LOCAL UNIQUENESS AND NON-DEGENERACY OF BLOW UP SOLUTIONS OF MEAN FIELD EQUATIONS WITH SINGULAR DATA

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ABSTRACT. We are concerned with the mean field equation with singular data on bounded domains. Under suitable non-degeneracy conditions we prove local uniqueness and non-degeneracy of bubbling solutions blowing up at singular points. The proof is based on sharp estimates for bubbling solutions of singular mean field equations and suitably defined Pohozaev-type identities.

Keywords: Mean field equations, uniqueness, non-degeneracy, blow up solutions, singular data.

1. INTRODUCTION

We are concerned with a sequence of solutions of the following mean field equation with singular data

$$\begin{cases} -\Delta u_n = \rho_n \dfrac{he^{u_n}}{\int_\Omega he^{u_n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathbf{P}_{\rho_n})$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $h = h_* \exp(-4\pi \sum_{i=1}^N \alpha_i G(x, p_i))$, p_i are distinct points in Ω, $\alpha_i \in (0, \infty) \setminus \mathbb{N}$, $h_* \in C^\infty(\overline{\Omega})$, and G is the Green function satisfying

$$\begin{cases} -\Delta G(x,p) = \delta_p & \text{in } \Omega, \\ G(x,p) = 0 & \text{on } \partial\Omega. \end{cases}$$

The mean field equation $(\mathbf{P}_{\rho_n})$ (and its counterpart on compact surfaces) have been widely discussed in the last decades because of their several applications in Mathematics and Physics, such as Electroweak and Chern-Simons self-dual vortices [47, 49, 53], conformal metrics on surfaces with [50] or without conical singularities [35], statistical mechanics of two-dimensional turbulence [20] and of self-gravitating systems [52] and cosmic strings [45], and the theory of hyperelliptic curves [22] and of the Painlevé equations [24]. There are by now many results concerning existence [1, 3, 4, 5, 15, 21, 26, 28, 30, 31, 32, 36, 42], multiplicity [5, 29], uniqueness [6, 7, 10, 11, 12, 13, 14, 23, 33, 34, 40, 41, 48] and blow up analysis [2, 9, 16, 18, 17, 19, 25, 27, 37, 38, 39, 51, 54].

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Our goal is to show that bubbling solutions of $(\mathbf{P}_{\rho_n})$ blowing up at singular points p_i are unique and non-degenerate for n large enough.

Definition 1.1. *Let u_n be a sequence of solutions of $(\mathbf{P}_{\rho_n})$. We say that u_n is a regular m-bubbling solution blowing up at the points $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, m$, if,*

$$\frac{he^{u_n}}{\int_\Omega he^{u_n} dx} \rightharpoonup 8\pi \sum_{j=1}^{m} \delta_{q_j},$$

weakly in the sense of measures in Ω.

We say that u_n is a singular m-bubbling solution blowing up at the points $p_j \in \{p_1, \cdots, p_N\}$, $j = 1, \cdots, m$, $m \leq N$ if,

$$\frac{he^{u_n}}{\int_\Omega he^{u_n} dx} \rightharpoonup 8\pi \sum_{j=1}^{m} (1+\alpha_j)\delta_{p_j},$$

weakly in the sense of measures in Ω.

To state the main result and to compare it with the existing literature we introduce some notation. Let $R(x,y) = \frac{1}{2\pi} \log|x-y| + G(x,y)$ be the regular part of $G(x,y)$. For what concerns regular bubbling solutions, for $\mathbf{q} = (q_1, \cdots, q_m) \in \overline{\Omega} \times \cdots \times \overline{\Omega}$, we let $G_j^*(x) = 8\pi R(x,q_j) + 8\pi \sum_{l \neq j}^{1,\cdots,m} G(x,q_l)$ and

$$\ell_{\rm reg}(\mathbf{q}) = \sum_{j=1}^{m} [\Delta \log h(q_j)] h(q_j) e^{G_j^*(q_j)}.$$

For $(x_1, \cdots, x_m) \in \overline{\Omega} \times \cdots \overline{\Omega}$, we also define the m-vortex Hamiltonian,

$$\mathcal{H}_m(x_1, x_2, \cdots, x_m) = \sum_{j=1}^{m} \big[\log(h(x_j)) + 4\pi R(x_j, x_j) \big] + 4\pi \sum_{l \neq j}^{1,\cdots,m} G(x_l, x_j). \quad (1.1)$$

Then, by assuming suitable non-degeneracy conditions the authors in [8, 9] proved that regular m-bubbling solutions are unique and non-degenerate (see also [10] for an analogous result for the Gelfand equation).

Theorem A ([8, 9]). *Let $u_n^{(1)}$ and $u_n^{(2)}$ be two regular m-bubbling solutions of $(\mathbf{P}_{\rho_n})$, with $\rho_n^{(1)} = \rho_n = \rho_n^{(2)}$, blowing up at the points $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, m$, where $\mathbf{q} = (q_1, \cdots, q_m)$ is a critical point of $\mathcal{H}_m$. Assume that,*

1. *$det(D^2 \mathcal{H}_m(\mathbf{q})) \neq 0$,*
2. *$\ell_{\rm reg}(\mathbf{q}) \neq 0$.*

Then there exists $n_0 \geq 1$ such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq n_0$. Moreover, the linearized problem at a m-bubbling solution u_n

$$\begin{cases} \Delta\phi + \rho_n \dfrac{he^{u_n}}{\int_\Omega he^{u_n} dx} \left(\phi - \dfrac{\int_\Omega he^{u_n} \phi \, dx}{\int_\Omega he^{u_n} \, dx} \right) = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

admits only the trivial solution $\phi \equiv 0$ for any $n \geq n_0$.

The above condition (2) can be relaxed by assuming $\ell_{\rm reg}(\mathbf{q}) = 0$ and $D(\mathbf{q}) \neq 0$, where $D(\mathbf{q})$ is a geometric quantity. Our aim is to extend the latter result to singular bubbling solutions. Even though the argument works out for more general situations we focus here on singular 1-bubbling solution blowing up at p_i for some $i \in \{1, \cdots, N\}$, see also Remark 1.3. More precisely, we assume without loss of generality that $\alpha_i \neq \alpha_j$ for $i \neq j$ and we study the case $\rho_n \to 8\pi(1+\alpha_i)$ for some fixed $i \in \{1, \cdots, N\}$ and

$$\|u_n\|_{L^\infty(\Omega)} \to +\infty \quad \text{as } n \to +\infty.$$

We define

$$\ell(p_i) = \frac{2\pi^2}{(1+\alpha_i)\sin\left(\frac{\pi}{1+\alpha_i}\right)} \left(\frac{(1+\alpha_i)}{\pi \overline{h}_1(p_i)}\right)^{\frac{1}{1+\alpha_i}} \Delta \log h_*(p_i), \tag{1.3}$$

where $h(x) = \overline{h}_1(x)|x-p_i|^{2\alpha_i}$. Moreover, we define the 'desingularized' 1-vortex Hamiltonian to be

$$\mathcal{H}_{p_i}(x) = 8\pi(1+\alpha_i)\big(R(x,p_i) - R(p_i,p_i)\big) + \big(\log \overline{h}_1(x) - \log \overline{h}_1(p_i)\big). \tag{1.4}$$

Our main results are the following.

Theorem 1.1. *Let $u_n^{(1)}$ and $u_n^{(2)}$ be two singular 1-bubbling solutions of* $(\mathbf{P}_{\rho_n})$*, with $\rho_n^{(1)} = \rho_n = \rho_n^{(2)}$, blowing up at the point p_i for some $i \in \{1, \cdots, N\}$, $\alpha_i \in (0, \infty) \setminus \mathbb{N}$. Assume that,*

(1) *p_i is a critical point of $\mathcal{H}_{p_i}$,*
(2) *$\ell(p_i) \neq 0$.*

Then there exists $n_0 \geq 1$ such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq n_0$.

Theorem 1.2. *Let u_n be a singular 1-bubbling solution of* $(\mathbf{P}_{\rho_n})$*, blowing up at the point p_i for some $i \in \{1, \cdots, N\}$, $\alpha_i \in (0, \infty) \setminus \mathbb{N}$. Assume that the conditions (1)-(2) of Theorem 1.1 hold true. Then there exists $n_0 \geq 1$ such that, for any $n \geq n_0$,* (1.2) *admits only the trivial solution $\phi \equiv 0$.*

Observe that we do not need the non-degeneracy of the Hamiltonian as in condition (1) of Theorem A. This is essentially due to the difference of the linearized problem, see (1.8) and the discussion later on. On the other hand, we do need to assume p_i to be a critical point of $\mathcal{H}_{p_i}$. For the regular blow up this is always the case since it is well-known [44] that for a regular m-bubbling solution blowing up at the points $q_j \notin \{p_1, \cdots, p_N\}$, then $\mathbf{q} = (q_1, \cdots, q_m)$ has to be a critical point of $\mathcal{H}_m$.

Remark 1.3. *The argument yielding Theorems 1.1 and 1.2 works out for more general situations and can be carried out to prove local uniqueness of singular m-bubbling and even for mixed scenarios of singular m-bubbling and regular m'-bubbling solutions. The decision to focus on singular* 1*-bubbling is twofold: on one side the latter case is very subtle since in general the singular blow up point does not need be a critical point of the Hamiltonian $\mathcal{H}_{p_i}$ and furthermore we are not assuming any non-degeneracy of $\mathcal{H}_{p_i}$, and on the other side we want to highlight the differences with respect to the regular case. We postpone the general situation to a future paper. The case $\alpha \in (-1, 0)$ will be treated in a separate paper since we first need to derive suitable sharp estimates for bubbling solutions,* *which are still missing in this case. Finally, the case $\alpha \in \mathbb{N}$ is by now out of reach due to the presence of non-simple (and non-radial) blow up* [18, 37].

To prove Theorem 1.1 we argue by contradiction and we analyze the asymptotic behavior of the (normalized) difference of two distinct solutions for $(\mathbf{P}_{\rho_n})$,

$$\xi_n = \frac{u_n^{(1)} - u_n^{(2)}}{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(\Omega)}}. \tag{1.5}$$

Near the blow up point p_i, and after a suitable scaling, ξ_n converges to an entire solution of the linearized problem of the Liouville equation

$$\Delta v + |x|^{2\alpha_i} e^v = 0 \quad \text{in } \mathbb{R}^2. \tag{1.6}$$

Solutions of (1.6) with finite mass are completely classified [46] and for $\alpha_i \in (0,\infty) \setminus \mathbb{N}$ take the form,

$$v\,(z) = v_\mu(z) = \log \frac{8(1+\alpha_i)^2 e^\mu}{(1+e^\mu |z|^{2(1+\alpha_i)})^2}, \quad \mu \in \mathbb{R}. \tag{1.7}$$

The freedom in the choice of μ is due to the invariance of equation (1.6) under dilations. The linearized operator L relative to v_0 is defined by,

$$L\phi := \Delta\phi + \frac{8(1+\alpha_i)^2 |z|^{2\alpha_i}}{(1+|z|^{2(1+\alpha_i)})^2}\phi \quad \text{in } \mathbb{R}^2. \tag{1.8}$$

It follows from [27, Corollary 2.2] that the L^∞-bounded kernel of L has one eigenfunction Y_0, where,

$$Y_0(z) = \frac{1-|z|^{2(1+\alpha_i)}}{1+|z|^{2(1+\alpha_i)}} = \frac{\partial v_\mu}{\partial \mu}\Big|_{\mu=0}.$$

The main part of the proof of Theorem 1.1 is to show that, after scaling and for large n, ξ_n is orthogonal to Y_0. This is done by a delicate analysis of a suitably defined Pohozaev-type identity first introduced in [43] and then exploited in [8, 10].

