CRITICAL METRICS FOR LOG-DETERMINANT FUNCTIONALS IN CONFORMAL GEOMETRY

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Key Words: functional determinants, singular solutions, blow-up analysis, min-max theory

AMS subject classification: 35G20, 35B44, 35J35, 58J50

Abstract. We consider critical points of a class of functionals on compact four-dimensional manifolds arising from *Regularized Determinants* for conformally covariant operators, whose explicit form was derived in [10], extending Polyakov's formula. These correspond to solutions of elliptic equations of Liouville type that are quasilinear, of mixed orders and of critical type. After studying existence, asymptotic behaviour and uniqueness of *fundamental solutions*, we prove a quantization property under blow-up, and then derive existence results via critical point theory.

1. Introduction

Consider a compact Riemannian manifold (M, g) without boundary of dimension n, with Laplace-Beltrami operator Δ_g. By *Weyl's asymptotic formula* it is known that the eigenvalues λ_j of $-\Delta_g$ obey the limiting law $\lambda_j \sim j^{2/n}$ as $j \to \infty$. The *determinant* of $-\Delta_g$ is formally the product of all its eigenvalues, with a rigorous definition that can be obtained via holomorphic extension of the *zeta function*

$$\zeta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}.$$

The behaviour of the λ_j's implies that $\zeta(s)$ is analytic for $\mathrm{Re}(s) > n/2$: it is possible anyway to meromorphically extend ζ so that it becomes regular near $s = 0$ (see [48]). From the formal calculation $\zeta'(0) = -\sum_{j=1}^{\infty} \log \lambda_j = -\log \det(-\Delta_g)$ one then defines

$$\det(-\Delta_g) = e^{-\zeta'(0)}.$$

Recall that in two dimensions the Laplace-Beltrami operator is conformally covariant in the sense that

$$\Delta_{\tilde{g}} = e^{-2w} \Delta_g, \qquad \tilde{g} = e^{2w} g. \tag{1.1}$$

This property, as well as the transformation law for the Gaussian curvature

$$-\Delta_g w + K_g = K_{\tilde{g}} e^{2w}, \tag{1.2}$$

allowed Polyakov in [47] to determine the logarithm of the ratio of regularized determinants of two conformally-equivalent metrics with the same area on a compact surface:

$$\log \frac{\det(-\Delta_{\tilde{g}})}{\det(-\Delta_g)} = -\frac{1}{12\pi} \int_\Sigma (|\nabla w|_g^2 + 2K_g w) \, dv_g. \tag{1.3}$$

The Gaussian curvature K_g appears in the above formula since it is possible to rewrite the zeta function as an integral of a trace

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty Tr \left(e^{\Delta_g t} - \frac{1}{\mathrm{Area}_g(\Sigma)} \right) dt,$$

where $\Gamma(s)$ is Euler's Gamma function and $e^{\Delta_g t}$ is the heat kernel on (Σ, g). The latter kernel, for t small, has the asymptotic profile of the Euclidean one, with next-order corrections involving the Gaussian curvature and its covariant derivatives, as shown in [41].

Using (1.2) and Polyakov's formula it is easy to show that critical points of the regularized determinant in a given conformal class give rise to constant Gaussian curvature metrics. In [45, 46] Osgood, Phillips and Sarnak proved existence of extremals for all given topologies: uniqueness holds for non-positive Euler characteristic, while in the positive case there are as many solutions as Möbius maps. The Möbius action is indeed employed to fix a *center of mass* gauge, in the spirit of [5], to exploit an *improved* Moser-Trudinger type inequality. Still in [45, 46] the authors used formula (1.3) in order to derive compactness of isospectral metrics on closed surfaces with a given topology. This result was then extended to the three-dimensional case in [14], for metrics within a fixed conformal class.

In four dimension formulas similar to (1.3) were obtained for regularized determinants of operators enjoying covariance properties analogous to (1.1). More precisely, a differential operator A_g (depending on the metric) is said to be *conformally covariant of bi-degree* (a,b) if

$$A_{\tilde{g}}\psi = e^{-bw}A_g(e^{aw}\psi), \quad \tilde{g} = e^{2w}g, \tag{1.4}$$

for each smooth function ψ (or even for a smooth section of a vector bundle). One such example is the *conformal Laplacian* in dimension $n \geq 3$

$$L_g = -\Delta_g + \frac{(n-2)}{4(n-1)}R_g,$$

where R_g is the scalar curvature: this operator satisfies (1.4) with $a = \frac{n-2}{2}$ and $b = \frac{n+2}{2}$. Other examples include the *Dirac operator* $\not{D}_g$, which satisfies (1.4) with $a = \frac{n-1}{2}$, $b = \frac{n+1}{2}$, and the *Paneitz operator* in four dimensions

$$P_g\psi = \Delta_g^2\psi - \operatorname{div}\left(\frac{2}{3}R_g\nabla\psi - 2Ric_g(\cdot,\nabla\psi)\right), \tag{1.5}$$

that satisfies (1.4) with $a = 0$ and $b = 4$.

Branson and Ørsted generalized in [10] Polyakov's formula to four-dimensional manifolds (M,g), proving the following result: the logarithmic ratio of two regularized determinants is the linear combination of three universal functionals, with coefficients depending on the specific operator. More precisely, if $A = A_g$ is conformally covariant and has no kernel (otherwise, see Remark 1.4), then one has

$$F_A[w] = \log\frac{\det A_{\tilde{g}}}{\det A_g} = \gamma_1(A)I[w] + \gamma_2(A)II[w] + \gamma_3(A)III[w], \quad \tilde{g} = e^{2w}g, \tag{1.6}$$

where $(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3$ and I, II, III are defined as

$$\begin{aligned}
I[w] &= 4\int_M w|W_g|_g^2\,dv_g - \Big(\int_M |W_g|_g^2\,dv_g\Big)\log \fint_M e^{4w}\,dv_g\\
II[w] &= \int_M wP_gw\,dv_g + 4\int_M Q_gw\,dv_g - \Big(\int_M Q_g\,dv_g\Big)\log\fint_M e^{4w}\,dv_g\\
III[w] &= 12\int_M(\Delta_gw + |\nabla w|_g^2)^2\,dv_g - 4\int_M(w\Delta_gR_g + R_g|\nabla w|_g^2)\,dv_g.
\end{aligned}$$

Here W_g is the Weyl curvature tensor, and Q_g the *Q-curvature* of (M,g)

$$Q_g = \frac{1}{12}(-\Delta_gR_g + R_g^2 - 3|Ric_g|_g^2).$$

The latter quantity is a natural higher-order counterpart of the Gaussian curvature, and transforms conformally via the Paneitz operator by the law

$$P_gw + 2Q_g = 2Q_{\tilde{g}}e^{4w}, \quad \tilde{g} = e^{2w}g,$$

totally analogous to (1.2). The above three functionals are geometrically natural as their critical points can be characterized by the conditions

$$\begin{aligned}
\tilde{g} = e^{2w}g \text{ is a critical point of } I &\iff |W_{\tilde{g}}|_{\tilde{g}}^2 = const.\\
\tilde{g} = e^{2w}g \text{ is a critical point of } II &\iff Q_{\tilde{g}} = const.\\
\tilde{g} = e^{2w}g \text{ is a critical point of } III &\iff \Delta_{\tilde{g}}R_{\tilde{g}} = 0.
\end{aligned}$$

Notice that, since M is compact, the last condition yields a Yamabe metric, with constant scalar curvature.

The Euler-Lagrange equation for F_A implies constancy of a scalar quantity U_g, which we call *U-curvature*, defined as

$$U_g = \gamma_1|W_g|_g^2 + \gamma_2Q_g - \gamma_3\Delta_gR_g. \tag{1.7}$$

In terms of the conformal factor the stationarity equation is

$$\mathcal{N}_g(w) + U_g = \mu e^{4w}; \tag{1.8}$$

$$\mathcal{N}(w) = \frac{\gamma_2}{2} P_g w + 6\gamma_3 \Delta_g(\Delta_g w + |\nabla w|_g^2) - 12\gamma_3 \mathrm{div}\big[(\Delta_g w + |\nabla w|_g^2)\nabla w\big] + 2\gamma_3 \mathrm{div}(R_g \nabla w), \tag{1.9}$$

where

$$\mu = -\frac{\kappa_A}{\int_M e^{4w} dv_g}; \qquad \kappa_A = -\gamma_1 \int_M |W_g|_g^2 \, dv_g - \gamma_2 \int_M Q_g \, dv_g.$$

We note that k_A is a conformal invariant, since $\int_M Q_g \, dv_g$ is, and that the above equation (1.8) corresponds to solving $U_{\tilde{g}} \equiv \mu$.

For example, one has

$$\gamma_1(L_g) = 1, \qquad \gamma_2(L_g) = -4, \qquad \gamma_3(L_g) = -2/3$$

for the conformal Laplacian and

$$\gamma_1(\not{D}_g^2) = -7, \qquad \gamma_2(\not{D}_g^2) = -88, \qquad \gamma_3(\not{D}_g^2) = -\frac{14}{3}$$

for the square of the Dirac operator $\not{D}_g$. For the Paneitz operator, instead, one has

$$\gamma_1(P_g) = -\frac{1}{4}, \qquad \gamma_2(P_g) = -14, \qquad \gamma_3(P_g) = 8/3.$$

Concerning extremality of functionals that are linear combinations of I, II and III, as in (1.6), Chang and Yang [13] proved an existence result (with a sign-reverse notation) under the conditions $\gamma_2, \gamma_3 > 0$ and $\kappa_A < 8\pi^2\gamma_2$.

The latter inequality (showed in [31] to hold in positive Yamabe class, except for manifolds conformal to the round sphere) was used with a geometric version of a Moser-Trudinger type inequality: in [1] an estimate on the (logarithmic) integral of the exponential of the conformal factor was derived in terms of the squared norm of the Laplacian, while in [13] in terms of the quadratic form induced by the Paneitz operator, which is conformally covariant. Uniqueness was also proved for the case $k_A < 0$, using the convexity of the functional F_A; see also [9] for the case of the round sphere, where extremals were classified as Möbius maps (and as unique critical points in [29]). Extremal properties of the round metric on S^n in general even dimension were studied in [42]. Regularity of arbitrary extremals was proved in [12], and extended in [55] to other critical points. The existence result in [13] was used in [30] to derive optimal bounds on the Weyl functional and to prove some rigidity results for Kähler-Einstein metrics.

Due to the above results, one has a satisfactory existence theory on manifolds of positive Yamabe class. It is the aim of this paper to derive it also for manifolds of more general type. One fact that distinguishes two and four dimensions from the conformal point of view is that in the latter case Gauss-Bonnet integrals can be larger than those on the round sphere of equal dimension. For example, the total integral of Q-curvature on four-manifolds of negative Yamabe class can be arbitrarily large. This fact causes the lack of one-side control on the functional II in terms of the Moser-Trudinger inequality, which was available in [13]. Nevertheless, in [20] conformal metrics with constant Q-curvature were found as saddle-type critical points of II. The main tool to produce these was a variational min-max scheme that used suitable improvements of the Moser-Trudinger inequality for conformal factors whose volume is macroscopically *spread* over the underlying manifold M. Such kind of improvement was derived in two dimensions in [5] for the case of the round sphere (see also [43]) and in [16] for general surfaces. With improved inequalities at hand, it was then possible in [20] to characterize low-sublevels of the functional II, showing that if $\int_M Q_g dv_g < 8(k+1)\pi^2$ for some $k \in \mathbb{N}$, and if $II(w)$ is sufficiently low, then the *conformal volume* e^{4w} approaches distributionally a measure supported on at most k points of M. This geometric characterization of the Euler-Lagrange functional II allowed to produce *Palais-Smale sequences*, namely approximate solutions to the prescribed Q-curvature equation. Using also a *monotonicity argument* from [53] one can replace Palais-Smale sequences by sequences of solutions to approximate equations, which might carry more information than general Palais-Smale sequences.

Here comes the other main aspect of the prescribed Q-curvature equation: compactness. One would like to show that the latter solutions converge to a solution of the original problem. This is actually the result of the two independent papers [23] and [40]: there it is proved that non-compact sequences of solutions develop after rescaling a finite number of *bubbles*, the conformal factors of the stereographic projection from $\mathbb{S}^4$ to $\mathbb{R}^4$. Each of them carries $8\pi^2$ in Q-curvature, and in the latter work it is shown that no other residual volume can occur. A contradiction to loss of compactness is then reached assuming that the initial total Q-curvature $\int_M Q_g dv_g$ is not a integer multiple of $8\pi^2$.

The first among our results is an analogous compactness property for log-determinant functionals.

Theorem 1.1. *Suppose M is a compact four-manifold and that $\gamma_2, \gamma_3 \neq 0$, with $\frac{\gamma_2}{\gamma_3} \geq 6$. Suppose also that $(w_n)_n$ is a sequence of smooth solutions of*

$$\mathcal{N}_g(w_n) + \tilde{U}_n = \mu_n e^{4w_n} \qquad \text{in } M,$$

where $\mathcal{N}_g$ is given by (1.9). Assume that $\int_M e^{4w_n} dv_g = 1$, $\mu_n = \int_M \tilde{U}_n dv_g$ and $\tilde{U}_n \to U_g$ C^1-uniformly in M as $n \to +\infty$. Up to a subsequence, we have one of the following two alternatives:

i) *$(w_n - \fint_M w_n \, dv_g)_n$ is uniformly bounded in $C^{4,\alpha}(M)$-norm;*

ii) *$(w_n)_n$ blows up, i.e. $\max_M w_n \to +\infty$, and one has that $\fint_M w_n \, dv_g \to -\infty$ and*

$$\mu_n e^{4w_n} \rightharpoonup \sum_{i=1}^{l} 8\pi^2 \gamma_2 \delta_{p_i}$$

in the weak sense of distributions for distinct points $p_1, \dots, p_l \in M$.

As a consequence, solutions stay compact if $\int_M U_g dv_g \notin 8\pi^2 \gamma_2 \mathbb{N}$.

Remark 1.2. *In Theorem 1.1, it is possible to replace the limit of $\tilde{U}_n$ by any smooth function $\tilde{U}$.*

Well-known results of the above type were proved for second-order Liouville equations in [11, 15, 36], in presence of singular sources in [6] and in the fourth-order case [2, 38, 39, 49, 50, 56]. The counterpart of Theorem 1.1 for Q-curvature in [23, 40] relied extensively on the Green's representation formula for the Paneitz operator, which is linear. A related quantization result was proved in [24] for a Liouville-type n-Laplace equation in $n-$dimensional euclidean domains, the equation there of second order allowing truncation techniques towards a-priori estimates (see also [25] for a classification result of entire solutions). Here, being our operator quasi-linear and of mixed type, none of these arguments can be applied and we need to devise new arguments.

In Section 2 we derive some uniform control of subcritical type on blowing-up solutions, followed by a Caccioppoli-type inequality and a uniform BMO estimate, which is a natural one since blow-up is expected to occur with a logarithmic profile. In Section 3 we develop a general *linear* theory for the operator $\mathcal{N}$ in (1.9), solving for arbitrary measures in the R.H.S.. Solutions will be found by a limiting procedure with smooth approximations (SOLA: see the terminology there), and the solvability theory will exploit in a crucial way a nonlinear Hodge decomposition technique. For a R.H.S. given as a linear combination of Dirac masses, a corresponding SOLA is referred to as a *fundamental solution* and uniqueness in general fails unless $\gamma_2 = 6\gamma_3$.

In Section 4 we show however that any *fundamental solution* satisfies weighted $W^{2,2}-$estimates, allowing via techniques developed in [55] to prove its logarithmic behaviour near the singularities.

There is a vast literature concerning existence and uniqueness issues for problems involving the $p-$Laplace operator, let us just quote [7, 8, 22, 27] and references therein. While for the latter both maximum principles and truncation arguments are available, it is not the case for our problem, and we had therefore to rely on different arguments.

With the asymptotics of fundamental solutions at hand, we can finally pass to the blow-up analysis of (1.8). First, via a Pohozaev type identity, scaling arguments and an epsilon-regularity result we prove a quantization for the volume accumulation at blow-up points. After this, we can then determine that there is no absolutely continuous part in the limit volume measure, after blow-up, leading to Theorem 1.1. We collect in an appendix some useful auxiliary results.

As an application of Theorem 1.1 we have the following existence theorem.

Theorem 1.3. *Assume $\gamma_2, \gamma_3 \neq 0$ and $\frac{\gamma_2}{\gamma_3} \geq 6$. Suppose M is a compact four-manifold such that $\int_M U_g dv_g \notin 8\pi^2 \gamma_2 \mathbb{N}$. Then there exists a conformal metric $\tilde{g}$ with constant U-curvature.*

Examples to which the latter theorem applies include (suitable) products of negatively-curved surfaces, hyperbolic manifolds or their perturbations.

Remark 1.4. *In case of trivial kernel, both log-determinants of L_g and $\not{D}_g^2$ fit in the assumptions of Theorems 1.1 and 1.3.*

In general, if a conformally-covariant operator A has a non-trivial kernel, some additional quantities appear in (1.6)*, see Remark 2.2 in* [10]*. If A has order 2ℓ, on the R.H.S. of* (1.6) *one should add the term*

$$2\ell \int_M \left(w \int_0^1 \Phi_t^2 e^{4tw} dt \right) dv_g - \frac{1}{2}\ell\, q[A] \log \frac{\int_M e^{4w} dv_g}{Vol_g(M)}. \tag{1.10}$$

Here $q[A]$ stands for the dimension of the kernel of A, while $\Phi_t^2(x) = \sum_{j=1}^{q[A]} \varphi_{j,t}^2(x)$, with $(\varphi_{j,t})_j$ an orthonormal basis of elements of the kernel with respect to the metric $e^{2tw}g$.

For example if $A = L$, the conformal Laplacian, and if the kernel is one-dimensional, denote by φ_1 an element of the kernel normalized in L^2 with respect to dv_g. Then, recalling that (1.4) *holds with $a = 1$, we find that*

$$\Phi_t^2(x) = \frac{e^{-2tw(x)}\varphi_1^2(x)}{\int_M e^{2tw(y)}\varphi_1^2(y) dv_g(y)}.$$

Therefore, the extra-term in (1.10) *becomes*

$$2\int_M \left(\int_0^1 \left(\frac{e^{2tw(x)}\varphi_1^2(x) w(x)}{\int_M e^{2tw(y)}\varphi_1^2(y) dv_g(y)} \right) dt \right) dv_g(x) - \frac{1}{2} \log \frac{\int_M e^{4w} dv_g}{Vol_g(M)}.$$

Noticing that

$$2\int_M \left(\frac{e^{2tw(x)}\varphi_1^2(x) w(x)}{\int_M e^{2tw(y)}\varphi_1^2(y) dv_g(y)} \right) dv_g(x) = \frac{d}{dt} \log \int_M e^{2tw(x)} \varphi_1^2(x) dv_g(x),$$

the expression in (1.10) *finally becomes*

$$\log \int_M e^{2w(x)}\varphi_1^2(x) dv_g(x) - \frac{1}{2} \log \frac{\int_M e^{4w} dv_g}{Vol_g(M)}.$$

We will not analyze this term in the present paper.

The proof of Theorem 1.3, given in Section 6 is variational and mainly inspired from [13, 20], where the Q-curvature problem was treated. First, using the results in Section 2, one can obtain a sharp Moser-Trudinger inequality involving combinations of the functionals I, II and III. The latter is then improved under suitable conditions on the distribution of conformal volume. This allows to apply a general min-max scheme, relying also on the construction of test functions with low energy and a prescribed (multiple) concentration behaviour of the conformal volume.

It would be interesting to consider on general manifolds cases with γ's of opposite signs, like for the determinant of the Paneitz operator (see [17], IV.4.γ). This issue is quite hard, as the two main terms in the nonlinear operator have competing effects. It is indeed studied so far only in particular cases with ODE techniques, see for example [28].

Notation. We will work on a compact four-dimensional Riemannian manifold M without boundary endowed with a background metric g. When considering this metric, the index g relative to it will be omitted in symbols like Δ_g, P_g, dv_g, etc. Spaces of L^p functions with respect to dv_g will be simply denoted by L^p, $p \geq 1$, with norm $\| \cdot \|_p$, and similarly for Sobolev spaces. When the domain of integration is omitted, we mean that it coincides with the whole M. The injectivity radius of (M, g) will be denoted by i_0 and B_r will denote a generic geodesic ball in M. The symbols $\overline{w}$, $\overline{w}^A$ and $\overline{w}^r$ will stand for $⨍_M w\, dv_g$, $⨍_A w\, dv_g$ and $⨍_{B_r} w\, dv_g$, respectively.

Acknowledgments. A.M. has been supported by the project *Geometric Variational Problems* and *Finanziamento a supporto della ricerca di base* from Scuola Normale Superiore and by MIUR Bando PRIN 2015 2015KB9WPT$_{001}$. P.E. has been supported by MIUR Bando PRIN 2015 2015KB9WPT$_{008}$. As members, they are both partially supported by GNAMPA as part of INdAM.

2. Some basic estimates

In this section we will derive some uniform estimates for smooth solutions of (1.8) with a general R.H.S. by just assuming $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. To this aim, recall the definition of the quasilinear differential operator $\mathcal{N}$ in (1.9). Integrating by parts, notice that the main order term in $\langle \mathcal{N}(w), w\rangle$ has the form

$$(\frac{\gamma_2}{2} + 6\gamma_3) \int (\Delta w)^2 dv + 18\gamma_3 \int \Delta w |\nabla w|^2 dv + 12\gamma_3 \int |\nabla w|^4 dv,$$

which can be easily seen to have a sign by a squares completion provided $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. In the next section, we will further strengthen the a-priori estimates when $\frac{\gamma_2}{\gamma_3} \geq 6$ and deduce uniqueness properties when $\frac{\gamma_2}{\gamma_3} = 6$.

In order to include also local estimates, test (1.9) against $\varphi = \chi^4 \psi(w - c)$, where $c \in \mathbb{R}$, $\psi \in C^2(\mathbb{R})$ (bounded, and with bounded first- and second-order derivatives) and $\chi \in C^\infty(M)$, to get

$$\begin{aligned}\langle \mathcal{N}(w), \varphi\rangle &= (\frac{\gamma_2}{2} + 6\gamma_3) \int \chi^4 \psi' (\Delta w)^2 dv + \int \chi^4 [18\gamma_3 \psi' + (\frac{\gamma_2}{2} + 6\gamma_3)\psi''] \Delta w |\nabla w|^2 dv \\ &+ 6\gamma_3 \int \chi^4 (2\psi' + \psi'')|\nabla w|^4 dv + \int \chi^4 \psi' [(\frac{\gamma_2}{3} - 2\gamma_3) R |\nabla w|^2 - \gamma_2 Ric(\nabla w, \nabla w)] dv + \mathcal{R},\end{aligned} \tag{2.1}$$

with

$$\begin{aligned}\mathcal{R} \quad = \quad & \int [(\frac{\gamma_2}{2} + 6\gamma_3)\Delta w + 6\gamma_3 |\nabla w|^2][\psi \Delta \chi^4 + 2\psi' \langle \nabla \chi^4, \nabla w\rangle] dv + 12\gamma_3 \int (\Delta w + |\nabla w|^2)\psi \langle \nabla w, \nabla \chi^4\rangle dv \\ & + \int \psi [(\frac{\gamma_2}{3} - 2\gamma_3) R \langle \nabla w, \nabla \chi^4 \rangle - \gamma_2 Ric(\nabla w, \nabla \chi^4)] dv,\end{aligned}$$

where the argument of ψ has been omitted for simplicity.

Remark 2.1. *When $\partial M \neq \emptyset$, (2.1) still holds for $\chi \in C_0^\infty(M)$: this will be useful in Section 5.*

The first use of (2.1) concerns global bounds for weighted $W^{2,2}-$norms in M:

Theorem 2.2. *Let $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. Assume $\overline{f} = 0$ and $\|f\|_1 \leq C_0$ for some $C_0 > 0$. Then there exists $C > 0$ so that*

$$\int \frac{(\Delta w)^2 + |\nabla w|^4}{[1 + (w - \overline{w})^2]^{\frac{2}{3}}} dv \leq C \tag{2.2}$$

for every smooth solution w of $\mathcal{N}(w) = f$ in M. Moreover, given $1 \leq q < 2$ there exists $C > 0$ so that

$$\|w - \overline{w}\|_{W^{2,q}} \leq C \tag{2.3}$$

for any such solution w.

Proof. Let $\chi \equiv 1$, $c = \overline{w}$ and $\psi \in C^2(\mathbb{R})$ be so that $2\psi' + \psi'' > 0$. Then $\mathcal{R} = 0$ and by a squares completion the (re-normalized) main order term in (2.1) satisfies, thanks to $\beta = \frac{\gamma_2}{\gamma_3} > \frac{3}{2}$, the inequality

$$\begin{aligned}&(\beta + 12) \int \psi' (\Delta w)^2 dv + \int [36\psi' + (\beta + 12)\psi''] \Delta w |\nabla w|^2 dv + 12 \int (2\psi' + \psi'')|\nabla w|^4 dv \\ &\geq \int \frac{48[2\beta - 3 - 2\delta(\beta + 12)](\psi')^2 - 24(1 + 2\delta)(\beta + 12)\psi'\psi'' - (\beta + 12)^2 (\psi'')^2}{48(1 - \delta)(2\psi' + \psi'')} (\Delta w)^2 dv \\ &+ 12\delta \int (2\psi' + \psi'')|\nabla w|^4 dv\end{aligned} \tag{2.4}$$

for any $0 < \delta < 1$, in view of the positivity of

$$\int [\frac{36\psi' + (\beta + 12)\psi''}{\sqrt{48(1 - \delta)(2\psi' + \psi'')}} \Delta w + \sqrt{12(1 - \delta)(2\psi' + \psi'')} |\nabla w|^2]^2 \ dv.$$

Fix $0 < \delta < \frac{2\beta - 3}{4(\beta + 12)}$ and set $\psi(t) = \int_{-\infty}^t \frac{ds}{(M_0 + s^2)^{\frac{2}{3}}}$, $M_0 \geq 1$. Since

$$|\frac{\psi''}{\psi'}| = \frac{4}{3} \frac{|t|}{M_0 + t^2} \leq \frac{2}{3\sqrt{M_0}}, \tag{2.5}$$

we can find $M_0 \geq 1$ large so that

$$\frac{48[2\beta-3-2\delta(\beta+12)](\psi')^2-24(1+2\delta)(\beta+12)\psi'\psi''-(\beta+12)^2(\psi'')^2}{48(1-\delta)(2\psi'+\psi'')},\ 12\delta(2\psi'+\psi'')\geq \delta^2\psi'. \tag{2.6}$$

Thanks to (2.6) we have that

$$\begin{aligned}(\beta+12)\int \psi'(\Delta w)^2 dv \quad &+ \quad \int [36\psi'+(\beta+12)\psi'']\Delta w|\nabla w|^2 dv + 12\int (2\psi'+\psi'')|\nabla w|^4 dv\\ &\geq \quad \delta^2\int \psi'[(\Delta w)^2+|\nabla w|^4]dv,\end{aligned}$$

and then

$$\int \frac{(\Delta w)^2+|\nabla w|^4}{[1+(w-\overline{w})^2]^{\frac{2}{3}}}dv \leq C_1\left(\|f\|_1+\int |\nabla w|^2 dv\right) \tag{2.7}$$

for some $C_1>0$ in view of $M_0^{-\frac{2}{3}}(1+t^2)^{-\frac{2}{3}} \leq \psi' \leq 1$ and $0 \leq \psi \leq \int_{\mathbb{R}} \frac{ds}{(1+s^2)^{\frac{2}{3}}}$. From (2.7) and Hölder's inequality we obtain

$$\begin{aligned}\int |\nabla w|^2 dv \quad &\leq \quad \int [1+|w-\overline{w}|^{\frac{2}{3}}]\frac{|\nabla w|^2}{[1+(w-\overline{w})^2]^{\frac{1}{3}}}dv \leq \|1+|w-\overline{w}|^{\frac{2}{3}}\|_2\left(\int \frac{|\nabla w|^4}{[1+(w-\overline{w})^2]^{\frac{2}{3}}}dv\right)^{\frac{1}{2}}\\ &\leq \quad C_1^{\frac{1}{2}}\left(|M|^{\frac{1}{2}}+\|w-\overline{w}\|_{\frac{4}{3}}^{\frac{2}{3}}\right)\left(\|f\|_1+\int |\nabla w|^2 dv\right)^{\frac{1}{2}} \leq C_2\left(1+\int |\nabla w|^2 dv\right)^{\frac{5}{6}}\end{aligned}$$

for some $C_2>0$ in view of Poincaré's inequality. By Young's inequality we then have $\int |\nabla w|^2 dv \leq C$ for some $C>0$, and in turn by (2.7) we deduce the validity of (2.2).

Similarly, since $W^{2,q}(M)$ embeds continuously into $L^{\frac{4q}{3(2-q)}}(M)$ by Sobolev's Theorem, for any $1 \leq q < 2$ there holds

$$\begin{aligned}\int |\Delta w|^q dv \quad &\leq \quad \int [1+|w-\overline{w}|^{\frac{2q}{3}}]\frac{|\Delta w|^q}{[1+(w-\overline{w})^2]^{\frac{q}{3}}}dv \leq \|1+|w-\overline{w}|^{\frac{2q}{3}}\|_{\frac{2}{2-q}}\left(\int \frac{(\Delta w)^2}{[1+(w-\overline{w})^2]^{\frac{2}{3}}}dv\right)^{\frac{q}{2}}\\ &\leq \quad C_3\left(1+\|w-\overline{w}\|_{W^{2,q}}^{\frac{2q}{3}}\right)\end{aligned}$$

for some $C_3>0$ in view of (2.2). Since $(\int |\Delta w|^q dv)^{\frac{1}{q}}$ is equivalent to the $W^{2,q}-$norm on the functions in $W^{2,q}(M)$ with zero average, by Young's inequality we then have the validity of (2.3) for some uniform $C>0$. ■

Once global bounds on $W^{2,q}-$norms have been derived for $1 \leq q < 2$, we will make use once more of (2.1) to establish Caccioppoli-type estimates:

Theorem 2.3. *Let $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. There exist $C>0$ and $k_0>0$ so that*

$$\int_{\{|w-c|<k\}\cap B_\rho} [(\Delta w)^2+|\nabla w|^4]dv \leq \frac{C}{(r-\rho)^4}\int_{B_r\setminus B_\rho}(1+(w-c)^4)\ dv + Ck\int_{B_r}|f|\ dv \tag{2.8}$$

for any $0<\rho<r<i_0$, $c\in\mathbb{R}$, $k\geq k_0$ and any smooth solution w of $\mathcal{N}(w)=f$ in M with $\overline{f}=0$. Here B_ρ and B_r are centered at the same point.

