \tag{31}$$

and therefore

$$\begin{aligned} \sum_{Q_\varepsilon(z) \in \mathcal{G}_\varepsilon} M_p^\varepsilon[f, Q_\varepsilon(z)] &\overset{(21)}{\leqslant} \tau_1^2 \gamma \sum_{Q_\varepsilon(z) \in \mathcal{G}_\varepsilon} \int_{Q_\varepsilon(z)} |\nabla f(x)|^p \, \mathrm{d}x \\ &\qquad + \sum_{Q_\varepsilon(z) \in \mathcal{G}_\varepsilon} \tau_2 (1 + \tau_1 \gamma) |\mathcal{O}(\varepsilon^{n+p})| \\ &\leqslant \tau_1^2 \gamma \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x + \tau_2 (1 + \tau_1 \gamma) \#\mathcal{G}_\varepsilon |\mathcal{O}(\varepsilon^{n+p})| \\ &\leqslant \tau_1^2 \gamma \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x + \tau_2 (1 + \tau_1 \gamma)\, |\Omega|\, |\hat{Q}|^{-1} |\mathcal{O}(\varepsilon^p)|. \end{aligned}$$

Hence, for every $\varepsilon > 0$ sufficiently small, we have

$$\kappa_p^\varepsilon[f, \Omega] \leqslant \tau_1^2 \gamma \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x + \tau_2 (1 + \tau_1 \gamma)\, |\Omega|\, |\hat{Q}|^{-1} |\mathcal{O}(\varepsilon^p)|.$$

By taking first the $\limsup_{\varepsilon \to 0}$ of both sides and then the limit for $\tau \to 0$, we end up with the relation

$$\limsup_{\varepsilon \to 0} \kappa_p^\varepsilon[f, \Omega] \leqslant \gamma \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x.$$

Step 4 (lower bound). It remains to show that

$$\liminf_{\varepsilon \to 0} \kappa_p^\varepsilon[f, \Omega] \geqslant \gamma \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x.$$

To this end, we consider, as before, the open set $A(f) := \{x \in \mathbb{R}^n : |\nabla f(x)| \neq 0\} \subseteq \operatorname{supp}_{\mathbb{R}^n} f$, and we fix $\eta > 0$. We make use of Lemma 3, according to which there exists a finite family

$(\Gamma_1,\ldots,\Gamma_{k(\eta)})$ of pairwise disjoint open subsets of $\mathbb{S}^{n-1}$, with $k(\eta)$ depending on η, such that $\operatorname{diam}(\Gamma_j)<\eta$ for every $j\in\mathbb{N}_{k(\eta)}$ and

$$|\{x\in A(f):\nabla f(x)/|\nabla f(x)|\in\mathcal{M}_0\}|=0,\quad \mathcal{M}_0:=\mathbb{S}^{n-1}\setminus\cup_{j\in\mathbb{N}_{k(\eta)}}\Gamma_j.$$

For the sets

$$\Omega_j:=\{x\in A(f)\cap\Omega:\nabla f(x)/|\nabla f(x)|\in\Gamma_j\}\qquad j\in\mathbb{N}_{k(\eta)}\,,$$

we claim that

Lemma 5. *For every $j\in\mathbb{N}_{k(\eta)}$ there exist $R_j\in O(n)$ such that*

$$\gamma\geqslant\gamma_f(z,Q^{(j)})\geqslant\gamma-|\mathcal{O}(\eta)|=:\gamma_\eta\qquad\forall z\in\Omega_j\,,\tag{32}$$

where $Q^{(j)}=Q_1^{(j)}(0):=R_j(\hat{Q})$ and $|\mathcal{O}(\eta)|/\eta$ is uniformly bounded with respect to $j\in\mathbb{N}_{k(\eta)}$ and $z\in\Omega_j$.