The proof of Theorem 1.2 follows the same strategy by analyzing the asymptotic behavior of

$$\Xi_n = \frac{\phi_n - \frac{\int_\Omega h e^{u_n}\phi_n\,dx}{\int_\Omega h e^{u_n}\,dx}}{\left\|\phi_n - \frac{\int_\Omega h e^{u_n}\phi_n\,dx}{\int_\Omega h e^{u_n}\,dx}\right\|_{L^\infty(\Omega)}},$$

for a non-trivial solution ϕ_n of (1.2), which plays the role of (1.5).

The paper is organized as follows. In section 2 we introduce some preliminary results, in section 3 we estimate the L^∞-norm of the difference of two solutions to $(\mathbf{P}_{\rho_n})$ and in section 4 we then deduce the first estimates of ξ_n, the normalized difference of two solutions, away from the blow up point. In section 5 we introduce a Pohozaev-type identity to get refined estimates on ξ_n and prove Theorem 1.1. Finally, in section 6 we give the sketch of the proof of Theorem 1.2.

2. Preliminary estimates about the blow up Phenomenon at the singular point

In this section we collect some preliminary results which will be used in the sequel. Let us assume that $i = 1$ and set $p = p_1$, $0 \neq \alpha = \alpha_1 \in (0, +\infty) \setminus \mathbb{N}$. We define

$$\tilde{u}_n = u_n - \log\left(\int_\Omega h e^{u_n} dx\right), \quad \lambda_n = \max_\Omega \tilde{u}_n, \quad \sigma_n^{2(1+\alpha)} = e^{-\lambda_n}, \tag{2.1}$$

and

$$U_n(x) = \lambda_n - 2\log(1 + \gamma_n e^{\lambda_n}|x-p|^{2+2\alpha}), \quad \gamma_n = \frac{\rho_n \overline{h}_1(p)}{8(1+\alpha)^2},$$

where

$$\overline{h}_1(x) = h_* \exp(-4\pi \overline{G}_1(x)) \quad \overline{G}_1(x) = \sum_{i=2}^{N} \alpha_i G(x, p_i) + R(x, p),$$

and $R(x,y) = G(x,y) + \frac{1}{2\pi}\log|x-y|$ is the regular part of the Green function. Therefore, we have

$$h(x) = \overline{h}_1(x)|x-p|^{2\alpha},$$

and in any small enough ball centered at p it holds that $\overline{h}_1 > 0$. It has been shown in [2] (for $\alpha \in (0, +\infty) \setminus \mathbb{N}$) and [17] (for $\alpha \in (-1, 0)$) that

$$|\tilde{u}_n(x) - U_n(x)| \leq C, \quad \forall x \in B_r(p). \tag{2.2}$$

Actually the proofs in [18, 2] show that this estimate holds locally near p, but then the global estimate follows by looking at the Definition 1 and the Green representation formula.

More recently, it has been proved in [27] that if $\alpha \in (0, +\infty) \setminus \mathbb{N}$, then

$$\rho_n - 8\pi(1+\alpha) = \ell(p)e^{-\frac{\lambda_n}{1+\alpha}} + O(e^{-\lambda_n \frac{1+\epsilon_0}{1+\alpha}}) \text{ as } n \to +\infty, \tag{2.3}$$

and

$$\rho_{n,1} - 8\pi(1+\alpha) = \ell(p)e^{-\frac{\lambda_n}{1+\alpha}} + O(e^{-\lambda_n \frac{1+\epsilon_0}{1+\alpha}}) \text{ as } n \to +\infty, \tag{2.4}$$

where

$$\rho_{n,1} = \int_{B(p,r_0)} h e^{\tilde{u}_n}, \quad \ell(p) = \frac{2\pi^2}{(1+\alpha)\sin\left(\frac{\pi}{1+\alpha}\right)} \left(\frac{(1+\alpha)}{\pi \overline{h}_1(p)}\right)^{\frac{1}{1+\alpha}} \Delta \log h_*(p),$$

and

$$\epsilon_0 = 2 - 2(1-\alpha)^+ = \begin{cases} 2, & \text{if } \alpha \geq 1, \\ 2\alpha, & \text{if } \alpha \in (0,1). \end{cases}$$

Next, we set

$$R_{n,1} = \rho_{n,1} R(x, p),$$

let $r_0 > 0$ be a small positive number and set

$$v_n = \tilde{u}_n - (R_{n,1}(x) - R_{n,1}(p)), \quad x \in B(p, r_0), \tag{2.5}$$

and as in [54], we denote ψ_n as the solution of

$$\begin{cases} \Delta\psi_n = 0, & \text{in } B_(p, 4r_0), \\ \\ \psi_n = v_n - \frac{1}{8\pi r_0}\int_{|x-p|=4r_0} v_n ds & \text{on } \partial B(p, 4r_0). \end{cases}$$

By the Mean value Therorem, we have $\psi(0) = 0$. It has been proved in [54] that

$$v_n - U_n - \psi_n(x) = \sigma_n \psi_{n,1}\left(\frac{x-p}{\sigma_n}\right) + \sigma_n^2 \psi_{n,2}\left(\frac{x-p}{\sigma_n}\right) + O(\sigma_n^2) \text{ in } B(p, 4r_0), \tag{2.6}$$

where

$$\psi_{n,1}(y) = -\frac{2(1+\alpha)a_{n,1}}{\alpha}\frac{y_1}{1+\gamma_n|y|^{2(1+\alpha)}}, \quad y = (y_1, y_2) \in \mathbb{R}^2, \tag{2.7}$$

and

$$\psi_{n,2}(y) = -a_n \log(2 + |y|) + a_{n,0} + O(|y|^{-\epsilon_0}), \quad y = (y_1, y_2) \in \mathbb{R}^2 \setminus B(0, R_0), \tag{2.8}$$

for suitable $R_0 \geq 1$. Here $a_{n,0}$ is a uniformly bounded sequence,

$$a_n = \frac{\pi}{(1+\alpha)\sin\left(\frac{\pi}{1+\alpha}\right)}\left(\frac{8(1+\alpha)^2}{\rho_n \overline{h}_1(p)}\right)^{\frac{1}{1+\alpha}} \Delta \log h_*(p),$$

and, composing with suitable rotations, we can assume that

$$(a_{n,1}, 0) = \nabla \log\left(\overline{h}_1(x)e^{R_{n,1}(x)+\psi_n(x)}\right)|_{x=p} . \tag{2.9}$$

Moreover, it has been shown in [27, Lemma 3.2] that

$$\psi_n(x) = O(\sigma_n^2) \quad , x \in B(p, 4r_0). \tag{2.10}$$

Since ψ_n is harmonic, then we also have

$$|\nabla\psi_n(x)| = O(\sigma_n^2) \quad , x \in B(p, 3r_0). \tag{2.11}$$

We also have, see [27, Lemma 3.1],

$$u_n(x) - \rho_n G(x, p) = O(\sigma_n), \quad x \in \overline{\Omega} \setminus B(p, r_0). \tag{2.12}$$

Also, we will need the fllowing improved estimate obtained by matching (2.6) and (2.12).

Lemma 2.1. *It holds,*

$$\lambda_n - \log\left(\int_\Omega he^{u_n}\right) + 2\log\gamma_n + 8\pi(1+\alpha)R(p,p) = O(\sigma_n). \tag{2.13}$$

Proof. Putting $c_n = \log\left(\int_\Omega he^{u_n}\right)$ and picking any $|x - p| = 2r_0$ in (2.6) and (2.12), we conclude that

$$\rho_n G(x, p) - c_n - (R_{n,1}(x) - R_{n,1}(p)) - U_n(x) = O(\sigma_n).$$

Clearly we have

$$U_n(x) = -\lambda_n - 4(1+\alpha)\log|x-p| - 2\log(\gamma_n) + O(\sigma_n^{2(1+\alpha)}),$$

and we find that

$$-\frac{\rho_n}{2\pi}\log|x-p|+\rho_n R(x,p)-c_n-\rho_{n,1}R(x,p)+\rho_{n,1}R(p,p)$$
$$+\lambda_n+4(1+\alpha)\log|x-p|+2\log(\gamma_n)=O(\sigma_n),$$

and then the desired conclusion easily follows from (2.3) and (2.4). □

Finally, similar arguments used in the estimate (2.12), yield

$$\nabla(\tilde{u}_n-\rho_n G(x,p))=O(\sigma_n),\quad x\in\overline{\Omega}\setminus B(p,r_0). \tag{2.14}$$

3. Estimate of the L^∞-norm

The proof of Theorem 1.1 is obtained by contradiction and we assume that two distinct solutions $u_n^{(i)}$, $i=1,2$, exist for **($\mathbf{P}_{\rho_n}$)**, whence in particular with the same ρ_n, which satisfy

$$\rho_n\to 8\pi(1+\alpha)\quad\text{as}\quad n\to+\infty,$$

where $\alpha=\alpha_1$. We also assume without loss of generality that

$$p_1=0\in\Omega.$$

Then we define

$$\tilde{u}_n^{(i)}=u_n^{(i)}(x)-\log\left(\int_\Omega he^{u_n^{(i)}}\right),\quad \lambda_n^{(i)}=\max_{\overline{\Omega}}\tilde{u}_n^{(i)},$$

and in particular $v_n^{(i)}$ defined as in (2.5). Also we set

$$U_n^{(i)}(x)=\lambda_n^{(i)}-2\log(1+\gamma_n e^{\lambda_n^{(i)}}|x-p|^{2(1+\alpha)}),\quad i=1,2,\quad \gamma_n=\frac{\rho_n\overline{h}_1(p)}{8(1+\alpha)^2}.$$

There is no loss of generality in assuming that

$$\lambda_n^{(1)}\le\lambda_n^{(2)}.$$

To simplify the notation, we set

$$\sigma_n^{2(1+\alpha)}=e^{-\lambda_n^{(1)}}.$$

Then we have

Lemma 3.1. (i) $|\lambda_n^{(1)}-\lambda_n^{(2)}|=O(\sum_{i=1}^2 e^{-\frac{\epsilon_0}{\alpha+1}\lambda_n^{(i)}})=O(e^{-\frac{\epsilon_0}{\alpha+1}\lambda_n^{(1)}})=O(\sigma_n^{2\epsilon_0})$.

(ii) $\|\tilde{u}_n^{(1)}-\tilde{u}_n^{(2)}\|_{L^\infty(B(0,r_0))}\le|\lambda_n^{(2)}-\lambda_n^{(1)}|+O(\lambda_n^{(1)}\sigma_n^2)$.

(iii) $\|\tilde{u}_n^{(1)}-\tilde{u}_n^{(2)}\|_{L^\infty(\Omega\setminus B(0,r_0))}\le O(\sigma_n)$.

Proof. (i) In view of (2.3), we find that

$$\ell(p)e^{-\frac{\lambda_n^{(1)}}{1+\alpha}}+O\left(e^{-\frac{1+\epsilon_0}{1+\alpha}\lambda_n^{(1)}}\right)=\ell(p)e^{-\frac{\lambda_n^{(2)}}{1+\alpha}}+O\left(e^{-\frac{1+\epsilon_0}{1+\alpha}\lambda_n^{(2)}}\right),$$

which immediately implies, since $\ell(p)\ne 0$,

$$\lambda_n^{(1)}-\lambda_n^{(2)}=O(e^{-\frac{\epsilon_0}{1+\alpha}\lambda_n^{(1)}})+e^{-\frac{\epsilon_0}{1+\alpha}\lambda_n^{(2)}},$$

as claimed.

