Non-convex functionals penalizing simultaneous oscillations along independent directions: rigidity estimates

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Abstract

We study a family of non-convex functionals $\{\mathcal{E}\}$ on the space of measurable functions $u : \Omega_1 \times \Omega_2 \subset \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \to \mathbf{R}$. These functionals vanish on the non-convex subset $S(\Omega_1 \times \Omega_2)$ formed by functions of the form $u(x_1, x_2) = u_1(x_1)$ or $u(x_1, x_2) = u_2(x_2)$. We investigate under which conditions the converse implication "$\mathcal{E}(u) = 0 \Rightarrow u \in S(\Omega_1 \times \Omega_2)$" holds. In particular, we show that the answer depends strongly on the smoothness of u. We also obtain quantitative versions of this implication by proving that (at least for some parameters) $\mathcal{E}(u)$ controls in a strong sense the distance of u to $S(\Omega_1 \times \Omega_2)$.

1 Introduction

Given two bounded non-empty connected and open sets $\Omega_1 \subset \mathbf{R}^{n_1}$, $\Omega_2 \subset \mathbf{R}^{n_2}$, we consider the *non-convex* set $S(\Omega)$ of measurable functions u defined on $\Omega = \Omega_1 \times \Omega_2 \subset \mathbf{R}^{n_1+n_2}$ which only depend on the first n_1 coordinates or on the last n_2 coordinates, that is $u(x) = u_1(x_1)$ or $u(x) = u_2(x_2)$. We present and study a family of non-convex functionals that detect whether a measurable function u defined on Ω belongs to $S(\Omega)$. As a first guess, we could expect that for u with Sobolev regularity the relation

$$|\nabla_1 u|\,|\nabla_2 u| \equiv 0 \quad \text{a.e. in } \Omega \tag{1.1}$$

would imply $u \in S(\Omega)$. As we will see later on, this is not the case even for $u \in Lip(\Omega)$ and therefore (1.1) cannot be the starting point for defining our functionals. However, our construction relies on a discrete version of (1.1). If $u \in S(\Omega)$ then at any point $x = (x_1, x_2) \in \Omega$, and any $z = (z_1, z_2) \in \mathbf{R}^{n_1+n_2}$ with $|z|$ small enough, the product

$$|u(x_1 + z_1, x_2) - u(x)|\,|u(x_1, x_2 + z_2) - u(x)| \tag{1.2}$$

is well defined and vanishes. The functionals that we consider are based on the integration of (1.2) in x and z with a weight depending on z and a limiting process that localizes the integration around $z = 0$.

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1.1 Definitions and context

Let us introduce some notation. For convenience, we write the Cartesian product as a sum: we decompose the n-dimensional Euclidean space $X = \mathbf{R}^n$ as,

$$\mathbf{R}^n = X_1 \oplus X_2, \quad \text{with} \quad n_l := \dim X_l \geq 1 \text{ for } l = 1, 2.$$

We will assume for simplicity that $X_1 \perp X_2$. The domain $\Omega \subset \mathbf{R}^n$ is $\Omega := \Omega_1 + \Omega_2$ where, for $l = 1, 2$, Ω_l is a non-empty bounded domain (i.e. open and connected) of the subspace X_l. We note $L(U)$ the space of Lebesgue-measurable functions defined on a measurable set U and for $x \in \mathbf{R}^n$, we note $x = x_1 + x_2$ its decomposition in $X_1 + X_2$. With this notation we define the set

$$S(\Omega) := \{u \in L(\Omega) : u(x) = u_l(x_l) \text{ in } \Omega \text{ with } l = 1 \text{ or } l = 2 \text{ and } u_l \in L(\Omega_l)\}.$$

(This set depends on Ω_1, Ω_2 but as no ambiguity arises we choose this short notation.) We also fix a radial non-negative kernel

$$\rho \in L^1(\mathbf{R}^n, \mathbf{R}_+) \quad \text{with} \quad \int \rho = 1, \text{ and } \operatorname{supp} \rho \subset B_1.$$

As usual, for $\varepsilon > 0$ we introduce the rescaled kernel $\rho_\varepsilon := \varepsilon^{-n} \rho(\varepsilon^{-1} \cdot)$ so that $\{\rho_\varepsilon\}_{\varepsilon > 0}$ forms a family of radial mollifiers. We introduce three real parameters $p, \theta_1, \theta_2 > 0$ and define for any measurable function $u : \Omega \to \mathbf{R}$ and any $\varepsilon > 0$, the quantity

$$\mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u; \Omega) = \mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u) := \int_{\mathbf{R}^n} \rho_\varepsilon(z) \int_{\Omega^\varepsilon} \frac{|u(x+z_1) - u(x)|^{\theta_1} |u(x+z_2) - u(x)|^{\theta_2}}{|z|^p} \, dx \, dz, \tag{1.3}$$

with

$$\Omega^\varepsilon := \Omega_1^\varepsilon + \Omega_2^\varepsilon, \quad \text{and} \quad \Omega_l^\varepsilon := \bigcap_{z_l \in X_l, |z_l| < \varepsilon} (\Omega_l - z_l) \quad \text{for } l = 1, 2.$$

Eventually, we send ε to 0 and define the functional

$$\mathcal{E}_p^{\theta_1,\theta_2}(u; \Omega) = \mathcal{E}_p^{\theta_1,\theta_2}(u) := \liminf_{\varepsilon \downarrow 0} \mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u).$$

Most of the time, we omit the dependency on the parameters θ_1, θ_2 and note

$$\mathcal{E}_{\varepsilon,p}(u) := \mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u), \quad \mathcal{E}_p(u) := \mathcal{E}_p^{\theta_1,\theta_2}(u).$$

We are interested in the qualitative and quantitative properties of functions with finite energy $\mathcal{E}_p(u)$. We first observe that u, θ_1 and θ_2 being fixed, there exists at most one value of p for which $0 < \mathcal{E}_p(u) < \infty$.

Remark A. From the properties of the kernel ρ, we have for $p < q$,

$$\mathcal{E}_{\varepsilon,p}(u) \leq \varepsilon^{q-p} \mathcal{E}_{\varepsilon,q}(u).$$

Sending ε to 0, we see that $[\mathcal{E}_p(u) > 0] \Rightarrow [\mathcal{E}_q(u) = \infty]$ and we deduce that there exists $p^*(u) \in [0, \infty]$ such that

$$\mathcal{E}_p(u) = \begin{cases} 0 & \text{for } p < p^*(u), \\ \infty & \text{for } p > p^*(u). \end{cases}$$

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By construction, $\mathcal{E}_{\varepsilon,p}(u) = \mathcal{E}_p(u) = 0$ for every $u \in S(\Omega)$. A first natural question is:

$$\text{``Does } \mathcal{E}_p(u) = 0 \text{ implies } u \in S(\Omega)?\text{''} \tag{1.4}$$

We will see that the answer is "yes" for p large enough depending on θ_1, θ_2. This question initially appeared (with u being a characteristic function and $n_2 = 1$) in the study of pattern formation in some variational models involving competition between a local attractive term and a non-local repulsive one. Indeed, in [GR19, DR19]), energies related to $\mathcal{E}_p$ are used to show that some sets $S \subset \mathbf{R}^n$ are union of stripes. The functionals $\mathcal{E}_p$ extend this setting to general functions and to general dimensions (n_1, n_2).

Our second main result may be seen as an answer to a quantitative version of Question (1.4). Indeed, we prove (at least for some values of θ_1 and θ_2), that the non-convex energy $\mathcal{E}_p(u)$ controls the distance from u to the non-convex set $S(\Omega)$ in a strong norm. Of course, as seen from Remark A, this is an interesting question only for the borderline exponent p for Question (1.4).

Remark B. Let us point out that when investigating Question (1.4), there is no loss of generality in assuming that $u \in L^\infty$. Indeed, $\arctan u \in S(\Omega)$ if and only if $u \in S(\Omega)$ and $\mathcal{E}_p(\arctan u) \leq \mathcal{E}_p(u)$.

Remark C. Notice that the functionals $\mathcal{E}_p^{\theta_1,\theta_2}$ can be seen as variants of the non-local functionals used by Brezis *et al* to characterize Sobolev spaces [BBM01, Bre02, DMMS08, Ign05, Pon04]. It turns out that the present *non-convex* setting is rather different but we do use their results in our analyses at some point: when $u(x)$ splits as $u_1(x_1) + u_2(x_2)$.

1.2 The vanishing energy case: Question (1.4)

To get some insight into the behavior of the functional $\mathcal{E}_p$, let us first consider the simple situation $u \in C^1(\overline{\Omega})$. Within this setting, the problem is rigid as soon as $p \geq \theta_1 + \theta_2$.

Proposition D (Proposition 2.1). *Let us note $\theta := \theta_1 + \theta_2$.*

(i) *For every $u \in \mathrm{Lip}(\Omega)$, $\mathcal{E}_\theta(u) < \infty$ (and therefore $\mathcal{E}_p(u) = 0$ for $p < \theta$);*

(ii) *If $u \in C^1(\overline{\Omega})$, then $[\mathcal{E}_\theta(u) = 0] \implies u \in S(\Omega)$.*

To obtain the point (i) we simply plug the inequality $|u(x + z_l) - u(x)| \leq \|\nabla u\|_\infty |z_l|$ in the definition of $\mathcal{E}_{\varepsilon,\theta}$. This first point shows that the parameter $p = \theta$ is sharp in (ii). The proof of (ii) runs as follows. Using the relation $u(x + z_l) - u(x) = \nabla u(x) \cdot z_l + o(|z|)$ in the definition of $\mathcal{E}_{\varepsilon,\theta}(u)$, we obtain that $\mathcal{E}_\theta(u) = 0$ implies that ∇u satisfies the differential inclusion (equivalent to (1.1))

$$\nabla u(x) \in X_1 \cup X_2 \quad \text{almost everywhere in } \Omega. \tag{1.5}$$

This differential inclusion is rigid for $u \in C^1(\Omega)$: it yields $u \in S(\Omega)$. On the contrary, since the convex hull of $X_1 \cup X_2$ is $\mathbf{R}^n$, the differential inclusion (1.5) is not rigid in the class of Lipschitz continuous functions. Indeed, the set of Lipschitz functions satisfying (1.5) is dense in $W^{1,1}(\Omega)$ (see [Dac08, Theorem 10.18]). As a consequence we cannot substitute Lipschitz continuity to the C^1-regularity assumption in Proposition D (ii). More precisely,

at least in the range $\min(\theta_1, \theta_2) \leq 1$, when we pass from C^1-regularity to Lipschitz continuity, the threshold jumps from $p = \theta$ to $p = 1 + \theta$ as shown by the two following propositions. The first one follows from Theorem I (with θ_1, θ_2 in the range (b) of (1.6) below).

Proposition E (Proposition 3.9). *Assume* $\min(\theta_1, \theta_2) \leq 1$ *then for* $u \in Lip(\Omega)$, $[\mathcal{E}_{1+\theta}(u) = 0] \Longrightarrow u \in S(\Omega)$.

Proposition F (Proposition 2.2 (i)). *There exists* $u \in Lip(\Omega) \backslash S(\Omega)$ *with* $0 < \mathcal{E}_{1+\theta}(u) < \infty$.

The typical example of a function $u \in Lip(\Omega) \backslash S(\Omega)$ with $\mathcal{E}_{1+\theta}(u) < \infty$ is the "roof" function $u(x) := \min(x_1, x_2)$ defined on $\Omega = (0,1) \times (0,1)$ with $n_1 = n_2 = 1$. This function is locally independent of x_1 or of x_2 away from the diagonal $\{x_1 = x_2\}$, so that the integrand in (1.3) vanishes outside $\{(x, z) : |x_1 - x_2| \leq |z| < \varepsilon\}$. With this remark, it is not difficult to guess that $\mathcal{E}_{1+\theta}(u)$ is finite and positive.

In this work we consider lower regularities than Lipschitz continuity, like merely measurable functions or L^1_{loc} or L^∞ functions for the finest results (recall however Remark B). In this setting, a second important example of a function which "almost" belongs to $S(\Omega)$ is given by the characteristic function of a "corner", $u := \mathbf{1}_{(0,1)^2}$ defined in $\Omega = (-1, 1)^2$. Here, the integrand of (1.3) vanishes outside $\{(x, z) : |x| < \varepsilon, |z| < \varepsilon\}$ and it is easy to check that $\mathcal{E}_2(u) < \infty$.

Proposition G (Proposition 2.2 (ii)). *There exists* $u \in L^\infty(\Omega) \backslash S(\Omega)$ *with* $0 < \mathcal{E}_2(u) < \infty$.

Propositions F and G show that the implication

$$\forall u \in L(\Omega), \quad \mathcal{E}_p(u) = 0 \Longrightarrow u \in S(\Omega)$$

could only hold under the condition $p \geq \max(1 + \theta, 2)$. Our first main result shows that in many cases, this bound is sharp.

Definition H. For $\theta_1, \theta_2 > 0$, using the notation $\theta := \theta_1 + \theta_2$, we define

$$P(\theta_1, \theta_2) := \begin{cases} 2 & \text{if } \theta \leq 1, & (a) \\ 1 + \theta & \text{if } \theta \geq 1 \text{ and } \min(\theta_1, \theta_2) \leq 1, & (b) \\ \min(\theta_1, \theta_2) + \theta & \text{if } \theta_1, \theta_2 > 1. & (c) \end{cases} \qquad (1.6)$$

To lighten notation, from now on, $\theta_1, \theta_2 > 0$ being given, we note

$$\mathcal{E}(u) := \mathcal{E}_{P(\theta_1,\theta_2)}(u) = \mathcal{E}^{\theta_1,\theta_2}_{P(\theta_1,\theta_2)}(u).$$

Theorem I (Theorem 3.7). *For every* $u \in L(\Omega)$, *there holds* $\mathcal{E}(u) = 0 \implies u \in S(\Omega)$.

Remark J. The counter-examples of Propositions F, G show that the exponent $P(\theta_1, \theta_2)$ cannot be improved in the cases (a) and (b) of (1.6). On the contrary, in case (c) we believe that the sharp exponent should still be $1 + \theta$ although we only succeed to prove that the optimal exponent was not larger than $P(\theta_1, \theta_2) = \min(\theta_1, \theta_2) + \theta > 1 + \theta$.

Since $\mathcal{E}(u) < \infty$ implies $\mathcal{E}_p(u) = 0$ for every $p < P(\theta_1, \theta_2)$ (see Remark A), we obtain as direct corollary of Theorem I:

Corollary K. *If $u \in L(\Omega)$ satisfies $\mathcal{E}_p(u) < \infty$ for some $p > P(\theta_1, \theta_2)$, then $u \in S(\Omega)$.*

Another consequence of the theorem is the following generalization of [GR19, Proposition 4.3].

Corollary L (Corollary 3.8). *For every $p \geq P(\theta_1, \theta_2)$, $r > 0$, and every $u \in L(\Omega)$, if*

$$\int_{B_r} \int_{\Omega^r} \frac{|u(x + z_1) - u(x)|^{\theta_1} |u(x + z_2) - u(x)|^{\theta_2}}{|z|^{n+p}} dx dz < \infty,$$

then $u \in S(\Omega^r)$.

This is indeed a far reaching generalization of [GR19, Proposition 4.3] since the same conclusion was obtained there under the assumptions that $n_2 = 1$, $u = \mathbf{1}_E$ for some set E of finite perimeter satisfying an extra technical assumption and $p > n$ (see also [DR19] where the condition $p > n$ was independently relaxed to $p \geq 2$). As opposed to [GR19] and [DR19] where the proofs are somewhat geometrical and based on slicing, our proof is purely analytical.

The main insight in the proof of Theorem I is that if $\mathcal{E}(u)$ does not control first order differential quotients it does control second order ones. Very roughly speaking (at least for $\theta \leq 1$), the main observation is that[1]

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^n} \rho_\varepsilon(z) \int_{\Omega^\varepsilon} \frac{|u(x + z_1 + z_2) - u(x + z_1) - u(x + z_2) + u(x)|^{\theta}}{|z|^{P(\theta_1, \theta_2)}} dx \, dz \lesssim \mathcal{E}(u).$$

This yields a quantitative control on the distribution[2]

$$\mu[u] := \nabla_1 \nabla_2 u.$$

Proposition M (Proposition 3.3). *Let $u \in L^1_{loc}(\Omega)$. For every $\varphi \in C_c^\infty(\Omega, \mathbf{R}^{n_1 \times n_2})$,*

$$\langle \mu[u], \varphi \rangle \lesssim \begin{cases} \mathcal{E}(u) \|\varphi\|_\infty & \text{if (a) holds with } \begin{cases} \theta = 1 \text{ or} \\ \theta < 1 \text{ and } \|u\|_\infty \leq 1, \end{cases} \\ \mathcal{E}(u)^{1/\theta} \|\nabla_1 \varphi\|_1^{1-1/\theta} \|\varphi\|_\infty^{1/\theta} & \text{if (b) holds and } \theta_1 \leq 1, \\ \mathcal{E}(u)^{1/\theta} \|\nabla_1 \varphi\|_1^{\theta_2/\theta} \|\varphi\|_\infty^{\theta_1/\theta} & \text{if (c) holds and } \theta_1 \leq \theta_2. \end{cases}$$

$$(1.7)$$

Remark N. For some estimates (as the first one above), u is required to be bounded. In these situations, we avoid complex formulas involving $\|u\|_\infty$ by assuming $\|u\|_\infty \leq 1$. The general case can be recovered by scaling.

