# EXISTENCE OF MINIMIZERS FOR POLYCONVEX AND NONPOLYCONVEX PROBLEMS

GIOVANNI CUPINI AND ELVIRA MASCOLO *

**Abstract.** We study the existence of Lipschitz minimizers of integral functionals

$$\mathcal{I}(u) = \int_\Omega \varphi(x, \det Du(x))\, dx$$

where $\Omega$ is an open subset of $\mathbb{R}^N$ with Lipschitz boundary, $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function and $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, $u(x) = x$ on $\partial\Omega$. We consider both the cases of $\varphi$ convex and nonconvex with respect to the last variable. The attainment results are obtained passing through the minimization of an auxiliary functional and the solution of a prescribed jacobian equation.

**Key words.** nonpolyconvex functional, existence of minimizers, Lipschitz regularity, prescribed jacobian equation

**AMS subject classifications.** 49J10, 35J60

## 1. Introduction.

In this paper we consider integral functionals

$$\mathcal{I}(u) = \int_\Omega \varphi(x, \det Du(x))\, dx\,, \tag{1.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary, $N \geq 2$, $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function and $u \in W^{1,N}(\Omega, \mathbb{R}^N)$.

We aim at proving the existence of Lipschitz solutions to the variational problem

$$\min \left\{ \mathcal{I}(u) \,:\, u \in W^{1,N}(\Omega, \mathbb{R}^N),\, \det Du > 0 \text{ a.e.},\, u(x) = x \text{ on } \partial\Omega \right\}. \tag{1.2}$$

Notice that even if a growth condition from below of the type $t^p \leq \varphi(x,t)$ (which is common in the theory of Calculus of Variations) is assumed, no coercivity of $\mathcal{I}$ follows in any Sobolev space, preventing from establishing the existence of minimizers via the Direct Method. Nevertheless many problems of this type have a solution and the question of fixing which conditions on $\varphi$ ensure the existence of solutions is worth of interest, also for its applications in physics, mainly in elasticity theory and in the problem of the equilibrium of gases (see [17], [5], [6] and [12]). For instance, (1.2) is the variational problem corresponding to a non homogeneous elastic material with reference configuration $\Omega$ whose stored energy $\varphi$ is a nonnegative, continuous function depending on the position $x$ in the reference configuration and the size of the deformation of the volume element $\det Du(x) > 0$.

It is well known that an important role is played by the convexity of $\varphi$ with respect to the last variable: when $\varphi$ is convex then $\mathcal{I}$ is said to be a polyconvex functional, if not then $\mathcal{I}$ is nonpolyconvex. The polyconvex case $\varphi = \varphi(t)$ has been studied by Dacorogna [5] and the nonpolyconvex case by Mascolo-Schianchi [14] and Cellina-Zagatti [4].

In order to solve (1.2) our strategy is the following: the first step is to look for solutions to the following variational problem (from now referred to as the *auxiliary* problem)

$$\min \left\{ \mathcal{J}(v) = \int_\Omega \varphi(x, v(x))\, dx \,:\, v \in L^1(\Omega),\, v > 0 \text{ a.e.},\, \int_\Omega v(x)\, dx = |\Omega| \right\}, \tag{1.3}$$

*Dipartimento di Matematica "U. Dini", Università degli Studi di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy (cupini@math.unifi.it, mascolo@math.unifi.it).

where $|\Omega|$ stands for the $N$-dimensional Lebesgue measure of $\Omega$. Then, if $v$ is a solution to (1.3), the second step is to solve in $W^{1,N}(\Omega)$ the boundary value problem

$$\begin{cases} \det Du(x) = v(x) & \text{for a.e. } x \text{ in } \Omega, \\ u(x) = x & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

A solution $u$ to (1.4) is a solution to (1.2), too. In fact, if $w \in W^{1,N}(\Omega)$, $w(x) = x$, on $\partial\Omega$, then $\det Dw \in L^1(\Omega)$ and $\int_\Omega \det Dw(x)\,dx = |\Omega|$; therefore if $\det Dw > 0$ a.e. then

$$\mathcal{I}(u) = \mathcal{J}(v) \leq \mathcal{J}(\det Dw) = \mathcal{I}(w)\,.$$

Following the above scheme, Mascolo in [13] proves the existence of minimizers of (1.2) for smooth domains $\Omega$ and $\varphi \in C^2(\overline{\Omega} \times (0, +\infty))$ strictly convex in the last variable.

As far as problem (1.3) is concerned, Ekeland and Temam in [8] prove a relaxation result and Ball and Knowles in [1] obtain an attainment result with the tool of the Young measures, see also Friesecke [10] for related results. The boundary value problem (1.4) may have no solution unless $v$ is sufficiently regular. For instance, the simple continuity of $v$ is not a sufficient condition to get Lipschitz solutions, see the counterexamples independently given by Burago and Kleiner [2] and by McMullen [15]. Thus, also the regularity properties of minimizers of the auxiliary problem have to be studied. The pioneering papers on (1.4) are due to Moser [16] and Dacorogna-Moser [7]. In particular in [7] the authors prove that if $v$ is in $C^{k,\alpha}(\overline{\Omega})$, $k \geq 0$, and $\partial\Omega \in C^{k+3,\alpha}$, then there exists a diffemorphism of class $C^{k+1,\alpha}(\overline{\Omega})$ solution to (1.4). Later results are due to Rivière and Ye, who prove in [18], Theorem 4, the existence of a bi-Lipschitz homeomorphism $u$ solution to (1.4) under less restrictive assumptions on $\Omega$, with $v$ satisfying a Dini-type continuity property. In [19] Ye proves existence results in the framework of the Sobolev spaces.

The plan of the paper is the following. In section 2 we introduce a class of open sets, invariant under bi-Lipschitz homeomorphisms, which is slightly larger than that of open sets with Lipschitz boundaries, see Definition 2.1. In Theorem 2.4 we state the existence of Lipschitz solutions to (1.4) with $\Omega$ in this class of open sets and Hölder continuous datum $v$. It is a variant of the above cited Theorem 4 in [18] and in Appendix we give the details of the proof. In section 3 we deal with polyconvex functionals. We consider the class of functions $\varphi$ strictly convex in the last variable satisfying, as a substitute for the growth conditions,

$$\lim_{t\to 0^+} D_t\varphi(x,t) = \lambda_0 \quad \text{with } \lambda_0 \in \mathbb{R} \cup \{-\infty\}, \qquad \lim_{t\to +\infty} D_t\varphi(x,t) = +\infty, \tag{1.5}$$

uniformly with respect to $x$. In Proposition 3.1 we prove that a unique solution $v$ to (1.3) exists and that $v$ is in $L^\infty(\Omega)$. In Proposition 3.5, under more regularity assumptions on $\varphi$, we prove that $v$ is Hölder continuous. Therefore, the Lipschitz solution $u$ to (1.4), which exists by Theorem 2.4, is a minimizer of (1.2), see Theorem 3.6. In section 4 we deal with a function $\varphi$ nonconvex with respect to $t$, satisfying (1.5). Denoting $\varphi^{**}$ the convex envelope of $\varphi$ with respect to $t$, we assume that there exist $\alpha, \beta \in L^\infty(\Omega)$, $\beta(x) > \alpha(x)$, $\inf \alpha > 0$, such that for every $x \in \Omega$

$$t \mapsto \varphi^{**}(x,t) \text{ is affine in } [\alpha(x), \beta(x)]$$

and

$$\varphi(x,\cdot) \equiv \varphi^{**}(x,\cdot) \quad \text{and} \quad \varphi(x,\cdot) \text{ is strictly convex} \qquad \text{in } (0,\alpha(x)] \text{ and } [\beta(x), +\infty).$$

Under these assumptions in Theorem 4.1 we prove the existence of a bounded solution $v$ to the auxiliary problem (1.3). In section 5 under regularity assumptions on $\varphi$ we get that $v$ is piecewise Hölder continuous, see Theorem 5.2. In section 6 first we prove that if in (1.4) the datum $v$ is piecewise Hölder continuous there exists a Lipschitz solution, see Proposition 6.2. Then, solving (1.4) with $v$ the piecewise Hölder continuous solution to the auxiliary problem, in Theorems 6.3 and 6.4 we get a Lipschitz continuous minimizer of functional (1.1). In section 7 we consider special classes of nonpolyconvex functionals. First we consider the class of functionals with a nonconvex $\varphi$ satisfying $\varphi(x,\alpha(x)) = \varphi(x,\beta(x)) = 0$. This class has been considered by Zagatti [20] (see also Celada-Perrotta [3] for the case $\varphi(x,u,t)$) with the assumption $\int_\Omega \alpha(x)\,dx < |\Omega| < \int_\Omega \beta(x)\,dx$. In [20] and [3] the attainment result is proved using different arguments, as the Baire category method and the convex integration method, respectively. Theorems 7.1 and 7.2 are attainment results including the cases $\int_\Omega \alpha\,dx \geq |\Omega|$ and $\int_\Omega \beta(x)\,dx \leq |\Omega|$. Theorem 7.4 deals with a *perturbation* of these functionals, see problem (7.2). We conclude the section considering functionals with $\varphi$ satisfying the structure condition $\varphi(x,t) = \tilde{\varphi}(|x|,t)$. In this case the existence of bounded radial solutions to (1.3) directly implies the existence of Lipschitz solutions to (1.4).

**2. Notations and preliminary results.** In the following if $\Omega$ is a measurable subset of $\mathbb{R}^N$ then $|\Omega|$ stands for its $N$-dimensional Lebesgue measure. We write $Q$ in place of $(0,1)^N$ and $B_r(x)$ denotes the ball in $\mathbb{R}^N$ with center at $x$ and radius $r$. If $\varphi : \Omega \times (0,+\infty) \to [0,+\infty)$ then $\varphi^{**}$ is the convex envelope of $\varphi$ with respect to the second variable, i.e. $t \mapsto \varphi^{**}(x,t)$ is the greatest convex function lower than $t \mapsto \varphi(x,t)$. For the sake of simplicity we write $\varphi(x,\cdot)$ instead of $t \mapsto \varphi(x,t)$,

$$D_t^- \varphi(x,s) := \lim_{t\to s^-} \frac{\varphi(x,t)-\varphi(x,s)}{t-s}, \qquad D_t^+ \varphi(x,s) := \lim_{t\to s^+} \frac{\varphi(x,t)-\varphi(x,s)}{t-s}$$

and $\partial\varphi(x,s) := \{d \in \mathbb{R} : \varphi(x,t) \geq \varphi(x,s) + d(t-s) \text{ for every } t \in (0,+\infty)\}$.

We define a class of bounded open subsets of $\mathbb{R}^N$.

Definition 2.1. *We say that a bounded open set $\Omega$ of $\mathbb{R}^N$ is of class $(L)$ if $\overline{\Omega}$ has a covering of finitely many open sets $\Omega_j$ such that for every $j$ there exists a bi-Lipschitz homeomorphism $\psi_j : \overline{\Omega_j \cap \Omega} \to \overline{Q}$, satisfying*

*(a) $\psi_j(\overline{\Omega}_j \cap \partial\Omega) = \{0\} \times [0,1]^{N-1}$, whenever $\overline{\Omega}_j \cap \partial\Omega$ is not empty,*

*(b) $\det D\psi_j$ is Lipschitz continuous and there exists $A \geq 1$ such that $\frac{1}{A} \leq \det D\psi_j \leq A$.*

The above definition describes a larger class than that of open sets with Lipschitz boundary, i.e. with the boundary which locally is the graph of a Lipschitz function. This result can be proved in a similar way as Proposition A.1 in [7].