(ii) By using $\lambda_n^{(1)} \leq \lambda_n^{(2)}$, it is not difficult to see that

$$\begin{aligned} U_n^{(2)} - U_n^{(1)} &= (\lambda_n^{(2)} - \lambda_n^{(1)}) \left(\frac{1 - e^{\lambda_n^{(1)}} \gamma_n |x-p|^{2(1+\alpha)}}{1 + e^{\lambda_n^{(1)}} \gamma_n |x-p|^{2(1+\alpha)}} \right) + O((\lambda_n^{(2)} - \lambda_n^{(1)})^2) \\ &\leq |\lambda_n^{(2)} - \lambda_n^{(1)}| + O((\lambda_n^{(2)} - \lambda_n^{(1)})^2), \end{aligned}$$

uniformly in $B(0,r)$ for any $r > 0$. Also, in view of (2.10), and since the $\psi_n^{(i)}$'s are harmonic, we find that

$$\psi_n^{(2)}(x) - \psi_n^{(1)}(x) = O(\sigma_n^2), \quad |\nabla(\psi_n^{(2)}(x) - \psi_n^{(1)}(x))| = O(\sigma_n^2),$$

uniformly in $B(0, 3r_0)$, we use this gradient estimate to evaluate the difference,

$$|a_{n,2} - a_{n,1}| = |\nabla(\psi_n^{(2)}(0) - \psi_n^{(1)}(0))| = O(\sigma_n^2),$$

which implies that

$$\begin{aligned} |\psi_{n,2}^{(2)}(x) - \psi_{n,1}^{(1)}(x)| &\leq \frac{2(1+\alpha)}{\alpha} |a_{n,1}^{(2)} - a_{n,1}^{(1)}||x_1| + O(\lambda_n^{(2)} - \lambda_n^{(1)}) \\ &= O(\sigma_n^2) + O(\lambda_n^{(2)} - \lambda_n^{(1)}), \end{aligned}$$

uniformly in $B(0, r_0)$. Also it is easy to see that

$$\left| \psi_{n,2}^{(2)} - \psi_{n,2}^{(1)} \right| = O(\lambda_n^{(1)} \sigma_n^2).$$

Therefore, in view of (2.6) and Lemma 3.1, we finally conclude that,

$$\|\tilde{u}_n^{(1)}(x) - \tilde{u}_n^{(2)}(x)\|_{L^\infty(B(0,r_0))} \leq |\lambda_n^{(2)} - \lambda_n^{(1)}| + O(\lambda_n^{(1)} \sigma_n^2),$$

which is (ii).

(iii) Next we obtain the estimate in $\Omega \setminus B(0, r_0)$, by using the Green's representation formula,

$$\begin{aligned} \tilde{u}_n^{(1)}(x) - \tilde{u}_n^{(2)}(x) &= \rho_n \int_\Omega G(y,x) h(y) (e^{\tilde{u}_n^{(1)}(y)} - e^{\tilde{u}_n^{(2)}(y)}) dy \\ &= \rho_n \int_{B(0,r_0)} (G(y,x) - G(0,x)) h(y) (e^{\tilde{u}_n^{(1)}(y)} - e^{\tilde{u}_n^{(2)}(y)}) dy \\ &\quad + G(0,x) \int_{B(0,r_0)} \rho_n h(y) (e^{\tilde{u}_n^{(1)}(y)} - e^{\tilde{u}_n^{(2)}(y)}) dy \\ &\quad + \rho_n \int_{\Omega \setminus B(0,r_0)} G(y,x) h(y) (e^{\tilde{u}_n^{(1)}(y)} - e^{\tilde{u}_n^{(2)}(y)}) dy. \end{aligned}$$

In view of (2.2) and since ρ_n is the same for the two solutions, then we have

$$\begin{aligned} \rho_{n,1}^{(1)} - \rho_{n,1}^{(2)} &= \rho_n \int_{B(0,r_0)} h(y) e^{\tilde{u}_n^{(1)}(y)} - \rho_n \int_{B(0,r_0)} h(y) e^{\tilde{u}_n^{(2)}(y)} \\ &= \rho_n \int_{\Omega \setminus B(0,r_0)} h(y) \left(e^{\tilde{u}_n^{(2)}(y)} - e^{\tilde{u}_n^{(1)}(y)} \right) dy = O(e^{-\lambda_n}). \end{aligned}$$

Then, by using (2.2) once more, for $x \in \Omega \setminus B(0, r_0)$ we have,

$$\begin{aligned}
&\tilde{u}_n^{(1)}(x) - \tilde{u}_n^{(2)}(x) \\
&= \rho_n \int_{B(0,r_0)} (G(y,x) - G(0,x))h(y) \left(e^{\tilde{u}_n^{(2)}(y)} - e^{\tilde{u}_n^{(1)}(y)} \right) dy \\
&\quad + G(0,x)(\rho_{n,1}^{(1)} - \rho_{n,2}^{(2)}) + O(e^{-\lambda_n}) \\
&= \rho_n \int_{B(0,r_0)} (G(y,x) - G(0,x))h(y) \left(e^{\tilde{u}_n^{(2)}(y)} - e^{\tilde{u}_n^{(1)}(y)} \right) dy + O(e^{-\lambda_n}) \\
&= \int_{B(0,r_0)} O(1) \left(\sum_{i=1,2} \frac{|y|^{2\alpha+1} e^{\lambda_n^{(i)}}}{(1 + \gamma_n e^{\lambda_n^{(i)}} |y|^{2+2\alpha})^2} \right) dy + O(e^{-\lambda_n}) \\
&= O(\sigma_n),
\end{aligned}$$

uniformly in $x \in \Omega \setminus B(0, r_0)$. Therefore we conclude that

$$\|\tilde{u}_n^{(1)}(x) - \tilde{u}_n^{(2)}(x)\|_{L^\infty(\Omega \setminus B(0,r_0))} \le O(\sigma_n),$$

as claimed. □

4. Estimate of the difference away from the blow up point

Let

$$\xi_n = \frac{\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}}{\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(\Omega)}}. \tag{4.1}$$

Clearly ξ_n satisfies

$$\begin{cases} \Delta \xi_n + \rho_n h(x) c_n(x) \xi_n(x) = 0 & \text{in } \Omega, \\ \xi_n = -d_n & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

for some constant d_n satisfying $|d_n| \le 1$ and

$$c_n(x) = \frac{e^{\tilde{u}_n^{(1)}} - e^{\tilde{u}_n^{(2)}}}{\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}}.$$

To simplify the notations, we set

$$\lambda_n = \lambda_n^{(1)} \quad \text{and} \quad \sigma_n^{2+2\alpha} = e^{-\lambda_n}.$$

Then by defining

$$\hat{\xi}_n(z) = \xi_n(\sigma_n z), \quad |z| < 4\sigma_n^{-1} r_0,$$

we prove the following

Lemma 4.1. *There exists a constant $b_0 \in \mathbb{R}$, such that $\hat{\xi}_n(z) \to b_0 \hat{\xi}_0(z)$ in $C^0_{\text{loc}}(\mathbb{R}^2)$, where*

$$\hat{\xi}_0(z) = \frac{1 - \gamma |z|^{2+2\alpha}}{1 + \gamma |z|^{2+2\alpha}}, \quad z \in \mathbb{R}^2,$$

where $\gamma = \frac{\pi \bar{h}_1(0)}{1+\alpha}$.

Proof. By Lemma 3.1, we see that

$$c_n(x) = e^{\tilde u_n^{(1)}(x)}\left(1 + O(\|\tilde u_n^{(1)} - \tilde u_n^{(2)}\|_{L^\infty(\Omega)})\right)$$
$$= e^{\tilde u_n^{(1)}(x)}(1 + O(|\lambda_n^{(2)} - \lambda_n^{(1)}| + \sigma_n)),$$

and then by (2.6), (2.10) and (2.11)

$$e^{-\lambda_n} c_n(\sigma_n z) = \frac{e^{C\sigma_n(1+O(|\lambda_n^{(2)}-\lambda_n^{(1)}|+\sigma_n))}}{(1+\gamma_n|z|^{2+2\alpha})^2} \to \frac{1}{(1+\gamma|z|^{2+2\alpha})^2} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2),$$

where $\gamma = \frac{\pi \overline{h}_1(0)}{1+\alpha}$.

We define

$$\Omega_{\sigma_n} = \left\{ z \in \mathbb{R}^2 \mid \sigma_n z \in \Omega \right\}.$$

By using (4.2), we have

$$\begin{cases} \Delta\hat\xi_n + \rho_n \overline{h}_1(\sigma_n z)|z|^{2\alpha}\sigma_n^{2+2\alpha} & \text{in} \quad \Omega_{\sigma_n}, \\ \hat\xi_n(z) = -d_n & \text{on} \quad \partial\Omega_{\sigma_n}, \end{cases}$$

and since $|\hat\xi_n| \leq 1$, then we conclude that $\hat\xi_n \to \hat\xi$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$, where $\hat\xi$ is a solution of

$$\Delta\hat\xi + \frac{8\gamma(1+\alpha)^2|z|^{2\alpha}}{(1+\gamma|z|^{2(1+\alpha)})^2}\hat\xi = 0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad |\hat\xi(z)| \leq 1 \text{ in } \mathbb{R}^2.$$

It follows from [27, Corollary 2.2] that $\hat\xi(z) = b_0\xi_0(z)$, for some constant b_0, as claimed. □

Next, we have

Lemma 4.2. *For any r_0 small enough we have*

$$\xi_n(x) = -b_0 + o(1), \quad x \in \Omega \setminus B(0, r_0),$$

where b_0 is defined by Lemma 4.1.

Proof. It follows from (2.2) that

$$c_n(x) \to 0 \quad \text{in} \quad C^0_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}).$$

Since $\|\xi_n\|_{L^\infty(\Omega)} \leq 1$, then (4.2) implies that

$$\xi_n \to \xi_0 \quad \text{in} \quad C^0_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}),$$

where

$$\Delta\xi_0 = 0 \text{ in } \Omega \setminus \{0\} \quad \text{and} \quad \|\xi_0\|_{L^\infty(\Omega)} \leq 1.$$

As a consequence, ξ_0 is smooth in $\Omega \setminus \{0\}$ and in particular

$$\Delta\xi_0 = 0 \quad \text{in} \quad \Omega.$$

Therefore $\xi_0 = -b$ in Ω for some constant b and

$$\xi_n \to -b \quad \text{in} \quad C^0_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}). \tag{4.3}$$

In particular $-\xi_n(x) = d_n \to b$ for $x \in \partial\Omega$. Let $\phi_n = \frac{1-\gamma_n e^{\lambda_n}|x|^{2+2\alpha}}{1+\gamma_n e^{\lambda_n}|x|^{2+2\alpha}}$ and let us fix $d \in (0, r_0)$. Then, by using (2.6), (2.10) and (2.11), we find that

$$\begin{aligned}
&\int_{\partial B(0,d)} \left(\phi_n \frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \phi_n}{\partial \nu}\right) d\sigma = \int_{B(0,d)} (\phi_n \Delta \xi_n - \xi_n \Delta \phi_n) dx \\
&= \int_{B(0,d)} \left\{ -\rho_n \xi_n \phi_n \overline{h}_1(x)|x|^{2\alpha} \left(\frac{e^{\tilde{u}_n^{(1)}} - e^{\tilde{u}_n^{(2)}}}{\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}} \right) + 8(1+\alpha)^2 \gamma_n \xi_n \phi_n |x|^{2\alpha} e^{U_n} \right\} dx \\
&= \int_{B(0,d)} \rho_n \xi_n \phi_n \left\{ -\overline{h}_1(x)|x|^{2\alpha} e^{\tilde{u}_n^{(1)}} (1 + O(|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}|)) + \overline{h}_1(0)|x|^{2\alpha} e^{U_n} \right\} dx \\
&= \int_{B(0,d)} \rho_n \xi_n \phi_n |x|^{2\alpha} e^{U_n} \left\{ -\overline{h}_1(x) e^{O(\sigma_n)} (1 + O(|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}|)) + \overline{h}_1(0) \right\} dx.
\end{aligned}$$

Therefore, by the scaling $x = \sigma_n z$, we see that,

$$\begin{aligned}
&\int_{\partial B(0,d)} \left(\phi_n \frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \phi_n}{\partial \nu}\right) d\sigma \\
&= \int_{B(0,d/\sigma_n)} \rho_n \hat{\xi}_n(z) \hat{\phi}_n(z) |z|^{2\alpha} \frac{O(1)(\sigma_n|z| + |\hat{u}_n^{(1)} - \hat{u}_n^{(2)}| + \sigma_n)}{(1+\gamma_n|z|^{2+2\alpha})^2} dz.
\end{aligned}$$

