

The total intrinsic curvature of curves in Riemannian surfaces & Erratum

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Abstract. We deal with irregular curves contained in smooth, closed, and compact surfaces. For curves with finite total intrinsic curvature, a weak notion of parallel transport of tangent vector fields is well-defined in the Sobolev setting. Also, the angle of the parallel transport is a function with bounded variation, and its total variation is equal to an energy functional that depends on the “tangential” component of the derivative of the tantrix of the curve. We show that the total intrinsic curvature of irregular curves agrees with such an energy functional. By exploiting isometric embeddings, the previous results are then extended to irregular curves contained in Riemannian surfaces. Finally, the relationship with the notion of displacement of a smooth curve is analyzed.

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Erratum

Our paper [14] appeared in *Rendiconti del Circolo Matematico di Palermo*. After publication, while working on [15], we realized that in the statements of the main results, Theorems from 1 to 9 and Proposition 3 (here Theorems from 1.1 to 6.3 and Proposition 6.4), one has to assume in addition that the curve \mathbf{c} is rectifiable. This Erratum will appear in *Rendiconti del Circolo Matematico di Palermo*. In this arXiv paper we added the appropriate hypothesis in said Theorems and Proposition.

The main point is that the equivalence in formula (2.7), namely:

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty$$

holds true for rectifiable curves \mathbf{c} , whereas it is false in general that if $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then $\mathrm{TC}(\mathbf{c}) < \infty$. If one e.g. takes a curve in \mathcal{S}^2 , the unit sphere in \mathbb{R}^3 , that winds around an equator infinitely many times, its total intrinsic curvature is zero but its length and total curvature are both infinite.

Our mistake goes back to a flaw that we recently found in [2, Thm. 6.3.1], where Alexandrov-Reshetnyak erroneously stated that if the geodesic turn of a spherical curve is finite, then its spatial turn is also finite. This is true if the spherical diameter of the curve is smaller than a dimensional constant δ_0 . In this case, in fact, for polygonal curves in \mathcal{S}^2 they obtain the inequality $\mathbf{k}^*(P) \leq \pi + 2\mathbf{k}_{\mathcal{S}^2}(P)$.

Therefore, their statement holds true provided that the curve can be divided in a finite number of arcs each one with spherical diameter smaller than δ_0 . However, the latter property is false, in general, if the curve fails to be rectifiable, as the previous example shows.

Dealing with rectifiable curves \mathbf{c} in \mathcal{M} , in fact, by the smoothness and compactness of \mathcal{M} , the normal curvature of the geodesic arcs of \mathcal{M} is uniformly bounded, and hence we recover the nontrivial implication \Rightarrow in the previous equivalence by arguing as in the model case $\mathcal{M} = \mathcal{S}^2$ considered in [2].

For that reason, all the main results in [14] hold true for rectifiable curves with finite total intrinsic curvature.

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1 Introduction

The theory of irregular curves goes back to A. D. Alexandrov and his collaborators in the 40's of the last century. His joint work with Yu G. Reshetnyak is collected in the book [2] published in 1989. We address to the survey paper [17] for detailed references.

A fundamental role in the theory of the Russian school is played by the class of *one-sidedly smooth* curves. Such a regularity is exhibited e.g. by rectifiable curves in the Euclidean space \mathbb{R}^N with finite *total curvature*. In fact, the unit tangent vector (or tantrix) exists almost everywhere, and it turns out to be a one-dimensional function of *bounded variation*. By exploiting arguments based on integral geometric formulas, Alexandrov-Reshetnyak were also able to study irregular curves with values in the unit N -sphere.

A parallel theory of curves with finite total curvature, say FTC curves, was introduced with a slightly different approach by J. W. Milnor [11, 12] in the 50's. More recently, J. M. Sullivan [18] analyzed variational problems and geometric knot theory in this framework, showing the interplay between discrete and differential geometry. For our purposes, we recall that the *total curvature* (i.e., the supremum of the *rotation* of the polygonals inscribed in the curve) of any FTC curve in \mathbb{R}^N turns out to be equal to the essential variation of the tantrix of the curve in the Gauss sphere \mathbb{S}^{N-1} , see (2.2). For smooth curves, it clearly agrees with the integral of the scalar curvature.

Differently to the Euclidean case, an *intrinsic* theory of FTC curves with values e.g. in a Riemannian manifold \mathcal{M} fails to be complete, even in the model case $\mathcal{M} = \mathcal{S}^2$, the unit 2-sphere in \mathbb{R}^3 .

A first problem comes with the good notion of *total intrinsic curvature* $\text{TC}_{\mathcal{M}}(\mathbf{c})$ of an irregular curve \mathbf{c} in \mathcal{M} , in terms of the best approximation with “curved” polygonals of \mathcal{M} inscribed in \mathbf{c} . In fact, for manifolds with positive sectional curvature (as e.g. $\mathcal{M} = \mathcal{S}^2$) the crucial monotonicity formula of the rotation of inscribed polygonals fails to hold.

In order to overcome this drawback, the good intrinsic notion turns out to be the one proposed by S. B. Alexander and R. L. Bishop [1], that goes back to the one considered by Alexandrov-Reshetnyak [2].

It involves the notion of *modulus* of an inscribed polygonal, that is, the greatest geodesic diameter of the arcs of the curve detected by the polygonal, see Definition 2.5.

With this notation, in fact, C. Maneesawarnng and Y. Lenbury [10] showed that the total intrinsic curvature of a FTC curve in \mathcal{M} is equal to the limit of the rotation of *any* sequence of inscribed polygonals whose modulus goes to zero, see Proposition 2.6.

Notwithstanding, to our knowledge an explicit representation formula for the total intrinsic curvature $\text{TC}_{\mathcal{M}}(\mathbf{c})$ is unknown in this general framework, for irregular curves \mathbf{c} .

A partial result in this direction has been obtained by M. Castrillón Lopez, V. Fernández Mateos, and J. Muñoz Masqué in [5] for the sub-class of (piecewise) smooth curves, see Theorem 2.3. Extending a result by Bishop [4], they showed that

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) = \int_{\mathbf{c}} |\mathfrak{K}_g| ds + \sum_i |\alpha_i| \tag{1.1}$$

where \mathfrak{K}_g is the *geodesic curvature* of the curve (that exists up to a finite number of points) and the second addendum is the finite sum of the “turning angles” at the corner points of \mathbf{c} .

CONTENT OF THE PAPER. We deal with irregular curves contained in 2-dimensional Riemannian manifolds and with finite total intrinsic curvature. We first consider curves \mathbf{c} contained in a smooth (at least of class C^3), closed, compact, and immersed surface \mathcal{M} in \mathbb{R}^N . Notice that \mathcal{M} is not assumed to be oriented.

For the sake of clearness, in the first three sections we deal with the case of surfaces \mathcal{M} in \mathbb{R}^3 , our model case being $\mathcal{M} = \mathcal{S}^2$, the standard unit sphere. The high codimension case, $N \geq 4$, is treated in Sec. 5.

We remark that the analysis of irregular curves in high dimension Riemannian manifolds needs some more work, and hence it will not be treated in this paper.

In Sec. 2, we collect the notation concerning one dimensional BV-functions, total curvature, geodesic curvature, and total intrinsic curvature, by discussing the previously cited properties.

Our first new result, Theorem 4.1, states that a notion of *weak parallel transport* is well-defined for curves with finite total intrinsic curvature. For that reason, in Sec. 3 we collect some well-known features concerning the classical parallel transport of tangent vector fields along smooth curves. We also deal with piecewise-smooth curves, having in mind the case of polygonals P_h in \mathcal{M} inscribed in the irregular curve \mathbf{c} .

Now, if the curve \mathbf{c} in \mathcal{M} has finite total intrinsic curvature, say $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then \mathbf{c} is rectifiable. We let $\mathbf{c} : \bar{I}_L \rightarrow \mathcal{M}$ be its arc-length parameterization, where $I_L := (0, L)$ and L is the length of \mathbf{c} . By Rademacher's theorem, the tantrix $\mathbf{t} := \dot{\mathbf{c}}$ is well-defined a.e. on I_L . Moreover, by smoothness and compactness of \mathcal{M} , it turns out that \mathbf{c} is also a FTC curve in \mathbb{R}^N . Therefore, the tantrix \mathbf{t} is a function with bounded variation.

We also denote by \mathbf{u} the unit conormal to \mathbf{c} obtained by means of a positive rotation of \mathbf{t} on the tangent space $T_{\mathbf{c}}\mathcal{M}$ along \mathbf{c} . If $\mathcal{M} \subset \mathbb{R}^3$, we let $\mathbf{u} := \mathbf{n} \times \mathbf{t}$, where \mathbf{n} is the (Lipschitz-continuous) outward unit normal to \mathcal{M} along the curve.

In the sequel, the polygonals $P_h : \bar{I}_L \rightarrow \mathcal{M}$ are parameterized with constant velocity, and we denote by $X_h : \bar{I}_L \rightarrow \mathbb{R}^N$ the parallel transport of the vector field $\mathbf{t}(0)$ along P_h . Our Theorem 4.1 states:

Theorem 1.1 *If \mathbf{c} is a rectifiable curve, $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, and $\{P_h\}$ is a sequence of inscribed polygonals whose modulus goes to zero, then a subsequence of $\{X_h\}$ strongly converges in $W^{1,1}$ to some function $X \in W^{1,1}(I_L, \mathbb{R}^N)$ satisfying*

$$X(s) = \cos \Theta(s) \mathbf{t}(s) - \sin \Theta(s) \mathbf{u}(s)$$

for a.e. $s \in I_L$. Furthermore, the angle function Θ has bounded variation, $\Theta \in \text{BV}(I_L)$.

For smooth curves \mathbf{c} on \mathcal{M} , the arc-length derivative $\dot{\Theta}$ of the angle function of the parallel transport is equal to the geodesic curvature \mathfrak{K}_g of the curve. In our second result, we shall compute the total variation of the three components of the derivative of the optimal angle function Θ , showing their relation with the three corresponding components of the ‘‘tangential derivative’’ of the tantrix $\mathbf{t} := \dot{\mathbf{c}}$.

For this purpose, we recall that the distributional derivative of a BV function $f : I_L \rightarrow \mathbb{R}^k$ is a finite measure given by the sum $Df = D^a f + D^C f + D^J f$ of its absolutely continuous, Cantor, and Jump components. The latter ones are mutually singular and the decomposition $|Df|(I_L) = |D^a f|(I_L) + |D^C f|(I_L) + |D^J f|(I_L)$ of the total variation holds.

The optimal angle is obtained by possibly minimizing the Jump of Θ , without affecting the definition of weak parallel transport X , due to the 2π -periodicity, see Remark 4.2. Our Theorem 4.3, in fact, states:

Theorem 1.2 *The optimal angle function Θ in Theorem 1.1 satisfies:*

$$|D^a \Theta|(I_L) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| ds, \quad |D^C \Theta|(I_L) = |D^C \mathbf{t}|(I_L), \quad |D^J \Theta|(I_L) = \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^{N-1}}(\mathbf{t}(s+), \mathbf{t}(s-))$$

where \bullet is the scalar product in \mathbb{R}^N and $\mathbf{t}(s\pm)$ denotes the right or left limit of \mathbf{t} at s .

As a consequence, the weak parallel transport X along \mathbf{c} is essentially unique. Notice, moreover, that for smooth curves \mathbf{c} , in the first integral from Theorem 1.2 one has $|\dot{\mathbf{t}} \bullet \mathbf{u}| = |\mathfrak{K}_g|$, whereas for piecewise smooth curves the Jump set $J_{\mathbf{t}}$ of the tantrix is finite, and the last term (where $\mathbf{t}(s\pm)$ denote the right and left limit of \mathbf{t} at the Jump points) agrees with the sum of the turning angles at the corner points.

For a curve \mathbf{c} with finite total intrinsic curvature, we are thus led to introduce the *energy functional*

$$\mathcal{F}(\mathbf{t}) := \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| ds + |D^C \mathbf{t}|(I_L) + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^{N-1}}(\mathbf{t}(s+), \mathbf{t}(s-)) \quad (1.2)$$

where, we recall, $\mathbf{t} := \dot{\mathbf{c}}$ is a function with bounded variation. In the cited Theorem 2.3 on piecewise smooth curves, in fact, formula (1.1) reads:

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) = |D\Theta|(I_L) = \mathcal{F}(\mathbf{t}), \quad \mathbf{t} := \dot{\mathbf{c}} \quad (1.3)$$

We also point out that the Cantor component $D^C \mathbf{t}$ of the derivative of the tantrix is tangential to \mathcal{M} . More precisely, recalling that the unit conormal satisfies $\mathbf{u}(s) \in T_{\mathbf{c}(s)}\mathcal{M}$ for a.e. $s \in I_L$, we have:

$$D^C \mathbf{t} = \mathbf{u}(\mathbf{u} \bullet D^C \mathbf{t}) = \mathbf{u} D^C \Theta.$$

We thus expect that *the total intrinsic curvature $\text{TC}_{\mathcal{M}}(\mathbf{c})$ agrees with the total variation $|D\Theta|(I_L)$ of the angle function*, and hence, by Theorem 1.2, that the explicit formula (1.3) holds true in full generality.

Now, denoting by Θ_h the angle function of the parallel transport X_h along an approximating sequence $\{P_h\}$ as in Theorem 1.1, on account of the cited Proposition 2.6, the representation formula (1.3) holds true as a consequence of the *strict convergence*

$$\lim_{h \rightarrow \infty} |D\Theta_h|(I_L) = |D\Theta|(I_L). \quad (1.4)$$

Obtaining the strict convergence (1.4) is a quite difficult task. We observe that if one considers planar curves in \mathbb{R}^2 , the above limit holds true provided that one replaces the angle of the parallel transport with the oriented angle w.r.t. a fixed direction. Therefore, in some sense, such a property relies on the validity of a “planar” version of Gauss-Bonnet theorem, for domains whose boundary is parameterized by a curve with finite total curvature, see Sec. 6.

Following this approach, we show that the classical Gauss-Bonnet theorem generalizes to domains U in \mathcal{M} bounded by simple and closed curves \mathbf{c} with finite total intrinsic curvature. Referring to Theorem 6.1 for the precise statement, we only remark here that the term given by the circuitation of the geodesic curvature along the boundary of U , see (7.6), is replaced by the integral $\int_0^L k(s) ds$, where $k(s) ds := D\Theta[0, s]$ and Θ is the angle function in Theorems 1.1 and 1.2, so that

$$\int_0^L k(s) ds = \Theta(L) - \Theta(0).$$

We point out that the class of curves with finite total intrinsic curvature seems to be the largest ambient in which the Gauss-Bonnet theorem makes sense. If $\text{TC}_{\mathcal{M}}(\mathbf{c}) = \infty$, in fact, we expect that there is no way to find a finite measure that contains the information (given by the derivative $D\Theta$ of the angle function of the parallel transport along the curve) on the “signed geodesic curvature” of the curve \mathbf{c} .

Our Lemma 6.5 on one-sidedly smooth curves, which is illustrated in Figure 1, allows to suitably exploit the generalized Gauss-Bonnet formulas from Theorem 6.1. In Proposition 6.4, in fact, we build up a sequence $\{\tilde{\Theta}_h\}$ of “modified” angle functions that allows us to recover the upper semicontinuity inequality in the strict convergence (1.4), the lower semicontinuity inequality being a trivial matter. We remark that a bit of care in the construction of the functions $\tilde{\Theta}_h$ has to be taken when the surface \mathcal{M} has positive Gauss curvature near the curve \mathbf{c} , as in the model case $\mathcal{M} = \mathcal{S}^2$. In conclusion, in Theorem 6.3 we obtain:

Theorem 1.3 *For every rectifiable curve \mathbf{c} in \mathcal{M} with finite total curvature, $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, the representation formula (1.3) holds true, where $\mathcal{F}(\mathbf{t})$ is the energy functional in (1.2) and $\mathbf{t} = \dot{\mathbf{c}}$ is the tantrix of the curve.*

In Sec. 7, we deal with the case of curves into any smooth, closed, and compact Riemannian surface $\tilde{\mathcal{M}}$. The notion of total intrinsic curvature, in fact, clearly extends to curves γ in $\tilde{\mathcal{M}}$, where it is denoted by $\text{TC}_{\tilde{\mathcal{M}}}(\gamma)$.

By means of an isometric embedding F of $\tilde{\mathcal{M}}$ into a surface $\mathcal{M} = F(\tilde{\mathcal{M}})$ in \mathbb{R}^N , we can apply our previous results to the curve $\mathbf{c} := F \circ \gamma$.

For this purpose, we shall focus in particular on the validity of the compactness theorem 1.1. In fact, by a quick inspection it turns out that the fundamental inequality (4.3) is the unique point of the previous theory where we used non-intrinsic quantities.

Moreover, we introduce geodesic polar coordinates, and write the local expression of the geodesic curvature of a smooth curve γ in $\tilde{\mathcal{M}}$. It turns out that length, angles and geodesics are preserved by isometries. In fact, we show that the geodesic curvature \mathfrak{K}_g of $\mathbf{c} := F \circ \gamma$ in $\mathcal{M} := F(\tilde{\mathcal{M}})$ agrees with the intrinsic local expression, and hence that the latter does not depend on the choice of the isometric embedding. In a similar way, we check that the rotation of a polygonal \tilde{P} in $\tilde{\mathcal{M}}$ is an intrinsic notion.

As a consequence, for piecewise smooth curves γ in $\tilde{\mathcal{M}}$ we obtain the equality:

$$\text{TC}_{\tilde{\mathcal{M}}}(\gamma) = \text{TC}_{\mathcal{M}}(\mathbf{c}) \quad \text{if} \quad \mathbf{c} := F \circ \gamma$$

independently of the chosen isometric embedding F . In conclusion, we obtain the following:

Theorem 1.4 *For every rectifiable curve γ in $\tilde{\mathcal{M}}$ with finite total intrinsic curvature, we have*

$$\text{TC}_{\tilde{\mathcal{M}}}(\gamma) = \mathcal{F}(\mathbf{t})$$

where the energy functional $\mathcal{F}(\mathbf{t})$ is defined by (1.2) in correspondence to the tantrix $\mathbf{t} = \dot{\mathbf{c}}$ of $\mathbf{c} = F \circ \gamma$, and F is any isometric embedding of $\widetilde{\mathcal{M}}$ as above.

In Sec. 8, we finally deal with the notion of *development of a smooth curve* γ in a surface \mathcal{M} of \mathbb{R}^3 , and analyze its relationship with the definition of total intrinsic curvature.

Namely, the *envelope of the tangent planes* to γ is a ruled surface Σ with zero Gauss curvature around the trace of the curve, and hence it is locally isometric to a planar domain. Moreover, the geodesic curvature \mathfrak{K}_g of the curve γ can be equivalently computed by using either local coordinates in \mathcal{M} or in Σ .

The total intrinsic curvature $\text{TC}_\Sigma(\gamma)$ of γ as a curve in Σ is well-defined, and in Proposition 8.1 we show that it can be recovered by means of the total curvature of the development of γ in \mathbb{R}^2 , yielding to the expected formula:

$$\text{TC}_\Sigma(\gamma) = \int_\gamma |\mathfrak{K}_g| ds.$$

Therefore, even if in general the rotation of a polygonal \widetilde{P}_h of Σ and inscribed in γ , is different from the rotation of the corresponding polygonal P_h in \mathcal{M} , see Example 8.2, by our previous results we infer that

$$\text{TC}_\mathcal{M}(\gamma) = \text{TC}_\Sigma(\gamma)$$

which yields that the limits of the rotation of P_h and of \widetilde{P}_h coincide, if the modulus goes to zero.

We finally point out that similar arguments, based on considering iterations of the displacement of the “complete tangent indicatrix”, are proposed by Reshetnyak [17] as a way to treat the “curvatures” of an irregular curve in \mathbb{R}^N . A first step in this direction has been obtained in our paper [13], where a weak notion of torsion is analyzed.

2 Total intrinsic curvature

In this section, we recall some properties concerning the total intrinsic curvature of smooth curves contained into surfaces. We thus let \mathcal{M} denote an immersed surface in \mathbb{R}^3 . We assume \mathcal{M} smooth (at least of class C^3), closed, and compact, our model case being $\mathcal{M} = \mathcal{S}^2$, the standard unit sphere in \mathbb{R}^3 .

BV-FUNCTIONS OF ONE VARIABLE. We refer to Secs. 3.1 and 3.2 of [3] for the following notation.