Lemma 2.2. *If a bounded open set $\Omega$ of $\mathbb{R}^N$ has a Lipschitz boundary then it is of class $(L)$.*

An easy consequence of the chain rule for Lipschitz functions is that Definition 2.1 is invariant under bi-Lipschitz homeomorphisms.

Lemma 2.3. *Let $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ be a bi-Lipschitz homeomorphism, with $\det Du_0$ Lipschitz continuous, $\frac{1}{A} \leq \det Du_0 \leq A$ for some $A$. If $\Omega$ is of class $(L)$ then $u_0(\Omega)$ is of class $(L)$, too.*

On the contrary, there are examples of bounded open sets of $\mathbb{R}^N$ with Lipschitz boundary which are mapped by a bi-Lipschitz homeomorphism $u : \mathbb{R}^N \to \mathbb{R}^N$ onto sets with a not (Lipschitz) continuous boundary, see e.g. [11], p.8-9. Therefore, the converse of Lemma 2.2 is not true.

Now, we state an existence result of Lipschitz solutions to

$$\begin{cases} \det Du = f & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with $f$ Hölder continuous.

THEOREM 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set of class $(L)$. Let $f$ be a Hölder continuous function, $\inf f > 0$, $\int_\Omega f(x)\,dx = |\Omega|$. Then there exists a bi-Lipschitz homeomorphism $u : \overline{\Omega} \to \overline{\Omega}$ solution to (2.1).*

A similar result is proved in [18], Theorem 4, with a weaker assumption on $v$, which is assumed to satisfy a Dini-type continuity property, and a regular domain $\Omega$. In [18] the proof is given for cubes only. The proof of Theorem 2.4, based upon the application to open sets of class $(L)$ of the partition method due to Moser [16], is in the Appendix.

**3. Polyconvex problems: an attainment result.** In this section we consider the variational problem

$$\min\left\{\int_\Omega \psi(x, \det Du(x))\,dx \;:\; u \in W^{1,N}(\Omega, \mathbb{R}^N),\ \det Du > 0 \text{ a.e., } u(x) = x \text{ on } \partial\Omega\right\} \tag{3.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary and $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function.

To get solutions to (3.1), we first consider the following variational problem

$$\min\left\{\int_\Omega \psi(x, v(x))\,dx \;:\; v \in L^1(\Omega),\ v > 0 \text{ a.e., } \int_\Omega v(x)\,dx = a\right\},\ a > 0. \tag{3.2}$$

As far as the problem (3.2) is concerned, the Lipschitz regularity of the boundary of $\Omega$ can be dropped.

We prove that there exists a (unique) bounded solution to (3.2) if

(H1) $t \mapsto \psi(x,t)$ is strictly convex for all $x \in \Omega$,

(H2) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to 0^+} D_t^+ \psi(x,t) = \lambda_0, \qquad \lim_{t \to +\infty} D_t^- \psi(x,t) = +\infty, \qquad \text{uniformly in } x.$$

PROPOSITION 3.1. *Assume that $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function satisfying (H1) and (H2). Then for every $\lambda > \lambda_0$ there exists a unique $u_\lambda \in L^\infty(\Omega)$, $\inf u_\lambda > 0$, such that*

$$\lambda \in \partial\psi(x, u_\lambda(x)) \qquad \forall\, x \in \Omega. \tag{3.3}$$

*Moreover, there exists $\lambda_a > \lambda_0$ such that $u_{\lambda_a}$ is the unique solution to (3.2).*

*Proof.* We proceed as follows: at first we prove that for every $\lambda > \lambda_0$ there exists a function $u_\lambda$ such that (3.3) holds. Then, we prove that $u_\lambda$ is in $L^\infty(\Omega)$, $\inf u_\lambda > 0$, and there exists $\lambda_a$ such that $\int_\Omega u_{\lambda_a}\,dx = a$. Thus, it turns out that $u_{\lambda_a}$ is a solution to (3.2) and it is unique, because of the strict convexity of the functional.

*Step 1. The definition of $u_\lambda$.* Fixed $x \in \Omega$, we define the sets

$$C(x) := \{s \in (0, +\infty) \;:\; D_t^- \psi(x,s) < D_t^+ \psi(x,s)\}, \quad \Omega_C := \{x \in \Omega \;:\; C(x) \neq \emptyset\}.$$

Notice that $\partial\psi(x,s) = [D_t^- \psi(x,s), D_t^+ \psi(x,s)]$ for all $(x,s) \in \Omega \times (0, +\infty)$.

Suppose that $x \in \Omega \backslash \Omega_C$. From (H1) and the definition of $\Omega_C$, the function $D_t\psi(x,\cdot) : (0,+\infty) \to (\lambda_0,+\infty)$ is well defined, continuous and strictly increasing. Moreover, it is a surjective function, because of (H2). Let $u(x,\cdot)$ be its inverse function, i.e. $u(x,\cdot) : (\lambda_0,+\infty) \to (0,+\infty)$ is such that $u(x,\lambda)$ (from now on denoted by $u_\lambda(x)$) is the unique positive number such that $\lambda = D_t\psi(x,u_\lambda(x))$. $u(x,\cdot)$ is a well defined, strictly increasing and continuous function.
Now let us consider $x \in \Omega_C$. From (H1), $C(x)$ is (at most) a countable set, so that we denote $C(x) = \{t_n(x)\}_{n\in J(x)}$, where $J(x) \subseteq \mathbb{N}$. As in the above case, if $\lambda \notin \cup_{n\in J(x)}\partial\psi(x,t_n(x))$ we define $u_\lambda(x)$ as the unique positive number such that $D_t\psi(x,u_\lambda(x)) = \lambda$. If instead $\lambda \in \partial\psi(x,t_n(x))$ for some $n \in J(x)$, then we set $u_\lambda(x) = t_n(x)$. Notice that if $u_\lambda(x)$ is chosen greater (less) than $t_n(x)$ then $\lambda < D_t^-\psi(x,u_\lambda(x))$ ($\lambda > D_t^+\psi(x,u_\lambda(x))$). It is easy to prove that for each $x \in \Omega_C$ the function $u(x,\cdot) : (\lambda_0,+\infty) \to (0,+\infty)$ is well defined, increasing and continuous.
Thus, $u_\lambda : \Omega \to (0,+\infty)$ is the unique function satisfying (3.3) and it is measurable, since

$$\{x \in \Omega \,:\, u_\lambda(x) < t\} = \{x \in \Omega \,:\, D_t^-\psi(x,t) > \lambda\}$$

and $D_t^-\psi(x,t) = \sup_{h<0}(\psi(x,t+h)-\psi(x,t))/h$. By the second limit in (H2) for every $\lambda > \lambda_0$ there exists $R > 0$ such that $D_t^-\psi(x,R) > \lambda$ for every $x \in \Omega$, which implies $u_\lambda(x) < R$ for every $x \in \Omega$. In fact, if $u_\lambda(x) \geq R$ for some $x$, then by the convexity of $\psi$ with respect to the second variable it would be $D_t^-\psi(x,R) \leq D_t^-\psi(x,u_\lambda(x))$ and by (3.3) we would obtain $D_t^-\psi(x,R) \leq \lambda$, which is a contradiction. Thus, $u_\lambda$ is in $L^\infty(\Omega)$. The first limit in (H2) implies that for each $\lambda > \lambda_0$ there exists $c(\lambda) > 0$ such that $\sup_{y\in\Omega} D_t^+\psi(y,t) < \lambda$ for every $t < c(\lambda)$. Therefore it cannot be $u_\lambda(x) < c(\lambda)$, because $\lambda \leq D_t^+\psi(x,u_\lambda(x))$, so that $\inf u_\lambda > 0$.

*Step 2. The definition of* $\lambda_a$. Define $\Psi : (\lambda_0,+\infty) \to (0,+\infty)$, $\Psi(\lambda) := \int_\Omega u_\lambda(x)\,dx$, where $u_\lambda(x) = u(x,\lambda)$ is defined as in step 1. By the monotonicity of $u$ with respect to $\lambda$, $\Psi$ is increasing. It holds true that $\lim_{\lambda\to\lambda_0^+} u_\lambda(x) = 0$. In fact, suppose that $\lim_{\lambda\to\lambda_0^+} u_\lambda(x) = \delta(x) > 0$. By (H1), the first limit in (H2) and (3.3) we get

$$\lambda_0 < D_t^-\psi(x,\delta(x)) \leq D_t^-\psi(x,u_\lambda(x)) \leq \lambda\,.$$

Therefore, letting $\lambda$ go to $\lambda_0^+$ we get a contradiction. Analogously it can be proved that $\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty$. Hence,

$$\lim_{\lambda\to\lambda_0^+} \Psi(\lambda) = 0, \qquad \lim_{\lambda\to+\infty} \Psi(\lambda) = +\infty. \tag{3.4}$$

From the previous step $\lambda \mapsto u_\lambda(x)$ is continuous and increasing for all $x$ and $u_\lambda \in L^\infty(\Omega)$ for all $\lambda$, therefore $\Psi$ is a continuous function. Thus, there exists $\lambda_a > \lambda_0$ such that $\Psi(\lambda_a) = a$. We claim that $u_{\lambda_a}$ is a solution to (3.2). In fact, from (H1) and (3.3) for every $w \in L^1(\Omega)$ such that $w > 0$ and $\int_\Omega w(x)\,dx = a$, we have that

$$\psi(x,w(x)) \geq \psi(x,u_{\lambda_a}(x)) + \lambda_a(w(x) - u_{\lambda_a}(x)) \qquad \forall\, x \in \Omega.$$

Thus,

$$\int_\Omega \psi(x,w(x))\,dx \geq \int_\Omega \psi(x,u_{\lambda_a}(x))\,dx + \lambda_a \int_\Omega (w(x)-u_{\lambda_a}(x))\,dx = \int_\Omega \psi(x,u_{\lambda_a}(x))\,dx\,.$$

□

Remark 3.2. *The growth conditions*

$$\lim_{t\to 0^+}\inf_{y\in\Omega}\psi(y,t)=+\infty\,,\qquad \lim_{t\to+\infty}\inf_{y\in\Omega}\frac{\psi(y,t)}{t}=+\infty,$$

*imply (H2). If the first limit in (H2) is not uniform with respect to $x$, then may be $\inf u_\lambda = 0$. Moreover, the proof of Proposition 3.1 works also if we replace $\lim_{t\to+\infty} D_t^-\psi(x,t)=+\infty$ with the more general*

$$\lim_{t\to+\infty} D_t^-\psi(x,t)=\lambda_\infty,\qquad \lambda_\infty\in\mathbb{R}\cup\{+\infty\}.$$

It is easy to prove the following refinement of Proposition 3.1.