In view of Lemma 3.1 we obtain

$$\int_{\partial B(0,d)} \left(\phi_n \frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \phi_n}{\partial \nu}\right) d\sigma = O(\sigma_n + \sigma_n^{2\epsilon_0}). \tag{4.4}$$

Let $\zeta_n = \int_0^{2\pi} \xi_n(r, \theta) d\theta$, where $r = |x|$. Then, for any fixed $R > 0$, (4.4) yields

$$(\zeta_n)'(r)\phi_n(r) - \zeta_n(r)\phi_n'(r) = \frac{O(\sigma_n + \sigma_n^{2\epsilon_0})}{r}, \quad \forall r \in (R\sigma_n, r_0].$$

Also for any $R > 0$ large enough, and for any $r \in (R\sigma_n, r_0]$, we also obtain that

$$\phi_n(r) = -1 + O\left(\frac{\sigma_n^{2+2\alpha}}{r^{2+2\alpha}}\right), \quad \phi_n'(r) = O\left(\frac{\sigma_n^{2+2\alpha}}{r^{3+2\alpha}}\right),$$

and so we conclude that

$$\zeta_n'(r) = \frac{O(\sigma_n + \sigma_n^{2\epsilon_0})}{r} + O\left(\frac{\sigma_n^{2+2\alpha}}{r^{3+2\alpha}}\right), \quad \forall r \in (R\sigma_n, r_0]. \tag{4.5}$$

Integrating (4.5) we obtain that

$$\zeta_n(r) = \zeta_n(R\sigma_n) + o(1) + O(R^{-(2+2\alpha)}), \quad \forall r \in (R\sigma_n, r_0]. \tag{4.6}$$

In view of Lemma 4.1, we also have

$$\zeta_n(R\sigma_n) = -2\pi b_0 + o_R(1) + o_n(1),$$

where $\lim_{R\to+\infty} o_R(1) = 0$ and $\lim_{n\to+\infty} o_n(1) = 0$. Then by (4.6) we have

$$\zeta_n(r) = -2\pi b_0 + o_R(1) + o_n(1)(1 + O(R)), \quad \forall r \in (R\sigma_n, r_0]. \tag{4.7}$$

In view of (4.3), we see that

$$\zeta_n = -2\pi b + o_n(1) \text{ in } C_{\text{loc}}(\Omega \setminus \{0\}),$$

which implies that $b = b_0$. Hence, we finish the proof. □

Next, we need a refined estimate about ξ_n which will be needed in next section.

Lemma 4.3.

$$\xi_n(x) = -d_n + A_n G(0,x) + o(\sigma_n) \quad \text{in} \quad C^1(\Omega \setminus B(0,2r_0)), \tag{4.8}$$

where

$$A_n = \int_\Omega f_n^*(x) \quad \text{and} \quad f_n^*(x) = \rho_n c_n(x) h(x) \xi_n(x).$$

Moreover, there is a constant $C > 0$, which does not depend on $R > 0$, such that

$$|\xi_n(x) + d_n - A_n G(0,x)| \le C\sigma_n \left(\frac{1_{B(0,2r_0)}(x)}{|x|} + 1_{\Omega \setminus B(0,2r_0)}(x) \right), \; x \in \Omega \setminus B(0, R\sigma_n). \tag{4.9}$$

Proof. By the Green representation formula we find that,

$$\begin{aligned} \xi_n(x) =& -d_n + \int_\Omega G(y,x) f_n^*(y) dy \\ =& -d_n + A_n G(0,x) + \int_\Omega (G(y,x) - G(0,x)) f_n^*(y) dy, \end{aligned} \tag{4.10}$$

while, by Lemma 3.1, we also find that

$$c_n(x)\xi_n(x) = \frac{e^{\tilde{u}_n^{(1)}} - e^{\tilde{u}_n^{(2)}}}{\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(\Omega)}} = e^{\tilde{u}_n^{(1)}} \xi_n(x)(1 + O(\lambda_n^{(2)} - \lambda_n^{(1)} + \sigma_n)). \tag{4.11}$$

Thus, for $x \in \Omega \setminus B(0,2r_0)$, we see from (2.2), (2.6) that

$$\begin{aligned} &\int_\Omega (G(y,x) - G(0,x)) f_n^*(y) dy = \int_{B(0,r_0)} (G(y,x) - G(0,x)) f_n^*(y) dy + O(e^{-\lambda_n}) \\ &= \int_{B(0,r_0)} \left\langle \partial_y G(y,x) \mid_{y=0}, y \right\rangle f_n^*(y) dy + O(1) \left(\frac{|y|^{2+2\alpha} e^{\lambda_n}}{(1 + e^{\lambda_n} |y|^{2+2\alpha})^2} dy \right) \\ &= \int_{B(0,r_0)} \left\langle \partial_y G(y,x) \mid_{y=0}, y \right\rangle f_n^*(y) dy + O(\sigma_n^2). \end{aligned} \tag{4.12}$$

By using (2.2), (2.6) and Lemma 3.1, after scaling we see that for $x \in \Omega \setminus B(0,2r_0)$, it holds

$$\begin{aligned} &\int_{B(0,r_0)} \left\langle \partial_y G(y,x) \mid_{y=0}, y \right\rangle f_n^*(y) dy \\ &= \sigma_n^{3+2\alpha} \int_{B(0,r_0/\sigma_n)} \left\langle \partial_y G(y,x) \mid_{y=0}, z \right\rangle \rho_n \overline{h}_1(\sigma_n z) |z|^{2\alpha} e^{\hat{U}_n + \hat{R}_{n,1}(z) - \hat{R}_{n,1}(0)} \hat{\xi}_n(z) dz \\ &\quad + O(\sigma_n^{1+2\epsilon_0} + \sigma_n^2) \\ &= \sigma_n \int_{B(0,r_0/\sigma_n)} \frac{\left\langle \partial_y G(y,x) \mid_{y=0}, z \right\rangle \rho_n \overline{h}_1(0) |z|^{2\alpha} \hat{\xi}_n(z)}{(1 + \gamma_n |z|^{2+2\alpha})^2} dz + O(\sigma_n^{1+2\epsilon_0} + \sigma_n^2). \end{aligned}$$

Therefore, in view of Lemma 4.1, for $x \in \Omega \setminus B(0,2r_0)$ we find that,

$$\begin{aligned} &\int_{B(0,r_0)} \left\langle \partial_y G(y,x) \mid_{y=0}, y \right\rangle f_n^*(y) dy \\ &= \sigma_n \sum_{h=1}^{2} \partial_{y_h} G(y,x) \mid_{y=0} \rho_n \overline{h}_1(0) b_0 \int_{B(0,r_0/\sigma_n)} \frac{z_h |z|^{2\alpha} \hat{\xi}_0(z)}{(1 + \gamma_n |z|^{2+2\alpha})^2} dz + o(\sigma_n) \\ &= \sigma_n \sum_{h=1}^{2} \partial_{y_h} G(y,x) \mid_{y=0} \rho_n \overline{h}_1(0) b_0 \int_{\mathbb{R}^2} \frac{z_h |z|^{2\alpha} \hat{\xi}_0(z)}{(1 + \gamma_n |z|^{2+2\alpha})^2} dz + o(\sigma_n). \end{aligned} \tag{4.13}$$

From (4.10)-(4.13), we see that the estimate (4.8) holds in $C^0(\Omega \setminus B(0,r_0))$. The proof of the fact that (4.8) holds in $C^1(\Omega \setminus B(0,r_0))$ is similar and we skip it here to avoid repetitions.

From (4.11), (2.6) and suitable scaling, we see that there exists $C > 0$, which is independent of $R > 0$ such that for $x \in B(0,2r_0) \setminus B(0,\sigma_n R)$, it holds that

$$\begin{aligned}
&|\xi_n(x) + d_n - A_n G(0,x)| \le |\int_{B(0,3r_0)} (G(y,x) - G(0,x)) f_n^*(y)dy| + O(e^{-\lambda_n}) \\
&\le |\frac{1}{2\pi} \int_{B(0,3r_0)} \log \frac{|x|}{|x-y|} f_n^*(y)dy| + O\left(\int_{B(0,3r_0)} \frac{e^{\lambda_n}|y|^{1+2\alpha}}{(1+e^{\lambda_n}|y|^{2+2\alpha})^2}dy\right) + O(e^{-\lambda_n}) \\
&\le O(1)\left(\int_{B(0,3r_0/\sigma_n)} \frac{|\log|x| - \log|x-\sigma_n z||\,|z|^{2\alpha}}{(1+|z|^{2+2\alpha})^2}dz\right) + O(\sigma_n) \\
&\le O(1)\left(\int_{\frac{|x/\sigma_n|}{2} \le |z| \le 2|x/\sigma_n|} \frac{|\log|x| - \log|x-\sigma_n z||\,|z|^{2\alpha}}{(1+|z|^{2+2\alpha})^2}dz\right) \\
&\quad + O(1)\left(\int_{B(0,3r_0/\sigma_n)} \frac{\sigma_n|z|^{2\alpha+1}}{|x|(1+|z|^{2+2\alpha})^2}dz\right) + O(\sigma_n) \\
&\le O(1)\left(\frac{\sigma_n}{|x|}\right) + O(1)(\log|z||z|^{-2}\mid_{|z|=|x|/\sigma_n}) + O(\sigma_n) \le C\left(\frac{\sigma_n}{|x|}\right).
\end{aligned} \tag{4.14}$$

By (4.10), (4.11) and (2.6), we also see that for $x \in \Omega \setminus B(0,2r_0)$, it holds that

$$|\xi_n(x) + d_n - A_n G(x,0)| = O\left(\int_{B(0,r_0)} \frac{e^{\lambda_n}|y|^{1+2\alpha}}{(1+e^{\lambda_n}|y|^{2+2\alpha})^2}dy\right) + O(e^{-\lambda_n}) = O(\sigma_n). \tag{4.15}$$

By (4.14) and (4.15) we obtain (4.9), which concludes the proof of Lemma 4.3. □

5. Estimates via Pohozaev identities

From now on, for a given function $f(y,x)$, we shall use ∂ and D to denote the partial derivatives with respect to y and x respectively. With a small abuse of notation, for a function $f(x)$ we will use both ∇ and D to denote its gradient.

We define

$$\varphi_n(y) = \rho_n(R(y,0) - R(0,0)), \tag{5.1}$$

and

$$v_n^{(i)} = \tilde{u}_n^{(i)} - \varphi_n(y), \quad i = 1,2. \tag{5.2}$$

Recall the definition of ξ_n which satisfies (4.2). Our aim is to show that the projection of ξ_n on the radial part kernel is zero, i.e., $b_0 = 0$. We shall accomplish it by exploiting the following Pohozaev identity to derive a more accurate estimate on ξ_n.