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[1] By convention $a \lesssim b$ means that there exists a non-negative constant C which may only depend on $\theta_1, \theta_2, n, \Omega$ or on the kernel ρ such that $a \leq Cb$.

[2] For $u \in L^1_{loc}(\Omega)$ we note ∇_l the distributional gradient with respect to the variables in X_l.

Remark O. If $n_1 = n_2 = 1$ and $\mathcal{E}(u) = 0$, (1.7) implies that u satisfies the wave equation $\partial_1\partial_2 u = 0$. As noticed in the context of linearized elasticity by Kohn (see [DM95]), this equation is rigid when coupled with the constraint $u \in \{0, 1\}$ i.e. if $u \in \{0, 1\}$ solves the wave equation then $u \in S(\Omega)$ (see also [GM20]).

The proof of Theorem I continues as follows. Proposition M implies that if $\mathcal{E}(u) = 0$ then $\mu[u] = 0$. Obviously, the space of functions $u \in L^1_{loc}(\Omega)$ with $\mu[u] = 0$ is

$$L^1_{loc}(\Omega) \cap \operatorname{span} S(\Omega) = \left\{ x \in \Omega \mapsto u_1(x_1) + u_2(x_2) : u_l \in L^1_{loc}(\Omega_l) \text{ for } l = 1, 2 \right\}.$$

Plugging the decomposition $u(x) = u_1(x_1) + u_2(x_2)$ into the definition of $\mathcal{E}_{\varepsilon,p}(u)$, the integrations over Ω_1 and Ω_2 decouple. Using again $\mathcal{E}(u) = 0$, we deduce that for $l = 1$ or $l = 2$, we have a control of the form

$$\int_{\Omega_{\tilde{l}}^\varepsilon} \int_{\Omega_{\tilde{l}}^\varepsilon} \tilde{\rho}_\varepsilon(x_l - y_l) \frac{|u_l(x_l) - u_l(y_l)|^{\theta_l}}{|x_l - y_l|^{\theta_l}} \, dx_l \, dy_l \xrightarrow{\varepsilon\downarrow 0} 0.$$

We then use ideas from [BBM01] in either Ω_1 or Ω_2 to obtain Theorem I.

Before developing further the consequences of Proposition M let us comment about its optimality. In the cases (a) and (b) of Proposition M, the estimates are optimal in the sense that for $p < P(\theta_1, \theta_2)$, (1.7) does not hold in general. Indeed, we can precise Propositions F, G as follows.

Proposition P (Proposition 2.2)**.**

  (i) *There exists $u \in Lip(\Omega)\backslash S(\Omega)$ with $0 < \mathcal{E}_{1+\theta}(u) < \infty$ and $\mu[u] \neq 0$ (typically, u is a "roof" function);*

  (ii) *There exists $u \in L^\infty(\Omega)\backslash S(\Omega)$ with $0 < \mathcal{E}_2(u) < \infty$ and $\mu[u] \neq 0$ (typically, u is a "corner" function).*

Let us make the important observation that Proposition M gives more information than $\mathcal{E}(u) = 0$ implies $u \in S(\Omega)$. Indeed, it shows that if $0 < \mathcal{E}(u) < \infty$, then $\mu[u]$ lies in some Sobolev space with null or negative regularity exponent. In particular, in case (a) ($\theta = 1$ or $\theta < 1$ and $u \in L^\infty(\Omega)$), if $\mathcal{E}(u) = \mathcal{E}_2(u)$ is finite then $\mu[u]$ is a finite Radon measure. By construction, we prove that it is not necessarily true in the other cases.

Proposition Q (Proposition 2.3)**.** *For every $\theta > 1$, there exists $u \in Lip(\mathbf{R}^2)$, compactly supported, with $\mathcal{E}_{1+\theta}(u) < \infty$ and for which $\mu[u] = \partial_1\partial_2 u$ is not a finite Radon measure.*

1.3 Control of the distance of u to $S(\Omega)$

We then focus on case (a) of (1.6) with the assumption $u \in L^\infty(\Omega)$. We prove that the energy gives a quantitative control on the distance of u to $S(\Omega)$. We obtain the strongest result in this direction for $n = 2$.

Theorem R (Theorem 4.1)**.** *Assume that $n_1 = n_2 = 1$ and $\theta \leq 1$. Then, for every $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leq 1$ and $\mathcal{E}(u) < \infty$, there exists $\bar{u} \in S(\Omega)$ such that $u - \bar{u} \in BV(\Omega) \cap L^\infty(\Omega)$ with the estimate*

$$\|u - \bar{u}\|_\infty + |\nabla[u - \bar{u}]|(\Omega) \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}.$$

The idea of the proof is to first decompose u as $u(x) = u_1(x_1) + u_2(x_2) + w(x)$ where w satisfies $\partial_1 \partial_2 w = \mu[u] = \partial_1 \partial_2 u$ and $\|w\|_\infty + |\nabla w|(\Omega) \lesssim \mathcal{E}(u)$. Using this (and in particular the L^∞ bound on w), we can quantify how much the integration with respect to x_1 and x_2 in the definition (1.3) of $\mathcal{E}_{\varepsilon,2}(u)$ decouples. In higher dimension, the failure of the Sobolev embedding $BV(\Omega_l) \subset L^\infty(\Omega_l)$ makes the situation more complex (and in particular the energy does not control the corresponding w in L^∞) and we were not able to obtain a BV estimate. Nevertheless, we have,

Theorem S (Theorem 4.5). *Assume that $\hat{n} := \max(n_1, n_2) \geq 2$, that Ω_1 and Ω_2 are bounded extension domains and that $\theta \leq 1$. Let $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leq 1$ and $\mathcal{E}(u) < \infty$. Then, there exists $\bar{u} \in L^\infty(\Omega) \cap S(\Omega)$ such that*

$$\|u - \bar{u}\|_{L^{\frac{\hat{n}}{\hat{n}-1}}(\Omega)} \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}.$$

Notice that the $L^{\frac{\hat{n}}{\hat{n}-1}}$ norm (which comes from the embedding of $BV(\Omega_l)$ into $L^{\frac{n_l}{n_l-1}}(\Omega_l)$) is stronger than the $L^{\frac{n}{n-1}}$ norm which would come from the embedding of $BV(\Omega)$.

1.4   Further results

In a second paper [GM20], we will focus on the case $\theta \leq 1$ with $u \in L^\infty(\Omega)$ and $\mathcal{E}(u) < \infty$ and study the structure of the defect measure $\mu = \mu[u]$ (which is then a Radon measure).

In dimension 2 (*i.e.* $n_1 = n_2 = 1$) we show that if $\theta = 1$, then μ concentrates on a set with Hausdorff dimension at most 1. Moreover, if u is Lipschitz continuous, then μ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^1$ (and satisfies the differential inclusion (1.5)). On the contrary, if $\theta < 1$, we show that μ concentrates on a countable set: there exist sequences $(x_k)_{k \geq 1} \subset \Omega$ and $(m_k)_{k \geq 1} \subset \mathbf{R}$ such that $\sum |m_k|^\theta \lesssim \mathcal{E}_2(u)$ and $\mu = \sum m_k \delta_{x_k}$. Moreover, if u is a characteristic function, then $m_k \in \pm\{1, 2\}$ for every $k \geq 1$. As a consequence of the estimate $\sum |m_k|^\theta \lesssim \mathcal{E}_2(u)$ there exists some $\eta > 0$ such that $\mathcal{E}_2(u) < \eta$ implies $\mu = 0$, which in turn leads to $u \in S(\Omega)$. This improves Theorem I and Theorem R in this particular case.

In higher dimensions, we assume $\theta < 1$. Using tools from Geometric Measure Theory (mainly the rectifiability criterion for flat chains of White [Whi99]), we prove that μ is a $(n-2)$-rectifiable measure with a tensor structure: for $l = 1, 2$, there exist $\Sigma_l \subset X_l$ $(n_l - 1)$−rectifiable and a Borel function $m : \Sigma_1 + \Sigma_2 \to \mathbf{R}$ such that $\mu = m\nu_1 \otimes \nu_2 \mathcal{H}^{n-2} \llcorner (\Sigma_1 + \Sigma_2)$, where for $l = 1, 2$, $\nu_l \in X_l$ is a normal to Σ_l. This gives a relatively good understanding of the case where the typical function u with $\mathcal{E}(u) < \infty$ is a "corner" (recall Proposition P).

In the case $\theta = 1$, in order to distinguish between "corners" and "roofs" (see Proposition P), one needs to impose more regularity on u. To understand this better, we plan to investigate in a future work, the set of Lipschitz continuous functions which satisfy the differential inclusion (1.5) and are such that $\mu[u] = \nabla_1 \nabla_2 u$ is a Radon measure.

1.5   Conventions and notation

In all the paper, we consider $\theta_1, \theta_2 > 0$ and note $\theta = \theta_1 + \theta_2$ their sum. For $x, z \in \mathbf{R}^n$, and $u : \Omega \to \mathbf{R}$ we note

$$Du(x, z) := u(x + z) - u(x).$$

In particular, the expression (1.3) of $\mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u)$ simplifies to

$$\mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}(u) = \int_{\mathbf{R}^n} \rho_\varepsilon(z) \int_{\Omega^\varepsilon} \frac{|Du(x,z_1)|^{\theta_1} |Du(x,z_2)|^{\theta_2}}{|z|^p} \, dx \, dz.$$

As already said, except at two points (case (c) in the proofs of Proposition M and Theorem I), we omit the superscript θ_1, θ_2 by writing $\mathcal{E}_{\varepsilon,p}$ for $\mathcal{E}_{\varepsilon,p}^{\theta_1,\theta_2}$ and $\mathcal{E}_p$ for $\mathcal{E}_p^{\theta_1,\theta_2}$. When $p = P(\theta_1, \theta_2)$ as defined in (1.6), we simply write $\mathcal{E}$ for $\mathcal{E}_p$.

If $v \in L(\Omega)$ and $U = U_1 + U_2$ with $U_l \subset \Omega_l$ a subdomain of Ω_l for $l = 1, 2$, we note,

$$\mathcal{E}_{\varepsilon,p}(v; U), \text{ for } \varepsilon > 0, \quad \text{and} \quad \mathcal{E}_p(v; U)$$

the energies of the restriction $v_{|U}$.

In the sequel $(e_1, \cdots, e_{n_1})$ (respectively $(f_1, \cdots, f_{n_2})$) denotes an orthonormal basis of X_1 (respectively of X_2).

Given $x \in \mathbf{R}^n$, we note x_l its orthogonal projection on X_l, for $l = 1, 2$, so that $x = x_1 + x_2$. Similarly, given a function $\varphi \in C_c^\infty(\Omega)$ and $x \in \Omega$, we note $\nabla_1\varphi(x) + \nabla_2\varphi(x)$ the decomposition of $\nabla\varphi(x)$ in $X_1 + X_2$. By duality, the distributional derivative of a function $u \in L_{loc}^1(\Omega)$ decomposes as $\nabla u = \nabla_1 u + \nabla_2 u$, where $\nabla_l u$ is a distribution on Ω with values into X_l.

For $r > 0$ and $x \in \mathbf{R}^n$, $B_r(x)$ denotes the open ball in $\mathbf{R}^n$ with center x and radius r. If $x = 0$, we simply write B_r. For $l = 1, 2$, we note $B_r^{X_l}(x_l)$ the open ball in X_l with center x_l and radius r.

We use standard notation for the function spaces $(L_{loc}^1(\Omega), L^q(\Omega), L^\infty(\Omega), Lip(\Omega), BV(\Omega))$.

For $a \in \mathbf{R}$, we note $a_+ := \max(a, 0)$ its non-negative part.

If $A \subset \mathbf{R}^n$ is a measurable subset, we note $|A|$ its volume and $\mathcal{H}^k(A)$ its $k-$dimensional Hausdorff measure.

We note $\fint_\Omega u(x) \, dx = |\Omega|^{-1} \int_\Omega u$ the mean value of a function u over Ω.

Eventually, by convention, $a \lesssim b$ means that there exists a non-negative constant C which may only depend on $\theta_1, \theta_2, n, \Omega$ or on the kernel ρ such that $a \leq Cb$.

1.6 Outline of the paper

In the first section, we prove Proposition D and we build all the examples and counter-examples introduced above. In Section 3, we consider the zero-energy case and prove Theorem I (and all the results from Proposition E to Q). Section 4 is dedicated to the quantitative control of the distance of u to $S(\Omega)$ in terms of $\mathcal{E}(u)$.

2 The case of C^1-functions and counter-examples

We first settle Question (1.4) in the setting of C^1-functions.

Proposition 2.1 (Proposition D)**.**

(i) *For every $u \in \text{Lip}(\Omega)$, $\mathcal{E}_\theta(u) < \infty$ (and therefore $\mathcal{E}_p(u) = 0$ for $p < \theta$);*

(ii) *For $u \in C^1(\overline{\Omega})$, we have $\mathcal{E}_\theta(u) = 0 \implies u \in S(\Omega)$.*

Proof. (i) Let $u \in Lip(\Omega)$ with Lipschitz constant λ. We have

$$\mathcal{E}_{\varepsilon,\theta}(u) \leq \lambda^\theta \int_{\Omega^\varepsilon} \left[\int_{\mathbf{R}^n} \frac{|z_1|^{\theta_1} |z_2|^{\theta_2}}{|z|^\theta} \rho_\varepsilon(z) \, dz \right] dx.$$

Using the change of variable $z = \varepsilon\tilde{z}$ and sending ε to 0, we get

$$\mathcal{E}_\theta(u) \leq \lambda^\theta |\Omega| \left(\int_{\mathbf{R}^n} \frac{|z_1|^{\theta_1} |z_2|^{\theta_2}}{|z|^\theta} \rho(z) \, dz \right),$$

which proves (i).

(ii) Let $u \in C^1(\overline{\Omega})$. Plugging the relation $u(x + z_l) - u(x) = \nabla_l u(x) \cdot z_l + o(|z_l|)$ into the definition of $\mathcal{E}_{\varepsilon,\theta}^{\theta_1,\theta_2}(u)$, we observe that

$$\mathcal{E}_{\varepsilon,\theta}^{\theta_1,\theta_2}(u) = \int_{\Omega^\varepsilon} \left[\int_{\mathbf{R}^n} \left(\frac{|\nabla_1 u(x) \cdot z_1|^{\theta_1} |\nabla_2 u(x) \cdot z_2|^{\theta_2}}{|z|^\theta} + o(|1|) \right) \rho_\varepsilon(z) \, dz \right] dx.$$

Sending ε to 0, we have

$$\mathcal{E}_\theta^{\theta_1,\theta_2}(u) = \left(\int_{\mathbf{R}^n} \frac{|v_1 \cdot z_1|^{\theta_1} |v_2 \cdot z_2|^{\theta_2}}{|z|^\theta} \rho(z) \, dz \right) \int_\Omega |\nabla_1 u(x)|^{\theta_1} |\nabla_2 u(x)|^{\theta_2} \, dx, \qquad (2.1)$$

where $v_j \in X_l$ are (arbitrary) unit vectors.

Hence, if $\mathcal{E}_\theta^{\theta_1,\theta_2}(u) = 0$, (2.1) implies that $|\nabla_1 u(x)||\nabla_2 u(x)| = 0$ in Ω.

Next, suppose that at some point $x^0 \in \Omega$, $w_1 := \nabla_1 u(x^0) \neq 0$. Let C_2 be the connected component of the set

$$\{y_2 \in \Omega_2 : \nabla_1 u(x_1^0 + y_2) = w_1\}$$

which contains x_2^0. By continuity of $\nabla_1 u$, C_2 is closed in Ω_2. Let us show that it is also open. Let $y_2 \in C_2$, by continuity of $\nabla_1 u$ we have $\nabla_1 u \not\equiv 0$ in some neighborhood $N_\varepsilon(x_1^0 + y_2) := B_\varepsilon^{X_1}(x_1^0) + B_\varepsilon^{X_2}(y_2)$. Since $|\nabla_1 u||\nabla_2 u|$ vanishes, we have $\nabla_2 u \equiv 0$ in $N_\varepsilon(x_1^0 + y_2)$. In particular, $\nabla_1 u(x_1^0 + y_2') = \nabla_1 u(x_1^0 + y_2) = w_1$ for every $y_2' \in B_\varepsilon^{X_2}(y_2)$. We conclude that $B_\varepsilon^{X_2}(y_2) \subset C_2$ which proves that C_2 is relatively open in Ω_2. Finally, the open set Ω_2 being connected, we have $C_2 = \Omega_2$ and we conclude that $\nabla_1 u(x_1^0 + y_2) \neq 0$ for every $y_2 \in \Omega_2$.