Let $I \subset \mathbb{R}$ be a bounded open interval, and $N \in \mathbb{N}^+$. A vector-valued summable function $u : I \rightarrow \mathbb{R}^N$ is said to be of *bounded variation* if its distributional derivative Du is a finite \mathbb{R}^N -valued measure in I .

The *total variation* $|Du|(I)$ of a function $u \in \text{BV}(I, \mathbb{R}^N)$ is given by

$$|Du|(I) := \sup \left\{ \int_I \varphi'(s) u(s) ds \mid \varphi \in C_c^\infty(I, \mathbb{R}^N), \quad \|\varphi\|_\infty \leq 1 \right\}$$

and hence it does not depend on the choice of the representative in the equivalence class of the functions that agree \mathcal{L}^1 -a.e. in I with u , where \mathcal{L}^1 is the Lebesgue measure in \mathbb{R} .

We say that a sequence $\{u_h\} \subset \text{BV}(I, \mathbb{R}^N)$ converges to $u \in \text{BV}(I, \mathbb{R}^N)$ *weakly** in BV if u_h converges to u strongly in $L^1(I, \mathbb{R}^N)$ and $\sup_h |Du_h|(I) < \infty$. In this case, the lower semicontinuity inequality holds:

$$|Du|(I) \leq \liminf_{h \rightarrow \infty} |Du_h|(I).$$

If in addition $|Du_h|(I) \rightarrow |Du|(I)$, we say that $\{u_h\}$ *strictly converges* to u .

The *weak** compactness theorem yields that if $\{u_h\} \subset \text{BV}(I, \mathbb{R}^N)$ converges \mathcal{L}^1 -a.e. on I to a function u , and if $\sup_h |Du_h|(I) < \infty$, then $u \in \text{BV}(I, \mathbb{R}^N)$ and a subsequence of $\{u_h\}$ weakly* converges to u .

Let $u \in \text{BV}(I, \mathbb{R}^N)$. Since each component of u is the difference of two monotone functions, it turns out that u is continuous outside an at most countable set, and that both the left and right limits $u(s\pm) := \lim_{t \rightarrow s\pm} u(t)$ exist for every $s \in I$. Also, u is an L^∞ function that is differentiable \mathcal{L}^1 -a.e. on I , with derivative \dot{u} in $L^1(I, \mathbb{R}^N)$.

The total variation of u agrees with the *essential variation* $\text{Var}_{\mathbb{R}^N}(u)$, which is equal to the pointwise variation of any *good representative* of u in its equivalence class. A good (or precise) representative is e.g.

given by choosing $u(s) = (u(s+) + u(s-))/2$ at the discontinuity points. Letting $u_{\pm}(s) := u(s\pm)$ for every $s \in I$, both the left- and right-continuous functions u_{\pm} are good representatives.

If $u \in \text{BV}(I, \mathbb{R}^N)$, the decomposition into the *absolutely continuous*, *Jump*, and *Cantor* parts holds:

$$Du = D^a u + D^J u + D^C u, \quad |Du|(I) = |D^a u|(I) + |D^J u|(I) + |D^C u|(I).$$

More precisely, one splits $Du = D^a u + D^s u$ into the absolutely continuous and singular parts w.r.t. the Lebesgue measure \mathcal{L}^1 . The Jump set J_u being the (at most countable) set of discontinuity points of any good representative of u , and δ_s denoting the unit Dirac mass at $s \in I$, one has:

$$D^a u = \dot{u} \mathcal{L}^1, \quad D^J u = \sum_{s \in J_u} [u(s+) - u(s-)] \delta_s, \quad D^C u = D^s u \llcorner (I \setminus J_u).$$

Also, any $u \in \text{BV}(I, \mathbb{R}^N)$ can be represented by $u = u^a + u^J + u^C$, where u^a is a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$, u^J is a Jump function, and u^C is a Cantor function, so that

$$|D^a u|(I) = |Du^a|(I), \quad |D^J u|(I) = |Du^J|(I), \quad |D^C u|(I) = |Du^C|(I).$$

Finally, we recall that if $u, v \in \text{BV}(I) := \text{BV}(I, \mathbb{R})$, the product $uv \in \text{BV}(I)$. In the particular case in which the Jump sets coincide, $J_u = J_v = J$, the chain rule formula (cf. [3, Sec. 3.10]) yields:

$$D^a(uv) = (\dot{u}v + u\dot{v}) \mathcal{L}^1, \quad D^J(uv) = \sum_{s \in J} [u(s+)v(s+) - u(s-)v(s-)] \delta_s, \quad D^C(uv) = uD^C v + vD^C u \quad (2.1)$$

where we can choose any good representatives of u and v in the third equality.

TOTAL CURVATURE. We recall that the rotation $\mathbf{k}^*(P)$ of a polygonal P in \mathbb{R}^3 is the sum of the exterior angles between consecutive segments. A polygonal P is said to be inscribed in a curve $\mathbf{c} : [a, b] \rightarrow \mathbb{R}^3$ if P is obtained by choosing a partition $a \leq t_0 < t_1 < \dots < t_n \leq b$ and connecting with segments the consecutive points $\mathbf{c}(t_i)$ of the curve. The mesh of the polygonal is $\text{mesh}(P) := \max_{1 \leq i \leq n} (t_i - t_{i-1})$. The *Euclidean total curvature* $\text{TC}(\mathbf{c})$ of a curve \mathbf{c} in \mathbb{R}^3 is defined by Milnor [11, 12] as the supremum of the rotation $\mathbf{k}^*(P)$ computed among all the polygonals P in \mathbb{R}^3 which are inscribed in \mathbf{c} . Then $\text{TC}(P) = \mathbf{k}^*(P)$ for each polygonal P .

Let \mathbf{c} have compact support and finite total curvature, $\text{TC}(\mathbf{c}) < \infty$. Then, \mathbf{c} is a rectifiable curve. In the sequel, we shall thus tacitly assume that \mathbf{c} is parameterized by arc-length, so that $\mathbf{c} = \mathbf{c}(s)$, with $s \in [0, L] = \bar{I}_L$, where $I_L := (0, L)$ and $L = \mathcal{L}(\mathbf{c})$, the length of \mathbf{c} . If \mathbf{c} is smooth and regular, one has $\text{TC}(\mathbf{c}) = \int_0^L |\mathbf{k}| ds$, where $\mathbf{k}(s) := \ddot{\mathbf{c}}(s)$ is the curvature vector. More generally, since \mathbf{c} is a Lipschitz function, by Rademacher's theorem (cf. [3, Thm. 2.14]) it is differentiable \mathcal{L}^1 -a.e. in I_L . Denoting by $\dot{\mathbf{f}} := \frac{d}{ds} \mathbf{f}$ the derivative w.r.t. the arc-length parameter s , the tantrix $\mathbf{t} = \dot{\mathbf{c}}$ exists a.e., and actually $\mathbf{t} : I_L \rightarrow \mathbb{R}^3$ is a function of bounded variation. Since moreover $\mathbf{t}(s) \in \mathbb{S}^2$ for a.e. s , where \mathbb{S}^2 is the Gauss 2-sphere, we shall write $\mathbf{t} \in \text{BV}(I_L, \mathbb{S}^2)$. The essential variation $\text{Var}_{\mathbb{S}^2}(\mathbf{t})$ of \mathbf{t} in \mathbb{S}^2 differs from $\text{Var}_{\mathbb{R}^3}(\mathbf{t})$, as its definition involves the geodesic distance $d_{\mathbb{S}^2}$ in \mathbb{S}^2 instead of the Euclidean distance in \mathbb{R}^3 . Therefore, $\text{Var}_{\mathbb{R}^3}(\mathbf{t}) \leq \text{Var}_{\mathbb{S}^2}(\mathbf{t})$, and equality holds if and only if \mathbf{t} has a continuous representative. More precisely, by decomposing $\mathbf{t} = \mathbf{t}^a + \mathbf{t}^J + \mathbf{t}^C$, one obtains:

$$\text{Var}_{\mathbb{S}^2}(\mathbf{t}) = \int_0^L |\dot{\mathbf{t}}| ds + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)) + |D^C \mathbf{t}|(I_L) \quad (2.2)$$

whereas in the formula for $\text{Var}_{\mathbb{R}^3}(\mathbf{t})$, that is equal to $|D\mathbf{t}|(I_L)$, one has to replace in (2.2) the geodesic distance $d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-))$ with the Euclidean distance $|\mathbf{t}(s+) - \mathbf{t}(s-)|$ at each Jump point $s \in J_{\mathbf{t}}$.

A REPRESENTATION FORMULA. The following facts hold:

- i) if P and P' are inscribed polygonals and P' is obtained by adding a vertex in \mathbf{c} to the vertexes of P , then $\mathbf{k}^*(P) \leq \mathbf{k}^*(P')$;

- ii) if \mathbf{c} has finite total curvature, for each point v in \mathbf{c} , small open arcs of \mathbf{c} with an end point equal to v have small total curvature.

As a consequence, compare [18], it turns out that $\text{TC}(\mathbf{c}) = \text{Var}_{\mathbb{S}^2}(\mathbf{t})$, see (2.2), and that the total curvature of \mathbf{c} is equal to the limit of $\mathbf{k}^*(P_h)$ for *any* sequence $\{P_h\}$ of polygonals in \mathbb{R}^3 inscribed in \mathbf{c} and such that $\text{mesh}(P_h) \rightarrow 0$. More precisely, if \mathbf{t}_h is the tantrix of P_h , then $\text{Var}_{\mathbb{S}^2}(\mathbf{t}_h) \rightarrow \text{Var}_{\mathbb{S}^2}(\mathbf{t})$, see Remark 6.6.

Remark 2.1 The Cantor component $D^C \mathbf{t}$ is non-trivial, in general. In fact, let e.g. $\gamma : \bar{I} \rightarrow \mathbb{R}^2$, where $I = (0, 1)$, denote the Cartesian curve $\gamma(t) := (t, u(t))$ in \mathbb{R}^2 given by the graph of the primitive $u(t) := \int_0^t v(\lambda) d\lambda$ of the classical Cantor-Vitali function $v : \bar{I} \rightarrow \mathbb{R}$ associated to the ‘‘middle thirds’’ Cantor set. It turns out that $\mathbf{t} = (1 + v^2)^{-1/2}(1, v)$, whence \mathbf{t} is a Cantor function, i.e., $D^a \mathbf{t} = D^J \mathbf{t} = 0$, and

$$D\mathbf{t}(I) = D^C \mathbf{t}(I) = \int_I \frac{1}{(1 + v^2)^{3/2}} (-v, 1) dD^C v.$$

Notice that the angle ω between the unit vectors $(1, 0)$ and \mathbf{t} satisfies $\omega = \arctan v \in \text{BV}(I)$. Therefore, $D\omega(I) = D^C \omega(I) = \int_I \frac{1}{1 + v^2} dD^C v$, which yields

$$|D\omega|(I) = \int_I \frac{1}{1 + v^2} d|D^C v| = |D\mathbf{t}|(I) = \text{TC}(\gamma) = \frac{\pi}{4}.$$

GEODESIC CURVATURE. Assume now that \mathbf{c} is a smooth and regular curve supported in \mathcal{M} . The Darboux frame along \mathbf{c} is the triad $(\mathbf{t}, \mathbf{n}, \mathbf{u})$, where $\mathbf{t}(s) := \dot{\mathbf{c}}(s)$ is the unit tangent vector, $\mathbf{n}(s) := \nu(\mathbf{c}(s))$, $\nu(p)$ being the unit normal to the tangent 2-space $T_p \mathcal{M}$, and $\mathbf{u}(s) := \mathbf{n}(s) \times \mathbf{t}(s)$, where \times denotes the vector product in \mathbb{R}^3 , is the unit conormal. Therefore, the tangent space $T_{\mathbf{c}(s)} \mathcal{M}$ is spanned by $(\mathbf{t}(s), \mathbf{u}(s))$. The curvature vector $\mathbf{k}(s) = \dot{\mathbf{t}}(s)$ is orthogonal to $\mathbf{t}(s)$, and thus decomposes as

$$\mathbf{k}(s) = \mathfrak{K}_g(s) \mathbf{u}(s) + \mathfrak{K}_n(s) \mathbf{n}(s)$$

where $\mathfrak{K}_g := \mathbf{k} \bullet \mathbf{u}$ and $\mathfrak{K}_n := \mathbf{k} \bullet \mathbf{n}$ denote the *geodesic* and *normal curvature* of \mathbf{c} , respectively, and \bullet is the scalar product in \mathbb{R}^3 . The projection $\mathfrak{K}_g \mathbf{u}$ of \mathbf{k} onto the tangent bundle of \mathcal{M} is an intrinsic object, see Sec. 7. Also, the Frenet formulas in \mathbb{R}^3 yield to the Darboux system:

$$\dot{\mathbf{t}} = \mathfrak{K}_g \mathbf{u} + \mathfrak{K}_n \mathbf{n}, \quad \dot{\mathbf{n}} = -\mathfrak{K}_n \mathbf{t} - \mathfrak{T}_g \mathbf{u}, \quad \dot{\mathbf{u}} = -\mathfrak{K}_g \mathbf{t} + \mathfrak{T}_g \mathbf{n} \quad (2.3)$$

where $\mathfrak{T}_g := \dot{\mathbf{n}} \bullet (\mathbf{t} \times \mathbf{n})$ is the geodesic torsion of the curve.

Remark 2.2 If \mathbf{c} is a geodesic on \mathcal{M} , we have $\mathfrak{K}_g \equiv 0$, whence the Darboux frame $(\mathbf{t}, \mathbf{n}, \mathbf{u})$ agrees (up to the sign) with the Frenet frame, and the conormal \mathbf{u} with the bi-normal vector. In particular, the normal curvature \mathfrak{K}_n and the geodesic torsion \mathfrak{T}_g are equal (up to the sign) to the scalar curvature and to the torsion of \mathbf{c} in \mathbb{R}^3 , respectively. Finally, the following estimate will be used in the proof of Theorem 4.1: as for \mathfrak{K}_n , both \mathfrak{T}_g and its arc-length derivative are uniformly bounded by a constant only depending on the maximum of the modulus of the principal curvatures of \mathcal{M} and of their derivatives, respectively.

TOTAL INTRINSIC CURVATURE. The (*intrinsic*) rotation $\mathbf{k}_{\mathcal{M}}(P)$ of a polygonal P in \mathcal{M} , where $\mathcal{M} \subset \mathbb{R}^3$, is the sum of the turning angles between the consecutive geodesic arcs of P . The polygonal P is said to be inscribed in a curve $\mathbf{c} : [a, b] \rightarrow \mathcal{M} \subset \mathbb{R}^3$ if P is obtained by choosing a partition $a \leq t_0 < t_1 < \dots < t_n \leq b$ and connecting with geodesic segments the consecutive points $\mathbf{c}(t_i)$ of the curve. For a general curve \mathbf{c} supported in $\mathcal{M} \subset \mathbb{R}^3$, we shall denote by $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ the class of polygonals in \mathcal{M} which are inscribed in \mathbf{c} . Also, if \mathbf{c} is rectifiable (and parameterized in arc-length) the mesh of a polygonal P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is equivalently given by the maximum of the length of the arcs of \mathbf{c} bounded by the consecutive vertexes of P . Notice that one clearly has $\mathbf{k}_{\mathcal{M}}(P) \leq \text{TC}(P)$, and that the difference $\text{TC}(P) - \mathbf{k}_{\mathcal{M}}(P)$ is equal to the sum of the integrals of the modulus of the normal curvature \mathfrak{K}_n of the geodesic arcs of P .

If e.g. $\mathcal{M} = \mathcal{S}^2$, then $\mathfrak{K}_n \equiv -1$ and hence $\text{TC}(P) = \mathbf{k}_{\mathcal{S}^2}(P) + \mathcal{L}(P)$. In general, by the smoothness and compactness of \mathcal{M} , the normal curvature of the geodesic arcs of \mathcal{M} is uniformly bounded, and hence there exists a real constant $c_{\mathcal{M}} > 0$ depending on \mathcal{M} such that for each polygonal P in \mathcal{M}

$$\text{TC}(P) \leq \mathbf{k}_{\mathcal{M}}(P) + c_{\mathcal{M}} \cdot \mathcal{L}(P). \quad (2.4)$$

The following property has been proved in [5].

Theorem 2.3 ([5, Thm. 3.4]) *Let \mathbf{c} be a regular curve in \mathcal{M} of class C^2 , parameterized by arc-length. Then, for any sequence $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ such that $\text{mesh}(P_h) \rightarrow 0$, one has*

$$\lim_{h \rightarrow \infty} \mathbf{k}_{\mathcal{M}}(P_h) = \int_{\mathbf{c}} |\mathfrak{K}_g| ds = \int_0^L |\mathfrak{K}_g(s)| ds.$$

As a consequence, for a curve \mathbf{c} in \mathcal{M} , one is tempted to define its total intrinsic curvature as in the Euclidean case, i.e., as the supremum of the intrinsic rotation $\mathbf{k}_{\mathcal{M}}(P)$ computed among all the polygonals P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$. However, as observed in [5], if \mathcal{M} has positive sectional (Gauss) curvature, as e.g. $\mathcal{M} = \mathcal{S}^2$, the latter definition does not work. In fact, if $P, P' \in \mathcal{P}_{\mathcal{M}}(\mathbf{c})$, and P' is obtained by adding a vertex in \mathbf{c} to the vertexes of P , then the monotonicity inequality $\mathbf{k}_{\mathcal{M}}(P) \leq \mathbf{k}_{\mathcal{M}}(P')$ holds true in general provided that \mathcal{M} has non-positive sectional curvature. In fact, it relies on the fact that in this case the sum of the interior angles of a geodesic triangle of \mathcal{M} is not greater than π , see [5, Lemma 4.1].

Example 2.4 If e.g. $\mathcal{M} = \mathcal{S}^2$, and \mathbf{c} is a parallel which is not a great circle, then the opposite inequality $\mathbf{k}_{\mathcal{S}^2}(P) \geq \mathbf{k}_{\mathcal{S}^2}(P')$ holds, and for any $P \in \mathcal{P}_{\mathcal{S}^2}(\mathbf{c})$ one has $\mathbf{k}_{\mathcal{S}^2}(P) > \int_{\mathbf{c}} |\mathfrak{K}_g| ds$, see Example 3.2.

Actually, the good definition turns out to be the one introduced by Alexandrov-Reshetnyak [2]. For this purpose, compare e.g. [10], we recall that the *modulus* $\mu_{\mathbf{c}}(P)$ of a polygonal P in $\mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is the maximum of the geodesic diameter of the arcs of \mathbf{c} determined by the consecutive vertexes in P . For $\varepsilon > 0$, we also let

$$\Sigma_{\varepsilon}(\mathbf{c}) := \{P \in \mathcal{P}_{\mathcal{M}}(\mathbf{c}) \mid \mu_{\mathbf{c}}(P) < \varepsilon\}.$$

Definition 2.5 The *total intrinsic curvature* of a curve \mathbf{c} in \mathcal{M} is

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) := \lim_{\varepsilon \rightarrow 0^+} \sup\{\mathbf{k}_{\mathcal{M}}(P) \mid P \in \Sigma_{\varepsilon}(\mathbf{c})\}.$$

Clearly, the above limit is equal to the infimum of $\sup\{\mathbf{k}_{\mathcal{M}}(P) \mid P \in \Sigma_{\varepsilon}(\mathbf{c})\}$ as $\varepsilon > 0$. Moreover, arguing as in [10, Prop. 2.1], for a polygonal P in \mathcal{M} we always have $\text{TC}_{\mathcal{M}}(P) = \mathbf{k}_{\mathcal{M}}(P)$. Also, since \mathcal{M} is compact, a curve with finite total curvature $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$ is rectifiable, too (cf. [10, Prop. 2.4]). Most importantly, making use of a result by Dekster [6], as a consequence of [10, Prop. 2.4] one obtains:

Proposition 2.6 *The total curvature $\text{TC}_{\mathcal{M}}(\mathbf{c})$ of any curve \mathbf{c} in \mathcal{M} is equal to the limit of the rotation $\mathbf{k}_{\mathcal{M}}(P_h)$ of any sequence of polygonals $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ such that $\mu_{\mathbf{c}}(P_h) \rightarrow 0$.*

Remark 2.7 Proposition 2.6 is proved in [2, Thm. 6.3.2], when $\mathcal{M} = \mathcal{S}^2$, and in [5, Prop. 4.3], when \mathcal{M} has non-positive Gauss curvature. The proof for general smooth surfaces \mathcal{M} is obtained by arguing as in [10, Prop. 2.4], where it is firstly proved for curves in $\text{CAT}(\mathbb{K})$ spaces. It suffices to observe that the Gauss curvature of \mathcal{M} is bounded, provided that \mathcal{M} is smooth and compact. A crucial step is the following result (cf. [2, Thm. 2.1.3]): if $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if γ is an arc of \mathbf{c} with geodesic diameter lower than δ , the length of γ is smaller than ε . As a consequence, if $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ is such that the modulus $\mu_{\mathbf{c}}(P_h) \rightarrow 0$, then also $\text{mesh}(P_h) \rightarrow 0$, the converse implication being trivial.