Lemma 5.1. ([43]) *For any fixed $r \in (0, r_0)$, it holds*

$$\begin{aligned}&\frac{1}{2}\int_{\partial B(0,r)} r\left\langle Dv_n^{(1)}+Dv_n^{(2)}, D\xi_n\right\rangle d\sigma-\int_{\partial B(0,r)} r\left\langle \nu, D(v_n^{(1)}+v_n^{(2)})\right\rangle\langle \nu, D\xi_n\rangle\, d\sigma\\&=\int_{\partial B(0,r)}\frac{r\rho_n h(x)}{\|v_n^{(1)}-v_n^{(2)}\|_{L^\infty(\Omega)}}(e^{v_n^{(1)}+\varphi_n}-e^{v_n^{(2)}+\varphi_n})d\sigma\\&\quad-\int_{B(0,r)}\frac{\rho_n h(x)(e^{v_n^{(1)}+\varphi_n}-e^{v_n^{(2)}+\varphi_n})}{\|v_n^{(1)}-v_n^{(2)}\|_{L^\infty(\Omega)}}\left(2+2\alpha+\left\langle D(\log\overline{h}_1(x)+\varphi_n(x)),x\right\rangle\right)dx.\end{aligned}\tag{5.3}$$

Proof. See [8] for a proof of this identity. □

Let

$$\begin{aligned}\Phi(y,0)=&-8\pi(1+\alpha)\log|y|+8\pi(1+\alpha)(R(y,0)-R(0,0))\\&+\log(\overline{h}_1(y))-\log(\overline{h}_1(0)).\end{aligned}\tag{5.4}$$

Recall the definition of A_n given in Lemma 4.3. Then we have

Lemma 5.2.

$$\begin{aligned}\text{L.H.S. of (5.3)}=&-4(1+\alpha)A_n-\frac{(8(1+\alpha)^2)^3 b_0 e^{-\lambda_n}}{2\rho_n\overline{h}_1(0)}\int_{\Omega\setminus B(0,r)}|y|^{2\alpha}e^{\Phi(y,0)}\\&+o(\sigma_n^2)+O(\sigma_n|A_n|)+O(r^{-3}\sigma_n^3).\end{aligned}$$

Proof. Let

$$G_n(x)=\rho_n G(x,0),\tag{5.5}$$

so that

$$\nabla(G_n(x)-\varphi_n)(x)=-\frac{\rho_n}{2\pi}\frac{x}{|x|^2}.\tag{5.6}$$

In view of (2.14), we have

$$\begin{aligned}\nabla v_n^{(i)}(x)&=\nabla(\tilde{u}_n^{(i)}-G_n(x))+\nabla(G_n(x)-\varphi_n(x))\\&=\nabla(G_n(x)-\varphi_n(x))+O(\sigma_n),\quad x\in\overline{\Omega}\setminus B(0,r_0),\end{aligned}$$

for any fixed small $r_0>0$. As a consequence, for fixed $r>r_0$, we find that

$$\begin{aligned}\text{L.H.S. of (5.3)}=&\int_{\partial B(0,r)} r\langle D(G_n-\varphi_n), D\xi_n\rangle\, d\sigma-2\int_{\partial B(0,r)} r\langle \nu, D(G_n-\varphi_n)\rangle\langle \nu, D\xi_n\rangle\, d\sigma\\&+O(\sigma_n\|D\xi_n\|_{L^\infty(\partial B(0,r))})\\=&\int_{\partial B(0,r)}\frac{\rho_n}{2\pi}\langle D\xi_n,\nu\rangle\, d\sigma+O(\sigma_n\|D\xi_n\|_{L^\infty(\partial B(0,r))})\\=&\int_{\partial B(0,r)}4(1+\alpha)\langle D\xi_n,\nu\rangle\, d\sigma+O(\sigma_n\|D\xi_n\|_{L^\infty(\partial B(0,r))}),\end{aligned}\tag{5.7}$$

where we used (2.3). Therefore, as a consequence of Lemma 4.3, we conclude that

$$\text{L.H.S. of (5.3)}=4(1+\alpha)\int_{\partial B(0,r)}\langle D\xi_n,\nu\rangle\, d\sigma+O(\sigma_n|A_n|)+o(\sigma_n^2).\tag{5.8}$$

In this particular case, we have $A_n=0$.

To estimate the right hand side of (5.8), we need a refined estimate about ξ_n on $\partial B(0,r)$. So, by the Green representation formula with $x \in \partial B(0,r)$, we find that

$$\begin{aligned}\xi_n(x) =& -d_n + \int_\Omega G(y,x) f_n^*(y)dy \\ =& -d_n + A_n G(0,x) + \sum_{h=1}^{2} B_{n,h}\partial_{y_h} G(y,x)\mid_{y=0} + \frac{1}{2}\sum_{h,k=1}^{2} C_{n,h,k}\partial^2_{y_h y_k} G(y,x)\mid_{y=0} \\ &+ \int_\Omega \Psi_n(y,x) f_n^*(y),\end{aligned} \tag{5.9}$$

where

$$A_n = \int_\Omega f_n^*(y)dy, \quad B_{n,h} = \int_{B(0,r)} y_h f_n^*(y)dy, \quad C_{n,h,k} = \int_{B(0,r)} y_h y_k f_n^*(y),$$

and

$$\begin{aligned}\Psi_n(y,x) = G(y,x) - G(0,x) - \langle \partial_y G(y,x)\mid_{y=0}, y\rangle 1_{\partial B(0,r)}(y) \\ - \frac{1}{2}\left\langle \partial_y^2 G(y,x)\mid_{y=0} y, y\right\rangle 1_{B(0,r)}(y).\end{aligned}$$

At this point, let us fix $\overline{\theta} \in (0, \frac{r}{2})$. By Lemma 3.1 and Lemma 4.2, we find that,

$$\begin{aligned}f_n^*(y) &= \rho_n \overline{h}_1 |y|^{2\alpha} e^{\tilde{u}_n^{(1)}}(\xi_n(y) + O(\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(\Omega)}) \\ &= \rho_n \overline{h}_1 |y|^{2\alpha} e^{\tilde{u}_n^{(1)}}(-b_0 + o(1)),\end{aligned} \tag{5.10}$$

for any $y \in \partial\Omega \setminus B(0,\overline{\theta})$. By (2.4), (2.12), (2.13) and (5.10), we conclude that

$$\begin{aligned}f_n^*(y) &= \rho_n \overline{h}_1 |y|^{2\alpha} e^{\rho_{n,1}^{(1)} G(y,0) - \lambda_n - 2\log(\gamma_n) - 8\pi(1+\alpha)R(0,0)}(-b_0 + o(1)) \\ &= (8(1+\alpha)^2)^2 \frac{e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} |y|^{2\alpha} e^{\Phi(y,0)}(-b_0 + o(1)) \quad \text{for} \quad y \in \Omega \setminus B(0,\overline{\theta}),\end{aligned} \tag{5.11}$$

where

$$\Phi(y,0) = -4(1+\alpha)\log|y| + 8\pi(1+\alpha)(R(y,0) - R(0,0)) + \log(\overline{h}_1(0)) - \log(\overline{h}_1(0)).$$

On the other hand, by (2.6), we have for $y \in B(0,\overline{\theta})$,

$$f_n^*(y) = \rho_n h e^{\tilde{u}_n^{(1)}}(\xi_n + O(\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(\Omega)})) = O\left(\frac{|y|^{2\alpha} e^{\lambda_n}}{(1 + e^{\lambda_n}|y|^{2+2\alpha})^2}\right). \tag{5.12}$$

Next, by (5.10), for $y \in B(0,\overline{\theta})$ and $x \in \partial B(0,r)$, we get

$$\Psi_n(y,x) = O\left(\frac{|y|^3}{|x|^3}\right), \quad \text{and} \quad \nabla_x \Psi_n(y,x) = O\left(\frac{|y|^3}{|x|^4}\right). \tag{5.13}$$

Let us define

$$\overline{G}_n(x) = A_n G(0,x) + \sum_{h=1}^{2} B_{n,h}\partial_{y_h} G(y,x)\mid_{y=0} + \frac{1}{2}\sum_{h,k=1}^{2} C_{n,h,k}\partial^2_{y_h y_k} G(y,x)\mid_{y=0}, \tag{5.14}$$

so that, by (5.11)-(5.13), we conclude that for $x \in \partial B(0,r)$, it holds

$$\begin{aligned}
\xi_n(x) + d_n - \overline{G}_n(x) &= \int_{\Omega \setminus B(0,\bar{\theta})} \Psi_n(y,x) f_n^*(y) dy + \int_{B(0,\bar{\theta})} \Psi_n(y,x) f_n^*(y) dy \\
&= -b_0 \int_{\Omega \setminus B(0,\bar{\theta})} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} \Psi_n(y,x) |y|^{2\alpha} e^{\Phi(y,0)} dy \\
&\quad + O(\int_{B(0,\bar{\theta})} \frac{|y|^3}{|x|^3} \frac{|y|^{2\alpha} e^{\lambda_n}}{(1+e^{\lambda_n}|y|^{2+2\alpha})^2} dy) + o(e^{-\lambda_n}) \\
&= -b_0 \int_{\Omega \setminus B(0,\bar{\theta})} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} \Psi_n(y,x) |y|^{2\alpha} e^{\Phi(y,0)} dy \\
&\quad + O\left(\frac{m_{n,\alpha}}{|x|^3}\right) + o(e^{-\lambda_n}) \text{ in } C^1(\partial B(0,r)),
\end{aligned} \tag{5.15}$$

where

$$m_{n,\alpha} = \begin{cases} \sigma_n^3, \quad \text{if } 2\alpha > 1, \\ \sigma_n^3 \log(\sigma_n^{-1}), \quad \text{if } 2\alpha = 1, \\ \sigma_n^{2+2\alpha} \overline{\theta}^{1-2\alpha}, \quad \text{if } 2\alpha < 1. \end{cases}$$

Let us set

$$\zeta_n^*(x) = -b_0 \int_{\Omega \setminus B(0,\bar{\theta})} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} \Psi_n(y,x) |y|^{2\alpha} e^{\Phi(y,0)} dy \tag{5.16}$$

and then subsititute (5.15) into (5.8), to derive that

$$\text{L.H.S. of (5.3)} = \int_{\partial B(0,r)} 4(1+\alpha) \left\langle \nu, D(\overline{G}_n + \zeta_n^*)(x) \right\rangle d\sigma + O(\sigma |A_n|) + O(\frac{m_{n,\alpha}}{r^3}) + o(\sigma_n^2). \tag{5.17}$$

To estimate the right hand side of (5.17), we notice that for any pair of (smooth enough) functions u and v, it holds

$$\begin{aligned}
&\Delta u(\nabla v \cdot x) + \Delta v(\nabla u \cdot x) \\
&= \operatorname{div}\left(\nabla u(\nabla v \cdot x) + \nabla v(\nabla u \cdot x) - \nabla u \cdot \nabla v(x)\right).
\end{aligned} \tag{5.18}$$

In view of (5.14), we also see that, for any $\underline{\theta} \in (0,r)$,

$$\Delta \overline{G}_n(x) = A_n = \int_\Omega f_n^* dy = \int_\Omega \frac{\rho_n h(e^{\tilde{u}_n^{(1)}} - e^{\tilde{u}_n^{(2)}})}{\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(\Omega)}} = 0 \text{ for } x \in B(0,r) \setminus B(0,\underline{\theta}), \tag{5.19}$$

and moreover, by using (5.5) and (5.1), we have

$$\Delta(G_n - \varphi_n)(x) = 0 \quad \text{for} \quad x \in B(0,r) \setminus B(0,\underline{\theta}). \tag{5.20}$$

By using (5.18)-(5.20) and (5.6), we conclude that

$$0 = \int_{B(0,r)\setminus B(0,\underline{\theta})} \left[\Delta \overline{G}_n (\nabla(G_n - \varphi_n)\cdot x) + \Delta(G_n - \varphi_n)(\nabla \overline{G}_n \cdot x)\right] dx$$

$$= \int_{\partial(B(0,r)\setminus B(0,\underline{\theta}))} \left(\frac{\partial \overline{G}_n}{\partial \nu}(\nabla(G_n - \varphi_n)\cdot x) + \frac{\partial(G_n - \varphi_n)}{\partial \nu}(\nabla \overline{G}_n \cdot x) - \nabla \overline{G}_n \cdot \nabla(G_n - \varphi_n) \langle x, \nu \rangle \right) d\sigma$$

$$= -\frac{\rho_n}{2\pi} \int_{\partial(B(0,r)\setminus B(0,\underline{\theta}))} \frac{\partial \overline{G}_n}{\partial \nu} d\sigma,$$

and thus,

$$\int_{\partial B(0,r)} \frac{\partial \overline{G}_n}{\partial \nu}(x) d\sigma = \int_{\partial B(0,\underline{\theta})} \frac{\partial \overline{G}_n}{\partial \nu}(x) d\sigma. \tag{5.21}$$

At this point, let us denote by $o_{\underline{\theta}}(1)$ any quantity which converges to 0 as $\underline{\theta} \to 0^+$, and then observe that,

$$4(1+\alpha) \int_{\partial B(0,\underline{\theta})} \langle \nu, A_n D_x G(0,x) \rangle \, d\sigma = -4(1+\alpha) A_n + o_{\underline{\theta}}(1). \tag{5.22}$$