Proof. Let $\chi \in C_0^\infty(B_r)$ be so that $0\leq \chi\leq 1$, $\chi=1$ in B_ρ and

$$(r-\rho)|\nabla\chi|+(r-\rho)^2|\Delta\chi|\leq C. \tag{2.9}$$

Letting Ψ be the odd extension to $\mathbb{R}$ of

$$\Psi(s)=\begin{cases} s & \text{if } 0\leq s\leq 1\\ 8-9s^{-\frac{1}{3}}+2s^{-1} & \text{if } s>1,\end{cases}$$

we have that $\Psi\in C^2(\mathbb{R})$ satisfies $|\Psi''|\leq 4\Psi'$, $0<\Psi'\leq 1$, $\Psi^2\leq 8^2s^2\Psi'$ and $\Psi^4\leq 8^4s^4(\Psi')^3$ in $\mathbb{R}$. Hence, $\psi(s)=k\Psi(\frac{s}{k})$ is a C^2-function so that $0<\psi'\leq 1$,

$$\sup_{s\in\mathbb{R}}\frac{|\psi''(s)|}{\psi'(s)}\leq \frac{4}{k} \tag{2.10}$$

and

$$\sup_{s\in\mathbb{R}} \frac{\psi^2(s)}{s^2\psi'(s)} \le 8^2, \qquad \sup_{s\in\mathbb{R}} \frac{\psi^4(s)}{s^4(\psi'(s))^3} \le 8^4. \tag{2.11}$$

By Young's inequality we have that

$$\begin{aligned}\int [|\Delta w| + |\nabla w|^2]|\psi||\Delta \chi^4|\,dv &\le \frac{C}{(r-\rho)^2}\int_{B_r\setminus B_\rho} [|\Delta w| + |\nabla w|^2]\chi^2|\psi| \\ &\le \epsilon \int \psi'\chi^4[(\Delta w)^2 + |\nabla w|^4]dv + \frac{C_\epsilon}{(r-\rho)^4}\int_{B_r\setminus B_\rho} |w-c|^2\,dv\end{aligned}$$

in view of (2.9) and (2.11), where ψ stands for $\psi(w-c)$. Similarly, there holds

$$\begin{aligned}&\int [|\Delta w| + |\nabla w|^2](\psi' + |\psi|)|\nabla\chi^4||\nabla w|dv \le \frac{C}{r-\rho}\int_{B_r\setminus B_\rho} [|\Delta w| + |\nabla w|^2](\psi' + |\psi|)\chi^3|\nabla w|dv \\ &\le \epsilon\int \psi'\chi^4[(\Delta w)^2 + |\nabla w|^4]dv + \epsilon\int \psi'\chi^4|\nabla w|^4 dv + \frac{C'_\epsilon}{(r-\rho)^4}\int_{B_r\setminus B_\rho} \frac{(\psi' + |\psi|)^4}{(\psi')^3}dv \\ &\le 2\epsilon\int \psi'\chi^4[(\Delta w)^2 + |\nabla w|^4]dv + \frac{C_\epsilon}{(r-\rho)^4}\int_{B_r\setminus B_\rho} (1 + (w-c)^4)dv,\end{aligned}$$

and

$$\int |\psi||\nabla w||\nabla\chi^4|dv \le \frac{C}{r-\rho}\int_{B_r\setminus B_\rho} |\psi||\nabla w|\chi^3 dv \le \epsilon\int \psi'\chi^4|\nabla w|^4 dv + \frac{C_\epsilon}{(r-\rho)^4}\int_{B_r\setminus B_\rho} (w-c)^4 dv + C_\epsilon$$

in view of (2.9) and (2.11). In conclusion, for all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that $\mathcal{R}$ in (2.1) satisfies

$$|\mathcal{R}| \le C\epsilon\int \psi'\chi^4[(\Delta w)^2 + |\nabla w|^4]dv + \frac{C_\epsilon}{(r-\rho)^4}\int_{B_r\setminus B_\rho} (1 + (w-c)^4)dv \tag{2.12}$$

for some $C > 0$. Since $\frac{|\psi''(s)|}{\psi'(s)}$ can be made as small as we need for k large thanks to (2.10), we are in the same situation as with (2.5) and, arguing as in the proof of Theorem 2.2, there exists $k_0 > 0$ large so that

$$\begin{aligned}\Big|(\frac{\gamma_2}{2} + 6\gamma_3)\int \chi^4\psi'(\Delta w)^2 dv &+ \int \chi^4[18\gamma_3\psi' + (\frac{\gamma_2}{2} + 6\gamma_3)\psi'']\Delta w|\nabla w|^2 dv \\ &+ 6\gamma_3\int \chi^4(2\psi' + \psi'')|\nabla w|^4 dv\Big| \ge \delta^2\int \psi'\chi^4[(\Delta w)^2 + |\nabla w|^4]dv\end{aligned} \tag{2.13}$$

for some $\delta > 0$ and all $k \ge k_0$. Since $\int \psi'\chi^4|\nabla w|^2 dv \le \epsilon\int \psi'\chi^4|\nabla w|^4 dv + C_\epsilon$ and $|\int f\chi^4\psi\,dv| \le 8k\int_{B_r} |f|\,dv$, by inserting (2.12)-(2.13) into (2.1) for $\epsilon > 0$ small we deduce the validity of (2.8) for all $k \ge k_0$ in view of $\chi^4\psi'(w-c) \ge \chi_{\{|w-c|<k\}\cap B_\rho}$. ■

The aim is now to control the mean oscillation

$$[w]_{BMO} = \left(\sup_{0<r<i_0} \fint_{B_r} (w - \overline{w}^r)^4 dv\right)^{\frac14}$$

of a solution w. Our approach in this step heavily relies on the ideas developed in [22], where Caccioppoli-type estimates like in Theorem 2.3 were crucial to establish BMO-bounds. We believe that $L^{4,\infty}-$estimates on ∇w are still true as in [22] but it is not clear which are the optimal bounds for Δw. We will not pursue more this line since the following BMO-estimates are enough for our purposes.

Theorem 2.4. *Let $\frac{\gamma_2}{\gamma_3} > \frac32$. Assume $\overline{f} = 0$ and $\|f\|_1 \le C_0$ for some $C_0 > 0$. There exists $C > 0$ such that for any smooth solution w of $\mathcal{N}(w) = f$ in M one has*

$$[w]_{BMO} \le C. \tag{2.14}$$

Proof. If (2.14) does not hold, we can find smooth solutions w_n of $\mathcal{N}(w_n) = f_n$ so that $[w_n]_{BMO} \to +\infty$ as $n \to +\infty$, with $\overline{f}_n = 0$ and $\|f_n\|_1 \le C_0$. By definition we can find $0 < r_n < i_0$, $x_n \in M$ so that

$$\fint_{B_{r_n}(x_n)} (w_n - \overline{w}_n^{r_n})^4\,dv \ge \frac12 [w_n]^4_{BMO}. \tag{2.15}$$

Since $[w_n]_{BMO} \to +\infty$ as $n \to +\infty$, up to a subsequence we can assume that $r_n \to 0$ as $n \to +\infty$ in view of

$$\sup_{n\in\mathbb{N}} \sup_{\delta<r<i_0} \fint_{B_r} (w_n - \overline{w}_n^r)^4 dv < +\infty$$

for all $0 < \delta \le i_0$, as it follows by the Poincaré-Sobolev's embedding

$$\left(\int_{B_r} |w_n - \overline{w}_n^r|^4 dv \right)^{\frac{1}{4}} \le C \left(\int_{B_r} |\nabla w_n|^2 dv \right)^{\frac{1}{2}}$$

and Theorem 2.2. Letting $exp_{x_n} : B_{i_0}(0) \to B_{i_0}(x_n)$ be the exponential map at x_n, for $|y| < \frac{i_0}{r_n}$ introduce the rescaled metric $g_n(y) = g(exp_{x_n}(r_n y))$ and the rescaled functions

$$u_n(y) = \frac{w_n(exp_{x_n}(r_n y)) - \overline{w}_n^{r_n}}{[w_n]_{BMO}}.$$

We have that

$$\int_{B_1(0)} u_n \, dv_{g_n} = 0, \quad \int_{B_1(0)} u_n^4 \, dv_{g_n} \ge \frac{\text{vol}(B_{r_n}(x_n))}{2r_n^4}, \quad \int_{B_r(0)} (u_n - \overline{u}_n^r)^4 \, dv_{g_n} \le \frac{\text{vol}(B_{rr_n}(x_n))}{r_n^4} \tag{2.16}$$

for all $r < \frac{i_0}{r_n}$ in view of (2.15), where $\overline{u}_n^r = \fint_{B_r(0)} u_n \, dv_{g_n}$ is the average of u_n on $B_r(0)$ w.r.t. g_n. Neglecting the term involving the Laplacian, we can rewrite the estimate (2.8) in terms of u_n as

$$\int_{\{|u_n-c|<k\}\cap B_\rho(0)} |\nabla u_n|_{g_n}^4 dv_{g_n} \le \frac{C}{(r-\rho)^4} \int_{B_r(0)\setminus B_\rho(0)} \left[\frac{1}{[w_n]_{BMO}^4} + (u_n - c)^4 \right] dv_{g_n} + \frac{Ck\|f_n\|_1}{[w_n]_{BMO}^3} \tag{2.17}$$

for any $0 < \rho < r < \frac{i_0}{r_n}$, $c \in \mathbb{R}$ and $k \ge \frac{k_0}{[w_n]_{BMO}}$. Since $\text{vol}(B_{rr_n}(x_n)) \le C(rr_n)^4$ for all $0 < r < \frac{i_0}{r_n}$ there holds

$$\int_{B_r(0)} (u_n - \overline{u}_n^r)^4 \, dv_{g_n} \le Cr^4 \qquad \forall\, 0 < r < \frac{i_0}{r_n} \tag{2.18}$$

thanks to (2.16), and we can apply (2.17) with $\rho = \frac{r}{2}$ and $c = \overline{u}_n^r$ to get

$$\int_{\{|u_n-\overline{v}_n^r|<k\}\cap B_{\frac{r}{2}}(0)} |\nabla u_n|_{g_n}^4 dv_{g_n} \le C(\frac{1}{[w_n]_{BMO}^4} + 1) + \frac{Ck\|f_n\|_1}{[w_n]_{BMO}^3} \tag{2.19}$$

in view of (2.18). Since

$$|\overline{u}_n^r| \int_{B_1(0)} dv_{g_n} \le \int_{B_1(0)} |u_n - \overline{u}_n^r| \, dv_{g_n} \le C \left(\int_{B_r(0)} (u_n - \overline{u}_n^r)^4 \, dv_{g_n} \right)^{\frac{1}{4}} \left(\int_{B_1(0)} dv_{g_n} \right)^{\frac{3}{4}} \le C_0 r \int_{B_1(0)} dv_{g_n}$$

for all $1 \le r < \frac{i_0}{r_n}$ in view of (2.16) and (2.18), we have that $\{|u_n| < k\} \subset \{|u_n - \overline{u}_n^r| < 2k\}$ and then

$$\int_{\{|u_n|<k\}\cap B_{\frac{r}{2}}(0)} |\nabla u_n|_{g_n}^4 dv_{g_n} \le C \left(1 + \frac{k\|f_n\|_1}{[w_n]_{BMO}^3} \right) \tag{2.20}$$

for all $1 \le r < \frac{i_0}{r_n}$ and $k > C_0 r$ in view of (2.19). From (2.20) and $\int_{B_1(0)} u_n \, dv_{g_n} = 0$ it is rather classical to derive that u_n is uniformly bounded in $W^{1,q}_{loc}(\mathbb{R}^4)$ for all $1 \le q < 4$, see for example Lemma 2.3 in [22] and the proof of Lemma 10 in [21]. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in $W^{1,q}_{loc}(\mathbb{R}^4)$ for all $1 \le q < 4$. Letting $\varphi_k \in C_0^\infty(-k, k)$ so that $\varphi_k(s) = s$ for $s \in [-\frac{k}{2}, \frac{k}{2}]$, by $|\varphi_k'| \le C_k$ and (2.17) we deduce that

$$\int_{B_\rho(0)} |\nabla \varphi_k(u_n - c)|^4 dx \le \frac{C}{(r-\rho)^4} \int_{B_r(0)\setminus B_\rho(0)} \left[\frac{1}{[w_n]_{BMO}^4} + (u_n - c)^4 \right] dx + \frac{Ck\|f_n\|_1}{[w_n]_{BMO}^3} \tag{2.21}$$

for any $0 < \rho < r < \frac{i_0}{r_n}$, $c \in \mathbb{R}$ and $k \ge \frac{k_0}{[w_n]_{BMO}}$. Since $\nabla \varphi_k(u_n - c) \rightharpoonup \nabla \varphi_k(u - c)$ in $L^4_{loc}(\mathbb{R}^4)$ in view of $u_n \to u$ in $L^q_{loc}(\mathbb{R}^4)$ for all $q \ge 1$ as $n \to +\infty$, by weak lower semi-continuity of the L^4-norm we can let $n \to +\infty$ in (2.21) to get

$$\int_{\{|u-c|<\frac{k}{2}\}\cap B_\rho(0)} |\nabla u|^4 dx \le \frac{C}{(r-\rho)^4} \int_{B_r(0)\setminus B_\rho(0)} (u - c)^4 \, dx$$

and then by the Monotone Convergence Theorem as $k \to +\infty$

$$\int_{B_\rho(0)} |\nabla u|^4 dx \le \frac{C}{(r-\rho)^4} \int_{B_r(0)\setminus B_\rho(0)} (u-c)^4 \, dx \tag{2.22}$$

for any $0<\rho<r$, $c \in \mathbb{R}$ and $k>0$. Similarly, by letting $n \to +\infty$ into (2.20) we deduce that

$$\int_{\{|u|<\frac{k}{2}\}\cap B_{\frac{r}{2}}(0)} |\nabla u|^4 dx \le C$$

for all $r \ge 1$ and $k > C_0 r$, and then by the Monotone Convergence Theorem we get $\int_{\mathbb{R}^4} |\nabla u|^4 dx < +\infty$ as $k, r \to +\infty$. Taking $\rho = \frac{r}{2}$ and $c = \fint_{B_r(0)\setminus B_{\frac{r}{2}}(0)} u\, dx$ in (2.22), by Poincaré's inequality one finally deduces

$$\int_{B_{\frac{r}{2}}(0)} |\nabla u|^4 dx \le \frac{C}{r^4} \int_{B_r(0)\setminus B_{\frac{r}{2}}(0)} (u-c)^4 \, dx \le C' \int_{B_r(0)\setminus B_{\frac{r}{2}}(0)} |\nabla u|^4 dx \to 0$$

as $r \to +\infty$ in view of $\int_{\mathbb{R}^4} |\nabla u|^4 dx < +\infty$, leading to $\nabla u = 0$ a.e. in $\mathbb{R}^4$. By (2.16) and $g_n \to \delta_{eucl}$ locally uniformly as $n \to +\infty$ we have that $u = 0$ a.e. in view $\int_{B_1(0)} u\, dx = 0$, in contradiction with $\int_{B_1(0)} u^4 dx \ge \frac{\omega_4}{6}$. □

3. General "linear" theory

We aim to develop a comprehensive theory for the operator $\mathcal{N}$ in (1.9) when $\frac{\gamma_2}{\gamma_3} \ge 6$. In this section we are interested in existence issues for a general Radon measure μ and Solutions will be Obtained as Limits of smooth Approximations, from now on referred to as SOLA (see [7, 8]). On the other hand since, as we will see, blow-up sequences give rise in the limit to a solution with a linear combination μ_s of Dirac masses as R.H.S., it will be crucial to establish in the next section the logarithmic behaviour of any of such singular solutions, referred to as a *fundamental solution* of $\mathcal{N}$ corresponding to μ_s. We will guarantee that SOLA's will be unique just when $\gamma_2 = 6\gamma_3$.

The assumption $\frac{\gamma_2}{\gamma_3} \ge 6$ is crucial to have some monotonicity property on $\mathcal{N}$, expressed by a sign for the main order term in expressions of the form $\langle \mathcal{N}(w_1) - \mathcal{N}(w_2), w_1 - w_2 \rangle$. When $\gamma_2 = 6\gamma_3$ the lower-order terms cancel out and uniqueness is in order, as already noticed in [13]. The operator $\mathcal{N}(w)$ in (1.9) is considered here in the following distributional sense:

$$\begin{aligned}\langle \mathcal{N}(w), \varphi \rangle \;=\;& \frac{\gamma_2}{2} \int \Delta w \Delta \varphi \, dv - \gamma_2 \int \mathrm{Ric}(\nabla w, \nabla \varphi) dv + 6\gamma_3 \int (\Delta w + |\nabla w|^2) \Delta \varphi \, dv \\ &+ 12\gamma_3 \int (\Delta w + |\nabla w|^2) \langle \nabla w, \nabla \varphi \rangle dv + (\frac{\gamma_2}{3} - 2\gamma_3) \int R \langle \nabla w, \nabla \varphi \rangle dv\end{aligned}$$

for all $\varphi \in C^\infty(M)$, provided $\nabla w \in L^3$ and $\nabla^2 w \in L^{\frac{3}{2}}$. We have the following result.

Proposition 3.1. *There holds*

$$\begin{aligned}\langle \mathcal{N}(w_1) - \mathcal{N}(w_2), \varphi \rangle \;=\;& 3\gamma_3 \int \Delta_{\hat g} p \; \Delta_{\hat g} \varphi \, dv_{\hat g} + 6\gamma_3 \int \langle \nabla^2_{\hat g} p, \nabla^2_{\hat g} \varphi \rangle_{\hat g} dv_{\hat g} + 3\gamma_3 \int |\nabla p|^2_{\hat g} \langle \nabla p, \nabla \varphi \rangle_{\hat g} dv_{\hat g} \\ &+ (\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p \Delta \varphi \, dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int [3Ric(\nabla p, \nabla \varphi) - R \langle \nabla p, \nabla \varphi \rangle] dv\end{aligned} \tag{3.1}$$

for all $\varphi \in C^\infty(M)$ provided $\mathcal{N}(w_1)$ and $\mathcal{N}(w_2)$ exist in a distributional sense, where $p = w_1 - w_2$, $q = w_1 + w_2$ and $\hat g = e^q g$.

Proof. Notice that when $w_1 = w_2$, $q = 2w_i$ and hence our notation for the conformal metric $\hat g = e^q g$ is consistent with out previous one. Since $\hat g = e^q g$ has derivatives in a weak sense up to order two, the Riemann tensor of $\hat g$ and all the geometric quantities which involve at most second-order derivatives make sense. One can easily check that

$$dv_{\hat g} = e^{2q} dv, \quad e^q \Delta_{\hat g} w = \Delta w + \langle \nabla q, \nabla w \rangle, \quad e^{2q} |\nabla w|^4_{\hat g} = |\nabla w|^4, \tag{3.2}$$

$$\nabla^2_{\hat g} w = \nabla^2 w - \frac{1}{2} dw \otimes dq - \frac{1}{2} dq \otimes dw + \frac{1}{2} \langle \nabla q, \nabla w \rangle g. \tag{3.3}$$

Since $w_1 = \frac{p+q}{2}$ and $w_2 = \frac{q-p}{2}$ we have that

$$\int [(\Delta w_1 + |\nabla w_1|^2) - (\Delta w_2 + |\nabla w_2|^2)] \Delta \varphi \, dv = \int (\Delta p + \langle \nabla p, \nabla q \rangle) \Delta \varphi \, dv, \tag{3.4}$$

and

$$\int \langle (\Delta w_1 + |\nabla w_1|^2)\nabla w_1 - (\Delta w_2 + |\nabla w_2|^2)\nabla w_2, \nabla\varphi\rangle dv = \frac{1}{2}\int (\Delta p + \langle \nabla p, \nabla q\rangle)\langle \nabla q, \nabla \varphi\rangle dv + \frac{1}{4}\int (2\Delta q + |\nabla p|^2 + |\nabla q|^2)\langle \nabla p, \nabla\varphi\rangle dv. \tag{3.5}$$

By (3.4)-(3.5) we deduce that

$$\begin{aligned} &2\int \langle (\Delta w_1 + |\nabla w_1|^2)\nabla w_1 - (\Delta w_2 + |\nabla w_2|^2)\nabla w_2, \nabla\varphi\rangle dv \\ &+\int [(\Delta w_1 + |\nabla w_1|^2) - (\Delta w_2 + |\nabla w_2|^2)]\Delta\varphi\, dv = \frac{1}{2}\int \Delta p \Delta\varphi\, dv - \int \langle \nabla^2 p, \nabla^2\varphi\rangle dv \\ &+\frac{1}{2}\int \Delta_{\hat g} p \Delta_{\hat g}\varphi\, dv_{\hat g} + \int \langle \nabla^2_{\hat g} p, \nabla^2_{\hat g}\varphi\rangle_{\hat g} dv_{\hat g} + \frac{1}{2}\int |\nabla p|^2_{\hat g}\langle \nabla p, \nabla\varphi\rangle_{\hat g} dv_{\hat g}, \end{aligned} \tag{3.6}$$

in view of (3.2)-(3.3) and the formula

$$\begin{aligned} &\int \langle \nabla^2_{\hat g} p, \nabla^2_{\hat g}\varphi\rangle_{\hat g} dv_{\hat g} - \int \langle \nabla^2 p, \nabla^2\varphi\rangle dv = \int \langle \nabla^2_{\hat g} p, \nabla^2_{\hat g}\varphi\rangle dv - \int \langle \nabla^2 p, \nabla^2\varphi\rangle dv \\ &= \int \left[\Delta q\langle \nabla p, \nabla\varphi\rangle + \frac{1}{2}\Delta p\langle \nabla q, \nabla\varphi\rangle + \frac{1}{2}\langle \nabla p, \nabla q\rangle\langle \nabla q, \nabla\varphi\rangle + \frac{1}{2}|\nabla q|^2\langle \nabla p, \nabla\varphi\rangle + \frac{1}{2}\langle \nabla p, \nabla q\rangle \Delta\varphi\right] dv. \end{aligned} \tag{3.7}$$

To establish (3.7) we simply use (3.3) and an integration by parts to get

$$\int \left[\nabla^2 p(\nabla q, \nabla\varphi) + \nabla^2\varphi(\nabla q, \nabla p)\right] dv = \int \langle \nabla q, \nabla\langle \nabla p, \nabla\varphi\rangle\rangle dv = -\int \Delta q\langle \nabla p, \nabla\varphi\rangle dv \tag{3.8}$$

for all $\varphi \in C^\infty(M)$, in view of $\nabla p, \nabla q \in L^3$ and $\nabla^2 p, \nabla^2 q \in L^{\frac{3}{2}}$. Thanks to Bochner's identity

$$\mathrm{Ric}(\nabla p, \nabla p) = -\langle \nabla p, \nabla\Delta p\rangle - |\nabla^2 p|^2 + \frac{1}{2}\Delta(|\nabla p|^2), \quad p \in C^3(M),$$

an integration by parts gives that $\int \mathrm{Ric}(\nabla p, \nabla p) dv = \int (\Delta p)^2 dv - \int |\nabla^2 p|^2 dv$ and by differentiation

$$\int \mathrm{Ric}(\nabla p, \nabla\varphi) dv = \int \Delta p \Delta\varphi\, dv - \int \langle \nabla^2 p, \nabla^2\varphi\rangle dv \tag{3.9}$$

for all $\varphi \in C^\infty(M)$, where by density it is enough to assume $\nabla p,\ \nabla^2 p \in L^1$. By inserting (3.9) into (3.6), we then deduce the validity of (3.1). ∎

Remark 3.2. *When $\partial M \neq \emptyset$ notice that the integrations by parts in* (3.8)-(3.9) *and then* (3.1) *are still valid for $\varphi \in C_0^\infty(M)$ as long as $\mathcal{N}(u),\ \mathcal{N}(v)$ exist in a distributional sense.*

The usefulness of assumption $\frac{\gamma_2}{\gamma_3} \geq 6$ becomes apparent from the choice $\varphi = p$ in (3.1) since it guarantees that the first four terms in the R.H.S. of (3.1) have all the same sign. When $\gamma_2 = 6\gamma_3$ there are no lower-order terms and uniqueness is expected. Since in general p is not an admissible function in (3.1), we will follow the strategy in [27, 32, 33] via a Hodge decomposition to build up admissible approximations of p to be used in (3.1).

Letting w_1 and w_2 be smooth functions, consider the *Hodge decomposition*

$$\frac{\nabla p}{(\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} = \nabla\varphi + h, \tag{3.10}$$

where $\epsilon > 0$, $0 < \delta \leq 1$ and φ, h satisfy $\Delta\,\mathrm{div}\ h = 0$ and $\overline{\varphi} = 0$. Notice that

$$\Delta\varphi = \frac{\Delta p}{(\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} - 4\epsilon\frac{\nabla^2 p(\nabla p, \nabla p) + \nabla^2 q(\nabla p, \nabla q)}{(\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{2\epsilon+1}} - \mathrm{div}\ h. \tag{3.11}$$

Even if div $h = 0$ when $\partial M = \emptyset$, we prefer to keep this term in order to include later the case $\partial M \neq \emptyset$. The function φ is uniquely determined as the smooth solution of

$$\Delta^2\varphi = \Delta\left[\frac{\Delta p}{(\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} - 4\epsilon\frac{\nabla^2 p(\nabla p, \nabla p) + \nabla^2 q(\nabla p, \nabla q)}{(\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{2\epsilon+1}}\right], \quad \overline{\varphi} = 0,$$

in view of (3.11), and then h is simply defined as $h = \frac{\nabla p}{(\delta^2+|\nabla p|^2+|\nabla q|^2)^{2\epsilon}} - \nabla\varphi$. Given distinct points $p_1,\dots,p_l \in M$ and $\alpha_1,\dots,\alpha_l \in \mathbb{R}$, we want to allow one between functions w_i, say w_2, to satisfy $w_2 \in C^\infty(M \setminus \{p_1,\dots,p_l\})$ and such that

$$\lim_{x\to 0} |x|^k |\nabla^{(k)}(w_2 - \alpha_i \log|x|)| = 0, \quad k = 1,2,3, \tag{3.12}$$

holds in geodesic coordinates near each p_i. Let us justify (3.10) more in general (i.e. for w_1 smooth and w_2 singular) by introducing the Green's function $G(x,y)$ of Δ^2 in M, i.e. the solution of

$$\begin{cases} \Delta^2 G(x,\cdot) = \delta_x - \frac{1}{|M|} & \text{in } M \\ \int G(x,y)dv(y) = 0. \end{cases}$$

For all $F \in C^\infty(M,TM)$ the solution of $\Delta^2\varphi = \Delta\text{div } F$ in M, $\overline{\varphi} = 0$, takes the form

$$\varphi(x) = \int G(x,y)\Delta\text{div } F(y)dv(y) = -\int \langle \nabla_y \Delta_y G(x,y), F(y)\rangle\, dv(y).$$

Hence $\nabla\varphi$ can be expressed as the singular integral

$$\nabla\varphi(x) = -\Big(\int \nabla_{xy}\Delta_y G(x,y)[F(y)]dv(y)\Big)^\sharp = \mathcal{K}(F),$$

where $\sharp$ stands for the sharp musical isomorphism. Since M is a smooth manifold, by the theory of singular integrals the operator $\mathcal{K}$ extends from $C^\infty(M,TM)$ to $L^s(M,TM)$ and $\nabla\varphi = \mathcal{K}(F)$, $h = F - \mathcal{K}(F)$ provide for the vector field F the Hodge decomposition $F = \nabla\varphi + h$ with

$$\|\nabla\varphi\|_s + \|h\|_s \le C(s)\|F\|_s \tag{3.13}$$

for all $s > 1$. The key point is that $C(s)$ is locally uniformly bounded in $(1,+\infty)$, see for example [34].

Since w_1 is smooth and w_2 satisfies (3.12), in geodesic coordinates near each p_i there holds

$$|x|^2(\delta^2 + |\nabla p|^2 + |\nabla q|^2) = 2\alpha_i^2 + o(1), \quad |\Delta p| + |\nabla^2 p| + |\nabla^2 q| = O(\frac{1}{|x|^2}) \qquad \text{as } x \to 0,$$

and then $F = \frac{\nabla p}{(\delta^2+|\nabla p|^2+|\nabla q|^2)^{2\epsilon}}$ satisfies div $F = O(\frac{1}{|x|^{2(1-2\epsilon)}})$ as $x \to 0$. Since w_2 is smooth away from $p_1,\dots,p_l$, we have that div $F \in L^{2(1+2\epsilon)}(M)$ and then by elliptic regularity theory the solution φ of $\Delta^2\varphi = \Delta\text{div } F$ in M, $\overline{\varphi} = 0$, is in $W^{2,2(1+2\epsilon)}(M)$. The Hodge decomposition (3.10) does hold with $h = \frac{\nabla p}{(\delta^2+|\nabla p|^2+|\nabla q|^2)^{2\epsilon}} - \nabla\varphi \in W^{1,2(1+2\epsilon)}(M)$ and by (3.13) φ satisfies

$$\|\nabla\varphi\|_{\frac{4(1-\epsilon)}{1-4\epsilon}} \le K\Big\|\frac{\nabla p}{(\delta^2+|\nabla p|^2+|\nabla q|^2)^{2\epsilon}}\Big\|_{\frac{4(1-\epsilon)}{1-4\epsilon}} \le K\|\nabla p\|_{4(1-\epsilon)}^{1-4\epsilon}. \tag{3.14}$$

To show the smallness of h in (3.10) for ϵ small, we follow the approach introduced in [32] based on a general estimate for commutators in Lebesgue spaces. For the sake of completeness we include it in the Appendix and we just make use here of the following estimate:

$$\|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}} \le K\epsilon\left(\delta^{1-4\epsilon} + \|\nabla p\|_{4(1-\epsilon)}^{1-4\epsilon} + \|\nabla q\|_{4(1-\epsilon)}^{1-4\epsilon}\right) \tag{3.15}$$

for all $0 < \epsilon \le \epsilon_0$ and $0 < \delta \le 1$, for some $K > 0$ and $\epsilon_0 > 0$ small. Thanks to the Hodge decomposition (3.10) we are now ready to show the following result.