Proposition 3.3. *Let $\psi:\Omega\times(0,+\infty)\to[0,+\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\psi\in C(\Omega\times(0,+\infty))$. If (H1) and (H2) hold then the functions $u_\lambda$ in Proposition 3.1 are continuous for every $\lambda>\lambda_0$.*

*Proof.* For every $\lambda>\lambda_0$ let $u_\lambda\in L^\infty(\Omega)$ be as in Proposition 3.1. $u_\lambda$ is lower semicontinuous. In fact, if

$$\liminf_{x\to x_0} u_\lambda(x)<\alpha<u_\lambda(x_0)\,, \tag{3.5}$$

then (H1) and (3.3) imply $D_t\psi(x_0,\alpha)<\lambda$. By continuity of $D_t\psi$ there exists $\delta>0$ such that $D_t\psi(x,\alpha)<\lambda$ for every $x\in(x_0-\delta,x_0+\delta)$. Then, from (3.3) again we have that $D_t\psi(x,\alpha)<D_t\psi(x,u_\lambda(x))$ for every $x\in(x_0-\delta,x_0+\delta)$, which implies $\alpha<u_\lambda(x)$, in contradiction with (3.5). Analogously the upper semicontinuity of $u_\lambda$ can be proved. □

To get Hölder continuous solutions to (3.2) we require more regularity on $\psi$:

(H3) there exists $0<\sigma\le 1$ such that for every compact $K\subset(0,+\infty)$ and for every $t\in K$ the function $x\mapsto D_t\psi(x,t)$ is of class $C^{0,\sigma}(\Omega)$, with $[D_t\psi(\cdot,t)]_{0,\sigma}\le k_K$,

(H4) for every $m>0$ there exists $c_m>0$ such that

$$\psi(x,t)\ge\psi(x,s)+D_t\psi(x,s)(t-s)+c_m|t-s|^{2+\varepsilon}$$

for every $t>s\ge m$, for every $x\in\Omega$ and some $\varepsilon\ge 0$.

Remark 3.4. *Assumption (H4) is equivalent to assume that for every $m>0$ there exists $\tilde c_m>0$ such that*

$$D_t\psi(x,t)-D_t\psi(x,s)\ge\tilde c_m|t-s|^{1+\varepsilon}\qquad\forall\, t>s\ge m,\quad\forall\, x\in\Omega. \tag{3.6}$$

*Roughly speaking, if $\psi\in C^2$ satisfies (H4), then $D_{tt}\psi$ may vanish provided that a suitable growth near the zeros is satisfied, see (3.a) below.*

*Notice that if $\psi_0$ satisfies (H4) and $\psi_1=\psi_1(x,t)$ is such that $\psi_1(x,\cdot)$ is convex and $C^1$, then $\psi=\psi_0+\psi_1$ satisfies (H4), too. Examples of functions $\psi_0$ satisfying (H4) are the following:*

*(1) $\psi_0(t):=(1+t^2)^{p/2}$, $p\ge 2$. See [9] for details.*

*(2) $\psi_0(x,t):=|t-a(x)|^p$, with $a:\Omega\to\mathbb{R}$ and $p\ge 2$,*

*(3) $\psi_0:\overline{\Omega}\times(0,+\infty)\to[0,+\infty)$ of class $C^2$, strictly convex with respect to $t$, such that for every $x$ there exists at most finitely many positive numbers $\{s_i(x)\}$, such that $D_{tt}\psi_0(x,s_i(x))=0$ and the following two hold:*

*(a) there exist $\varepsilon,c>0$ such that $D_{tt}\psi_0(x,t)\ge c|t-s_i(x)|^\varepsilon$, for every $t$ in a neighborhood of $s_i(x)$,*

*(b) there exists* $M > 0$ *such that* $\inf\{D_{tt}\psi_0(x,t) \,:\, (x,t) \in \Omega \times [M,+\infty)\} > 0$.

PROPOSITION 3.5. *Let* $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ *be a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4). Then for each* $\lambda > \lambda_0$*, the function* $u_\lambda$ *in Proposition 3.1 is in* $C^{0,\sigma/(1+\varepsilon)}(\Omega)$*. In particular, for every* $a > 0$ *the unique solution* $u_{\lambda_a}$ *to (3.2) is Hölder continuous.*

*Proof.* Fix $\lambda$ and let $u_\lambda$, from now on referred to as $u$, be the correspondent function as described in Proposition 3.1. From the strict convexity of $\psi$ with respect to the last variable and since $\lambda = D_t\psi(x,u(x))$ for every $x \in \Omega$ it is easy to check that $u$ is $\gamma$-Hölder continuous with Hölder constant $[u]_\gamma$ if and only if

$$(3.7) \qquad D_t\psi(y,u(x)+[u]_{0,\gamma}|x-y|^\gamma) - D_t\psi(x,u(x)) \geq 0\,, \qquad \forall\, x,y \in \Omega.$$

Fix $x, y \in \Omega$. By (H4) and (3.6) there exist $\varepsilon \geq 0$ and $\tilde{c} > 0$ such that

$$(3.8) \quad D_t\psi(x,t) - D_t\psi(x,s) \geq \tilde{c}(t-s)^{1+\varepsilon} \qquad \forall\, t > s \geq \inf u > 0, \quad \forall\, x \in \Omega.$$

Consider the compact interval $K = [\inf u, \|u\|_\infty]$ and let $s$ and $t$ be equal to $u(x)$ and $u(x) + \left(\frac{k}{\tilde{c}}|x-y|^\sigma\right)^{1/(1+\varepsilon)}$, respectively, with $\sigma$ and $k_K$ as in (H3). Using (3.8) and (H3) to estimate $D_t\psi(y,t) - D_t\psi(y,s)$ and $D_t\psi(y,s) - D_t\psi(x,s)$, respectively, we get

$$D_t\psi(y,t) - D_t\psi(x,s) = D_t\psi(y,t) - D_t\psi(y,s) + D_t\psi(y,s) - D_t\psi(x,s) \geq 0.$$

Then $u$ is $\gamma$-Hölder continuous, with $\gamma = \frac{\sigma}{1+\varepsilon}$.
Thus, fixed $a > 0$, the solution $u_{\lambda_a}$ to (3.2), that exists by Proposition 3.1, is Hölder continuous. □

Now we are ready to state an existence result of Lipschitz solutions to the polyconvex problem (3.1).

THEOREM 3.6. *Suppose that* $\Omega$ *is a bounded open subset of* $\mathbb{R}^N$ *with Lipschitz boundary and let* $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ *be a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4). Then there exists a Lipschitz continuous solution to (3.1).*

*Proof.* Set $a = |\Omega|$ and consider the variational problem (3.2). From Propositions 3.1 and 3.5 such a problem has a (unique) solution $u_{\lambda_a} \in C^{0,\gamma}(\Omega)$, $\gamma > 0$, and $\inf u_{\lambda_a} > 0$. Hence, from Theorem 2.4 there exists a bi-Lipschitz homeomorphism $u$ solving

$$\begin{cases} \det Du = u_{\lambda_a} & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases}$$

and $u$ is a solution to (3.1), too. □

**4. Nonpolyconvex problems: attainment result for the auxiliary problem.** In this section we consider the variational problem

$$(4.1) \qquad \min\left\{\int_\Omega \varphi(x,v(x))\,dx \,:\, v \in L^1(\Omega),\, v > 0 \text{ a.e.}, \int_\Omega v(x)\,dx = a\right\},\ a > 0$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $\varphi : \Omega \times (0,+\infty) \to [0,+\infty)$ is a continuous function, nonconvex with respect to the last variable $t$.
Let $\varphi^{**}$ be the convex envelope of $\varphi$ with respect to the second variable and define

$$\Omega_A := \{x \in \Omega \,:\, t \to \varphi(x,t) \text{ is not strictly convex}\}.$$

We assume that the following assumptions hold:

(K1) $\Omega_A$ is a (not empty) measurable set and there exist $\alpha, \beta \in L^\infty(\Omega_A)$, $\beta(x) > \alpha(x)$ for all $x$, $\inf \alpha > 0$, such that $\varphi(x,\cdot)$ and $\varphi^{**}(x,\cdot)$ both coincide and are strictly convex in $(0,\alpha(x)]$ and $[\beta(x),+\infty)$, for every $x \in \Omega_A$,

(K2) $\varphi^{**}(x,\cdot)$ is affine in $[\alpha(x),\beta(x)]$ for all $x \in \Omega_A$, i.e. for every $\alpha(x) \le t \le \beta(x)$

$$\varphi^{**}(x,t) = h(x)t + q(x), \qquad \text{with } h(x) = \frac{\varphi(x,\beta(x)) - \varphi(x,\alpha(x))}{\beta(x)-\alpha(x)},$$

(K3) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t\to 0^+} D_t^+\varphi(x,t) = \lambda_0, \qquad \lim_{t\to+\infty} D_t^-\varphi(x,t) = +\infty, \qquad \text{uniformly in } x.$$

Theorem 4.1. *Assume (K1), (K2) and (K3). Then there exist* $\lambda_a > \lambda_0$ *and* $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, *such that*

*(i)* $v_{\lambda_a}(x) \notin (\alpha(x),\beta(x))$ *for every* $x \in \Omega_A$,

*(ii)* $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}(x))$ *for every* $x \in \Omega$,

*(iii)* $\int_\Omega v_{\lambda_a}(x)\,dx = a$.

*In particular,* $v_{\lambda_a}$ *is a solution to (4.1). Moreover, if* $\Omega = B_1(0)$ *and* $\varphi(x,t) = \tilde\varphi(|x|,t)$ *then* $v_{\lambda_a}$ *is a radial function.*

We postpone the proof of Theorem 4.1 to the following lemma.

Lemma 4.2. *Let* $O$ *be a bounded measurable subset of* $\mathbb{R}^N$. *Let* $\alpha,\beta \in L^1(O)$ *be such that* $\alpha(x) \le \beta(x)$ *for a.e.* $x$ *and suppose*

$$\int_O \alpha(x)\,dx < \kappa < \int_O \beta(x)\,dx\,. \tag{4.2}$$

*Then there exists* $r > 0$ *such that* $\Theta : O \to \mathbb{R}$, $\Theta(x) := \alpha(x)$ *if* $x \in O \cap B_r(0)$ *and* $\Theta(x) := \beta(x)$ *else, satisfying* $\int_O \Theta(x)\,dx = \kappa$.

*Proof.* Let $R$ be such that $O \subset B_R(0)$. Consider the functions $\theta_\rho : O \to \mathbb{R}$, $0 \le \rho \le R$, defined as follows: $\theta_0 := \beta$ and if $\rho \ne 0$ then $\theta_\rho(x) := \alpha(x)$ if $x \in O \cap B_r(0)$ and $\theta_\rho(x) := \beta(x)$ else. The continuity of $\rho \to \int_O \theta_\rho(x)\,dx$ and (4.2) imply that there exists $0 < r < R$ such that $\int_O \theta_r(x)\,dx = \kappa$. □

We are now ready to prove Theorem 4.1.