Since, $D_i D_h \log|x| = \frac{\delta_{ih}|x|^2 - 2x_i x_h}{|x|^4}$, then we find that,

$$\int_{\partial B(0,\underline{\theta})} \langle \nu, D_x \partial_{y_h} (\log|y-x|) \mid_{y=0} \rangle \, d\sigma = -\int_{\partial B(0,\underline{\theta})} \sum_{i=1}^{2} \frac{x_i}{|x|} \left(\frac{\delta_{ih}|x|^2 - 2x_i x_h}{|x|^4} \right) d\sigma = 0. \tag{5.23}$$

We observe that, if $h = k$ then $D_i \log|x| = \frac{x_i}{|x|^2}$,

$$D_i D^2_{hh} \log|x| = -\frac{2x_i}{|x|^4} - \frac{4x_h \delta_{ih}}{|x|^4} + \frac{8x_h^2 x_i}{|x|^6},$$

and thus,

$$\int_{\partial B(0,\underline{\theta})} \left\langle \nu, D_x \frac{\partial^2}{\partial y_h^2} \log \frac{1}{|y-x|} \mid_{y=0} \right\rangle d\sigma = \int_{\partial B(0,\underline{\theta})} \left(\frac{2}{|x|^3} - \frac{4x_h^2}{|x|^5} \right) d\sigma = 0. \tag{5.24}$$

If $h \neq k$, then

$$D_i D^2_{hk} \log|x| = -\frac{2(x_h \delta_{ki} + x_k \delta_{hi})}{|x|^4} + \frac{8x_k x_i x_h}{|x|^6},$$

which implies that

$$\int_{\partial B(0,\underline{\theta})} \left\langle \nu, D_x \frac{\partial^2}{\partial y_h \partial y_k} \log \frac{1}{|y-x|} \mid_{y=0} \right\rangle d\sigma = \int_{\partial B(0,\underline{\theta})} \left(\frac{4x_h x_k}{|x|^5} - \frac{8x_h x_k}{|x|^5} \right) d\sigma = 0. \tag{5.25}$$

By (5.21)-(5.25), we conclude that

$$4(1+\alpha) \int_{\partial B(0,r)} \langle \nu, D_x \overline{G}_n(x) \rangle \, d\sigma = -4(1+\alpha) A_n + o_{\underline{\theta}}(1). \tag{5.26}$$

Next we estimate the other terms in (5.17), that is $4(1+\alpha) \int_{\partial B(0,r)} \langle \nu, D_x \zeta_n^*(x) \rangle \, d\sigma$, where ζ_n^* is defined in (5.16). Clearly we have

$$D_x \Psi_n(y,x) = D_x \left(G(y,x) - G(0,x) - \langle \partial_y G(y,x) \mid_{y=0}, y \rangle 1_{B(0,r)}(y) \right)$$
$$- \frac{1}{2} D_x \left(\frac{1}{2} \left\langle \partial_y^2 G(y,x) \mid_{y=0} (y), y \right\rangle 1_{B(0,r)}(y) \right).$$

If $y \in \Omega \setminus B(0,\overline{\theta})$ and $x \in \partial B(0,\theta)$ with $\theta \ll (\overline{\theta})^2$, then we find that

$$|D_x G(x,y)| \le \frac{C}{\sqrt{\theta}} \text{ for some constant } C > 0, \tag{5.27}$$

which implies

$$\int_{\partial B(0,\theta)} \langle \nu, D_x G(y,x)\rangle \, dx = o_\theta(1).$$

Thus (5.23)-(5.25) and (5.27) imply that

$$\begin{aligned}
&4(1+\alpha)\int_{\partial B(0,\theta)} \langle \nu, D_x \Psi_n(y,x)\rangle \, dx \\
&= -4(1+\alpha)\int_{\partial B(0,\theta)} \langle \nu, D_x G(0,x)\rangle \, dx \\
&\quad - 4(1+\alpha)\int_{\partial B(0,\theta)} \Big\langle \nu, D_x \left\langle \partial_y G(y,x)\mid_{y=0}, y\right\rangle 1_{B(0,r_0)}(y)\Big\rangle \, dx \\
&\quad - 2(1+\alpha)\int_{\partial B(0,\theta)} \Big\langle \nu, D_x \left\langle \partial_y^2 G(y,x)\mid_{y=0} y, y\right\rangle 1_{B(0,r_0)}(y)\Big\rangle \, dx + o_\theta(1) \\
&= 4(1+\alpha) + o_\theta(1) \quad \text{for} \quad y \in \Omega\setminus B(0,\overline{\theta}), \quad \text{and} \quad x \in \partial B(0,\theta).
\end{aligned} \tag{5.28}$$

We observe that

$$-\Delta_x \Psi_n(y,x) = \delta_y \quad \text{for} \quad x \in B(0,r)\setminus B(0,\theta)$$

and let us choose $u(x) = \Psi_n(y,x)$ and $v(x) = G_n(x) - \varphi_n(x)$ in (5.18). Then we consider the following two cases:

(i) If $y \in \partial B(0,r)\setminus B(0,\overline{\theta})$, then from (5.18) and (5.28), we obtain that

$$\begin{aligned}
&4(1+\alpha)\int_{\partial B(0,r)} \langle \nu, D_x \Psi_n(y,x)\rangle \, dx \\
&= 4(1+\alpha)\int_{\partial B(0,\theta)} \langle \nu, D_x \Psi_n(y,x)\rangle \, dx - 4(1+\alpha) = o_\theta(1).
\end{aligned} \tag{5.29}$$

(ii) If $y \in \Omega\setminus B(0,r)$, then we see from (5.18) and (5.28) that

$$\begin{aligned}
4(1+\alpha)\int_{\partial B(0,r)} \langle \nu, D_x \Psi_n(y,x)\rangle \, dx &= 4(1+\alpha)\int_{\partial B(0,\theta)} \langle \nu, D_x \Psi_n(y,x)\rangle \, dx \\
&= 4(1+\alpha) + o_\theta(1).
\end{aligned} \tag{5.30}$$

and by (5.16), and (5.29)-(5.30), we finally conclude that

$$\begin{aligned}
&4(1+\alpha)\int_{\partial B(0,r)} \langle \nu, D_x \zeta_n^*(x)\rangle \, dx \\
&= -\frac{(8(1+\alpha)^2)^3 b_0 e^{-\lambda_n}}{2\rho_n \overline{h}_1(0)} \int_{\Omega\setminus B(0,\overline{\theta})} \left(\int_{\partial B(0,r)} \langle \nu, D_x \Psi_n\rangle\right) |y|^{2\alpha} e^{\Phi(y,0)} dx dy \\
&= -\frac{(8(1+\alpha)^2)^3 b_0 e^{-\lambda_n}}{2\rho_n \overline{h}_1(0)} \int_{\Omega\setminus B(0,r)} |y|^{2\alpha} e^{\Phi(y,0)} dy + o(e^{-\lambda_n}).
\end{aligned} \tag{5.31}$$

Obviously from (5.17), (5.26) and (5.31) we get the conclusion of Lemma 5.2 □

To estimate the right hand side of (5.3) of Lemma 5.1, we recall, see for example (5.10), that

$$f_n^*(x) = \rho_n h(x) e^{\tilde{u}_n^{(1)}} (\xi_n + o(1)).$$

Recall also the definitions of $\Phi(x,0)$ and $\mathcal{H}_p = \mathcal{H}_0$ in (5.4) and (1.4), respectively and the definition of $\ell(p)$ after (2.4). A crucial point in our proof is the following estimate.

Lemma 5.3. (i)

$$\int_{\partial B(0,r)} r f_n^* d\sigma = -\frac{128(1+\alpha)^4 b_0 \pi e^{-\lambda_n}}{\rho_n \overline{h}_1(0) r^{2+2\alpha}} - \frac{32(1+\alpha)^4 b_0 \pi e^{-\lambda_n}}{\rho_n \overline{h}_1(0) r^{2\alpha}} \Delta \log h_*(0) + O(r^{1-2\alpha} e^{-\lambda_n}) + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}},$$

(ii)

$$\int_{B(0,r)} f_n^*(x)dx = \frac{64(1+\alpha)^4 b_0 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} \int_{\Omega\setminus B(0,r)} |x|^{2\alpha} e^{\Phi(x,0)} dx + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}},$$

(iii)

$$\begin{aligned}
&\int_{B(0,r)} f_n^* \left\langle D(\log \overline{h}_1 + \varphi_n), x \right\rangle dx \\
&= -2b_0 \ell(p) \sigma_n^2 + o(\sigma_n)|\nabla \mathcal{H}_0(0)| + O(m_{n,1}(\alpha)) \\
&\quad + O(\sigma_n^{2\epsilon_0} + \lambda_n \sigma_n^2)\left(|\nabla \mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) + O(\sigma_n^{2+\epsilon_0}) \\
&\quad + \left(O(R^{-2\alpha}) + O(\lambda_n)|A_n| + O(\frac{1}{R})\right)\left(|\nabla \mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right).
\end{aligned}$$

where $O(m_{n,1}(\alpha))$ is defined after (5.36) *and $O(1)$ is used to denote any quantity uniformly bounded with respect to r, R and n.*

Proof. (i) We first observe that (5.11) implies that

$$\int_{\partial B(0,r)} r f_n^*(x) d\sigma = \int_{\partial B(0,r)} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n}(-b_0 + o(1))|x|^{2\alpha} e^{\mathcal{H}_0(x)}}{\rho_n \overline{h}_1(0)|x|^{3+4\alpha}} d\sigma. \quad (5.32)$$

Clearly we have

$$\mathcal{H}_0(x) = \langle D\mathcal{H}_0(0), x\rangle + \frac{1}{2}\left\langle D_x^2 \mathcal{H}_0 \mid_{x=0} x, x\right\rangle + O(|x|^3). \quad (5.33)$$

By (5.32) and (5.33), we obtain,

$$\begin{aligned}
&\int_{\partial B(0,r)} r f_n^*(x) d\sigma \\
&= \int_{\partial B(0,r)} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)|x|^{3+3\alpha}} \left(b_0(1 + \langle D\mathcal{H}_0, x\rangle + \frac{1}{2}\left\langle D_x^2 \mathcal{H}_0 \mid_{x=0} x, x\right\rangle) + O(|x|^3) + o(1)\right) d\sigma \\
&= -\int_{\partial B(0,r)} \frac{(8(1+\alpha)^2)^2 e^{-\lambda_n} b_0 (1 + \frac{\Delta \mathcal{H}_0}{4}|x|^2)}{\rho_n \overline{h}_1(0)|x|^{3+2\alpha}} d\sigma + O(r^{1-2\alpha} e^{-\lambda_n}) + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}} \\
&= -\frac{128(1+\alpha)^4 b_0 \pi e^{-\lambda_n}}{\rho_n \overline{h}_1(0) r^{2+2\alpha}} - \frac{32(1+\alpha)^4 b_0 \pi e^{-\lambda_n}}{\rho_n \overline{h}_1(0) r^{2\alpha}} \Delta \log(h_*(0)) + O(r^{1-2\alpha} e^{-\lambda_n}) + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}},
\end{aligned}$$

which proves (i).

(ii) We notice that $A_n = \int_\Omega f_n^* = 0$, and thus

$$\int_{B(0,r)} f_n^*(x)dx = -\int_{\Omega\setminus B(0,r)} f_n^*(x)dx. \quad (5.34)$$

By (5.11) we see that

$$
\begin{aligned}
-\int_{\Omega\setminus B(0,r)} f_n^* dx &= \int_{\Omega\setminus B(0,r)} \frac{64(1+\alpha)^4 b_0 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} |x|^{2\alpha} e^{\Phi(x,0)} dx + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}} \\
&= \frac{64(1+\alpha)^4 b_0 e^{-\lambda_n}}{\rho_n \overline{h}_1(0)} \int_{\Omega\setminus B(0,r)} |x|^{2\alpha} e^{\Phi(x,0)} dx + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}},
\end{aligned} \tag{5.35}
$$

which proves (ii).