Proposition 3.3. *Let* $\frac{\gamma_2}{\gamma_3} \ge 6$ *and set*

$$\eta = |\gamma_2 - 6\gamma_3|\sup_M(|R| + \|Ric\|). \tag{3.16}$$

There exist $\epsilon_0 > 0$ *and* $C > 0$ *so that*

$$\int \frac{|\nabla^2_{\hat g} p|^2_{\hat g} + |\nabla p|^4_{\hat g}}{(|\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} dv_{\hat g} \le C(\|F_1 - F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \eta\|\nabla p\|_{2-4\epsilon}^{2-4\epsilon} + \epsilon^{\frac43}\|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \epsilon^{\frac43}\|F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \epsilon^{\frac23}) \tag{3.17}$$

for all $0 < \epsilon \le \epsilon_0$ *and all distributional solutions* w_i *of* $\mathcal{N}(w_i) = $ *div* F_i, $i = 1,2$, *provided that* w_1 *is smooth and either* w_2 *is smooth or satisfies* (3.12). *Here* $p = w_1 - w_2$, $q = w_1 + w_2$ *and* $\hat g = e^q g$.

Proof. As already observed, we have that $\varphi \in W^{1,\frac{4(1-\epsilon)}{1-4\epsilon}}(M) \cap W^{2,2(1+2\epsilon)}(M)$. Letting $\varphi_k \in C^\infty(M)$ so that $\varphi_k \to \varphi$ in $W^{1,\frac{4(1-\epsilon)}{1-4\epsilon}}(M) \cap W^{2,2(1+2\epsilon)}(M)$ as $k \to +\infty$, we can use (3.1) with φ_k: thanks to (3.2)-(3.3) and

$$|\nabla p|^2 + |\nabla q|^2 + |\Delta p| + |\nabla^2 p| \in \bigcap_{1 \le q < 2} L^q(M),$$

let $k \to +\infty$ to get the validity of

$$\begin{aligned}(3.18)\quad & 3\gamma_3 \int \Delta_{\hat g} p\ \Delta_{\hat g}\varphi\, dv_{\hat g} + 6\gamma_3 \int \langle \nabla^2_{\hat g} p, \nabla^2_{\hat g}\varphi\rangle_{\hat g} dv_{\hat g} + 3\gamma_3 \int |\nabla p|^2_{\hat g} \langle \nabla p, \nabla \varphi\rangle_{\hat g} dv_{\hat g} \\ &+(\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p \Delta \varphi\, dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int [3\mathrm{Ric}(\nabla p, \nabla\varphi) - R\langle \nabla p, \nabla \varphi\rangle] dv = -\int \langle F_1 - F_2, \nabla \varphi\rangle\ dv.\end{aligned}$$

Notice that such a Sobolev regularity of φ might fail for a general solution $w_2 \in W^{\theta,2,2)}(M)$, see the definition in (3.35), and this explains why, even tough SOLA lie in $W^{\theta,2,2)}(M)$, in Theorem 3.6 we will not prove uniqueness in such a *grand Sobolev space*.

Setting $\rho = (\delta^2 + |\nabla p|^2 + |\nabla q|^2)^{-\epsilon}$, by (3.10)-(3.11) we deduce that

$$\begin{aligned}(3.19)\quad & |\Delta_{\hat g}\varphi - (\rho^2 \Delta_{\hat g} p - e^{-q} \mathrm{div}\, h)| + |\nabla^2_{\hat g}\varphi - (\rho^2 \nabla^2_{\hat g} p - \nabla h^\flat)|_{\hat g} = \\ &= \epsilon \rho^2 O\left(|\nabla p|_{\hat g} |\nabla q|_{\hat g} + |\nabla q|^2_{\hat g} + |\nabla^2_{\hat g} p|_{\hat g} + |\nabla^2_{\hat g} q|_{\hat g}\right) + O\left(|\nabla q|_{\hat g} |h|_{\hat g}\right)\end{aligned}$$

and

$$(3.20)\quad |\Delta \varphi - (\rho^2 \Delta p - \mathrm{div}\, h)| = \epsilon \rho^2 O\left(|\nabla p||\nabla q| + |\nabla q|^2 + |\nabla^2_{\hat g} p| + |\nabla^2_{\hat g} q|\right),$$

in view of (3.2)-(3.3), where $\flat$ stands for the flat musical isomorphism. By (3.10) and (3.19)-(3.20) let us re-write (3.18) as

$$\begin{aligned}(3.21)\quad & 3\gamma_3 \int \rho^2 (\Delta_{\hat g} p)^2 dv_{\hat g} + 6\gamma_3 \int \rho^2 |\nabla^2_{\hat g} p|^2_{\hat g} dv_{\hat g} + 3\gamma_3 \int \rho^2 |\nabla p|^4_{\hat g} dv_{\hat g} + (\frac{\gamma_2}{2} - 3\gamma_3) \int \rho^2 (\Delta p)^2 dv \\ & -3\gamma_3 \int e^{-q} \Delta_{\hat g} p\, \mathrm{div}\, h\, dv_{\hat g} - 6\gamma_3 \int \langle \nabla^2_{\hat g} p, \nabla h^\flat\rangle_{\hat g} dv_{\hat g} - (\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p\, \mathrm{div}\, h\, dv \\ & = -\int \langle F_1 - F_2, \nabla \varphi\rangle dv + \mathfrak{R},\end{aligned}$$

where by (3.2)-(3.3) and Hölder's inequality $\mathfrak{R}$ satisfies

$$\begin{aligned}\mathfrak{R} \ &= \ \epsilon \left(\|\rho \nabla^2_{\hat g} p\|_{2,\hat g} + (\int \rho^2 |\nabla p|^4_{\hat g} dv_{\hat g})^{\frac14} (\int \rho^2 |\nabla q|^4_{\hat g} dv_{\hat g})^{\frac14} \right) O\left[(\int \rho^2 |\nabla p|^4_{\hat g} dv_{\hat g})^{\frac14} (\int \rho^2 |\nabla q|^4_{\hat g} dv_{\hat g})^{\frac14} \right. \\ & \left. + (\int \rho^2 |\nabla q|^4_{\hat g} dv_{\hat g})^{\frac12} + \|\rho \nabla^2_{\hat g} p\|_{2,\hat g} + \|\rho \nabla^2_{\hat g} q\|_{2,\hat g} \right] + O\Big(\int [|\nabla^2_{\hat g} p||\nabla q| + |\nabla p|^3] |h| dv \Big) \\ (3.22)\quad & + O\left(\eta \int [|\nabla p|^{2-4\epsilon} + |\nabla p||h|] dv \right).\end{aligned}$$

Notice that by (3.2) and Hölder's inequality

$$\begin{aligned}(3.23)\quad & \int [|\nabla^2_{\hat g} p||\nabla q| + |\nabla p|^3] |h| dv \\ & = O\Big(\|\rho \nabla^2_{\hat g} p\|_{2,\hat g} (\int \rho^2 |\nabla q|^4_{\hat g} dv_{\hat g})^{\frac14} + (\int \rho^2 |\nabla p|^4_{\hat g} dv_{\hat g})^{\frac34} \Big) \|\rho^{-1}\|^{\frac32}_{\frac{2(1-\epsilon)}{\epsilon}} \|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}} \\ & = \epsilon\, O\Big(\|\rho \nabla^2_{\hat g} p\|_{2,\hat g} (\int \rho^2 |\nabla q|^4_{\hat g} dv_{\hat g})^{\frac14} + (\int \rho^2 |\nabla p|^4_{\hat g} dv_{\hat g})^{\frac34} \Big) (\delta^{1-\epsilon} + \|\nabla p\|^{1-\epsilon}_{4(1-\epsilon)} + \|\nabla q\|^{1-\epsilon}_{4(1-\epsilon)}),\end{aligned}$$

thanks to (3.15) and

$$(3.24)\quad \|\rho^{-1}\|_{\frac{2(1-\epsilon)}{\epsilon}} \le \|\delta + |\nabla p| + |\nabla q|\|^{2\epsilon}_{4(1-\epsilon)} = O(\delta^{2\epsilon} + \|\nabla p\|^{2\epsilon}_{4(1-\epsilon)} + \|\nabla q\|^{2\epsilon}_{4(1-\epsilon)}).$$

The difficult term to handle is

$$\begin{aligned} & 3\gamma_3 \int e^{-q} \Delta_{\hat g} p\, \mathrm{div}\, h\, dv_{\hat g} + 6\gamma_3 \int \langle \nabla^2_{\hat g} p, \nabla h^\flat\rangle_{\hat g} dv_{\hat g} + (\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p\, \mathrm{div}\, h\, dv \\ & = 3\gamma_3 \int \langle \nabla q, \nabla p\rangle \mathrm{div}\, h\, dv + 6\gamma_3 \int \langle \nabla^2_{\hat g} p, \nabla h^\flat\rangle dv + \frac{\gamma_2}{2} \int \Delta p\, \mathrm{div}\, h\, dv\end{aligned}$$

in view of (3.2)-(3.3). For smooth functions w_1 and w_2, integrating by parts we have that

$$3\gamma_3\int\langle\nabla q,\nabla p\rangle\mathrm{div}\,h\,dv+\frac{\gamma_2}{2}\int\Delta p\,\mathrm{div}\,h\,dv=-3\gamma_3\int\langle\nabla\langle\nabla q,\nabla p\rangle,h\rangle\,dv+\frac{\gamma_2}{2}\int\Delta p\,\mathrm{div}\,h\,dv, \tag{3.25}$$

and

$$\begin{aligned}\int\langle\nabla^2_{\hat g}p,\nabla h^\flat\rangle dv &= -\int g^{ij}h^k(\nabla^2_{\hat g}p)_{kj;i}\,dv\\ &= -\int[\langle h,\nabla\Delta p\rangle+\mathrm{Ric}\ (h,\nabla p)]dv+\frac{1}{2}\int[\Delta p\langle\nabla q,h\rangle+\Delta q\langle\nabla p,h\rangle]dv\\ &= \int[\Delta p\,\mathrm{div}\,h-\mathrm{Ric}\ (h,\nabla p)+\frac{1}{2}\Delta p\langle\nabla q,h\rangle+\frac{1}{2}\Delta q\langle\nabla p,h\rangle]dv\end{aligned} \tag{3.26}$$

in view of (3.3) and

$$g^{ij}h^kp_{;jki}=g^{ij}h^kp_{;jik}+R_{sk}h^k(\nabla p)^s=\langle h,\nabla\Delta p\rangle+\mathrm{Ric}\ (h,\nabla p).$$

Since $\Delta\,\mathrm{div}\ h=0$, by Hölder's inequality and (3.25)-(3.26) we then have

$$\begin{aligned}&3\gamma_3\int e^{-q}\Delta_{\hat g}p\,\mathrm{div}\,h\,dv_{\hat g}+6\gamma_3\int\langle\nabla^2_{\hat g}p,\nabla h^\flat\rangle_{\hat g}dv_{\hat g}+(\frac{\gamma_2}{2}-3\gamma_3)\int\Delta p\,\mathrm{div}\,h\,dv\\ =\ & O\Big(\int|h||\nabla p|dv+\int[|\nabla^2_{\hat g}p||\nabla q|+|\nabla^2_{\hat g}q||\nabla p|+|\nabla q|^2|\nabla p|]|h|dv\Big)\\ =\ & O\Big(\|\nabla p\|_{\frac{4(1-\epsilon)}{3}}\|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}}\Big)+O\Big(\|\rho\nabla^2_{\hat g}q\|_{2,\hat g}(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})^{\frac14}\|\rho^{-1}\|^{\frac32}_{\frac{2(1-\epsilon)}{\epsilon}}\|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}}\Big)\\ &+O\Big(\|\rho\nabla^2_{\hat g}p\|_{2,\hat g}+(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})^{\frac14}(\int\rho^2|\nabla q|^4_{\hat g}dv_{\hat g})^{\frac14}\Big)(\int\rho^2|\nabla q|^4_{\hat g}dv_{\hat g})^{\frac14}\|\rho^{-1}\|^{\frac32}_{\frac{2(1-\epsilon)}{\epsilon}}\|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}}\\ =\ & \epsilon\,O(\delta^{2-4\epsilon}+\|\nabla p\|^{2-4\epsilon}_{4(1-\epsilon)}+\|\nabla q\|^{2-4\epsilon}_{4(1-\epsilon)})+\epsilon(\delta^{1-\epsilon}+\|\nabla p\|^{1-\epsilon}_{4(1-\epsilon)}+\|\nabla q\|^{1-\epsilon}_{4(1-\epsilon)})\times\\ &\times O[\|\rho\nabla^2_{\hat g}q\|_{2,\hat g}(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})^{\frac14}+\|\rho\nabla^2_{\hat g}p\|_{2,\hat g}(\int\rho^2|\nabla q|^4_{\hat g}dv_{\hat g})^{\frac14}+(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})^{\frac14}(\int\rho^2|\nabla q|^4_{\hat g}dv_{\hat g})^{\frac12}]\end{aligned} \tag{3.27}$$

in view of (3.2)-(3.3), (3.15) and (3.24). When w_2 satisfies (3.12), notice that $p,q\in\bigcap_{1\le q<2}W^{2,q}(M)$ and $h\in L^{\frac{4(1-\epsilon)}{1-4\epsilon}}(M)\cap W^{1,2(1+2\epsilon)}(M)$. By an approximation argument we see that (3.25)-(3.26) and $\int\Delta p\ \mathrm{div}h\ dv=0$ still hold for p, q and h also in this case, and then (3.27) again follows.

As

$$\|\nabla\varphi\|^{\frac{4(1-\epsilon)}{1-4\epsilon}}_{\frac{4(1-\epsilon)}{1-4\epsilon}}=O\left(\int(\rho^2|\nabla p|)^{\frac{4(1-\epsilon)}{1-4\epsilon}}dv\right)=O\left(\int\rho^2|\nabla p|^4(\frac{|\nabla p|^2}{\delta^2+|\nabla p|^2+|\nabla q|^2})^{\frac{6\epsilon}{1-4\epsilon}}dv\right)=O(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})$$

in view of (3.14), notice that

$$\int\langle F_1-F_2,\nabla\varphi\rangle dv=O\left(\|F_1-F_2\|_{\frac{4(1-\epsilon)}{3}}(\int\rho^2|\nabla p|^4_{\hat g}dv_{\hat g})^{\frac{1-4\epsilon}{4(1-\epsilon)}}\right). \tag{3.28}$$

Since

$$\eta\int|\nabla p||h|dv=O(\eta^{2-4\epsilon}\|\nabla p\|^{2-4\epsilon}_{2-4\epsilon}+\epsilon^{\frac83}+\frac{1}{\epsilon^{\frac83}}\|h\|^{\frac{4(1-\epsilon)}{1-4\epsilon}}_{\frac{4(1-\epsilon)}{1-4\epsilon}}),$$

inserting (3.22)-(3.23) and (3.27)-(3.28) into (3.21), by Young's inequality and (3.15) one finally gets that

$$\begin{aligned}\int\rho^2\Big[|\nabla^2_{\hat g}p|^2_{\hat g}+|\nabla p|^4_{\hat g}\Big]dv_{\hat g} &= O\big(\|F_1-F_2\|^{\frac{4(1-\epsilon)}{3}}_{\frac{4(1-\epsilon)}{3}}+\eta\|\nabla p\|^{2-4\epsilon}_{2-4\epsilon}\big)\\ &\quad+\epsilon^{\frac43}O\big(\|\rho\nabla^2_{\hat g}q\|^2_{2,\hat g}+\|\nabla p\|^{4-4\epsilon}_{4(1-\epsilon)}+\|\nabla q\|^{4-4\epsilon}_{4(1-\epsilon)}+\epsilon^{-\frac23}\big)\end{aligned} \tag{3.29}$$

for all $0<\epsilon\le\epsilon_0$ and $0<\delta\le1$, for some $\epsilon_0>0$ small.

Since (3.29) holds for any smooth functions w_1 and w_2, if we choose $w_2=F_2=0$ then $w_1=p=q$ satisfies

$$\int\frac{|\nabla^2_{\tilde g}w_1|^2_{\tilde g}+|\nabla w_1|^4_{\tilde g}}{(\delta^2+|\nabla w_1|^2)^{2\epsilon}}dv_{\tilde g}=O\Big(\|F_1\|^{\frac{4(1-\epsilon)}{3}}_{\frac{4(1-\epsilon)}{3}}+\|\nabla w_1\|^{2-4\epsilon}_{2-4\epsilon}+\epsilon^{\frac43}\|\nabla w_1\|^{4-4\epsilon}_{4(1-\epsilon)}+\epsilon^{\frac23}\Big) \tag{3.30}$$

for all $0<\epsilon \leq \epsilon_0$ and $0<\delta\leq 1$, where $\tilde g = e^{w_1}g$. Letting $\delta \to 0^+$ in (3.30), by Fatou's Lemma we deduce that

$$\int \frac{|\nabla^2_{\tilde g} w_1|^2_{\tilde g}+|\nabla w_1|^4_{\tilde g}}{|\nabla w_1|^{4\epsilon}} dv_{\tilde g} = O\Big(\|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+\|\nabla w_1\|_{2-4\epsilon}^{2-4\epsilon}+\epsilon^{\frac{4}{3}}\|\nabla w_1\|_{4(1-\epsilon)}^{4-4\epsilon}+\epsilon^{\frac{2}{3}}\Big)$$

for all $0<\epsilon\leq\epsilon_0$. Since $\int \frac{|\nabla w_1|^4_{\tilde g}}{|\nabla w_1|^{4\epsilon}}dv_{\tilde g}=\int |\nabla w_1|^{4(1-\epsilon)}dv$, by Young's inequality we obtain that

$$\int \frac{|\nabla^2_{\tilde g} w_1|^2_{\tilde g}}{|\nabla w_1|^{4\epsilon}} dv_{\tilde g}+\|\nabla w_1\|_{4(1-\epsilon)}^{4(1-\epsilon)} = O(\|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+1). \tag{3.31}$$

If w_2 is either smooth or satisfies (3.12), we can still apply (3.29) with $w_1=F_1=0$ and get

$$\int \frac{|\nabla^2_{g^\#} w_2|^2_{g^\#}}{|\nabla w_2|^{4\epsilon}} dv_{g^\#}+\|\nabla w_2\|_{4(1-\epsilon)}^{4(1-\epsilon)} = O(\|F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+1) \tag{3.32}$$

for all $0<\epsilon\leq\epsilon_0$, where $g^\#=e^{w_2}g$. Since $\rho\leq |\nabla w_1|^{-2\epsilon}, |\nabla w_2|^{-2\epsilon}$ and

$$\begin{aligned} e^{2q}[|\nabla^2_{\hat g}p|^2_{\hat g}+|\nabla^2_{\hat g}q|^2_{\hat g}] &= 2e^{2w_1}|\nabla^2_{\tilde g}w_1|^2_{\tilde g}+2e^{2w_2}|\nabla^2_{g^\#}w_2|^2_{g^\#}+|dw_1\otimes dw_2+dw_2\otimes dw_1-\langle\nabla w_1,\nabla w_2\rangle g|^2\\ &\quad -2\langle \nabla^2_{\tilde g}w_1+\nabla^2_{g^\#}w_2, dw_1\otimes dw_2+dw_2\otimes dw_1-\langle\nabla w_1,\nabla w_2\rangle g\rangle \end{aligned}$$

in view of (3.2)-(3.3), by (3.31)-(3.32) we deduce that

$$\|\nabla p\|_{4(1-\epsilon)}^{4(1-\epsilon)}+\|\nabla q\|_{4(1-\epsilon)}^{4(1-\epsilon)} = O(\|\nabla w_1\|_{4(1-\epsilon)}^{4(1-\epsilon)}+\|\nabla w_2\|_{4(1-\epsilon)}^{4(1-\epsilon)}) = O(\|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+\|F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+1) \tag{3.33}$$

and

$$\begin{aligned} \|\rho\nabla^2_{\hat g}p\|^2_{2,\hat g}+\|\rho\nabla^2_{\hat g}q\|^2_{2,\hat g} &= O\Big(\int \frac{|\nabla^2_{\tilde g} w_1|^2_{\tilde g}}{|\nabla w_1|^{4\epsilon}} dv_{\tilde g}+\int \frac{|\nabla^2_{g^\#} w_2|^2_{g^\#}}{|\nabla w_2|^{4\epsilon}} dv_{g^\#}+\int |\nabla w_1|^{2-2\epsilon}|\nabla w_2|^{2-2\epsilon}dv\Big)\\ &= O(\|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+\|F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}}+1) \end{aligned} \tag{3.34}$$

for all $0<\epsilon\leq\epsilon_0$. Inserting (3.33)-(3.34) into (3.29) and letting $\delta\to 0^+$, estimate (3.17) follows by Fatou's Lemma for some $\epsilon_0>0$ small. ■

Remark 3.4. *When $\partial M\neq\emptyset$, re-consider $G(x,y)$ as the Green function of Δ^2 in M with boundary conditions $G(x,\cdot)=\partial_\nu G(x,\cdot)=0$ on ∂M. The Hodge decomposition* (3.10) *does hold with $\varphi\in W_0^{2,2(1+2\epsilon)}(M)$ and $h\in W_0^{1,2(1+2\epsilon)}(M)$. Letting $\varphi_k\in C_0^\infty(M)$ so that $\varphi_k\to\varphi$ in $W_0^{1,\frac{4(1-\epsilon)}{1-4\epsilon}}(M)\cap W_0^{2,2(1+2\epsilon)}(M)$ as $k\to+\infty$, thanks to Remark 3.2 we can use* (3.1) *with φ_k and let $k\to+\infty$ to get the validity of* (3.18) *for φ. The integrations by parts* (3.25)-(3.26) *are still valid since $h\in W_0^{1,2(1+2\epsilon)}(M)$, while $\int \Delta p \, div\, h\, dv=0$ does hold provided $w_1-w_2\in W_0^{2,1}(M)$. Hence, Proposition 3.3 does hold when $\partial M\neq\emptyset$ provided that we assume $w_1-w_2\in W_0^{2,1}(M)$.*

Let $L^{\theta,q)}(M,TM)$ be the *grand Lebesgue space* of all vector fields $F\in\bigcup\limits_{1\leq\tilde q<q}L^{\tilde q}(M,TM)$ with

$$\|F\|_{\theta,q)}=\sup_{0<\epsilon\leq\epsilon_0}\epsilon^{\frac{\theta}{q}}\|F\|_{q(1-\epsilon)}<+\infty$$

and $W^{\theta,2,2)}$ be the *grand Sobolev space*

$$W^{\theta,2,2)}=\{w\in W^{2,1}(M):\ \overline{w}=0, \|w\|_{W^{\theta,2,2)}}:=\|\Delta w\|_{\theta,2)}+\|\nabla w\|_{\theta,4)}<+\infty\}. \tag{3.35}$$

Let $\mathcal M=\{\mu \text{ Radon measure in } M:\ \mu(M)=0\}$. For $\mu\in\mathcal M$ we say that a distributional solution w of $\mathcal N(w)=\mu$ in M is a SOLA if $w=\lim\limits_{n\to+\infty}w_n$ a.e., where w_n are smooth solutions of $\mathcal N(w_n)=f_n$ with $f_n\in C^\infty(M)$, $\overline{w}_n=\overline{f}_n=0$ and $f_n dv\rightharpoonup\mu$ as $n\to+\infty$. Letting G_2 be the Green's function of Δ in M, the function

$$H(\mu)=\int\nabla_x G_2(x,y)d\mu(y)$$

for $\mu\in\mathcal M$ satisfies by Jensen's inequality

$$\epsilon^{\frac{3}{4}}\|H(\mu)\|_{\frac{4(1-\epsilon)}{3}}\leq\epsilon^{\frac{3}{4}}|d\mu|\sup_{y\in M}\Big(\int|\nabla_x G_2(x,y)|^{\frac{4(1-\epsilon)}{3}}dv(x)\Big)^{\frac{3}{4(1-\epsilon)}}\leq C|d\mu| \tag{3.36}$$

for all $0 < \epsilon \le \epsilon_0$. Therefore, we have that $H : \mathcal{M} \to L^{1,\frac{4}{3})}(M, TM)$ is a linear bounded operator satisfying the property $\mu = \operatorname{div} H(\mu)$, and we can now re-phrase Proposition 3.3 as the following main a-priori estimate.

Proposition 3.5. *Let $\frac{\gamma_2}{\gamma_3} \ge 6$, $\frac{2}{3} \le \theta < \frac{4}{3}$ and η be given as in (3.16). There exists $C > 0$ such that*

$$\begin{aligned} \|w_1 - w_2\|_{W^{\theta,2,2)}} \le\ & C\|F_1 - F_2\|_{\theta,\frac{4}{3})}^{\frac{4-3\theta}{6}} (\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{\theta}{2}} \\ & + C\|F_1 - F_2\|_{\theta,\frac{4}{3})}^{\frac{4-3\theta}{12}} (\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{4+3\theta}{12}} \\ & + \eta(\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{1}{3}}\, O(\|\nabla(w_1 - w_2)\|_2 + \|\nabla(w_1 - w_2)\|_2^{\frac{1}{4}}) \end{aligned} \tag{3.37}$$

for all SOLA's w_1, w_2 of $\mathcal{N}(w_1) = \mu_1 \in \mathcal{M}$, $\mathcal{N}(w_2) = \mu_2 \in \mathcal{M}$, where $F_1 = H(\mu_1)$ and $F_2 = H(\mu_2)$. Estimate (3.37) *holds even if w_2 is a distributional solution which satisfies* (3.12).

Proof. Since w_1 is a SOLA, by definition let $f_{1,n}$ be the corresponding approximating sequence of $\mu_1 = \operatorname{div} F_1$. Letting $u_{1,n}$ be the smooth solution of $\Delta u_{1,n} = f_{1,n}$ in M, $\overline{u}_{1,n} = 0$, we have that $u_{1,n}$ is pre-compact in $W^{1,q}(M)$ for all $1 \le q < \frac{4}{3}$, see for example Lemma 1 in [8] in the Euclidean context, and then the following property does hold:

$$\sup_n \|f_{1,n}\|_1 < +\infty \quad \Rightarrow \quad H(f_{1,n}dv) \text{ pre-compact in } L^q(M),\ 1 \le q < \frac{4}{3}, \tag{3.38}$$

in view of $H(f_{1,n}dv) = \nabla u_{1,n}$. Up to a subsequence, we have that $u_{1,n} \to u_1$ in $W^{1,q}(M)$ for all $1 \le q < \frac{4}{3}$, where u_1 is a distributional solution of $\Delta u_1 = \mu_1$ in M, $\overline{u}_1 = 0$. By uniqueness $\nabla u_1 = H(\mu_1)$ and therefore $w_1 = \lim\limits_{n\to+\infty} w_{1,n}$ a.e., where $\mathcal{N}(w_{1,n}) = \operatorname{div} F_{1,n}$ with $F_{1,n} = \nabla u_{1,n} \to F_1$ in $L^q(M)$ for all $1 \le q < \frac{4}{3}$.