*Proof.* [Proof of Theorem 4.1.] We divide the proof into three steps. In step 1 we define a family of functions $v_\lambda^- : \Omega \to (0,+\infty)$, $\lambda > \lambda_0$, such that

$$v_\lambda^-(x) \notin (\alpha(x),\beta(x)) \qquad \forall x \in \Omega_A, \quad \forall \lambda > \lambda_0, \tag{4.3}$$

and

$$\lambda \in \partial\varphi^{**}(x, v_\lambda^-(x)) \qquad \forall x \in \Omega, \quad \forall \lambda > \lambda_0. \tag{4.4}$$

In step 2 we define a function $v_{\lambda_a}$ satisfying (i), (ii) and (iii). Finally, in step 3 we consider the case $\varphi(x,t) = \tilde\varphi(|x|,t)$.

*Step 1. The definition of* $v_\lambda^-$. Let us define the function $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$, such that $\psi \equiv \varphi$ in $(\Omega \setminus \Omega_A) \times (0,+\infty)$ and

$$\psi(x,t) := \begin{cases} \varphi(x,t) & \text{if } x \in \Omega_A,\ 0 < t \le \alpha(x), \\ \varphi(x, t+\beta(x)-\alpha(x)) - & \\ -\varphi(x,\beta(x)) + \varphi(x,\alpha(x)) & \text{if } x \in \Omega_A,\ t > \alpha(x). \end{cases} \tag{4.5}$$

(K1) and (K2) imply that for every $x \in \Omega_A$

$$D_t^- \varphi(x, \alpha(x)) \le h(x) = \frac{\varphi(x, \beta(x)) - \varphi(x, \alpha(x))}{\beta(x) - \alpha(x)} \le D_t^+ \varphi(x, \beta(x)) \tag{4.6}$$

and that $\psi$ satisfies (H1). Moreover, for every $x \notin \Omega_A$ and every $t > 0$ we have $\partial\psi(x,t) = \partial\varphi(x,t) = \partial\varphi^{**}(x,t)$. If instead $x \in \Omega_A$ then

$$\partial\psi(x,t) = \begin{cases} \partial\varphi(x,t) & \text{if } \ 0 < t < \alpha(x), \\ \partial\varphi^{**}(x,\alpha(x)) \cup \partial\varphi^{**}(x,\beta(x)) & \text{if } \ t = \alpha(x), \\ \partial\varphi(x, t + \beta(x) - \alpha(x)) & \text{if } \ t > \alpha(x). \end{cases} \tag{4.7}$$

We claim that (K3) implies that $\psi$ satisfies (H2).
The first limit in (K3) and the assumption $\inf \alpha > 0$ imply $\lim_{t \to 0^+} D_t^+ \psi(x,t) = \lambda_0$, uniformly. Let us prove that $\psi$ satisfies the property on the second limit in (H2). Since $\alpha, \beta \in L^\infty(\Omega_A)$ then for every $x \in \Omega$ and $t > \|\alpha\|_{L^\infty(\Omega_A)}$

$$\begin{aligned} \inf_{y \in \Omega} D_t^- \varphi(y,t) &\le \min\{ \inf_{y \in \Omega_A} D_t^- \varphi(y, t + \beta(y) - \alpha(y)), \inf_{y \in \Omega \setminus \Omega_A} D_t^- \varphi(y,t) \} = \\ &= \inf_{y \in \Omega} D_t^- \psi(y,t) \le D_t^- \psi(x,t) \le D_t^- \varphi(x, t + \|\beta - \alpha\|_{L^\infty(\Omega_A)}), \end{aligned}$$

so that by (K3) as $t$ goes to $+\infty$ we get

$$\lim_{t \to +\infty} \inf_{y \in \Omega} D_t^- \psi(y,t) = \lim_{t \to +\infty} D_t^- \psi(x,t) = +\infty\,, \qquad \forall x \in \Omega.$$

Since $\psi$ satisfies the assumptions of Proposition 3.1 then for every $\lambda > \lambda_0$ there exists $u_\lambda \in L^\infty(\Omega)$, $\inf u_\lambda > 0$, satisfying (3.3). Moreover, for every $x \in \Omega_A$

$$\begin{array}{lll} u_\lambda(x) < \alpha(x) & \text{if} & \lambda < D_t^- \varphi(x,\alpha(x)), \\ u_\lambda(x) = \alpha(x) & \text{if} & \lambda \in [D_t^- \varphi(x,\alpha(x)), D_t^+ \varphi(x,\beta(x))], \\ u_\lambda(x) > \alpha(x) & \text{if} & \lambda > D_t^+ \varphi(x,\beta(x)). \end{array} \tag{4.8}$$

Let us define $v_\lambda^- : \Omega \to (0, +\infty)$,

$$v_\lambda^-(x) := u_\lambda(x) + (\beta(x) - \alpha(x)) \chi_{\{y \in \Omega_A \,:\, h(y) < \lambda\}}(x).$$

Since $u_\lambda \in L^\infty(\Omega)$ and $\alpha, \beta \in L^\infty(\Omega_A)$, then $v_\lambda^- \in L^\infty(\Omega)$. From (3.3), (4.6), (4.7) and (4.8) if $x \in \Omega_A$ the following implications hold:

- if $\lambda < D_t^- \varphi(x,\alpha(x))$ then $v_\lambda^-(x) = u_\lambda(x) < \alpha(x)$ and $\lambda \in \partial\psi(x, u_\lambda(x)) = \partial\varphi(x, v_\lambda^-(x))$,
- if $\lambda \in [D_t^- \varphi(x,\alpha(x)), h(x)]$ then $v_\lambda^-(x) = u_\lambda(x) = \alpha(x)$ and $\lambda \in \partial\varphi^{**}(x,\alpha(x))$,
- if $\lambda \in (h(x), D_t^+ \varphi(x,\beta(x))]$ then $v_\lambda^-(x) = \beta(x)$ and $\lambda \in \partial\varphi^{**}(x,\beta(x))$,
- if $\lambda > D_t^+ \varphi(x,\beta(x))$ then $v_\lambda^-(x) = u_\lambda(x) + \beta(x) - \alpha(x) > \beta(x)$ and $\lambda \in \partial\psi(x, u_\lambda(x)) = \partial\varphi(x, v_\lambda^-(x))$.

Thus (4.3) holds and

$$\lambda \in \partial\varphi^{**}(x, v_\lambda^-(x)), \tag{4.9}$$

for every $x \in \Omega_A$ and $\lambda > \lambda_0$. When $x \notin \Omega_A$, the equality $v_\lambda^-(x) = u_\lambda(x)$ and (3.3) imply (4.9). Therefore (4.4) holds true.

*Step 2. The definition of* $\lambda_a$ *and* $v_{\lambda_a}$. Let us define $\Phi : (\lambda_0, +\infty) \to (0, +\infty)$,

$$\Phi(\lambda) := \int_\Omega v_\lambda^-(x)\, dx = \int_\Omega \big(u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\{y\in\Omega_A\,:\,h(y)<\lambda\}}(x)\big)\, dx.$$

As in the proof of (3.4) we have that $\lim_{\lambda\to\lambda_0^+}\Phi(\lambda) = 0$ and $\lim_{\lambda\to+\infty}\Phi(\lambda) = +\infty$. For each $\lambda > \lambda_0$ define $v_\lambda^+ : \Omega \to (0, +\infty)$,

$$v_\lambda^+(x) := u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\{y\in\Omega_A\,:\,h(y)\le\lambda\}}(x).$$

For every $\mu > \lambda_0$

$$\lim_{\lambda\to\mu^-}\Phi(\lambda) = \Phi(\mu)\,, \qquad \lim_{\lambda\to\mu^+}\Phi(\lambda) = \int_\Omega v_\mu^+(x)\, dx.$$

Thus, $\Phi$ is discontinuous at $\mu$ if and only if $|\{y \in \Omega_A \,:\, h(y) = \mu\}| > 0$.
Only one of the following cases is possible:

1. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) = a$,
2. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a = \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$,
3. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$.

*Case 1.* As proved in step 1, $v_{\lambda_a}^-$ satisfies (i), (ii) and $\inf v_{\lambda_a}^- \ge \inf u_{\lambda_a} > 0$. Moreover, by definition of $\lambda_a$, (iii) holds. Thus, define $v_{\lambda_a} = v_{\lambda_a}^-$.

*Case 2.* As above, $v_{\lambda_a}^-$ satisfies (i), (ii) and $\inf v_{\lambda_a}^- \ge \inf u_{\lambda_a} > 0$. It is easy to check that a property analogous to (i) is satisfied by $v_{\lambda_a}^+$ and that $\inf v_{\lambda_a}^+ \ge \inf v_{\lambda_a}^- > 0$. By the very definition of $v_{\lambda_a}^+$ we have also $\int_\Omega v_{\lambda_a}^+\, dx = a$.
Let us prove that $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$ for every $x$. If $x \notin \Omega_A$ or if $x \in \Omega_A$ and $h(x) \ne \lambda_a$ then $v_{\lambda_a}^-(x) = v_{\lambda_a}^+(x)$ and the above inclusion follows. Suppose that $x \in \Omega_A$ and $h(x) = \lambda_a$. Then $v_{\lambda_a}^-(x) = \alpha(x) < \beta(x) = v_{\lambda_a}^+(x)$ and (K2) implies $\lambda_a \in \partial\varphi^{**}(x, \beta(x)) = \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$.
We have so proved that $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$ for every $x \in \Omega$. Thus, define $v_{\lambda_a} := v_{\lambda_a}^+$.

*Case 3.* Define $O := \{x \in \Omega_A \,:\, \lambda_a = h(x)\}$ and $\kappa := a - \int_{\Omega\setminus O} v_{\lambda_a}^-(x)\, dx$. The assumption $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$ implies

$$\int_O \alpha(x)\, dx = \int_O v_{\lambda_a}^-(x)\, dx < \kappa < \int_\Omega v_{\lambda_a}^+(x)\, dx - \int_{\Omega\setminus O} v_{\lambda_a}^-(x)\, dx = \int_O \beta(x)\, dx.$$

From Lemma 4.2 there exists $\Theta : O \to \mathbb{R}$, $\Theta(x) \in \{\alpha(x), \beta(x)\}$ such that $\int_O \Theta(x)\, dx = \kappa$. Define $v_{\lambda_a} : \Omega \to \mathbb{R}$, $v_{\lambda_a}(x) = v_{\lambda_a}^-(x)$ if $x \notin O$ and $v_{\lambda_a}(x) = \Theta(x)$ else.
It is easy to prove that $v_{\lambda_a}$ satisfies (i), (ii), (iii) and $\inf v_{\lambda_a} > 0$.
Since $\varphi \ge \varphi^{**}$, then for every $v \in L^1(\Omega)$, such that $v > 0$ a.e. and $\int_\Omega v\, dx = a$, we have that

$$\begin{aligned}(4.10)\qquad &\int_\Omega \varphi(x, v(x))\, dx \ge \int_\Omega \varphi^{**}(x, v(x))\, dx \ge \\ &\ge \int_\Omega \varphi^{**}(x, v_{\lambda_a}(x))\, dx + \lambda_a \int_\Omega (v(x) - v_{\lambda_a}(x))\, dx = \int_\Omega \varphi(x, v_{\lambda_a}(x))\, dx.\end{aligned}$$

Thus, $v_{\lambda_a}$ is a solution to (4.1).