(iii) By (2.3) and (2.6), we see that

$$\tilde{u}_n(x) = U_n(x) + 8\pi(1+\alpha)(R(x,0) - R(0,0)) + \eta_n(x), \quad x \in B(0,r),$$

where

$$\eta_n(x) = \sigma_n \psi_{n,1}(\sigma_n^{-1} x) + \sigma_n^2 \psi_{n,2}(\sigma_n^{-1} x) + O(\sigma_n^2),$$

see (2.7), (2.8) and (2.10). Thus, we set

$$\omega_n(r) = \| \tilde{u}_n^{(1)} - \tilde{u}_n^{(2)} \|_{L^\infty(B(0,r))},$$

and use Lemma 3.1 and (2.4), we deduce that

$$
\begin{aligned}
&\int_{B(0,r)} f_n^* \left\langle D(\log \overline{h}_1(x) + \varphi_n), x \right\rangle dx \\
&= \int_{B(0,r)} \frac{\rho_n \overline{h}_1(0)|x|^{2\alpha} e^{\lambda_n + \mathcal{H}_0(x) + \eta_n(x)}}{(1+\gamma_n e^{\lambda_n}|x|^{2+2\alpha})^2} (\xi_n - \frac{\omega_n(r)}{2}\xi_n^2 + O(\omega_n^2(r))) \langle D\mathcal{H}_0(x), x \rangle dx \\
&= \int_{B(0,r)} \frac{\rho_n \overline{h}_1(0)|x|^{2\alpha} e^{\lambda_n + \mathcal{H}_0(x) + \eta_n(x)}}{(1+\gamma_n e^{\lambda_n}|x|^{2+2\alpha})^2} (\xi_n - \frac{\omega_n(r)}{2}\xi_n^2 + O(\omega_n^2(r))) \\
&\quad \times \left\langle D\mathcal{H}_0(0) + D^2\mathcal{H}_0(0)x + O(|x|^2), x \right\rangle dx \\
&= \int_{B(0,\sigma_n^{-1}r)} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}}{(1+\gamma_n |z|^{2+2\alpha})^2} \left(\hat{\xi}_n - \frac{\omega_n}{2}\hat{\xi}_n^2 + \langle D\mathcal{H}_0(0), \sigma_n z \rangle + \eta_n + O(\sigma_n^2|z|^2) + O(\omega_n^2) \right) \\
&\quad \times \left\langle D\mathcal{H}_0(0) + D^2\mathcal{H}_0(0)\cdot \sigma_n z + O(\sigma_n^2|z|^2), \sigma_n z \right\rangle dz =: K_{n,r}.
\end{aligned} \tag{5.36}
$$

Set

$$
m_{n,1}(\alpha) = \begin{cases} \sigma_n^3 \text{ if } \alpha > \frac{1}{2} \\ \log(r\sigma_n^{-1})\sigma_n^3 \text{ if } \alpha = \frac{1}{2} \\ r^{1-2\alpha}\sigma_n^{2(1+\alpha)} \text{ if } \alpha \in (0,\frac{1}{2}) \end{cases}, \quad m_{n,2}(\alpha) = \begin{cases} \sigma_n^4 \text{ if } \alpha > 1 \\ \log(r\sigma_n^{-1})\sigma_n^4 \text{ if } \alpha = 1 \\ r^{2-2\alpha}\sigma_n^{2(1+\alpha)} \text{ if } \alpha \in (0,1) \end{cases},
$$

$$
m_{n,3}(\alpha) = \begin{cases} \sigma_n^5 \text{ if } \alpha > \frac{3}{2} \\ \log(r\sigma_n^{-1})\sigma_n^5 \text{ if } \alpha = \frac{3}{2} \\ r^{3-2\alpha}\sigma_n^{2(1+\alpha)} \text{ if } \alpha \in (0,\frac{3}{2}) \end{cases}.
$$

Using (5.36) together with (2.5), (2.7) and Lemma 3.1, we conclude that

$$
\begin{aligned}
K_{n,r} &= \int_{B(0,r\sigma_n^{-1})} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}}{(1+\gamma_n|z|^{2(1+\alpha)})^2} \\
&\quad \times \left(\hat{\xi}_n - \frac{\omega_n(r)}{2}(\hat{\xi}_n)^2 + \sigma_n(\psi_{n,1}(z) + < D\mathcal{H}_0(0), z >) + O(\sigma_n^2|z|^2) + O((\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)^2) + \sigma_n^2 \tilde{\psi}_{n,2}(z) \right) \\
&\quad \times \left\langle D\mathcal{H}_0(0) + D^2\mathcal{H}_0(0)\cdot\sigma_n z + O(\sigma_n^2|z|^2), \sigma_n z \right\rangle dz
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{B(0,r\sigma_n^{-1})} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}\left(\hat{\xi}_n - \frac{\omega_n(r)}{2}(\hat{\xi}_n)^2 + \sigma_n(\psi_{n,1}(z) + < D\mathcal{H}_0(0), z >)\right)}{(1+\gamma_n|z|^{2(1+\alpha)})^2} \\
&\times < D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z > \sigma_n dz + O(m_{n,1}(\alpha) + m_{n,2}(\alpha) + m_{n,3}(\alpha)) \\
&+ \left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) O((\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)^2) \\
&= I_{n,1} + I_{n,2} + O(m_{n,1}(\alpha)) + \left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) O(\sigma_n^{4\epsilon_0}),
\end{aligned}
$$

where

$$
I_{n,1} = \int_{B(0,r\sigma_n^{-1})} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}(\hat{\xi}_n - \frac{\omega_n}{2}\hat{\xi}_n^2)}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z\right\rangle \sigma_n dz,
$$

and

$$
I_{n,2} = \int_{B(0,r\sigma_n^{-1})} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}(\psi_{n,1}(z) + \langle D\mathcal{H}_0(0), z\rangle)}{(1+\gamma_n|z|^{2+2\alpha})^2} \langle D\mathcal{H}_0(0), z\rangle \sigma_n^2 dz.
$$

In view of (2.9), (2.10), (2.11) and (2.4), we have

$$
\langle D\mathcal{H}_0(0), z\rangle = \partial_{x_1}\mathcal{H}_0(0)z_1 + O(\sigma_n^2)(z_1+z_2) = a_{n,1}z_1 + O(\sigma_n^2)(z_1+z_2),
$$

and then, putting $a_1 = \partial_{x_1}\mathcal{H}_0$ and $\Lambda(z) = \rho_n\overline{h}_1(0)|z|^{2\alpha}$, we conclude that

$$
\begin{aligned}
\sigma_n^{-2} I_{n,2} &= \int_{B(0,r\sigma_n^{-1})} \frac{\Lambda(z)(\psi_{n,1}(z) + a_1 z_1 + O(\sigma_n^2)(z_1+z_2))}{(1+\gamma_n|z|^{2+2\alpha})^2}(a_1 z_1 + O(\sigma_n^2)(z_1+z_2))dz \\
&= \int_{B(0,r\sigma_n^{-1})} \frac{\Lambda(z)}{(1+\gamma_n|z|^{2+2\alpha})^2}\left(-\frac{2(1+\alpha)a_1 z_1}{\alpha(1+\gamma_n|z|^{2+2\alpha})} + a_1 z_1 + O(\sigma_n^2)(z_1+z_2)\right) \\
&\quad \times (a_1 z_1 + O(\sigma_n^2)(z_1+z_2))dz \\
&= -\int_{B(0,r\sigma_n^{-1})} \frac{2(1+\alpha)\Lambda(z)a_1^2 z_1^2}{\alpha(1+\gamma_n|z|^{2+2\alpha})^3}dz + \int_{B(0,r\sigma_n^{-1})} \frac{\Lambda(z)a_1^2 z_1^2}{(1+\gamma_n|z|^{2+2\alpha})^2}dz + O(\sigma_n^2) \\
&= -\rho_n\overline{h}_1(0)\frac{a_1^2\pi^2}{2(1+\alpha)^2\gamma_n^{\frac{2+\alpha}{1+\alpha}}}\frac{1}{\sin\frac{\pi}{1+\alpha}} + \rho_n\overline{h}_1(0)\frac{a_1^2\pi}{2\alpha\gamma_n^{\frac{2+\alpha}{1+\alpha}}}\Gamma(\frac{2+\alpha}{1+\alpha})\Gamma(\frac{1+2\alpha}{1+\alpha}) \\
&\quad + O(\sigma_n^{2\alpha}) + O(\sigma_n^2) \\
&= O(\sigma_n^{2\alpha}) + O(\sigma_n^2),
\end{aligned}
$$

where we used the properties of $\Gamma(x)$, and thus

$$
I_{n,2} = O(\sigma_n^{2+\epsilon_0}). \tag{5.37}
$$

On the other hand, in view of Lemma 4.1, for any fixed $R \geq 1$ large, we have

$$
\begin{aligned}
&\int_{B(0,R)} \frac{\Lambda(z)(\hat{\xi}_n - \frac{\omega_n}{2}\hat{\xi}_n^2)}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z \right\rangle \sigma_n dz \\
&= \int_{B(0,R)} \frac{\Lambda(z)(b_0\hat{\xi}_0(z) + o(1) + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2))}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z \right\rangle \sigma_n dz \\
&= \sigma_n^2 \int_{B(0,R)} \frac{\Lambda(z)(b_0\hat{\xi}_0(z) + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2))}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D^2\mathcal{H}_0(0)\cdot z, z \right\rangle dz \\
&\quad + o(\sigma_n)|\nabla\mathcal{H}_0(0)| + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&= \sigma_n^2 \int_{B(0,R)} \frac{\Lambda(z)(b_0\hat{\xi}_0(z)}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D^2\mathcal{H}_0(0)\cdot z, z \right\rangle dz + o(\sigma_n)|\nabla\mathcal{H}_0(0)| \\
&\quad + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&= 8\pi(1+\alpha)\overline{h}_1(0)b_0\sigma_n^2 \int_{B(0,R)} \frac{|z|^{2\alpha}\hat{\xi}_0(z)}{(1+\gamma|z|^{2+2\alpha})^2} \left\langle D^2\mathcal{H}_0(0)\cdot z, z \right\rangle dz \\
&\quad + o(\sigma_n)|\nabla\mathcal{H}_0(0)| + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right).
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
&\int_{B(0,R)} \frac{|z|^{2\alpha}\hat{\xi}_0(z)}{(1+\gamma|z|^{2+2\alpha})^2} \left\langle D^2\mathcal{H}_0(0)\cdot z, z \right\rangle dz = \frac{\Delta\mathcal{H}_0(0)}{2} \int_{B(0,R)} \frac{1-\gamma|z|^{2+2\alpha}}{(1+\gamma|z|^{2+2\alpha})^3}|z|^{2\alpha+2} dz \\
&= -\frac{\Delta\mathcal{H}_0(0)}{2} \frac{\pi^2}{(1+\alpha)^3\gamma^{\frac{2+\alpha}{1+\alpha}} \sin\frac{\pi}{1+\alpha}} + O(R^{-2\alpha}) \\
&= -\frac{\pi}{2(1+\alpha)^2\overline{h}_1(0)\gamma^{\frac{1}{1+\alpha}} \sin\frac{\pi}{1+\alpha}} \Delta\log(h_*(0)) + O(R^{-2\alpha}).
\end{aligned}
$$

On the other side, in view of (4.9), we also see that if $R \leq |z| \leq r/\sigma_n$, then it holds

$$
\hat{\xi}_n(z) = -d_n + O(\lambda_n)|A_n| + O(\frac{1}{|z|}), \tag{5.38}
$$

and thus

$$
\hat{\xi}_n(z)^2 = d_n^2 + O(\lambda_n^2 + \frac{\lambda_n}{|z|})|A_n| + O(\frac{1}{|z|^2}).
$$