Assuming by contradiction that there exists also some point $\tilde{x}^0 \in \Omega$ with $\nabla_2 u(\tilde{x}^0) \neq 0$ we obtain similarly that $\nabla_2 u(y_1 + \tilde{x}_2^0) \neq 0$ for every $y_1 \in \Omega_1$. Choosing $y_1 = x_1^0$ and $y_2 = \tilde{x}_2^0$ we obtain at the point $x = x_1^0 + \tilde{x}_2^0$, the contradiction

$$\nabla_1 u(x) \neq 0 \text{ and } \nabla_2 u(x) \neq 0.$$

We conclude that $\nabla_l u \equiv 0$ for $l = 1$ or $l = 2$: in short, $u \in S(\Omega)$. This proves (ii). $\qquad\square$

We now, give lower-bounds for p in Question (1.4) in the setting of Lipschitz continuous and of bounded functions.

Proposition 2.2 (Proposition P)**.**

(i) *There exists* $u \in Lip(\Omega) \backslash S(\Omega)$ *with* $0 < \mathcal{E}_{1+\theta}(u) < \infty$ *and* $\mu[u] \neq 0$;

(ii) *There exists* $u \in L^\infty(\Omega) \backslash S(\Omega)$ *with* $0 < \mathcal{E}_2(u) < \infty$ *and* $\mu[u] \neq 0$.

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Proof. (i) Let us assume that $n_1 = n_2 = 1$ and let us introduce the "roof" function defined on $\mathbf{R}^2$ by $v(x_1, x_2) := \min(x_1, x_2)$. We have $\partial_1 \partial_2 v = (1/\sqrt{2}) \mathcal{H}^1 \llcorner \{x_1 = x_2\}$. We now consider $\Omega = (0,1)^2$ and $u(x) := v(x)$ in Ω. We have $\mu[u] = (1/\sqrt{2}) \mathcal{H}^1 \llcorner J$, with $J = \{(t,t) : t \in (0,1)\}$ and $\mu[u]$ is a Radon measure with

$$0 < |\mu[u]|(\Omega) < \infty.$$

Next, using the changes of variable $x = \varepsilon x'$, $z = \varepsilon z'$ and the 1-homogeneity of u, we obtain for $\varepsilon > 0$,

$$\mathcal{E}_{\varepsilon, 1+\theta}(u) = \varepsilon\, \mathcal{E}_{1,1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right).$$

Let us set $w(s) := \min(s, 0)$. We have

$$v(x_1 + z_1, x_2) - v(x) = Dw(x_1 - x_2, z_1), \qquad v(x_1, x_2 + z_2) - v(x) = Dw(x_2 - x_1, z_2)$$

and thus

$$\mathcal{E}_{1,1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right) = \int_{\left[\frac{1}{\varepsilon}\Omega\right]^1} \left[\int_{\mathbf{R}^2} \rho(z) \frac{|Dw(x_1 - x_2, z_1)|^{\theta_1}\, |Dw(x_2 - x_1, z_2)|^{\theta_2}}{|z|^{1+\theta}}\, dz \right]\, dx. \quad (2.2)$$

We claim that $\mathcal{E}_{1,1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right) = c_1/\varepsilon - c_2$ for some $c_1, c_2 > 0$. The idea is that since w is constant in $\mathbf{R}_+$, the integrand vanishes when $|z| \le |x_1 - x_2|$ and since $\operatorname{supp}\rho \subset B_1$, the integrand vanishes away from the $\sqrt{2}/2$-neighborhood of the diagonal segment $(1/\varepsilon)J$. Since this segment has length $\sqrt{2}/\varepsilon$, we expect $\mathcal{E}_{1,1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right)$ to be of order $1/\varepsilon$ and therefore, $\mathcal{E}_{\varepsilon, 1+\theta}(u)$ to be of order 1.

Let us note $f(x_1 - x_2)$ the integral inside the brackets in (2.2). The function f takes values in $[0, \infty]$, is measurable and even. We first show that f is integrable on $(-1, 1)$. For $s \ge 0$ and $z \in \mathbf{R}^2$, we have

$$|Dw(s, z_1)|^{\theta_1}\, |Dw(-s, z_2)|^{\theta_2} = \begin{cases} |s + z_1|^{\theta_1} |z_2|^{\theta_2} & \text{if } z_2 \le s \le -z_1, \\ |s + z_1|^{\theta_1} s^{\theta_2} & \text{if } 0 < s < \min(z_2, -z_1), \\ 0 & \text{in the other cases.} \end{cases} \quad (2.3)$$

Integrating in s over $(0, 1)$, we deduce

$$\int_0^1 |Dw(s, z_1)|^{\theta_1}\, |Dw(-s, z_2)|^{\theta_2}\, ds \lesssim |z|^{1+\theta}.$$

Using Fubini and the symmetries of the problem and the above estimate, we get,

$$\int_{-1}^1 f(s)ds = 2 \int_{\mathbf{R}^2} \frac{\rho(z)}{|z|^{1+\theta}} \left[\int_0^1 |Dw(s, z_1)|^{\theta_1}\, |Dw(-s, z_2)|^{\theta_2}\, ds \right] dz \lesssim \int_{\mathbf{R}^2} \rho(z)\, dz = 1.$$

Therefore, $f \in L^1(-1, 1)$. We also observe from (2.3) that the integral is strictly positive. In summary,

$$c_1 := \int_{-1}^1 f(s)ds \in (0, \infty). \quad (2.4)$$

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Let us return to (2.2). Since $\operatorname{supp}\rho \subset B_1$, the identity (2.3) shows that we can reduce the integration with respect to x to the set

$$A_\varepsilon := \left\{(x_1, x_2) \in \mathbf{R}^2 : 1 < x_1, x_2 < 1/\varepsilon - 1, \ |x_1 - x_2| < 1\right\}.$$

Let us introduce the following rectangle which contains A_ε,

$$R_\varepsilon := \left\{x \in \mathbf{R}^2; 0 < x_1 + x_2 < 2/\varepsilon, \ |x_1 - x_2| < 1\right\}$$

We also set $C_\varepsilon := R_\varepsilon \backslash \overline{A}_\varepsilon$, see Figure 1.

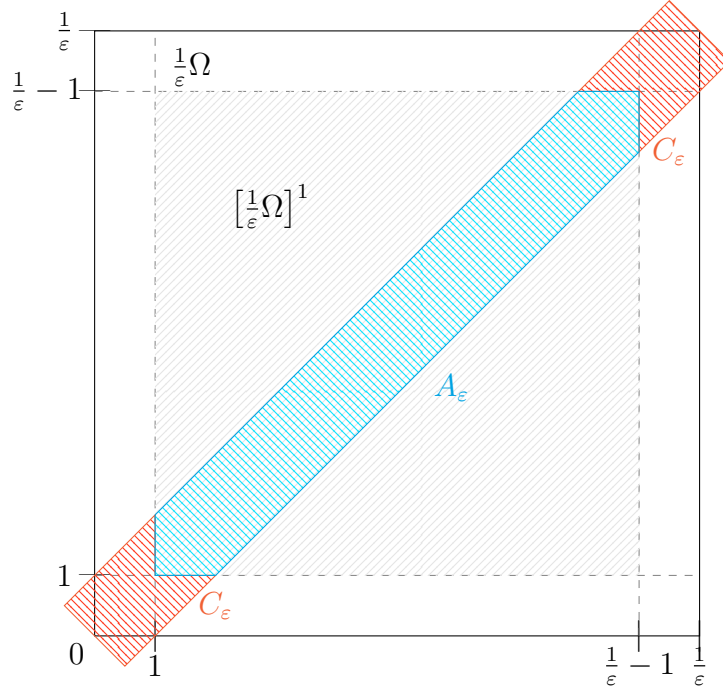Figure 1: The sets A_ε and $C_\varepsilon = R_\varepsilon \backslash \overline{A}_\varepsilon$.

We decompose,

$$\mathcal{E}_{1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right) = \int_{R_\varepsilon} f(x_1 - x_2)\, dx - \int_{C_\varepsilon} f(x_1 - x_2)\, dx.$$

Obviously the second integral does not depend on ε and is positive, we note c_2 its value. For the first integral, we perform the change of variable $s = x_1 - x_2$, $t = (1/2)(x_1 + x_2)$ to obtain

$$\int_{R_\varepsilon} f(x_1 - x_2)\, dx = \varepsilon^{-1} \int_{-1}^{1} f(s)\, ds = c_1/\varepsilon.$$

This gives $\mathcal{E}_{1+\theta}\left(v; \frac{1}{\varepsilon}\Omega\right) = c_1/\varepsilon - c_2$, which leads to

$$\mathcal{E}_{\varepsilon, 1+\theta}(u) = c_1 - \varepsilon c_2.$$

Sending ε to 0, we get $\mathcal{E}_{1+\theta}(u) = c_1 \in (0, \infty)$ (recall (2.4)) and conclude the proof of (i) in the case $n_1 = n_2 = 1$ and $\Omega = (0, 1)^2$. We obtain counter-examples for any non

empty and bounded two dimensional domain by translation and scaling of this example and in higher dimensions by extending the constructions trivially in the complementary directions.

(ii) Again, we only treat the two-dimensional case $n_1 = n_2 = 1$, since higher dimensional cases may be obtained by tensorisation. Let $\Omega = (-1,1)^2$ and set $u := \mathbf{1}_\omega$ with $\omega := (0,1)^2$. Obviously, $\mu = \partial_1 \partial_2 u = \delta_0$ is a non-trivial finite Radon measure. Let us compute $\mathcal{E}_2(u)$. First, we have

$$
|Du(x, z_1)|^{\theta_1} |Du(x, z_2)|^{\theta_2} = \begin{cases} 1 & \text{if } x_1 > 0, \ x_2 > 0, \ x_1 + z_1 \leq 0 \text{ and } x_2 + z_2 \leq 0. \\ 0 & \text{in the other cases.} \end{cases}
$$

We compute for $0 < \varepsilon < 1$,

$$
\mathcal{E}_{\varepsilon,2}(u) = \int_{\mathbf{R}_- \times \mathbf{R}_-} \frac{\rho_\varepsilon(z)}{|z|^2} \left[\int_0^{-z_1} \int_0^{-z_2} dx_1 \, dx_2 \right] dz = \frac{1}{4} \int_{\mathbf{R}^2} \frac{\rho_\varepsilon(z) |z_1| \, |z_2|}{|z|^2} dz = \frac{1}{4\pi}.
$$

For the last identity, we have used the radial symmetry of ρ_ε and $\int_{\mathbf{R}^2} \rho_\varepsilon = 1$. We conclude that $\mathcal{E}_2(u) = 1/(4\pi) \in (0, \infty)$ as required. $\qquad \square$

In Proposition 3.3 below, we will prove that in the case $\theta \leq 1$, if $\mathcal{E}(u)$ is finite then $\mu[u]$ is a Radon measure. We show that this is no longer true when $\theta > 1$.

Proposition 2.3 (Proposition Q). *For every $\theta_1, \theta_2 > 0$ with $\theta = \theta_1 + \theta_2 > 1$, there exists $u \in L^\infty(\mathbf{R}^2)$ compactly supported, with $\mathcal{E}_{1+\theta}(u) < \infty$ but for which $\mu[u] = \partial_1 \partial_2 u$ is not a finite Radon measure.*

Proof. We will build on the example of Proposition 2.2 (i) and consider the "hat" function defined on $(0,1)^2$ as $w(x_1, x_2) := \min(x_1, x_2, 1 - x_1, 1 - x_2)$ and extended by zero outside $(0,1)^2$. Arguing as in the proof of Proposition 2.2, we have $0 < \mathcal{E}_{1+\theta}(u) < \infty$. For $h, \ell > 0$, let $w_{h,\ell}(x) := hw(\ell^{-1}x)$ be defined on $\mathbf{R}^2$. We then have

$$
|\partial_1 \partial_2 w_{h,\ell}|((0,\ell)^2) = h|\partial_1 \partial_2 w|((0,1)^2) = 2h \quad \text{and} \quad \mathcal{E}_{1+\theta}(w_{h,\ell}) = h^\theta \ell^{1-\theta} \mathcal{E}_{1+\theta}(w). \quad (2.5)
$$

Let $(h_k)_{k \geq 1}$, $(\ell_k)_{k \geq 1} \subset (0, \infty)$ be two decreasing sequences such that

$$
\left(\frac{h_k}{\ell_k} \right) \text{ is decreasing,} \quad \frac{h_k^\theta}{\ell_k^{1+\theta}} \sum_{j \geq k} \ell_j^2 \xrightarrow{k \uparrow \infty} 0, \quad \sum_k h_k^\theta \ell_k^{1-\theta} < \infty \quad \text{but} \quad \sum_k h_k = \infty. \quad (2.6)
$$

We also require that there exists some constant $c > 0$ such that

$$
\ell_{k+1} \geq c \, \ell_k \quad \text{for } k \geq 1, \qquad (2.7)
$$

and that

$$
\ell_0 \leq 1/2. \qquad (2.8)
$$

For instance, the sequences defined by $h_k := k^{-(4\theta+1)/(5\theta)}$ and $\ell_k := k^{-3/4}/2$ satisfy (2.6), (2.7) and (2.8). Next, we build a bounded sequence $(x^k)_{k \geq 0} \subset \mathbf{R}^2$ such that

$$
\inf_{j \neq k} |x^j - x^k| \geq 2\ell_k \quad \text{for every } k \geq 0. \qquad (2.9)
$$

For this, we observe that $\sum \ell_k \geq (\ell_1/h_1) \sum h_k = \infty$. We set $k_0 = 1$ and we define recursively k_{m+1} for $m = 0, 1, 2, \cdots$, as the unique integer to be such that

$$1 - 2\ell_{k_{m+1}} < \sum_{k=k_m}^{k_{m+1}-1} 2\ell_k \leq 1.$$

Notice that by (2.8), the sequence $(k_m)_{m\geq0}$ is not stationary (and thus converges towards $+\infty$). We then define $L_0 := 0$ and for $q \geq 1$, $L_q := \sum_{m=0}^{q-1} 2\ell_{k_m}$. The sequence $(L_q)_{q\geq0}$ is increasing and bounded, indeed

$$L_\infty := \lim_{q\uparrow\infty} L_q = 2\ell_1 + \sum_{m\geq1}^{\infty} 2\ell_{k_m} \leq 2\ell_1 + \sum_{m\geq0}^{\infty} 2\ell_{k_{m+1}} \left(\sum_{k=k_m}^{k_{m+1}-1} 2\ell_k \right)$$

$$\leq 2\ell_1 + 4\sum_{m\geq0}^{\infty} \left(\sum_{k=k_m}^{k_{m+1}-1} \ell_k^2 \right) = 2\ell_1 + 4\sum_{k\geq0} \ell_k^2 \overset{(2.6)}{<} \infty.$$

Now, we set for $m \geq 0$ and $k_m \leq k < k_{m+1}$,

$$x^k := \left(L_m, \sum_{k_m \leq j < k} 2\ell_j \right).$$

By construction the sequence (x^k) satisfies (2.9) and $(x^k) \subset (0, L_\infty) \times (0, 1)$. Eventually, we define our candidate (see Figure 2),

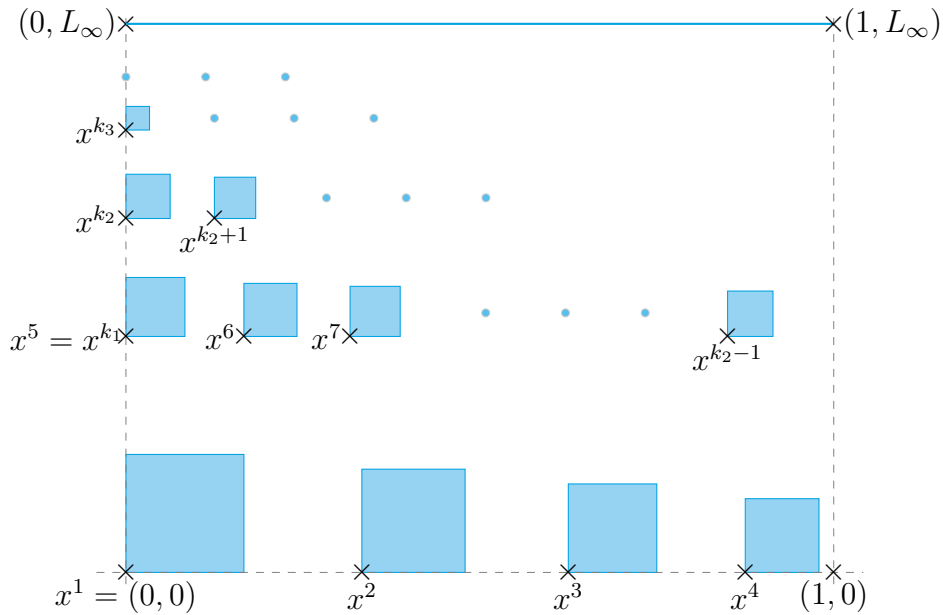$$u(x) := \sum_{k\geq1} w_{h_k, \ell_k}(x - x^k) \quad \text{for } x \in \mathbf{R}^2.$$

Figure 2: The (blue) closed square with bottom left corners x^k and side length ℓ_k is the support of $w_{h_k, \ell_k}(\cdot - x^k)$.