Proposition 2.6 fills the gap given by the lack of monotonicity observed e.g. in Example 2.4, yielding to the conclusion that Definition 2.5 involves a control on the modulus and not on the mesh, at least when the sectional curvature of \mathcal{M} fails to be non-negative.

As a consequence, by Theorem 2.3 one infers that for smooth curves \mathbf{c} in \mathcal{M} one has $\text{TC}_{\mathcal{M}}(\mathbf{c}) = \int_{\mathbf{c}} |\mathfrak{K}_g| ds$. By [5, Cor. 3.6], for piecewise smooth curves \mathbf{c} in \mathcal{M} one similarly obtains that

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\mathfrak{K}_g(s)| ds + \sum_i |\alpha_i|. \quad (2.5)$$

In this formula, the integral is computed separately outside the corner points of \mathbf{c} , where the geodesic curvature \mathfrak{K}_g is well-defined, and the second addendum denotes the finite sum of the absolute value of the

oriented turning angles α_i between the incoming and outgoing unit tangent vectors at each corner point of \mathbf{c} . Therefore, for piecewise smooth curves we can rewrite formula (2.5) as

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| ds + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)). \quad (2.6)$$

PROPERTIES. For a curve \mathbf{c} in \mathcal{M} , we clearly have $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) \leq \mathrm{TC}(\mathbf{c})$. On account of the inequality (2.4), arguing as in [2, Thm. 6.3.1], where the following property is proved for curves into \mathcal{S}^2 , it turns out that if $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then also $\mathrm{TC}(\mathbf{c}) < \infty$, and hence that we definitely have:

$$\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty. \quad (2.7)$$

Therefore, if $\mathrm{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then \mathbf{c} is rectifiable and the tantrix $\mathbf{t} := \dot{\mathbf{c}} \in \mathrm{BV}(I_L, \mathbb{S}^2)$. Moreover, the curve is *one-sidedly smooth* in the sense of [2, Sec. 3.1], i.e., the curve has a left and a right tangent $\mathbf{T}_{\pm}(s)$ at all the points $\mathbf{c}(s)$ in the “strong sense”.

Remark 2.8 This implies that for each $s \in [0, L[$ and $\delta > 0$ we can find $\varepsilon > 0$ such that any secant inscribed in the arc $\mathbf{c}|_{[s, s+\varepsilon]}$ forms with the straight line $\mathbf{T}_+(s)$ an angle less than δ , and similarly for the left tangent.

As in the smooth case, we let $\mathbf{n} := \nu \circ \mathbf{c}$ denote the unit normal to $T_{\mathbf{c}}\mathcal{M}$ along \mathbf{c} . Since \mathcal{M} is smooth and compact, and \mathbf{c} is Lipschitz-continuous, it turns out that $\mathbf{n} \in \mathrm{Lip}([0, L], \mathbb{S}^2)$. Therefore, the *weak conormal* $\mathbf{u} := \mathbf{n} \times \mathbf{t}$ belongs to $\mathrm{BV}(I_L, \mathbb{S}^2)$, with $J_{\mathbf{u}} = J_{\mathbf{t}}$. Since moreover $\mathbf{t} \bullet \mathbf{t} = 0$ a.e. in I_L , we may decompose $\dot{\mathbf{t}} = (\dot{\mathbf{t}} \bullet \mathbf{u})\mathbf{u} + (\dot{\mathbf{t}} \bullet \mathbf{n})\mathbf{n}$.

Remark 2.9 We finally see that if \mathbf{c} is a curve in \mathcal{M} with finite total curvature, the Cantor component $D^C\mathbf{t}$ is tangential to \mathcal{M} , namely:

$$D^C\mathbf{t} = \mathbf{u}(\mathbf{u} \bullet D^C\mathbf{t})$$

where $\mathbf{u}(s) \in T_{\mathbf{c}(s)}\mathcal{M}$ for a.e. $s \in I_L$. In fact, using that $|\dot{\mathbf{t}}|^2 = |\mathbf{u}|^2 = 1$ and $\mathbf{t} \bullet \mathbf{u} = \mathbf{t} \bullet \mathbf{n} = 0$ a.e., whereas both $\mathbf{t} \bullet \mathbf{u}$ and $\mathbf{t} \bullet \mathbf{n}$ are functions of bounded variation, and $D^C\mathbf{n} = 0$, by (2.1) we infer that $\mathbf{t} \bullet D^C\mathbf{t} = 0$, $\mathbf{u} \bullet D^C\mathbf{u} = 0$, $\mathbf{u} \bullet D^C\mathbf{t} = -\mathbf{t} \bullet D^C\mathbf{u}$, and $\mathbf{n} \bullet D^C\mathbf{t} = D^C(\mathbf{t} \bullet \mathbf{n}) = 0$. Since $(\mathbf{t}, \mathbf{n}, \mathbf{u})$ is an orthonormal frame to \mathbb{R}^3 , the tangential property follows.

3 Parallel transport

In this section, we collect some well-known facts concerning the parallel transport of tangent vector fields X along smooth curves in \mathcal{M} . We then also analyze the case of piecewise smooth curves. Finally, we give some more detail in the model case $\mathcal{M} = \mathcal{S}^2$.

Let \mathbf{c} be a smooth, regular, and rectifiable curve in \mathcal{M} . Then $X : [0, L] \rightarrow \mathbb{R}^3$ is a parallel transport along \mathbf{c} if for each $s \in [0, L]$ one has $X(s) \in T_{\mathbf{c}(s)}\mathcal{M}$ and $\dot{X}(s) \perp T_{\mathbf{c}(s)}\mathcal{M}$, i.e., $\dot{X}(s) \parallel \mathbf{n}(s)$. We recall that since $\frac{d}{ds}|X(s)|^2 = 2X(s) \bullet \dot{X}(s) = 0$ for every s , the parallel transport preserves the length of the initial tangent vector $X(0)$.

The proof of the following well-known property is taken from [16, 13.6.1].

Proposition 3.1 *Let $\Theta(s)$ denote the oriented angle from the parallel transport $X(s)$ to the tangent vector $\mathbf{t}(s)$ to \mathbf{c} . Then, the geodesic curvature of \mathbf{c} satisfies $\mathfrak{K}_g(s) = \dot{\Theta}(s)$ for each $s \in [0, L]$.*

PROOF: Assume $|X(0)| = 1$, so that $|X(s)| = 1$ for every s . Writing

$$X(s) = \cos \Theta(s) \mathbf{t}(s) - \sin \Theta(s) \mathbf{u}(s), \quad s \in [0, L] \quad (3.1)$$

we find for each s

$$0 = \mathbf{t} \bullet \dot{X} = \mathbf{t} \bullet [(\cos \Theta \dot{\mathbf{t}} - \sin \Theta \dot{\mathbf{u}}) - \dot{\Theta}(\sin \Theta \mathbf{t} + \cos \Theta \mathbf{u})] = -\sin \Theta (\mathbf{t} \bullet \dot{\mathbf{u}} + \dot{\Theta})$$

where we used that $\mathbf{t} \bullet \dot{\mathbf{t}} = \mathbf{t} \bullet \mathbf{u} = 0$. Similarly, condition $\mathbf{u} \bullet \dot{X} = 0$ implies

$$0 = \cos \Theta (\mathbf{u} \bullet \dot{\mathbf{t}} - \dot{\Theta}).$$

Since $\mathbf{k} = \dot{\mathbf{t}}$, we have $\mathfrak{K}_g = \dot{\mathbf{t}} \bullet \mathbf{u}$. Using that $\mathbf{t} \bullet \mathbf{u} = 0$, we also get $\mathbf{t} \bullet \dot{\mathbf{u}} = -\dot{\mathbf{t}} \bullet \mathbf{u} = -\mathfrak{K}_g$. Therefore, the above centered equations become

$$(\mathfrak{K}_g(s) - \dot{\Theta}(s)) \sin \Theta(s) = 0 = (\mathfrak{K}_g(s) - \dot{\Theta}(s)) \cos \Theta(s) \quad \forall s \in [0, L]$$

which yields $\mathfrak{K}_g = \dot{\Theta}$. □

We thus get the formula for the total intrinsic curvature of a smooth regular curve \mathbf{c} in \mathcal{M}

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) = \int_0^L |\mathfrak{K}_g(s)| ds = \int_0^L |\dot{\Theta}(s)| ds \quad (3.2)$$

compare e.g. [5]. Finally, notice that when $X(s) \bullet \mathbf{t}(s) \neq 0$, by (3.1) one has

$$\tan \Theta(s) = -\frac{X(s) \bullet \mathbf{u}(s)}{X(s) \bullet \mathbf{t}(s)}. \quad (3.3)$$

PIECEWISE SMOOTH CURVES. The parallel transport (3.1) is a well-defined smooth vector field for each regular and piecewise smooth curve \mathbf{c} , once the initial position $X(0)$ is prescribed. If e.g. the curve is rectifiable and its arc-length parameterization is piecewise C^k , then the parallel transport is of class C^k . Moreover, the angle Θ is a function of bounded variation, with a finite number of Jump points in correspondence to the values $\{s_i \mid i = 1, \dots, n\}$ of the arc-length parameter $s \in I_L$ where $\mathbf{c}(s)$ fails to be smooth, the corner points $\mathbf{c}(s_i)$ of \mathbf{c} . More precisely, Θ is a special function of bounded variation in $\text{SBV}(I_L)$, i.e., $D^C \Theta = 0$, and its distributional derivative decomposes as $D\Theta = \dot{\Theta} \mathcal{L}^1 + D^J \Theta$.

By Proposition 3.1, it turns out that the derivative $\dot{\Theta}$ agrees with the geodesic curvature \mathfrak{K}_g outside the corner points of \mathbf{c} , and the Jump component $D^J \Theta$ is a sum of Dirac masses centered at the points s_i , with weight given by the oriented turning angles α_i between the incoming and outgoing unit tangent vectors at each corner point of \mathbf{c} . We thus have

$$D\Theta = \mathfrak{K}_g \mathcal{L}^1 + \sum_{i=1}^n \alpha_i \delta_{s_i}, \quad |D\Theta|(I_L) = \int_0^L |\mathfrak{K}_g| ds + \sum_{i=1}^n |\alpha_i|$$

and hence by (2.5) one infers that

$$|D\Theta|(I_L) = \text{TC}_{\mathcal{M}}(\mathbf{c}).$$

In particular, if \mathbf{c} is a polygonal P in \mathcal{M} , the angle function is piecewise constant and

$$D\Theta = \sum_{i=1}^n \alpha_i \delta_{s_i}, \quad |D\Theta|(I_L) = \sum_{i=1}^n |\alpha_i| = \mathbf{k}_{\mathcal{M}}(P).$$

Moreover, denoting by $(\mathbf{t}, \mathbf{n}, \mathbf{u})$ the Darboux frame of \mathbf{c} , so that formulas (2.3) hold true outside the points s_i , by the smoothness of X in general we have

$$\dot{X} = -\sin \Theta \dot{\Theta} \mathbf{t} - \cos \Theta \dot{\Theta} \mathbf{u} + \cos \Theta \dot{\mathbf{t}} - \sin \Theta \dot{\mathbf{u}}$$

and hence the parallel transport of piecewise smooth curves satisfies, for $s \neq s_i$,

$$\dot{X} = (\cos \Theta \mathfrak{K}_n - \sin \Theta \mathfrak{T}_g) \mathbf{n}. \quad (3.4)$$

CURVES INTO THE 2-SPHERE. Assume now $\mathcal{M} = \mathcal{S}^2$. Taking polar coordinates

$$\mathbf{r}(\theta, \varphi)^T = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi]$$

the curve \mathbf{c} may thus be parameterized by $\mathbf{c}(s) = \mathbf{r}(\theta(s), \varphi(s))^T$ for some smooth angle functions $\theta(s)$ and $\varphi(s)$. Consider the frame

$$\mathbf{e}_\theta(\theta, \varphi) := \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix}, \quad \mathbf{e}_\varphi(\theta, \varphi) := \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}, \quad \mathbf{n}(\theta, \varphi) := \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$$

where $\mathbf{n} = \mathbf{e}_\theta \times \mathbf{e}_\varphi$ is the outward unit normal. The partial derivatives of the tangent frame $(\mathbf{e}_\theta, \mathbf{e}_\varphi)$ satisfy

$$\partial_\theta \mathbf{e}_\theta = -\mathbf{n}, \quad \partial_\varphi \mathbf{e}_\theta = \cos \theta \mathbf{e}_\varphi, \quad \partial_\theta \mathbf{e}_\varphi \equiv 0, \quad \partial_\varphi \mathbf{e}_\varphi = -\sin \theta \mathbf{n} - \cos \theta \mathbf{e}_\theta.$$

Letting

$$\mathbf{e}_\theta(s) := \mathbf{e}_\theta(\theta(s), \varphi(s)), \quad \mathbf{e}_\varphi(s) := \mathbf{e}_\varphi(\theta(s), \varphi(s)), \quad \mathbf{n}(s) := \mathbf{n}(\theta(s), \varphi(s))$$

we thus have

$$\mathbf{t}(s) := \dot{\mathbf{c}}(s) = \dot{\theta}(s) \mathbf{e}_\theta(s) + \sin \theta(s) \dot{\varphi}(s) \mathbf{e}_\varphi(s), \quad \dot{\theta}(s)^2 + \sin^2 \theta(s) \dot{\varphi}(s)^2 = 1 \quad \forall s \in [0, L]. \quad (3.5)$$

Consider a tangent vector field X along \mathbf{c} , so that

$$X(s) := \alpha(s) \mathbf{e}_\theta(s) + \beta(s) \mathbf{e}_\varphi(s), \quad s \in [0, L]$$

for some smooth unknown functions $\alpha(s)$ and $\beta(s)$. We compute for each $s \in [0, L]$

$$\begin{aligned} \dot{X} &= \dot{\alpha} \mathbf{e}_\theta + \alpha (\partial_\theta \mathbf{e}_\theta \dot{\theta} + \partial_\varphi \mathbf{e}_\theta \dot{\varphi}) + \dot{\beta} \mathbf{e}_\varphi + \beta (\partial_\theta \mathbf{e}_\varphi \dot{\theta} + \partial_\varphi \mathbf{e}_\varphi \dot{\varphi}) \\ &= \dot{\alpha} \mathbf{e}_\theta + \alpha (-\mathbf{n} \dot{\theta} + \cos \theta \mathbf{e}_\varphi \dot{\varphi}) + \dot{\beta} \mathbf{e}_\varphi + \beta (-\sin \theta \mathbf{n} \dot{\theta} - \cos \theta \mathbf{e}_\theta \dot{\varphi}) \\ &= (\dot{\alpha} - \beta \cos \theta \dot{\varphi}) \mathbf{e}_\theta + (\dot{\beta} + \alpha \cos \theta \dot{\varphi}) \mathbf{e}_\varphi + (-\alpha \dot{\theta} - \beta \sin \theta \dot{\varphi}) \mathbf{n}. \end{aligned}$$

Condition for a parallel transport is $\dot{X}(s) \parallel X(s)$ for each s . This is equivalent to the first order system for the unknown coefficients $\alpha(s)$ and $\beta(s)$:

$$\begin{cases} \dot{\alpha}(s) = \cos \theta(s) \dot{\varphi}(s) \beta(s) \\ \dot{\beta}(s) = -\cos \theta(s) \dot{\varphi}(s) \alpha(s) \end{cases} \quad s \in [0, L] \quad (3.6)$$

which turns out to have a unique solution for any given initial position $X(0) \in T_{\mathbf{c}(0)} \mathcal{S}^2$.

Since the parallel transport preserves the length, assuming $X(0) = \mathbf{t}(0)$, we have

$$\alpha^2(s) + \beta^2(s) = 1 \quad \forall s \in [0, L].$$

Therefore, from (3.6) one also obtains the identity:

$$\dot{\alpha}(s) \beta(s) - \alpha(s) \dot{\beta}(s) = \cos \theta(s) \dot{\varphi}(s) \quad \forall s \in [0, L]. \quad (3.7)$$

On account of (3.3), and since by (3.5) the unit conormal along \mathbf{c} is

$$\mathbf{u}(s) := \mathbf{n}(s) \times \mathbf{t}(s) = -\sin \theta(s) \dot{\varphi}(s) \mathbf{e}_\theta(s) + \dot{\theta}(s) \mathbf{e}_\varphi(s) \quad (3.8)$$

one infers that for each $s \in [0, L]$ such that $\alpha \dot{\theta} + \beta \sin \theta \dot{\varphi} \neq 0$,

$$\tan \Theta = \frac{\alpha \sin \theta \dot{\varphi} - \beta \dot{\theta}}{\alpha \dot{\theta} + \beta \sin \theta \dot{\varphi}}.$$

Using repeatedly that $\alpha^2 + \beta^2 \equiv \dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \equiv 1$, one has

$$\begin{aligned} \dot{\Theta} &= \frac{d}{ds} (\alpha \sin \theta \dot{\varphi} - \beta \dot{\theta}) \cdot (\alpha \dot{\theta} + \beta \sin \theta \dot{\varphi}) - \frac{d}{ds} (\alpha \dot{\theta} + \beta \sin \theta \dot{\varphi}) \cdot (\alpha \sin \theta \dot{\varphi} - \beta \dot{\theta}) \\ &= \dot{\alpha} \beta - \alpha \dot{\beta} + \sin \theta (\dot{\varphi} \dot{\theta} - \ddot{\theta} \dot{\varphi}) + \cos \theta \dot{\theta}^2 \dot{\varphi} \\ &= \sin \theta (\dot{\varphi} \dot{\theta} - \ddot{\theta} \dot{\varphi}) + \cos \theta \dot{\varphi} (\sin^2 \theta \dot{\varphi}^2 + 2\dot{\theta}^2) \end{aligned}$$

where the last equality follows from the identity (3.7).

On the other hand, recalling formula (3.5), the curvature vector of \mathbf{c} is

$$\mathbf{k} = \dot{\mathbf{t}} = (\ddot{\theta} - \sin \theta \cos \theta \dot{\varphi}^2) \mathbf{e}_\theta + (2 \cos \theta \dot{\theta} \dot{\varphi} + \sin \theta \ddot{\varphi}) \mathbf{e}_\varphi - \mathbf{n} \quad (3.9)$$

and hence by (3.8) the geodesic curvature becomes

$$\mathfrak{K}_g = \mathbf{k} \bullet \mathbf{u} = \sin \theta (\dot{\varphi} \dot{\theta} - \ddot{\theta} \dot{\varphi}) + \cos \theta \dot{\varphi} (\sin^2 \theta \dot{\varphi}^2 + 2\dot{\theta}^2) \quad (3.10)$$

where $(\sin^2 \theta \dot{\varphi}^2 + 2\dot{\theta}^2) = (1 + \dot{\theta}^2)$, so that one recovers the equality $\mathfrak{K}_g = \dot{\Theta}$ from Proposition 3.1.

Example 3.2 If $\mathbf{c} = \mathbf{c}_{\theta_0}$ is the parallel with constant co-latitude $\theta_0 \in]0, \pi/2]$, we choose $\theta(s) \equiv \theta_0$ and $\varphi(s) = s/\sin \theta_0$, where $s \in [0, L]$, with $L := \mathcal{L}(\mathbf{c}_{\theta_0}) = 2\pi \sin \theta_0$. By (3.5) and (3.8), one has

$$\mathbf{t}(s) = \mathbf{e}_\varphi(\theta_0, s/\sin \theta_0), \quad \mathbf{u}(s) = -\mathbf{e}_\theta(\theta_0, s/\sin \theta_0) \quad \forall s$$

and by solving the system (3.6) as above, on account of (3.9) and (3.10) one obtains

$$\Theta(s) = \cot \theta_0 \cdot s, \quad \mathfrak{K}_g = \dot{\Theta} \equiv \cot \theta_0 \quad \forall s.$$

Therefore, according to (3.2) one recovers for any $\theta_0 \in]0, \pi/2]$ the formula

$$\text{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0}) = \int_0^{2\pi \sin \theta_0} |\dot{\Theta}(s)| ds = 2\pi \cos \theta_0$$

for the total intrinsic curvature of the parallel, compare e.g. [5]. In particular, $\text{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0})$ is equal to zero when $\theta_0 = \pi/2$, i.e., when \mathbf{c}_{θ_0} is a great circle, whence a geodesic in \mathcal{S}^2 .