*Step 3. The case* $\varphi(x,t)=\tilde{\varphi}(|x|,t)$*.* Assume that $\Omega$ is the unit ball $B_1(0)$ and that $\varphi$ has the radial structure $\varphi(x,t)=\tilde{\varphi}(|x|,t)$. It is easy to prove that $\varphi^{**}(x,t)=(\tilde{\varphi})^{**}(|x|,t)$ and that $\alpha$, $\beta$, $h$ are radial functions. Moreover, the sets $\Omega_A$, $\{y\in\Omega_A : h(y)<\lambda\}$ and $\{y\in\Omega_A : h(y)=\lambda\}$ are symmetric sets with respect to the origin. If $\psi$ is defined as in step 1 above, then it immediately follows that $\psi(x,t)=\tilde{\psi}(|x|,t)$. Looking at the first step of the proof of Proposition 3.1, it turns out that $u_\lambda$, satisfying $\partial\psi(x,u_\lambda(x))=\lambda$, is a radial function for all $\lambda$. All these facts allow us to conclude that whenever the cases 1 or 2 in step 2 hold, i.e. $\Phi(\lambda_a)=a$ or $\Phi(\lambda_a)<a=\lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$, respectively, then $v_{\lambda_a}$ is a radial function. To prove that $v_{\lambda_a}$ is radial in the third case it is sufficient to notice that the sets $O$, $O\cap B_r(0)$ and $O\setminus B_r(0)$ are symmetric with respect to the origin and consequently the function $\Theta$ is radial. □

**5. Nonpolyconvex problems: regularity result for the auxiliary problem.** In this section we prove a regularity result for solutions to the nonconvex variational problem (4.1). Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $\varphi:\Omega\times(0,+\infty)\to[0,+\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi\in C^{0,\delta}(\Omega\times K)$, $0<\delta\le 1$, for every compact $K$ in $(0,+\infty)$, such that

(A1) there exist $\alpha,\beta\in C^{0,\delta}(\Omega)$, $\beta(x)>\alpha(x)$ for every $x$, $\inf\alpha>0$, such that $\varphi(x,\cdot)$ and $\varphi^{**}(x,\cdot)$ both coincide and are strictly convex in $(0,\alpha(x)]$ and $[\beta(x),+\infty)$, for every $x\in\Omega$,

(A2) $t\to\varphi^{**}(x,t)$ is affine in $[\alpha(x),\beta(x)]$ for every $x\in\Omega$, i.e. for every $\alpha(x)\le t\le\beta(x)$

$$\varphi^{**}(x,t)=h(x)t+q(x),\qquad \text{with } h(x)=\frac{\varphi(x,\beta(x))-\varphi(x,\alpha(x))}{\beta(x)-\alpha(x)}.$$

Moreover,

$$|\partial\{x : h(x)=\lambda\}|=0\,,\qquad \forall\,\lambda\in\mathbb{R},$$

(A3) there exists $\lambda_0\in\mathbb{R}\cup\{-\infty\}$ such that

$$\lim_{t\to0^+}D_t\varphi(x,t)=\lambda_0\,,\qquad \lim_{t\to+\infty}D_t\varphi(x,t)=+\infty,\qquad \text{uniformly in } x.$$

(A4) for every $m>0$ there exists $c_m>0$ such that

$$\varphi(x,t)\ge\varphi(x,s)+D_t\varphi(x,s)(t-s)+c_m|t-s|^{2+\varepsilon}$$

for every $s,t\ge m$, such that $s<t\le\alpha(x)$ or $\beta(x)\le s<t$, for every $x\in\Omega$ and some $\varepsilon\ge0$.

The following result is in the same spirit of Lemma 4.2.

Lemma 5.1. *Let $O$ be an open set in $\mathbb{R}^N$. Let $\alpha,\beta\in L^1(O)$ be such that $\alpha(x)\le\beta(x)$ for a.e. $x$ and suppose that*

$$\int_O\alpha(x)\,dx<\kappa<\int_O\beta(x)\,dx. \tag{5.1}$$

*Then there exists a finite number of balls $B_{\rho_j}(y_j)$, $j=1,...,m$, satisfying*

*(1)* $\overline{B_{\rho_j}(y_j)}\subset\subset O$, $j=1,...,m$,

*(2)* $\overline{B_{\rho_i}(y_i)}\cap\overline{B_{\rho_j}(y_j)}=\emptyset$ *for every* $i\neq j$,

*(3)* $\int_O\Theta(x)\,dx=\kappa$,

*where* $\Theta(x) := \alpha(x)$ *if* $x \in \cup_{1\le j\le m} B_{\rho_j}(y_j)$ *and* $\Theta(x) := \beta(x)$ *else.*

*Proof.* Since $O$ is open, there exist (at most) countably many pairwise disjoint balls $\{B_{R_j}(y_j)\}_{j\in J}$ in $O$, and a negligible set $\mathcal{N}$ such that $O = \mathcal{N} \cup \left(\bigcup_{j\in J} B_{R_j}(y_j)\right)$. Without loss of generality we assume $J = \{1, 2, ..., m\}$ if $\operatorname{card} J = m \in \mathbb{N}$ and $J = \mathbb{N}$ if $J$ is countable. For every $n \in J$, let us define the function $\theta_n : O \to \mathbb{R}$,

$$\theta_n(x) := \begin{cases} \alpha(x), & \text{if } x \in \displaystyle\bigcup_{1\le j\le n} B_{R_j}(y_j), \\ \beta(x), & \text{else.} \end{cases}$$

If $J$ is finite then (5.1) implies $\displaystyle\int_O \theta_m(x)\,dx < \kappa$. If $J = \mathbb{N}$, it is easy to check that $\displaystyle\lim_{n\to+\infty} \int_O \theta_n(x)\,dx < \kappa$, thus, there exists $m \in \mathbb{N}$ such that

$$\int_O \theta_m(x)\,dx = \int_{\cup_{1\le j\le m} B_{R_j}(y_j)} \alpha(x)\,dx + \int_{O\setminus \cup_{1\le j\le m} B_{R_j}(y_j)} \beta(x)\,dx < \kappa\,.$$

Aiming at (1) and (2), we slightly reduce the radius of the previously selected balls $\{B_{R_j}(y_j)\}_{1\le j\le m}$. This can easily be done by noticing that

$$\lim_{\varepsilon\to 0^+} \int_{\cup_{j=1}^m B_{R_j}(y_j)\setminus B_{R_j-\varepsilon}(y_j)} (\beta(x) - \alpha(x))\,dx = 0.$$

Thus there exists $0 < \varepsilon < \min\{R_j : 1 \le j \le m\}$ such that

$$\int_{\cup_{1\le j\le m} B_{R_j-\varepsilon}(y_j)} \alpha(x)\,dx + \int_{O\setminus \cup_{1\le j\le m} B_{R_j-\varepsilon}(y_j)} \beta(x)\,dx < \kappa\,. \tag{5.2}$$

Set $R := \max\{R_j - \varepsilon : 1 \le j \le m\}$ and define $\theta : O \times [0, R] \to \mathbb{R}$, $\theta(x, 0) := \beta(x)$ and

$$\theta(x, \rho) := \begin{cases} \alpha(x) & \text{if } x \in \displaystyle\bigcup_{1\le j\le m} \left(B_{R_j-\varepsilon}(y_j) \cap B_\rho(y_j)\right), \\ \beta(x) & \text{else,} \end{cases}$$

for every $\rho > 0$. From (5.2) we have that

$$\int_O \theta(x, R)\,dx < \kappa < \int_O \theta(x, 0)\,dx = \int_O \beta(x)\,dx.$$

Since $\rho \to \int_O \theta(x, \rho)\,dx$ is a continuous function, there exists $\overline{\rho}$ such that $\displaystyle\int_O \theta(x, \overline{\rho})\,dx = \kappa$. The claim of the theorem follows by defining $\Theta(x) := \theta(x, \overline{\rho})$ and $\rho_j := \min\{R_j - \varepsilon, \overline{\rho}\}$, $1 \le j \le m$. □

Let $h$ be as in (A2). For every $\lambda > \lambda_0$ we define

$$\Omega_\lambda^+ := \{x : h(x) > \lambda\}, \quad \Omega_\lambda^- := \{x : h(x) < \lambda\}, \quad \Omega_\lambda^= := \{x : h(x) = \lambda\}. \tag{5.3}$$

Under (A1)–(A4) there exists a piecewise Hölder continuous solution to (4.1).

THEOREM 5.2. *Let* $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ *be a continuous function, differentiable with respect to the last variable,* $D_t\varphi(x, t)$ *in* $C^{0,\delta}(\Omega \times K)$ *for every compact* $K \subset (0, +\infty)$*. Suppose that (A1)–(A4) hold. Then, fixed* $a > 0$ *there exist* $\lambda_a > \lambda_0$ *and* $v_{\lambda_a} \in L^\infty(\Omega)$*,* $\inf v_{\lambda_a} > 0$*, satisfying the following properties:*

*(i)* $D_t\varphi^{**}(x, v_{\lambda_a}(x)) = \lambda_a$ *for every* $x \in \Omega$,
*(ii)* $\int_\Omega v_{\lambda_a}(x)\, dx = a$
*(iii)* $v_{\lambda_a}$ *is Hölder continuous in* $\Omega^+_{\lambda_a} \cup \Omega^-_{\lambda_a}$,
*(iv)* $v_{\lambda_a}(x) < \alpha(x)$ *for all* $x \in \Omega^+_{\lambda_a}$, *and* $v_{\lambda_a}(x) > \beta(x)$ *for all* $x \in \Omega^-_{\lambda_a}$,
*(v)* *in* $\Omega^=_{\lambda_a}$ *either* $v_{\lambda_a} \equiv \alpha$ *or* $v_{\lambda_a} \equiv \beta$ *or*

$$v_{\lambda_a}(x) = \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ \beta(x) & \text{if } x \in \Omega^=_{\lambda_a} \setminus \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \end{cases} \tag{5.4}$$

*with* $B_{\rho_j}(y_j) \subset\subset \mathrm{int}\,\Omega^=_{\lambda_a}$, $j = 1, ..., m$, *such that* $\overline{B_{\rho_i}(y_i)} \cap \overline{B_{\rho_j}(y_j)} = \emptyset$, *if* $i \neq j$.