Assume that w_2 is either a SOLA or a distributional solution satisfying (3.12) of $\mathcal{N}(w_2) = \mu_2 = \operatorname{div} F_2$. In the first case, let $f_{2,n}$ and $F_{2,n}$ be the corresponding sequences for w_2 so that $w_2 = \lim\limits_{n\to+\infty} w_{2,n}$ a.e., where $\mathcal{N}(w_{2,n}) = \operatorname{div} F_{2,n}$ with $F_{2,n} \to F_2$ in $L^q(M)$ for all $1 \le q < \frac{4}{3}$. In the second case, consider $w_{2,n} = w_2$ for all $n \in \mathbb{N}$. Apply (3.17) to $w_{1,n}$ and $w_{2,n}$ to get by (3.33)

$$\int \frac{|\nabla^2_{\hat g_n} p_n|^2_{\hat g_n} + |\nabla p_n|^4_{\hat g_n}}{(|\nabla p_n|^2 + |\nabla q_n|^2)^{2\epsilon}} dv_{\hat g_n} \le C$$

in terms of $p_n = w_{1,n} - w_{2,n}$, $q_n = w_{1,n} + w_{2,n}$ and $\hat g_n = e^{q_n} g$. Notice that for $1 \le q < 2$ by Hölder's estimate there holds

$$\int |\Delta p_n|^q dv \le C \left(\int \frac{(\Delta_{\hat g_n} p_n)^2 + |\nabla q_n|^2_{\hat g_n} |\nabla p_n|^2_{\hat g_n}}{(|\nabla p_n|^2 + |\nabla q_n|^2)^{2\epsilon}} dv_{\hat g_n} \right)^{\frac{q}{2}} \left(\int (|\nabla p_n|^2 + |\nabla q_n|^2)^{\frac{2\epsilon q}{2-q}} dv \right)^{\frac{2-q}{2}}$$

in view of (3.2), and then p_n is uniformly bounded in $W^{2,q}(M)$ for all $1 \le q < 2$ thanks to (3.33). By Rellich's Theorem we deduce that $p_n \to w_1 - w_2$ in $W^{1,q}(M)$ for all $1 \le q < 4$. Letting $n \to +\infty$ into (3.17) applied to $w_{1,n}$ and $w_{2,n}$, by Fatou's Lemma we get the validity of

$$\int \frac{|\nabla^2_{\hat g} p|^2_{\hat g} + |\nabla p|^4_{\hat g}}{(|\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} dv_{\hat g} \le C(\|F_1 - F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \eta\|\nabla p\|_{2-4\epsilon}^{2-4\epsilon} + \epsilon^{\frac{4}{3}} \|F_1\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \epsilon^{\frac{4}{3}} \|F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \epsilon^{\frac{2}{3}}) \tag{3.39}$$

for all $0 < \epsilon \le \epsilon_0$ and for all distributional solutions w_i of $\mathcal{N}(w_i) = \operatorname{div} F_i$, $i = 1, 2$, provided w_1 is a SOLA and w_2 is either a SOLA or satisfies (3.12), where $p = w_1 - w_2$, $q = w_1 + w_2$ and $\hat g = e^q g$. Re-written (3.39) as

$$\begin{aligned} \int \frac{(\Delta p + \langle \nabla q, \nabla p\rangle)^2 + |\nabla p|^4}{(|\nabla p|^2 + |\nabla q|^2)^{2\epsilon}} dv \le\ & C(\epsilon^{-\theta} \|F_1 - F_2\|_{\theta,\frac{4}{3})}^{\frac{4(1-\epsilon)}{3}} + \eta\|\nabla p\|_{2-4\epsilon}^{2-4\epsilon}) \\ & + C\epsilon^{\frac{4}{3}-\theta} (\|F_1\|_{\theta,\frac{4}{3})}^{\frac{4(1-\epsilon)}{3}} + \|F_2\|_{\theta,\frac{4}{3})}^{\frac{4(1-\epsilon)}{3}} + \epsilon^{\theta-\frac{2}{3}}) \end{aligned}$$

in view of (3.2), by Young's inequality we deduce that

$$\begin{aligned}(3.40)\qquad &\int |\Delta p+\langle \nabla q,\nabla p\rangle|^{2(1-\epsilon)}dv+\int |\nabla p|^{4(1-\epsilon)}dv \le C\int [(\Delta p+\langle \nabla q,\nabla p\rangle)^2+|\nabla p|^4]^{1-\epsilon}dv\\ &=O\left(\int \frac{(\Delta p+\langle \nabla q,\nabla p\rangle)^2+|\nabla p|^4}{(|\nabla p|^2+|\nabla q|^2)^{2\epsilon}}dv\right)+\epsilon O\left(\|\nabla p\|_{4(1-\epsilon)}^{4(1-\epsilon)}+\|\nabla q\|_{4(1-\epsilon)}^{4(1-\epsilon)}\right)\le C\eta\|\nabla p\|_{2-4\epsilon}^{2-4\epsilon}\\ &+C\epsilon^{-\theta}\|F_1-F_2\|_{\theta,\frac43)}^{\frac{4(1-\epsilon)}{3}}+C\epsilon^{\frac43-\theta}(\|F_1\|_{\theta,\frac43)}^{\frac{4(1-\epsilon)}{3}}+\|F_2\|_{\theta,\frac43)}^{\frac{4(1-\epsilon)}{3}}+\epsilon^{\theta-\frac23})\end{aligned}$$

for $0<\epsilon\le\epsilon_0$ in view of (3.33). If $F_1\neq F_2$, let $\epsilon_\delta>0$ be defined as

$$\epsilon_\delta=\delta(\frac{\|F_1-F_2\|_{\theta,\frac43)}}{\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1})$$

for $0<\delta\le\epsilon_0$. Since $0<\epsilon_\delta\le\delta\le\epsilon_0$ and $\|\cdot\|_{q(1-\delta)}=O(\|\cdot\|_{q(1-\epsilon_\delta)})$ by Hölder's inequality, inserting ϵ_δ into (3.40) we deduce that

$$\begin{aligned}(3.41)\quad &\|\Delta p+\langle\nabla q,\nabla p\rangle\|_{\theta,2)}=\sup_{0<\delta\le\epsilon_0}\delta^{\frac\theta2}\|\Delta p+\langle\nabla q,\nabla p\rangle\|_{2(1-\delta)}=O(\sup_{0<\delta\le\epsilon_0}\delta^{\frac\theta2}\|\Delta p+\langle\nabla q,\nabla p\rangle\|_{2(1-\epsilon_\delta)})\\ &=\|F_1-F_2\|_{\theta,\frac43)}^{\frac{4-3\theta}{6}}O(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac\theta2}+\eta\, O(\|\nabla p\|_2+\|\nabla p\|_2^{\frac12})\end{aligned}$$

and

$$\begin{aligned}(3.42)\qquad \|\nabla p\|_{\theta,4)} &= \sup_{0<\delta\le\epsilon_0}\delta^{\frac\theta4}\|\nabla p\|_{4(1-\delta)}=O(\sup_{0<\delta\le\epsilon_0}\delta^{\frac\theta4}\|\nabla p\|_{4(1-\epsilon_\delta)})\\ &= \|F_1-F_2\|_{\theta,\frac43)}^{\frac{4-3\theta}{12}}O(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac\theta4}+\eta\, O(\|\nabla p\|_2^{\frac12}+\|\nabla p\|_2^{\frac14}).\end{aligned}$$

Considering as above the two cases $w_1=F_1=0$ and $w_2=F_2=0$ by (3.42) and Young's inequality we obtain that

$$\|\nabla q\|_{\theta,4)}=O(\|\nabla w_1\|_{\theta,4)}+\|\nabla w_2\|_{\theta,4)})=O(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac13},$$

which inserted into (3.41) by Hölder's inequality gives

$$\begin{aligned}&\|\Delta p\|_{\theta,2)}=O(\|\Delta p+\langle\nabla q,\nabla p\rangle\|_{\theta,2)}+\|\nabla p\|_{\theta,4)}\|\nabla q\|_{\theta,4)})\\ &=\|F_1-F_2\|_{\theta,\frac43)}^{\frac{4-3\theta}{6}}O(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac\theta2}+\|F_1-F_2\|_{\theta,\frac43)}^{\frac{4-3\theta}{12}}O(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac{4+3\theta}{12}}\\ &+\eta(\|F_1\|_{\theta,\frac43)}+\|F_2\|_{\theta,\frac43)}+1)^{\frac13}\,O(\|\nabla p\|_2+\|\nabla p\|_2^{\frac14}).\end{aligned}$$

Therefore (3.37) has been established. ■

We have the following general result of independent interest.

Theorem 3.6. *Let $\frac{\gamma_2}{\gamma_3}\ge 6$. For any $\mu\in\mathcal M$ there exists a SOLA w of $\mathcal N(w)=\mu$ in M so that $w\in W^{1,2,2)}$. When $\gamma_2=6\gamma_3$ such a SOLA is unique.*

Proof. Since $\eta=0$ when $\gamma_2=6\gamma_3$, uniqueness directly follows from estimate (3.37) and we are just concerned with the existence issue. Letting ρ_n be a sequence of mollifiers in $[0,+\infty)$, define the approximate measures $\mu_n=(f_n-\overline{f}_n)dv$, where $f_n(x)=\int\rho_n(d(x,y))d\mu(y)$ are smooth functions. Since $\mu_n\rightharpoonup\mu$, by (3.36) and (3.38) we have that $F_n=H(\mu_n)$ is uniformly bounded in $L^{1,\frac43)}(M,TM)$ and is pre-compact in $L^q(M)$ for all $1\le q<\frac43$. Up to a subsequence, it is easily seen that F_n is a Cauchy sequence in $L^{\theta,\frac43)}(M,TM)$ for all $\theta>1$. In order to solve $\mathcal N(w_n)=f_n$ in M, notice that $\mathcal N(w)=\frac{J'(w)}{4}$, where

$$\begin{aligned}J(w) &= \gamma_2\int(\Delta w)^2dv-2\gamma_2\int \mathrm{Ric}(\nabla w,\nabla w)dv+12\gamma_3\int(\Delta w+|\nabla w|^2)^2dv\\ &\quad+(\frac23\gamma_2-4\gamma_3)\int R|\nabla w|^2dv,\quad w\in W^{2,2}(M).\end{aligned}$$

Since by squares completion

$$\beta\int(\Delta w)^2dv+12\int(\Delta w+|\nabla w|^2)^2dv\ge\frac{24+\beta-\sqrt{576+\beta^2}}{2}\int[(\Delta w)^2+|\nabla w|^4]dv$$

with $\beta = \frac{\gamma_2}{\gamma_3} > 0$, the functional $J(w) - 4\int fw\, dv$ is easily seen to attain a minimizer in $W^{2,2}(M) \cap \{\overline{w} = 0\}$ as long as $f \in L^q(M)$ for some $q > 1$. So we can construct $w_n \in W^{2,2}(M)$ solutions of $\mathcal{N}(w_n) = f_n$ in M, $\overline{w}_n = 0$, which are smooth thanks to [55]. Estimate (3.42) provides by Young's inequality

$$\|\nabla w_n\|_{1,4)} = O\Big(\|F_n\|_{1,\frac{4}{3})}^{\frac{1}{12}} (\|F_n\|_{1,\frac{4}{3})} + 1)^{\frac{1}{4}} + 1\Big).$$

Therefore, by (3.37) w_n is a bounded sequence in $W^{1,2,2)}$. In particular, w_n is uniformly bounded in $W^{2,q}(M)$ for all $1 \le q < 2$ and by Rellich's Theorem we deduce that, up to a subsequence, $w_n \to w$ in $W^{1,q}(M)$ for all $1 \le q < 4$. Since $\|\nabla(w_n - w_m)\|_2 \to 0$ as $n, m \to +\infty$, we can use again (3.37) to show that w_n is a Cauchy sequence in $W^{\theta,2,2)}$ for $1 < \theta < \frac{4}{3}$. Then w is a SOLA of $\mathcal{N}(w) = \mu$ in M with $w \in W^{1,2,2)}$ by the boundedness of w_n in $W^{1,2,2)}$. ■

Remark 3.7. *Let $\partial M \neq \emptyset$ and $\Phi \in C^\infty(\overline{M})$. For a Radon measure μ on M we say that a distributional solution w of $\mathcal{N}(w) = \mu$ in M, $w = \Phi$ and $\partial_\nu w = \partial_\nu \Phi$ on ∂M, is a SOLA if $w = \lim_{n\to+\infty} w_n$ a.e., where w_n are smooth solutions of $\mathcal{N}(w_n) = f_n$ in M, $w_n = \Phi$ and $\partial_\nu w_n = \partial_\nu \Phi$ on ∂M, for $f_n \in C^\infty(\overline{M})$ so that $f_n dv \rightharpoonup \mu$ as $n \to +\infty$. The map H is defined by using as $G_2(x,y)$ the Green function of Δ in M with zero Dirichlet boundary condition on ∂M. By Remark 3.4 we have that Proposition 3.5 still holds in this context provided $w_1 - w_2 \in W^{2,1}_0(M)$ and Theorem 3.6 does hold providing a SOLA $w \in W^{1,2,2)}(M)$ of $\mathcal{N}(w) = \mu$ in M, $w = \Phi$ and $\partial_\nu w = \partial_\nu \Phi$ on ∂M for any Radon measure μ.*

4. Fundamental solutions

Let $\mu_s = \sum_{i=1}^{l} \beta_i \delta_{p_i}$ be a linear combination of Dirac masses centred at distinct points $p_1, \dots, p_l \in M$. Given U as in (1.7), the parameters $\beta_1, \dots, \beta_l \neq 0$ are chosen to satisfy

$$\sum_{i=1}^{l} \beta_i = \int U dv. \tag{4.1}$$

Since (4.1) guarantees that $\mu_s - U \in \mathcal{M}$, for $\frac{\gamma_2}{\gamma_3} \ge 6$ we can apply Theorem 3.6 to find a SOLA $w_s \in W^{1,2,2)}(M)$ (recall (3.35)) of $\mathcal{N}(w_s) = \sum_{i=1}^{l} \beta_i \delta_{p_i} - U$ in M, referred to as a fundamental solution corresponding to μ_s. Unless $\gamma_2 = 6\gamma_3$, fundamental solutions w_s corresponding to μ_s are not unique and the aim now is to establish a logarithmic behaviour of each w_s, no matter whether uniqueness holds or not.

Since

$$\frac{d}{dx}[(\gamma_2 + 12\gamma_3)x + 18\gamma_3 x^2 + 6\gamma_3 x^3] = (\gamma_2 + 12\gamma_3) + 36\gamma_3 x + 18\gamma_3 x^2$$

has a given sign in view of $\Delta = -72\gamma_3^2(\frac{\gamma_2}{\gamma_3} - 6) \le 0$, let $\alpha_i = \alpha(\beta_i) \neq 0$ be the unique solution of

$$-4\pi^2[(\gamma_2 + 12\gamma_3)\alpha + 18\gamma_3\alpha^2 + 6\gamma_3\alpha^3] = \beta_i. \tag{4.2}$$

The function

$$w_0(x) = \sum_{i=1}^{l} \alpha_i \log \tilde{d}(x, p_i) \tag{4.3}$$

is an approximate solution of $\mathcal{N}(w) = \sum_{i=1}^{l} \beta_i \delta_{p_i} - U$ in M, where $\tilde{d}(x, p_i)$ stands for the distance function smoothed away from p_i. Since w_0 satisfies (3.12) and $\mathcal{N}(w_s) - \mathcal{N}(w_0)$ is sufficiently integrable, we can let $\epsilon \to 0$ in estimate (3.39) to obtain $W^{2,2}-$estimates w.r.t. $\hat{g} = e^{w_s + w_0} g$. Once re-written as $W^{2,2}-$estimates w.r.t. $g_0 = e^{2w_0} g$, the argument in [55] can be adapted to annular regions around the singularities to show that such weighted $W^{2,2}$-estimates imply the validity of (3.12) for w_s too.

Concerning the role of w_0 we have the following result.

Lemma 4.1. *The function w_0 in (4.3) is a distributional solution of*

$$\mathcal{N}(w_0)=\sum_{i=1}^{l}\beta_i\delta_{p_i}+f_0 \tag{4.4}$$

with $f_0-\gamma_2\,div[Ric(\cdot,\nabla w_0)]-(2\gamma_3-\frac{\gamma_2}{3})div(R\nabla w_0)\in L^\infty(M)$.

Proof. w_0 is a radial function in a neighbourhood of p_i, so in geodesic coordinates it satisfies

$$\Delta w_0=\frac{2\alpha_i}{|x|^2},\quad |\nabla w_0|^2=\frac{\alpha_i^2}{|x|^2},\quad (\Delta w_0+|\nabla w_0|^2)\nabla w_0=(2+\alpha_i)\alpha_i^2\frac{x}{|x|^4}$$

for all $x\neq 0$. Since

$$\begin{aligned}\mathcal{N}(w_0)&=(\frac{\gamma_2}{2}+6\gamma_3)\Delta^2 w_0+6\gamma_3\Delta(|\nabla w_0|^2)-12\gamma_3\text{div}[(\Delta w_0+|\nabla w_0|^2)\nabla w_0]+\gamma_2\text{div}[\text{Ric}(\cdot,\nabla w_0)]\\&\quad+(2\gamma_3-\frac{\gamma_2}{3})\text{div}(R\nabla w_0)=\gamma_2\text{div}[\text{Ric}(\cdot,\nabla w_0)]+(2\gamma_3-\frac{\gamma_2}{3})\text{div}(R\nabla w_0)\end{aligned}$$

near p_i and $\mathcal{N}(w_0)$ is a bounded function away from $p_1,\dots,p_l$, we have that w_0 solves $\mathcal{N}(w_0)=f_0$ in $M\setminus\{p_1,\dots,p_l\}$, with $f_0-\gamma_2\text{div}[\text{Ric}(\cdot,\nabla w_0)]-(2\gamma_3-\frac{\gamma_2}{3})\text{div}(R\nabla w_0)\in L^\infty(M)$.

Given $\epsilon>0$ small and $\varphi\in C^\infty(M)$, we have that

$$\begin{aligned}&\int_{M\setminus\cup_{i=1}^l B_\epsilon(p_i)}f_0\varphi dv=\int_{M\setminus\cup_{i=1}^l B_\epsilon(p_i)}\mathcal{N}(w_0)\varphi dv\\&=-\sum_{i=1}^{l}\oint_{\partial B_\epsilon(p_i)}\left[(\frac{\gamma_2}{2}+6\gamma_3)\partial_\nu\Delta w_0+6\gamma_3\partial_\nu|\nabla w_0|^2-12\gamma_3(\Delta w_0+|\nabla w_0|^2)\partial_\nu w_0\right]\varphi d\sigma\\&+\int_{M\setminus\cup_{i=1}^l B_\epsilon(p_i)}\left[(\frac{\gamma_2}{2}+6\gamma_3)\Delta w_0\Delta\varphi+6\gamma_3|\nabla w_0|^2\Delta\varphi+12\gamma_3(\Delta w_0+|\nabla w_0|^2)\langle\nabla w_0,\nabla\varphi\rangle\right]dv\\&-\int_{M\setminus\cup_{i=1}^l B_\epsilon(p_i)}\left[\gamma_2\text{Ric}(\nabla w_0,\nabla\varphi)+(2\gamma_3-\frac{\gamma_2}{3})R\langle\nabla w_0,\nabla\varphi\rangle\right]dv+o_\epsilon(1),\end{aligned}$$

where $o_\epsilon(1)\to 0$ as $\epsilon\to 0^+$. Since

$$\partial_\nu\left[(\frac{\gamma_2}{2}+6\gamma_3)\Delta w_0+6\gamma_3|\nabla w_0|^2\right]=-\frac{2\alpha_i}{\epsilon^3}[\gamma_2+12\gamma_3+6\alpha_i\gamma_3],\quad (\Delta w_0+|\nabla w_0|^2)\partial_\nu w_0=\frac{2\alpha_i^2+\alpha_i^3}{\epsilon^3}$$

on $\partial B_\epsilon(p_i)$, as $\epsilon\to 0^+$ we get that

$$\begin{aligned}&\int[(\frac{\gamma_2}{2}+6\gamma_3)\Delta w_0\Delta\varphi+6\gamma_3|\nabla w_0|^2\Delta\varphi+12\gamma_3(\Delta w_0+|\nabla w_0|^2)\langle\nabla w_0,\nabla\varphi\rangle]dv\\&-\int[\gamma_2\text{Ric}(\nabla w_0,\nabla\varphi)+(2\gamma_3-\frac{\gamma_2}{3})R\langle\nabla w_0,\nabla\varphi\rangle]dv=\sum_{i=1}^{l}\beta_i\varphi(p_i)+\int f_0\varphi\,dv\end{aligned}$$

for all $\varphi\in C^\infty(M)$ in view of (4.2), i.e. w_0 is a distributional solution of (4.4). ■

Remark 4.2. *Let $\Phi\in C^\infty(\overline{B_r(p_i)})$, $i=1,\dots,l$, so that $\Phi=0$ near p_i and assume that $\{p_1,\dots,p_l\}\cap\overline{B_r(p_i)}=\{p_i\}$. Letting $-4\pi^2[(\gamma_2+12\gamma_3)\alpha_i+18\gamma_3\alpha_i^2+6\gamma_3\alpha_i^3]=\beta_i$, choose $w_0(x)=\alpha_i\log\tilde d(x,p_i)$ in such a way that $w_0=0$ near $\partial B_r(p_i)$. We have that $w_0+\Phi$ is a distributional solution of (4.4) in $B_r(p_i)$ such that $f_0-\gamma_2\,div[Ric(\cdot,\nabla w_0)]-(2\gamma_3-\frac{\gamma_2}{3})div(R\nabla w_0)\in L^\infty(B_r(p_i))$. Moreover, thanks to Remark 3.7 there exists a fundamental solution w_s corresponding to μ_s and Φ, namely a SOLA $w_s\in W^{1,2,2)}(B_r(p_i))$ of $\mathcal{N}(w_s)=\beta_i\delta_{p_i}-U$ in $B_r(p_i)$, $w_s=\Phi$ and $\partial_\nu w_s=\partial_\nu\Phi$ on $\partial B_r(p_i)$.*

The aim now is to show that any fundamental solution w_s has a logarithmic behaviour near $p_1,\dots,p_l$. For problems involving the $p-$Laplace operator an extensive study on isolated singularities is available, see [35, 51, 52] (see also [37] for some fully nonlinear equations in conformal geometry). We adapt the argument in [55] to our situation and in presence of singularities to show the following result.

Theorem 4.3. *Let $\frac{\gamma_2}{\gamma_3}\geq 6$. Any fundamental solution w_s corresponding to μ_s satisfies $w_s\in C^\infty(M\setminus\{p_1,\dots,p_l\})$ and (3.12) with α_i given by (4.2).*

Proof. Recall that w_s is a SOLA of $\mathcal{N}(w_s)=\mu_s-U:=\operatorname{div}F$ and w_0 is a distributional solution of $\mathcal{N}(w_0)=\mu_s+f_0:=\operatorname{div}F_0$. Since $F,F_0\in L^{1,\frac{4}{3})}(M,TM)$ with $\operatorname{div}(F-F_0)=-(f_0+U)\in L^q(M)$ for all $1\le q<2$ in view of

$$f_0-\gamma_2\operatorname{div}[\operatorname{Ric}(\cdot,\nabla w_0)]-(2\gamma_3-\frac{\gamma_2}{3})\operatorname{div}(R\nabla w_0)\in L^\infty(M) \tag{4.5}$$

by Lemma 4.1, we can let $\epsilon\to 0^+$ in (3.39) and by Fatou's lemma end up with

$$\int[|\nabla^2_{\hat g}p|^2+|\nabla p|^4]dv\le C(\|F-F_0\|^{\frac{4}{3}}_{\frac{4}{3}}+\eta\|\nabla p\|_2^2+1)<+\infty \tag{4.6}$$

in view of (3.2), where $p=w_s-w_0$ and $\hat g=e^{w_s+w_0}g$. Setting $g_0=e^{2w_0}g$, by $2w_0=w_s+w_0-p$ we deduce that $\nabla^2_{g_0}p=\nabla^2_{\hat g}p+O(|\nabla p|^2)$ in view of (3.3) and then (4.6) re-writes as

$$\int[|\nabla^2_{g_0}p|^2+|\nabla p|^4]dv<+\infty. \tag{4.7}$$

Notice that w_s and w_0 satisfy

$$\langle\mathcal{N}(w_s)-\mathcal{N}(w_0),\varphi\rangle=-\int(f_0+U)\varphi\,dv,\quad \varphi\in C^\infty(M), \tag{4.8}$$

and it is crucial to properly re-write the L.H.S. in terms of g_0 and not $\hat g$ as in (3.1). We can argue exactly as in Proposition 3.1 to get

$$\begin{aligned}\langle\mathcal{N}(w_s)-\mathcal{N}(w_0),\varphi\rangle &= 3\gamma_3\int(\Delta_{g_0}p+2|\nabla p|^2_{g_0})\Delta_{g_0}\varphi\,dv_{g_0}+6\gamma_3\int\langle\nabla^2_{g_0}p,\nabla^2_{g_0}\varphi\rangle_{g_0}dv_{g_0}\\ &\quad+12\gamma_3\int(\Delta_{g_0}p+|\nabla p|^2_{g_0})\langle\nabla p,\nabla\varphi\rangle_{g_0}dv_{g_0}+(\frac{\gamma_2}{2}-3\gamma_3)\int\Delta p\Delta\varphi\,dv\\ &\quad+(2\gamma_3-\frac{\gamma_2}{3})\int[3\operatorname{Ric}(\nabla p,\nabla\varphi)dv-R\langle\nabla p,\nabla\varphi\rangle]dv\end{aligned} \tag{4.9}$$

for all $\varphi\in C^\infty(M)$. Setting $\Delta_0 p=\Delta p+2\langle\nabla w_0,\nabla p\rangle$, by (3.2) we can re-write (4.8)-(4.9) as

$$\begin{aligned}&3\gamma_3\int[\Delta_0p+2|\nabla p|^2]\Delta_0\varphi\,dv+6\gamma_3\int\langle\nabla^2_{g_0}p,\nabla^2_{g_0}\varphi\rangle dv+12\gamma_3\int[\Delta_0p+|\nabla p|^2]\langle\nabla p,\nabla\varphi\rangle dv\\ &+(\frac{\gamma_2}{2}-3\gamma_3)\int\Delta p\Delta\varphi\,dv+(2\gamma_3-\frac{\gamma_2}{3})\int[3\operatorname{Ric}(\nabla p,\nabla\varphi)dv-R\langle\nabla p,\nabla\varphi\rangle]dv=-\int(f_0+U)\varphi\,dv\end{aligned} \tag{4.10}$$

for all $\varphi\in C^\infty(M)$.

Given $p=p_i$, $i=1,\dots,l$, set $\alpha=\alpha_i$, $A=\{x\in M:\ d(x,p)\in[\frac{r}{4},8r]\}$, $r>0$ small, and fix $2\le q<4$. Through geodesic coordinates at p and the change of variable $x=ry$, notice that

$$\begin{aligned}\int_A|\Delta_0\varphi|^q dv &= \int_{B_{8r}\setminus B_{\frac{r}{4}}}|\Delta\varphi+\frac{2\alpha}{|x|}\partial_{|x|}\varphi|^q\sqrt{|g|}dx=r^{4-2q}\int_{B_8\setminus B_{\frac14}}|\Delta_{g^r}\varphi^r+\frac{2\alpha}{|y|}\partial_{|y|}\varphi^r|^q\sqrt{|g^r|}dy\\ &\le Cr^{4-2q}\int_{B_8\setminus B_{\frac14}}|\Delta_{g^r}\varphi^r|^q\sqrt{|g^r|}dy=C\int_A|\Delta\varphi|^q dv\end{aligned} \tag{4.11}$$

for all $\varphi\in W^{2,q}_0(A)$, where $\varphi^r(y)=\varphi(\exp_p(ry))\in W^{2,q}_0(B_8\setminus B_{\frac14})$ and $g^r(y)=g(\exp_p(ry))\to\delta_{\text{eucl}}$ C^2−uniformly in $B_8\setminus B_{\frac14}$ as $r\to 0^+$. We have used that

$$\int_{B_8\setminus B_{\frac14}}|\nabla\varphi^r|^q\sqrt{|g^r|}dy\le C\int_{B_8\setminus B_{\frac14}}|\Delta_{g^r}\varphi^r|^q\sqrt{|g^r|}dy$$

in view of Poincaré's inequality. Arguing in the same way, one can also show that

$$\int_A|\nabla^2_{g_0}\varphi|^q dv\le C'\int_A|\nabla^2\varphi|^q dv\le C\int_A|\Delta\varphi|^q dv \tag{4.12}$$

for all $\varphi\in W^{2,q}_0(A)$, and

$$(\int_A|\psi|^{\frac{4q}{4-q}}dv)^{\frac{4-q}{4q}}\le C(\int_A|\nabla\psi|^q dv)^{\frac1q},\quad (\int_A|\nabla\varphi|^{\frac{4q}{4-q}}dv)^{\frac{4-q}{4q}}\le C(\int_A|\Delta\varphi|^q dv)^{\frac1q} \tag{4.13}$$

for all $\psi\in W^{1,q}(A)$ such that either $\psi|_{\partial A}=0$ or $\overline{\psi}^A=0$ and for all $\varphi\in W^{2,q}_0(A)$.