4 Weak parallel transport

In this section, we show that a weak notion of parallel transport holds true for curves \mathbf{c} in \mathcal{M} with finite total intrinsic curvature, see Theorem 4.1. The parallel transport turns out to be a Sobolev function satisfying (3.1), where the unit tangent \mathbf{t} and conormal \mathbf{u} are functions of bounded variation, and the angle function Θ is of bounded variation, too. As a consequence, we infer that *the optimal angle function Θ is essentially unique*, and that the *weak transport X* along the non-smooth curve \mathbf{c} is well-defined by the $W^{1,1}$ tangent vector field in Theorem 4.1. In fact, it turns out that the distributional derivative of the angle function Θ is strongly related to the tangential component of the derivative of the tantrix \mathbf{t} , see Theorem 4.3.

A COMPACTNESS RESULT. We first prove the following

Theorem 4.1 *Let \mathbf{c} be a rectifiable curve in \mathcal{M} with finite total intrinsic curvature, parameterized by arclength $\mathbf{c} : [0, L] \rightarrow \mathcal{M}$, with $L = \mathcal{L}(\mathbf{c})$. Let $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ be such that the modulus $\mu_{\mathbf{c}}(P_h) \rightarrow 0$. For each h , let $P_h : [0, L] \rightarrow \mathcal{M}$ be parameterized with constant velocity, and let $X_h : [0, L] \rightarrow \mathbb{R}^3$ be the parallel transport along P_h , with constant initial condition $X_h(0) = \mathbf{t}(0) \in \mathbb{S}^2$. Then, possibly passing to a subsequence, the sequence $\{X_h\}$ strongly converges in $W^{1,1}$ to some function $X \in W^{1,1}(I_L, \mathbb{R}^3)$ satisfying*

$$X(s) = \cos \Theta(s) \mathbf{t}(s) - \sin \Theta(s) \mathbf{u}(s) \quad (4.1)$$

for \mathcal{L}^1 -a.e. $s \in I_L$, where $\mathbf{t} = \dot{\mathbf{c}}$ is the unit tangent vector, \mathbf{n} the normal to $T_{\mathbf{c}}\mathcal{M}$ along \mathbf{c} , and $\mathbf{u} := \mathbf{n} \times \mathbf{t}$ is the unit conormal. Furthermore, \mathbf{t} and \mathbf{u} are functions in $\text{BV}(I_L, \mathbb{S}^2)$, and the angle function Θ has bounded variation in $\text{BV}(I_L)$.

PROOF: Write for each h

$$X_h(s) = \cos \Theta_h(s) \mathbf{t}_h(s) - \sin \Theta_h(s) \mathbf{u}_h(s) \quad (4.2)$$

and recall that $|D\Theta_h|(I_L) = \mathbf{k}_{\mathcal{M}}(P_h)$, whereas the difference $\text{TC}(P_h) - \mathbf{k}_{\mathcal{M}}(P_h)$ is equal to the sum of the integrals of the modulus of the normal curvature \mathfrak{K}_n of the geodesic arcs of P_h , so that the inequality (2.4) holds. Using that $\mathbf{k}_{\mathcal{M}}(P_h) \rightarrow \text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$ and $\mathcal{L}(P_h) \rightarrow L$, we thus obtain the bounds:

$$\sup_h |D\Theta_h|(I_L) < \infty, \quad \sup_h \text{Var}_{\mathbb{S}^2}(\mathbf{t}_h) = \sup_h \text{TC}(P_h) < \infty.$$

Therefore, by the weak-* compactness, and by using the strong convergence of P_h to \mathbf{c} , possibly passing to a subsequence it turns out that $\{\mathbf{t}_h\}$ and $\{\mathbf{u}_h\}$ converge weakly-* in the BV-sense to \mathbf{t} and \mathbf{u} , respectively, and that the sequence $\{\Theta_h\}$ converges weakly-* in the BV-sense to some function $\Theta \in \text{BV}(I_L)$.

We claim that for each $s \in [0, L]$ and for $\delta > 0$ small

$$\int_0^L |\dot{X}_h(s+\delta) - \dot{X}_h(s)| ds \leq C_{\mathcal{M}} \cdot \delta \cdot [\mathcal{L}(P_h) + |D\Theta_h|(I_L)] \quad (4.3)$$

where the real constant $C_{\mathcal{M}}$ only depends on \mathcal{M} . As a consequence, the sequences $\{\mathcal{L}(P_h)\}$ and $\{|D\Theta_h|(I_L)\}$ being bounded, it turns out that

$$\lim_{|\delta| \rightarrow 0} \sup_h \int_0^L |\dot{X}_h(s+\delta) - \dot{X}_h(s)| ds = 0$$

whereas $|X_h(s)| \equiv 1$ for each h . Therefore, by Kolmogorov-Riesz-Frechet compactness theorem, a further subsequence of $\{X_h\}$ strongly converges in $W^{1,1}$ to some function $X \in W^{1,1}(I_L, \mathbb{R}^3)$. Finally, by the L^1 convergence of \mathbf{t}_h , \mathbf{u}_h and Θ_h to \mathbf{t} , \mathbf{u} , and Θ , respectively, we conclude that (4.1) holds \mathcal{L}^1 -a.e. on I_L .

In order to prove the inequality (4.3), for each h we first smoothly extend the transport X_h to an interval $[-\delta_0, L + \delta_0]$ along the extreme geodesic arcs of P_h , where $\delta_0 > 0$ is fixed. For $0 < |\delta| < \delta_0$, using formula (3.4) for $X = X_h$, and omitting for simplicity to write the dependence on h , for each $s \in [0, L]$ we have:

$$\begin{aligned} \dot{X}(s+\delta) - \dot{X}(s) = & (\cos \Theta(s+\delta) - \cos \Theta(s)) \mathfrak{K}_n(s+\delta) \mathbf{n}(s+\delta) \\ & + \cos \Theta(s) (\mathfrak{K}_n(s+\delta) - \mathfrak{K}_n(s)) \mathbf{n}(s+\delta) \\ & + \cos \Theta(s) \mathfrak{K}_n(s) (\mathbf{n}(s+\delta) - \mathbf{n}(s)) \\ & - (\sin \Theta(s+\delta) - \sin \Theta(s)) \mathfrak{T}_g(s+\delta) \mathbf{n}(s+\delta) \\ & - \sin \Theta(s) (\mathfrak{T}_g(s+\delta) - \mathfrak{T}_g(s)) \mathbf{n}(s+\delta) \\ & - \sin \Theta(s) \mathfrak{T}_g(s) (\mathbf{n}(s+\delta) - \mathbf{n}(s)). \end{aligned}$$

On account of Remark 2.2, we first estimate the three terms depending on \mathfrak{K}_n as follows:

$$|(\cos \Theta(s+\delta) - \cos \Theta(s)) \mathfrak{K}_n(s+\delta) \mathbf{n}(s+\delta)| \leq |\mathfrak{K}_n(s+\delta)| \cdot |D\Theta|(s, s+\delta),$$

where by Fubini-Tonelli's theorem

$$\int_0^L |\mathfrak{K}_n(s+\delta)| \cdot |D\Theta|(s, s+\delta) ds \leq c_{\mathcal{M}} \cdot |D\Theta|(I_L) \cdot \delta,$$

$c_{\mathcal{M}}$ being the maximum of the modulus of the principal curvatures of \mathcal{M} . Moreover,

$$|\cos \Theta(s) (\mathfrak{K}_n(s+\delta) - \mathfrak{K}_n(s)) \mathbf{n}(s+\delta)| \leq c'_{\mathcal{M}} \delta$$

$c'_{\mathcal{M}}$ being the maximum of the modulus of the derivative of the principal curvatures of \mathcal{M} , and similarly, since $|\mathbf{n}(s+\delta) - \mathbf{n}(s)| \leq c_{\mathcal{M}} \cdot \delta$, we get:

$$|\cos \Theta(s) \mathfrak{K}_n(s) (\mathbf{n}(s+\delta) - \mathbf{n}(s))| \leq c_{\mathcal{M}}^2 \delta.$$

As to the three terms depending on \mathfrak{T}_g , we infer as above:

$$\int_0^L |(\sin \Theta(s+\delta) - \sin \Theta(s)) \mathfrak{T}_g(s+\delta) \mathbf{n}(s+\delta)| ds \leq K_{\mathcal{M}} \cdot |D\Theta|(I_L) \cdot \delta$$

$K_{\mathcal{M}}$ being a uniform bound, only depending on \mathcal{M} , of the maximum of the modulus of the geodesic torsion of P_h , outside the corner points. Moreover,

$$|\sin \Theta(s) (\mathfrak{T}_g(s+\delta) - \mathfrak{T}_g(s)) \mathbf{n}(s+\delta)| \leq K'_{\mathcal{M}} \delta$$

$K'_{\mathcal{M}}$ being a uniform bound, only depending on \mathcal{M} , of the maximum of the modulus of the derivative of the geodesic torsion of P_h , outside the corner points. Finally,

$$|\sin \Theta(s) \mathfrak{T}_g(s) (\mathbf{n}(s+\delta) - \mathbf{n}(s))| \leq K_{\mathcal{M}} c_{\mathcal{M}} \delta.$$

Therefore, inequality (4.3) readily follows, and the proof is complete. \square

THE ANGLE FUNCTION. In principle, the angle function Θ depends on the subsequence corresponding to the approximating sequence $\{P_h\}$. We now show that the *optimal* angle function Θ , see Remark 4.2, is essentially unique and hence that the parallel transport X along irregular curves \mathbf{c} with finite total curvature

is well-defined in the $W^{1,1}$ setting. In fact, in Theorem 4.3 we write the total variation of the optimal angle function in terms of the tangential weak derivative of the tantrix \mathbf{t} .

For this purpose, recalling the decomposition $\dot{\mathbf{t}} = (\dot{\mathbf{t}} \bullet \mathbf{u}) \mathbf{u} + (\dot{\mathbf{t}} \bullet \mathbf{n}) \mathbf{n}$ of the differential of the tantrix $\mathbf{t} := \dot{\mathbf{c}}$ into the tangential and normal components, we introduce the *energy functional*

$$\mathcal{F}(\mathbf{t}) := \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| ds + |D^C \mathbf{t}|(I_L) + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)). \quad (4.4)$$

Notice that since $|\dot{\mathbf{t}}| \geq |\dot{\mathbf{t}} \bullet \mathbf{u}|$, on account of (2.2) we clearly have $\mathcal{F}(\mathbf{t}) \leq \text{Var}_{\mathbb{S}^2}(\mathbf{t})$, where the strict inequality holds in general, as $\dot{\mathbf{t}} \bullet \mathbf{n} \neq 0$ a.e. on I_L , when \mathcal{M} has no “flat” parts.

Remark 4.2 In Theorem 4.1, we may and do assume that at each Jump point $s \in J_{\Theta}$, the Jump

$$[\Theta]_s := \Theta(s+) - \Theta(s-)$$

is bounded by π , i.e., $|\llbracket \Theta \rrbracket_s| \leq \pi$. For this purpose, we consider the BV function $u = e^{i\Theta} : I_L \rightarrow \mathbb{S}^1$ and build up an optimal lifting $\tilde{\Theta} : I_L \rightarrow \mathbb{R}$ of u as in [9]. Roughly speaking, we replace the Jump component Θ^J with a Jump function $\tilde{\Theta}^J$ which has Jump set contained in J_{Θ} and such that for each $s \in J_{\Theta}$

$$|\llbracket \tilde{\Theta}^J \rrbracket_s| \leq \pi, \quad \llbracket \tilde{\Theta}^J \rrbracket_s = \llbracket \Theta^J \rrbracket_s + 2k\pi, \quad k \in \mathbb{Z}.$$

The optimal angle function is such that for a.e. $s \in I_L$ there exists $k \in \mathbb{Z}$ such that $\tilde{\Theta}(s) = \Theta(s) + 2k\pi$, whence $\cos \tilde{\Theta} = \cos \Theta$ and $\sin \tilde{\Theta} = \sin \Theta$ a.e. on I_L . This yields that formula (4.1) remains unchanged if we replace Θ with the optimal angle $\tilde{\Theta}$.

Theorem 4.3 *Under the hypotheses of Theorem 4.1, and on account of Remark 4.2, we have*

$$|D\Theta|(I_L) = \mathcal{F}(\mathbf{t}).$$

More precisely, in the decomposition formula $|D\Theta|(I_L) = |D^a\Theta|(I_L) + |D^J\Theta|(I_L) + |D^C\Theta|(I_L)$ we have:

$$|D^a\Theta|(I_L) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}| ds, \quad |D^C\Theta|(I_L) = |D^C \mathbf{t}|(I_L), \quad |D^J\Theta|(I_L) = \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)). \quad (4.5)$$

PROOF: Let $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ as in Theorem 4.1, with transport vector fields X_h and Darboux frames $(\mathbf{t}_h, \mathbf{n}_h, \mathbf{u}_h)$, and let X be the $W^{1,1}$ transport vector field given by (4.1).

THE A.C. COMPONENTS. Recalling that $|D^a\Theta|(I_L) = \int_0^L |\dot{\Theta}| ds$, the first equality in (4.5) follows provided that we show that for \mathcal{L}^1 -a.e. $s \in I_L$

$$\dot{\Theta}(s) = \dot{\mathbf{t}}(s) \bullet \mathbf{u}(s). \quad (4.6)$$

For this purpose, we first observe that from (4.1), using that X is a Sobolev function, and hence that it has a continuous representative, see eq. (4.10) below, it turns out that the Jump set of Θ agrees with the Jump set of \mathbf{t} (and hence of \mathbf{u}). By the chain rule formula (2.1) we infer that for a.e. s

$$\dot{X} = \dot{\Theta}(-\sin \Theta \mathbf{t} - \cos \Theta \mathbf{u}) + \cos \Theta \dot{\mathbf{t}} - \sin \Theta \dot{\mathbf{u}}.$$

On the one hand, passing to the limit in the identities $\dot{X}_h \bullet \mathbf{t}_h = 0$ and $\dot{X}_h \bullet \mathbf{u}_h = 0$, by the a.e. convergences $X_h \rightarrow X$, $\mathbf{t}_h \rightarrow \mathbf{t}$, and $\mathbf{u}_h \rightarrow \mathbf{u}$, that hold true along subsequences, due to the L^1 convergences, we deduce that $\dot{X} \bullet \mathbf{t} = 0$ and $\dot{X} \bullet \mathbf{u} = 0$ a.e. on I_L . On the other hand, using that $|\mathbf{t}_h| = 1$, $|\mathbf{u}_h| = 1$, and $\mathbf{t}_h \bullet \mathbf{u}_h = 0$, we also infer that $\dot{\mathbf{t}} \bullet \mathbf{t} = 0$, $\dot{\mathbf{u}} \bullet \mathbf{u} = 0$, and $\dot{\mathbf{u}} \bullet \mathbf{t} = -\dot{\mathbf{t}} \bullet \mathbf{u}$ a.e. on I_L . As in the proof of Proposition 3.1, by the above properties we obtain for a.e. s the equations

$$0 = \dot{X} \bullet \mathbf{t} = -\sin \Theta (\dot{\Theta} - \dot{\mathbf{t}} \bullet \mathbf{u}), \quad 0 = \dot{X} \bullet \mathbf{u} = -\cos \Theta (\dot{\Theta} - \dot{\mathbf{t}} \bullet \mathbf{u})$$

that clearly imply (4.6).

THE CANTOR COMPONENTS. The second equality in (4.5) holds true if we show that

$$D^C \mathbf{t} = \mathbf{u} D^C \Theta. \quad (4.7)$$

To this aim, using again the chain rule formula (2.1), and since $X \in W^{1,1}$, we have

$$0 = D^C X = -\sin \Theta \mathbf{t} D^C \Theta - \cos \Theta \mathbf{u} D^C \Theta + \cos \Theta D^C \mathbf{t} - \sin \Theta D^C \mathbf{u}$$

(where we choose good representatives of \mathbf{t} , \mathbf{u} , and Θ) which is equivalent to the equation:

$$\cos \Theta (D^C \mathbf{t} - \mathbf{u} D^C \Theta) = \sin \Theta (D^C \mathbf{u} + \mathbf{t} D^C \Theta). \quad (4.8)$$

Now, by taking the scalar products with \mathbf{t} and \mathbf{u} in equation (4.8), and observing that by (2.1) we also have $\mathbf{t} \bullet D^C \mathbf{t} = 0$, $\mathbf{u} \bullet D^C \mathbf{u} = 0$, and $\mathbf{t} \bullet D^C \mathbf{u} = -\mathbf{u} \bullet D^C \mathbf{t}$, we obtain

$$0 = \sin \Theta (-\mathbf{u} \bullet D^C \mathbf{t} + D^C \Theta), \quad \cos \Theta (\mathbf{u} \bullet D^C \mathbf{t} - D^C \Theta) = 0$$

which yields that $\mathbf{u} \bullet D^C \mathbf{t} = D^C \Theta$. But we have seen in Remark 2.9 that $D^C \mathbf{t}$ is tangential, namely, $D^C \mathbf{t} = \mathbf{u} (\mathbf{u} \bullet D^C \mathbf{t})$. Therefore, formula (4.7) is proved.

THE JUMP COMPONENTS. Recalling that $J_{\mathbf{t}} = J_{\mathbf{u}} = J_{\Theta}$, the third equality in (4.5) holds true if we show that for every $s \in J_{\Theta}$

$$|\Theta(s+) - \Theta(s-)| = d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)). \quad (4.9)$$

Now, again by the chain rule formula (2.1) we infer that

$$0 = D^J X = D^J (\cos \Theta \mathbf{t} - \sin \Theta \mathbf{u}) = \sum_{s \in J_{\Theta}} [\cos \Theta \mathbf{t} - \sin \Theta \mathbf{u}]_s \delta_s \quad (4.10)$$

where for each $s \in J_{\Theta}$

$$[\cos \Theta \mathbf{t} - \sin \Theta \mathbf{u}]_s := [\cos \Theta(s+) \mathbf{t}(s+) - \sin \Theta(s+) \mathbf{u}(s+)] - [\cos \Theta(s-) \mathbf{t}(s-) - \sin \Theta(s-) \mathbf{u}(s-)].$$

For any fixed $s \in J_{\Theta}$, up to a rotation in the target space we may and do assume that $\mathbf{n}(s) = (0, 0, 1)$, and hence we can write

$$\mathbf{t}(s\pm) = (\cos \alpha_{\pm}, \sin \alpha_{\pm}, 0), \quad \mathbf{u}(s\pm) = \mathbf{n}(s) \times \mathbf{t}(s\pm) = (-\sin \alpha_{\pm}, \cos \alpha_{\pm}, 0)$$

for some real numbers α_{\pm} satisfying $|\alpha_+ - \alpha_-| \leq \pi$. Condition $[\cos \Theta \mathbf{t} - \sin \Theta \mathbf{u}]_s = 0$ yields to the system

$$\begin{cases} \cos(\alpha_+ - \Theta(s+)) = \cos(\alpha_- - \Theta(s-)) \\ \sin(\alpha_+ - \Theta(s+)) = \sin(\alpha_- - \Theta(s-)) \end{cases}$$

which gives $\Theta(s+) - \Theta(s-) = \alpha_+ - \alpha_- \bmod 2\pi$. By Remark 4.2, the optimal angle function satisfies $|\Theta(s+) - \Theta(s-)| \leq \pi$. Since $|\alpha_+ - \alpha_-| \leq \pi$, we thus conclude that $\Theta(s+) - \Theta(s-) = \alpha_+ - \alpha_-$. Therefore, equality (4.9) follows by observing that $d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)) = |\alpha_+ - \alpha_-|$, as required. \square

5 The high codimension case

In this section, we extend the previous results to the high codimension case of curves \mathbf{c} in \mathcal{M} , where \mathcal{M} is a smooth (at least of class C^3), closed, and compact immersed surface in \mathbb{R}^N , with $N \geq 4$. We remark that \mathcal{M} is not assumed to be oriented.