*Moreover,* $v_{\lambda_a}$ *is a solution to (4.1).*

*Proof.* Let $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ be defined as

$$\psi(x,t) := \begin{cases} \varphi(x,t) & \text{if } \ 0 < t \le \alpha(x)\,,\ x \in \Omega, \\ \varphi(x, t + \beta(x) - \alpha(x)) - & \\ -\varphi(x, \beta(x)) + \varphi(x, \alpha(x)) & \text{if } \ t > \alpha(x)\,,\ x \in \Omega. \end{cases} \tag{5.5}$$

It holds true that $\psi$ is a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4) in section 3, with possibly different constants. By Proposition 3.5 for every $\lambda > \lambda_0$, there exists $u_\lambda$ such that $u_\lambda \in C^{0,\gamma}(\Omega)$ for some $0 < \gamma \le 1$, $\inf u_\lambda > 0$ and

$$D_t\psi(x, u_\lambda(x)) = \lambda, \qquad \forall\, x \in \Omega. \tag{5.6}$$

Moreover, see (4.6) and (4.8)

$$u_\lambda < \alpha \ \text{ in } \Omega^+_\lambda, \qquad u_\lambda = \alpha \ \text{ in } \Omega^=_\lambda, \qquad u_\lambda > \alpha \ \text{ in } \Omega^-_\lambda. \tag{5.7}$$

Let $\Phi : (\lambda_0, +\infty) \to \mathbb{R}$ be the left-continuous function defined as

$$\Phi(\lambda) := \int_\Omega \Big( u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\Omega^-_\lambda}(x) \Big)\, dx\,, \qquad \lambda > \lambda_0.$$

We have three different cases:

*1.* there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) = a$,
*2.* there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a = \lim_{\lambda\to\lambda_a^+} \Phi(\lambda)$,
*3.* there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+} \Phi(\lambda)$.

Let us consider the first two cases: since (A1)–(A3) imply (K1)–(K3) then by proceeding as in Theorem 4.1 there exists $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, which satisfies (i) and (ii). Moreover if case 1 holds then $v_{\lambda_a} := u_{\lambda_a} + (\beta - \alpha)\chi_{\{h<\lambda_a\}}$, i.e.

$$v_{\lambda_a} := u_{\lambda_a} \ \text{ in } \Omega^+_{\lambda_a}, \qquad v_{\lambda_a} := \alpha \ \text{ in } \Omega^=_{\lambda_a}, \qquad v_{\lambda_a} := u_{\lambda_a} + \beta - \alpha \ \text{ in } \Omega^-_{\lambda_a},$$

if instead case 2 holds then $v_{\lambda_a} := u_{\lambda_a} + (\beta - \alpha)\chi_{\{h\le\lambda\}}$, i.e.

$$v_{\lambda_a} := u_{\lambda_a} \ \text{ in } \Omega^+_{\lambda_a}, \qquad v_{\lambda_a} := \beta \ \text{ in } \Omega^=_{\lambda_a}, \qquad v_{\lambda_a} := u_{\lambda_a} + \beta - \alpha \ \text{ in } \Omega^-_{\lambda_a},$$

Therefore, from the Hölder continuity of $\alpha$ and $\beta$, (5.6) and (5.7) it follows that $v_{\lambda_a}$ satisfies (iii), (iv) and (v). Moreover, reasoning as in (4.10) we get that $v_{\lambda_a}$ is a solution to (4.1).

Suppose the third case holds. We define $v_{\lambda_a}$ as in the proof of Theorem 4.1, but using Lemma 5.1 instead of Lemma 4.2. Precisely, since

$$\int_{\Omega^=_{\lambda_a}} \alpha(x)\,dx < \kappa < \int_{\Omega^=_{\lambda_a}} \beta(x)\,dx,$$

with

$$\kappa := a - \int_{\Omega\setminus\Omega^=_{\lambda_a}} \Big(u_{\lambda_a}(x) + (\beta(x)-\alpha(x))\chi_{\Omega^-_{\lambda_a}}(x)\Big)\,dx,$$

then from Lemma 5.1 there exist $m$ balls $B_{\rho_j}(y_j) \subset\subset \operatorname{int}\Omega^=_{\lambda_a}$, $j=1,...,m$, $\overline{B_{\rho_i}(y_i)} \cap \overline{B_{\rho_j}(y_j)} = \emptyset$ for every $i \neq j$, such that $\Theta : \operatorname{int}\Omega^=_{\lambda_a} \to \mathbb{R}$,

$$\Theta := \alpha \quad \text{in } \textstyle\bigcup_{1\le j\le m} B_{\rho_j}(y_j), \qquad \Theta := \beta \quad \text{in } \operatorname{int}\Omega^=_{\lambda_a} \setminus \textstyle\bigcup_{1\le j\le m} B_{\rho_j}(y_j),$$

satisfies $\int_{\operatorname{int}\Omega^=_{\lambda_a}} \Theta(x)\,dx = \kappa$.

Define $v_{\lambda_a}$ as follows:

$$v_{\lambda_a}(x) : \begin{cases} u_{\lambda_a}(x) & \text{if } x \in \Omega^+_{\lambda_a}, \\ \alpha(x) & \text{if } x \in \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ \beta(x) & \text{if } x \in \Omega^=_{\lambda_a} \setminus \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ u_{\lambda_a}(x) + \beta(x) - \alpha(x) & \text{if } x \in \Omega^-_{\lambda_a}. \end{cases}$$

We have that $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, and it satisfies (i)–(v). Moreover $v_{\lambda_a}$ is a solution to (4.1). □

**6. Nonpolyconvex problems: attainment result in a general setting.** In this section we consider the variational problem

$$\min\left\{\int_\Omega \varphi(x,\det Du(x))\,dx \;:\; u \in W^{1,N}(\Omega,\mathbb{R}^N),\ \det Du > 0 \text{ a.e.},\ u(x) = x \text{ on } \partial\Omega\right\}, \tag{6.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $\varphi : \Omega \times (0,+\infty) \to [0,+\infty)$ is a nonconvex function with respect to the second variable. Before stating an attainment result for (6.1), we need some preliminary results.

LEMMA 6.1. *Let $\Omega$ be a bounded open set with Lipschitz boundary and let $\overline{\Omega} = \cup_{i=1}^m \overline{\Omega}_i$, with $\{\Omega_i\}$ pairwise disjoint open connected sets with Lipschitz boundary. Consider $\alpha_i > 0$, $i = 1,...,m$, with $\sum_{i=1}^m \alpha_i = |\Omega|$. Then there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that $\det Du_0 \in C^\infty(\overline{\Omega})$, $\inf \det Du_0 > 0$ and*

$$u_0(x) = x \quad \text{on } \partial\Omega, \qquad |u_0(\Omega_i)| = \alpha_i, \quad i = 1,...,m. \tag{6.2}$$

*Moreover, $u_0(\Omega_i)$ is an open set of class $(L)$ for every $i$.*

*Proof.* Fix $0 < \delta < \min\{\alpha_i/|\Omega_i| : i = 1,..,m\}$. For every $1 \le i \le m$ let $\eta_i \in C_c^\infty(\Omega_i)$ be such that $\int_{\Omega_i} \eta_i(x)\,dx = 1$. Define

$$f(x) = \delta + \sum_{i=1}^m (\alpha_i - \delta|\Omega_i|)\eta_i(x), \qquad x \in \overline{\Omega}.$$

Hence, $f \in C^\infty(\overline{\Omega})$, $\inf f > 0$, $\int_{\Omega_i} f(x)\,dx = \alpha_i$ for every $i$, and $\int_\Omega f(x)\,dx = |\Omega|$. From Theorem 2.4 there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that

$$\det Du_0 = f \text{ in } \Omega, \qquad u_0(x) = x \text{ on } \partial\Omega.$$

Therefore

$$|u_0(\Omega_i)| = \int_{\Omega_i} \det Du_0(x)\, dx = \int_{\Omega_i} f(x)\, dx = \alpha_i, \qquad i = 1, ..., m;$$

moreover Lemma 2.3 implies that $u_0(\Omega_i)$ is an open set of class $(L)$ for each $i$. □

PROPOSITION 6.2. *Let $\Omega$ and $\Omega_i$, $i = 1, ..., m$, be as in Lemma 6.1. Suppose that $g_i : \overline{\Omega_i} \to [c_0, +\infty)$, with $c_0 > 0$, $i = 1, ..., m$, are Hölder continuous functions satisfying*

$$\sum_{i=1}^{m} \int_{\Omega_i} g_i(x)\, dx = |\Omega|.$$

*Then there exists a Lipschitz continuous function $u : \overline{\Omega} \to \overline{\Omega}$, such that*

$$u(x) = x \quad \text{on } \partial\Omega, \qquad \det Du(x) = g_i(x) \quad \forall x \in \Omega_i,\ \forall i = 1, ..., m. \tag{6.3}$$

*Proof.* By Lemma 6.1 there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that

$$u_0(x) = x \ \text{ on } \partial\Omega, \qquad |u_0(\Omega_i)| = \int_{\Omega_i} g_i(x)\, dx,$$

and $u_0(\Omega_i)$ is of class $(L)$, for each $i = 1, ..., m$. Moreover $f := \det Du_0$ is of class $C^\infty(\overline{\Omega})$ and $\inf f > 0$. Since $\frac{g_i}{f} \circ u_0^{-1}$ is Hölder continuous in $\overline{u_0(\Omega_i)}$ and it satisfies

$$\int_{u_0(\Omega_i)} \frac{g_i}{f} \circ u_0^{-1}(y)\, dy = \int_{\Omega_i} g_i(x)\, dx = |u_0(\Omega_i)|,$$

then from Theorem 2.4 there exists a bi-Lipschitz homeomorphism $z_i : \overline{u_0(\Omega_i)} \to \overline{u_0(\Omega_i)}$ such that

$$\begin{cases} \det Dz_i = \dfrac{g_i}{f} \circ u_0^{-1} & \text{in } u_0(\Omega_i), \\ z_i(y) = y & \text{on } \partial u_0(\Omega_i), \end{cases}$$

Thus, $u_i = z_i \circ u_0$ is a Lipschitz homeomorphism such that

$$\begin{cases} \det Du_i = g_i & \text{in } \Omega_i, \\ u_i = u_0 & \text{on } \partial\Omega_i. \end{cases}$$

Hence, the Lipschitz continuous function $u : \overline{\Omega} \to \overline{\Omega}$ such that $u(x) = u_i(x)$ for every $x \in \overline{\Omega}_i$, $i = 1, ..., m$, satisfies (6.3). □

We are in position to state an existence result for the nonpolyconvex problem (6.1). The sets $\Omega_\lambda^+$, $\Omega_\lambda^-$ and $\Omega_\lambda^=$ are defined in (5.3).