Let us show that u has the desired properties. First, by construction supp $u \subset [0,1] \times [0, L_\infty]$ so that u is compactly supported. Next, from the first identity of (2.5),

$$|\mu[u]|(\Omega) = |\partial_1 \partial_2 u|(\Omega) = 4 \sum_{k \geq 1} h_k \stackrel{(2.6)}{=} \infty. \tag{2.10}$$

Let us now establish that $\mathcal{E}_{1+\theta}(u)$ is finite. We emphasize that

$$\text{supp } w_{h_k, \ell_k}(\cdot - x^k) = [x_1^k, x_1^k + \ell_k] \times [x_2^k, x_2^k + \ell_k].$$

so that, from (2.9), the functions $w_{h_k, \ell_k}(\cdot, x^k)$ have disjoint supports and for $j \neq k$,

$$d\left(\text{supp } w_{h_j, \ell_j}(\cdot - x^j), \text{supp } w_{h_k, \ell_k}(\cdot - x^k)\right) \geq (2 - \sqrt{2})\ell_k. \tag{2.11}$$

By (2.7) for $\varepsilon > 0$ small enough, there exists an integer k_ε such that

$$(2 - \sqrt{2})c\,\ell_{k_\varepsilon} \leq (2 - \sqrt{2})\ell_{k_\varepsilon + 1} < \varepsilon \leq (2 - \sqrt{2})\ell_{k_\varepsilon}. \tag{2.12}$$

Using supp $\rho_\varepsilon \subset B_\varepsilon$ and (2.11), we can write

$$\mathcal{E}_{\varepsilon, 1+\theta}(u) = \sum_{j=1}^{k_\varepsilon} \mathcal{E}_{\varepsilon, 1+\theta}(w_{h_j, \ell_j}) + \mathcal{E}_{\varepsilon, 1+\theta}(u_\varepsilon), \tag{2.13}$$

where we note

$$u_\varepsilon(x) := \sum_{j > k_\varepsilon} w_{h_j, \ell_j}(x - x^j).$$

Let us first bound the remaining term $\mathcal{E}_{\varepsilon, 1+\theta}(u_\varepsilon)$ in (2.13). We notice that $w_{h_j, \ell_j}(\cdot, x^j)$ is Lipschitz continuous with Lipschitz constant $\|\nabla w_{h_j, \ell_j}\|_\infty = h_j / \ell_j \leq h_{k_\varepsilon} / \ell_{k_\varepsilon}$ for $j \geq k_\varepsilon$. Since these functions have disjoint supports, we conclude that their sum u_ε is Lipschitz continuous with

$$\|\nabla u_\varepsilon\|_\infty \leq h_{k_\varepsilon} / \ell_{k_\varepsilon}.$$

Using $|Du_\varepsilon(x, z_l)| \leq (h_{k_\varepsilon} / \ell_{k_\varepsilon})|z|$ and $|Du_\varepsilon(x, z_1)||Du_\varepsilon(x, z_2)| = 0$ if the three points x, $x + z_1$, $x + z_2$ lie in the complement of supp u_ε, we compute,

$$\mathcal{E}_{\varepsilon, 1+\theta}(u_\varepsilon) = \int_{B_\varepsilon} \frac{\rho_\varepsilon(z)}{|z|^{1+\theta}} \int_{\mathbf{R}^2} |Du_\varepsilon(x, z_1)|^{\theta_1} |Du_\varepsilon(x, z_2)|^{\theta_2}\, dx\, dz$$

$$\leq \left(\frac{h_{k_\varepsilon}}{\ell_{k_\varepsilon}}\right)^\theta \int_{B_\varepsilon} \frac{\rho_\varepsilon(z)}{|z|} |A_\varepsilon(z)|\, dz,$$

where $A_\varepsilon(z)$ is the set of points $x \in \mathbf{R}^2$ such that at least one of the three points $\{x, x + z_1, x + z_2\}$ belongs to supp u_ε. We have

$$|A_\varepsilon(z)| \leq 3 \sum_{j \geq k_\varepsilon} \ell_j^2.$$

This leads to the estimate

$$\mathcal{E}_{\varepsilon, 1+\theta}(u_\varepsilon) \lesssim \frac{h_{k_\varepsilon}^\theta}{\ell_{k_\varepsilon}^\theta \varepsilon} \sum_{j \geq k_\varepsilon} \ell_j^2 \stackrel{(2.12)}{\lesssim} \frac{h_{k_\varepsilon}^\theta}{\ell_{k_\varepsilon}^{1+\theta}} \sum_{j \geq k_\varepsilon} \ell_j^2 \stackrel{(2.6)}{\longrightarrow} 0 \quad \text{as } \varepsilon \downarrow 0. \tag{2.14}$$

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We now pass to the limit in the terms $\mathcal{E}_{\varepsilon,1+\theta}(w_{h_j,\ell_j})$ for $j \leq k_\varepsilon$. We have

$$\mathcal{E}_{\varepsilon,1+\theta}(w_{h_j,\ell_j}) = h_j^\theta \ell_j^{1-\theta} \mathcal{E}_{\varepsilon/\ell_j,1+\theta}(w).$$

As in the proof of Proposition 2.2 (i), we have

$$\mathcal{E}_{\varepsilon/\ell_j,1+\theta}(w) \uparrow \mathcal{E}_{1+\theta}(w) \quad \text{as } \varepsilon \downarrow 0.$$

Summing over $1 \leq j \leq k_\varepsilon$, and sending $\varepsilon \downarrow 0$, we get by monotone convergence theorem,

$$\lim_{\varepsilon \downarrow 0} \sum_{j=1}^{k_\varepsilon} \mathcal{E}_{\varepsilon,1+\theta}(w_{h_j,\ell_j}) = \mathcal{E}_{1+\theta}(w) \sum_{j \geq 1} h_j^\theta \ell_j^{1-\theta}.$$

Combining this together with (2.13) and (2.14), we obtain $\mathcal{E}_{1+\theta}(u) = \mathcal{E}_{1+\theta}(w) \sum_j h_j^\theta \ell_j^{1-\theta}$ and by (2.6) we conclude that $\mathcal{E}_{1+\theta}(u)$ is finite whereas from (2.10), $\mu[u]$ is not a finite measure. $\qquad\square$

3 The zero energy case

In order to present the main ideas of the proof of Theorem I, we start by considering the simplest possible setting. We restrict ourselves to $n_1 = n_2 = 1$, $\theta_1 + \theta_2 = 1$ and work on the torus $\Omega = \mathbf{T} = (\mathbf{R}/\mathbf{Z})^2$ to avoid boundary effects (in particular, $\mathbf{T}^\varepsilon = \mathbf{T}$). In this periodic setting, we need to distinguish the ambient manifold $\mathbf{T}$ from the space of tangent vectors $X = \mathbf{R}^2$, we define:

$$\Omega_1 = \mathbf{R}/\mathbf{Z} \times \{0\}, \quad \Omega_2 = \{0\} \times \mathbf{R}/\mathbf{Z}, \quad X_1 = \mathbf{R} \times \{0\}, \quad X_2 = \{0\} \times \mathbf{R}.$$

With this notation, the definitions of $S(\mathbf{T})$ and of the energy are unchanged.

Proposition 3.1. *Let θ_1, $\theta_2 > 0$ be such that $\theta_1 + \theta_2 = 1$ and let $u \in L(\mathbf{T})$ be such that*

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{\mathbf{T}} \frac{|Du(x,z_1)|^{\theta_1} |Du(x,z_2)|^{\theta_2}}{|z|^2} \, dx \, dz = 0, \tag{3.1}$$

then $u \in S(\mathbf{T})$.

Proof. As noticed in Remark B, we may assume without loss of generality that $u \in L^\infty(\Omega)$.
Step 1. In this first step we prove that (3.1) allows us to find a sequence $z^k = \varepsilon_k(\sigma_1 + \sigma_2)$ with $\sigma_1 \in X_1\backslash\{0\}$, $\sigma_2 \in X_2\backslash\{0\}$ and $\varepsilon_k \downarrow 0$ such that

$$\lim_{k \uparrow \infty} \int_{\mathbf{T}} \frac{q(x,z^k)}{\varepsilon_k^2} \, dx = 0, \tag{3.2}$$

where we have set

$$q(x,z) := \left(|Du(x+z_2,z_1)|^{\theta_1} + |Du(x,z_1)|^{\theta_1}\right)\left(|Du(x+z_1,z_2)|^{\theta_2} + |Du(x,z_2)|^{\theta_2}\right). \tag{3.3}$$

Let us digress slightly and first derive a consequence of (3.2). For this, first notice that

$$\begin{aligned}
Du(x+z_1,z_2) - Du(x,z_2) &= u(x+z_1+z_2) - u(x+z_2) - u(x+z_1) + u(x) \\
&= Du(x+z_2,z_1) - Du(x,z_1),
\end{aligned}$$

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so that by the triangle inequality we have

$$|Du(x + z_1, z_2) - Du(x, z_2)| \leq |Du(x, z_1)| + |Du(x + z_2, z_1)|$$
$$|Du(x + z_1, z_2) - Du(x, z_2)| \leq |Du(x, z_2)| + |Du(x + z_1, z_2)|.$$

Hence,

$$|Du(x + z_1, z_2) - Du(x, z_2)|^{\theta_1} \lesssim |Du(x, z_1)|^{\theta_1} + |Du(x + z_2, z_1)|^{\theta_1}, \tag{3.4}$$

and similarly for θ_2 so that since $\theta_1 + \theta_2 = 1$,

$$|D[Du(\cdot, z_2)](x, z_1)| = |Du(x + z_1, z_2) - Du(x, z_2)|$$
$$\lesssim \left(|Du(x + z_2, z_1)|^{\theta_1} + |Du(x, z_1)|^{\theta_1}\right)\left(|Du(x + z_1, z_2)|^{\theta_2} + |Du(x, z_2)|^{\theta_2}\right). \tag{3.5}$$

Therefore, a consequence of (3.2) would be the maybe more suggestive

$$\lim_{k \uparrow \infty} \int_{\mathbf{T}} \frac{|D[Du(\cdot, z_2^k)](x, z_1^k)|}{\varepsilon_k^2} \, dx = 0. \tag{3.6}$$

Observe that if u were smooth , the integrand would converge to $|\sigma_1| \, |\sigma_2| \, |\partial_1 \partial_2 u(x)|$ and (3.6) would directly imply $\partial_1 \partial_2 u = 0$.

We now prove (3.2). Making the change of variable $\tilde{x} = x + z_1$, $\tilde{z} = -z_1 + z_2$ in (3.1), we have

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{\mathbf{T}} \frac{|Du(\tilde{x}, \tilde{z}_1)|^{\theta_1} |Du(\tilde{x} + \tilde{z}_1, \tilde{z}_2)|^{\theta_2}}{|\tilde{z}|^2} d\tilde{x} d\tilde{z} = 0.$$

Similarly we get

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{\mathbf{T}} \frac{|Du(x + z_2, z_1)|^{\theta_1} |Du(x, z_2)|^{\theta_2}}{|z|^2} \, dx \, dz = 0$$

and

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{\mathbf{T}} \frac{|Du(x + z_2, z_1)|^{\theta_1} |Du(x + z_1, z_2)|^{\theta_2}}{|z|^2} \, dx \, dz = 0.$$

Summing these limits and (3.1) we obtain,

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{\mathbf{T}} \frac{q(x, z)}{|z|^2} \, dx \, dz \overset{(3.3)}{=} 0, \tag{3.7}$$

with q given by (3.3). Using polar coordinates and the change of variables $r = \varepsilon s$, we obtain

$$\liminf_{\varepsilon \downarrow 0} \int_{\partial B_1} \int_0^\infty \left[\int_{\mathbf{T}} \frac{q(x, s\varepsilon\sigma)}{s^2 \varepsilon^2} \, dx \right] s\rho(s) \, ds d\mathcal{H}^1(\sigma) = 0.$$

Applying Fatou Lemma and then Markov inequality, we may find for $l = 1, 2$, $\hat{\sigma}_l \in X_l \backslash \{0\}$ with $\frac{1}{4} \leq |\hat{\sigma}_l|^2 \leq \frac{3}{4}$ and $s \in (0, 1)$ such that setting $\sigma := s(\hat{\sigma}_1 + \hat{\sigma}_2)$,

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{T}} \frac{q(x, \varepsilon\sigma)}{\varepsilon^2} \, dx = 0.$$

Passing to a subsequence and noting $z^k := \varepsilon_k \sigma$, we get (3.2).

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Step 2. Let us show that (3.2) implies $\partial_1\partial_2 u = 0$ in $\mathcal{D}'(\mathbf{T})$. Let $\varphi \in C^\infty(\mathbf{T})$. For every $x \in \mathbf{T}$,

$$\partial_1\partial_2\varphi(x) = \pm\frac{1}{|\sigma_1|\,|\sigma_2|}\partial_{\sigma_1}\partial_{\sigma_2}\varphi(x) = \pm\frac{1}{|\sigma_1|\,|\sigma_2|}\lim_{k\uparrow\infty}\frac{D[D\varphi(\cdot, -z_1^k)](x, -z_2^k)}{\varepsilon_k^2},$$

with uniform convergence in $\mathbf{T}$. Multiplying by u, using the dominated convergence theorem and two discrete integration by parts, we compute

$$
\begin{aligned}
\left|\int_{\mathbf{T}}\partial_1\partial_2\varphi u\,dx\right| &= \frac{1}{|\sigma_1|\,|\sigma_2|}\lim_{k\uparrow\infty}\frac{1}{\varepsilon_k^2}\left|\int_{\mathbf{T}}D[D\varphi(\cdot, -z_1^k)](x, -z_2^k)u(x)\,dx\right| \\
&= \frac{1}{|\sigma_1|\,|\sigma_2|}\lim_{k\uparrow\infty}\frac{1}{\varepsilon_k^2}\left|\int_{\mathbf{T}}\varphi D[Du(\cdot, z_1^k)](x, z_2^k)\,dx\right| \quad (3.8)\\
&\leq \frac{\|\varphi\|_\infty}{|\sigma_1|\,|\sigma_2|}\lim_{k\uparrow\infty}\int_{\mathbf{T}}\frac{|D[Du(\cdot, z_1^k)](x, z_2^k)|}{\varepsilon_k^2}\,dx \overset{(3.6)}{=} 0.
\end{aligned}
$$

Therefore, $\partial_1\partial_2 u = 0$ in the sense of distributions in $\mathbf{T}$.