Proof. Let $R_j\in O(n)$ be arbitrary. By Remark 2.7 we have, setting $g(z):=\nabla f(z)/\,|\nabla f(z)|$,

$$\begin{aligned}\gamma-\gamma_f(z,Q^{(j)}) &= \int_{R_j(\hat{Q})}|\sigma_{R_j(\hat{Q})}\cdot x|^p-|g(z)\cdot x|^p\,\mathrm{d}x\\ &= \int_{\hat{Q}}|R_j^\top\sigma_{R_j(\hat{Q})}\cdot x|^p-|R_j^\top g(z)\cdot x|^p\,\mathrm{d}x.\end{aligned}$$

Note that, by definition, $\gamma-\gamma_f(z,Q^{(j)})\geqslant 0$. Also, again by Remark 2.7,

$$R^\top\sigma_{R(\hat{Q})}\in\operatorname*{argmax}_{\sigma\in\mathbb{S}^{n-1}}\int_{\hat{Q}}|\sigma\cdot x|^p\,\mathrm{d}x,$$

hence (see, e.g., [14, (2.15.1)])

$$0\;\leqslant\;\gamma-\gamma_f(z,Q^{(j)})\leqslant p\cdot|R_j\sigma_{\hat{Q}}-g(z)|\cdot\|x\|^p_{L^p(\hat{Q})}\,.\tag{33}$$

Note that, $g(z)\in\Gamma_j$ because of $z\in\Omega_j$. Therefore, if we choose $R_j\in O(n)$ such that $R_j\sigma_{\hat{Q}}\in\Gamma_j$, we get

$$\gamma-\gamma_f(z,Q^{(j)})\leqslant c_{\hat{Q}}(p)\operatorname{diam}(\Gamma_j)<c_{\hat{Q}}(p)\eta,\quad c_{\hat{Q}}(p):=p\|x\|^p_{L^p(\hat{Q})}\,,$$

and therefore the lemma follows. □

Now for each $j\in\mathbb{N}_{k(\eta)}$ fix a regular tessellation family $(\mathcal{Q}_\varepsilon^{(j)}(\Omega_j))_{\varepsilon\in I}$, where each $\mathcal{Q}_\varepsilon^{(j)}(\Omega_j)$ consists of cells whose orientation coincides with the one of $R_j(\hat{Q})$, i.e.,

$$Q_\varepsilon^{(j)}(z)\in\mathcal{Q}_\varepsilon^{(j)}(\Omega_j)\quad\Rightarrow\quad Q_\varepsilon^{(j)}(z)=z+\varepsilon R_j(\hat{Q})\quad\text{or}\quad Q_\varepsilon^{(j)}(z)=z-\varepsilon R_j(\hat{Q})\,,$$

where $R_j\in O(n)$ is given by Lemma 5. Note that, the existence of such regular tessellation families is guaranteed by Proposition 1 (see also Remark 2.4). Since

$$\bigcup_{j\in\mathbb{N}_{k(\eta)}}\mathcal{Q}_\varepsilon^{(j)}\left(\Omega_j\right)\in\mathfrak{S}_\varepsilon\left(\Omega\right)\,,\tag{34}$$

denoting by $\mathcal{Q}_\varepsilon^{(j)\cup}(\Omega_j)\subseteq\Omega$ the union of all the disjoint open sets of the family $\mathcal{Q}_\varepsilon^{(j)}\left(\Omega_j\right)$, and setting $Q^{(j)}:=Q_1^{(j)}(0)=\varepsilon^{-1}(-z+Q_\varepsilon^{(j)}(z))$, we have

$$\kappa_p^\varepsilon\left[f,\Omega\right]=\sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}M_p^\varepsilon[f,Q_\varepsilon(z)]\overset{(34)}{\geqslant}\sum_{j=1}^{k(\eta)}\sum_{Q_\varepsilon^{(j)}(z)\in\mathcal{Q}_\varepsilon^{(j)}(\Omega_j)}M_p^\varepsilon[f,Q_\varepsilon^{(j)}(z)].\tag{35}$$