We will only sketch the proofs: further details can be obtained by arguing in a way very similar to the codimension one case previously considered. Moreover, when referring to analogous results from the previous sections, we shall tacitly assume that one has to replace \mathbb{S}^2 and \mathbb{R}^3 with \mathbb{S}^{N-1} and \mathbb{R}^N , respectively, where \mathbb{S}^{N-1} is the unit hyper-sphere in \mathbb{R}^N .

TOTAL CURVATURE. The Euclidean total curvature $\text{TC}(\mathbf{c})$ of a curve \mathbf{c} in \mathbb{R}^N is defined as in the case $N = 3$, and similar features hold. Namely, if \mathbf{c} is smooth and regular, and $\mathbf{c} : [0, L] \rightarrow \mathbb{R}^N$ is its arc-length parameterization, one has $\text{TC}(\mathbf{c}) = \int_0^L |\mathbf{k}| ds$, where $\mathbf{k}(s)$ is the curvature vector of \mathbf{c} . More generally, if \mathbf{c} has compact support and finite total curvature, then \mathbf{c} is rectifiable, and the tantrix $\mathbf{t} = \dot{\mathbf{c}}$ exists a.e., with $\mathbf{t} \in \mathbb{S}^{N-1}$. Moreover, the function $\mathbf{t} : I_L \rightarrow \mathbb{S}^{N-1}$ has bounded variation, and its essential variation in \mathbb{S}^{N-1} is equal to the total curvature of \mathbf{c} , whereas formula (2.2) continues to hold for $\text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t})$. Furthermore, the total curvature of \mathbf{c} is equal to the limit of any sequence of polygonals $\{P_h\}$ in \mathbb{R}^N inscribed in \mathbf{c} and such that $\text{mesh}(P_h) \rightarrow 0$, see Remark 6.6.

TOTAL INTRINSIC CURVATURE. If \mathbf{c} is a smooth and regular curve in \mathcal{M} , using that $\dot{\mathbf{t}} \bullet \mathbf{t} \equiv 0$, where \bullet is the scalar product in \mathbb{R}^N , the curvature vector $\mathbf{k}(s) := \dot{\mathbf{t}}(s)$ again decomposes as

$$\mathbf{k}(s) = \mathfrak{K}_g(s) \mathbf{u}(s) + \mathfrak{K}_n(s) \mathbf{n}(s). \quad (5.1)$$

The unit conormal $\mathbf{u} : [0, L] \rightarrow \mathbb{S}^{N-1}$ is the unit vector orthogonal to \mathbf{t} and obtained by means of a positive rotation of \mathbf{t} on the tangent space $T_{\mathbf{c}}\mathcal{M}$ along \mathbf{c} , so that $\mathbf{t} \bullet \mathbf{u} = 0$ and the tangent space $T_{\mathbf{c}(s)}\mathcal{M}$ is spanned by $(\mathbf{t}(s), \mathbf{u}(s))$. Also, $\mathbf{n} : [0, L] \rightarrow \mathbb{S}^{N-1}$ is a smooth normal unit vector field (a section of the normal bundle).

The total intrinsic curvature $\text{TC}_{\mathcal{M}}(\mathbf{c})$ of a curve \mathbf{c} in \mathcal{M} is defined as in the case $N = 3$, see Definition 2.5. For a polygonal P in \mathcal{M} , we have $\text{TC}_{\mathcal{M}}(P) = \mathbf{k}_{\mathcal{M}}(P)$, and Proposition 2.6 continues to hold.

Since inequality (2.4) is verified through the assumptions on \mathcal{M} , it turns out that a curve \mathbf{c} in \mathcal{M} has finite total intrinsic curvature if and only if it has finite Euclidean total curvature, see (2.7). Whence, if $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, then \mathbf{c} is rectifiable and one-sidedly smooth, see Remark 2.8, and the tantrix $\mathbf{t} \in \text{BV}(I_L, \mathbb{S}^{N-1})$.

As in the smooth case, we define the weak conormal \mathbf{u} in $\text{BV}(I_L, \mathbb{S}^{N-1})$ by the unit vector orthogonal to \mathbf{t} and obtained by means of a positive rotation of \mathbf{t} on the tangent space $T_{\mathbf{c}}\mathcal{M}$ along \mathbf{c} .

Finally, formula $D^C \mathbf{t} = \mathbf{u}(\mathbf{u} \bullet D^C \mathbf{t})$ is obtained by arguing as in Remark 2.9, but this time observing that $\mathbf{n}_i \bullet D^C \mathbf{t} = D^C(\mathbf{t} \bullet \mathbf{n}_i) = 0$ for each $i = 3, \dots, N$, where $s \mapsto (\mathbf{n}_3, \dots, \mathbf{n}_N)(s)$ is a Lipschitz-continuous orthonormal frame that spans the normal space to \mathcal{M} along \mathbf{c} .

WEAK PARALLEL TRANSPORT. Proposition 3.1 clearly extends to smooth and regular curves in $\mathcal{M} \subset \mathbb{R}^N$. Also, on account of (3.1) and (5.1), by decomposing the derivative of the unit conormal

$$\dot{\mathbf{u}} = (\dot{\mathbf{u}} \bullet \mathbf{t}) \mathbf{t} + \dot{\mathbf{u}}^\perp$$

into the tangential and normal component to \mathcal{M} , and recalling that $\dot{\mathbf{u}} \bullet \mathbf{t} = -\mathbf{t} \bullet \mathbf{u} = -\dot{\Theta}$, the parallel transport of (piecewise) smooth curves this time satisfies

$$\dot{X} = \cos \Theta \mathfrak{K}_n \mathbf{n} - \sin \Theta \dot{\mathbf{u}}^\perp, \quad (5.2)$$

where $\dot{\mathbf{u}}^\perp = \dot{\mathbf{u}}$ when \mathbf{c} is a geodesic arc.

Moreover, a compactness property as in Theorem 4.1 holds true: the limit function $X \in W^{1,1}(I_L, \mathbb{R}^N)$ satisfies (4.1) for \mathcal{L}^1 -a.e. $s \in I_L$, where $\mathbf{t} = \dot{\mathbf{c}}$ is the unit tangent vector and the conormal \mathbf{u} agrees with the weak-* BV-limit of the sequence $\{\mathbf{u}_h\}$ of the conormals to a subsequence of $\{P_h\}$.

In fact, compactness in $W^{1,1}$ is based on the validity of the estimate (4.3), where the real constant $C_{\mathcal{M}}$ only depends on \mathcal{M} . Now, using this time the formula (5.2) for the derivative of $X = X_h$, where $\dot{\mathbf{u}}_h^\perp = \dot{\mathbf{u}}_h$, as P_h is a polygonal in \mathcal{M} , the inequality (4.3) is checked by arguing as in the proof of Theorem 4.1, but this time observing that:

- i) the normal curvatures \mathfrak{K}_n of the geodesics in \mathcal{M} , and their derivatives w.r.t. the arc-length parameter, are equibounded by a constant only depending on \mathcal{M} ;
- ii) if \mathbf{u} is the unit conormal of a geodesic arc in \mathcal{M} , and $\dot{\mathbf{u}}$ is its derivative w.r.t. the arc-length parameter, both $|\mathbf{u}|$ and $|\dot{\mathbf{u}}|$ are equibounded by a constant only depending on \mathcal{M} .

Remark 5.1 The above properties follow from the smoothness and compactness of the surface \mathcal{M} in \mathbb{R}^N , and they will be discussed in Sec. 7, see also Example 7.6.

THE ANGLE FUNCTION. We now see that Theorem 4.3 continues to hold. For this purpose, by the structure (4.1) of the $W^{1,1}$ transport X we again infer that $J_\Theta = J_{\mathbf{t}} = J_{\mathbf{u}}$. Therefore, the equalities (4.5) hold true if we check the validity of the three formulas (4.6), (4.7), and (4.9).

The equality (4.6) involving the a.c. components is readily proved by means of the same argument.

As to the Cantor components, using that $\mathbf{t} \bullet D^C \mathbf{t} = 0$, $\mathbf{u} \bullet D^C \mathbf{u} = 0$, and $\mathbf{t} \bullet D^C \mathbf{u} = -\mathbf{u} \bullet D^C \mathbf{t}$, we similarly obtain that $\mathbf{u} \bullet D^C \mathbf{t} = D^C \Theta$, whence the equality (4.7) follows since we have already checked the tangential property $D^C \mathbf{t} = \mathbf{u}(\mathbf{u} \bullet D^C \mathbf{t})$.

As to the Jump components, for any $s \in J_\Theta$, up to a rotation we may and do assume that the tangent space $T_{\mathbf{c}(s)}\mathcal{M}$ is spanned by the first two vectors of the canonical basis in \mathbb{R}^N . Therefore, we can write

$$\mathbf{t}(s_\pm) = (\cos \alpha_\pm, \sin \alpha_\pm, 0_{\mathbb{R}^{N-2}}), \quad \mathbf{u}(s_\pm) = (\sin \alpha_\pm, -\cos \alpha_\pm, 0_{\mathbb{R}^{N-2}})$$

for some real numbers α_\pm satisfying $|\alpha_+ - \alpha_-| \leq \pi$, whence $d_{\mathbb{S}^{N-1}}(\mathbf{t}(s_+), \mathbf{t}(s_-)) = |\alpha_+ - \alpha_-| \leq \pi$. Condition $[\cos \Theta \mathbf{t} - \sin \Theta \mathbf{u}]_s = 0$ implies again that $\Theta(s_+) - \Theta(s_-) = \alpha_- - \alpha_+ \bmod 2\pi$, whereas Remark 4.2 on the optimal angle function Θ continues to hold, whence equation (4.9) is satisfied and the proof is complete.

6 Gauss-Bonnet theorem and representation formula

In this section, we discuss the validity of Gauss-Bonnet formula in the setting of domains in \mathcal{M} bounded by simple and closed curves with finite total curvature, Theorem 6.1. As a consequence, we shall obtain an explicit representation formula for the total intrinsic curvature of curves in immersed surfaces, Theorem 6.3.

THE GAUSS-BONNET THEOREM. We have:

Theorem 6.1 *Let \mathcal{M} be a smooth, closed, compact, and immersed surface in \mathbb{R}^N , where $N \geq 3$. Let $\mathbf{c} : [0, L] \rightarrow \mathcal{M}$ be a simple and closed rectifiable curve with finite total curvature, $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$. Let $k(s) ds := D\Theta[0, s]$, where Θ is the left-continuous representative of the optimal angle function of the parallel transport along \mathbf{c} , see Theorems 4.1 and 4.3, so that*

$$\int_0^L k(s) ds = \Theta(L) - \Theta(0).$$

Let U be the open set in \mathcal{M} enclosed by the oriented curve \mathbf{c} . Moreover, assume that U is simply connected, and that for a.e. $s \in I_L$ the tangent vector $\mathbf{t}(s)$ is positively oriented w.r.t. the natural orientation on the boundary of U at $\mathbf{c}(s)$. Finally, let \mathbf{K} denote the Gauss curvature of \mathcal{M} , and α the oriented angle from $\mathbf{t}(L-)$ to $\mathbf{t}(0+)$ at the junction point $\mathbf{c}(0) = \mathbf{c}(L)$. Then we have:

$$\int_U \mathbf{K} dA = 2\pi - \int_0^L k(s) ds - \alpha.$$

Notice that if \mathbf{c} is smooth, by Proposition 3.1 we know that $D\Theta = \dot{\Theta} \mathcal{L}^1$, with $\dot{\Theta}(s) = \mathfrak{K}_g(s)$ for each s , so that we recover the classical formula, as $\int_0^L k(s) ds = \int_{\mathbf{c}} \mathfrak{K}_g(s) ds$, see (7.6). In a similar way one may proceed in the case of piecewise smooth curves, this time obtaining an extra term given by the sum of the oriented turning angles at the corner points of \mathbf{c} , in correspondence to the Jump points of the angle function Θ in I_L , plus a possible extra term at the junction point $\mathbf{c}(0) = \mathbf{c}(L)$. Therefore, our Theorem 6.1 extends the classical Gauss-Bonnet theorem to the wider class of curves with finite total curvature.

If $\text{TC}_{\mathcal{M}}(\mathbf{c}) = \infty$, in fact, we expect that there is no way to find a finite measure that contains the information (given by the derivative $D\Theta$ of the angle function of the parallel transport along the curve) on the “signed geodesic curvature” of the curve \mathbf{c} .

Finally, a more general result could be obtained if U fails to be simply-connected, assuming \mathcal{M} oriented. This time, the term $2\pi \cdot \chi(U)$ appears, $\chi(U)$ being the Euler-Poincaré characteristic of U .

PROOF OF THEOREM 6.1: Let $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ as in Theorem 4.1, with transport vector fields $X_h : [0, L] \rightarrow \mathcal{M}$ given by (4.2). Let U_h be the open set in \mathcal{M} enclosed by the oriented closed polygonal P_h , and $\mathbf{i}_h(x)$

the index of P_h at the point $x \in \mathcal{M}$. By uniform convergence, for h sufficiently large we can choose a simply-connected and open set U_h in \mathcal{M} such that the index i_h is equal to zero outside U_h . By applying the classical Gauss-Bonnet theorem, and recalling that by our assumptions $P_h(0) = P_h(L) = \mathbf{c}(0) = \mathbf{c}(L)$, it is readily checked that the equality

$$\int_{U_h} i_h \mathbf{K} dA = 2\pi - \int_0^L k_h(s) ds - \alpha_h$$

holds true, where $k_h(s) := D\Theta_h[0, s]$, so that $\int_0^L k_h(s) ds = \Theta_h(L) - \Theta_h(0)$, and α_h is the oriented angle from $\mathbf{t}_h(L)$ to $\mathbf{t}_h(0)$ at the junction point $P_h(0) = P_h(L)$. By the weak-* convergence of $D\Theta_h$ to $D\Theta$, we infer that $\int_0^L k_h(s) ds \rightarrow \int_0^L k(s) ds$ as $h \rightarrow \infty$. On the other hand, by the uniform convergence of P_h to \mathbf{c} we obtain that $\int_{U_h} i_h \mathbf{K} dA \rightarrow \int_U \mathbf{K} dA$. Finally, since \mathbf{c} is one-sidedly smooth, we also infer that $\alpha_h \rightarrow \alpha$, as required. \square

THE REPRESENTATION FORMULA. In general, by the sequential lower-semicontinuity of the total variation w.r.t. the weak-* convergence, in Theorem 4.1 (that holds true for curves contained in surfaces \mathcal{M} of \mathbb{R}^N) we only have

$$|D\Theta|(I_L) \leq \liminf_{h \rightarrow \infty} |D\Theta_h|(I_L) = \lim_{h \rightarrow \infty} \mathbf{k}_{\mathcal{M}}(P_h) = \text{TC}_{\mathcal{M}}(\mathbf{c})$$

where the last equality follows from Proposition 2.6.

As a consequence, by Theorem 4.3 we obtain the inequality

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) \geq \mathcal{F}(\mathbf{t}) \tag{6.1}$$

where $\mathcal{F}(\mathbf{t})$ is the energy functional given by (4.4), and we expect that equality holds in (6.1) in full generality.

In fact, for piecewise smooth and regular curves \mathbf{c} in \mathcal{M} , one has:

$$\mathcal{F}(\mathbf{t}) = \int_0^L |\mathfrak{K}_g(s)| ds + \sum_i |\alpha_i|$$

so that it suffices to apply Theorem 2.3 and (2.5).

Remark 6.2 We now readily check that *equality holds in (6.1) for convex or concave curves* with finite total intrinsic curvature, i.e., for simple and closed curves \mathbf{c} such that the right-hand (or left-end) side region with boundary the trace of \mathbf{c} is a geodesically-convex subset of \mathcal{M} . For non-closed curves, this means that all the length minimizing arcs connecting two points of the curve lie on the same side w.r.t. the tantrix of the curve.

In this case, in fact, for any polygonal P_h in \mathcal{M} inscribed in \mathbf{c} , the angle Θ_h of the parallel transport along P_h is a monotone function. Therefore, for each $(a, b) \subset I_L$ we have $|D\Theta_h|(a, b) = |\Theta_h(b-) - \Theta_h(a+)|$. The a.e. convergence of Θ_h to Θ , that holds true for a subsequence, yields that the angle Θ is a monotone function, too, whence $|D\Theta|(a, b) = |\Theta(b-) - \Theta(a+)|$. As a consequence, we obtain the strict convergence $|D\Theta_h|(I) \rightarrow |D\Theta|(I)$, which implies the equality sign in (6.1), on account of Theorem 4.3.

By exploiting (in Proposition 6.4) the generalized Gauss-Bonnet theorem 6.1, we are able to prove that equality holds in (6.1), even in the non trivial case of surfaces \mathcal{M} with positive Gauss curvature.

Theorem 6.3 *Let \mathcal{M} be a smooth (at least of class C^3), closed, and compact (not necessarily oriented) immersed surface in \mathbb{R}^N . Then, for every rectifiable curve \mathbf{c} in \mathcal{M} with finite total curvature, $\text{TC}_{\mathcal{M}}(\mathbf{c}) < \infty$, we have*

$$\text{TC}_{\mathcal{M}}(\mathbf{c}) = \mathcal{F}(\mathbf{t})$$

where $\mathcal{F}(\mathbf{t})$ is given by (4.4) and $\mathbf{t} = \dot{\mathbf{c}}$ is the tantrix of the curve.

We first observe that Theorem 6.3 holds true as a consequence of the following proposition, that will be proved in the second part of this section.

Proposition 6.4 *Let $\mathbf{c} : [0, L] \rightarrow \mathcal{M}$ be a rectifiable curve with finite total curvature (parameterized by arc-length), and let Θ denote the left-continuous representative of the optimal angle of the parallel transport X along \mathbf{c} , with initial condition $X(0) = \mathbf{t}(0)$. Let $\{P_h\} \subset \mathcal{P}_{\mathcal{M}}(\mathbf{c})$ with modulus $\mu_{\mathbf{c}}(P_h) \rightarrow 0$. Assume that P_h is generated by the consecutive vertexes $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \dots < s_n = L$ (with $\{s_i\}$ and n depending on h), and that every s_i is not a Jump point of the angle function Θ . Also, let Θ_h denote the angle of the parallel transport X_h along P_h , with initial condition $X_h(0) = \mathbf{t}(0)$. Then, for h sufficiently large there exists a piecewise constant function $\tilde{\Theta}_h : I_L \rightarrow \mathbb{R}$ such that:*

- (a) *for each $i = 1, \dots, n$, there exists a parameter $\tilde{s}_i \in [s_{i-1}, s_i[$ such that $\tilde{\Theta}_h(s) = t_i \Theta(\tilde{s}_i+) + (1-t_i)\Theta(\tilde{s}_i-)$ for any $s \in]s_{i-1}, s_i[$, where $t_i \in [0, 1]$;*
- (b) *$\text{Var}(\Theta_h) \leq \text{Var}(\tilde{\Theta}_h) + \varepsilon_h$, where $\varepsilon_h \rightarrow 0^+$ as $h \rightarrow \infty$.*

PROOF OF THEOREM 6.3: We first notice that the assumption on the continuity of the angle function Θ at the points s_i is assumed without loss of generality, as the Jump set J_{Θ} is at most countable.

Property (a) in Proposition 6.4 implies that the modified angle $\tilde{\Theta}_h$ is a competitor to the computation of the essential variation of Θ , compare [3, Sec. 3.2], whence $\text{Var}(\tilde{\Theta}_h) \leq \text{Var}(\Theta)$. By property (b) in Proposition 6.4, we deduce that $\limsup_h \text{Var}(\Theta_h) \leq \text{Var}(\Theta)$. The weak convergence of Θ_h to Θ , see Theorem 4.1, yields that $\text{Var}(\Theta) \leq \liminf_h \text{Var}(\Theta_h)$, whence we obtain the strict convergence $\text{Var}(\Theta_h) \rightarrow \text{Var}(\Theta)$. Since $\text{Var}(\Theta_h) = \mathbf{k}_{\mathcal{M}}(P_h)$, whereas by Proposition 2.6 we know that $\mathbf{k}_{\mathcal{M}}(P_h) \rightarrow \text{TC}_{\mathcal{M}}(\mathbf{c})$, and by Theorem 4.3 that $\text{Var}(\Theta) = |D\Theta|(I_L) = \mathcal{F}(\mathbf{t})$, we conclude that $\text{TC}_{\mathcal{M}}(\mathbf{c}) = \mathcal{F}(\mathbf{t})$, as required. \square

A LOCALIZATION LEMMA. Proposition 6.4 will be proved by exploiting Theorem 6.1, see formulas (6.2) and (6.6). For this purpose, we shall make use of the following result, which is illustrated in Figure 1.