Step 3. Integrating the relation $\partial_1\partial_2 u = 0$ we find that $u(x) = u_1(x_1) + u_2(x_2)$ with u_1 and u_2 periodic functions on $\mathbf{R}$. Moreover, since u is bounded, so are u_1 and u_2. Let us finally prove that $\partial_1 u_1 \equiv 0$ or $\partial_2 u_2 \equiv 0$. From (3.2), we have for some $\sigma_1 \in X_1\backslash\{0\}$, $\sigma_2 \in X_2\backslash\{0\}$ and a sequence $\varepsilon_k \downarrow 0$,

$$
\begin{aligned}
0 &= \lim_{k\uparrow\infty}\int_{\mathbf{T}}\frac{|Du_1(x, \varepsilon_k\sigma_1)|^{\theta_1}|Du_2(x, \varepsilon_k\sigma_2)|^{\theta_2}}{\varepsilon_k^2}\,dx \\
&= \lim_{k\uparrow\infty}\left(\int_0^1\frac{|Du_1(x_1, \varepsilon_k\sigma_1)|^{\theta_1}}{\varepsilon_k}\,dx_1\right)\left(\int_0^1\frac{|Du_1(x_2, \varepsilon_k\sigma_2)|^{\theta_1}}{\varepsilon_k}\,dx_2\right),
\end{aligned}
$$

so that up to extraction for $l = 1$ or $l = 2$, there holds

$$\lim_{k\uparrow\infty}\int_0^1\frac{|Du_l(x_l, \varepsilon_k\sigma_l)|^{\theta_l}}{\varepsilon_k}\,dx_l = 0.$$

In particular, since u_l is bounded and $0 < \theta_l < 1$,

$$\lim_{k\uparrow\infty}\int_0^1\frac{|Du_l(x_l, \varepsilon_k\sigma_l)|}{\varepsilon_k}\,dx_l \leq \|u_l\|_\infty^{1-\theta_l}\lim_{k\uparrow\infty}\int_0^1\frac{|Du_l(x_l, \varepsilon_k\sigma_l)|^{\theta_l}}{\varepsilon_k}\,dx_l = 0.$$

Arguing as in (3.8), we obtain that $\partial_l u_l \equiv 0$ in the sense of distributions in $\mathbf{R}/\mathbf{Z}$. We conclude that $u \in S(\mathbf{T})$ which ends the proof of the proposition. $\qquad\square$

We turn to the proof of Theorem I, which extends Proposition 3.1 in several directions by considering general space dimensions and powers θ_1, θ_2.

Let us recall some notation. For $\Omega = \Omega_1 + \Omega_2$ with $\Omega_1 \subset X_1$ and $\Omega_2 \subset X_2$ and a function $u \in L^1_{loc}(\Omega)$, the matrix valued distribution $\mu[u]$ is defined as

$$\mu = \mu[u] := \nabla_1\nabla_2 u = (\partial_{x_1^i}\partial_{x_2^j}u)_{1\leq i\leq n_1, 1\leq j\leq n_2},$$

where $\dim X_1 = n_1$ and $\dim X_2 = n_2$. For θ_1, $\theta_2 > 0$, we also recall that the critical exponent $P(\theta_1, \theta_2)$ has been introduced in Definition H and that when $p = P(\theta_1, \theta_2)$, we simply write $\mathcal{E}(u)$ for $\mathcal{E}_p(u)$.

Remark 3.2. We will use the following inequality to reduce the case $\theta < 1$ to the case $\theta = 1$. Assume that $\theta < 1$ and let us note $\theta_l' := \theta_l + (1 - \theta)/2$ for $l \in \{1, 2\}$. We have $\theta' := \theta_1' + \theta_2' = 1$ and for $u \in L^\infty(\Omega)$ and x, z such that $x, x + z \in \Omega$, we have

$$|Du(x, z_1)|^{\theta_1'}|Du(x, z_2)|^{\theta_2'} \leq 2^{1-\theta}\|u\|_\infty^{1-\theta}|Du(x, z_1)|^{\theta_1}|Du(x, z_2)|^{\theta_2},$$

so that

$$\mathcal{E}_2^{\theta_1', \theta_2'}(u) \leq 2^{1-\theta}\|u\|_\infty^{1-\theta}\mathcal{E}_2^{\theta_1, \theta_2}(u).$$

We now prove that the energy $\mathcal{E}(u)$ controls the cross derivatives $\mu[u] = \nabla_1 \nabla_2 u$.

Proposition 3.3 (Proposition M). *Let $u \in L^1_{loc}(\Omega)$. We have the following estimates, for every $\varphi \in C_c^\infty(\Omega, \mathbf{R}^{n_1 \times n_2})$,*

$$\langle \mu[u], \varphi \rangle \lesssim \begin{cases} \mathcal{E}(u)\|\varphi\|_\infty & \text{if } \theta = 1 \text{ or } [\theta < 1 \text{ and } \|u\|_\infty \leq 1], \\ \mathcal{E}(u)^{1/\theta}\|\nabla_1\varphi\|_1^{1-1/\theta}\|\varphi\|_\infty^{1/\theta} & \text{if } \theta \geq 1 \text{ with } \theta_1 \leq 1, \\ \mathcal{E}(u)^{1/\theta}\|\nabla_1\varphi\|_1^{\theta_2/\theta}\|\varphi\|_\infty^{\theta_1/\theta} & \text{if } 1 \leq \theta_1 \leq \theta_2. \end{cases} \tag{3.9}$$

Because of the applications we have in mind in the last part of the paper (and in [GM20]), it will actually be more convenient to derive Proposition 3.3 as a consequence of Lemma 3.4 and Lemma 3.6 below.

We may now state a first lemma which is the extension of Step 1 in the proof of Proposition 3.1 to the more general setting. We recall that we defined $q(x, z)$ in (3.3) as

$$q(x, z) = \left(|Du(x + z_2, z_1)|^{\theta_1} + |Du(x, z_1)|^{\theta_1}\right)\left(|Du(x + z_1, z_2)|^{\theta_2} + |Du(x, z_2)|^{\theta_2}\right).$$

We also recall that $(e_1, \cdots, e_{n_1})$ and $(f_1, \cdots, f_{n_2})$ denote orthonormal bases of X_1 and X_2.

Lemma 3.4. *If u is such that $\mathcal{E}_p(u) < \infty$, then there exist sequences $0 < r_k \leq \varepsilon_k$ tending to 0, two numbers $\lambda_1, \lambda_2 \in (1/2, \sqrt{3}/2)$ and two rotations $R_l \in SO(X_l)$ such that*

$$\limsup_{k\uparrow\infty} \sum_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \int_{\Omega^{\varepsilon_k}} \frac{q(x, r_k(\lambda_1 R_1 e_i + \lambda_2 R_2 f_j))}{r_k^p} \, dx \lesssim \mathcal{E}_p(u). \tag{3.10}$$

Proof. For $l = 1, 2$, let ν_l denotes the normalized Haar measure on $SO(X_l)$ and let $\mathbf{S}(X_l)$ denote the unit sphere of X_l. We also set $\nu := \nu_1 \otimes \nu_2$. We first claim that for every $g \in L^1(\mathbf{R}^n)$, there holds

$$\int_{\mathbf{R}^n} g(z) \, dz$$

$$= \frac{1}{n_1 n_2} \int_{SO(X_1) \times SO(X_2)} \int_{\mathbf{R}^n} \left[\sum_{1 \leq i \leq n_1, 1 \leq j \leq n_2} g(|z_1|R_1 e_i + |z_2|R_2 f_j)\right] dz \, d\nu(R_1, R_2). \tag{3.11}$$

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To prove (3.11), let us notice first that for every function $v \in L^1(\mathbf{S}(X_1))$, the integral $\int_{SO(X_1)} v(R_1 e) d\nu_1(R_1)$ does not depend on the choice of $e \in \mathbf{S}(X_1)$ and therefore

$$\int_{SO(X_1)} v(R_1 e) d\nu_1(R_1) = \frac{1}{\mathcal{H}^{n_1-1}(\mathbf{S}(X_1))} \int_{\mathbf{S}(X_1)} \int_{SO(X_1)} v(R_1 \sigma) d\nu_1(R_1) d\mathcal{H}^{n_1-1}(\sigma)$$

$$= \frac{1}{\mathcal{H}^{n_1-1}(\mathbf{S}(X_1))} \int_{SO(X_1)} \int_{\mathbf{S}(X_1)} v(R_1 \sigma) d\mathcal{H}^{n_1-1}(\sigma) d\nu_1(R_1)$$

$$\overset{\sigma'=R_1\sigma}{=} \frac{1}{\mathcal{H}^{n_1-1}(\mathbf{S}(X_1))} \int_{SO(X_1)} \left(\int_{\mathbf{S}(X_1)} v(\sigma') d\mathcal{H}^{n_1-1}(\sigma') \right) d\nu_1(R_1)$$

$$= \frac{1}{\mathcal{H}^{n_1-1}(\mathbf{S}(X_1))} \int_{\mathbf{S}(X_1)} v(\sigma) d\mathcal{H}^{n_1-1}(\sigma).$$

For $g_1 \in L^1(X_1)$, we deduce from the above formula and of a decomposition in polar coordinates that

$$\int_{X_1} g_1 = \int_0^\infty \mathcal{H}^{n_1-1}(\mathbf{S}(X_1)) \left\{ \int_{SO(X_1)} g_1(r R_1 e) \, d\nu_1(R_1) \right\} r^{n_1-1} \, dr$$

$$\overset{r=|z_1|}{=} \int_{X_1} \left\{ \int_{SO(X_1)} g_1(|z_1| R_1 e) \, d\nu_1(R_1) \right\} dz_1$$

$$\overset{\text{Fubini}}{=} \int_{SO(X_1)} \left(\int_{X_1} g_1(|z_1| R_1 e) \, dz_1 \right) d\nu_1(R_1).$$

Combining this with the analog formula for $f \in \mathbf{S}(X_2)$, $g_2 \in L^1(X_2)$ and using Fubini, we have for $g \in L^1(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} g(z) \, dz = \int_{SO(X_1) \times SO(X_2)} \left[\int_{\mathbf{R}^n} g(|z_1| R_1 e + |z_2| R_2 f) \, dz \right] d\nu(R_1, R_2).$$

Taking $e = e_i$ and $f = f_j$ in the above identity and summing over $1 \le i \le n_1$, $1 \le j \le n_2$, we obtain (3.11).

Define

$$g(z) := \frac{\rho_\varepsilon(z)}{|z|^p} \int_{\Omega^{2\varepsilon}} q(x, z) \, dx,$$

so that arguing as for (3.7) we have

$$\int_{\mathbf{R}^n} g(z) \, dz \lesssim \mathcal{E}_{\varepsilon,p}(u).$$

Using (3.11), polar coordinates $z = r\sigma$, with $r > 0$, $\sigma \in \partial B_1$ and the change of variables $r = \varepsilon s$, we obtain

$$\int_{SO(X_1) \times SO(X_2)} \int_{\partial B_1} \int_0^\infty Q_\varepsilon(s, \sigma, R_1, R_2) s^{n-1} \rho(s) \, ds \, d\mathcal{H}^{n-1}(\sigma) \, d\nu(R_1, R_2) \lesssim \mathcal{E}_{\varepsilon,p}(u),$$

where we note

$$Q_\varepsilon(s, \sigma, R_1, R_2) := \sum_{1 \le i \le n_1, \, 1 \le j \le n_2} \int_{\Omega^{2\varepsilon}} \frac{q(x, s\varepsilon|\sigma_1| R_1 e_i + s\varepsilon|\sigma_2| R_2 f_j)}{s^p \, \varepsilon^p} \, dx.$$

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Passing to the infimum limit in ε, using Fatou Lemma and then Markov inequality, we find that there exist $R_1 \in \mathrm{SO}(X_1)$, $R_2 \in \mathrm{SO}(X_2)$, $\sigma \in \partial B_1$ with $1/2 \leq |\sigma_1|, |\sigma_2| \leq \sqrt{3}/2$ and $s \in [c, 1)$, where $c > 0$ only depends on the kernel ρ, such that

$$\liminf_{\varepsilon \downarrow 0} \sum_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \int_{\Omega^{2\varepsilon}} \frac{q(x, s\varepsilon |\sigma_1| R_1 e_i + s\varepsilon |\sigma_2| R_2 f_j)}{s^p \, \varepsilon^p} \, dx \lesssim \mathcal{E}_p(u).$$

Extracting a subsequence ε_k' realizing the liminf we find

$$\limsup_{k \uparrow \infty} \sum_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \int_{\Omega^{2\varepsilon_k'}} \frac{q(x, (s\varepsilon_k')[|\sigma_1| R_1 e_i + |\sigma_2| R_2 f_j])}{(s\varepsilon_k')^p} \, dx \lesssim \mathcal{E}_p(u).$$

Noting $\lambda_l := |\sigma_l|$ for $l = 1, 2$, $\varepsilon_k := 2\varepsilon_k'$ and $r_k := s\varepsilon_k'$ we conclude the proof of (3.10). $\qquad \square$

Remark 3.5. With the notation of the lemma, we define the map $A \in \mathrm{GL}(\mathbf{R}^n)$ by $Az := \lambda_1 R_1 z_1 + \lambda_2 R_2 z_2$. Making the change of variables $x = A\hat{x}$ and $\hat{u}(\hat{x}) = u(A\hat{x})$ in (3.10), defining $\hat{\varepsilon}_k := \max(\lambda_1^{-1}, \lambda_2^{-1}) \varepsilon_k$ and observing that $[A^{-1}(\Omega)]^{\hat{\varepsilon}_k} \subset A^{-1}(\Omega^{\varepsilon_k})$, we find

$$\limsup_{k \uparrow \infty} \sum_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \int_{[A^{-1}(\Omega)]^{\hat{\varepsilon}_k}} \frac{\hat{q}(x, r_k(e_i + f_j))}{r_k^p} \, dx \lesssim \mathcal{E}_p(u) \sim \mathcal{E}_p(\hat{u}),$$

with the definition of $\hat{q}$ modeled on the definition of q:

$$\hat{q}(x, z) := \left(|D\hat{u}(x + z_2, z_1)|^{\theta_1} + |D\hat{u}(x, z_1)|^{\theta_1} \right) \left(|D\hat{u}(x + z_1, z_2)|^{\theta_2} + |D\hat{u}(x, z_2)|^{\theta_2} \right).$$

Since, X_1 and X_2 are stable by A, we have $u \in S(\Omega) \iff \hat{u} \in S(A^{-1}\Omega)$. Therefore, up to this change of variables, we may always assume that (3.10) holds true with $\lambda_l R_l = \mathrm{Id}_{X_l}$ for $l = 1, 2$.

Lemma 3.6. *Let $u \in L^1_{loc}(\Omega)$ and let (ε_k), (r_k) be two sequences with $0 < r_k \leq \varepsilon_k \downarrow 0$. For $i \in \{1, \cdots, n_1\}$ and $j \in \{1, \cdots, n_2\}$, we define*

$$\mathcal{F}_{i,j}(u) := \liminf_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} \frac{q(x, r_k(e_i + f_j))}{r_k^{P(\theta_1, \theta_2)}} \, dx.$$

(i) If $\theta = 1$ or $[\theta < 1$ and $\|u\|_\infty \leq 1]$ (in both cases, $P(\theta_1, \theta_2) = 2$), we have for $i \in \{1, \cdots, n_1\}$, $j \in \{1, \cdots, n_2\}$,

$$\mathcal{G}_{i,j}(u) := \liminf_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} \frac{|D[Du(\cdot, r_k f_j)](x, r_k e_i)|}{r_k^2} \, dx \lesssim \mathcal{F}_{i,j}(u). \tag{3.12}$$

(ii) Noting $\mu = \mu[u]$, we have for $i \in \{1, \cdots, n_1\}$, $j \in \{1, \cdots, n_2\}$ and every $\varphi \in C_c^\infty(\Omega)$,

$$\langle \mu_{i,j}, \varphi \rangle \lesssim \begin{cases} \mathcal{G}_{i,j}(u) \|\varphi\|_\infty & \text{in any cases,} & (a) \\ (\mathcal{F}_{i,j}(u))^{1/\theta} \|\nabla_1 \varphi\|_1^{1-1/\theta} \|\varphi\|_\infty^{1/\theta} & \text{if } \theta \geq 1 \text{ and } \theta_1 \leq 1, & (b) \\ (\mathcal{F}_{i,j}(u))^{1/\theta} \|\nabla_1 \varphi\|_1^{\theta_2/\theta} \|\varphi\|_\infty^{\theta_1/\theta} & \text{if } 1 < \theta_1 \leq \theta_2. & (c) \end{cases} \tag{3.13}$$

Proof. Part (i). Let us first consider the case $\theta = 1$, so that $P(\theta_1, \theta_2) = 2$. Let $i \in \{1, \cdots, n_1\}$, $j \in \{1, \cdots, n_2\}$ and let $k \geq 1$ and $x \in \Omega^{\varepsilon_k}$. We note $z^k = r_k(e_i + f_j)$. As in (3.5) we have

$$|D[Du(\cdot, z_2^k)](x, z_1^k)|$$
$$\lesssim \left(|Du(x + z_2^k, z_1^k)|^{\theta_1} + |Du(x, z_1^k)|^{\theta_1}\right)\left(|Du(x + z_1^k, z_2^k)|^{\theta_2} + |Du(x, z_2^k)|^{\theta_2}\right) = q(x, z^k).$$

Integrating in x over Ω^{ε_k} we get (3.12).