THEOREM 6.3. *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi \in C^{0,\delta}(\Omega \times K)$, $0 < \delta \leq 1$, for every compact $K \subset (0, +\infty)$. Suppose that (A1)–(A4) hold and assume that, for every $\lambda > \lambda_0$, $\Omega_\lambda^+$, $\Omega_\lambda^-$ and $\operatorname{int} \Omega_\lambda^=$ are either empty or connected open sets with Lipschitz boundary. Then the variational problem (6.1) has a Lipschitz continuous solution.*

*Proof.* From Theorem 5.2, applied with $a = |\Omega|$, there exist $\lambda_a > \lambda_0$ and a solution $v_{\lambda_a}$ to (4.1), with $\inf v_{\lambda_a} > 0$. Throughout we write $v$ instead of $v_{\lambda_a}$.
From Theorem 5.2 $v$ is Hölder continuous in $\Omega^+_{\lambda_a} \cup \Omega^-_{\lambda_a}$. If $\operatorname{int}\Omega^=_{\lambda_a}$ is empty we get the thesis applying Proposition 6.2 with $\Omega_1 = \Omega^+_{\lambda_a}$, $\Omega_2 = \Omega^-_{\lambda_a}$ and replacing $g_1$ and $g_2$ with the continuous extension of $v$ to $\Omega^+_{\lambda_a}$ and to $\Omega^-_{\lambda_a}$, respectively.
If $\operatorname{int}\Omega^=_{\lambda_a}$ is not empty, correspondingly to (v) of Theorem 5.2 we have to consider three cases.
If $v = \alpha$ in $\Omega^=_{\lambda_a}$ the thesis follows by applying Proposition 6.2 with $m = 3$, choosing $\Omega_1 = \Omega^+_{\lambda_a}$, $\Omega_2 = \Omega^-_{\lambda_a}$, $\Omega_3 = \operatorname{int}\Omega^=_{\lambda_a}$ and replacing, as above, $g_1$ and $g_2$ with the continuous extension of $v$ to $\Omega^+_{\lambda_a}$ and $\Omega^-_{\lambda_a}$, respectively, and $g_3$ with $\alpha$. Analogously, we proceed if $v = \beta$ in $\Omega^=_{\lambda_a}$, but defining $g_3 = \beta$.
Now suppose that (5.4) holds. In this case the thesis follows by Proposition 6.2 choosing $\Omega_1 = \Omega^+_{\lambda_a}$, $\Omega_2 = \Omega^-_{\lambda_a}$, $\Omega_3 = \operatorname{int}\Omega^=_{\lambda_a} \setminus \cup_{1\le j\le n} B_{\rho_j}(y_j)$, $\Omega_{3+i} = B_{\rho_i}(y_i)$ for every $i = 1, ..., n$ and $g_1 = v$, $g_2 = v$, $g_3 = \beta$, $g_{3+i} = \alpha$, for every $i = 1, ..., n$. □

With obvious changes in the proof above, it follows:

Theorem 6.4. *Let $\Omega$ and $\varphi$ be as in Theorem 6.3. Suppose that (A1)–(A4) hold and assume that for every $\lambda > \lambda_0$*

$$\overline{\Omega^+_\lambda} = \bigcup_{i=1}^{h} \overline{A_i}\,, \qquad \overline{\Omega^-_\lambda} = \bigcup_{i=h+1}^{k} \overline{A_i}\,, \qquad \operatorname{int}\Omega^=_\lambda = \bigcup_{i=k+1}^{l} A_i\,, \tag{6.4}$$

*with $A_i$ either empty or pairwise disjoint open connected sets with Lipschitz boundary. Then the variational problem (6.1) has a Lipschitz continuous solution.*

Remark 6.5. *Examples of sets $\Omega$ and functions $h : \Omega \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$ (6.4) holds with either empty or disjoint open sets $\{A_i\}$ with Lipschitz boundary are the following:*

*(a) $\Omega$ is a bounded and convex set and $h$ is strictly convex in $\Omega$ and constant on $\partial\Omega$,*

*(b) $\Omega = B_1(0)$ and $h$ is a radial function, $h(x) = \tilde{h}(|x|)$, with $\tilde{h}$ piecewise monotone, i.e. there exists $0 = s_0 < s_1 < ... < s_m = 1$ such that $\tilde{h}|_{[s_i,s_{i+1}]}$ is monotone for all $i$.*

**7. Nonpolyconvex problems: some special cases.** In this section we consider particular classes of the variational problem (6.1), where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function satisfying (A1) and (A2). We begin considering the case of functions $\varphi$ such that $h$ in (A2) is a constant. See [20] and [3] for related results.

Theorem 7.1. *Let $\varphi : \Omega\times(0, +\infty) \to [0, +\infty)$ be a continuous function satisfying (A1) and (A2), with $h$ constant. If $\int_\Omega \alpha(x)\, dx \le |\Omega| \le \int_\Omega \beta(x)\, dx$, then (6.1) has a Lipschitz continuous solution.*

*Proof.* Consider the auxiliary problem (4.1) with $a = |\Omega|$. If $\int_\Omega \alpha(x)\, dx$ is equal to $|\Omega|$ then $\alpha$ solves (4.1). Then from Theorem 2.4 there exists a Lipschitz homeomorphism $u$ solution to (2.1) with $f = \alpha$. Moreover, $u$ is a solution of (6.1). The same argument works if $\int_\Omega \beta(x)\, dx$ is equal to $|\Omega|$. Of course in this case choose $f = \beta$.

Suppose $\int_\Omega \alpha(x)\, dx < |\Omega| < \int_\Omega \beta(x)\, dx$. Then using Lemma 5.1 with $O = \Omega$, we get that a Lipschitz continuous solution $u$ to (4.1) exists, with $u \equiv \alpha$ on pairwise disjoint balls $B_{\rho_j}(y_j) \subset\subset \Omega$, $j = 1, ..., n$, and with $u \equiv \beta$ outside these balls. The thesis follows by Proposition 6.2 with $m = n + 1$, $\Omega_j = B_{\rho_j}(y_j)$ and $g_j = \alpha$ if $j = 1, ..., m - 1$, and with $\Omega_m = \Omega \setminus \bigcup_{j=1}^n B_{\rho_j}(y_j)$, $g_m = \beta$. □

THEOREM 7.2. *Let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi \in C^{0,\delta}(\Omega \times K)$, $0 < \delta \le 1$, for every compact $K \subset (0, +\infty)$. Suppose that (A1), (A2) with $h$ constant, (A3) and (A4) hold. If $\int_\Omega \alpha(x)\, dx > |\Omega|$ or $\int_\Omega \beta(x)\, dx < |\Omega|$ then (6.1) has a Lipschitz continuous solution.*

*Proof.* Let $a = |\Omega|$. From Theorem 5.2 there exist $\lambda_a > \lambda_0$ and $v_{\lambda_a} \in L^\infty(\Omega)$ satisfying

$$v_{\lambda_a}(x) \notin (\alpha(x), \beta(x)), \quad D_t\varphi^{**}(x, v_{\lambda_a}(x)) = \lambda_a, \quad \int_\Omega v_{\lambda_a}(x)\, dx = |\Omega|. \tag{7.1}$$

(A1), (A2) and (A3) imply $h = D_t\varphi(x, \alpha(x)) = D_t\varphi(x, \beta(x))$ and the definition of $\{v_\lambda\}$ (see the proofs of Theorems 4.1 and 5.2) gives that $\lambda < h$ if and only if $v_\lambda(x) < \alpha(x)$ for all $x$, $\lambda > h$ if and only if $v_\lambda(x) > \beta(x)$ for all $x$. Therefore, if $\int_\Omega \alpha(x)\, dx > |\Omega|$, then $\lambda_a < h$ and $v_{\lambda_a}(x) < \alpha(x)$. Thus, using the notations in (5.3), $\Omega^+_{\lambda_a} = \Omega$. Analogously, if $\int_\Omega \beta(x)\, dx < |\Omega|$ then $\lambda_a > h$ and $v_{\lambda_a}(x) > \beta(x)$, so that $\Omega^-_{\lambda_a} = \Omega$. Therefore Theorem 5.2 implies that $v_{\lambda_a}$ is Hölder continuous in $\Omega$. A Lipschitz continuous solution $u$ to

$$\begin{cases} \det Du = v_{\lambda_a} & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases}$$

solution also to (6.1), exists because of Theorem 2.4. $\square$

In Propositions 7.3 and 7.4 we deal with a variant of functionals considered above, precisely

$$\min\left\{\int_\Omega \Phi(x, \det Du(x))\, dx : \ u \in W^{1,N}(\Omega, \mathbb{R}^N),\ \det Du > 0\, a.e.,\ u(x) = x \text{ on } \partial\Omega\right\}. \tag{7.2}$$

with $\Phi(x,t) = \varphi(x,t) + f(x)t$.

PROPOSITION 7.3. *Let $\Omega$ be a bounded open convex set in $\mathbb{R}^N$ and let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ satisfy the assumptions of Theorem 7.2, with $\lambda_0 = -\infty$ in (A3). Suppose that $f : \mathbb{R}^N \to (0, +\infty)$ is a strictly convex function, constant on $\partial\Omega$. Then there exists a Lipschitz solution to (7.2).*

*Proof.* It is easy to see that $\Phi$ satisfies the assumptions of Theorem 6.3. Since $\Phi^{**}(x,t) = \varphi^{**}(x,t) + f(x)t$ for every $x \in \Omega$, then in $(0, \alpha(x)]$ and in $[\beta(x), +\infty)$ we have that $\Phi(x, \cdot) = \Phi^{**}(x, \cdot)$. Moreover, for every $t \in [\alpha(x), \beta(x)]$ it holds true that $\Phi^{**}(x,t) = H(x)t + q(x)$, with $H(x) := \mu + f(x)$ and the super-level, sub-level and level sets of $H$ satisfy the assumptions in Theorem 6.3. (A3) implies that $D_t\Phi(x,t) = D_t\varphi(x,t) + f(x)$ goes to $-\infty$ as $t \to -\infty$ and goes to $+\infty$ as $t \to +\infty$, uniformly with respect to $x$. The thesis easily follows from Theorem 6.3. $\square$

From now on, $\Omega$ is the unit ball $B$ in $\mathbb{R}^N$ centered at the origin.

PROPOSITION 7.4. *Let $\varphi : B \times (0, +\infty) \to [0, +\infty)$ satisfy the assumptions of Theorem 7.2, with $\lambda_0 = -\infty$ in (A3). Let $f \in C^{0,\gamma}([0,1])$, $0 < \gamma \le 1$, $f(s) > 0$ for every $s$, $f$ piecewise monotone. Then there exists a Lipschitz continuous solution to (7.2), with $\Phi(x,t) = \varphi(x,t) + f(|x|)t$.*

*Proof.* Proceeding as in the proof of Proposition 7.3, the thesis easily follows from Remark 6.5 (b) and from Theorem 6.4 applied to $\Phi(x,t) = \varphi(x,t) + f(|x|)t$. $\square$

Now, we deal with one more class of nonpolyconvex functionals, characterized by an integrand $\varphi$ with radial structure $\varphi(x,t) = \tilde{\varphi}(|x|, t)$. Precisely, we deal with the variational problem

$$\min\left\{\int_B \tilde\varphi(|x|,\det Du(x))\,dx\ :\ u\in W^{1,N}(B,\mathbb{R}^N),\ \det Du>0 \text{ a.e., } u(x)=x \text{ on } \partial B\right\}, \tag{7.3}$$

and $\tilde\varphi:[0,1)\times(0,+\infty)\to[0,+\infty)$ is a continuous function.