Lemma 6.5 *Given any one-sidedly smooth curve $\gamma : [0, L] \rightarrow \mathcal{M}$, parameterized in arc length, there is $\varepsilon_0 > 0$ such that for any $[a, b] \subset [0, L]$ satisfying $b - a < \varepsilon_0$ we can find a simply-connected closed set $\Omega \subset \mathcal{M}$ for which $\gamma([a, b]) \subset \Omega$ and $\gamma(a), \gamma(b) \in \partial\Omega$, in such a way that the minimal geodesic arcs connecting any couple of points in the curve $\gamma([a, b])$ are contained in Ω . In particular, the geodesic arc connecting $\gamma(a)$ and $\gamma(b)$ divides Ω in two connected components.*

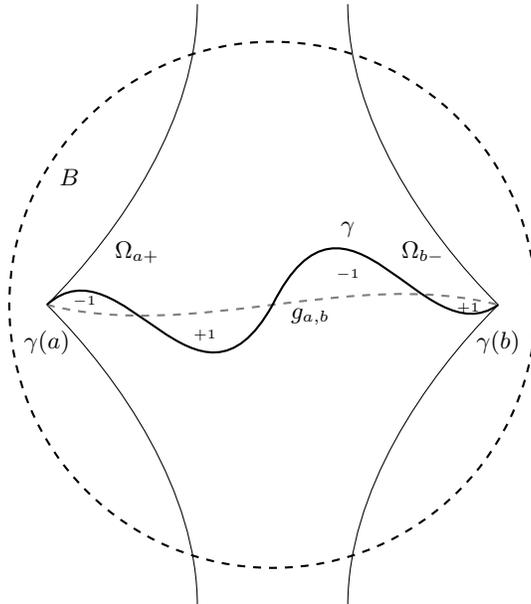


Figure 1: The simply-connected closed set $\Omega = B \cap \Omega_{a+} \cap \Omega_{b-}$ of Lemma 6.5. The arc γ is drawn with a continuous line, and the geodesic arc connecting $\gamma(a)$ and $\gamma(b)$ with a dashed line.

PROOF OF LEMMA 6.5: Let us fix $s \in [0, L]$. Let $\varepsilon_1(s)$ be half of the injectivity radius of \mathcal{M} at $\gamma(s)$, and let $\varepsilon_1 := \inf_{s \in [0, L]} \varepsilon_1(s)$, so that by compactness of the curve γ and smoothness of \mathcal{M} , which implies a uniform

bound on the sectional curvature of \mathcal{M} , we get

$$\varepsilon_1 = \min_{s \in [0, L]} \varepsilon_1(s), \quad \varepsilon_1 > 0.$$

From now on we will consider only points $a, b \in [0, L]$ at most ε_1 apart, so that the geodesic $g_{a,b}$ from $\gamma(a)$ to $\gamma(b)$ can be uniquely defined as the shortest path connecting $\gamma(a)$ and $\gamma(b)$.

We first extend γ to a one-sidedly smooth curve defined in a neighborhood of $[0, L]$. Now, if $a \in [0, L]$, the right geodesic tangent at $\gamma(a)$, i.e., the geodesic g_{a+} starting in $\gamma(a)$ with tangent vector $\mathbf{t}(a+)$, is well-defined. Moreover, by Remark 2.8, where we fix e.g. $\delta = \pi/4$, and by the smoothness and compactness on \mathcal{M} , it turns out that for any $s \in [0, L]$, there is $\varepsilon_2(s) \in (0, \varepsilon_1]$ such that if $a, b \in [s, s + \varepsilon_2(s)]$, with $a < b$, then the angle in $\gamma(a)$ between g_{a+} and $g_{a,b}$ is less than $\delta/2$. Let

$$\varepsilon_2 := \inf_{s \in [0, L]} \varepsilon_2(s),$$

so that ε_2 is a positive minimum, $\varepsilon_2 > 0$, by continuity of the function $\varepsilon_2(s)$ in $[0, L]$.

By the previous construction, if $0 \leq a < c < b \leq L$ are chosen so that $b - a \leq \varepsilon_2$, then the angle between $g_{a,b}$ and $g_{a,c}$ in $\gamma(a)$ is smaller than δ . As a consequence, the curve $\gamma([a, b])$ is contained in the geodesic sector Ω_{a+} bounded by the geodesics from $\gamma(a)$ with starting direction tilted by $\pm\delta$ from the one of $g_{a,b}$.

With the same reasoning applied to $b \in (0, L]$ and to the left geodesic g_{b-} , we can find a positive number $\varepsilon_3 \in (0, \varepsilon_1]$ such that if $0 \leq a < c < b \leq L$ satisfy $b - a \leq \varepsilon_3$, then the angle between $g_{a,b}$ and $g_{c,b}$ in $\gamma(b)$ is smaller than δ . Hence the curve $\gamma([a, b])$ is contained in the geodesic sector Ω_{b-} bounded by the geodesics from $\gamma(b)$ with starting direction tilted by $\pm\delta$ from the one of $g_{a,b}$.

Let then $\varepsilon_0 := \min\{\varepsilon_2, \varepsilon_3\}$, and let $\Omega := \Omega_{a+} \cap \Omega_{b-} \cap B$, where B is the intersection of the geodesic balls of radii ε_0 centered in $\gamma(a)$ and $\gamma(b)$, see Figure 1. We thus conclude that if $a, b \in [0, L]$ are such that $0 < b - a < \varepsilon_0$, then $\gamma([a, b]) \subset \Omega$, the closed set Ω is simply-connected, and $\gamma(a), \gamma(b) \in \partial\Omega$. Moreover, the minimal geodesic arcs connecting any couple of points in the curve $\gamma([a, b])$ are contained in Ω . Finally, the arc $g_{a,b}$ divides Ω in two connected components, as required. \square

THE ROLE OF GAUSS-BONNET THEOREM. In order to make the proof of Proposition 6.4 more clear, we first recall how the equality $\text{TC}(\mathbf{c}) = \text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t})$ is checked for curves \mathbf{c} in \mathbb{R}^N with finite total curvature, and then deal with the case $N = 2$, where we apply a “planar” version of the Gauss-Bonnet theorem 6.1.

Remark 6.6 Let P_h be an inscribed polygonal to the curve $\mathbf{c} : [0, L] \rightarrow \mathbb{R}^N$ (parameterized by arc-length) and generated by the consecutive vertexes $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \dots < s_n = L$, and let \mathbf{v}_i be the oriented segment of P_h from $\mathbf{c}(s_{i-1})$ to $\mathbf{c}(s_i)$. If \mathbf{t}_h is the tantrix of P_h in \mathbb{S}^{N-1} , the value of \mathbf{t}_h in \mathbf{v}_i is an average of the values of the restriction of the tantrix \mathbf{t} of \mathbf{c} to (s_{i-1}, s_i) , when completed to a continuous curve in \mathbb{S}^{N-1} by connecting with geodesic arcs the points $\mathbf{t}(s-)$ and $\mathbf{t}(s+)$ for each $s \in J_{\mathbf{t}} \cap (s_{i-1}, s_i)$, compare [2]. This property implies that $\text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t}_h) \leq \text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t})$. If $\{P_h\}$ is an inscribed sequence satisfying $\text{mesh}(P_h) \rightarrow 0$, the weak BV convergence of \mathbf{t}_h to \mathbf{t} implies the lower semicontinuity inequality $\text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t}) \leq \liminf_h \text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t}_h)$, yielding the strict convergence $\text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t}_h) \rightarrow \text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t})$. Using that $\text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t}_h) \rightarrow \text{TC}(\mathbf{c})$, one concludes that $\text{TC}(\mathbf{c}) = \text{Var}_{\mathbb{S}^{N-1}}(\mathbf{t})$.

When \mathbf{c} is a *planar curve*, i.e., when $N = 2$, the value of $\mathbf{t}_h \in \mathbb{S}^1$ on the segment \mathbf{v}_i is equal to one of the values of the “completion” in \mathbb{S}^1 of the restriction of the tantrix \mathbf{t} to the interval $]s_{i-1}, s_i[$.

We now see that this property can be rewritten in terms of angle functions, and hence of the “planar” version of the Gauss-Bonnet theorem 6.1, where of course $\mathbf{K} \equiv 0$. This is the starting point to treat the case of curves on surfaces. In the proof of Proposition 6.4, moreover, we have to consider the angle of the parallel transport, and to deal with the extra term given by the integral of the Gauss curvature.

We thus denote by $\omega(s)$ the oriented angle from $\mathbf{t}(s)$ to the fixed direction $\mathbf{t}(0)$, where we choose \mathbf{t} equal to the left-continuous representative of the BV-function $\dot{\mathbf{c}}$. We assume moreover that $P_h : [0, L] \rightarrow \mathbb{R}^2$ is parameterized with constant velocity on each interval $]s_{i-1}, s_i[$, in such a way that $P_h(s_i) = \mathbf{c}(s_i)$ for each i , and that every s_i is not a Jump point of \mathbf{t} .

If $\omega_h(s)$ is the oriented angle from $\mathbf{t}_h(s)$ to $\mathbf{t}(0)$, then $\omega_h(s)$ is constant on each interval $]s_{i-1}, s_i[$. In order to show that $\text{Var}(\omega_h) \rightarrow \text{Var}(\omega)$, by [17, Lemma 1] we may and do assume that \mathbf{c} is a simple arc. Also, by Lemma 6.5 we can reduce to the following situation, for h large enough.

Denote by $\angle \mathbf{t}(s)\mathbf{v}_i$ the oriented angle from $\mathbf{t}(s)$ to \mathbf{v}_i , where $s \in]s_{i-1}, s_i[$, and \mathbf{v}_i is the oriented segment of P_h from $\mathbf{c}(s_{i-1})$ to $\mathbf{c}(s_i)$. For $i = 1, \dots, n$, letting $\alpha_i := \angle \mathbf{t}(s_{i-1})\mathbf{v}_i$, if $\alpha_i \neq 0$, we choose the *first* parameter \bar{s}_i in the interval $]s_{i-1}, s_i]$ such that $\mathbf{c}(\bar{s}_i) \in \mathbf{v}_i$. Then, by Lemma 6.5, the angle $\bar{\beta}_i := \angle \mathbf{t}(\bar{s}_i)\mathbf{v}_i$ cannot have the same sign as α_i , i.e., $\alpha_i \cdot \bar{\beta}_i \leq 0$. Moreover, denoting by γ_i the oriented closed curve given by the join of the arc $\mathbf{c}_i := \mathbf{c}_{]s_{i-1}, \bar{s}_i]}$ plus the segment of P_h from $\mathbf{c}(\bar{s}_i)$ to $\mathbf{c}(s_{i-1})$, the index of γ_i on the open set U_i enclosed by γ_i is equal to the sign of α_i , see Figure 1. We thus have

$$\omega(\bar{s}_i) - \omega(s_{i-1}) = \alpha_i - \bar{\beta}_i, \quad \alpha_i \neq 0, \quad \alpha_i \cdot \bar{\beta}_i \leq 0.$$

Letting now $f_i(s) := \omega(s) - \omega(s_{i-1})$, we get $f_i(s_{i-1}) < \alpha_i$ and $f_i(\bar{s}_i) \geq \alpha_i$, when $\alpha_i > 0$ and $\bar{\beta}_i \leq 0$, whereas $f_i(s_{i-1}) > \alpha_i$ and $f_i(\bar{s}_i) \leq \alpha_i$, when $\alpha_i < 0$ and $\bar{\beta}_i \geq 0$. Therefore, using that ω is a function with bounded variation, we find $\tilde{s}_i \in]s_{i-1}, \bar{s}_i[$ such that either $\alpha_i = t_i f_i(\tilde{s}_i+) + (1 - t_i) f_i(\tilde{s}_i-)$ for some $t_i \in [0, 1]$, if \tilde{s}_i is a Jump point of f_i , or $\alpha_i = f_i(\tilde{s}_i)$, otherwise. When $\alpha_i = 0$, we clearly have $\alpha_i = f_i(0)$.

Recall that $\omega(s_0) = 0$ and $\alpha_i := \angle \mathbf{t}(s_{i-1})\mathbf{v}_i$. Setting $\beta_i := \angle \mathbf{t}(s_i)\mathbf{v}_i$, by the previous discussion based on Lemma 6.5, we also get:

$$\omega(s_j) - \omega(s_{j-1}) = \alpha_j - \beta_j \quad \forall j = 1, \dots, n.$$

Moreover, for $j = 1, \dots, n-1$, the oriented turning angle of the polygonal P_h at the corner point $\mathbf{c}(s_j)$ is equal to $\alpha_{j+1} - \beta_j$. We thus have $\omega_h(s) = \alpha_1$ if $s \in]s_0, s_1[$, whereas if $s \in]s_{i-1}, s_i[$, and $i = 2, \dots, n$, then

$$\omega_h(s) = \alpha_1 + \sum_{j=1}^{i-1} (\alpha_{j+1} - \beta_j) = \alpha_i + \sum_{j=1}^{i-1} (\alpha_j - \beta_j) = \alpha_i + \sum_{j=1}^{i-1} (\omega(s_j) - \omega(s_{j-1})) = \alpha_i + \omega(s_{i-1}).$$

We thus conclude that for each $i = 1, \dots, n$ there exists $\tilde{s}_i \in]s_{i-1}, s_i[$ and $t_i \in [0, 1]$ such that

$$\omega_h(s) = t_i \omega(\tilde{s}_i+) + (1 - t_i) \omega(\tilde{s}_i-) \quad \forall s \in]s_{i-1}, s_i[.$$

The above property, that actually expresses the parallelism condition in term of angle functions, implies that ω_h is a competitor to the computation of the essential variation of ω , whence $\text{Var}(\omega_h) \leq \text{Var}(\omega)$. By the weak-* BV convergence of ω_h to ω , which ensures that $\text{Var}(\omega) \leq \liminf_h \text{Var}(\omega_h)$, we obtain the strict convergence $\text{Var}(\omega_h) \rightarrow \text{Var}(\omega)$.

PROOF OF PROPOSITION 6.4: By [17, Lemma 1], the curve \mathbf{c} being one-sidedly smooth, it consists of finitely many simple arcs. Therefore, we clearly may and do assume that \mathbf{c} is a simple arc.

We let $P_h : [0, L] \rightarrow \mathcal{M}$ be parameterized with constant velocity on each interval $]s_{i-1}, s_i[$, in such a way that $P_h(s_i) = \mathbf{c}(s_i)$ for each i . Notice that by the uniform convergence of P_h to \mathbf{c} , for h sufficiently large the subset of \mathcal{M} enclosed by the curves \mathbf{c} and P_h is a simply-connected domain U_h of \mathcal{M} with small surface area. In particular, U_h can be equipped with an orientation, that is inherited by the tangent space $T_{\mathbf{c}(s)}\mathcal{M}$ along the curve. If $\mathbf{v}_0, \mathbf{v}_1 \in T_{\mathbf{c}(s)}\mathcal{M}$ are non-trivial vectors, we shall thus denote by $\angle \mathbf{v}_0\mathbf{v}_1$ the oriented angle in $T_{\mathbf{c}(s)}\mathcal{M}$ from \mathbf{v}_0 to \mathbf{v}_1 , for any $s \in [0, L]$. The rest of the proof is divided into three steps.

STEP 1: We prove property (a) in Proposition 6.4.

Choose h large enough so that $\mu_{\mathbf{c}}(P_h) \leq \varepsilon_0$, where the positive constant $\varepsilon_0 > 0$ is given by Lemma 6.5 in correspondence to the curve \mathbf{c} . We are now in a situation similar to the one described in the planar case.

For $i = 1, \dots, n$, letting $\alpha_i := \angle \mathbf{t}(s_{i-1})\mathbf{t}_h(s_{i-1}+)$, if $\alpha_i \neq 0$, we choose the *first* parameter \bar{s}_i in the interval $]s_{i-1}, s_i]$ such that $\mathbf{c}(\bar{s}_i) = P_h(\hat{s}_i)$ for some $\hat{s}_i \in]s_{i-1}, s_i]$, and let $\bar{\beta}_i := \angle \mathbf{t}(\bar{s}_i)\mathbf{t}_h(\hat{s}_i-)$.

By Lemma 6.5, the angle $\bar{\beta}_i$ cannot have the same sign as α_i , i.e., $\alpha_i \cdot \bar{\beta}_i \leq 0$, see Figure 1. Also, denoting by $\tilde{\gamma}_i$ the oriented closed curve given by the join of the arc $\mathbf{c}_i := \mathbf{c}_{]s_{i-1}, \bar{s}_i]}$ plus the geodesic arc of P_h reversely oriented from $\mathbf{c}(\bar{s}_i)$ to $\mathbf{c}(s_{i-1})$, the index of $\tilde{\gamma}_i$ on the open set \tilde{U}_i enclosed by $\tilde{\gamma}_i$ is equal to ± 1 , in concordance with the sign of the initial angle α_i , see Figure 1.

Therefore, the Gauss-Bonnet theorem 6.1 yields:

$$\begin{cases} \Theta(\bar{s}_i) - \Theta(s_{i-1}) = \alpha_i - \bar{\beta}_i - \int_{\tilde{U}_i} \mathbf{K} dA & \text{if } \alpha_i > 0 \\ \Theta(\bar{s}_i) - \Theta(s_{i-1}) = \alpha_i - \bar{\beta}_i + \int_{\tilde{U}_i} \mathbf{K} dA & \text{if } \alpha_i < 0 \end{cases} \quad (6.2)$$

where, we recall, $\alpha_i \cdot \bar{\beta}_i \leq 0$.

We first consider the easier case when $\mathbf{K} \leq 0$. Letting $\bar{\alpha}_i := \alpha_i - \int_{\bar{U}_i} \mathbf{K} dA$, if $\alpha_i > 0$, and $\bar{\alpha}_i := \alpha_i + \int_{\bar{U}_i} \mathbf{K} dA$, if $\alpha_i < 0$, in both cases the sign of $\bar{\alpha}_i$ is concordant with the sign of α_i , and definitely:

$$\Theta(\bar{s}_i) - \Theta(s_{i-1}) = \bar{\alpha}_i - \bar{\beta}_i, \quad \bar{\alpha}_i \neq 0, \quad \bar{\alpha}_i \cdot \bar{\beta}_i \leq 0.$$

Denoting $f_i(s) := \Theta(s) - \Theta(s_{i-1})$, we get $f_i(s_{i-1}) < \bar{\alpha}_i$ and $f_i(\bar{s}_i) \geq \bar{\alpha}_i$, when $\bar{\alpha}_i > 0$ and $\bar{\beta}_i \leq 0$, whereas $f_i(s_{i-1}) > \bar{\alpha}_i$ and $f_i(\bar{s}_i) \leq \bar{\alpha}_i$, when $\bar{\alpha}_i < 0$ and $\bar{\beta}_i \geq 0$. Therefore, recalling that the angle function Θ has bounded variation, and setting $\Theta_{h,i} := \bar{\alpha}_i + \Theta(s_{i-1})$, in both cases we find $\tilde{s}_i \in]s_{i-1}, \bar{s}_i[$ such that

$$\Theta_{h,i} = t_i \Theta(\tilde{s}_i+) + (1 - t_i) \Theta(\tilde{s}_i-)$$

for some $t_i \in [0, 1]$, if \tilde{s}_i is a Jump point of Θ , or $\Theta_{h,i} = \Theta(\tilde{s}_i)$, otherwise. When $\alpha_i = 0$, we clearly have $\alpha_i = f_i(0)$, and we obviously choose $\Theta_{h,i} := \Theta(s_{i-1})$.

In order to treat the general case, where the Gauss curvature \mathbf{K} may possibly take positive values, in Step 2 we shall prove the following:

Claim. *For each $i = 1, \dots, n$, we can find a coefficient $\lambda_i \in [-1, 1]$ such that with $\bar{\alpha}_i := \alpha_i + \lambda_i \int_{\bar{U}_i} \mathbf{K} dA$ and $\Theta_{h,i} := \bar{\alpha}_i + \Theta(s_{i-1})$, we have*

$$\Theta_{h,i} = t_i \Theta(\tilde{s}_i+) + (1 - t_i) \Theta(\tilde{s}_i-)$$

for some $\tilde{s}_i \in [s_{i-1}, s_i[$ and $t_i \in [0, 1]$.