In the case $\theta < 1$ and $u \in L^\infty(\Omega)$, noting $\theta_l' = \theta_l/\theta$ for $l = 1, 2$, by triangle inequality, we have for every $x \in \Omega^\varepsilon$, $z \in B_\varepsilon$ and $l \in \{1, 2\}$,

$$|Du(x, z_l)|^{\theta_l'} = |u(x + z_l) - u(x)|^{\theta_l'} = |u(x + z_l) - u(x)|^{(1-\theta)\theta_l/\theta}|Du(x, z_l)|^{\theta_l}$$
$$\leq 2^{(1-\theta)\theta_l/\theta}\|u\|_\infty^{(1-\theta)\theta_l/\theta}|Du(x, z_l)|^{\theta_l}.$$

Since $\theta_1' + \theta_2' = 1$, the case $[\theta < 1$ and $\|u\|_\infty \leq 1]$ follows form this inequality and from the case $\theta = 1$ applied to the pair (θ_1', θ_2').

Part (ii). Let $u \in L^1_{loc}(\Omega)$, $i \in \{1, \cdots, n_1\}$, $j \in \{1, \cdots, n_2\}$ and $\varphi \in C_c^\infty(\Omega)$. For $k \geq 1$, we note $z^k = r_k(e_i + f_j)$. We treat the three cases (a), (b), (c) of (3.13) separately.

Case (a). We proceed exactly as in (3.8). Using discrete integration by parts, we compute

$$\langle \mu_{i,j}, \varphi \rangle = \int_\Omega \partial_{e_i}\partial_{f_j}\varphi\, u\, dx = \lim_{k\uparrow\infty}\frac{1}{r_k^2}\left|\int_\Omega D[D\varphi(\cdot, -z_1^k)](x, -z_2^k)u(x)\, dx\right|$$
$$= \lim_{k\uparrow\infty}\frac{1}{r_k^2}\left|\int_\Omega \varphi D[Du(\cdot, z_1^k)](x, z_2^k)\, dx\right|$$
$$\leq \|\varphi\|_\infty \liminf_{k\uparrow\infty}\int_{\Omega^{\varepsilon_k}}\frac{|D[Du(\cdot, z_1^k)](x, z_2^k)|}{r_k^2}\, dx = \|\varphi\|_\infty \mathcal{G}_{i,j}(u).$$

This proves the claim.

Case (c). Let us show that this case follows from case (b). Let us assume that $1 \leq \theta_1 \leq \theta_2$. By Hölder inequality,

$$\int_{\Omega^{\varepsilon_k}}\frac{\left(|Du(x + z_2^k, z_1^k)| + |Du(x, z_1^k)|\right)\left(|Du(x + z_1^k, z_2^k)|^{\theta_2/\theta_1} + |Du(x, z_2^k)|^{\theta_2/\theta_1}\right)}{r_k^{2+\theta_2/\theta_1}}\, dx$$
$$\lesssim |\Omega|^{\frac{\theta_1-1}{\theta_1}}\left(\int_{\Omega^{\varepsilon_k}}\frac{\left(|Du(x + z_2^k, z_1^k)|^{\theta_1} + |Du(x, z_1^k)|^{\theta_1}\right)\left(|Du(x + z_1^k, z_2^k)|^{\theta_2} + |Du(x, z_2^k)|^{\theta_2}\right)}{r_k^{2\theta_1+\theta_2}}\, dx\right)^{\frac{1}{\theta_1}}.$$

Applying (3.13.b) with $\theta_1' = 1$, $\theta_2' = \theta_2/\theta_1$ and $P(\theta_1', \theta_2') = 1 + \theta_1' + \theta_2' = 2 + \theta_2/\theta_1$ yields (3.13.c).

Case (b). From now on we assume $\theta_1 \leq 1$, $\theta \geq 1$ and $p = P(\theta_1, \theta_2) = 1 + \theta$. Let $x \in \Omega$ and $z \in \mathbf{R}^n$ be such that $x \in \Omega^{|z|}$. Arguing as in (3.4), we obtain by triangle inequality,

$$|D[Du(\cdot, z_2)](x, z_1)|^{\theta_1} \lesssim |Du(x + z_2, z_1)|^{\theta_1} + |Du(x, z_1)|^{\theta_1},$$

and[3]

$$|D[Du(\cdot, z_2)](x, z_1)|^{1-\theta_1} \lesssim \left(|Du(x + z_1, z_2)|^{\theta_2} + |Du(x, z_2)|^{\theta_2}\right)^{(1-\theta_1)/\theta_2}.$$

---

[3]Remark that this is where we used the hypothesis $\theta_1 \leq 1$: in case (c) $(\theta_1, \theta_2 > 1)$, this inequality fails and the present method breaks down.

Taking the product of the last two inequalities with $(|Du(x+z_1, z_2)|^{\theta_2} + |Du(x, z_2)|^{\theta_2})^{(\theta-1)/\theta_2}$, we get

$$(|Du(x+z_1, z_2)|^{\theta_2} + |Du(x, z_2)|^{\theta_2})^{\frac{\theta-1}{\theta_2}} |D[Du(\cdot, z_2)](x, z_1)| \lesssim q(x, z).$$

We use this estimate in the form

$$(|Du(x+z_1, z_2)|^{\theta-1} + |Du(x, z_2)|^{\theta-1}) |D[Du(\cdot, z_2)](x, z_1)| \lesssim q(x, z). \tag{3.14}$$

We now set $\phi(s) := |s|^{\theta-1}s$ for $s \in \mathbf{R}$. Using the estimate

$$\left| |s|^{\theta-1}s - |t|^{\theta-1}t \right| \lesssim |s-t| \left(|s|^{\theta-1} + |t|^{\theta-1} \right),$$

and (3.14), we have

$$|D[\phi(Du(\cdot, z_2))](x, z_1)| \lesssim |D[Du(\cdot, z_2)](x, z_1)| (|Du(x+z_1, z_2)| + |Du(x, z_2)|)^{\theta-1}$$
$$\lesssim q(x, z).$$

Applying this estimate with $z = z^k$, integrating in $x \in \Omega^{\varepsilon_k}$ and sending k to ∞, we have:

$$\liminf_{k\to\infty} \int_{\Omega^{\varepsilon_k}} \frac{|D[\phi(Du(\cdot, z_2^k))](x, z_1^k)|}{r_k^{1+\theta}} \, dx \lesssim \mathcal{F}_{i,j}(u). \tag{3.15}$$

For $v : \Omega \to \mathbf{R}$, $e \in \mathbf{S}(X_1) \cup \mathbf{S}(X_2)$, $r > 0$ and $x \in \Omega^r$, we introduce the discrete derivative,

$$D_r v(x, e) := \frac{Dv(x, re)}{r} = \frac{v(x+re) - v(x)}{r}.$$

With this notation, (3.15) rewrites as

$$\liminf_{k\to\infty} \int_{\Omega^{\varepsilon_k}} |D_{r_k}[\phi(D_{r_k} u(\cdot, f_j))](x, e_i)| \, dx \lesssim \mathcal{F}_{i,j}(u). \tag{3.16}$$

For smooth functions u, this inequality would provide a control on the L^1-norm of the function $\partial_{e_i} \phi(\partial_{f_j} u)$. Here, we only assume $u \in L^1_{loc}(\Omega)$ and it is difficult to give a meaning to the nonlinear term $\phi(\partial_{f_j} u)$. For this reason, we linearize ϕ away from 0. For $\eta > 0$, we introduce the function ϕ_η given by

$$\phi_\eta(s) := \begin{cases} \frac{|s|^{\theta-1}s}{\eta^{\theta-1}} & \text{if } |s| \leq \eta, \\ s & \text{if } |s| \geq \eta, \end{cases}$$

so that ϕ_η is an odd, Lipschitz continuous function satisfying $0 \leq \phi'_\eta \leq \eta^{1-\theta}\phi'$ on $\mathbf{R}_+$. As a consequence,

$$|D_{r_k}[\phi_\eta(v)](x, e_i)| \leq \eta^{1-\theta}|D_{r_k}[\phi(v)](x, e_i)|.$$

Thus, from (3.16) we have,

$$\liminf_{k\to\infty} \int_{\Omega^{\varepsilon_k}} |D_{r_k}[\phi_\eta(D_{r_k} u(\cdot, f_j))](x, e_i)| \, dx \lesssim \eta^{1-\theta}\mathcal{F}_{i,j}(u). \tag{3.17}$$

For $\varphi \in C_c^\infty(\Omega, \mathbf{R})$, using the dominated convergence theorem and a discrete integration by parts, we compute

$$\langle \mu_{i,j}, \varphi \rangle = \lim_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} u \frac{\partial^2 \varphi}{\partial e_i \partial f_j} \, dx = \lim_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} u(x) D_{r_k}[D_{r_k} \varphi(\cdot, -e_i)](x, -f_j) \, dx$$

$$= \lim_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} D_{r_k} u(x, f_j) D_{r_k} \varphi(x, -e_i) \, dx. \tag{3.18}$$

We introduce the decomposition $D_{r_k} u(x, f_j) = \phi_\eta(D_{r_k} u(x, f_j)) - \chi_\eta(D_{r_k} u(x, f_j))$ where $\chi_\eta(s) := \phi_\eta(s) - s$ satisfies

$$\|\chi_\eta\|_\infty = \sup_{|s| \leq \eta} \left| s \left(1 - \frac{|s|^{\theta-1}}{\eta^{\theta-1}} \right) \right| \leq \eta.$$

For k large enough (so that $\operatorname{supp} \varphi \subset \Omega^{\varepsilon_k}$), we have

$$\left| \int_{\Omega^{\varepsilon_k}} D_{r_k} u(x, f_j) \, D_{r_k} \varphi(x, -e_i) \, dx \right|$$

$$\leq \left| \int_{\Omega^{\varepsilon_k}} \phi_\eta(D_{r_k} u(x, f_j)) \, D_{r_k} \varphi(x, -e_i) \, dx \right| + \left| \int_{\Omega^{\varepsilon_k}} \chi_\eta(D_{r_k} u(x, f_j)) \, D_{r_k} \varphi(x, -e_i) \, dx \right|$$

$$\leq \|\varphi\|_\infty \int_{\Omega^{\varepsilon_k}} |D_{r_k}[\phi_\eta(D_{r_k} u(\cdot, f_j))](x, e_i)| \, dx + \eta \|\partial_{e_i} \varphi\|_{L^1},$$

where we used a discrete integration by parts to treat the first term and the bound $\|\chi_\eta\|_\infty \leq \eta$ for the second term. Using (3.18) and (3.17), we obtain,

$$\langle \mu_{i,j}, \varphi \rangle \lesssim \|\varphi\|_\infty \eta^{1-\theta} \mathcal{F}_{i,j}(u) + \eta \|\nabla_1 \varphi\|_1.$$

Eventually, optimizing in η by choosing $\eta^\theta = \mathcal{F}_{i,j}(u) \|\varphi\|_\infty / \|\nabla_1 \varphi\|_1$, we get (3.13.b). $\qquad \square$

Proof of Proposition 3.3. For $\theta > 1$ the proposition corresponds to (3.13.b) and (3.13.c). For $\theta \leq 1$, the proposition follows from (3.12) and (3.13.a). $\qquad \square$

We can now show that if $\mathcal{E}(u) = 0$ then u depends only on the variables in X_1 or only on the variables in X_2.

Theorem 3.7 (Theorem I). *If $u \in L(\Omega)$ is such that $\mathcal{E}(u) = 0$, then $u \in S(\Omega)$.*

Proof. Let $u \in L(\Omega)$ with $\mathcal{E}(u) = 0$. As in the proof of Proposition 3.1, we may assume that u is bounded. Applying Lemma 3.4 (see also Remark 3.5) we find sequences $0 < r_k \leq \varepsilon_k \downarrow 0$ such that up to a change of coordinates,

$$\lim_{k \uparrow \infty} \sum_{i,j} \int_{\Omega^{\varepsilon_k}} \frac{q(x, r_k(e_i + f_j))}{r_k^{P(\theta_1, \theta_2)}} \, dx = 0. \tag{3.19}$$

By Lemma 3.6, we get

$$\mu := \mu[u] = \nabla_1 \nabla_2 u = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Integrating twice this identity, since Ω_1 and Ω_2 are connected, there exist two distributions $u_l \in \mathcal{D}'(\Omega_l), l \in \{1, 2\}$ such that $u = u_1 \otimes \mathbf{1}_{\Omega_2} + \mathbf{1}_{\Omega_1} \otimes u_2$ in $\mathcal{D}'(\Omega)$. Let $\varphi_2 \in C_c^\infty(\Omega_2, \mathbf{R}_+)$

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with $\int \varphi_2 = 1$. Using test functions of the form $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$, we have, since u is bounded,

$$\langle u_1, \varphi_1 \rangle \leq (\|u\|_\infty + |\langle u_2, \varphi_2 \rangle|) \|\varphi_1\|_{L^1} \qquad \text{for every } \varphi_1 \in C_c^\infty(\Omega_1).$$

We deduce that $u_1 \in L^\infty(\Omega_1)$ and similarly $u_2 \in L^\infty(\Omega_2)$. In conclusion, we have two functions $u_l \in L^\infty(\Omega_l)$, $l \in \{1, 2\}$ with $u(x) = u_1(x_1) + u_2(x_2)$ for $x \in \Omega$.
Since $q(x, r_k(e_i + f_j)) \geq |Du(x, r_k e_i)|^{\theta_1} |Du(x, r_k f_j)|^{\theta_2}$, (3.19) implies that

$$\lim_{k \uparrow \infty} \sum_{i,j} \int_{\Omega^{\varepsilon_k}} \frac{|Du(x, r_k e_i)|^{\theta_1} |Du(x, r_k f_j)|^{\theta_2}}{r_k^{P(\theta_1, \theta_2)}} \, dx = 0.$$

It is easy to check from the formula of $P(\theta_1, \theta_2)$ of Definition H that $\max(1, \theta_1) + \max(1, \theta_2) \leq P(\theta_1, \theta_2)$ for $\theta_1, \theta_2 > 0$. Moreover, for $l = 1, 2$, we have $|Du|^{\max(1, \theta_l)} \lesssim |Du|^{\theta_l} \|u\|_\infty^{(1-\theta_l)+}$. These inequalities lead to

$$\lim_{k \uparrow \infty} \left(\sum_i \int_{\Omega_1^{\varepsilon_k}} \frac{|Du_1(x_1, r_k e_i)|^{\max(1, \theta_1)}}{r_k^{\max(1, \theta_1)}} \, dx_1 \right) \left(\sum_j \int_{\Omega_2^{\varepsilon_k}} \frac{|Du_2(x_2, r_k f_j)|^{\max(1, \theta_2)}}{r_k^{\max(1, \theta_2)}} \, dx_2 \right) = 0.$$

Therefore, up to a subsequence, either

$$\lim_{k \uparrow \infty} \sum_i \int_{\Omega_1^{\varepsilon_k}} \frac{|Du_1(x_1, r_k e_i)|^{\max(1, \theta_1)}}{r_k^{\max(1, \theta_1)}} \, dx_1 = 0$$

$$\text{or} \qquad \lim_{k \uparrow \infty} \sum_j \int_{\Omega_2^{\varepsilon_k}} \frac{|Du_2(x_2, r_k f_j)|^{\max(1, \theta_2)}}{r_k^{\max(1, \theta_2)}} \, dx_2 = 0.$$

Let us assume without loss of generality that the former holds. Using Hölder inequality, this yields, for $i \in \{1, \cdots, n_1\}$,

$$\lim_{k \uparrow \infty} \int_{\Omega_1^{\varepsilon_k}} \frac{|Du_1(x_1, r_k e_i)|}{r_k} \, dx_1 = 0.$$

We deduce that the distributions $\partial_{e_i} u_1$ vanish for $i \in \{1, \cdots, n_1\}$ and thus (since Ω_1 is connected) u_1 is constant in Ω_1. This concludes the proof of the theorem. $\qquad \square$

Corollary 3.8 (Corollary L)**.** *For every $p \geq P(\theta_1, \theta_2)$, $r > 0$, and every measurable function u, if*

$$\int_{B_r} \int_{\Omega^r} \frac{|u(x + z_1) - u(x)|^{\theta_1} |u(x + z_2) - u(x)|^{\theta_2}}{|z|^{n+p}} dx dz < \infty, \tag{3.20}$$

then $u \in S(\Omega^r)$.

Proof of Corollary 3.8. Let $\rho := \frac{1}{|B_r \setminus B_{r/2}|} \chi_{B_r \setminus B_{r/2}}$. Then, letting $\varepsilon_k := 2^{-k} r$, it is readily seen that (3.20) implies that

$$\sum_{k \geq 0} \int_{\mathbf{R}^n} \rho_{\varepsilon_k}(z) \int_{(\Omega^r)^{\varepsilon_k}} \frac{|u(x + z_1) - u(x)|^{\theta_1} |u(x + z_2) - u(x)|^{\theta_2}}{|z|^p} \, dx \, dz = 0.$$

In particular, $\mathcal{E}_p(u; \Omega^r) = 0$ and we get $u \in S(\Omega^r)$ from Theorem 3.7. $\qquad \square$

In the space of Lipschitz continuous functions, we know from Proposition 2.2 (i) that the critical exponent p is larger than $1 + \theta$. We deduce from Theorem 3.7 that, as soon as $\theta_1 \leq 1$ or $\theta_2 \leq 1$, this critical exponent is indeed $1 + \theta$.