Therefore, by Lemma 4 and 5 we infer that

$$
\begin{aligned}
\kappa_p^{\varepsilon}[f,\Omega] &\overset{(22)}{\geqslant} \sum_{j=1}^{k(\eta)} \sum_{Q_\varepsilon^{(j)}(z)\in \mathcal{Q}_\varepsilon^{(j)}(\Omega_j)} \Bigg[\tau_1^{-2}\gamma_f(z,Q^{(j)})\left(\int_{Q_\varepsilon^{(j)}(z)} |\nabla f(x)|^p \,\mathrm{d}x\right) \\
&\qquad\qquad - \tau_3^{-1}(1+\tau_1^{-1}\gamma_f(z,Q^{(j)}))|\mathcal{O}(\varepsilon^{n+p})|\Bigg] \\
&= \sum_{j=1}^{k(\eta)} \Bigg[-\tau_3^{-1}(1+\tau_1^{-1}\gamma_f(z,Q^{(j)}))|\mathcal{O}(\varepsilon^{n+p})|\#\mathcal{Q}_\varepsilon^{(j)}(\Omega_j) \\
&\qquad + \sum_{Q_\varepsilon^{(j)}(z)\in \mathcal{Q}_\varepsilon^{(j)}(\Omega_j)} \tau_1^{-2}\gamma_f(z,Q^{(j)})\left(\int_{Q_\varepsilon^{(j)}(z)} |\nabla f(x)|^p \,\mathrm{d}x\right)\Bigg] \\
&\overset{(31)}{\geqslant} \sum_{j=1}^{k(\eta)} \Bigg[-\tau_3^{-1}(1+\tau_1^{-1}\gamma_f(z,Q^{(j)}))|\mathcal{O}(\varepsilon^{p})|\,|\Omega_j| \\
&\qquad + \tau_1^{-2}\gamma_f(z,Q^{(j)})\left(\sum_{Q_\varepsilon^{(j)}(z)\in \mathcal{Q}_\varepsilon^{(j)}(\Omega_j)} \int_{Q_\varepsilon^{(j)}(z)} |\nabla f(x)|^p \,\mathrm{d}x\right)\Bigg] \\
&\overset{(32)}{\geqslant} -\tau_3^{-1}(1+\tau_1^{-1}\gamma)|\mathcal{O}(\varepsilon^{p})|\,|\Omega| + \tau_1^{-2}\gamma_\eta\left(\sum_{j=1}^{k(\eta)} \int_{\mathcal{Q}_\varepsilon^{(j)\cup}(\Omega_j)} |\nabla f(x)|^p \,\mathrm{d}x\right) \qquad (36)
\end{aligned}
$$

On the other hand, by Lemma 1, and the hypotheses on the regular tessellation families, we get that

$$
\liminf_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^{(j)\cup}(\Omega_j)} |\nabla f(x)|^p \,\mathrm{d}x \geqslant \int_{\Omega_j} |\nabla f(x)|^p \,\mathrm{d}x,
$$

from which there holds

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \sum_{j=1}^{k(\eta)} \int_{\mathcal{Q}_\varepsilon^{(j)\cup}(\Omega_j)} |\nabla f(x)|^p \,\mathrm{d}x &\geqslant \sum_{j=1}^{k(\eta)} \int_{\Omega_j} |\nabla f(x)|^p \,\mathrm{d}x \\
&= \int_{A(f)\cap\Omega} |\nabla f(x)|^p \,\mathrm{d}x = \int_{\Omega} |\nabla f(x)|^p \,\mathrm{d}x. \qquad (37)
\end{aligned}
$$

Hence, taking first the $\liminf_{\varepsilon\to 0}$ of both sides of (36), and then the limit for $\tau \to 0$, we end up with the relation

$$
\liminf_{\varepsilon\to 0} \kappa_p^{\varepsilon}[f,\Omega] \geqslant \gamma_\eta \int_{\Omega} |\nabla f(x)|^p \,\mathrm{d}x \overset{(32)}{=} (\gamma - |\mathcal{O}(\eta)|)\int_{\Omega} |\nabla f(x)|^p \,\mathrm{d}x\,.
$$

Finally, taking the limit for $\eta \to 0$, we conclude the proof.