Setting in fact

$$\tilde{\Theta}_h(s) := \Theta_{h,i} \quad \text{if } s \in]s_{i-1}, s_i[, \quad \forall i = 1, \dots, n \quad (6.3)$$

property (a) in Proposition 6.4 holds true.

STEP 2: We prove the Claim, by generalizing the previous argument. Denote for simplicity

$$\Delta\Theta_i := \Theta(\bar{s}_i) - \Theta(s_{i-1}), \quad K_i := \int_{\bar{U}_i} \mathbf{K} dA.$$

If $K_i \leq 0$, we argue exactly as in Step 1, so that we now assume $K_i > 0$.

We first consider the case $\alpha_i > 0$ and $\bar{\beta}_i \leq 0$, so that the first equation in (6.2) becomes

$$\Delta\Theta_i = \alpha_i - K_i - \bar{\beta}_i \quad (6.4)$$

and we can write $\alpha_i = \lambda K_i$ for some $\lambda > 0$. We now distinguish among the possible values of the term $\Delta\Theta_i$.

- i) If $\Delta\Theta_i = 0$, then by (6.4) we get $\lambda \in]0, 1[$ and $\bar{\beta}_i = (\lambda - 1) K_i$. Letting $\bar{\alpha}_i := \alpha_i - \lambda K_i$, we clearly have $\bar{\alpha}_i = 0 = f_i(0)$, where $f_i(s)$ is defined as in Step 1.
- ii) If $\Delta\Theta_i > 0$, then $\bar{\beta}_i = -\mu K_i$ for some $\mu \geq 0$, so that (6.4) becomes $\Delta\Theta_i = (\lambda + \mu - 1) K_i$, whence $\lambda + \mu > 1$. If $\lambda \geq 1$, letting $\bar{\alpha}_i := \alpha_i - K_i$, we have

$$f_i(0) \leq \bar{\alpha}_i, \quad f_i(s_{i-1}) = \Delta\Theta_i = \bar{\alpha}_i + \mu K_i \geq \bar{\alpha}_i.$$

If $\lambda \in]0, 1[$, instead, letting $\bar{\alpha}_i := \alpha_i - \lambda K_i$ we again have $\bar{\alpha}_i = 0 = f_i(0)$.

- iii) If $\Delta\Theta_i < 0$, by (6.4) we have $\alpha_i - \bar{\beta}_i < K_i$, hence $\lambda \in [0, 1[$, so that we again let $\bar{\alpha}_i := \alpha_i - \lambda K_i = f_i(0)$.

We now deal with the case $\alpha_i < 0$ and $\bar{\beta}_i \geq 0$, so that the second equation in (6.2) becomes

$$\Delta\Theta_i = \alpha_i + K_i - \bar{\beta}_i \quad (6.5)$$

and hence this time $\alpha_i = -\lambda K_i$ for some $\lambda > 0$.

- i) If $\Delta\Theta_i = 0$, then by (6.5) we get $\lambda \in]0, 1[$ and $\bar{\beta}_i = (1 - \lambda) K_i$. Letting $\bar{\alpha}_i := \alpha_i + \lambda K_i$, we have $\bar{\alpha}_i = 0 = f_i(0)$.

ii) If $\Delta\Theta_i < 0$, there exist $\mu \geq 0$ such that $\bar{\beta}_i = \mu K_i$, so that (6.5) becomes $\Delta\Theta_i = -(\lambda + \mu - 1)K_i$, whence $\lambda + \mu > 1$. If $\lambda \geq 1$, letting $\bar{\alpha}_i := \alpha_i + K_i$, this time we have

$$f_i(0) \geq \bar{\alpha}_i, \quad f_i(s_{i-1}) = \Delta\Theta_i = \bar{\alpha}_i - \mu K_i \leq \bar{\alpha}_i.$$

If $\lambda \in]0, 1[$, letting $\bar{\alpha}_i := \alpha_i + \lambda K_i$ we again have $\bar{\alpha}_i = 0 = f_i(0)$.

iii) If $\Delta\Theta_i > 0$, by (6.5) we have $(1 - \lambda)K_i > \bar{\beta}_i$, hence $\lambda \in [0, 1[$, so that we again let $\bar{\alpha}_i := \alpha_i + \lambda K_i$.

Finally, when $\alpha_i = 0$, we have $\alpha_i = f_i(0)$, and we choose $\bar{\alpha}_i := 0$.

Recalling that $\bar{s}_i \in]s_{i-1}, s_i]$, and setting $\Theta_{h,i} := \bar{\alpha}_i + \Theta(s_{i-1})$, the proof of the Claim is completed as in the easier case $\mathbf{K} \leq 0$ previously considered in Step 1.

STEP 3: We now check property (b) in Proposition 6.4. Denoting $\beta_j := \angle \mathbf{t}(s_j) \mathbf{t}_h(s_j -)$, again by Lemma 6.5 and Theorem 6.1, for each $j = 1, \dots, n$ we have

$$\Theta(s_j) - \Theta(s_{j-1}) = \alpha_j - \beta_j - \int_{U_j} \mathbf{i}_{\Gamma_j} \mathbf{K} dA. \quad (6.6)$$

In this formula, Γ_j is the oriented closed curve given by the join of the arc of \mathbf{c} from $\mathbf{c}(s_{j-1})$ to $\mathbf{c}(s_j)$ and the geodesic arc of P_h from $\mathbf{c}(s_j)$ to $\mathbf{c}(s_{j-1})$, and \mathbf{i}_{Γ_j} is the index of the curve Γ_j on \mathcal{M} . Also, U_j is the open subset of \mathcal{M} enclosed by the curve Γ_j .

Notice that by our construction, see Figure 1, we deduce that the index \mathbf{i}_{Γ_j} is well-defined and actually $\mathbf{i}_{\Gamma_j} = \pm 1$ in the interior of each component of U_j , whereas $\mathbf{i}_{\Gamma_j} = 0$ outside U_j . Since moreover \mathcal{M} is assumed smooth and compact, the Gauss curvature \mathbf{K} is uniformly bounded on \mathcal{M} . By (6.6), we thus get:

$$\left| \int_{U_j} \mathbf{i}_{\Gamma_j} \mathbf{K} dA \right| \leq \int_{U_j} |\mathbf{K}| dA \leq \|\mathbf{K}\|_\infty \cdot \text{meas}(U_j) < \infty. \quad (6.7)$$

Now, for $j = 1, \dots, n-1$, the oriented turning angle of the polygonal P_h at the corner point $\mathbf{c}(s_j)$ is equal to $\alpha_{j+1} - \beta_j$, whereas by (6.3) we correspondingly get:

$$\Theta_{h,j+1} - \Theta_{h,j} = (\alpha_{j+1} - \beta_j) - \lambda_{j+1} \int_{\tilde{U}_{j+1}} \mathbf{K} dA + \lambda_j \int_{\tilde{U}_j} \mathbf{K} dA - \int_{U_j} \mathbf{i}_{\Gamma_j} \mathbf{K} dA,$$

where $\lambda_j \in [-1, 1]$, by our Claim, and $\tilde{U}_j = \emptyset$, if $\alpha_j = 0$. By (6.7) we can thus estimate:

$$|\alpha_{j+1} - \beta_j| \leq |\Theta_{h,j+1} - \Theta_{h,j}| + \|\mathbf{K}\|_\infty \cdot (\text{meas}(\tilde{U}_{j+1}) + \text{meas}(\tilde{U}_j) + \text{meas}(U_j)). \quad (6.8)$$

We now observe that the Jumps of the piecewise constant function Θ_h are the turning angles $(\alpha_{j+1} - \beta_j)$, whereas by (6.3), the corresponding Jumps of the modified angle function $\tilde{\Theta}_h$ are equal to $(\Theta_{h,j+1} - \Theta_{h,j})$. By summing on $j = 1, \dots, n-1$ in (6.8), and using that $\tilde{U}_j \subset U_j$ for each j , we then infer:

$$\text{Var}(\Theta_h) \leq \text{Var}(\tilde{\Theta}_h) + \varepsilon_h, \quad \varepsilon_h := 3 \|\mathbf{K}\|_\infty \cdot \sum_{j=1}^n \text{meas}(U_j).$$

Finally, by the uniform convergence of P_h to \mathbf{c} , we deduce that $\varepsilon_h \rightarrow 0$ as $h \rightarrow \infty$, whence property (b) in Proposition 6.4 holds true. \square

7 Curves into Riemannian surfaces

In this section, we extend the previous results to the more general case of curves into Riemannian surfaces, i.e., 2-dimensional Riemannian manifolds $(\tilde{\mathcal{M}}, g)$.

We assume that $\tilde{\mathcal{M}}$ is smooth (at least of class C^3), closed, and compact. Recall that we can always find a smooth isometric embedding $F : \tilde{\mathcal{M}} \hookrightarrow \mathbb{R}^N$ of $\tilde{\mathcal{M}}$ into a surface $\mathcal{M} = F(\tilde{\mathcal{M}})$ immersed in the N -dimensional

Euclidean space, for some $N \geq 4$. Since the total intrinsic curvature of piecewise smooth curves involves the geodesic curvature and the turning angles at corner points, we do not need $\widetilde{\mathcal{M}}$ to be oriented.

TOTAL INTRINSIC CURVATURE. We first extend Definition 2.5, by saying that the *total intrinsic curvature* of any curve γ in $\widetilde{\mathcal{M}}$ is

$$\text{TC}_{\widetilde{\mathcal{M}}}(\gamma) := \lim_{\varepsilon \rightarrow 0^+} \sup \{ \mathbf{k}_{\widetilde{\mathcal{M}}}(\tilde{P}) \mid \tilde{P} \in \Sigma_\varepsilon(\gamma) \}$$

where $\Sigma_\varepsilon(\gamma)$ is the class of polygons \tilde{P} in $\widetilde{\mathcal{M}}$ inscribed in γ and with modulus $\mu_\gamma(\tilde{P}) < \varepsilon$, and $\mathbf{k}_{\widetilde{\mathcal{M}}}(\tilde{P})$ is the rotation of \tilde{P} , both modulus and rotation being defined as in the case of surfaces \mathcal{M} in \mathbb{R}^N .

RESULTS. We extend the representation formula in Theorem 6.3, by the following:

Theorem 7.1 *Let $\widetilde{\mathcal{M}}$ be any smooth, closed, and compact Riemannian surface. For every rectifiable curve γ in $\widetilde{\mathcal{M}}$ with finite total intrinsic curvature, we have*

$$\text{TC}_{\widetilde{\mathcal{M}}}(\gamma) = \mathcal{F}(\mathbf{t})$$

where the energy functional $\mathcal{F}(\mathbf{t})$ is defined by (4.4) in correspondence to the tangent indicatrix $\mathbf{t} = \dot{\mathbf{c}}$ of $\mathbf{c} = F \circ \gamma$, and F is any isometric embedding of $\widetilde{\mathcal{M}}$ as above.

In order to prove Theorem 7.1, we shall first introduce geodesic polar coordinates, and write the local expression (7.4) of the geodesic curvature of a smooth curve γ in $\widetilde{\mathcal{M}}$. It turns out that length, angles and geodesics are preserved by isometries. Letting then $\mathbf{c} := F \circ \gamma$, we shall compute the geodesic curvature \mathfrak{K}_g of \mathbf{c} in $\mathcal{M} := F(\widetilde{\mathcal{M}})$, an immersed surface in \mathbb{R}^N , showing that \mathfrak{K}_g agrees with the intrinsic local expression (7.4), and hence that the latter does not depend on the choice of isometric embedding. In a similar way, we will check that the rotation of a polygonal \tilde{P} in $\widetilde{\mathcal{M}}$ is an intrinsic notion.

As a consequence, we readily obtain:

Proposition 7.2 *For any piecewise smooth curve γ in $\widetilde{\mathcal{M}}$, we have*

$$\text{TC}_{\widetilde{\mathcal{M}}}(\gamma) = \text{TC}_{\mathcal{M}}(\mathbf{c}) \quad \text{if} \quad \mathbf{c} := F \circ \gamma$$

independently of the chosen isometric embedding F .

Moreover, all the previous results obtained for curves \mathbf{c} in surfaces \mathcal{M} of \mathbb{R}^N extend to curves γ in a Riemannian surface $(\widetilde{\mathcal{M}}, g)$. In fact, it suffices to work with $\mathbf{c} = F \circ \gamma$ for any isometric embedding F , and to use standard arguments based on local geodesic coordinates and partition of unity.

For this purpose, we shall focus in particular on the validity of the compactness theorem 4.1. In fact, by a quick inspection it turns out that the fundamental inequality (4.3) is the unique point of the previous theory where we used non-intrinsic quantities.

On account of Proposition 7.2 and Theorem 6.3, we thus conclude with the validity of Theorem 7.1.

GEODESIC POLAR COORDINATES. Following e.g. [8, Sec. 4.12], on small open domains U of $\widetilde{\mathcal{M}}$ homeomorphic to a disk, we introduce geodesic polar coordinates $ds^2 = dr^2 + g(r, \phi) d\phi^2$, where g is a non-negative smooth function on U . We shall denote by $f_{,r}$, $f_{,\phi}$, $f_{,rr}$, $f_{,r\phi}$, and $f_{,\phi\phi}$ the partial first and second derivatives of a function $f(r, \phi)$ on U . The coefficient g of the Riemannian metric satisfies

$$\lim_{r \rightarrow 0} g = 0, \quad \lim_{r \rightarrow 0} (\sqrt{g})_{,r} = 1 \quad \forall \phi \tag{7.1}$$

compare [7, Sec. 4.6]. Also, in coordinates the non-trivial Christoffel coefficients of the Levi-Civita connection ∇_g of the Riemannian metric are

$$\Gamma_{22}^1 = -\frac{1}{2} g_{,r}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2g} g_{,r}, \quad \Gamma_{22}^2 = \frac{1}{2g} g_{,\phi\phi}. \tag{7.2}$$

Let $\gamma : I \rightarrow \widetilde{\mathcal{M}}$ be a smooth and regular curve parameterized by arc-length. Assume that $\gamma(\widetilde{I}) \subset U$ for some open interval $\widetilde{I} \subset I$. Also, we choose the pole of the coordinates not lying on the trace $\gamma(\widetilde{I})$ of the curve. Therefore, there exists a positive real constant c such that $g(r, \phi) \geq c > 0$ for every $(r, \phi) \in \gamma(\widetilde{I})$.

In coordinates, we thus have $\gamma(s) = (r(s), \phi(s))$ for some smooth functions $r(s)$ and $\phi(s)$ satisfying $\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_g = \dot{r}^2 + g(r, \phi) \dot{\phi}^2 = 1$ for every $s \in \widetilde{I}$. Therefore, the unit tangent vector and unit conormal are

$$\dot{\gamma} = (\dot{r}, \dot{\phi}), \quad \dot{\gamma}^\perp := (-g^{1/2} \dot{\phi}, g^{-1/2} \dot{r}).$$

The acceleration vector $\nabla_{\dot{\gamma}} \dot{\gamma}$ can be written in components as $(\nabla_{\dot{\gamma}} \dot{\gamma})^k = \ddot{\gamma}^k + \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j$, for $k = 1, 2$, so that in the previous local coordinates we get

$$(\nabla_{\dot{\gamma}} \dot{\gamma})^1 = \ddot{r} - \frac{1}{2} g_{,r} \dot{\phi}^2, \quad (\nabla_{\dot{\gamma}} \dot{\gamma})^2 = \ddot{\phi} + \frac{1}{g} g_{,r} \dot{r} \dot{\phi} + \frac{1}{2g} g_{,\phi} \dot{\phi}^2. \quad (7.3)$$

We have $\langle \nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_g = 0$, whence $\nabla_{\dot{\gamma}} \dot{\gamma} = \mathfrak{K}_g \dot{\gamma}^\perp$, where $\mathfrak{K}_g := \langle \nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}^\perp \rangle_g$ is the geodesic curvature of γ , so that $|\mathfrak{K}_g| = |\nabla_{\dot{\gamma}} \dot{\gamma}|_g$. This yields to the local expression:

$$\begin{aligned} \mathfrak{K}_g &= \sqrt{g} \left[-\dot{\phi} (\nabla_{\dot{\gamma}} \dot{\gamma})^1 + \dot{r} (\nabla_{\dot{\gamma}} \dot{\gamma})^2 \right] \\ &= \sqrt{g} \left[(\dot{r} \ddot{\phi} - \dot{\phi} \ddot{r}) + \frac{1}{2} \left(g_{,r} \dot{\phi}^3 + 2 \frac{g_{,r}}{g} \dot{r}^2 \dot{\phi} + \frac{g_{,\phi}}{g} \dot{r} \dot{\phi}^2 \right) \right]. \end{aligned} \quad (7.4)$$

Example 7.3 If e.g. $\widetilde{\mathcal{M}} = \mathcal{M} = \mathcal{S}^2$ and $g(r, \phi) = \sin^2 r$, with $r = \theta$ and $\phi = \varphi$, using that

$$\Gamma_{22}^1 = -\sin \theta \cos \theta, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \cot \theta, \quad \Gamma_{22}^2 = 0$$

we recover the formula (3.10) for \mathfrak{K}_g .

Remark 7.4 We also recall that if ω denotes the angle between $\dot{\gamma}$ and the fixed direction $(1, 0)$, we find

$$\tan \omega = \sqrt{g} \frac{\dot{\phi}}{\dot{r}}, \quad \dot{\omega} = \mathfrak{K}_g - (\sqrt{g})_{,r} \dot{\phi}.$$

Therefore, if the curve γ parameterizes the positively oriented boundary of the smooth domain U , by Stokes theorem, compare [8, Sec. 4.12], one has

$$\oint_{\partial U} (\sqrt{g})_{,r} \dot{\phi} ds = - \int_U \mathbf{K} dA, \quad \mathbf{K} = -\frac{1}{\sqrt{g}} (\sqrt{g})_{,rr} \quad (7.5)$$

where \mathbf{K} is the Gauss curvature of (\mathcal{M}, g) , yielding to the local formula of Gauss-Bonnet theorem:

$$\int_U \mathbf{K} dA = 2\pi - \oint_{\partial U} \mathfrak{K}_g ds. \quad (7.6)$$

EMBEDDINGS. Given an isometric embedding $F : \widetilde{\mathcal{M}} \hookrightarrow \mathcal{M} \subset \mathbb{R}^N$, we let \bar{g} and $\bar{\nabla}$ denote the (Gaussian) metric and (Levi-Civita) connection induced by the Euclidean metric of \mathbb{R}^N on \mathcal{M} . The pull-back of \bar{g} and of $\bar{\nabla}$ through F agree with the metric g and Levi-Civita connection ∇_g on \mathcal{M} , respectively. Therefore, in local coordinates as above, writing $F = F(r, \phi) : U \rightarrow \mathbb{R}^N$, we have

$$F_{,r} \bullet F_{,r} = 1, \quad F_{,r} \bullet F_{,\phi} = 0, \quad F_{,\phi} \bullet F_{,\phi} = g. \quad (7.7)$$

By computing the partial second derivatives, we thus obtain the six formulas for the scalar products in \mathbb{R}^N

$$\begin{aligned} F_{,r} \bullet F_{,rr} &= 0, & F_{,r} \bullet F_{,r\phi} &= 0, & F_{,r} \bullet F_{,\phi\phi} &= -\frac{1}{2} g_{,r}, \\ F_{,\phi} \bullet F_{,rr} &= 0, & F_{,\phi} \bullet F_{,r\phi} &= \frac{1}{2} g_{,r}, & F_{,\phi} \bullet F_{,\phi\phi} &= \frac{1}{2} g_{,\phi}. \end{aligned} \quad (7.8)$$

Letting $\mathbf{c}(s) := F \circ \gamma(s)$, where $s \in \tilde{I}$, the unit tangent vector and conormal corresponding to $\dot{\gamma}$ and $\dot{\gamma}^\perp$ take the expression

$$\mathbf{t} = \dot{r} F_{,r} + \dot{\phi} F_{,\phi}, \quad \mathbf{u} = -g^{1/2} \dot{\phi} F_{,r} + g^{-1/2} \dot{r} F_{,\phi}. \quad (7.9)$$

The curvature vector of the curve \mathbf{c} in \mathbb{R}^N then becomes

$$\mathbf{k} = \dot{\mathbf{t}} = \ddot{r} F_{,r} + \ddot{\phi} F_{,\phi} + \dot{r}^2 F_{,rr} + 2 \dot{r} \dot{\phi} F_{,r\phi} + \dot{\phi}^2 F_{,\phi\phi}. \quad (7.10)$$

We compute the geodesic curvature of \mathbf{c} in \mathcal{M} through the formula $\mathfrak{K}_g := \dot{\mathbf{t}} \bullet \mathbf{u}$, obtaining by (7.7) and (7.8)

$$\begin{aligned} \mathfrak{K}_g &= -g^{1/2} \dot{\phi} \left(\ddot{r} + \dot{\phi}^2 \left(-\frac{1}{2} g_{,r} \right) \right) + g^{-1/2} \dot{r} \left(g \ddot{\phi} + 2 \dot{r} \dot{\phi} \left(\frac{1}{2} g_{,r} \right) + \dot{\phi}^2 \left(\frac{1}{2} g_{,\phi} \right) \right) \\ &= \sqrt{g} \left[(\dot{r} \ddot{\phi} - \dot{\phi} \ddot{r}) + \frac{1}{2} \left(g_{,r} \dot{\phi}^3 + 2 \frac{g_{,r}}{g} \dot{r}^2 \dot{\phi} + \frac{g_{,\phi}}{g} \dot{r} \dot{\phi}^2 \right) \right] \end{aligned}$$

which agrees with the local expression (7.4) for the geodesic curvature of γ in $\tilde{\mathcal{M}}$.