Proposition 3.9 (Proposition E)**.** *Assume that* $\min(\theta_1, \theta_2) \leq 1$*. Then, for* $u \in Lip(\Omega)$*,* $[\mathcal{E}_{1+\theta}(u) = 0] \implies u \in S(\Omega)$.

Proof. Let us assume without loss of generality that $\theta_1 \leq 1$. Let $u \in Lip(\Omega)$ with $\mathcal{E}_{1+\theta}(u) = 0$. If $\theta \geq 1$ then $P(\theta_1, \theta_2) = 1 + \theta$ and by Theorem 3.7, we have $u \in S(\Omega)$. If $\theta < 1$, by Lipschitz continuity of u, we have, for $x \in \Omega$ and $z \in \mathbf{R}^n \backslash \{0\}$ such that $x + z \in \Omega$,

$$\frac{|Du(x, z_1)|\,|Du(x, z_2)|^{\theta_2}}{|z|^{2+\theta_2}} \leq \|\nabla_1 u\|_\infty^{1-\theta_1} |z_1|^{1-\theta_1} \frac{|Du(x, z_1)|^{\theta_1}\,|Du(x, z_2)|^{\theta_2}}{|z|^{2+\theta_2}}$$

$$\leq \|\nabla_1 u\|_\infty^{1-\theta_1} \frac{|Du(x, z_1)|^{\theta_1}\,|Du(x, z_2)|^{\theta_2}}{|z|^{1+\theta}}.$$

This yields $\mathcal{E}_{2+\theta_2}^{1,\theta_2}(u) \leq \|\nabla_1 u\|_\infty^{1-\theta_1} \mathcal{E}_{1+\theta}^{\theta_1,\theta_2}(u) = 0$. We notice that $P(1, \theta_2) = 2 + \theta_2$, so that applying Theorem 3.7 with $\theta_1' = 1$, $\theta_2' = \theta_2$, we get again $u \in S(\Omega)$. $\qquad\square$

4 Quantitative control of the distance to $S(\Omega)$ in terms of $\mathcal{E}$

The aim of this section is to give a quantitative version of Theorem 3.7 by proving that for $u \in L^\infty(\Omega)$, $\mathcal{E}(u)$ controls the distance to $S(\Omega)$ in a strong sense. In order to obtain such a strong control, we use the fact that $\mu = \nabla_1 \nabla_2 u$ is a measure and thus restrict ourselves to the case (a) of (1.6), i.e. $\theta_1 + \theta_2 \leq 1$. We start by investigating the two dimensional case where the proof is simpler and the result stronger.

Theorem 4.1 (Theorem R)**.** *Assume that* $n_1 = n_2 = 1$ *and* $\theta = \theta_1 + \theta_2 \leq 1$*. Then, for every* $u \in L^\infty(\Omega)$ *with* $\|u\|_\infty \leq 1$ *and* $\mathcal{E}(u) < \infty$*, there exists* $\bar{u} \in S(\Omega)$ *such that* $u - \bar{u} \in BV(\Omega) \cap L^\infty(\Omega)$ *with the estimate*

$$\|u - \bar{u}\|_\infty + |\nabla[u - \bar{u}]|(\Omega) \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}. \tag{4.1}$$

Proof. To set notation, we assume that $\Omega = (0, \ell_1) \times (0, \ell_2)$, so that $(0, 0)$ is the bottom left corner of Ω. By Proposition 3.3, $\mu := \partial_1 \partial_2 u$ is a finite Radon measure with $|\mu|(\Omega) \lesssim \mathcal{E}(u)$. For $x \in \Omega$, we set $w(x) := \mu((0, x_1] \times (0, x_2])$ and notice that

$$\|w\|_\infty + |\nabla w|(\Omega) \leq 3|\mu|(\Omega) \lesssim \mathcal{E}(u). \tag{4.2}$$

Since $\partial_1 \partial_2(u - w) = 0$, arguing as in the proof of Theorem 3.7, we find two functions $u_l \in L^\infty((0, \ell_l))$, $l \in \{1, 2\}$ such that

$$u(x) = u_1(x_1) + u_2(x_2) + w(x) \qquad \text{for } x \in \Omega.$$

Thanks to (4.2), we only need to control u_1 or u_2. Let x_1 and $z_1 > 0$ be such that $0 < x_1 < x_1 + z_1 < \ell_1$, we have for $x_2 \in (0, \ell_2)$,

$$Du(x, z_1) = Du_1(x_1, z_1) + Dw(x, z_1) = Du_1(x_1, z_1) + \mu((x_1, x_1 + z_1] \times (0, x_2]).$$

In particular, for $x_2 \in (0, \ell_2)$,

$$|Du(x, z_1)| \geq \Big[|Du_1(x_1, z_1)| - |\mu|((x_1, x_1 + z_1] \times (0, \ell_2))\Big]_+ =: \psi_1(x_1, z_1). \qquad (4.3)$$

Similarly, for $x_1 \in (0, \ell_1)$ and $x_2, z_2 > 0$ such that $0 < x_2 < x_2 + z_2 < \ell_2$,

$$|Du(x, z_2)| \geq \Big[|Du_2(x_2, z_2)| - |\mu|((0, \ell_1) \times (x_2, x_2 + z_2])\Big]_+ =: \psi_2(x_2, z_2). \qquad (4.4)$$

Notice for later use that for $l \in \{1, 2\}$,

$$|\psi_l(x_l, z_l)| \leq |Du(x, z_l)| \leq 2\|u\|_\infty \leq 2, \quad \text{for every } x, z \in X \text{ with } x, x + z \in \Omega. \qquad (4.5)$$

Plugging inequalities (4.3) and (4.4) in the definition (1.3) of $\mathcal{E}_\varepsilon(u)$, we get,

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbf{R}^2} \rho_\varepsilon(z) \int_{(\varepsilon, \ell_1 - \varepsilon) \times (\varepsilon, \ell_2 - \varepsilon)} \frac{[\psi_1(x_1, z_1)]^{\theta_1} [\psi_2(x, z_2)]^{\theta_2}}{|z|^2} \, dx \, dz \leq \mathcal{E}(u).$$

Arguing as in the proof of Lemma 3.4, we may select $\sigma_1 + \sigma_2 = \sigma \in \partial B_1$, with $1/2 \leq |\sigma_l| \leq \sqrt{3}/2$ for $l = 1, 2$ and two sequences sequence (r_k), (ε_k) with $0 < r_k \leq \varepsilon_k \downarrow 0$ such that letting $z^k := r_k \sigma$,

$$\lim_{k \uparrow \infty} \left(\int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} \frac{[\psi_1(x_1, z_1^k)]^{\theta_1}}{r_k} \, dx_1 \right) \left(\int_{\varepsilon_k}^{\ell_2 - \varepsilon_k} \frac{[\psi_2(x_2, z_2^k)]^{\theta_2}}{r_k} \, dx_2 \right) \lesssim \mathcal{E}(u).$$

Extracting a further subsequence, there exists $l \in \{1, 2\}$ such that

$$\limsup_{k \uparrow \infty} \int_{\varepsilon_k}^{\ell_l - \varepsilon_k} \frac{[\psi_l(x_l, z_l^k)]^{\theta_l}}{r_k} \, dx_l \lesssim \mathcal{E}(u)^{\frac{1}{2}}. \qquad (4.6)$$

Without loss of generality, we assume $l = 1$. Using the definition of ψ_1 and Fubini, we write

$$\int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} |Du_1(x_1, z_1^k)| \, dx_1 \leq \int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} |\mu|((x_1, x_1 + z_1^k] \times (0, \ell_2)) \, dx_1 + \int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} \psi_1(x_1, z_1^k) \, dx_1$$

$$\leq z_1^k |\mu|(\Omega) + \left(\sup_{(\varepsilon_k, \ell_1 - \varepsilon_k)} \psi_1\left(\cdot, z_1^k\right) \right)^{1 - \theta_1} \int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} \left[\psi_1\left(x_1, z_1^k\right)\right]^{\theta_1} \, dx_1$$

$$\overset{(4.5)}{\lesssim} r_k \, \mathcal{E}(u) + \int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} \left[\psi_1\left(x_1, z_1^k\right)\right]^{\theta_1} \, dx_1.$$

Dividing by r_k and letting $k \uparrow \infty$, we deduce from (4.6),

$$\limsup_{k \uparrow \infty} \int_{\varepsilon_k}^{\ell_1 - \varepsilon_k} \frac{|Du_1(x_1, z_1^k)|}{r_k} \, dx_1 \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}.$$

We conclude that $u_1 \in BV(0, \ell_1)$ with $|\partial_1 u_1|(0, \ell_1) \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}$. This implies $\mathrm{osc}(u_1) \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}$, which together with (4.2) concludes the proof of the theorem. $\qquad \square$

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We now turn to the higher dimensional case $n > 2$. In order to obtain the analog of (4.1), we must define the higher dimensional counterpart of w. As will be clear from the proofs, the main requirements are that $\nabla_1 \nabla_2 w = \mu[u]$ and that for almost every $x = x_1 + x_2 \in \Omega$,

$$\int_{\Omega_2} w(x_1 + y_2)dy_2 = 0 \qquad \text{and} \qquad \int_{\Omega_1} w(y_1 + x_2)dy_1 = 0. \tag{4.7}$$

The unique function which satisfies these conditions is given by the formula:

$$w(x) := u(x) - \fint_{\Omega_2} u(x_1 + y_2) \, dy_2 - \fint_{\Omega_1} u(y_1 + x_2) \, dy_1 + \fint_{\Omega} u(y) \, dy. \tag{4.8}$$

For $x = x_1 + x_2 \in \Omega$, we have the decomposition $u(x) = u_1(x_1) + u_2(x_2) + w(x)$ where u_1 and u_2 are explicitly given by:

$$u_1(x_1) := \fint_{\Omega_2} u(x_1 + y_2) \, dy_2 - \frac{1}{2}\fint_{\Omega} udy, \qquad u_2(x_2) := \fint_{\Omega_1} u(y_1 + x_2) \, dy_1 - \frac{1}{2}\fint_{\Omega} udy. \tag{4.9}$$

We start by establishing some bounds on u_1, u_2 and w.

Proposition 4.2. *We assume that Ω_1 and Ω_2 are bounded extension domains. Let $u \in L^\infty(\Omega)$ and w, u_1, u_2 be given by (4.8)(4.9). Then, $w \in L^\infty(\Omega)$, $u_l \in L^\infty(\Omega_l)$ for $l = 1, 2$ with $u(x) = u_1(x_1) + u_2(x_2) + w(x)$ and we have the estimates*

$$\|w\|_\infty \le 4\|u\|_\infty, \qquad \|u_l\|_\infty \le (3/2)\|u\|_\infty \quad \text{for } l = 1, 2. \tag{4.10}$$

Moreover, if $\theta \le 1$ and $\mathcal{E}(u) < \infty$ then $w \in BV(\Omega)$ and denoting $\hat{n} := \max(n_1, n_2) \ge 2$,

$$\|w\|_{L^{\hat{n}/(\hat{n}-1)}(\Omega)} + |\nabla w|(\Omega) \lesssim \|u\|_\infty^{1-\theta} \mathcal{E}(u). \tag{4.11}$$

Proof. The estimates of (4.10) follow from the definitions (4.8), (4.9) and the triangle inequality.

We turn to (4.11). By (3.9) and the definition of w, $\nabla_1 \nabla_2 w = \mu$ with $|\mu|(\Omega) \lesssim \mathcal{E}(u)\|u\|_\infty^{1-\theta}$. It is thus enough to prove

$$\|w\|_{L^{\hat{n}/(\hat{n}-1)}(\Omega)} + |\nabla w|(\Omega) \lesssim |\mu|(\Omega). \tag{4.12}$$

By density (Ω is a bounded extension domain) we may assume that $u \in C^\infty(\overline{\Omega})$. For every $x_1 \in \Omega_1$, we have $W_{x_1} := \nabla_1 w(x_1 + \cdot) \in W^{1,1}(\Omega_2)$ with,

$$\int_{\Omega_1} \|\nabla_2 W_{x_1}\|_{L^1(\Omega_2)} \, dx_1 = \int_{\Omega_1} \int_{\Omega_2} |\nabla_2 \nabla_1 w| = |\mu|(\Omega).$$

By (4.7), we have $\fint_{\Omega_2} W_{x_1} = 0$ for every $x_1 \in \Omega_1$ and by Poincaré-Wirtinger inequality in $W^{1,1}(\Omega_2)$, this yields

$$\int_{\Omega} |\nabla_1 w| \, dx = \int_{\Omega_1} \|W_{x_1}\|_{L^1(\Omega_2)} \, dx_1 \lesssim \int_{\Omega_1} \|\nabla_2 W_{x_1}\|_{L^1(\Omega_2)} \, dx_1 = |\mu|(\Omega).$$

The analog bound on $\nabla_2 w$ shows that

$$\|\nabla w\|_{L^1(\Omega)} \lesssim |\mu|(\Omega). \tag{4.13}$$

We finally establish the $L^{\frac{\hat{n}}{\hat{n}-1}}$ bound. Assume without loss of generality that $\hat{n} = n_1 \geq n_2$. We then have by (4.7) and Poincaré-Wirtinger inequality in $W^{1,1}(\Omega_1)$ that

$$\|w\|_{L^{\frac{n_1}{n_1-1}}(\Omega)} \lesssim \left(\int_{\Omega_2} \left(\int_{\Omega_1} |\nabla_1 w| \, dx_1 \right)^{\frac{n_1}{n_1-1}} dx_2 \right)^{\frac{n_1-1}{n_1}} = \left\| \int_{\Omega_1} |\nabla_1 w(x_1 + \cdot)| \, dx_1 \right\|_{L^{\frac{n_1}{n_1-1}}(\Omega_2)}.$$

By Minkowski inequality, this leads to

$$\|w\|_{L^{\frac{n_1}{n_1-1}}(\Omega)} \lesssim \int_{\Omega_1} \|\nabla_1 w(x_1 + \cdot)\|_{L^{\frac{n_1}{n_1-1}}(\Omega_2)} \, dx_1.$$

We now apply the Poincaré-Wirtinger inequality to $\nabla_1 w(x_1 + \cdot)$ in $W^{1,1}(\Omega_2)$ to obtain (since $n_2 \leq n_1$),

$$\|w\|_{L^{\frac{n_1}{n_1-1}}(\Omega)} \lesssim \int_{\Omega_1} \|\nabla_2 \nabla_1 w(x_1 + \cdot)\|_{L^1(\Omega_2)} \, dx_1 = |\nabla_1 \nabla_2 u|(\Omega) = |\mu|(\Omega).$$

Together with (4.13) this proves (4.12). $\qquad\square$

Remark 4.3. We point out that the Poincaré-Wirtinger inequality gives a slightly stronger result than (4.11), namely that $\nabla_1 w \in \mathcal{M}(\Omega_1, BV(\Omega_2))$ (and similarly for $\nabla_2 w$).

We turn to the higher dimensional analog of Theorem 4.1.

Proposition 4.4. *Assume that Ω_1 and Ω_2 are bounded extension domains and that $\theta \leq 1$. Let $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leq 1$ and $\mathcal{E}(u) < \infty$. Using the notation $u = u_1 + u_2 + w$ of Proposition 4.2, there exist $l \in \{1, 2\}$ and $c \in \mathbf{R}$ such that,*

$$\|u_l - c\|_{L^{\frac{n_l}{n_l-1}}(\Omega)} \lesssim \mathcal{E}(u)^{\frac{1}{2}}. \tag{4.14}$$

Combining this estimate, Hölder inequality $\|u_l - c\|_{L^{\frac{\hat{n}}{\hat{n}-1}}(\Omega)} \lesssim \|u_l - c\|_{L^{\frac{n_l}{n_l-1}}(\Omega_l)}$ and Proposition 4.2, we deduce the following result.

Theorem 4.5 (Theorem S). *Let Ω_1 and Ω_2 be bounded extension domains and $\theta \leq 1$. Let $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leq 1$ and $\mathcal{E}(u) < \infty$. Noting $\hat{n} := \max(n_1, n_2)$, there exists $\bar{u} \in L^\infty(\Omega) \cap S(\Omega)$ such that,*

$$\|u - \bar{u}\|_{L^{\frac{\hat{n}}{\hat{n}-1}}(\Omega)} \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}$$

Proof of Proposition 4.4. Since $\|u\|_\infty \leq 1$, by (4.10) it is enough to establish (4.14) under the additional assumption that
$$\mathcal{E}(u) \leq \eta, \tag{4.15}$$
for some $\eta > 0$ only depending on Ω, n_1, n_2 and ρ.