Given $\tilde{\chi} \in C_0^\infty(\frac{1}{4}, 8)$ so that $0 \leq \tilde{\chi} \leq 1$ and $\tilde{\chi} = 1$ on $[\frac{1}{2}, 4]$, set $\chi(x) = \tilde{\chi}(\frac{d(x,p)}{r})$ and let

$$\epsilon_r^2 = \int_A (\Delta_0 p)^2 dv + \int_A |\nabla_{g_0}^2 p|^2 dv + (\int_A |\nabla p|^4 dv)^{\frac{1}{2}}. \tag{4.14}$$

We can assume that $0 < \epsilon_r \leq 1$ for $r > 0$ small since

$$\lim_{r \to 0} \epsilon_r = 0$$

in view of (4.7). By (4.11), (4.13) and Hölder's estimate we have that

$$\begin{aligned} &|\int [\Delta_0 p + 2|\nabla p|^2][\varphi \Delta_0 \chi + 2\langle \nabla \chi, \nabla \varphi \rangle] dv| + |\int [2\langle \nabla \chi, \nabla p \rangle (1 + p - \overline{p}^A) + \Delta_0 \chi (p - \overline{p}^A)] \Delta_0 \varphi \, dv| \\ &\leq \frac{C' \epsilon_r}{r^{\frac{2(q-2)}{q}}} [(\int_A |\varphi|^{\frac{2q}{q-2}} dv)^{\frac{q-2}{2q}} + (\int_A |\nabla \varphi|^{\frac{4q}{3q-4}} dv)^{\frac{3q-4}{4q}} + (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}}] \\ &\leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}} (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}} \end{aligned}$$

for all $\varphi \in W_0^{2,\frac{q}{q-1}}(A)$, taking into account that

$$\begin{aligned} &|\int \langle \nabla \chi, \nabla p \rangle (p - \overline{p}^A) \Delta_0 \varphi \, dv| + |\int \Delta_0 \chi (p - \overline{p}^A) \Delta_0 \varphi \, dv| \\ &\leq \frac{C''}{r^2} \left[r \epsilon_r (\int_A |p - \overline{p}^A|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}} + (\int_A |p - \overline{p}^A|^q dv)^{\frac{1}{q}} \right] (\int_A |\Delta_0 \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}} \\ &\leq \frac{C'}{r^2} \left[r \epsilon_r (\int_A |\nabla p|^q dv)^{\frac{1}{q}} + (\int_A |\nabla p|^{\frac{4q}{q+4}} dv)^{\frac{q+4}{4q}} \right] (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}} \leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}} (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}}. \end{aligned}$$

Since

$$\begin{aligned} (\Delta_0 p + 2|\nabla p|^2) \Delta_0 (\chi \varphi) &= (\Delta_0 h + 2\langle \nabla h, \nabla p \rangle) \Delta_0 \varphi + (\Delta_0 p + 2|\nabla p|^2)(\varphi \Delta_0 \chi + 2\langle \nabla \chi, \nabla \varphi \rangle) \\ &\quad -[2\langle \nabla \chi, \nabla p \rangle (1 + p - \overline{p}^A) + \Delta_0 \chi (p - \overline{p}^A)] \Delta_0 \varphi, \end{aligned}$$

where $h = \chi(p - \overline{p}^A)$, we have that for some $\mathcal{L}_1 \in W^{-2,q}(A)$:

$$\int (\Delta_0 p + 2|\nabla p|^2) \Delta_0 (\chi \varphi) dv = \int_A (\Delta_0 h + 2\langle \nabla h, \nabla p \rangle) \Delta_0 \varphi dv + \mathcal{L}_1, \quad \|\mathcal{L}_1\| \leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}}. \tag{4.15}$$

Analogously, there holds

$$\begin{aligned} &6\gamma_3 \int \langle \nabla_{g_0}^2 p, \nabla_{g_0}^2 (\chi \varphi) \rangle dv + (\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p \Delta (\chi \varphi) dv = 6\gamma_3 \int_A \langle \nabla_{g_0}^2 h, \nabla_{g_0}^2 \varphi \rangle dv \\ &+(\frac{\gamma_2}{2} - 3\gamma_3) \int_A \Delta h \Delta \varphi dv + \mathcal{L}_2, \quad \|\mathcal{L}_2\| \leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}}, \end{aligned} \tag{4.16}$$

thanks to

$$\begin{aligned} &|\int \langle O(|\nabla p||\nabla \chi|) + \nabla_{g_0}^2 \chi (p - \overline{p}^A), \nabla_{g_0}^2 \varphi \rangle dv| + |\int \langle \nabla_{g_0}^2 p, \varphi \nabla_{g_0}^2 \chi + O(|\nabla \chi||\nabla \varphi|) \rangle dv| \\ &\leq \frac{C' \epsilon_r}{r^{\frac{2(q-2)}{q}}} [(\int_A |\varphi|^{\frac{2q}{q-2}} dv)^{\frac{q-2}{2q}} + (\int_A |\nabla \varphi|^{\frac{4q}{3q-4}} dv)^{\frac{3q-4}{4q}} + (\int_A |\nabla_{g_0}^2 \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}}] \\ &\leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}} (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}} \end{aligned}$$

and

$$\begin{aligned} \langle \nabla_{g_0}^2 p, \nabla_{g_0}^2 (\chi \varphi) \rangle &= \langle \nabla_{g_0}^2 h - d\chi \otimes dp - dp \otimes d\chi - \nabla_{g_0}^2 \chi (p - \overline{p}^A), \nabla_{g_0}^2 \varphi \rangle \\ &\quad + \langle \nabla_{g_0}^2 p, \varphi \nabla_{g_0}^2 \chi + d\chi \otimes d\varphi + d\varphi \otimes d\chi \rangle, \end{aligned}$$

in view of (3.3) and (4.12)-(4.13). Since in a similar way

$$|\int |\nabla \chi| \Big(|\Delta_0 p| + |\nabla p|^2 + 1 \Big) \Big(|\nabla \varphi||p - \overline{p}^A| + |\nabla p||\varphi| \Big) dv| \leq \frac{C \epsilon_r}{r^{\frac{2(q-2)}{q}}} (\int_A |\Delta \varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}}$$

for all $\varphi \in W_0^{2,\frac{q}{q-1}}(A)$, there holds

$$12\gamma_3 \int [\Delta_0 p + |\nabla p|^2]\langle \nabla p, \nabla(\chi\varphi)\rangle dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int [3\mathrm{Ric}(\nabla p, \nabla(\chi\varphi))dv - R\langle \nabla p, \nabla(\chi\varphi)\rangle]dv \tag{4.17}$$
$$= 12\gamma_3 \int_A [\Delta_0 p + |\nabla p|^2]\langle \nabla h, \nabla\varphi\rangle dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int_A [3\mathrm{Ric}(\nabla h, \nabla\varphi)dv - R\langle \nabla h, \nabla\varphi\rangle]dv$$
$$+\mathcal{L}_3, \quad \|\mathcal{L}_3\| \le \frac{C\epsilon_r}{r^{\frac{2(q-2)}{q}}}.$$

Since by density and (4.7) we can use $\chi\varphi$, $\varphi \in W_0^{2,2}(A)$, into (4.10), by collecting (4.15)-(4.17) one has that

$$3\gamma_3 \int_A [\Delta_0 h + 2\langle \nabla h, \nabla p\rangle]\Delta_0\varphi\, dv + 6\gamma_3 \int_A \langle \nabla^2_{g_0} h, \nabla^2_{g_0}\varphi\rangle dv + 12\gamma_3 \int_A [\Delta_0 p + |\nabla p|^2]\langle \nabla h, \nabla\varphi\rangle dv \tag{4.18}$$
$$+(\frac{\gamma_2}{2} - 3\gamma_3) \int_A \Delta h \Delta\varphi dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int_A [3\mathrm{Ric}(\nabla h, \nabla\varphi)dv - R\langle \nabla h, \nabla\varphi\rangle]dv = \mathcal{L}(\varphi)$$

for some $\mathcal{L} \in W^{-2,2}(A)$, which can also be regarded as $\mathcal{L} \in W^{-2,q}(A)$ satisfying

$$\|\mathcal{L}\| \le \frac{C\epsilon_r}{r^{\frac{2(q-2)}{q}}} + (\int_A |f_0 + U|^{\frac{2q}{q+2}} dv)^{\frac{q+2}{2q}}, \tag{4.19}$$

in view of

$$|\int (f_0 + U)\chi\varphi\, dv| \le (\int_A |f_0 + U|^{\frac{2q}{q+2}} dv)^{\frac{q+2}{2q}} (\int_A |\varphi|^{\frac{2q}{q-2}} dv)^{\frac{q-2}{2q}}.$$

Since

$$|\int_A \langle \nabla\tilde{h}, \nabla p\rangle \Delta_0\varphi\, dv| + |\int_A [\Delta_0 p + |\nabla p|^2]\langle \nabla\tilde{h}, \nabla\varphi\rangle dv|$$
$$\le \epsilon_r (\int_A |\nabla\tilde{h}|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}} (\int_A |\Delta\varphi|^{\frac{q}{q-1}} dv)^{\frac{q-1}{q}} + C\epsilon_r (\int_A |\nabla\tilde{h}|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}} (\int_A |\nabla\varphi|^{\frac{4q}{3q-4}} dv)^{\frac{3q-4}{4q}}$$

and

$$|\int_A [3\mathrm{Ric}(\nabla\tilde{h}, \nabla\varphi)dv - R\langle \nabla\tilde{h}, \nabla\varphi\rangle]dv| \le Cr^2 (\int_A |\nabla\tilde{h}|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}} (\int_A |\nabla\varphi|^{\frac{4q}{3q-4}} dv)^{\frac{3q-4}{4q}}$$

for all $\varphi \in W_0^{2,\frac{q}{q-1}}(A)$, equation (4.18) written in $\tilde{h}$ is equivalent to

$$3\gamma_3 \int_A \Delta_0\tilde{h}\Delta_0\varphi\, dv + 6\gamma_3 \int_A \langle \nabla^2_{g_0}\tilde{h}, \nabla^2_{g_0}\varphi\rangle dv + (\frac{\gamma_2}{2} - 3\gamma_3) \int_A \Delta\tilde{h}\Delta\varphi dv + T[\tilde{h}](\varphi) = \mathcal{L}(\varphi),$$

where $T : W_0^{2,q}(A) \to W^{-2,q}(A)$ is a linear operator which satisfies $\|T\| \le C(\epsilon_r + r^2)$. The crucial point is that the linear operator $\Delta_0^2 : W_0^{2,q}(A) \to W^{-2,q}(A)$ is an isomorphism with uniformly bounded inverse, where

$$\Delta_0^2\tilde{h}(\varphi) = \int_A \Delta_0\tilde{h}\Delta_0\varphi\, dv + 2\int_A \langle \nabla^2_{g_0}\tilde{h}, \nabla^2_{g_0}\varphi\rangle dv + (\frac{\gamma_2}{6\gamma_3} - 1)\int_A \Delta\tilde{h}\Delta\varphi dv.$$

Since $\epsilon_r + r^2 \to 0$ as $r \to 0$ we have that $3\gamma_3\Delta_0^2 + T : W_0^{2,q}(A) \to W^{-2,q}(A)$ is still an isomorphism with uniformly bounded inverse. Then $3\gamma_3\Delta_0^2\tilde{h} + T[\tilde{h}] = \mathcal{L}$ is uniquely solvable in $W_0^{2,q}(A)$ for all $2 \le q < 4$ and such a solution $\tilde{h}$ coincides with $h \in W_0^{2,2}(A)$ by uniqueness in $W_0^{2,2}(A)$. So for all $2 \le q < 4$ we have shown that

$$\|h\|_{W_0^{2,q}(A)} \le C'\|\mathcal{L}\|_{W^{-2,q}(A)} \le C\left[\frac{\epsilon_r}{r^{\frac{2(q-2)}{q}}} + (\int_A |f_0 + U|^{\frac{2q}{q+2}} dv)^{\frac{q+2}{2q}}\right] \tag{4.20}$$

for some $C > 0$ thanks to (4.19).

In order to show that $\Delta_0^2 : W_0^{2,q}(A) \to W^{-2,q}(A)$ is an isomorphism with uniformly bounded inverse, notice first that

$$\delta_A := \inf\left\{\int_A (\Delta_0 h)^2 dv : \ h \in W_0^{2,2}(M), \ \int_A (\Delta h)^2 dv = 1\right\} > 0. \tag{4.21}$$

Indeed, letting h_n be a minimizing sequence in (4.21), we can assume that $h_n \rightharpoonup h$ in $W_0^{2,2}(A)$ and $h_n \to h$ in $W_0^{1,2}(A)$ as $n \to +\infty$ thanks to Sobolev's embedding Theorem. When $h = 0$ we have that $\int_A (\Delta_0 h_n)^2 dv \to 1$ as $n \to +\infty$ and then $\delta_A = 1$. If $h \ne 0$, we need to show that $\Delta_0 h \ne 0$ since by weak lower semi-continuity $\delta_A \geq \int_A (\Delta_0 h)^2 \, dv$. Observe that $\Delta_0 h = \Delta h + 2\langle \nabla w_0, \nabla h\rangle = 0$ has only the trivial solution in $W_0^{2,2}(A)$ as it follows by testing $\Delta_0 h$ against $e^{2w_0}h$ and integrating by parts:

$$0 = \int_A (\Delta h + 2\langle \nabla w_0, \nabla h\rangle) e^{2w_0} h dv = -\int_A e^{2w_0} |\nabla h|^2 dv.$$

Since every $\mathcal{L} \in W^{-2,q}(A)$ can be viewed as an element in $W^{-2,2}(A)$ in view of $\frac{q}{q-1} \leq 2$ and by (4.21) there holds

$$\Delta_0^2 h(h) = \int_A (\Delta_0 h)^2 dv + 2\int_A |\nabla_{g_0}^2 h|^2 dv + (\frac{\gamma_2}{6\gamma_3} - 1)(\int_A \Delta h)^2 dv \geq \delta_A \int_A (\Delta h)^2 dv$$

due to $\frac{\gamma_2}{\gamma_3} \geq 6$, we can minimize $\frac{1}{2}\Delta_0^2 h(h) - \mathcal{L}(h)$ in $W_0^{2,2}(A)$ and find a solution $h \in W_0^{2,2}(A)$ of $\Delta_0^2 h = \mathcal{L}$ in $W^{-2,2}(A)$. Thanks to (3.2)-(3.3) and (3.9) let us now rewrite $\Delta_0^2 h(\varphi)$ as

$$\Delta_0^2 h(\varphi) = (2 + \frac{\gamma_2}{6\gamma_3}) \int_A \Delta h \Delta \varphi \, dv + \tilde{\mathcal{L}}(\varphi),$$

where $\tilde{\mathcal{L}}$ satisfies $|\tilde{\mathcal{L}}(\varphi)| \leq \frac{C}{r} \|h\|_{W_0^{2,2}(A)} \|\varphi\|_{W_0^{2,\frac{4}{3}}(A)}$. Since $\mathcal{L} \in W^{-2,q}(A)$ and $\tilde{\mathcal{L}} \in W^{-2,4}(A)$, we can use elliptic estimates for the bi-Laplacian operator in [3] to show that $h \in W_0^{2,q}(A)$. Moreover, by the inverse mapping theorem we know that $\|\Delta_0^2 h\|_{W^{-2,q}(A)} \geq \delta \|h\|_{W_0^{2,q}(A)}$ for some $\delta = \delta(r) > 0$. To see that $\delta > 0$ can be chosen independent of $r > 0$, through geodesic coordinates at p and the change of variable $x = ry$ as in (4.11) we simply observe that

$$\|\Delta_0^2 h\|_{W^{-2,q}(A)} = r^{\frac{4-2q}{q}} \sup\{\Delta_0^{2,r} h^r(\psi) : \ \psi \in W_0^{2,\frac{q}{q-1}}(B_8 \setminus B_{\frac{1}{4}}), \ \int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta_{g^r} \psi|^{\frac{q}{q-1}} dv_{g^r} \leq 1\}$$

and $\|h\|_{W_0^{2,q}(A)} = r^{\frac{4-2q}{q}} (\int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta_{g^r} h^r|^q dv_{g^r})^{\frac{1}{q}}$, where $\nabla w_0^r(y) = \frac{\alpha y}{|y|^2}$ and

$$\begin{aligned} \Delta_0^{2,r} h^r(\psi) \quad = \quad & \int_{B_8 \setminus B_{\frac{1}{4}}} (\Delta_{g^r} h^r + 2\langle \nabla w_0^r, \nabla h^r\rangle_{g^r})(\Delta_{g^r}\psi + 2\langle \nabla w_0^r, \nabla \psi\rangle_{g^r}) dv_{g^r} \\ & + 2\int_{B_8 \setminus B_{\frac{1}{4}}} \langle \nabla_{g_0^r}^2 h^r, \nabla_{g_0^r}^2 \psi\rangle_{g^r} dv_{g^r} + (\frac{\gamma_2}{6\gamma_3} - 1)\int_{B_8 \setminus B_{\frac{1}{4}}} \Delta_{g^r} h^r \Delta_{g^r} \psi \, dv_{g^r}. \end{aligned}$$

Since $g_r(y) = g(ry) \to \delta_{\text{eucl}}$ C^2-uniformly in $B_8 \setminus B_{\frac{1}{4}}$ as $r \to 0^+$, we have that

$$\sup\{\Delta_0^{2,r} \tilde{h}(\psi) : \ \psi \in W_0^{2,\frac{q}{q-1}}(B_8 \setminus B_{\frac{1}{4}}), \ \int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta_{g^r} \psi|^{\frac{q}{q-1}} dv_{g^r} \leq 1\} \geq \delta(\int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta_{g^r} \tilde{h}|^q dv_{g^r})^{\frac{1}{q}} \tag{4.22}$$

uniformly in $\tilde{h}$ for some $\delta > 0$, and then $\|\Delta_0^2 h\|_{W^{-2,q}(A)} \geq \delta \|h\|_{W_0^{2,q}(A)}$. We have used that the desired inequality $\|\Delta_{0,eucl}^2 \tilde{h}\|_{W^{-2,q}(B_8 \setminus B_{\frac{1}{4}})} \geq \delta \|\tilde{h}\|_{W_0^{2,q}(B_8 \setminus B_{\frac{1}{4}})}$ does hold in the euclidean case with some $\delta > 0$ and the following convergences:

$$\text{L.H.S. in (4.22)} \to \sup\{\Delta_{0,eucl}^2 \tilde{h}(\psi) : \ \psi \in W_0^{2,\frac{q}{q-1}}(B_8 \setminus B_{\frac{1}{4}}), \ \int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta \psi|^{\frac{q}{q-1}} dx \leq 1\} = \|\Delta_{0,eucl}^2 \tilde{h}\|_{W^{-2,q}(B_8 \setminus B_{\frac{1}{4}})}$$

and

$$\text{R.H.S. in (4.22)} \to (\int_{B_8 \setminus B_{\frac{1}{4}}} |\Delta \tilde{h}|^q dx)^{\frac{1}{q}}$$

as $r \to 0^+$ uniformly in $\tilde{h}$.

Set $\tilde{A} = \{x \in M : \ d(x,p) \in [\frac{r}{2}, 4r]\}$. Notice that by (3.3) and (4.13) it follows that

$$(\int_{\tilde{A}} |\Delta_0 p|^q dv)^{\frac{1}{q}} + (\int_{\tilde{A}} |\nabla_{g_0}^2 p|^q dv)^{\frac{1}{q}} + (\int_{\tilde{A}} |\nabla p|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}} + r^{\frac{2(2-q)}{q}} \|p - \overline{p}^{\tilde{A}}\|_{\infty, \tilde{A}} \leq C \|h\|_{W_0^{2,q}(A)} \tag{4.23}$$

for some $C > 0$, in view of $(\int_A |\nabla w_0|^q |\nabla h|^q dv)^{\frac{1}{q}} \leq C(\int_A |\nabla h|^{\frac{4q}{4-q}} dv)^{\frac{4-q}{4q}}$ and through geodesic coordinates

$$\|\psi\|_{\infty, \tilde{A}} = \|\psi^r\|_{\infty, B_4 \setminus B_{\frac{1}{2}}} \leq C(\int_{B_4 \setminus B_{\frac{1}{2}}} |\Delta_{g^r} \psi^r|^q \sqrt{|g^r|} dy)^{\frac{1}{q}} = Cr^{\frac{2(q-2)}{q}} (\int_{\tilde{A}} |\Delta \psi|^q dv)^{\frac{1}{q}} \tag{4.24}$$

for all $\psi \in W^{1,q}(\tilde A)$ with $\overline{\psi}^{\tilde A}$ and for $q>2$. To get stronger estimates, let $\tilde\chi \in C_0^\infty(\frac12,4)$ with $0 \le \tilde\chi \le 1$ and $\tilde\chi = 1$ on $[1,2]$, and define now $\chi(x) = \tilde\chi(\frac{d(x,p)}{r})$ and $h = \chi(p-\overline{p}^{\tilde A})$. Thanks to (4.20) and (4.23) we can repeat the above argument and, integrating by parts all the terms involving second-order derivatives of φ, get that:

$$\begin{aligned}
&|\int [\Delta_0 p + 2|\nabla p|^2][\varphi \Delta_0 \chi + 2\langle \nabla \chi, \nabla \varphi\rangle] dv| + |\int [2\langle \nabla\chi, \nabla p\rangle (1+p-\overline{p}^{\tilde A}) + \Delta_0 \chi (p-\overline{p}^{\tilde A})]\Delta_0 \varphi\, dv| \\
&+|\int \langle d\chi \otimes dp + dp \otimes d\chi + \nabla^2_{g_0}\chi (p-\overline{p}^{\tilde A}), \nabla^2_{g_0}\varphi\rangle dv| + |\int \langle \nabla^2_{g_0} p, \varphi \nabla^2_{g_0}\chi + d\chi\otimes d\varphi + d\varphi \otimes d\chi\rangle dv| \\
&+|\int \Delta p[\varphi \Delta\chi + 2\langle\nabla\chi,\nabla\varphi\rangle]dv| + |\int [2\langle\nabla\chi,\nabla p\rangle + \Delta\chi(p-\overline{p}^{\tilde A})]\Delta\varphi\, dv| \\
&+\int |\nabla\chi|(|\Delta_0 p| + |\nabla p|^2 + 1)(|\nabla\varphi||p-\overline{p}^{\tilde A}| + |\nabla p||\varphi|)dv| \\
&\le \frac{C}{r}[\frac{\epsilon_r}{r^{\frac{2(q-2)}{q}}} + (\int_A |f_0+U|^{\frac{2q}{q+2}}dv)^{\frac{q+2}{2q}}](\int_{\tilde A} |\nabla\varphi|^{\frac{q}{q-1}}dv)^{\frac{q-1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&|\int_{\tilde A} \langle \nabla\tilde h, \nabla p\rangle \Delta_0\varphi\, dv| + |\int_{\tilde A}[\Delta_0 p + |\nabla p|^2]\langle\nabla\tilde h,\nabla\varphi\rangle dv| + |\int_{\tilde A}[3\mathrm{Ric}(\nabla\tilde h,\nabla\varphi)dv - R\langle\nabla\tilde h,\nabla\varphi\rangle]dv| \\
&\le C(\tilde\epsilon_r + r^2)\|\tilde h\|_{W_0^{3,q}(\tilde A)}(\int_{\tilde A}|\nabla\varphi|^{\frac{q}{q-1}}dv)^{\frac{q-1}{q}}
\end{aligned}$$

for all $\varphi \in W_0^{2,\frac{q}{q-1}}(\tilde A)$, where $\tilde\epsilon_r$ is given by (4.14) on $\tilde A$. Notice that quadratic or cubic terms in p have been estimated in the above expression by using (4.23) on p and (4.14) for the remaining powers of p. Hence, equation (4.18) in $\tilde h$ is equivalent to

$$3\gamma_3 \Delta_0^2 \tilde h(\varphi) + T[\tilde h](\varphi) = \mathcal{L}(\varphi),$$

where $T: W_0^{3,q}(\tilde A) \to W^{-1,q}(\tilde A)$ is a linear operator so that $\|T\| \le C(\tilde\epsilon_r + r^2)$ and $\mathcal{L} \in W^{-1,q}(\tilde A)$ satisfies

$$\|\mathcal{L}\| \le \frac{C}{r}[\frac{\epsilon_r}{r^{\frac{2(q-2)}{q}}} + (\int_A |f_0+U|^{\frac{2q}{q+2}}dv)^{\frac{q+2}{2q}}] + (\int_{\tilde A}|f_0+U|^{\frac{4q}{q+4}}dv)^{\frac{q+4}{4q}}$$

in view of

$$|\int_{\tilde A}(f_0+U)\chi\varphi\, dv| \le (\int_{\tilde A}|f_0+U|^{\frac{4q}{q+4}}dv)^{\frac{q+4}{4q}}(\int_{\tilde A}|\nabla\varphi|^{\frac{q}{q-1}}dv)^{\frac{q-1}{q}}.$$

Arguing as before, since the operator $\Delta_0^2 : W_0^{3,q}(\tilde A) \to W^{-1,q}(\tilde A)$ is an isomorphism with uniformly bounded inverse, $3\gamma_3\Delta_0^2\tilde h + T[\tilde h] = \mathcal{L}$ is uniquely solvable in $W_0^{3,q}(A)$, $2<q<4$, and such a solution $\tilde h$ coincides with $h \in W_0^{2,2}(\tilde A)$ by uniqueness in $W_0^{2,2}(\tilde A)$. Then, for all $2<q<4$ there holds

$$\|h\|_{W_0^{3,q}(\tilde A)} \le C\left[\frac{\epsilon_r}{r^{\frac{3q-4}{q}}} + \frac{1}{r}(\int_A|f_0+U|^{\frac{2q}{q+2}}dv)^{\frac{q+2}{2q}} + (\int_{\tilde A}|f_0+U|^{\frac{4q}{q+4}}dv)^{\frac{q+4}{4q}}\right] \tag{4.25}$$

for some $C>0$. Since arguing as in (4.24) there holds

$$r\|\nabla h\|_{\infty,\tilde A} = \|\nabla h^r\|_{\infty, B_4\setminus B_{\frac12}} \le C(\int_{B_4\setminus B_{\frac12}}|\Delta_{g^r}h^r|^{\frac{4q}{4-q}}\sqrt{|g^r|}dy)^{\frac{4-q}{4q}} = Cr^{\frac{3q-4}{q}}(\int_{\tilde A}|\Delta h|^{\frac{4q}{4-q}}dv)^{\frac{4-q}{4q}}$$

in view of $\frac{4q}{4-q} > 4$, by (4.13) and (4.25) for all $2<q<4$ we finally deduce that

$$r\|\nabla p\|_{\infty, B_{2r}\setminus B_r} \le C\left[\epsilon_r + r^{\frac{2(q-2)}{q}}(\int_A |f_0+U|^{\frac{2q}{q+2}}dv)^{\frac{q+2}{2q}} + r^{\frac{3q-4}{q}}(\int_{\tilde A}|f_0+U|^{\frac{4q}{q+4}}dv)^{\frac{q+4}{4q}}\right] \tag{4.26}$$

for some $C>0$. Estimate (4.26) establishes the validity of (3.12) when $k=1$ in view of (4.5). Iterating the argument one shows that (3.12) does hold for $k=2,3$ too.

When $p \in M\setminus\{p_1,\dots,p_l\}$, there is no need to work on annuli as in the previous argument, and it is therefore possible to show that $w \in W_0^{3,q}(B_r(p))$, $2<q<4$. Then $w \in C^\infty(M\setminus\{p_1,\dots,p_l\})$ by an iteration. ■

Remark 4.4. *According to the terminology in Remark 4.2, any fundamental solution corresponding to* $\mu_s = \beta_i\delta_{p_i}$ *and* $\Phi \in C^\infty(\overline{B_r(p_i)})$ *satisfies the conclusions of Theorem 4.3 in* $\overline{B_r(p_i)}$.

5. Blow-up analysis

In this section we are concerned with the asymptotic analysis of sequences of solutions w_n to (1.8). The first issue is to determine a minimal volume quantization in the blow-up scenario, as it will follow by Adams' inequality and (2.1). The blow-up threshold is not optimal but it can be sharpened by using a Pohozaev identity along with the logarithmic behaviour of the singular limit for $w_n - \overline{w}_n$. However, it is not clear whether $\overline{w}_n$ tends to minus infinity or not, determining whether the limiting measure of $\mu_n e^{4w_n}$ is purely concentrated or presents some residual L^1-part. The latter is usually excluded by comparison with the purely concentrated case.

In our setting *maximum principles* are not available for the fourth-order operator $\mathcal{N}$ and a new approach has to be devised, based only on the scaling invariance of the PDE: we apply asymptotic analysis and Pohozaev's identity to a slightly rescaled sequence u_n for which the limiting measure is purely concentrated, getting the optimal blow-up threshold; since the concentrated part is sufficiently strong, the fundamental solution in the purely-concentrated case has a low exponential integrability and, by using $W^{1,2,2)}$-bounds to make a comparison, the same remains true for $\lim\limits_{n\to+\infty}(w_n - \overline{w}_n)$ when $\inf_n \overline{w}_n > -\infty$, in contrast to $\int e^{4w_n}dv = 1$ (which is assumed in Theorem 1.1). In order to have an asymptotic description of u_n, observe that scaling-invariant uniform estimates on w_n are needed, which is precisely the content of Theorem 2.4.

Let g_n be a metric on B_r with volume element dv_{g_n}, $U_n \in C^\infty(\overline{B_r})$ and $\mathcal{N}_n$ be the operator associated to g_n through (1.9). We consider a sequence of solutions u_n to

$$\mathcal{N}_n(u_n) + U_n = \mu_n e^{4u_n} \qquad \text{in } B_r. \tag{5.1}$$

We assume that $\mu_n \to \mu_0$,

$$\sup_n \int_{B_r} e^{4u_n} dv_{g_n} < +\infty, \qquad \sup_n \int_{B_r} (u_n - c_n)^4 dv_{g_n} < +\infty, \tag{5.2}$$

and

$$U_n \to U_\infty \text{ in } C^1(\overline{B_r}), \qquad g_n \to g_\infty \text{ in } C^4(\overline{B_r}) \tag{5.3}$$

for some $U_\infty \in C^\infty(\overline{B_r})$, a metric g_∞ and $c_n \in \mathbb{R}$. Notice that (5.2) implies

$$\sup_n \int_{B_r} (u_n - \overline{u}_n^r)^4 dv_{g_n} < +\infty \tag{5.4}$$

in terms of the average $\overline{u}_n^r = \fint_{B_r} u_n dv_{g_n}$ of u_n on B_r w.r.t. g_n, since by Hölder's inequality

$$|\overline{u}_n^r - c_n| \le \fint_{B_r} |u_n - c_n| dv_{g_n} \le \frac{C}{r}\Big(\int_{B_r} (u_n - c_n)^4 dv_{g_n}\Big)^{\frac{1}{4}}.$$

We have the following local result on minimal volume quantization.

Proposition 5.1. *Let $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. There exists $\epsilon_0 > 0$ so that*

$$\sup_n \int_{B_{\frac{r}{2}}} [(\Delta_{g_n} u_n)^2 + |\nabla u_n|_{g_n}^4] \, dv_{g_n} < +\infty \tag{5.5}$$

provided $|\mu_n| \int_{B_r} e^{4u_n} dv_{g_n} \le \epsilon_0$. Moreover, assuming $u_n - c_n \rightharpoonup u_0$ in $W^{2,2}_{g_\infty}(B_{\frac{r}{2}})$ and $\frac{\gamma_2}{\gamma_3} \ge 6$, there exists $0 < r_0 \le \frac{r}{4}$ so that

$$\sup_n \|u_n - c_n\|_{C^{4,\alpha}(B_{r_0})} < +\infty \tag{5.6}$$

for any $\alpha \in (0,1)$.