Theorem 7.5. *Let $\tilde\varphi:[0,1)\times(0,+\infty)\to[0,+\infty)$ be a continuous function satisfying the following assumptions:*

*(i) there exist $a,b\in L^\infty(0,1)$, $b(s)>a(s)>0$ for every $s$, $\inf a>0$, such that $\tilde\varphi(s,\cdot)$ and $\tilde\varphi^{**}(s,\cdot)$ both concide and are strictly convex in $(0,a(s)]$ and $[b(s),+\infty)$, for all $s\in[0,1)$,*

*(ii) $\tilde\varphi^{**}(x,\cdot)$ is affine in $[a(s),b(s)]$ for all $s\in[0,1)$,*

*(iii) there exists $\lambda_0\in\mathbb{R}\cup\{-\infty\}$ such that*

$$\lim_{t\to0^+}D_t^+\tilde\varphi(s,t)=\lambda_0,\qquad \lim_{t\to+\infty}D_t^-\tilde\varphi(s,t)=+\infty,\qquad \textit{uniformly in } s.$$

*Then there exists a Lipschitz solution to (7.3).*

*Proof.* Let us define $\varphi(x,t):=\tilde\varphi(|x|,t)$ for every $x\in B$. Notice that $\varphi^{**}(x,t)=\tilde\varphi^{**}(|x|,t)$ and that assumptions (K1), (K2) and (K3) of Theorem 4.1 holds, with $\Omega=\Omega_A=B$, $\alpha(x)=a(|x|)$ and $\beta(x)=b(|x|)$. Let $v\in L^\infty(B)$, $\inf v>0$, be the radial solution of (4.1). It is a known fact (see e.g. [15]) that there exists a bi-Lipschitz solution $u$ to (2.1) with $f=v$ and $\Omega=B$. Thus, $u$ is a solution to (7.3), too. □

**Appendix A. Proof of Theorem 2.4.** In the following we use the arguments of the proof of Lemma 1 in [16] and the fact, proved in [18], that if $\Omega=(0,1)^N$ and $f$ is Hölder continuous then there exists a bi-Lipschitz homeomorphism solution to (2.1). We divide the proof into steps.

*Step 1.*
Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^N$ of class $(L)$. Thus, there exist $m$ open sets $\Omega_j$ such that $\overline\Omega\subset\cup_j\Omega_j$ and $m$ bi-Lipschitz homeomorphisms $\psi_j:\overline{\Sigma}_j\to\overline{Q}$, with $\Sigma_j=\Omega\cap\Omega_j$ and $Q=(0,1)^N$, such that $\det D\psi_j\in \mathrm{Lip}(\overline\Sigma_j)$ and $\frac{1}{A}<\det D\psi_j<A$ for some $A\ge1$. Consider a partition of unity $\{\phi_j\}_{j=1}^m$ subordinate to such a covering of $\overline\Omega$: $\{\phi_j\}_{j=1}^m$ is a family of smooth and nonnegative functions, $\sum_j\phi_j(x)=1$ for every $x\in\overline\Omega$ and

$$\operatorname{supp}\phi_j\subset\subset\Omega_j,\qquad \forall\, j=1,...,m. \tag{A.1}$$

Since $\Omega=\cup_{j=1}^m\Sigma_j$ and $\Omega$ is connected, we can assume that for every $k=2,...,m$ there exists $\rho(k)<k$ such that $\Sigma_k\cap\Sigma_{\rho(k)}$ is not empty. Define the matrix $(\alpha_{hk})$, $1\le h\le m$, $2\le k\le m$,

$$\alpha_{hk}=\begin{cases}1 & \text{if } h=k,\\ -1 & \text{if } h=\rho(k),\\ 0 & \text{else.}\end{cases}$$

Each of the $m-1$ columns contains exactly one pair $+1$, $-1$ so that $\sum_{k=2}^m\alpha_{hk}=0$ for every $h$.
Define $\eta_k\in C_c^\infty(\Sigma_k\cap\Sigma_{\rho(k)})$, such that $\int_\Omega\eta_k(x)\,dx=1$. Let $g\in C^{0,\alpha}(\overline\Omega)$ be such that $\int_\Omega g(x)\,dx=0$. Define the Hölder continuous functions $g_h:\overline\Omega\to\mathbb{R}$, $1\le h\le m$,

$$g_h:=g\phi_h|_{\overline\Omega}-\sum_{k=2}^m\lambda_k\alpha_{hk}\eta_k,$$

where $\lambda_2, ..., \lambda_m$ are real numbers solutions of the following system of $m$ equations

$$\sum_{k=2}^{m} \lambda_k \alpha_{hk} = \int_\Omega g\phi_h \, dx, \qquad h = 1, ..., m. \tag{A.2}$$

Since the rank of $(\alpha_{hk})$ is $m-1$ and both $\sum_{h=1}^m \sum_{k=2}^m \lambda_k \alpha_{hk}$ and $\sum_{h=1}^m \int_\Omega g\phi_h \, dx$ are equal to 0, then system (A.2) is uniquely solvable.
We claim that $\operatorname{supp} g_h \subseteq \overline{\Sigma}_h$. In fact $\operatorname{supp} \ \phi_h|_{\overline{\Omega}} \subseteq \overline{\Sigma}_h$ and, since $\alpha_{hk} \neq 0$ if and only if $h = k$ or $h = \rho(k)$,

$$\operatorname{supp} \lambda_k \alpha_{hk} \eta_k \subset \Sigma_k \cap \Sigma_{\rho(k)} \subseteq \Sigma_h,$$

for every $k = 2, ..., m$. Moreover, from (A.2) there exists $M > 0$ depending on $\Omega$, $\{\phi_j\}_j$ and $\{\eta_j\}_j$ only, such that $\sup |g_h| \leq M \sup |g|$.

*Step 2.*
Let $\Omega$, $\{\Sigma_j\}_j$, $\{\psi_j\}_j$, $\{\phi_j\}_j$, $\{\eta_j\}_j$, $m$ and $M$ be as above. Let $f$ in (2.1) be such that $\sup |f-1| < m^{-1}M^{-1}$. Define $m$ Hölder continuous functions $g_h$ reasoning as in the previous step, with $g$ replaced by $f-1$. For every $j = 1, ..., m+1$ define $f_j : \overline{\Omega} \to (0, +\infty)$,

$$f_j(x) := \begin{cases} 1 & \text{if } j = 1, \\ 1 + \sum_{h=1}^{j-1} g_h(x) & \text{if } j > 1. \end{cases}$$

In particular $f_{m+1} = f$. Notice that each $f_j$ is a Hölder continuous function and, since $\sup |f-1| < m^{-1}M^{-1}$, then $\inf f_j > 0$. Fixed $j = 1, ..., m$ we have that

$$f_{j+1} - f_j = 0 \quad \text{in } \overline{\Omega} \setminus \overline{\Sigma}_j, \qquad \int_\Omega f_j(x) \, dx = |\Omega|, \qquad \int_{\Sigma_j} f_{j+1}(x) \, dx = \int_{\Sigma_j} f_j(x) \, dx. \tag{A.3}$$

Define $f_j^*, f_{j+1}^* : \overline{Q} \to (0, +\infty)$,

$$f_j^* := f_j(\psi_j^{-1}) \det D\psi_j^{-1}, \quad f_{j+1}^* := f_{j+1}(\psi_j^{-1}) \det D\psi_j^{-1},$$

so that $f_j^*, f_{j+1}^* \in C^{0,\alpha}(\overline{Q})$ and $\int_Q f_j^* \, dx = \int_Q f_{j+1}^* \, dx$.
As proved in [18] there exist two bi-Lipschitz homeomorphisms $v_j, w_j : \overline{Q} \to \overline{Q}$ solutions to

$$\begin{cases} \det Dv_j = \frac{f_j^*}{\int_Q f_j^* \, dx} & \text{in } Q, \\ v_j(y) = y & \text{on } \partial Q, \end{cases} \qquad \text{and} \qquad \begin{cases} \det Dw_j = \frac{f_{j+1}^*}{\int_Q f_j^* \, dx} & \text{in } Q, \\ w_j(y) = y & \text{on } \partial Q, \end{cases}$$

respectively. Let us consider $\varphi_j : \overline{Q} \to \overline{Q}$, $\varphi_j(y) := (v_j^{-1} \circ w_j)(y)$. Then

$$\det D\varphi_j(y) = \det Dv_j^{-1}(w_j(y)) \det Dw_j(y) = \frac{f_{j+1}^*(y)}{f_j^*(\varphi_j(y))}, \qquad \forall\, y \in \overline{Q}$$

so that

$$f_j((\psi_j^{-1} \circ \varphi_j)(y)) \det D\psi_j^{-1}(\varphi_j(y)) \det D\varphi_j(y) = f_{j+1}(\psi_j^{-1}(y)) \det D\psi_j^{-1}(y), \qquad \forall\, y \in \overline{Q}.$$

Using the the invertibility of $\psi_j$ the equality above implies that

$$f_j(u_j(x)) \det Du_j(x) = f_{j+1}(x), \qquad \forall\, x \in \overline{\Sigma}_j, \tag{A.4}$$

where $u_j : \overline{\Sigma}_j \to \overline{\Sigma}_j$ is the Lipschitz continuous function defined as $u_j(x) := (\psi_j^{-1} \circ \varphi_j \circ \psi_j)(x)$.
Since $\varphi_j(\psi_j(x)) = \psi_j(x)$ for all $x \in \partial\Sigma_j$, we have that $u_j(x) = x$, for every $x \in \partial\Sigma_j$. Then $\tilde{u}_j : \overline{\Omega} \to \mathbb{R}$, $j = 1, ..., m$,

$$\tilde{u}_j(x) := \begin{cases} u_j(x) & \text{if } x \in \overline{\Sigma}_j, \\ x & \text{else,} \end{cases}$$

is Lipschitz continuous and from (A.3) and (A.4)

$$f_j(\tilde{u}_j(x)) \det D\tilde{u}_j(x) = f_{j+1}(x), \qquad \forall\, x \in \overline{\Omega}.$$

Iterating this argument on $j$ and recalling that $f_1 = 1$ and $f_{m+1} = f$, we get that $\tilde{u}_1 \circ \cdots \circ \tilde{u}_m$ is a Lipschitz solution to (2.1).

*Step 3.*
Now we suppose that $f$ in (2.1) satisfies $\sup |f - 1| \geq m^{-1}M^{-1}$. There exists $c_1 > 0$ and $0 < t_1 < 1$ such that $\int_\Omega c_1 f^{t_1}(x)\,dx = |\Omega|$ and $\sup |c_1 f^{t_1} - 1| < m^{-1}M^{-1}$. Applying the same arguments described in step 2 to $g := c_1 f^{t_1} - 1$, we obtain a Lipschitz function $u_1$ satisfying (2.1) with $f$ replaced by $c_1 f^{t_1}$. Applying again this procedure to $g := c_2 f^{t_2} - c_1 f^{t_1}$, with a suitable choice of $c_2$ and $t_2$ in such a way that $t_1 < t_2 \leq 1$, $\int_\Omega c_2 f^{t_2}\,dx = |\Omega|$ and $\sup |c_2 f^{t_2} - c_1 f^{t_1}| < m^{-1}M^{-1}$, we get $u_2$ Lipschitz solution to

$$\begin{cases} c_1 f^{t_1}(u_2) \det Du_2 = c_2 f^{t_2} & \text{in } \Omega, \\ u_2(x) = x & \text{on } \partial\Omega. \end{cases}$$

Hence, $u_1 \circ u_2$ solves (2.1) with $f$ replaced by $c_2 f^{t_2}$. It can be proved that the exponents $\{t_i\}$ can be chosen such that in finitely many steps, say $n$, we get $t_n = 1$. The existence of a Lipschitz continuous solution to (2.1) follows.

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