As a consequence, by Lemma 3.1, we find that

$$\begin{aligned}
&\int_{B(0,r/\sigma_n)\setminus B(0,R)} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}(\hat{\xi}_n - \frac{\omega_n}{2}\hat{\xi}_n^2)}{(1+\gamma_n|z|^{2+2\alpha})^2} \left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z\right\rangle \sigma_n dz \\
&= \left(-d_n - \frac{\omega_n(r)}{2}d_n^2\right)\int_{B(0,r/\sigma_n)\setminus B(0,R)} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}}{(1+\gamma_n|z|^{2+2\alpha})^2}\left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z\right\rangle \sigma_n dz \\
&\quad + \int_{B(0,r/\sigma_n)\setminus B(0,R)} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha}(O(\lambda_n)|A_n| + O(\frac{1}{|z|}))}{(1+\gamma_n|z|^{2+2\alpha})^2}\left\langle D\mathcal{H}_0(0) + \sigma_n D^2\mathcal{H}_0(0)\cdot z, z\right\rangle \sigma_n dz \\
&= -b_0\Delta\log h_*(0)\int_{B(0,r/\sigma_n)\setminus B(0,R)} \frac{\rho_n \overline{h}_1(0)|z|^{2\alpha+2}}{(1+\gamma_n|z|^{2+2\alpha})^2}\sigma_n^2 dz + O(\sigma_n^2(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)) \\
&\quad + \left(O(|\lambda_n|)|A_n| + O\left(\frac{1}{R}\right)\right)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&= O(R^{-2\alpha})\sigma_n^2 + \left(O(R^{-2\alpha}) + O(\lambda_n)|A_n| + O\left(\frac{1}{R}\right)\right)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&\quad + O(\sigma_n^{2+2\epsilon_0} + \lambda_n\sigma_n^4)
\end{aligned}$$

Collecting the above estimates we conclude that

$$\begin{aligned}
&\int_{B(0,r)} f_n^*\left\langle D(\log\overline{h}_1 + \varphi_n), x\right\rangle dx \\
&= -2b_0\ell(p)\sigma_n^2 + o(\sigma_n)|\nabla\mathcal{H}_0(0)| + O(m_{n,1}(\alpha)) + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&\quad + O(\sigma_n^{2+\epsilon_0}) + \left(O(R^{-2\alpha}) + O(\lambda_n)|A_n| + O(\frac{1}{R})\right)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right).
\end{aligned}$$

□

Recall that $p = 0$. Using the assumptions $\ell(p) \neq 0$ and $\nabla\mathcal{H}_0(0) = 0$ we can now prove that $b_0 = 0$.

Lemma 5.4. $b_0 = 0$.

Proof. By (5.3) and Lemmas 5.2-5.3, we have for any $r \in (0,1)$ and $R > 1$,

$$\begin{aligned}
&-4(1+\alpha)A_n - \frac{(8(1+\alpha)^2)^3 b_0 e^{-\lambda_n}}{2\rho_n\overline{h}_1(0)}\int_{\Omega\setminus B(0,r)}|y|^{2\alpha}e^{\Phi(y,0)}dy + o(\sigma_n^2) + O(\sigma_n|A_n| + \frac{\sigma_n^3}{r^3}) \\
&= -\frac{128(1+\alpha)^4 b_0\pi e^{-\lambda_n}}{\rho_n\overline{h}_1(0)r^{2+2\alpha}} - \frac{32(1+\alpha)^4 b_0\pi e^{-\lambda_n}}{\rho_n\overline{h}_1(0)r^{2\alpha}}\Delta\log h_*(0) + O(r^{1-2\alpha}e^{-\lambda_n}) \\
&\quad - \frac{128(1+\alpha)^5 b_0 e^{-\lambda_n}}{\rho_n\overline{h}_1(0)}\int_{\Omega\setminus B(0,r)}|y|^{2\alpha}e^{\Phi(y,0)}dy + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}} + 2b_0\ell(p)\sigma_n^2 \\
&\quad + o(\sigma_n)|\nabla\mathcal{H}_0(0)| + O(m_{n,1}(\alpha)) + O(\sigma_n^{2\epsilon_0} + \lambda_n\sigma_n^2)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right) \\
&\quad + O(\sigma_n^{2+\epsilon_0}) + \left(O(R^{-2\alpha}) + O(\lambda_n)|A_n| + O(\frac{1}{R})\right)\left(|\nabla\mathcal{H}_0(0)|\sigma_n + \sigma_n^2\right).
\end{aligned}$$

Recall $A_n = 0$. Since $\nabla \mathcal{H}_0(0) = 0$ by assumption, after some manipulations, for $r \in (0, r_0)$ and any $R > 1$, we find that

$$b_0 \ell(p) \sigma_n^2 = o(\sigma_n^2) + O(m_{n,1}(\alpha)) + O(R^{-2\alpha} + R^{-1})\sigma_n^2 + O(\sigma_n^{2+2\epsilon_0} + \lambda_n \sigma_n^4) + O\left(\frac{\sigma_n^3}{r^3}\right) + \frac{o(e^{-\lambda_n})}{r^{2+2\alpha}},$$

which implies

$$b_0 = 0.$$

provided $\ell(p) \neq 0$. Hence we finish the proof. □

Proof of Theorem 1.1. Let x_n^* be a maximum point of ξ_n, then we have,

$$|\xi_n(x_n^*)| = 1. \tag{5.39}$$

By Lemma 4.2 and Lemma 5.4 we have that $x_n^* \to p$. By Lemma 5.4, it holds that

$$\lim_{n \to +\infty} e^{\frac{\lambda_n^{(1)}}{2(1+\alpha)}} s_n = +\infty, \quad \text{where } s_n = |x_n^* - p|. \tag{5.40}$$

Setting $\bar{\xi}_n(x) = \xi(s_n x + p)$, then we have $\bar{\xi}_n$ satisfies

$$\begin{aligned} 0 &= \Delta \bar{\xi}_n + \rho_n s_n^2 h(s_n x + p) c_n(s_n x + p) \bar{\xi}_n \\ &= \Delta \bar{\xi}_n + \frac{\rho_n \bar{h}_1(p) |x|^{2\alpha} s_n^{2+2\alpha} e^{\lambda_n^{(1)}} (1 + O(s_n |x|) + o(1)) \bar{\xi}_n}{(1 + \frac{\rho_n \bar{h}_1(p)}{8(1+\alpha)^2} e^{\lambda_n^{(1)}} |s_n x|^{2+2\alpha})^2}. \end{aligned}$$

On the other hand, by (5.39), we also have

$$\left| \bar{\xi}_n \left(\frac{x_n^* - p}{s_n} \right) \right| = |\xi_n(x_n^*)| = 1. \tag{5.41}$$

In view of (5.40) and $|\bar{\xi}_n| \leq 1$ we see that $\bar{\xi}_n \to \bar{\xi}_0$ on any compact subset of $\mathbb{R}^2 \setminus \{0\}$, where $\bar{\xi}_0$ satisfies $\Delta \bar{\xi}_0 = 0$ in $\mathbb{R}^2 \setminus \{0\}$. Since $|\bar{\xi}_0| \leq 1$, we have $\Delta \bar{\xi}_0 = 0$ in $\mathbb{R}^2$, which implies $\bar{\xi}_0$ is a constant. At this point, since $\frac{|x_n^* - p|}{s_n} = 1$ and in view of (5.41), we find $\bar{\xi}_0 = 1$ or $\bar{\xi}_0 = -1$. From which we have $|\bar{\xi}_n(x)| \geq \frac{1}{2}$ when $s_n \leq |x - p| \leq \frac{1}{2} s_n$, which contradicts to (4.6)-(4.8) since $e^{-\frac{\lambda_n^{(1)}}{2(1+\alpha)}} \ll s_n$ and $\lim_{n \to +\infty} s_n = 0$ and $b_0 = 0$. This fact concludes the proof of Theorem 1.1. □

6. The proof of Theorem 1.2

In this section we give the proof of the non-degeneracy result stated in Theorem 1.2. Since the argument is similar to the one yielding local uniqueness of bubbling solutions we will be sketchy to avoid repetitions, referring to [9] for full details.

Suppose by contradiction the linearized problem (1.2) admits a non-trivial solution ϕ_n, where u_n is a singular 1-bubbling solution of $(\mathbf{P}_{\rho_n})$ blowing up at the point

p_i for some $i \in \{1, \cdots, N\}$. We suppose with no loss of generality that $p_i = 0 \in \Omega$, set $\alpha_i = \alpha$ and

$$\tilde{u}_n = u_n - \log\left(\int_\Omega h e^{u_n} dx\right), \quad \lambda_n = \max_\Omega \tilde{u}_n, \quad \sigma_n^{2(1+\alpha)} = e^{-\lambda_n},$$

Define

$$\Xi_n = \frac{\phi_n - \frac{\int_\Omega h e^{u_n}\phi_n\,dx}{\int_\Omega h e^{u_n}\,dx}}{\left\|\phi_n - \frac{\int_\Omega h e^{u_n}\phi_n\,dx}{\int_\Omega h e^{u_n}\,dx}\right\|_{L^\infty(\Omega)}},$$

which plays the role of the difference of two bubbling solutions, see (4.1) in the proof of Theorem 1.1. Then, Ξ_n satisfies

$$\begin{cases} \Delta\,\Xi_n + \rho_n h(x) c_n(x)\,\Xi_n(x) = 0 & \text{in } \Omega, \\ \\ \Xi_n = -d_n & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

for some constant d_n satisfying $|d_n| \le 1$ and $c_n(x) = e^{\tilde{u}_n(x)}$.

Step 1. We start by considering the asymptotic behavior of Ξ_n near the blow up point p_i. After a suitable scaling, Ξ_n converges in $C^0_{\rm loc}(\mathbb{R}^2)$ to a solution $\hat{\xi}$ of the linearized problem

$$\Delta\hat{\xi} + \frac{8\gamma(1+\alpha)^2|z|^{2\alpha}}{(1+\gamma|z|^{2(1+\alpha)})^2}\hat{\xi} = 0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad |\hat{\xi}(z)| \le 1 \text{ in } \mathbb{R}^2,$$

where $\gamma = \frac{\pi\bar{h}_1(0)}{1+\alpha}$, see for example Lemma 4.1. It follows from [27, Corollary 2.2] that there exists a constant $b_0 \in \mathbb{R}$ such that

$$\Xi_n(\sigma_n z) \to b_0 \frac{1-\gamma|z|^{2+2\alpha}}{1+\gamma|z|^{2+2\alpha}} \quad \text{in } C^0_{\rm loc}(\mathbb{R}^2). \tag{6.2}$$

Step 2. We next consider the global behavior of Ξ_n away from the blow up point p_i. It follows from (2.2) that

$$c_n(x) \to 0 \quad \text{in} \quad C^0_{\rm loc}(\overline{\Omega} \setminus \{0\}).$$

Using then $\|\Xi_n\|_{L^\infty(\Omega)} \le 1$ and (6.1) it is not difficult to see that

$$\Xi_n \to \xi_0 \quad \text{in } C^0_{\rm loc}(\overline{\Omega} \setminus \{0\}), \quad \Delta\xi_0 = 0 \quad \text{in } \Omega.$$

Therefore $\xi_0 = -b$ in Ω for some constant b and

$$\Xi_n \to -b \quad \text{in} \quad C^0_{\rm loc}(\overline{\Omega} \setminus \{0\}). \tag{6.3}$$

Finally, by an O.D.E. argument as in Lemma 4.2 one can show $b = b_0$.

Step 3. We then study the asymptotic in the Pohozaev-type identity given by Lemma 5.1 (with suitable minor modifications, see for example [9]). Using the assumption $\nabla\mathcal{H}_{p_i}(p_i) = 0$ it is possible to prove that

$$b_0\ell(p_i) = o(1) \quad \text{for } n \text{ large},$$

see section 5. Since by assumption $\ell(p_i) \ne 0$ we deduce $b_0 = 0$.

Step 4. The contradiction is then obtained by a blow up argument using $b = b_0 = 0$ jointly with (6.2) and (6.3) exactly as in the proof of Theorem 1.1, see the end of section 5. The proof of Theorem 1.2 is completed.

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