Proof. By (5.4), it is enough to establish the proposition with $c_n = \overline{u}_n^r$. For simplicity we omit the dependence on n and the dependence of geometric quantities on g_n. Let $\chi \in C_0^\infty(B_r)$ be so that $0 \le \chi \le 1$, $\chi = 1$ in $B_{\frac{r}{2}}$ and $|\Delta\chi| + |\nabla\chi| = O(1)$. In view of Remark 2.1, re-write (2.1) with $\psi(s) = s$:

$$\int_{B_r} \chi^4[\mu e^{4u} - U](u-c)\, dv = \int_{B_r} \chi^4[(\frac{\gamma_2}{2} + 6\gamma_3)(\Delta u)^2 + 18\gamma_3 \Delta u |\nabla u|^2 + 12\gamma_3|\nabla u|^4]\, dv$$
$$+O\left(\int_{B_r} [\chi^4 + \chi^2|u-c| + \chi^3|\nabla u|(1+|u-c|)][1 + |\Delta u| + |\nabla u|^2]\, dv\right).$$

By Young's inequality and (5.2) we have that

$$O(\int_{B_r} [\chi^4 + \chi^2 |u-c| + \chi^3 |\nabla u|(1+|u-c|)][1+|\Delta u| + |\nabla u|^2]\, dv) \le \epsilon \int_{B_r} \chi^4 [(\Delta u)^2 + |\nabla u|^4]\, dv + C_\epsilon$$

for all $\epsilon > 0$, with some $C_\epsilon > 0$. Setting $\beta = \frac{\gamma_2}{\gamma_3}$, arguing as in (2.4) when $\psi(s) = s$ we have that

$$\begin{aligned} (5.7) \qquad & \int \chi^4 \left[(\beta+12)(\Delta u)^2 + 36 \Delta u |\nabla u|^2 + 24 |\nabla u|^4\right] dv \\ & \ge (\beta + 12 - \frac{27}{2(1-\delta)}) \int \chi^4 (\Delta u)^2 dv + 24\delta \int \chi^4 |\nabla u|^4 dv \ge \frac{2\delta_0}{|\gamma_3|} \int \chi^4 [(\Delta u)^2 + |\nabla u|^4] dv \end{aligned}$$

for some $\delta_0 > 0$, thanks to $\beta > \frac{3}{2}$ and for a suitable choice of $\delta \in (0,1)$. Since $\Delta[\chi^2(u-c)] = \chi^2 \Delta u + O(|\nabla \chi^2||\nabla u| + |u-c|)$ and $\nabla[\chi(u-c)] = \chi \nabla u + O(|u-c|)$, by Young's inequality we obtain

$$\int_{B_r} [\Delta(\chi^2(u-c))]^2 + |\nabla(\chi(u-c))|^4]\, dv \le (1+\epsilon) \int_{B_r} \chi^4 [(\Delta u)^2 + |\nabla u|^4]\, dv + C_\epsilon$$

for all $\epsilon > 0$ with some $C_\epsilon > 0$, thanks to (5.2). Re-collecting all the above estimates we proved that

$$(5.8) \qquad \int_{B_r} [\Delta(\chi^2(u-c))]^2 + |\nabla(\chi(u-c))|^4]\, dv \le C_\epsilon + \frac{(1+\epsilon)|\mu|}{\delta_0 - \epsilon} \int_{B_r} \chi^4 e^{4u} |u-c|\, dv$$

for all $0 < \epsilon < \delta_0$ and some $C_\epsilon > 0$. To estimate the R.H.S. we use the inequality

$$\chi^4 |s| e^s \le \frac{2}{\lambda} e^s + e^{\lambda \chi^4 s^2}$$

with $s = 4(u-c)$ and $\lambda = \frac{\pi^2}{\|\Delta(\chi^2(u-c))\|^2_{L^2(B_r)}}$, to get by Jensen's inequality that

$$\int_{B_r} \chi^4 e^{4u} |u-c| dv \le \frac{\int_{B_r} e^{4u} dv}{2\pi^2} \int_{B_r} [\Delta(\chi^2(u-c))]^2\, dv + \frac{⨍_{B_r} e^{4u}\, dv}{4} \int_{B_r} e^{\frac{16\pi^2 \chi^4 (u-c)^2}{\|\Delta(\chi^2(u-c))\|^2_{L^2(B_r)}}}\, dv.$$

Setting $\epsilon_0 = \pi^2 \delta_0$, we can find $\epsilon > 0$ small so that $\frac{(1+\epsilon)|\mu|}{2\pi^2(\delta_0 - \epsilon)} \int_{B_r} e^{4u} dv \le \frac{3}{4}$ and then (5.8) produces

$$\int_{B_r} [\Delta(\chi^2(u-c))]^2 + |\nabla(\chi(u-c))|^4]\, dv \le C + C ⨍_{B_r} e^{4u} dv \int_{B_r} e^{\frac{16\pi^2 \chi^4 (u-c)^2}{\|\Delta(\chi^2(u-c))\|^2_{L^2(B_r)}}}\, dv$$

for some $C > 0$. Thanks to (5.3) and $16\pi^2 < 32\pi^2$ we can apply Adams' inequality in [1, 26] to $\chi^2(u-c)$ and finally get the validity of (5.5).

We are now in the case $u - c \rightharpoonup u_0$ in $W^{2,2}_{g_\infty}(B_{\frac{r}{2}})$ and $\frac{\gamma_2}{\gamma_3} \ge 6$. By contradiction, assume that for all $0 < r_0 \le \frac{r}{4}$ there holds, up to a subsequence,

$$\|u - c\|_{C^{4,\alpha}(B_{r_0})} \to +\infty$$

for some $\alpha \in (0,1)$ and $c \to c_0$, where $c_0 \in [-\infty, +\infty)$ thanks to Jensen's inequality and (5.2). By Adams' inequality it is straightforward to show that

$$(5.9) \qquad \mu e^{4u} \to \mu_0 e^{4u_0 + 4c_0} \qquad \text{in } L^q_{g_\infty}(B_{\frac{r}{2}}),\ q \ge 1.$$

Since the limiting function $u_0 \in W^{2,2}_{g_\infty}(B_{\frac{r}{2}})$ solves $\mathcal{N}_{g_\infty}(u_0) = \mu_0 e^{4u_0+4c_0} - U_\infty$ in $B_{\frac{r}{2}}$ in view of (5.9), by the regularity result in [55] we have that $u_0 \in C^\infty(B_{\frac{r}{2}})$ and then $\mathcal{N}(u_0) \to \mu_0 e^{4u_0+4c_0} - U_\infty$ holds locally uniformly in $B_{\frac{r}{2}}$ in view of (5.3). We can make use of (3.1) with $w_1 = u - c$, $w_2 = u_0$ and $\varphi \in C_0^\infty(B_{\frac{r}{2}})$ thanks to Remark 3.2. Setting $p = u - c - u_0$ and $q = u - c + u_0$, (3.1) re-writes as

$$\begin{aligned} (5.10) \qquad & 3\gamma_3 \int (\Delta p + \langle \nabla q, \nabla p \rangle)(\Delta \varphi + \langle \nabla q, \nabla \varphi \rangle) dv + 6\gamma_3 \int \langle \nabla^2_{\tilde g} p, \nabla^2_{\tilde g} \varphi \rangle dv + 3\gamma_3 \int |\nabla p|^2 \langle \nabla p, \nabla \varphi \rangle dv \\ & + (\frac{\gamma_2}{2} - 3\gamma_3) \int \Delta p \Delta \varphi dv + (2\gamma_3 - \frac{\gamma_2}{3}) \int [3 \mathrm{Ric}(\nabla p, \nabla \varphi) - R \langle \nabla p, \nabla \varphi \rangle] dv \\ & = \int [\mu e^{4u} - U - \mathcal{N}(u_0)] \varphi\, dv \end{aligned}$$

for all $\varphi \in C_0^\infty(B_{\frac{r}{2}})$ in view of (3.2), where $\hat{g} = e^q g$. Take $\varphi = \chi^4 p$ and $\chi \in C_0^\infty(B_{\frac{r}{2}})$ in (5.10) to get

$$\int \chi^4 \left[3\gamma_3(\Delta p + \langle \nabla q, \nabla p\rangle)^2 + 6\gamma_3|\nabla^2_{\hat{g}} p|^2 + 3\gamma_3|\nabla p|^4 + (\frac{\gamma_2}{2} - 3\gamma_3)(\Delta p)^2\right] dv \tag{5.11}$$
$$= O(\int_{B_{\frac{r}{2}}} \chi^4 |\mu e^{4u} - U - \mathcal{N}(u_0)||p| dv)$$
$$+ O(\int_{B_{\frac{r}{2}}} |p||\nabla p|^3 dv + \int_{B_{\frac{r}{2}}} (|p| + |\nabla p| + |p||\nabla q|)(|\nabla p| + |\nabla p||\nabla q| + |\nabla^2 p|) dv).$$

Since $p \rightharpoonup 0$ in $W^{2,2}_{g_\infty}(B_{\frac{r}{2}})$, by (5.3) we have that

$$\int \left[|\nabla p|^4 + |\nabla q|^4 + |\nabla^2 p|^2\right] dv = O(1), \quad \int \left[|p|^4 + |\nabla p|^{\frac{8}{3}}\right] dv \to 0. \tag{5.12}$$

Inserting (5.9) and (5.12) into (5.11) we deduce that

$$\int \chi^4 (\Delta_{g_\infty} p)^2 dv_{g_\infty} \to 0,$$

and by taking $\chi = 1$ on $B_{\frac{r}{4}}$ we end up with $u - c \to u_0$ in $W^{2,2}_{g_\infty}(B_{\frac{r}{4}})$. Since $u_0 \in C^\infty(B_{\frac{r}{2}})$, for all $\delta > 0$ we can find $0 < r_0 \le \frac{r}{4}$ so that

$$\int_{B_{r_0}} [(\Delta u)^2 + |\nabla u|^4] \, dv \le \delta :$$

this is the crucial assumption in [55] to derive upper bounds in strong norms on u which do not depend on g. Then $u - c$ is uniformly bounded in $C^{4,\alpha}(B_{r_0})$ for any $\alpha \in (0,1)$, which is a contradiction, and the proof is thereby complete. ■

Hereafter we assume $\frac{\gamma_2}{\gamma_3} \ge 6$. Let w_n be as in Theorem 1.1 and let us restrict our attention to the case $\|w_n - \overline{w}_n\|_{C^{4,\alpha}(M)} \to +\infty$ as $n \to +\infty$ for some $\alpha \in (0,1)$. Thanks to Theorem 2.4 we have that $[w_n]_{BMO} \le C$, which implies the validity of (5.2)-(5.3) for w_n with $c_n = \overline{w}_n$, $\tilde{U}_n$ and $g_n \equiv g$. Up to a subsequence, assume that $e^{4w_n} \rightharpoonup \hat{\mu}$ as $n \to +\infty$ in the weak sense of distributions on M, where $\hat{\mu}$ is a probability measure on M. Consider the finite set

$$S = \{p \in M : \ |\mu_0|\hat{\mu}(B_r(p)) \ge \epsilon_0 \ \forall \ 0 < r \le i_0\},$$

where $\epsilon_0 > 0$ is given by Proposition 5.1. For any compact set $K \subset M \setminus S$, by (5.5) we deduce

$$\sup_n \int_K [(\Delta w_n)^2 + |\nabla w_n|^4] \, dv < +\infty. \tag{5.13}$$

By (2.3) and (5.13) we have that $w_n - \overline{w}_n$ is uniformly bounded in $W^{2,2}(K)$ and then, up to a subsequence and a diagonal process, $w_n - \overline{w}_n \rightharpoonup w_0$ weakly in $W^{2,2}_{loc}(M \setminus S)$. For any $p \in M \setminus S$ by (5.6) we can find $r(p) > 0$ small so that $\|w_n - \overline{w}_n\|_{C^{4,\alpha}(B_{r(p)})} \le C(p)$. By compactness $w_n - \overline{w}_n$ is uniformly bounded in $C^{4,\alpha}_{loc}(M \setminus S)$ and then, up to a further subsequence, $w_n - \overline{w}_n \to w_0$ in $C^4_{loc}(M \setminus S)$. In particular $S \ne \emptyset$, $\mu_0 \ne 0$ and $\max_M w_n \to +\infty$ as $n \to +\infty$.

Since $e^{4\overline{w}_n} \le \frac{1}{\text{vol}M}$ by Jensen's inequality, up to a subsequence assume that $\overline{w}_n \to c \in [-\infty, +\infty)$ as $n \to +\infty$. Since $e^{4w_n} \to e^{4w_0+4c}$ locally uniformly in $M \setminus S$, we have that

$$e^{4w_n} \rightharpoonup e^{4w_0+4c} dv + \sum_{i=1}^l \tilde{\beta}_i \delta_{p_i} \qquad \text{as } n \to +\infty$$

weakly in the sense of measures, where $S = \{p_1, \dots, p_l\}$ and $\tilde{\beta}_i \ge \frac{\epsilon_0}{|\mu_0|}$. The function w_0 is a SOLA of

$$\mathcal{N}(w_0) = \mu_0 e^{4w_0+4c} + \sum_{i=1}^l \beta_i \delta_{p_i} - U \qquad \text{in } M \tag{5.14}$$

for $\beta_i = \mu_0 \tilde{\beta}_i$.

We aim to compute the values of the β_i's, and we will prove below a quantization result in a suitable general form. In particular, it will apply to the following scaling of w_n, $\tilde{U}_n$ and g:

$$u_n(y) = w_n[\exp_p(r_n y)] + \log r_n, \quad U_n(y) = r_n^4 \tilde{U}_n[\exp_p(r_n y)], \quad g_n(y) = g[\exp_p(r_n y)] \tag{5.15}$$

for $|y| \le \frac{i_0}{r_n}$, where $p \in M$ and $r_n \to 0^+$. The function u_n is a solution of (5.1) for $|y| \le \frac{i_0}{r_n}$ which satisfies

$$\int_{B_1(0)} |u_n - \overline{u}_n^1|^4 dv_{g_n} = \frac{1}{r_n^4} \int_{B_{r_n}(p)} |w_n - \overline{w}_n^{r_n}|^4 dv \le C' \fint_{B_{r_n}(p)} |w_n - \overline{w}_n^{r_n}|^4 dv \le C$$

in view of $[w_n]_{BMO} \le C$. Therefore u_n satisfies (5.2)-(5.3) on any $B_r \subset B_1(0)$ with $c_n = \overline{u}_n^1$, $U_n \to 0$ in $C^1(\overline{B_1(0)})$ and $g_n \to \delta_{eucl}$ in $C^4(\overline{B_1(0)})$. The result we have is the following.

Lemma 5.2. *Let u_n be a solution of (5.1) which satisfies (5.2)-(5.3) in $B_1(0)$. Suppose that*

$$\mu_n e^{4u_n} dv_{g_n} \rightharpoonup \beta\, \delta_0 \tag{5.16}$$

weakly in the sense of measures in $B_1(0)$ as $n \to +\infty$, for some $\beta \ne 0$. Then $\beta = 8\pi^2 \gamma_2$.

Proof. Arguing as we did for w_n, we can apply Proposition 5.1 to u_n to get that $u_n - \overline{u}_n^1$ is uniformly bounded in $W^{2,2}_{loc}(B_1 \setminus \{0\})$ in view of (5.4). Up to a subsequence and a diagonal process, we have that $u_n - \overline{u}_n^1 \rightharpoonup u_0$ weakly in $W^{2,2}_{loc}(B_1(0) \setminus \{0\})$ and in turn

$$u_n - \overline{u}_n^1 \to u_0 \quad \text{in } C^4_{loc}(B_1(0) \setminus \{0\}), \quad \overline{u}_n^1 \to -\infty, \tag{5.17}$$

as $n \to +\infty$ in view of (5.16). According to Remark 3.7 u_0 is a SOLA of $\mathcal{N}_{g_\infty} u_0 + U_\infty = \beta\, \delta_0$ in $B_{\frac12}(0)$, $u_0 = \Phi$ and $\partial_\nu u_0 = \partial_\nu \Phi$ on $\partial B_{\frac12}(0)$, where Φ is a smooth extension in $B_{\frac12}(0)$ of $u_0\Big|_{\partial B_{\frac12}(0)}$. We continue the proof dividing it into the following steps.

Step 1. Up to a subsequence, there exist $p_n^1, \dots, p_n^J$, $J \in \mathbb{N}$, such that $p_n^1, \dots, p_n^J \to 0$ as $n \to +\infty$ and

$$d_n(y)^4 e^{4u_n} \le C_1 \qquad \text{in } B_1(0) \tag{5.18}$$

where $d_n(y) = \min\{d_{g_n}(y, p_n^1), \dots, d_{g_n}(y, p_n^J)\}$.

To prove (5.18), we first take $p_n^1 \to 0$ as the maximum point of u_n in $B_1(0)$. Let z_n^1 be the scaling of u_n around p_n^1 with scale $\mu_n^1 = \exp[-u_n(p_n^1)] \to 0$ in view of $u_n(p_n^1) \to +\infty$. Since $z_n^1 \le z_n^1(0) = 0$, by Proposition 5.1 we deduce that

$$z_n^1 \to z^1 \qquad \text{in } C^4_{loc}(\mathbb{R}^4). \tag{5.19}$$

Given $r_n^1 >> \mu_n^1$ we have that the scaling $\tilde z_n^1$ of u_n around p_n^1 with scale r_n^1 still blows up and by Proposition 5.1 $|\mu_n| \int_{B_1(0)} e^{4\tilde z_n^1} dv_{\tilde g_n} \ge \epsilon_0$, where $\tilde g_n = g_n(r_n^1 y + p_n^1)$, or equivalently $|\mu_n| \int_{B_{r_n^1}(p_n^1)} e^{4u_n} dv_{g_n} \ge \epsilon_0$.

We now proceed as follows. If (5.18) were not valid with $d_n(y) = d_{g_n}(y, p_n^1)$, by (5.17) we would find a sequence $p_n^2 \to 0$ of maximum points for $d_{g_n}(y, p_n^1) e^{u_n}$ in $B_1(0)$ so that

$$d_{g_n}(p_n^1, p_n^2) e^{u_n(p_n^2)} \to +\infty. \tag{5.20}$$

Let z_n^2 be the scaling of u_n around p_n^2 with scale $\mu_n^2 = \exp[-u_n(p_n^2)] \to 0$ in view of (5.20). Thanks to (5.19)-(5.20) we have that

$$\frac{d_{g_n}(p_n^1, p_n^2)}{\mu_n^1} \to +\infty, \qquad \frac{d_{g_n}(p_n^1, p_n^2)}{\mu_n^2} \to +\infty.$$

By the maximality property of p_n^2, z_n^2 is bounded from above and then by Proposition 5.1

$$z_n^2 \to z^2 \qquad \text{in } C^4_{loc}(\mathbb{R}^4).$$

Arguing as above, for $r_n^2 >> \mu_n^2$ we have that $|\mu_n| \int_{B_{r_n^2}(p_n^2)} e^{4u_n} dv_{g_n} \ge \epsilon_0$. Iterating as long as (5.18) is not valid, we can find points $p_n^1, \dots, p_n^J \to 0$ so that

$$\frac{\mu_n^i + \mu_n^j}{d_{g_n}(p_n^i, p_n^j)} \to 0 \qquad \forall\, i \ne j \tag{5.21}$$

and $|\mu_n| \int_{B_{r_n^i}(p_n^i)} e^{4u_n} dv_{g_n} \ge \epsilon_0$ for all $i = 1, \dots, J$, for a choice $r_n^i >> \mu_n^i$. Now we define radii r_n^i by $r_n^i = \frac12 \min\{d_{g_n}(p_n^i, p_n^j) :\ j \ne i\}$, in such a way that $B_{r_n^i}(p_n^i) \cap B_{r_n^j}(p_n^j)$ for all $i \ne j$ and $r_n^i >> \mu_n^i$ thanks to (5.21). Since

$$|\mu_n| \int_{B_1(0)} e^{4u_n} dv_{g_n} \ge J\epsilon_0,$$

by $|\mu_n| \int_{B_1(0)} e^{4u_n} dv_{g_n} \to |\beta|$ we have that such an iterative procedure must stop after J times, and then (5.18) does hold with $p_n^1, \dots, p_n^J$.

Step 2. Assume that $d_{g_n}(y,p_n)^4 e^{4u_n} \le C_1$ does hold in $B_1(0)$ for some $p_n \to 0$. Then $\beta = 8\pi^2\gamma_2$.

To show this, first notice that by Proposition 5.1 and $d_{g_n}(y,p_n)^4 e^{4u_n} \le C_1$ in $B_1(p_n)$ there exists $\tilde C_1 > 0$ such that for all $s \in (0,1/4)$ one has

$$\int_{B_{2s}(p_n)\setminus B_s(p_n)} [(\Delta_{g_n} u_n)^2 + |\nabla u_n|^4_{g_n}]\, dv_{g_n} \le \tilde C_1 \tag{5.22}$$

for all n. Since by (5.22) the remainder volume integrals in the Pohozaev identity (7.13) converge to zero as $r \to 0$ uniformly in n, we can apply Proposition 7.2 in $B_r(p_n)$ and letting $n \to +\infty$ get that

$$-\beta = \mathcal{B}_{g_0}(0, B_r(0), u_0) + o_r(1),$$

in view of (5.3) and (5.16)-(5.17). By Remark 4.4 u_0 satisfies (3.12) at 0, and a straightforward computation for the boundary integrals in (7.15) leads as $r \to 0^+$ to the identity

$$-[9\gamma_3\alpha^4 + (\gamma_2 + 12\gamma_3)\alpha^2 + 24\gamma_3\alpha^3]2\pi^2 = -\beta = 4\pi^2[(\gamma_2+12\gamma_3)\alpha + 18\gamma_3\alpha^2 + 6\gamma_3\alpha^3]$$

in view of (4.2), which has a unique solution in $\mathbb{R}\setminus\{0\}$ given by $\alpha = -2$. Hence we have shown that $\beta = 8\pi^2\gamma_2$, as claimed.

Since (5.18) does not allow the direct use of Step 2 when $J \ge 2$, the idea is to properly group the points $p_n^1, \dots, p_n^J$ in clusters and substitute the corresponding points by a representative in the cluster. Up to re-ordering, assume that $d_{g_n}(p_n^1, p_n^2) = \inf\{d_{g_n}(p_n^i, p_n^j) : i \ne j\}$ and $d_{g_n}(p_n^i, p_n^j) \le C d_{g_n}(p_n^1, p_n^2)$ for all $i,j = 1,\dots,I$, $i \ne j$, for some $C > 0$, where $2 \le I \le J$. Setting $s_n = \frac{C d_{g_n}(p_n^1,p_n^2)}{2}$, as in the previous step by (5.18) the remainder volume integrals in (7.13)-(7.14) are well controlled on the disjoint balls $B_{s_n}(p_n^j)$, $j = 1,\dots,I$, leading to

$$\mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), u_n) = -\mu_n \int_{B_{s_n}(p_n^j)} e^{4u_n} dv_{g_n} + \frac{\mu_n}{4}\oint_{\partial B_{s_n}(p_n^j)} e^{4u_n}(x_{n,p_n^j})^i \nu_i d\sigma_{g_n} + o(1); \tag{5.23}$$

$$\mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), a_n, u_n) = \frac{\mu_n}{4}\oint_{\partial B_{s_n}(p_n^j)} e^{4u_n} a_n^i \nu_i\, d\sigma_{g_n} + o(1) \tag{5.24}$$

as $n \to +\infty$, for any infinitesimal vector field $(a_n^i)_i$ with constant components in a g_n-geodesic coordinate system $(x^i_{n,p_n^j})_i$ centred at p_n^j. The key point is to replace $p_n^1, \dots, p_n^I$ by the representative p_n^1 in such a way that (5.23)-(5.24) continue to hold for p_n^1 with $r_n >> d_{g_n}(p_n^1, p_n^2)$, as it follows by Step 3 below.

Step 3. Assume that

$$d_{g_n}(p_n^1,p_n^2) \le d_{g_n}(p_n^i,p_n^j) \le C d_{g_n}(p_n^1,p_n^2) \qquad \forall i,j = 1,\dots,I,\ i \ne j$$

for some $C > 1$ and (5.23)-(5.24) are valid in $B_{s_n}(p_n^j)$, $j = 1,\dots,I$, for $s_n = \frac{C d_{g_n}(p_n^1,p_n^2)}{2}$. Then (5.23)-(5.24) are valid in $B_{r_n}(p_n^1)$ for any $r_n >> d_{g_n}(p_n^1,p_n^2)$ provided (5.18) does hold in $A_n := B_{r_n}(p_n^1) \setminus B_n$ with $d_n(y) = \min\{d_{g_n}(y,p_n^1), d_{g_n}(y,p_n^{I+1}), \dots, d_{g_n}(y,p_n^J)\}$, where $B_n := \bigcup_{j=1}^I B_{s_n}(p_n^j)$.

To see this, by (5.18) in A_n with $d_n(y) = \min\{d_{g_n}(y,p_n^1), d_{g_n}(y,p_n^{I+1}), \dots, d_{g_n}(y,p_n^J)\}$ we deduce that the remainder volume integrals in (7.13)-(7.14) tend to zero in A_n:

$$\mathcal{B}_{g_n}(p_n^1, A_n, u_n) = -\mu_n \int_{A_n} e^{4u_n} dv_{g_n} + \frac{\mu_n}{4}\oint_{\partial A_n} e^{4u_n}(x_{n,p_n^1})^i \nu_i d\sigma_{g_n} + o(1) \tag{5.25}$$

$$\mathcal{B}_{g_n}(p_n^1, A_n, a_n, u_n) = \frac{\mu_n}{4}\oint_{\partial A_n} e^{4u_n} a_n^i \nu_i d\sigma_{g_n} + o(1) \tag{5.26}$$

for any infinitesimal vector field $(a_n^i)_i$ which is constant in a g_n-geodesic coordinate system $(x^i_{n,p_n^1})_i$ centred at p_n^1. Letting $a_{n,j} = \left(x_{n,p_n^1}(p_n^j)\right)^i$, we have that $a_{n,j} \to 0$ as $n \to +\infty$ and by the validity of (5.23)-(5.24) in $B_{s_n}(p_n^j)$, $j = 1,\dots,I$, we can deduce that

$$\sum_{j=1}^J \left[\mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), u_n) + \mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), a_{n,j}, u_n)\right] = -\mu_n \int_{B_n} e^{4u_n} dv_{g_n} \tag{5.27}$$
$$+\frac{\mu_n}{4}\sum_{j=1}^J \oint_{\partial B_{s_n}(p_n^j)} e^{4u_n}[a^i_{n,j} + (x_{n,p_{n,j}})^i]\nu_i d\sigma_{g_n} + o(1),$$

and

$$\sum_{j=1}^{J} \mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), a_n, u_n) = \frac{\mu_n}{4} \sum_{j=1}^{J} \oint_{\partial B_{s_n}(p_n^j)} e^{4u_n} a_n^i \nu_i d\sigma_{g_n} + o(1). \tag{5.28}$$

It is possible to orient the geodesic coordinates both at p_n^1 and at p_j^n so that the coordinates of $y \in \partial B_n$ in these systems satisfy (with exact equality for the Euclidean metric)

$$(x_{n,p_n^1})^i(y) = a_{n,j}^i + (x_{n,p_n^j})^i(y) + o(s_n).$$

By Proposition 5.1 and a scaling argument, there exists $\tilde{C} > 0$ such that

$$|\nabla u_n| \leq \frac{\tilde{C}}{s_n}; \quad |\nabla^2 u_n| \leq \frac{\tilde{C}}{s_n^2}; \quad |\nabla^3 u_n| \leq \frac{\tilde{C}}{s_n^3} \qquad \text{on } \partial B_n.$$

The last two formulas then imply that there is approximate compensation for the boundary integrals on ∂B_n and on the inner boundaries of ∂A_n. More precisely, one has

$$\mathcal{B}_{g_n}(p_n^1, A_n, u_n) + \sum_{j=1}^{J} \left[\mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), u_n) + \mathcal{B}_{g_n}(p_n^j, B_{s_n}(p_n^j), a_{n,j}, u_n)\right] = \mathcal{B}_{g_n}(p_n^1, B_{r_n}(p_n^1), u_n) + o(1),$$

and

$$\oint_{\partial A_n} e^{4u_n}(x_{n,p_n^1})^i \nu_i d\sigma_{g_n} + \sum_{j=1}^{J} \oint_{\partial B_{s_n}(p_n^j)} e^{4u_n}[a_{n,j}^i + (x_{n,p_{n,j}})^i]\nu_i d\sigma_{g_n} = \oint_{\partial B_{r_n}(p_n^1)} e^{4u_n}(x_{n,p_n^1})^i \nu_i d\sigma_{g_n} + o(1).$$

The latter formulas, together with (5.25) and (5.27), imply the validity of (5.23) for r_n and p_n^1. Summing up (5.26) and (5.28), we also deduce the validity of (5.24) for r_n and p_n^1.

Conclusion. We arrange the remaining points $p_n^{I+1}, \ldots, p_n^J$, if any, in clusters in a similar way and substitute them by a representative. We continue to arrange the representative points in clusters and to perform a substitution thanks to Step 3. At the end, we find a unique cluster which we collapse again to a single point p_n, obtaining the validity of (5.23) for p_n and $r > 0$ with $o_n(1) + o_r(1)$ as in Step 2. Letting $n \to +\infty$ and then $r \to 0^+$ we get that

$$-\beta = -[9\gamma_3\alpha^4 + (\gamma_2 + 12\gamma_3)\alpha^2 + 24\gamma_3\alpha^3]2\pi^2.$$

Comparing with (4.2) we deduce that $\alpha = -2$ and $\beta = 8\pi^2\gamma_2$, completing the proof of Lemma 5.2. ∎

Remark 5.3. *By studying the point-wise limiting behaviour of the rescaled blowing-up solutions, it should be possible to obtain the spherical profiles classified in* [29]*. Even without this information, in Lemma 5.2 we proved that such profiles would exhaust the volume accumulating near each blow-up point.*

We next have the following result.

Lemma 5.4. *In the above notation, there holds $c = -\infty$.*

Proof. By contradiction assume $c > -\infty$, and fix some $p = p_i \in S$, $\tilde{\beta} = \tilde{\beta}_{p_i}$. Given $0 < R \leq \min\{i_0, \frac{1}{2}\text{dist}(p_i, p_j) : \; j \neq i\}$, we have that

$$e^{4w_n} \rightharpoonup e^{4w_0+4c}dv + \tilde{\beta}\,\delta_p$$

weakly in the sense of measures on the ball $B_R = B_R(p)$ as $n \to +\infty$. Since

$$\int_{B_r} e^{4w_n}dv \to \int_{B_r} e^{4w_0+4c}dv + \tilde{\beta} > \tilde{\beta}$$

as $n \to +\infty$ for all $0 < r \leq R$, we can find a sequence $r_n \to 0$ so that

$$\int_{B_{r_n^2}} e^{4w_n}dv = \tilde{\beta}. \tag{5.29}$$

Since $\int_{B_r} e^{4w_0+4c}dv \to 0$ as $r \to 0$ and

$$0 \leq \int_{B_{r_n} \setminus B_{r_n^2}} e^{4w_n}dv \leq \int_{B_r} e^{4w_n}dv - \tilde{\beta} \to \int_{B_r} e^{4w_0+4c}dv$$

for all $r > 0$, notice that

$$\int_{B_{r_n} \setminus B_{r_n^2}} e^{4w_n} dv \to 0 \tag{5.30}$$

as $n \to +\infty$. We consider now the scaling u_n of w_n as given by (5.15), which satisfies, as already observed there, the assumptions (5.2)-(5.3) in $B_1(0)$ with $c_n = \overline{u}_n^1$, $U_\infty = 0$ and $g_\infty = \delta_{eucl}$. By (5.29)-(5.30) we have

$$\int_{B_1} e^{4u_n} dv_{g_n} = \int_{B_{r_n}} e^{4w_n} dv \to \tilde{\beta},$$

and

$$\begin{aligned}\int_{B_1} e^{4u_n} \phi\, dv_{g_n} &= \phi(0) \int_{B_{r_n}} e^{4u_n} dv_{g_n} + \int_{B_{r_n}} e^{4u_n} [\phi - \phi(0)] dv_{g_n} + \int_{B_1 \setminus B_{r_n}} e^{4u_n} \phi\, dv_{g_n} \\ &= \phi(0) \int_{B_{r_n^2}} e^{4w_n} dv + o(\int_{B_{r_n^2}} e^{4w_n} dv) + O(\int_{B_{r_n} \setminus B_{r_n^2}} e^{4w_n} dv) \to \tilde{\beta}\phi(0)\end{aligned}$$

for all $\phi \in C(B_1)$ as $n \to +\infty$. Hence we deduce that

$$e^{4u_n} dv_{g_n} \rightharpoonup \tilde{\beta}\delta_0$$

weakly in the sense of measures on B_1 as $n \to +\infty$. We now apply Lemma 5.2 to deduce that $\beta = \mu_0 \tilde{\beta} = 8\pi^2 \gamma_2$, or equivalently $\alpha = -2$ in view of (4.2).