4. A short proof of Theorem 1.*i?*

Keeping in mind the notation introduced, as a byproduct of the argument of the previous section, we can assert that the following equalities hold. Intuitively, the equalities hold because the functions of which limits are computed coincide or because in any step one gets rid of infinitesimal terms, which do not contribute to the final result.

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] &= \liminf_{\varepsilon\to 0} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} |f(x)-\langle f\rangle_{Q_\varepsilon(z)}|^p \mathrm{d}x \\
&= \liminf_{\varepsilon\to 0} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \varepsilon^{n-p} \fint_{Q_\varepsilon(z)} |\nabla f(z)\cdot(x-z)|^p \mathrm{d}x \\
&= \liminf_{\varepsilon\to 0} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \varepsilon^{n} \gamma_f(z,\varepsilon^{-1}(-z+Q_\varepsilon(z)))\,|\nabla f(z)|^p \\
&= \liminf_{\varepsilon\to 0} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \gamma_f(z,\varepsilon^{-1}Q_\varepsilon(0)) \int_{Q_\varepsilon(z)} |\nabla f(x)|^p \,\mathrm{d}x \\
&= \liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \sum_{j=1}^{k(\eta)} \sum_{Q_\varepsilon^{(j)}(z)\in\mathcal{Q}_\varepsilon^{(j)}(\Omega_j)} \gamma_f(z,Q^{(j)}) \int_{Q_\varepsilon^{(j)}(z)} |\nabla f(x)|^p \,\mathrm{d}x \\
&= \liminf_{\eta\to 0} \left[\gamma_f(z,Q^{(j)}) \cdot \liminf_{\varepsilon\to 0} \sum_{j=1}^{k(\eta)} \sum_{Q_\varepsilon^{(j)}(z)\in\mathcal{Q}_\varepsilon^{(j)}(\Omega_j)} \int_{Q_\varepsilon^{(j)}(z)} |\nabla f(x)|^p \,\mathrm{d}x \right] \\
&= \gamma\cdot \liminf_{\eta\to 0} \sum_{j=1}^{k(\eta)} \liminf_{\varepsilon\to 0} \int_{\mathcal{Q}_\varepsilon^{(j)\cup}(\Omega_j)} |\nabla f(x)|^p \,\mathrm{d}x \\
&= \gamma \liminf_{\eta\to 0} \sum_{j=1}^{k(\eta)} \int_{\Omega_j} |\nabla f(x)|^p \,\mathrm{d}x \\
&= \gamma \int_\Omega |\nabla f(x)|^p \,\mathrm{d}x\,.
\end{aligned}
$$

In principle, the previous relations represent a short proof of Theorem 1.*i* that highlights the main steps behind our argument. Its correctness, a fortiori, is justified by the results of Section 3.

5. Proof of Theorem 1.*ii*

We subdivide the proof in two steps. In what follows we write $U \Subset \Omega$ to denote an open subset U strictly contained in Ω, that is, such that $\bar{U}$ is compact and contained in Ω.

Step 1. First, let us suppose that $f \in L_c^p(\Omega)$, i.e., $\operatorname{supp}_\Omega f$ is a compact subset of Ω. We denote by U an open subset of Ω such that $\operatorname{supp}_\Omega f \subseteq U \Subset \Omega$. For any $\sigma > 0$ we consider $f_\sigma := f * \rho_\sigma$ with $\rho_\sigma(y) := \sigma^{-n}\rho(y/\sigma)$, ρ standard mollifier supported on the unit ball $B_1 \subseteq \mathbb{R}^n$. We choose $\sigma < \operatorname{dist}(\operatorname{supp}_\Omega f, \partial U)$ so that $\operatorname{supp}_\Omega f_\sigma \subseteq U$.

By Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned}
|f_\sigma(x) - \langle f_\sigma\rangle_{Q_\varepsilon(z)}|^p &= \left|\int_{B_1}\rho(w)\left[f(x-\sigma w) - \fint_{Q_\varepsilon(z)} f(y-\sigma w)\mathrm{d}y\right]\mathrm{d}w\right|^p \\
&\leqslant \int_{B_1}\rho(w)\left|f(x-\sigma w) - \fint_{Q_\varepsilon(z)} f(y-\sigma w)\mathrm{d}y\right|^p\mathrm{d}w \\
&= \int_{B_1}\rho(w)|f(x-\sigma w) - \langle f\rangle_{Q_\varepsilon(z-\sigma w)}|^p\mathrm{d}w
\end{aligned}$$

and therefore

$$\mathcal{I}_{Q_\varepsilon(z)}(f_\sigma) \leqslant \int_{B_1}\rho(w)\fint_{Q_\varepsilon(z-\sigma w)}|f(x) - \langle f\rangle_{Q_\varepsilon(z-\sigma w)}|^p\mathrm{d}x\mathrm{d}w,$$

where we set

$$\mathcal{I}_{Q_\varepsilon(z)}(f_\sigma) := \fint_{Q_\varepsilon(z)}|f_\sigma(x) - \langle f_\sigma\rangle_{Q_\varepsilon(z)}|^p\mathrm{d}x\,.$$

Now, for any $w\in B_1$ and any $\mathcal{G}_\varepsilon \in \mathfrak{S}_\varepsilon(U)$, we have that the family of disjoint cells

$$\mathcal{Q}_\varepsilon^{\sigma,\omega}(\Omega) := \{Q_\varepsilon^{\sigma,w} := Q_\varepsilon(z-\sigma w)\}_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}$$

belongs to $\mathfrak{S}_\varepsilon(\Omega)$ whenever $\sigma < d(U,\partial\Omega)$. Therefore, taking

$$\sigma < \min\{d(\mathrm{supp}_\Omega f, \partial U), d(U,\partial\Omega)\}$$

we get $\mathrm{supp}_\Omega f_\sigma \subseteq U$ and $\mathcal{Q}_\varepsilon^{\sigma,\omega}(\Omega)\in\mathfrak{S}_\varepsilon(\Omega)$ if $\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(U)$. Thus, for any $w\in B_1$ we have

$$\begin{aligned}
\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\mathcal{I}_{Q_\varepsilon(z)}(f_\sigma) &\leqslant \int_{B_1}\rho(w)\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\fint_{Q_\varepsilon(z-\sigma w)}|f(x) - \langle f\rangle_{Q_\varepsilon(z-\sigma w)}|^p\mathrm{d}x\mathrm{d}w \\
&= \int_{B_1}\rho(w)\sum_{Q_\varepsilon^{\sigma,w}\in\mathcal{Q}_\varepsilon^{\sigma,\omega}(\Omega)}\fint_{Q_\varepsilon^{\sigma,w}}|f(x) - \langle f\rangle_{Q_\varepsilon^{\sigma,w}}|^p\mathrm{d}x\mathrm{d}w \\
&\leqslant \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\mathcal{I}_{Q_\varepsilon(z)}(f) \qquad (38)
\end{aligned}$$

Overall, from (38) we get

$$\begin{aligned}
\kappa_p^\varepsilon[f_\sigma, U] &= \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(U)}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\varepsilon^{n-p}\mathcal{I}_{Q_\varepsilon(z)}(f_\sigma) \\
&\overset{(38)}{\leqslant} \varepsilon^{n-p}\sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)}\sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon}\mathcal{I}_{Q_\varepsilon(z)}(f) \\
&= \kappa_p^\varepsilon[f,\Omega]\,. \qquad (39)
\end{aligned}$$