Remark 7.5 If γ is a geodesic in $\tilde{\mathcal{M}}$, the curve $\mathbf{c} = F \circ \gamma$ is a geodesic in \mathcal{M} , whence the curvature vector $\dot{\mathbf{t}}$ is orthogonal to both $F_{,r}$ and $F_{,\phi}$. By (7.10), (7.7) and (7.8) we have

$$0 = \dot{\mathbf{t}} \bullet F_{,r} = \ddot{r} - \frac{1}{2} g_{,r} \dot{\phi}^2, \quad 0 = \dot{\mathbf{t}} \bullet F_{,\phi} = g \ddot{\phi} + g_{,r} \dot{r} \dot{\phi} + \frac{1}{2} g_{,\phi} \dot{\phi}^2$$

and hence for a geodesic \mathbf{c} one recovers the local expressions of the equations $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ from (7.3):

$$\ddot{r} = \frac{1}{2} g_{,r} \dot{\phi}^2, \quad \ddot{\phi} = -\frac{1}{2g} (2g_{,r} \dot{r} \dot{\phi} + g_{,\phi} \dot{\phi}^2). \quad (7.11)$$

ROTATION OF POLYGONALS. We now check that the rotation of a polygonal \tilde{P} in $\tilde{\mathcal{M}}$ is an intrinsic notion. Assume in fact that two geodesic arcs γ_i of \tilde{P} meet at a point (r_0, ϕ_0) in U . Denoting by $(\dot{r}_i, \dot{\phi}_i)$ the direction of the arc γ_i at the point (r_0, ϕ_0) , where $i = 1, 2$, the rotation of \tilde{P} at (r_0, ϕ_0) is equal to

$$\arccos \langle (\dot{r}_1, \dot{\phi}_1), (\dot{r}_2, \dot{\phi}_2) \rangle_g = \arccos (\dot{r}_1 \dot{r}_2 + g(r_0, \phi_0) \dot{\phi}_1 \dot{\phi}_2).$$

On the other hand, if $P = F(\tilde{P})$ is the corresponding polygonal in $\mathcal{M} = F(\tilde{\mathcal{M}})$, the direction of the geodesic arc $\bar{\gamma}_i := F \circ \gamma_i$ at the point $F(r_0, \phi_0)$ is

$$\mathbf{v}_i = \dot{r}_i F_{,r}(r_0, \phi_0) + \dot{\phi}_i F_{,\phi}(r_0, \phi_0)$$

and hence, using (7.7), the corresponding rotation angle is

$$\arccos(\mathbf{v}_1 \bullet \mathbf{v}_2) = \arccos(\dot{r}_1 \dot{r}_2 + g(r_0, \phi_0) \dot{\phi}_1 \dot{\phi}_2).$$

Therefore, the rotation of \tilde{P} is equal to the rotation of P , i.e., $\mathbf{k}_{\tilde{\mathcal{M}}}(\tilde{P}) = \mathbf{k}_{\mathcal{M}}(P)$, independently of the chosen isometric embedding $F : \tilde{\mathcal{M}} \hookrightarrow \mathcal{M} \subset \mathbb{R}^N$.

THE COMPACTNESS THEOREM. Going back to Theorem 4.1 on the $W^{1,1}$ compactness of the transport vector fields, it turns out that the fundamental inequality (4.3) actually involves a constant factor $C_{\mathcal{M}}$ which depends on the surface \mathcal{M} , see Remark 5.1. Therefore, in the case of curves in a Riemannian surface $(\tilde{\mathcal{M}}, g)$, the constant $C_{\mathcal{M}}$ definitely depends on the chosen embedding F .

However, since $\tilde{\mathcal{M}}$ is assumed to be of class C^3 and compact, all the derivatives of F up to the third order are equibounded on U , independently of the local chart on $\tilde{\mathcal{M}}$. Moreover, with the previous notation, we may and do assume that $g(r, \phi) \geq c > 0$ on $\gamma(\tilde{I})$, where the positive constant (that depends on the choice of the poles of the polar geodesic coordinates) is independent of the normal neighborhood of the partition of $\tilde{\mathcal{M}}$, by the smoothness and compactness of $\tilde{\mathcal{M}}$. Therefore, if \tilde{P} is a polygonal of $\tilde{\mathcal{M}}$ inscribed in γ , by choosing the modulus $\mu_\gamma(\tilde{P})$ sufficiently small, it turns out that $g(r, \phi) \geq c > 0$ for each (r, ϕ) in \tilde{P} . Setting then $P := F \circ \tilde{P}$, the above properties imply that (outside the corner points):

- i) both the normal curvatures \mathfrak{K}_n of the polygonals P in \mathcal{M} , and their derivatives w.r.t. the arc-length parameter, are equibounded by a constant only depending on $\mathcal{M} := F(\widetilde{\mathcal{M}})$;
- ii) if \mathbf{u} is the unit conormal of P , parameterized by arc-length, then both $|\dot{\mathbf{u}}|$ and $|\ddot{\mathbf{u}}|$ are equibounded by a constant only depending on $\mathcal{M} := F(\widetilde{\mathcal{M}})$, see Example 7.6.

By the above construction, we deduce that our compactness result continues to hold.

Example 7.6 We finally check that the local expressions of the arc-length derivatives $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ of the unit conormal to the curve $\mathbf{c} := F \circ \gamma$ do not depend on the second order derivatives of r and ϕ , when γ is a geodesic arc in $\widetilde{\mathcal{M}}$.

By formula (7.9), in fact, in general we obtain

$$\dot{\mathbf{u}} = a F_{,r} + b F_{,\phi} + c F_{,rr} + d F_{,r\phi} + e F_{,\phi\phi}$$

where

$$a := -\frac{1}{2\sqrt{g}}(g_{,r}\dot{r} + g_{,\phi}\dot{\phi})\dot{\phi} - \sqrt{g}\ddot{\phi}, \quad b := -\frac{1}{2g^{3/2}}(g_{,r}\dot{r} + g_{,\phi}\dot{\phi})\dot{r} + \frac{1}{\sqrt{g}}\ddot{r}$$

and

$$c := -\sqrt{g}\dot{r}\dot{\phi}, \quad d := \frac{1}{\sqrt{g}}(\dot{r}^2 - g\dot{\phi}^2), \quad e := \frac{1}{\sqrt{g}}\dot{r}\dot{\phi}.$$

When γ is a geodesic in \mathcal{M} , using the formulas (7.11) we can rewrite the coefficients a and b as

$$a = \frac{1}{2\sqrt{g}}g_{,r}\dot{r}\dot{\phi}, \quad b = \frac{1}{2g^{3/2}}(g_{,r}(g\dot{\phi}^2 - \dot{r}^2) - g_{,\phi}\dot{r}\dot{\phi}).$$

Therefore, when computing the second derivative $\ddot{\mathbf{u}}$, using again the formulas (7.11) it turns out that its local expression only depends on the first derivatives of (r, ϕ) and on the partial derivatives of g and F up to the third order, where, we recall, $g(r, s) \geq c > 0$ along the given geodesic arc γ , as required.

8 Development of curves

The original idea of parallel transport by Tullio Levi-Civita involves the concept of *development* of a curve on a surface. If e.g. $\mathcal{M} = \mathcal{S}^2$, it corresponds to drawing in a plane the points of the trace of the oriented curve in \mathcal{S}^2 as the 2-sphere rolls without slipping or spinning in the plane, while staying tangent to the plane at the points of the curve. The above construction implies that the scalar curvature of the developed curve on \mathbb{R}^2 is equal to the modulus of the geodesic curvature of the given curve in \mathcal{S}^2 , see Example 8.2.

In this final section, we analyze the relationship between the definition of total intrinsic curvature and the notion of development of a smooth curve, see Proposition 8.1. We point out that similar arguments, based on considering iterations of the development of the “complete tangent indicatrix”, are proposed by Reshetnyak [17] as a way to treat the “curvatures” of an irregular curve in \mathbb{R}^N .

DEVELOPMENT OF CURVES. Following e.g. [7], if $\gamma : I \rightarrow \mathcal{M}$ is a regular, smooth, and simple curve on a surface $\mathcal{M} \subset \mathbb{R}^3$, and $\dot{\mathbf{n}}(s) \neq 0$, where, we recall, $\mathbf{n}(s)$ is the unit normal $\mathbf{n}(s) := \dot{\gamma}(s)/\|\dot{\gamma}(s)\|$, then the *envelope of the tangent planes* is the ruled surface Σ parameterized by

$$X(s, v) := \gamma(s) + v \frac{\mathbf{n}(s) \times \dot{\mathbf{n}}(s)}{|\dot{\mathbf{n}}(s)|}$$

that in the case $\mathcal{M} = \mathcal{S}^2$ clearly becomes $X(s, v) := \gamma(s) + v \mathbf{u}(s)$. Around the trace of the curve, the ruled surface Σ has zero Gauss curvature, and hence, by Minding’s theorem, it is locally isometric to a planar domain. The parallel transport of tangent fields $X(s)$ along the curve is the same, when considering γ either as a curve on \mathcal{M} or as a curve on Σ . In particular, when $X(s) = \mathbf{t}(s)$, one can use either local coordinates on \mathcal{M} or on Σ in order to obtain the geodesic curvature \mathfrak{K}_g of the curve γ . As a consequence, the parallel transport can be computed locally by pulling back the parallel transport along the development of the curve on the plane \mathbb{R}^2 , see (8.1).

Moreover, we can define a tubular neighborhood (a strip) Σ of the envelope of the tangent planes to \mathcal{M} along γ , in such a way that Σ is a surface with Gauss curvature equal to zero. As a consequence, the total curvature $\text{TC}_\Sigma(\gamma)$ of γ as a curve in Σ is well-defined, according to Definition 2.5, by taking inscribed polygonals \tilde{P} in Σ with modulus sufficiently small (according to the width of the strip Σ , which actually depends on the maximum of the modulus of the geodesic curvature of the curve).

By means of the same vertexes as for \tilde{P} , we may correspondingly consider the polygonal P in \mathcal{M} inscribed in γ . However, in general the rotation of P in \mathcal{M} is different from the rotation of \tilde{P} in Σ , i.e.,

$$\mathbf{k}_\mathcal{M}(P) \neq \mathbf{k}_\Sigma(\tilde{P}).$$

In fact, if e.g. γ is a parallel of the 2-sphere $\mathcal{M} = \mathcal{S}^2$, and the vertexes of P are taken at equidistant points along γ , then the angles between \tilde{P} and γ are equal to the angles between the developed curve in \mathbb{R}^2 and the corresponding polygonal, whence they are smaller than the angles between P and γ , see Example 8.2.

A REPRESENTATION FORMULA. Notwithstanding, we shall see that the total curvature $\text{TC}_\Sigma(\gamma)$ of γ in the strip Σ can be computed by means of its development:

Proposition 8.1 *Let γ be a regular, smooth, and simple curve on a smooth surface $\mathcal{M} \subset \mathbb{R}^3$, with $\dot{\mathbf{n}} \neq 0$ everywhere. We have: $\text{TC}_\Sigma(\gamma) = \int_\gamma |\mathfrak{K}_g| ds$.*

Now, for any smooth curve γ as in Proposition 8.1, Theorem 2.3 says that the total curvature $\text{TC}_\mathcal{M}(\gamma)$ agrees with the integral on the right-hand side of the previous formula, whence we get:

$$\text{TC}_\mathcal{M}(\gamma) = \text{TC}_\Sigma(\gamma).$$

In particular, if $\{P_h\} \subset \mathcal{P}_\mathcal{M}(\gamma)$ satisfies $\mu_\gamma(P_h) \rightarrow 0$, and $\{\tilde{P}_h\}$ is (for h large enough) the corresponding sequence of inscribed polygonals in Σ , even if in general one has $\mathbf{k}_\mathcal{M}(P_h) \neq \mathbf{k}_\Sigma(\tilde{P}_h)$, we conclude that

$$\lim_{h \rightarrow \infty} \mathbf{k}_\mathcal{M}(P_h) = \lim_{h \rightarrow \infty} \mathbf{k}_\Sigma(\tilde{P}_h) = \int_\gamma |\mathfrak{K}_g| ds.$$

PROOF OF PROPOSITION 8.1: By a standard covering argument, we can reduce to the case in which the trace of γ is contained in a normal neighborhood U of Σ , and we equip U with geodesic polar coordinates where the pole does not lay on the trace of the curve γ . By the local formula (7.5) for the Gauss curvature, using that $\mathbf{K} = 0$ on Σ it turns out that the coefficient g of the Riemannian metric on Σ satisfies $(\sqrt{g})_{,rr} = 0$. By using the limits (7.1), this yields that $g(r, \phi) = r^2$, compare [7, Sec. 4.6].

We thus have $\gamma(s) = (r(s), \phi(s))$ for some smooth functions $r(s)$ and $\phi(s)$ satisfying $\dot{r}^2 + r^2 \dot{\phi}^2 = 1$, where $r(s) \geq c > 0$ for each $s \in I$. As a consequence, the acceleration vector in (7.3) takes the form

$$(\nabla_{\dot{\gamma}} \dot{\gamma})^1 = \ddot{r} - r \dot{\phi}^2, \quad (\nabla_{\dot{\gamma}} \dot{\gamma})^2 = \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi}$$

and the local expression (7.4) of the geodesic curvature \mathfrak{K}_g of γ becomes:

$$\mathfrak{K}_g = r (\dot{r} \ddot{\phi} - \dot{\phi} \ddot{r}) + (1 + \dot{r}^2) \dot{\phi}.$$

By Remark 7.4, the angle ω between $\dot{\gamma}$ and the fixed direction $(1, 0)$ in the vector bundle $T\Sigma$ satisfies

$$\tan \omega = r \frac{\dot{\phi}}{\dot{r}}, \quad \dot{\omega} = \mathfrak{K}_g - \dot{\phi},$$

whence $\frac{d}{ds}(\omega + \phi) = \mathfrak{K}_g$. By Proposition 3.1, this yields that $\Theta := \omega + \phi$ agrees (up to an additive constant) with the angle of the parallel transport along γ . Therefore, any curve $\tilde{\gamma}$ in \mathbb{R}^2 with unit tangent vector

$$\mathbf{T}(s) = (\cos \Theta(s), \sin \Theta(s)), \quad \Theta(s) := \omega(s) + \phi(s)$$

is such that its scalar curvature agrees with $|\mathfrak{K}_g|$. Moreover, using that $\cos(\arctan x) = (1 + x^2)^{-1/2}$ and $\sin(\arctan x) = x(1 + x^2)^{-1/2}$, and that $(1 + x^2) = \dot{r}^{-2}(\dot{r}^2 + r^2 \dot{\phi}^2) = \dot{r}^{-2}$ when $x = r \dot{\phi} / \dot{r}$, we infer:

$$\begin{aligned}\cos(\omega + \phi) &= \dot{r} \cos \phi - r \dot{\phi} \sin \phi = \frac{d}{ds}(r \cos \phi) \\ \sin(\omega + \phi) &= \dot{r} \sin \phi + r \dot{\phi} \cos \phi = \frac{d}{ds}(r \sin \phi).\end{aligned}$$

As a consequence, up to a rigid motion in \mathbb{R}^2 , the local expression of the developed curve $\tilde{\gamma}$ is:

$$\tilde{\gamma}(s) = r(s) \cdot (\cos \phi(s), \sin \phi(s)), \quad s \in I. \quad (8.1)$$

Denoting then by $\mathbf{T}(s) = \dot{\tilde{\gamma}}(s)$ the unit tangent vector to $\tilde{\gamma}$, by the previous computation we have

$$|\dot{\mathbf{T}}(s)| = |\dot{\Theta}(s)| = |\dot{\omega}(s) + \dot{\phi}(s)| = |\mathfrak{K}_g(s)| \quad \forall s \in I$$

and hence we deduce that

$$\text{TC}(\tilde{\gamma}) = \int_I |\mathfrak{K}_g(s)| ds \quad (8.2)$$

where, we recall, \mathfrak{K}_g is the geodesic curvature of the given curve γ as a curve in \mathcal{M} .

Now, for any sequence $\{\tilde{P}_h\} \subset \mathcal{P}_\Sigma(\gamma)$, condition $\mu_\gamma(\tilde{P}_h) \rightarrow 0$ implies that $\mathbf{k}_\Sigma(\tilde{P}_h) \rightarrow \text{TC}_\Sigma(\gamma)$. Moreover, by the above computation it turns out that if \hat{P}_h is the polygonal in \mathbb{R}^2 inscribed in $\tilde{\gamma}$ and with vertexes corresponding to the vertexes of \tilde{P}_h in γ , then $\mathbf{k}_\Sigma(\hat{P}_h) = \text{TC}(\hat{P}_h)$ for each h . Also, property $\mu_\gamma(\tilde{P}_h) \rightarrow 0$ yields that $\text{mesh}(\hat{P}_h) \rightarrow 0$. Since $\tilde{\gamma}$ is a planar curve, we infer that $\text{TC}(\hat{P}_h) \rightarrow \text{TC}(\tilde{\gamma})$. In conclusion, we get

$$\text{TC}_\Sigma(\gamma) = \lim_{h \rightarrow \infty} \mathbf{k}_\Sigma(\tilde{P}_h) = \lim_{h \rightarrow \infty} \text{TC}(\hat{P}_h) = \text{TC}(\tilde{\gamma})$$

and Proposition 8.1 holds true on account of formula (8.2). \square

Example 8.2 Following Example 3.2, if $\mathcal{M} = \mathcal{S}^2$ and $\gamma = \mathbf{c}_{\theta_0}$ is the parallel with constant co-latitude $\theta_0 \in]0, \pi/2]$, the geodesic polar coordinates on \mathcal{S}^2 give $g = \sin^2 r$, so that $r(s) \equiv \theta_0$ and $\phi(s) = s / \sin \theta_0$, where $s \in [0, 2\pi \sin \theta_0]$. The geodesic polar coordinates on Σ give instead $g = r^2$, whence $r(s) \equiv \tan \theta_0$ and $\phi(s) = \cot \theta_0 \cdot s$, where again $s \in [0, 2\pi \sin \theta_0]$. Therefore, according to (8.1), the corresponding developed curve $\tilde{\gamma}$ in \mathbb{R}^2 is the arc of a circle of radius $\tan \theta_0$ and length $2\pi \sin \theta_0$, i.e.,

$$\tilde{\gamma}(s) = \tan \theta_0 (\cos(\cot \theta_0 \cdot s), \sin(\cot \theta_0 \cdot s)), \quad s \in [0, 2\pi \sin \theta_0].$$

The pointwise scalar curvature of $\tilde{\gamma}$ is the reciprocal of the curvature radius of $\tilde{\gamma}$, and hence it is equal to the pointwise geodesic curvature $\mathfrak{K}_g \equiv \cot \theta_0$ of the parallel $\mathbf{c} = \mathbf{c}_{\theta_0}$, whereas the total curvature of $\tilde{\gamma}$ is equal to $2\pi \cot \theta_0$, i.e., to the total curvature $\text{TC}_{\mathcal{S}^2}(\mathbf{c}_{\theta_0})$ of the parallel.

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