Let $w_0 = \lim\limits_{n \to +\infty} w_n - c$ be a SOLA of (5.14). Given $0 < r \le i_0$, thanks to Remark 4.2 let w_s be a fundamental solution in $B_r(p)$ corresponding to $\mu_s = \beta\delta_p$ and the boundary values as w_0, namely w_s solves $\mathcal{N}(w_s) + U = \beta\delta_p$ in $B_r(p)$, $w_s = w_0$ and $\partial_\nu w_s = \partial_\nu w_0$ on $\partial B_r(p)$. In order to compare w_0 and w_s, consider the following scaling of w_0, w_s and g:

$$w_{0,r}(y) = w_0[\exp_p(ry)] + \log r, \quad w_{s,r}(y) = w_s[\exp_p(ry)] + \log r, \quad g_r(y) = g[\exp_p(ry)]$$

for $|y| \le 1$. Letting U_r the $U-$curvature and $\mathcal{N}_r$ be the operator associated to g_r, we have that

$$\mathcal{N}_r(w_{0,r}) + U_r = \mu_0 e^{4w_{0,r}+4c} + \beta\delta_p \text{ and } \mathcal{N}_r(w_{s,r}) + U_r = \beta\delta_p \quad \text{in } B_1(0)$$

with $w_{0,r} = w_{s,r}$ and $\partial_\nu w_{0,r} = \partial_\nu w_{s,r}$ on $\partial B_1(0)$. According to Remark 3.7 we have the validity of (3.37) on $w_{0,r} - w_{s,r}$, with constants which are uniform in r in view of $g_r \to \delta_{eucl}$ in $C^4(\overline{B_1(0)})$ as $r \to 0^+$. The constant η_r corresponding to g_r through (3.16) satisfies $\eta_r \to 0$ as $r \to 0^+$, and then (3.37) simply reduces to

$$\|w_{0,r} - w_{s,r}\|_{W^{1,2,2)}} \le C_0(\|\mu_0 e^{4w_{0,r}+4c}\|_1^{\frac{1}{12}} + \eta_r^{\frac{4}{3}}) \qquad \text{(w.r.t. } g_r)$$

for some $C_0 > 0$ in view of (3.36) and

$$\int_{B_1(0)} e^{4w_{0,r}} dv_{g_r} = \int_{B_r(p)} e^{4w_0} dv \le C, \quad \int_{B_1(0)} |U_r|\, dv_{g_r} = \int_{B_r(p)} |U|\, dv \le C.$$

In particular, there holds

$$\epsilon^{\frac{1}{4}} (\int_{B_1(0)} |\nabla(w_{0,r} - w_{s,r})|^{4(1-\epsilon)} dv_{g_r})^{\frac{1}{4(1-\epsilon)}} \le C_0(\|\mu_0 e^{4w_{0,r}+4c}\|_1^{\frac{1}{12}} + \eta_r^{\frac{4}{3}}) \tag{5.31}$$

for some $C_0 > 0$ and for all $0 < \epsilon \le \epsilon_0$.

In order to derive exponential estimates from (5.31), let us recall the optimal Euclidean inequality

$$(\int_{\mathbb{R}^4} |U|^k dx)^{\frac{1}{k}} \le C(k)(\int_{\mathbb{R}^4} |\nabla U|^{\frac{4k}{4+k}} dx)^{\frac{4+k}{4k}} \qquad U \in C_0^\infty(\mathbb{R}^4) \tag{5.32}$$

for all $k \ge 1$, where

$$C(k) = \pi^{-\frac{1}{2}} 4^{-\frac{4+k}{4k}} (\frac{3k-4}{16})^{\frac{3k-4}{4k}} \left[\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{4+k}{k})\Gamma(\frac{15k-20}{4k})}\right]^{\frac{1}{4}}$$

see [4, 54]. One has the following behaviour

$$\frac{C(k)}{k^{\frac{3}{4}}} \to C_1 = \frac{3}{8}\pi^{-\frac{1}{2}}\Gamma^{-\frac{1}{4}}(\frac{15}{4}) \quad \text{as } k \to +\infty. \tag{5.33}$$

Since $w_{0,r} - w_{s,r} \in W_0^{1,2}(B_1(0))$, we can extend it as zero outside $B_1(0)$ into a function $U \in D^{1,4}(\mathbb{R}^4)$. Since by density (5.32) does hold for U, by (5.31) we have that

$$\begin{aligned}\int_B e^{q|w_{0,r}-w_{s,r}|}dv_{g_r} &\leq 2\sum_{k=0}^{\infty}\frac{q^k}{k!}\int_{\mathbb{R}^4}|U|^k dx \leq 2\sum_{k=0}^{\infty}\frac{q^k C(k)^k}{k!}\Big(\int_{\mathbb{R}^4}|\nabla U|^{4(1-\frac{4}{4+k})}dx\Big)^{\frac{4+k}{4}} \\ &\leq 4\sum_{k=0}^{\infty}\frac{2^{\frac{k}{4}}q^k C(k)^k}{k!}\left(\int_B |\nabla(w_{0,r}-w_{s,r})|^{4(1-\frac{4}{4+k})}dv_{g_r}\right)^{\frac{4+k}{4}} \\ &\leq 4\sum_{k=0}^{\infty}\frac{q^k C(k)^k}{k!}\Big(\frac{4+k}{2}\Big)^{\frac{k}{4}}C_0^k\big(\|\mu_0 e^{4w_{0,r}+4c}\|_1^{\frac{1}{12}}+\eta_r^{\frac{4}{3}}\big)^k\end{aligned}$$

in view of $\frac{dx}{2} \leq dv_{g_r} \leq 2dx$ for $r > 0$ small. Thanks to (5.33) we have that

$$\frac{C(k)^k}{k!}\Big(\frac{4+k}{2}\Big)^{\frac{k}{4}} \sim \frac{C_1^k k^k}{2^{\frac{k}{4}}k!} \sim \frac{e^k C_1^k}{\sqrt{k}2^{\frac{k}{4}}}$$

in view of Stirling's formula. Then $e^{|w_{0,r}-w_{s,r}|} \in L^q(B_1(0))$ w.r.t. g_r for all $q < q_r$, where

$$q_r = \frac{2^{\frac{1}{4}}}{eC_0C_1(\|\mu_0 e^{4w_{0,r}+4c}\|_1^{\frac{1}{12}}+\eta_r^{\frac{4}{3}})}.$$

Since $q_r \to 0$ as $r \to 0^+$, we deduce that

$$r^{-4}\int_{B_r(p)} e^{q|w_0-w_s|}dv = \int_B e^{q|w_{0,r}-w_{s,r}|}dv_{g_r} < +\infty \tag{5.34}$$

for all $q \geq 1$ provided $r > 0$ is sufficiently small. Since w_s satisfies (3.12) in $B_r(p)$ with $\alpha = -2$ in view of Remark 4.4, we have that $w_s = -2(1+o(1))\log|x|$ as $x \to 0$ in geodesic coordinates near p and then $\int_{B_r(p)} e^{\gamma w_s}dv = +\infty$ for $\gamma > 2$. This is in contradiction for $\gamma < 4$ to

$$\int_{B_r(p)} e^{\gamma w_s}dv = \int_{B_r(p)} e^{\gamma(w_s-w_0)}e^{\gamma w_0}dv \leq \Big(\int_{B_r(p)} e^{\frac{4\gamma}{4-\gamma}(w_s-w_0)}dv\Big)^{\frac{4-\gamma}{4}}\Big(\int_{B_r(p)} e^{4w_0}, dv\Big)^{\frac{\gamma}{4}} < +\infty,$$

in view of $\int e^{4w_0}dv < +\infty$ and (5.34). This concludes the proof that $c = -\infty$. ■

Once we established that $c = -\infty$, we have that

$$\mu_n e^{4w_n} \rightharpoonup \sum_{i=1}^{l}\beta_i\delta_{p_i} \qquad \text{as } n \to +\infty$$

weakly in the sense of measures. We apply Lemma 5.2 near each p_i, ending up with $\beta_i = 8\pi^2\gamma_2$ for all $i = 1, \dots, l$. The proof of Theorem 1.1 is now complete.

6. Moser-Trudinger inequalities and existence results

In this section we show first a sharp Moser-Trudinger inequality of independent interest. We also derive an improved version of Adams' inequality involving also the functional III, a crucial ingredient for the existence of critical metrics in Theorem 1.3 via a variational and topological argument.

6.1. **Sharp and improved Moser-Trudinger inequalities.** In [13] (see also [1]), the following inequality was proved

$$\log\int e^{4w}dv \leq \frac{1}{8\pi^2}\int(\Delta u)^2 dv + 4\overline{w} + C_g \qquad \text{for all } w \in W^{2,2}(M). \tag{6.1}$$

If the Paneitz operator is positive-definite (see (1.5)), the integral of $(\Delta u)^2$ in the R.H.S. of the above formula can be replaced by the quadratic form induced by P. We have the following sharp inequality despite of the sign of the Paneitz operator, see also [18, 44] for related results.

Theorem 6.1. *Suppose $\int U dv \leq 8\pi^2\gamma_2$. Then, if $F_\gamma = \gamma_1 I + \gamma_2 II + \gamma_3 III$ with $\gamma_2, \gamma_3 > 0$ and $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$, then for all functions $w \in W^{2,2}(M)$ one has the lower bound*

$$F_\gamma(w) \geq -C$$

for some constant C.

PROOF. For $\varepsilon > 0$, consider the following functional

$$F_\varepsilon(w) := F_\gamma(w) + \varepsilon \log\left(\int e^{4(w-\overline{w})} dv\right).$$

Supposing by contradiction that F_γ is unbounded from below, we then have that

$$m_\varepsilon := \inf_{W^{2,2}} F_\varepsilon \to -\infty \qquad \text{as } \varepsilon \searrow 0.$$

By (6.1) (and some easy reasoning, exploiting the quartic gradient terms, if the Paneitz operator has negative eigenvalues) we know that F_ε admits a global minimum, which we call w_ε. Hence we have that $F_\varepsilon(w_\varepsilon) = m_\varepsilon \to -\infty$ as $\varepsilon \searrow 0$.

Looking at the Euler-Lagrange equation satisfied by w_ε, by Theorem 2.2 it follows that $\int |\nabla w_\varepsilon|^2 dv \leq C$. W.l.o.g., assume also that $\overline{w}_\varepsilon = 0$. Therefore, from the explicit form of F_ε and Poincaré's inequality, we have that

$$m_\varepsilon = F_\varepsilon(w_\varepsilon) \geq \gamma_2 \int (\Delta w_\varepsilon)^2 dv - (8\pi^2\gamma_2 - \varepsilon) \log\left(\int e^{4(w-\overline{w})} dv\right) - C.$$

Inequality (6.1) and the last formula imply that $F_\varepsilon(w_\varepsilon) \geq -2C$, which contradicts $F_\varepsilon(w_\varepsilon) \to -\infty$ as $\varepsilon \searrow 0$. ■

Next, we show that if e^{4w} has integral bounded from below into $(\ell+1)$ distinct regions of M, the Moser-Trudinger constant can be basically divided by $(\ell+1)$. When dealing with the functional II only, such an inequality was proved in [20], relying on some previous argument in [16].

Lemma 6.2. *Suppose $\gamma_2, \gamma_3 > 0$. For a fixed integer ℓ, let $\Omega_1, \dots, \Omega_{\ell+1}$ be subsets of M satisfying $dist(\Omega_i, \Omega_j) \geq \delta_0$ for $i \neq j$, where δ_0 is a positive real number, and let $\gamma_0 \in \left(0, \frac{1}{\ell+1}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\ell, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$8(\ell+1)\pi^2 \log \int e^{4(w-\overline{w})} dv \leq \ (1+\tilde{\varepsilon})\left(\langle w, Pw\rangle + \frac{\gamma_3}{\gamma_2} III(w)\right) + C$$

for all the functions $w \in W^{2,2}(M)$ satisfying

$$\frac{\int_{\Omega_i} e^{4w} dv}{\int e^{4w} dv} \geq \gamma_0, \qquad \forall\, i \in \{1, \dots, \ell+1\}.$$

PROOF. Assume without loss of generality that $\overline{w} = 0$. With the same argument as in the proof of Lemma 2.2 in [20], it is possible to show under the above conditions that

$$8(\ell+1)\pi^2 \log \int e^{4(w-\overline{w})} dv \leq \ (1+\tfrac{\tilde{\varepsilon}}{2}) \int (\Delta u)^2 dv + C.$$

Relabelling C, it is then enough to prove the inequality

$$(1+\tfrac{\tilde{\varepsilon}}{2}) \int (\Delta u)^2 dv \leq (1+\tilde{\varepsilon})\left(\langle w, Pw\rangle + \frac{\gamma_3}{\gamma_2} III(w)\right) + C. \tag{6.2}$$

However, using Poincaré's inequality and the expressions of P and III we can write that

$$\langle w, Pw\rangle + \frac{\gamma_3}{\gamma_2} III(w) \geq \int (\Delta u)^2 dv + 12\frac{\gamma_3}{\gamma_2} \int (\Delta u + |\nabla u|^2)^2 dv - C\int |\nabla w|^2 - C.$$

For $\varsigma > 0$ sufficiently small, one has that

$$\int (\Delta u)^2 dv + 12\frac{\gamma_3}{\gamma_2} \int (\Delta u + |\nabla u|^2)^2 dv \geq (1-2\varsigma) \int (\Delta u)^2 dv + \varsigma \int |\nabla u|^4 dv$$

Choosing ς small compared to $\tilde{\varepsilon}$ and using Young's inequality, from the last two formulas we obtain (6.2), yielding the conclusion. ■

For $j \in \mathbb{N}$, we define the family of probability measures

$$M_j = \{\mu \in \mathcal{P}(M) \ : \ card(supp(\mu)) \leq j\}.$$

We define the *distance* of an L^1- function f in M from M_j, $j \leq k$, as

$$\mathbf{d}(f, M_j) = \inf_{\sigma \in M_j} \sup \left\{ \left| \int f\,\psi\, dv - \langle \sigma, \psi\rangle \right| \ : \ \|\psi\|_{C^1(M)} \leq 1 \right\},$$

where $\langle \sigma, \psi \rangle$ stands for the duality product between $\mathcal{P}(M)$ and the space of C^1 functions. From Lemma 6.2 and Poincaré's inequality (to treat linear terms in w) we deduce immediately the following result.

Proposition 6.3. *Suppose that $\gamma_2, \gamma_3 > 0$ and that $\int U dv < 8(k+1)\gamma_2 \pi^2$ with $k \geq 1$. Then for any $\varepsilon > 0$ there exists a large positive $\Xi = \Xi(\varepsilon)$ such that for every $w \in W^{2,2}(M)$ with $F_\gamma(w) \leq -\Xi$ and $\int e^{4w} dv = 1$, we have $\mathbf{d}(\frac{e^{4w}}{\int e^{4w}}, M_k) \leq \varepsilon$.*

From the result in Section 3 of [20], one can deduce a further continuity property from $W^{2,2}(M)$ into $\mathcal{P}(M)$, endowed with the above distance $\mathbf{d}$.

Proposition 6.4. *For $\gamma_2, \gamma_3 > 0$ and $\int U dv < 8(k+1)\gamma_2 \pi^2$ there exist a large positive number Ξ and a continuous map $\Psi_k : \{F_\gamma \leq -\Xi\} \to M_k$ such that, if $e^{2w_n} \rightharpoonup \sigma \in M_k$, then $\Psi_k(w_n) \rightharpoonup \sigma$.*

6.2. **The topological argument.** The proof essentially follows the lines of Section 4 in [20], so we will mainly recall the principal steps. We first map M_k into some low sub-levels of F_γ and finally, once we map back onto M_k using Proposition 6.4, we obtain a map homotopic to the identity. The main difference with respect to the above reference is the energy estimate in Lemma 6.6, where we need to estimate the functional III on suitable test functions. We first recall a topological characterization of M_k.

Lemma 6.5. *([20]) For any $k \geq 1$, the set M_k is a* stratified set*, namely union of open manifolds of different dimensions, whose maximal one is $3k-1$. Furthermore M_k is non-contractible.*

For $\delta > 0$ small, consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{cases} \chi_\delta(t) = t & \text{for } t \in [0, \delta] \\ \chi_\delta(t) = 2\delta & \text{for } t \geq 2\delta \\ \chi_\delta(t) \in [\delta, 2\delta] & \text{for } t \in [\delta, 2\delta]. \end{cases}$$

Then, given $\sigma \in M_k$ $\left(\text{i.e. } \sigma = \sum_{i=1}^k t_i \delta_{x_i}\right)$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ as

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{4} \log \sum_{i=1}^k t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2\left(d(y, x_i)\right)} \right)^4, \qquad y \in M. \tag{6.3}$$

We prove next an energy estimate on the above test functions.

Lemma 6.6. *Suppose that $\gamma_2, \gamma_3 > 0$ and that $\varphi_{\lambda,\sigma}$ is as in (6.3). Then as $\lambda \to +\infty$ one has*

$$F_\gamma(\varphi_{\lambda,\sigma}) \leq \left(32 k \pi^2 \gamma_2 + o_\delta(1)\right) \log \lambda + C_\delta$$

uniformly in $\sigma \in M_k$, where $o_\delta(1) \to 0$ as $\delta \to 0$ and C_δ is a constant independent of λ and $x_1, \dots, x_k$.

Proof. In [20] it was proven that

$$\langle P \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle \leq \left(32 k \pi^2 + o_\delta(1)\right) \log \lambda + C_\delta$$

does hold uniformly in $\sigma \in M_k$, and moreover, as for formula (40) in [20], one has that

$$\left| \int U (\varphi_{\lambda,\sigma} - \overline{\varphi}_{\lambda,\sigma}) dv \right| \leq o_\delta(1) \log \lambda + C_\delta.$$

Therefore it is sufficient to show that the following estimate

$$|III(\varphi_{\lambda,\sigma})| = o_\lambda(1) \log \lambda \tag{6.4}$$

does hold uniformly in $\sigma \in M_k$. In order to do this, we can focus on the term $(\Delta \varphi_{\lambda,\sigma} + |\nabla \varphi_{\lambda,\sigma}|^2)^2$, since the others are shown in [20] to be of lower order. Setting

$$\mathcal{F}_i(y) := \frac{2\lambda}{1 + \lambda^2 \chi_\delta^2\left(d(y, x_i)\right)},$$

we compute explicitly $\Delta \varphi_{\lambda,\sigma} + |\nabla \varphi_{\lambda,\sigma}|^2$:

$$\Delta \varphi_{\lambda,\sigma} + |\nabla \varphi_{\lambda,\sigma}|^2 = \frac{\sum_i t_i \mathcal{F}_i^3 \Delta \mathcal{F}_i}{\sum_j t_j \mathcal{F}_j^4} + 3 \frac{\sum_i t_i \mathcal{F}_i^2 |\nabla \mathcal{F}_i|^2}{\sum_j t_j \mathcal{F}_j^4} - 3 \frac{\left|\sum_i t_i \mathcal{F}_i^3 \nabla \mathcal{F}_i\right|^2}{\left(\sum_j t_j \mathcal{F}_j^4\right)^2}.$$

This can be rewritten as

$$\Delta\varphi_{\lambda,\sigma} + |\nabla\varphi_{\lambda,\sigma}|^2 = \frac{\sum_i t_i \mathcal{F}_i^3 \Delta\mathcal{F}_i}{\sum_j t_j \mathcal{F}_j^4} + 3\frac{\sum_{i,k} t_i t_k \mathcal{F}_i^2 \mathcal{F}_k^2 \left(\mathcal{F}_k^2 |\nabla\mathcal{F}_i|^2 - \mathcal{F}_i\mathcal{F}_k \nabla\mathcal{F}_i\cdot\nabla\mathcal{F}_k\right)}{\left(\sum_j t_j \mathcal{F}_j^4\right)^2}.$$

At this point, symmetrizing in i, k and playing with elementary inequalities, it is enough to uniformly estimate in terms of $o_\lambda(1)\log\lambda$ the square L^2-norm of the following quantities

$$\frac{\Delta\mathcal{F}_i}{\mathcal{F}_i}; \qquad \mathcal{G}_{i,k} := \frac{|\mathcal{F}_k^2\mathcal{F}_i\nabla\mathcal{F}_i - \mathcal{F}_i^2\mathcal{F}_k\nabla\mathcal{F}_k|^2}{(\mathcal{F}_i^4+\mathcal{F}_k^4)^2}. \tag{6.5}$$

For the first term, working in normal coordinates y at x_i one finds

$$\Delta\mathcal{F}_i(y) = \Delta_{\delta_{eucl}}\mathcal{F}_i(y) + O(|y|)|\nabla\mathcal{F}_i|(y) + O(|y|^2)|\nabla^2\mathcal{F}_i|(y).$$

Using also the fact that

$$\Delta_{\delta_{eucl}}\left(\frac{1}{1+\lambda^2|x|^2}\right) = -\frac{8\lambda^2}{(1+\lambda^2|x|^2)^3},$$

one gets the following bounds

$$\mathcal{F}_i(y) \geq \begin{cases} C^{-1}\lambda & d(y,x_i)\leq \frac{1}{\lambda}; \\ \frac{C}{\lambda d^2(y,x_i)} & \frac{1}{\lambda}\leq d(y,x_i)\leq\delta, \end{cases} \qquad |\Delta\mathcal{F}_i(y)| \leq \begin{cases} C\lambda^3 & d(y,x_i)\leq\frac{1}{\lambda}; \\ \frac{C}{\lambda^3 d^6(y,x_i)} & \frac{1}{\lambda}\leq d(y,x_i)\leq\delta. \end{cases}$$

These imply

$$\int\left(\frac{\Delta\mathcal{F}_i}{\mathcal{F}_i}\right)^2 dv \leq \int_{B_{\frac{1}{\lambda}}(x_i)} C\lambda^4\, dv + \int_{B_\delta(x_i)\setminus B_{\frac{1}{\lambda}}(x_i)} \frac{C}{\lambda^4 d^8(y,x_i)} dv + C \leq C.$$

For the latter quantity in (6.5) we distinguish between two cases.

Case 1: $d(x_i,x_k)\geq\frac{\delta}{2}$. When we integrate near x_i, $\mathcal{F}_k$ and its gradient are bounded by $\frac{C_\delta}{\lambda}$. Using also the fact that

$$|\nabla\mathcal{F}_i| \leq \frac{C\lambda^3 d(y,x_i)}{(1+\lambda^2 d^2(y,x_i))^2},$$

we find the upper bound

$$\int_{B_{\frac{\delta}{4}}(x_i)} \mathcal{G}_{i,k}^2 dv \leq C\int_{B_{\frac{\delta}{4}}(x_i)}\left[\frac{d(y,x_i)^4(1+\lambda^2 d^2(y,x_i))^4}{\lambda^8} + \frac{(1+\lambda^2d^2(y,x_i))^8}{\lambda^{16}}\right] dv \leq C,$$

where the latter inequality follows from a change of variable. In the same way, one finds a similar bound on $B_{\frac{\delta}{4}}(x_k)$. In the exterior of these two balls, it is easily seen that $\mathcal{G}_{i,k}$ is uniformly bounded, and therefore $\mathcal{G}_{i,k}$ is uniformly bounded also in $L^2(M)$. In particular, there holds $\int\mathcal{G}_{i,k}^2 dv = o_\lambda(1)\log\lambda$.

Case 2: $d(x_i,x_k)\leq\frac{\delta}{2}$. In this case the functions $\mathcal{F}_i$ and $\mathcal{F}_k$ can be simultaneously *large* at the same point. By symmetry, it is sufficient to estimate $\mathcal{G}_{i,k}$ in the set

$$M_{i,k} := \{y\in M \;:\; d(y,x_i)\leq d(y,x_k)\}.$$

Set $\eta_{i,k} = \max\{\frac{1}{\lambda}, d(x_i,x_k)\}$. In $(M_{i,k}\cap B_\delta(x_i))\setminus B_{\mathfrak{C}\eta_{i,k}}(x_i)$, $\mathfrak{C}\geq 1$, one has the estimates

$$\mathcal{F}_k = \mathcal{F}_i(1+o_{\mathfrak{C}}(1)), \quad \nabla\mathcal{F}_k = \nabla\mathcal{F}_i + o_{\mathfrak{C}}(1)|\nabla\mathcal{F}_i|$$

with $o_{\mathfrak{C}}(1)\to 0$ as $\mathfrak{C}\to+\infty$, in view of

$$1\leq\frac{d(y,x_k)}{d(y,x_i)}\leq 1+\frac{d(x_i,x_k)}{d(y,x_i)}\leq 1+\frac{1}{\mathfrak{C}}.$$

Since these estimates imply some cancellations in the numerator of $\mathcal{G}_{i,k}$, we have that

$$\mathcal{G}_{i,k}^2 \leq \frac{o_{\mathfrak{C}}(1)}{|y-x_i|^4} \quad \text{in } (M_{i,k}\cap B_\delta(x_i))\setminus B_{\mathfrak{C}\eta_{i,k}}(x_i),$$

and therefore we find

$$\int_{(M_{i,k}\cap B_\delta(x_i))\setminus B_{\mathfrak{C}\eta_{i,k}(x_i)}} \mathcal{G}_{i,k}^2 dv = o_{\mathfrak{C}}(1)\log\lambda. \tag{6.6}$$

In $\left(M_{i,k} \cap B_\delta(x_i) \cap B_{\mathfrak{C}\eta_{i,k}}(x_i)\right) \setminus B_{\frac{1}{\lambda}}(x_i)$ we next have the following inequalities

$$\frac{1}{\lambda d^2(y,x_i)} \leq \mathcal{F}_i \leq \frac{2}{\lambda d^2(y,x_i)}, \quad |\nabla \mathcal{F}_i| \leq \frac{C}{\lambda d^3(y,x_i)}, \quad |\mathcal{F}_k| \leq \frac{C}{\lambda \eta_{i,k}^2}, \quad |\nabla \mathcal{F}_k| \leq \frac{C}{\lambda \eta_{i,k}^3}$$

in view of

$$d(y,x_k) \geq \begin{cases} d(x_i,x_k) - d(y,x_i) \geq \frac{1}{2}\eta_{i,k} & \text{if } \frac{1}{\lambda} \leq d(y,x_i) \leq \frac{1}{2} d(x_i,x_k) \\ d(y,x_i) \geq \frac{1}{2}\eta_{i,k} & \text{if } y \in M_{i,k},\ d(y,x_i) \geq \frac{1}{2}\eta_{i,k}, \end{cases}$$

which imply

$$\mathcal{G}_{i,k}^2 \leq C \left(\frac{d^{12}(y,x_i)}{\eta_{i,k}^{16}} + \frac{d^{16}(y,x_i)}{\eta_{i,k}^{20}} \right),$$

and therefore

$$\int_{\left(M_{i,k} \cap B_\delta(x_i) \cap B_{\mathfrak{C}\eta_{i,k}}(x_i)\right) \setminus B_{\frac{1}{\lambda}}(x_i)} \mathcal{G}_{i,k}^2 dv \leq C \mathfrak{C}^{20}. \tag{6.7}$$

Finally the estimate $\frac{|\nabla \mathcal{F}_i|}{\mathcal{F}_i} + \frac{|\nabla \mathcal{F}_k|}{\mathcal{F}_k} \leq C\lambda$ implies

$$\int_{M_{i,k} \cap B_{\frac{1}{\lambda}}(x_i)} \mathcal{G}_{i,k}^2 dv \leq C. \tag{6.8}$$

By first choosing $\mathfrak{C}$ and then λ large, by (6.6)-(6.8) we have shown that $\int_{M_{i,k} \cap B_\delta(x_i)} \mathcal{G}_{i,k}^2 dv = o_\lambda(1) \log \lambda$. By the symmetry of $\mathcal{G}_{i,k}$, exchanging i and k we also have that

$$\int_{M_{k,i} \cap B_{\frac{\delta}{2}}(x_i)} \mathcal{G}_{i,k}^2 dv \leq \int_{M_{k,i} \cap B_\delta(x_k)} \mathcal{G}_{k,i}^2 = o_\lambda(1) \log \lambda,$$

which combines with

$$\int_{M \setminus B_{\frac{\delta}{2}}(x_i) \cup B_{\frac{\delta}{2}}(x_k)} \mathcal{G}_{i,k}^2 dv \leq C$$

to show that also in **Case 2** there holds $\int \mathcal{G}_{i,k}^2 dv = o_\lambda(1) \log \lambda$. ■

The above results can be collected into the following proposition.

Proposition 6.7. *Suppose that $\gamma_2, \gamma_3 > 0$, $\int U dv \in (8k\gamma_2\pi^2, 8(k+1)\gamma_2\pi^2)$, and let $\varphi_{\lambda,\sigma}$ be defined as in (6.3). Then, as $\lambda \to +\infty$ the following properties hold true*

(i) $e^{4\varphi_{\lambda,\sigma}} \rightharpoonup \sigma$ weakly in the sense of distributions;

(ii) $F_\gamma(\varphi_{\lambda,\sigma}) \to -\infty$ uniformly in $\sigma \in M_k$;

(iii) if Ψ_k is given by Proposition 6.4 and if $\varphi_{\lambda,\sigma}$ is as in (6.3), then for λ sufficiently large the map $\sigma \mapsto \Psi_k(\varphi_{\lambda,\sigma})$ is homotopic to the identity on M_k.