Step 2. Next, we consider a cutoff function $\eta_\delta\in C_c^\infty(\Omega)$, $0\leqslant\eta_\delta\leqslant 1$ in Ω, such that $\eta_\delta\equiv 1$ in $\Omega_\delta := \{x\in\Omega : \mathrm{dist}(x,\partial\Omega) > \delta\}$ and $\sup_{x\in\Omega}|\nabla\eta_\delta(x)| \leqslant c_\Omega\delta^{-1}$. We denote by U_δ an open subset of Ω such that $\mathrm{supp}_\Omega\,\eta_\delta\subseteq U_\delta\Subset\Omega$. Overall we have

$$\Omega_\delta\subseteq\mathrm{supp}_\Omega\,\eta_\delta\subseteq U_\delta\Subset\Omega.$$

Assume $f \in L^p_{\rm loc}(\Omega)$. We have

$$\begin{aligned}\|f\eta_\delta - \langle f\eta_\delta\rangle_{Q_\varepsilon(z)}\|^p_{L^p(Q_\varepsilon(z))} &= \int_{Q_\varepsilon(z)} \left| f(x)\eta_\delta(x) - \fint_{Q_\varepsilon(z)} f(y)\eta_\delta(y)\mathrm{d}y\right|^p \mathrm{d}x \\ &= \int_{Q_\varepsilon(z)} \left|\fint_{Q_\varepsilon(z)} f(x)\eta_\delta(x) - f(y)\eta_\delta(y)\mathrm{d}y\right|^p \mathrm{d}x\,. \end{aligned} \tag{40}$$

Therefore, adding and subtracting $\eta_\delta(y)$ and then using triangular inequality we get

$$\begin{aligned}\|f\eta_\delta - \langle f\eta_\delta\rangle_{Q_\varepsilon(z)}\|^p_{L^p(Q_\varepsilon(z))} &\leqslant \tau_1 \int_{Q_\varepsilon(z)} \eta^p_\delta(x)\left|\fint_{Q_\varepsilon(z)} [f(x)-f(y)]\mathrm{d}y\right|^p \mathrm{d}x \\ &\qquad + \frac{\tau_2}{|\hat{Q}|\varepsilon^n}\int_{Q^2_\varepsilon(z)} |f(y)|^p|\eta_\delta(x)-\eta_\delta(y)|^p\mathrm{d}y\mathrm{d}x \\ &\leqslant \tau_1\|f-\langle f\rangle_{Q_\varepsilon(z)}\|^p_{L^p(Q_\varepsilon(z))} \\ &\qquad + \frac{\tau_2}{|\hat{Q}|\varepsilon^n}\int_{Q^2_\varepsilon(z)} |f(y)|^p\chi^p_\delta(x,y)\mathrm{d}y\mathrm{d}x \end{aligned} \tag{41}$$

with $\chi_\delta(x,y) := |\eta_\delta(x)-\eta_\delta(y)|$. We choose $\varepsilon \ll \varepsilon_\delta := \min\{\mathrm{dist}(\mathrm{supp}_\Omega\,\eta_\delta, \partial U_\delta), \mathrm{dist}(U_\delta,\partial\Omega)\}$. In this way, since $\mathrm{supp}_\Omega\,\eta_\delta \subseteq U_\delta$, we have

$$\chi_\delta(x,y) = 0 \quad \text{if } x,y \in Q_\varepsilon(z) \text{ and } Q_\varepsilon(z)\cap\Omega\backslash U_\delta \neq \emptyset, \tag{42}$$

and

$$\chi_\delta(x,y) \leqslant c_\Omega\delta^{-1}|x-y| \leqslant c_\Omega\delta^{-1}\varepsilon \quad \text{if } x,y\in Q_\varepsilon(z) \text{ and } Q_\varepsilon(z)\subseteq U_\delta\,. \tag{43}$$