Step 1. Let w, u_1 and u_2 be given by (4.8) and (4.9) so that $u(x) = u_1(x_1) + u_2(x_2) + w(x)$. Let us first recall that by (3.9), $\nabla_1 \nabla_2 w = \nabla_1 \nabla_2 u = \mu$ is a measure with $|\nabla_1 \nabla_2 w|(\Omega) = |\mu|(\Omega) \lesssim \mathcal{E}(u)$. Let (w_k) be a sequence of mollifications of w with $w_k \to w$ in $L^1(\Omega)$ and almost everywhere, $\nabla w_k \to \nabla w$ weakly star in $\mathcal{M}(\Omega)$,

$\|\nabla w_k\|_{L^1(\Omega)} \to |\nabla w|(\Omega)$ and $\|\nabla_1 \nabla_2 w_k\|_{L^1(\Omega)} \to |\mu|(\Omega)$ (which is possible since Ω_1 and Ω_2 are extension domains). For $k \geq 0$ and $x_1 \in \Omega_1$, we set

$$\psi_1^k(x_1) := \int_{\Omega_2} |\nabla_2 w_k|(x_1 + y_2) dy_2.$$

We have $\|\psi_1^k\|_{L^1(\Omega_1)} = \|\nabla_2 w_k\|_{L^1(\Omega)}$ and $\|\nabla_1 \psi_1^k\|_{L^1(\Omega)} \leq \|\nabla_1 \nabla_2 w_k\|_{L^1(\Omega)}$. Hence, by (4.11)

$$\limsup_{k\uparrow\infty} \|\psi_1^k\|_{L^1(\Omega_1)} + \|\nabla_1 \psi_1^k\|_{L^1(\Omega_1)} \lesssim \mathcal{E}(u).$$

Therefore, ψ_1^k is bounded in $W^{1,1}(\Omega_1)$ and up to extraction, ψ_1^k converges in $L^1(\Omega_1)$ to some function $\psi_1 \in BV(\Omega_1)$ with

$$\|\psi_1\|_{L^1(\Omega_1)} + |\nabla_1 \psi_1|(\Omega_1) \lesssim \mathcal{E}(u). \tag{4.16}$$

Let then for $\lambda > 0$,
$$\Omega_{1,\lambda} := \{x_1 \in \Omega_1 : \psi_1(x_1) \leq \lambda\}.$$

In the sequel we use that for a.e. $x_1 \in \Omega_{1,\lambda}$ and every $z_2 \in X_2$ such that $|(\Omega_2 - z_2) \cap \Omega_2| > 0$,

$$\begin{aligned}
\int_{(\Omega_2 - z_2)\cap\Omega_2} \frac{|Dw|(x, z_2)}{|z_2|} dx_2 &= \lim_{k\uparrow\infty} \int_{(\Omega_2 - z_2)\cap\Omega_2} \frac{|Dw_k|(x, z_2)}{|z_2|} dx_2 \\
&\leq \limsup_{k\uparrow\infty} \int_{(\Omega_2 - z_2)\cap\Omega_2} \int_0^1 \frac{|\nabla w_k(x + tz_2) \cdot z_2|}{|z_2|} dt \, dx_2 \\
&\leq \lim_{k\uparrow\infty} \int_{\Omega_2} |\nabla_2 w_k(x)| \, dx_2 = \psi_1(x_1) \leq \lambda. \tag{4.17}
\end{aligned}$$

Now, by co-area formula [AFP00, Theorem 3.40] for almost every $\lambda > 0$ the set $\Omega_{1,\lambda}$ is of finite perimeter in Ω_1 and denoting by $\partial\Omega_{1,\lambda}$ the measure-theoretic boundary of $\Omega_{1,\lambda}$,

$$\int_0^\infty \mathcal{H}^{n_1-1}(\partial\Omega_{1,\lambda} \cap \Omega_1) \, d\lambda = |\nabla_1 \psi_1|(\Omega_1) \overset{(4.16)}{\lesssim} \mathcal{E}(u).$$

We thus pick $\lambda_1 > 0$ such that

$$\frac{\mathcal{E}(u)^{\frac{1}{2}}}{4} \leq \lambda_1 \leq \frac{\mathcal{E}(u)^{\frac{1}{2}}}{2} \quad \text{and} \quad \mathcal{H}^{n_1-1}(\partial\Omega_{1,\lambda} \cap \Omega_1) \lesssim \mathcal{E}(u)^{\frac{1}{2}}. \tag{4.18}$$

We set $\widetilde{\Omega}_1 := \Omega_{1,\lambda_1}$ and define $\omega_1 := \Omega_1 \setminus \widetilde{\Omega}_1$ (so that $\mathcal{H}^{n_1-1}(\partial\Omega_{1,\lambda} \cap \Omega_1) = \mathcal{H}^{n_1-1}(\partial\omega_1 \cap \Omega_1)$). We notice for later use that

$$\lambda_1 |\omega_1| \leq \int_{\omega_1} |\psi_1| \, dx_1 \leq \|\psi_1\|_{L^1(\Omega_1)} \overset{(4.16)}{\lesssim} \mathcal{E}(u),$$

Therefore, since $\lambda_1 \geq \mathcal{E}(u)^{\frac{1}{2}}/4$, choosing η small enough in (4.15), we have,

$$|\omega_1| \lesssim \mathcal{E}(u)^{\frac{1}{2}} \leq |\Omega_1|/2. \tag{4.19}$$

We define ψ_2, λ_2, $\widetilde{\Omega}_2$ and ω_2 similarly.

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Step 2. Let (r_k), (ε_k) with $0 < r_k \leq \varepsilon_k \downarrow 0$ be given by Lemma 3.4 (recall Remark 3.5) and such that for every $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$,

$$\limsup_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} \frac{|Du(x, r_k e_i)|^{\theta_1} |Du(x, r_k f_j)|^{\theta_2}}{r_k^2} \, dx \lesssim \mathcal{E}(u). \tag{4.20}$$

Fix for the moment $k \geq 1$, $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$ and let $z^k := r_k(e_i + f_j)$. We define

$$A_1^k := \left\{ x \in \Omega^{\varepsilon_k} : |Du|(x, z_1^k) \geq \frac{1}{2}|Du_1|(x_1, z_1^k) \right\}$$

and similarly for A_2^k. Now we write

$$\left(\int_{\widetilde{\Omega}_1 \cap \Omega^{\varepsilon_k}} \frac{|Du_1|(x_1, z_1^k)}{r_k} \, dx_1 \right) \left(\int_{\widetilde{\Omega}_2 \cap \Omega^{\varepsilon_k}} \frac{|Du_2|(x_2, z_2^k)}{r_k} \, dx_2 \right)$$
$$= \int_{(\widetilde{\Omega}_1 \times \widetilde{\Omega}_2) \cap \Omega^{\varepsilon_k}} \frac{|Du_1|(x_1, z_1^k)|Du_2|(x_2, z_2^k)}{r_k^2} \, dx, \quad (4.21)$$

and we split the domain of integration as

$$(\widetilde{\Omega}_1 \times \widetilde{\Omega}_2) \cap \Omega^{\varepsilon_k} \subset \left[A_1^k \cap A_2^k \right] \cup \left[\left[(\Omega_1 \times \widetilde{\Omega}_2) \cap \Omega^{\varepsilon_k} \right] \setminus A_1^k \right] \cup \left[\left[(\widetilde{\Omega}_1 \times \Omega_2) \cap \Omega^{\varepsilon_k} \right] \setminus A_2^k \right].$$

In the first subdomain, we use the inequalities $|Du_1|(x_1, z_1^k) \leq 2|Du|(x, z_1^k)$ and $|Du_2|(x_2, z_2^k) \leq 2|Du|(x, z_2^k)$ valid by definition for $x \in A_1^k \cap A_2^k$. Then, since $\|u\|_\infty \leq 1$, we get

$$\limsup_{k \uparrow \infty} \int_{A_1^k \cap A_2^k} \frac{|Du_1|(x_1, z_1^k)|Du_2|(x_2, z_2^k)}{r_k^2} \, dx$$
$$\leq 4 \limsup_{k \uparrow \infty} \int_{\Omega^{\varepsilon_k}} \frac{|Du|^{\theta_1}(x, z_1^k)|Du|^{\theta_2}(x, z_2^k)}{r_k^2} \, dx \overset{(4.20)}{\lesssim} \mathcal{E}(u). \quad (4.22)$$

In the second subdomain, we use the triangle inequality to get for $x \notin A_1^k$,

$$|Du_1(x_1, z_1^k)| = |Du(x, z_1^k) - Dw(x, z_1^k)| \overset{x \notin A_1^k}{\leq} \frac{|Du_1(x_1, z_1^k)|}{2} + |Dw(x, z_1^k)|,$$

so that for $x \notin A_1^k$, there holds $|Du_1(x_1, z_1^k)| \leq 2|Dw|(x, z_1^k)$. This leads to

$$\int_{((\Omega_1 \times \widetilde{\Omega}_2) \cap \Omega^{\varepsilon_k}) \setminus A_1^k} \frac{|Du_1|(x_1, z_1^k)|Du_2|(x_2, z_2^k)}{r_k^2} \, dx$$
$$\leq 2 \int_{\widetilde{\Omega}_2 \cap \Omega_2^{\varepsilon_k}} \frac{|Du_2|(x_2, z_2^k)}{r_k} \left(\int_{\Omega_1^{\varepsilon_k}} \frac{|Dw|(x, z_1^k)}{r_k} \, dx_1 \right) \, dx_2.$$

By definition of $\widetilde{\Omega}_2$ and (4.17), the inner integral is bounded by $\lambda_2 \overset{(4.18)}{\leq} \mathcal{E}(u)^{\frac{1}{2}}/2$. Hence,

$$\limsup_{k \uparrow \infty} \int_{((\Omega_1 \times \widetilde{\Omega}_2) \cap \Omega^{\varepsilon_k}) \setminus A_1^k} \frac{|Du_1|(x_1, z_1^k)|Du_2|(x_2, z_2^k)}{r_k^2} \, dx$$
$$\leq \mathcal{E}(u)^{\frac{1}{2}} \limsup_{k \uparrow \infty} \int_{\widetilde{\Omega}_2 \cap \Omega_2^{\varepsilon_k}} \frac{|Du_2|(x_2, z_2^k)}{r_k} \, dx_2. \quad (4.23)$$

The third term is bounded similarly and we deduce from (4.21),(4.22) and (4.23),

$$\left[\limsup_{k\uparrow\infty} \int_{\widetilde{\Omega}_1 \cap \Omega_1^{\varepsilon k}} \frac{|Du_1|(x_1, z_1^k)}{r_k} \, dx_1 - \mathcal{E}(u)^{\frac{1}{2}}\right]_+$$

$$\left[\limsup_{k\uparrow\infty} \int_{\widetilde{\Omega}_2 \cap \Omega_2^{\varepsilon k}} \frac{|Du_2|(x_2, z_2^k)}{r_k} \, dx_2 - \mathcal{E}(u)^{\frac{1}{2}}\right]_+ \lesssim \mathcal{E}(u).$$

Recalling the notation $z_1^k = r_k e_i$, $z_2^k = r_k f_j$ and summing over $i \in \{1, \cdots, n_1\}$ and $j \in \{1, \cdots, n_2\}$, we see that up to extraction, we have

$$\text{For } i \in \{1, \cdots, n_1\}, \qquad \limsup_{k\uparrow\infty} \int_{\widetilde{\Omega}_1 \cap \Omega_1^{\varepsilon k}} \frac{|Du_1|(x_1, r_k e_i)}{r_k} \, dx_1 \lesssim \mathcal{E}(u)^{\frac{1}{2}}, \qquad (4.24)$$

$$\text{or for } j \in \{1, \cdots, n_2\}, \qquad \limsup_{k\uparrow\infty} \int_{\widetilde{\Omega}_2 \cap \Omega_2^{\varepsilon k}} \frac{|Du_2|(x_2, r_k f_j)}{r_k} \, dx_2 \lesssim \mathcal{E}(u)^{\frac{1}{2}}.$$

Step 3. Let us assume without loss of generality that the first possibility occurs and let us define the function

$$\tilde{u}_1 := u_1 \quad \text{in } \widetilde{\Omega}_1 \qquad \tilde{u}_1 := 0 \quad \text{in } \omega_1.$$

We fix again $i \in \{1, \cdots, n_1\}$ and use the short-hand notation $z_1^k := r_k e_i$. we may now estimate, $\int_{\Omega_1^{\varepsilon k}} |D\tilde{u}_1|(x_1, z_1^k)/r_k \, dx_1$. For this we use $|D\tilde{u}_1(x_1, z_1^k)| = |Du_1(x_1, z_1^k)|$ when $x_1, x_1 + z_1^k \in \widetilde{\Omega}_1$, $D\tilde{u}_1(x_1, z_1^k) = 0$ when $x_1, x_1 + z_1^k \in \omega_1$ and $|D\tilde{u}_1(x_1, z_1^k)| \leq \|u\|_\infty \leq 1$ when x_1 or $x_1 + z_1^k$ belongs to $\widetilde{\Omega}_1$ but not both. In the later case we thus have $|D\tilde{u}_1(x_1, z_1^k)| \leq |D\mathbf{1}_{\omega_1}(x_1, z_1^k)|$. This leads to

$$\liminf_{k\uparrow\infty} \int_{\Omega_1^{\varepsilon k}} \frac{|D\tilde{u}_1|(x_1, z_1^k)}{r_k} \, dx_1$$

$$\leq \limsup_{k\uparrow\infty} \int_{\widetilde{\Omega}_1 \cap \Omega_1^{\varepsilon k}} \frac{|Du_1|(x_1, z_1^k)}{r_k} \, dx_1 + \limsup_{k\uparrow\infty} \int_{\Omega_1^{\varepsilon k}} \frac{|D\mathbf{1}_{\omega_1}(x_1, z_1^k)|}{r_k} \, dx_1$$

$$\overset{(4.24)}{\lesssim} \mathcal{E}(u)^{\frac{1}{2}} + |e_i \cdot \nabla_1 \mathbf{1}_{\omega_1}|(\Omega_1).$$

By (4.18), we have $|\nabla_1 \mathbf{1}_{\omega_1}|(\Omega_1) = \mathcal{H}^{n_1-1}(\partial\omega_1 \cap \Omega_1) \lesssim \mathcal{E}(u)^{\frac{1}{2}}$ and we obtain that $e_i \cdot \nabla \tilde{u}_1$ is a measure in Ω_1 which satisfies $|e_i \cdot \nabla \tilde{u}_1|(\Omega_1) \lesssim \mathcal{E}(u)^{\frac{1}{2}}$. Summing over i we get $\tilde{u}_1 \in BV(\Omega_1)$ with

$$|\nabla \tilde{u}_1|(\Omega_1) \lesssim \mathcal{E}(u)^{\frac{1}{2}}. \qquad (4.25)$$

Let us note c the mean value of $\tilde{u}_1$ (in particular $|c| \leq \|u_1\|_\infty$). Using the Sobolev injection of $BV(\Omega_1)$ into $L^{n_1/(n_1-1)}(\Omega_1)$, we compute

$$\|u_1 - c\|_{L^{\frac{n_1}{n_1-1}}(\Omega_1)} \leq \|\tilde{u}_1 - c\|_{L^{\frac{n_1}{n_1-1}}(\Omega_1)} + \|u_1 - c\|_{L^{\frac{n_1}{n_1-1}}(\omega_1)}$$

$$\lesssim |\nabla_1 \tilde{u}_1|(\Omega_1) + 2\|u_1\|_\infty |\omega_1|^{\frac{n_1-1}{n_1}} \overset{(4.25)\&(4.10)}{\lesssim} \mathcal{E}(u)^{\frac{1}{2}} + |\omega_1|^{\frac{n_1-1}{n_1}}.$$

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By (4.19), we have $|\omega_1| \leq |\Omega_1|/2$ and thus by the relative isoperimetric inequality in Ω_1 (see for instance [Fed69, Lemma 4.5.2]),

$$|\omega_1|^{\frac{n_1-1}{n_1}} \lesssim \mathcal{H}^{n_1-1}(\partial \omega_1 \cap \Omega_1) \overset{(4.18)}{\lesssim} \mathcal{E}(u)^{\frac{1}{2}}$$

so that our final estimate is

$$\|u_1 - c\|_{L^{\frac{n_1}{n_1-1}}(\Omega_1)} \lesssim \mathcal{E}(u)^{\frac{1}{2}}.$$

This concludes the proof of the proposition. $\qquad\square$

Acknowledgments

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