We next introduce a variational scheme for obtaining existence of solutions of the Euler-Lagrange equation. Let $\hat{M}_k$ be the *topological cone* over M_k, which can be represented as $\hat{M}_k = M_k \times [0,1]$ with $M_k \times \{0\}$ identified to a single point. Let first Ξ be so large that Proposition 6.4 applies with $\frac{\Xi}{4}$, and then let $\overline{\lambda}$ be so large that $F_\gamma(\varphi_{\overline{\lambda},\sigma}) \leq -\Xi$ uniformly for $\sigma \in M_k$ (see Proposition 6.7 *(ii)*). Fixing this value of $\overline{\lambda}$, we define the family of maps

$$\Pi_{\overline{\lambda}} = \left\{ \varpi : \hat{M}_k \to W^{2,2}(M) \ : \ \varpi \text{ is continuous and } \varpi(\cdot \times \{1\}) = \varphi_{\overline{\lambda},\cdot} \text{ on } M_k \right\}. \tag{6.9}$$

Lemma 6.8. *$\Pi_{\overline{\lambda}}$ is non-empty and moreover, letting*

$$\overline{\Pi}_{\overline{\lambda}} = \inf_{\varpi \in \Pi_{\overline{\lambda}}} \sup_{m \in \hat{M}_k} F_\gamma(\varpi(m)), \qquad \textit{one has} \qquad \overline{\Pi}_{\overline{\lambda}} > -\frac{\Xi}{2}.$$

PROOF. To show that $\Pi_{\overline{\lambda}} \neq \emptyset$, it suffices to consider the map

$$\overline{\varpi}(\sigma, t) = t\varphi_{\overline{\lambda},\sigma}, \qquad (\sigma, t) \in \hat{M}_k. \tag{6.10}$$

Arguing by contradiction, suppose that $\overline{\Pi}_{\overline{\lambda}} \leq -\frac{\Xi}{2}$. Then there would exist a map $\varpi \in \Pi_{\overline{\lambda}}$ with $\sup_{m\in\hat{M}_k} F_\gamma(\varpi(m)) \leq -\frac{3}{8}\Xi$. Since by our choice of Ξ Proposition 6.4 applies with $\frac{\Xi}{4}$, writing $m = (\sigma, t)$, with $\sigma \in M_k$, the map

$$t \mapsto \Psi \circ \varpi(\cdot, t)$$

realizes a homotopy in M_k between $\Psi \circ \varphi_{\overline{\lambda},\cdot}$ and a constant map. However this cannot be, as M_k is non-contractible (see Lemma 6.5) and since $\Psi \circ \varphi_{\overline{\lambda},\cdot}$ is homotopic to the identity on M_k, by Proposition 6.7 (iii). Hence we deduce $\overline{\Pi}_{\overline{\lambda}} > -\frac{\Xi}{2}$. ■

By the statement of Lemma 6.8 and standard variational arguments, one can find a *Palais-Smale sequence* $(w_n)_n$ for F_γ at level $\overline{\Pi}_{\overline{\lambda}}$, namely a sequence for which

$$F_\gamma(w_n) \to \overline{\Pi}_{\overline{\lambda}}; \qquad \nabla F_\gamma(w_n) \to 0.$$

Unfortunately it is not known whether Palais-Smale sequences converge. To show this property, from the fact that $w \mapsto e^{4w}$ is compact from $W^{2,2}(M)$ to $L^1(M)$, it would be sufficient to show that any Palais-Smale sequence is bounded in $W^{2,2}$.

This is in fact proven indirectly, following an argument in [53], by making in the functional F_γ the substitutions $\int Qdv \mapsto t\int Qdv$, $\gamma_1 \mapsto t\gamma_1$, $\mu \mapsto t\mu$ and $II \mapsto II - \Theta(t-1)\gamma_2 \int |\nabla w|^2 dv$ for t close to 1, where Θ is a large positive constant (Θ can be taken zero if P has no negative eigenvalues). We choose a small $t_0 > 0$, and allow t to vary in the interval $[1-t_0, 1+t_0]$. We consider then the functional F_γ for these values of t, denoting it by $(F_\gamma)_t$. If t_0 is sufficiently small, the interval $\left[(1-t_0)\int Udv, (1+t_0)\int Udv\right]$ will be compactly contained in $(8k\gamma_2\pi^2, 8(k+1)\gamma_2\pi^2)$. Following the previous estimates with minor changes, one easily checks that the min-max scheme applies uniformly for $t \in [1-t_0, 1+t_0]$ and for $\overline{\lambda}$ sufficiently large. Precisely, given any large $\Xi > 0$, there exist t_0 sufficiently small and $\overline{\lambda}$ so large that for $t \in [1-t_0, 1+t_0]$

$$\sup_{m\in\partial\hat{M}_k} (F_\gamma)_t(\varpi(m)) < -2\,\Xi; \quad \overline{\Pi}_t := \inf_{\varpi\in\Pi_{\overline{\lambda}}} \sup_{m\in\hat{M}_k} (F_\gamma)_t(\varpi(m)) > -\frac{\Xi}{2},$$

where $\Pi_{\overline{\lambda}}$ is defined in (6.9). Moreover, using for example the test map (6.10), one shows that for t_0 sufficiently small there exists a large constant $\overline{\Xi}$ such that

$$\overline{\Pi}_t \leq \overline{\Xi} \qquad \text{for every } t \in [1-t_0, 1+t_0].$$

If the above constant Θ is chosen large enough (compared to the negative values of the Paneitz operator), it is easy to show that $t \mapsto \frac{\overline{\Pi}_t}{t}$ is non-increasing in $[1-t_0, 1+t_0]$. From this we deduce that the function $t \mapsto \frac{\overline{\Pi}_t}{t}$ is differentiable almost everywhere, and we obtain the following corollary.

Corollary 6.9. *Let $\overline{\lambda}$ and t_0 be as above, and let $\Lambda \subset [1-t_0, 1+t_0]$ be the (dense) set of t for which the function $\frac{\overline{\Pi}_t}{t}$ is differentiable. Then for $t \in \Lambda$ the functional F_γ possesses a bounded Palais-Smale sequence $(w_l)_l$ at level $\overline{\Pi}_t$, weakly converging to a solution of*

$$\mathcal{N}(w) + 2\gamma_2\,\Theta\,(t-1)\,\Delta w + t\,U = t\,\mu\frac{e^{4w}}{\int e^{4w}dv}.$$

PROOF. The existence of a Palais-Smale sequence $(w_l)_l$ follows from Lemma 6.8, and the boundedness is proved exactly as in [19], Lemma 3.2. ■

We can finally prove our second main result.

PROOF OF THEOREM 1.3. We assume that $\gamma_2, \gamma_3 > 0$: obvious changes have to be made for opposite signs. From the above result we obtain a sequence $t_n \to 1$ and a sequence w_n solving

$$\mathcal{N}(w_n) + 2\gamma_2\,\Theta\,(t_n-1)\,\Delta w_n + t_nU = t_n\mu\frac{e^{4w_n}}{\int e^{4w_n}dv},$$

which can be chosen to satisfy $\int e^{4w_n} dv = 1$ for all n. Since the extra term $2\gamma_2\,\Theta\, t_n\,\Delta w_n$ does not affect the analysis in Theorem 1.1, we can then pass to the limit using assumption $\int U dv \notin 8\pi^2\gamma_2\mathbb{N}$. This concludes the proof. ■

7. Appendix

In this appendix we collect a commutator estimate, useful in Section 3, and a Pohozaev-type identity that is used in Section 5.

Given $Q \in L^r(M, TM)$ and $\delta > 0$, define S^x as

$$\begin{array}{rlll} S^x: & L^r(M,TM) & \to & L^{\frac{r}{1-x}}(M,TM) \\ & F & \to & S^x F = \left(\frac{\|F\|_r^2+\|Q\|_r^2}{\delta^2+|F|^2+|Q|^2}\right)^{\frac{x}{2}} F. \end{array}$$

We have the following result:

Theorem 7.1. *Let $r > 1$, $0 < \rho < \min\{1, r-1\}$ and $\Lambda : L^s(M,TM) \to L^s(M,TM)$, $\frac{r}{1+\rho} \le s \le \frac{r}{1-\rho}$, be a linear operator so that*

$$K_0 = \sup_{\frac{r}{1+\rho} \le s \le \frac{r}{1-\rho}} \|\Lambda\|_{\mathcal{L}(L^s)} < +\infty.$$

There exists $K > 0$ so that

$$\|\Lambda(S^x F) - S^x(\Lambda F)\|_{\frac{r}{1-x}} \le K|x| \left(\delta^2 + \|F\|_r^2 + \|Q\|_r^2\right)^{\frac{\rho}{2}} \|F\|_r^{1-\rho} \tag{7.11}$$

for all $|x| \le \rho$, $\delta > 0$ and $Q \in L^r(M,TM)$.

Proof. Let $T = \{z = x + iy : \ |x| \le \rho\}$ and $r_x = \frac{r}{1-x}$, $q_x = \frac{r}{r-1+x}$ be conjugate exponents. Set

$$R_z : F \in L^r(M, TM + iTM) \to R_z F = \left(\frac{\|F\|_r^2 + \|Q\|_r^2}{\delta^2 + |F|^2 + |Q|^2}\right)^{\frac{z}{2}} F \in L^{r_x}(M, TM + iTM)$$

$$Q_z : G \in L^q(M, TM + iTM) \to Q_z G = \left(\frac{|G|}{\|G\|_q}\right)^{\frac{\bar z}{r-1}} G \in L^{q_x}(M, TM + iTM)$$

for all $z \in T$, where $q = \frac{r}{r-1}$. The map Q_z satisfies $\|Q_z G\|_{q_x} = \|G\|_q$ and is invertible with inverse $(Q_z)^{-1} H = \left(\frac{|H|}{\|H\|_{q_x}}\right)^{-\frac{q_x \bar z}{r}} H$. Given $F, G \in L^r(M, TM + iTM)$ define the map $\phi : T \to \mathbb{C}$ as

$$\phi(z) = \int \mathrm{Re}\, \langle \Lambda(R_z F) - R_z(\Lambda F), \overline{Q_z G} \rangle dv.$$

Notice that $\phi(z)$ is a well defined holomorphic function in T in view of $r_x \in [\frac{r}{1+\rho}, \frac{r}{1-\rho}]$. Since by Hölder's estimate there holds

$$\begin{array}{rcl} \|R_z F\|_{r_x} & = & (\|F\|_r^2 + \|Q\|_r^2)^{\frac{x}{2}} \|(\delta^2 + |F|^2 + |Q|^2)^{-\frac{x}{2}} |F|\|_{r_x} \\ & \le & (\|F\|_r^2 + \|Q\|_r^2)^{\frac{x}{2}} \times \begin{cases} \|F\|_r \|(\delta^2 + |F|^2 + |Q|^2)\|_{\frac{r}{2}}^{-\frac{x}{2}} & \text{if } x < 0 \\ \|F\|_r^{1-x} & \text{if } x > 0, \end{cases} \end{array}$$

we have that

$$\|R_z F\|_{r_x} \le \begin{cases} \left[\frac{\delta^2 |M|^{\frac{2}{r}}}{\|F\|_r^2 + \|Q\|_r^2} + 1\right]^{-\frac{x}{2}} \|F\|_r & \text{if } x < 0 \\ \left(1 + \frac{\|Q\|_r^2}{\|F\|_r^2}\right)^{\frac{x}{2}} \|F\|_r & \text{if } x > 0 \end{cases} \le \|F\|_r^{1-\rho} \left(\delta^2 |M|^{\frac{2}{r}} + \|F\|_r^2 + \|Q\|_r^2\right)^{\frac{\rho}{2}}.$$

Hence we can deduce the following estimate on ϕ:

$$|\phi(z)| \le 2K_0 c_0^{\frac{\rho}{2}} \left(\delta^2 + \|F\|_r^2 + \|Q\|_r^2\right)^{\frac{\rho}{2}} \|F\|_r^{1-\rho} \|G\|_q,$$

where $c_0 = \max\{1, K_0^{-2}, |M|^{\frac{2}{r}}, |M|^{\frac{2}{r}} K_0^{-2}\}$. Since $\phi(0) = 0$, Schwartz's lemma on $B_\rho(0) \subset T$ gives that

$$|\phi(z)| \le \frac{2K_0 c_0^{\frac{\rho}{2}}}{\rho} \left(\delta^2 + \|F\|_r^2 + \|Q\|_r^2\right)^{\frac{\rho}{2}} \|F\|_r^{1-\rho} \|G\|_q |z|,$$

and then

$$\begin{aligned}\|\Lambda(R_zF)-R_z(\Lambda F)\|_{r_x} &= \sup_{\|H\|_{q_x}\le 1}\left|\int \mathrm{Re}\,\langle \Lambda(R_zF)-R_z(\Lambda F),\overline{H}\rangle dv\right| \\ &= \sup_{\|G\|_q\le 1}\left|\int \mathrm{Re}\,\langle \Lambda(R_zF)-R_z(\Lambda F),\overline{Q_zG}\rangle dv\right| \\ &\le \frac{2K_0c_0^{\frac{\rho}{2}}}{\rho}\left(\delta^2+\|F\|_r^2+\|Q\|_r^2\right)^{\frac{\rho}{2}}\|F\|_r^{1-\rho}|z|.\end{aligned}$$

Setting $K=\frac{2K_0}{\rho}\max\{1,K_0^{-2},|M|^{\frac{2}{r}},|M|^{\frac{2}{r}}K_0^{-2}\}^{\frac{\rho}{2}}$, we have established the validity of (7.11) for all $|x|\le\rho$ in view of $R_x=S^x$. ■

Notice that (3.15) follows by Theorem 7.1 applied with $\Lambda=\mathrm{Id}-\mathcal{K}$, $F=\nabla p$, $Q=\nabla q$, $x=4\epsilon$ and $r=4(1-\epsilon)$ thanks to (3.13). We next prove a Pohozaev identity, useful to characterize volume quantization in Theorem 1.1.

Proposition 7.2. *Let $p\in M$ and let $\Omega\subseteq M$ be contained in a normal neighbourhood of p. Suppose u solves*

$$\mathcal{N}_g(u)+\tilde{U}=\mu\, e^{4u} \qquad \text{in } \Omega. \tag{7.12}$$

Let $(x^i)_i$ be a system of geodesic coordinates centred at p, and consider in these coordinates a vector field $a=a^i\frac{\partial}{\partial x^i}$ with constant components $(a^i)_i$. Then the following identities hold

$$\begin{aligned}\mathcal{B}_g(p,\Omega,u) &= -\mu\int_\Omega e^{4u}(1+O(|x|^2))dv+\frac{\mu}{4}\oint_{\partial\Omega}e^{4u}x^i\nu_i d\sigma+O\Big(\int_\Omega|\nabla u|(|x|+|\nabla u|)dv\Big) \\ &+ O\left(\int_\Omega|x|(|\nabla^2u|\,|\nabla u|+|\nabla u|^3)dv+\int_\Omega|x|^2(|\nabla^2u|^2+|\nabla u|^4)dv\right)\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}\mathcal{B}_g(p,\Omega,a,u) &= \frac{\mu}{4}\oint_{\partial\Omega}e^{4u}a^i\nu_i d\sigma-\mu\int_\Omega e^{4u}O(|x||a|)dv+O\Big(\int_\Omega|x||\nabla u|(1+|a||\nabla u|)dv\Big) \\ &+ O\left(\int_\Omega|a|(|\nabla^2u|\,|\nabla u|+|\nabla u|^3)dv+\int_\Omega|x||a|(|\nabla^2u|^2+|\nabla u|^4)dv\right),\end{aligned} \tag{7.14}$$

where

$$\begin{aligned}\mathcal{B}_g(p,\Omega,u) &= \left(\frac{\gamma_2}{2}+6\gamma_3\right)\oint_{\partial\Omega}(x^iu_{;i}\frac{\partial\Delta u}{\partial\nu}-\Delta u\frac{\partial(x^iu_{;i})}{\partial\nu}+\frac{1}{2}(\Delta u)^2x^j\nu_j)d\sigma \\ &- 12\gamma_3\oint_{\partial\Omega}(|\nabla u|^2u_{;k}\nu^kx^ju_{;j}-\frac{1}{4}|\nabla u|^4x^j\nu_j)d\sigma \\ &+ 6\gamma_3\oint_{\partial\Omega}\left[x^iu_{;i}\left(\frac{\partial}{\partial\nu}|\nabla u|^2-2\Delta u\frac{\partial u}{\partial\nu}\right)+|\nabla u|^2\left(x^i\nu_i\Delta u-\frac{\partial u}{\partial\nu}-\nabla^2u[x,\nu]\right)\right]d\sigma\end{aligned} \tag{7.15}$$

and

$$\begin{aligned}\mathcal{B}_g(p,\Omega,a,u) &= \left(\frac{\gamma_2}{2}+6\gamma_3\right)\oint_{\partial\Omega}(a^iu_{;i}\frac{\partial\Delta u}{\partial\nu}-\Delta u\frac{\partial(a^iu_{;i})}{\partial\nu}+\frac{1}{2}(\Delta u)^2a^j\nu_j)d\sigma \\ &- 12\gamma_3\oint_{\partial\Omega}(|\nabla u|^2u_{;k}\nu^ka^ju_{;j}-\frac{1}{4}|\nabla u|^4a^j\nu_j)d\sigma \\ &+ 6\gamma_3\oint_{\partial\Omega}\left[a^iu_{;i}\left(\frac{\partial}{\partial\nu}|\nabla u|^2-2\Delta u\frac{\partial u}{\partial\nu}\right)+|\nabla u|^2\left(a^i\nu_i\Delta u-\nabla^2u[a,\nu]\right)\right]d\sigma.\end{aligned} \tag{7.16}$$

Proof. Multiply (7.12) by $x^iu_{;i}$ and integrate by parts: starting with the bi-Laplacian of u we find

$$\int_\Omega x^iu_{;i}\Delta^2u\,dv=\int_\Omega(x^iu_{;ij}{}^j+2x^i_{;j}u_{;i}{}^j+x^i_{;j}{}^ju_{;i})\Delta u\,dv+\oint_{\partial\Omega}(x^iu_{;i}\frac{\partial\Delta u}{\partial\nu}-\Delta u\frac{\partial(x^iu_{;i})}{\partial\nu})d\sigma.$$

Using the fact that in normal coordinates $g_{ij}=\delta_{ij}+O(|x|^2)$ one has that

$$x^j_{;k}=\delta^j_k+O(|x|^2); \qquad x^j_{;k}{}^k=O(|x|); \qquad u_{;jk}{}^k=(\Delta u)_j+O(|\nabla u|).$$

From these we deduce that the L.H.S. in the above formula becomes

$$2\int_\Omega (\Delta u)^2 dv + \int_\Omega \Delta u\, x^j (\Delta u)_{;j} dv + \int_\Omega \left(|x|^2 |\nabla^2 u|^2 + |x||\nabla^2 u||\nabla u| \right) dv.$$

Integrating by parts the second term, the whole expression transforms into

$$\int_\Omega x^i u_{;i} \Delta^2 u\, dv = \oint_{\partial\Omega} (x^i u_{;i} \frac{\partial \Delta u}{\partial \nu} - \Delta u \frac{\partial (x^i u_{;i})}{\partial \nu} + \frac{1}{2} (\Delta u)^2 x^j \nu_j) d\sigma + \int_\Omega \left(|x|^2 |\nabla^2 u|^2 + |x||\nabla^2 u||\nabla u| \right) dv.$$

Similarly, we obtain that

$$\int_\Omega \mathrm{div}(|\nabla u|^2 \nabla u) x^j u_{;j} dv = \oint_{\partial\Omega} (|\nabla u|^2 u_{;k} \nu^k x^j u_{;j} - \frac{1}{4} |\nabla u|^4 x^j \nu_j) d\sigma + \int_\Omega O(|x|^2 |\nabla u|^4) dv.$$

On the other hand, we can also multiply the equation by $a^i u_{;i}$ and using the relations

$$a^j_{;k} = O(|x||a|); \qquad a^j_{;kk} = O(|a|)$$

we find that

$$\int_\Omega a^i u_{;i} \Delta^2 u dv = \oint_{\partial\Omega} (a^i u_{;i} \frac{\partial \Delta u}{\partial \nu} - \Delta u \frac{\partial (a^i u_{;i})}{\partial \nu} + \frac{1}{2} (\Delta u)^2 a^j \nu_j) d\sigma + \int_\Omega \left(|x||a||\nabla^2 u|^2 + |a||\nabla^2 u||\nabla u| \right) dv$$

and

$$\int_\Omega \mathrm{div}(|\nabla u|^2 \nabla u) a^j u_{;j} dv = \oint_{\partial\Omega} (|\nabla u|^2 u_{;k} \nu^k a^j u_{;j} - \frac{1}{4} |\nabla u|^4 a^j \nu_j) d\sigma + \int_\Omega O(|x||a||\nabla u|^4) dv.$$

Analogously, we have the following two formulas

$$\int_\Omega x^i u_{;i} \mathrm{div}(\nabla |\nabla u|^2 - 2\Delta u \nabla u) dv = \int_\Omega (|x||\nabla u|^3 + |x|^2 |\nabla u|^2 |\nabla^2 u|) dv$$
$$+ \quad \oint_{\partial\Omega} \left[x^i u_{;i} \left(\frac{\partial}{\partial \nu} |\nabla u|^2 - 2\Delta u \frac{\partial u}{\partial \nu} \right) + |\nabla u|^2 \left(x^i \nu_i \Delta u - \frac{\partial u}{\partial \nu} - \nabla^2 u[x, \nu] \right) \right] d\sigma$$

and

$$\int_\Omega a^i u_{;i} \mathrm{div}(\nabla |\nabla u|^2 - 2\Delta u \nabla u) dv = \int_\Omega (|a||\nabla u||\nabla^2 u| + |a||x||\nabla u|^2 |\nabla^2 u|) dv$$
$$+ \quad \oint_{\partial\Omega} \left[a^i u_{;i} \left(\frac{\partial}{\partial \nu} |\nabla u|^2 - 2\Delta u \frac{\partial u}{\partial \nu} \right) + |\nabla u|^2 \left(a^i \nu_i \Delta u - \nabla^2 u[a, \nu] \right) \right] d\sigma.$$

Finally, integrating by parts the exponential terms we find

$$\int_\Omega \mu e^{4u} x^i u_{;i} dv = \frac{1}{4} \mu \oint_{\partial\Omega} x^i \nu_i e^{4u} d\sigma - \mu \int_\Omega e^{4u} (1 + O(|x|^2)) dv$$

and

$$\int_\Omega \mu e^{4u} a^i u_{;i} dv = \frac{1}{4} \mu \oint_{\partial\Omega} a^i \nu_i e^{4u} d\sigma - \mu \int_\Omega e^{4u} O(|x||a|) dv.$$

Putting together all the above formulas, recalling the expression of the Paneitz operator and taking into account the lower-order terms, we obtain the conclusion. ■

References

[1] David R. Adams. A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math. (2)*, 128(2):385–398, 1988.

[2] Adimurthi, Frédéric Robert, and Michael Struwe. Concentration phenomena for Liouville's equation in dimension four. *J. Eur. Math. Soc. (JEMS)*, 8(2):171–180, 2006.

[3] Shmuel Agmon. The L_p approach to the Dirichlet problem. I. Regularity theorems. *Ann. Scuola Norm. Sup. Pisa (3)*, 13:405–448, 1959.

[4] Thierry Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geometry*, 11(4):573–598, 1976.

[5] Thierry Aubin. Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire. *J. Funct. Anal.*, 32(2):148–174, 1979.

[6] D. Bartolucci and G. Tarantello. Liouville type equations with singular data and their applications to periodic multi-vortices for the electroweak theory. *Comm. Math. Phys.*, 229(1):3–47, 2002.

[7] Philippe Bénilan, Lucio Boccardo, Thierry Gallouët, Ron Gariepy, Michel Pierre, and Juan Luis Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[8] L. Boccardo and T. Gallouët. Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations*, 17(3-4):641–655, 1992.

[9] Thomas P. Branson, Sun-Yung A. Chang, and Paul C. Yang. Estimates and extremals for zeta function determinants on four-manifolds. *Comm. Math. Phys.*, 149(2):241–262, 1992.
[10] Thomas P. Branson and Bent Ørsted. Explicit functional determinants in four dimensions. *Proc. Amer. Math. Soc.*, 113(3):669–682, 1991.
[11] Haim Brezis and Frank Merle. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Comm. Partial Differential Equations*, 16(8-9):1223–1253, 1991.
[12] Sun-Yung A. Chang, Matthew J. Gursky, and Paul C. Yang. Regularity of a fourth order nonlinear PDE with critical exponent. *Amer. J. Math.*, 121(2):215–257, 1999.
[13] Sun-Yung A. Chang and Paul C. Yang. Extremal metrics of zeta function determinants on 4-manifolds. *Ann. of Math. (2)*, 142(1):171–212, 1995.
[14] Sun-Yung A. Chang and Paul C.-P. Yang. Isospectral conformal metrics on 3-manifolds. *J. Amer. Math. Soc.*, 3(1):117–145, 1990.
[15] Chiun-Chuan Chen and Chang-Shou Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. *Comm. Pure Appl. Math.*, 55(6):728–771, 2002.
[16] Wen Xiong Chen and Congming Li. Prescribing Gaussian curvatures on surfaces with conical singularities. *J. Geom. Anal.*, 1(4):359–372, 1991.
[17] Alain Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.
[18] Weiyue Ding, Jürgen Jost, Jiayu Li, and Guofang Wang. The differential equation $\Delta u = 8\pi - 8\pi he^u$ on a compact Riemann surface. *Asian J. Math.*, 1(2):230–248, 1997.
[19] Weiyue Ding, Jürgen Jost, Jiayu Li, and Guofang Wang. Existence results for mean field equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(5):653–666, 1999.
[20] Zindine Djadli and Andrea Malchiodi. Existence of conformal metrics with constant Q-curvature. *Ann. of Math. (2)*, 168(3):813–858, 2008.
[21] Georg Dolzmann, Norbert Hungerbühler, and Stefan Müller. Non-linear elliptic systems with measure-valued right hand side. *Math. Z.*, 226(4):545–574, 1997.
[22] Georg Dolzmann, Norbert Hungerbühler, and Stefan Müller. Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side. *J. Reine Angew. Math.*, 520:1–35, 2000.
[23] O. Druet and F. Robert. Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth. *Proc. Amer. Math. Soc.*, 134(3):897–908, 2006.
[24] P. Esposito and F. Morlando. On a quasilinear mean field equation with an exponential nonlinearity. *J. Math. Pures Appl. (9)*, 104(2):354–382, 2015.
[25] Pierpaolo Esposito. A classification result for the quasi-linear Liouville equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(3):781–801, 2018.
[26] Luigi Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.*, 68(3):415–454, 1993.
[27] Luigi Greco, Tadeusz Iwaniec, and Carlo Sbordone. Inverting the p-harmonic operator. *Manuscripta Math.*, 92(2):249–258, 1997.
[28] Matthew Gursky and Andrea Malchiodi. Non-uniqueness results for critical metrics of regularized determinants in four dimensions. *Comm. Math. Phys.*, 315(1):1–37, 2012.
[29] Matthew J. Gursky. Uniqueness of the functional determinant. *Comm. Math. Phys.*, 189(3):655–665, 1997.
[30] Matthew J. Gursky. The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics. *Ann. of Math. (2)*, 148(1):315–337, 1998.
[31] Matthew J. Gursky. The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE. *Comm. Math. Phys.*, 207(1):131–143, 1999.
[32] T. Iwaniec and C. Sbordone. Weak minima of variational integrals. *J. Reine Angew. Math.*, 454:143–161, 1994.
[33] Tadeusz Iwaniec. p-harmonic tensors and quasiregular mappings. *Ann. of Math. (2)*, 136(3):589–624, 1992.
[34] Tadeusz Iwaniec and Gaven Martin. Riesz transforms and related singular integrals. *J. Reine Angew. Math.*, 473:25–57, 1996.
[35] Satyanad Kichenassamy and Laurent Véron. Singular solutions of the p-Laplace equation. *Math. Ann.*, 275(4):599–615, 1986.
[36] Yan Yan Li and Itai Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.*, 43(4):1255–1270, 1994.
[37] Yanyan Li and Luc Nguyen. To appear.
[38] Chang-Shou Lin and Juncheng Wei. Sharp estimates for bubbling solutions of a fourth order mean field equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 6(4):599–630, 2007.
[39] Changshou Lin, Juncheng Wei, and Liping Wang. Topological degree for solutions of fourth order mean field equations. *Math. Z.*, 268(3-4):675–705, 2011.
[40] Andrea Malchiodi. Compactness of solutions to some geometric fourth-order equations. *J. Reine Angew. Math.*, 594:137–174, 2006.
[41] S. Minakshisundaram and Å . Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math.*, 1:242–256, 1949.
[42] Niels Martin Möller. Extremal metrics for spectral functions of Dirac operators in even and odd dimensions. *Adv. Math.*, 229(2):1001–1046, 2012.
[43] J. Moser. On a nonlinear problem in differential geometry. pages 273–280, 1973.
[44] Margherita Nolasco and Gabriella Tarantello. On a sharp Sobolev-type inequality on two-dimensional compact manifolds. *Arch. Ration. Mech. Anal.*, 145(2):161–195, 1998.

[45] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of surfaces. *J. Funct. Anal.*, 80(1):212–234, 1988.
[46] B. Osgood, R. Phillips, and P. Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 80(1):148–211, 1988.
[47] Alexander M. Polyakov. Quantum geometry of fermionic strings. *Phys. Lett. B*, 103(3):211–213, 1981.
[48] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Advances in Math.*, 7:145–210, 1971.
[49] Frédéric Robert. Quantization effects for a fourth-order equation of exponential growth in dimension 4. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(3):531–553, 2007.
[50] Frédéric Robert and Juncheng Wei. Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition. *Indiana Univ. Math. J.*, 57(5):2039–2060, 2008.
[51] James Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964.
[52] James Serrin. Isolated singularities of solutions of quasi-linear equations. *Acta Math.*, 113:219–240, 1965.
[53] Michael Struwe. The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.*, 160(1-2):19–64, 1988.
[54] Giorgio Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.
[55] Karen K. Uhlenbeck and Jeff A. Viaclovsky. Regularity of weak solutions to critical exponent variational equations. *Math. Res. Lett.*, 7(5-6):651–656, 2000.
[56] Juncheng Wei. Asymptotic behavior of a nonlinear fourth order eigenvalue problem. *Comm. Partial Differential Equations*, 21(9-10):1451–1467, 1996.

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