We now focus on the right-hand side of (41). From the previous considerations, it follows that

$$\begin{aligned}\sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q^2_\varepsilon(z)} |f(y)|^p\chi^p_\delta(x,y) &\overset{(42)}{=} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(U_\delta)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q^2_\varepsilon(z)} |f(y)|^p\chi^p_\delta(x,y) \\ &\overset{(43)}{\leqslant} c^p_\Omega\frac{|\hat{Q}|\varepsilon^{n+p}}{\delta^p} \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(U_\delta)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \int_{Q_\varepsilon(z)} |f(y)|^p\mathrm{d}y \\ &\leqslant c^p_\Omega\frac{|\hat{Q}|\varepsilon^{n+p}}{\delta^p}\|f\|^p_{L^p(U_\delta)}. \end{aligned} \tag{44}$$

Combining the previous estimates, we get

$$\begin{aligned}\kappa^\varepsilon_p[(f\eta_\delta)_\sigma, U_\delta] &\overset{(39)}{\leqslant} \kappa^\varepsilon_p[f\eta_\delta,\Omega] \\ &= \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \frac{1}{\varepsilon^p}\|f\eta_\delta - \langle f\eta_\delta\rangle_{Q_\varepsilon(z)}\|^p_{L^p(Q_\varepsilon(z))} \\ &\overset{(41)}{\leqslant} \tau_1 \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \frac{1}{\varepsilon^p}\|f-\langle f\rangle_{Q_\varepsilon(z)}\|^p_{L^p(Q_\varepsilon(z))} \\ &\qquad + \tau_2 \sup_{\mathcal{G}_\varepsilon\in\mathfrak{S}_\varepsilon(\Omega)} \sum_{Q_\varepsilon(z)\in\mathcal{G}_\varepsilon} \frac{1}{|\hat{Q}|\varepsilon^{n+p}} \int_{Q^2_\varepsilon(z)} |f(y)|^p\chi^p_\delta(x,y)\mathrm{d}y\mathrm{d}x \\ &\overset{(44)}{\leqslant} \tau_1\kappa^\varepsilon_p[f,\Omega] + \tau_2\frac{c^p_\Omega}{\delta^p}\|f\|^p_{L^p(U_\delta)}. \end{aligned} \tag{45}$$

Summarizing, if $f \in L^p_{\mathrm{loc}}(\Omega)$ then $\kappa_p^\varepsilon[(f\eta_\delta)_\sigma, U_\delta] \leqslant \tau_1 \kappa_p^\varepsilon[f,\Omega] + \tau_2 \frac{c_\Omega^p}{\delta^p}\|f\|^p_{L^p(U_\delta)}$. Hence, by Theorem 1.$i$, we get

$$\gamma\,\|\nabla[(f\eta)_\sigma]\|^p_{L^p(U_\delta)} \quad\leqslant\quad \tau_1 \liminf_{\varepsilon\to 0} \kappa_p^\varepsilon[f,\Omega] + \tau_2 \frac{c_\Omega^p}{\delta^p}\|f\|^p_{L^p(U_\delta)}.$$

If $\liminf_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega] < \infty$, then by the convergence $(f\eta)_\sigma \rightharpoonup f\eta$ weakly in $W^{1,p}(\Omega)$ we get

$$\gamma\,\|\nabla f\|^p_{L^p(\Omega_\delta)} \leqslant \gamma\,\|\nabla(f\eta)\|^p_{L^p(U_\delta)} \leqslant \tau_1 \liminf_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega] + \tau_2 \frac{c_\Omega^p}{\delta^p}\|f\|^p_{L^p(U_\delta)}$$

This proves that if $\liminf_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega] < \infty$ then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and for every $U \Subset \Omega$ we have

$$\lim_{\varepsilon\to 0}\kappa_p^\varepsilon[f,U] = \gamma\,\|\nabla f\|^p_{L^p(U)} \leqslant \tau_1 \liminf_{\varepsilon\to 0}\kappa_p^\varepsilon[f,\Omega] + \tau_2 \frac{c_\Omega^p}{\mathrm{dist}\,(U,\partial\Omega)^p}\|f\|^p_{L^p(U)}.$$

This concludes the proof.

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