Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics

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Abstract

We study the existence and uniqueness for an elliptic problem with a nonlinear dynamic boundary condition, relating the conormal derivative of the unknown to the time derivative of its jump across an internal interface. We firstly prove the well-posedness of a suitable linear version of this problem, by means of a classical result in abstract parabolic theory; then, we study the nonlinear case using a fixed point technique.
Our mathematical scheme is of interest in the modelling of electrical conduction in biological tissues.

1 Introduction

We are interested in the existence and uniqueness of the solution u of the problem

$$-\mathrm{div}(\sigma_1 \nabla u) = g(x,t,u)\,, \qquad \text{in } \Omega_1; \tag{1.1}$$

$$-\mathrm{div}(\sigma_2 \nabla u) = g(x,t,u)\,, \qquad \text{in } \Omega_2; \tag{1.2}$$

$$\sigma_1 \nabla u^{(\text{int})} \cdot \nu = \sigma_2 \nabla u^{(\text{out})} \cdot \nu\,, \qquad \text{on } \Gamma; \tag{1.3}$$

$$\alpha \frac{\partial}{\partial t}[u] + f([u]) = \sigma_2 \nabla u^{(\text{out})} \cdot \nu\,, \qquad \text{on } \Gamma; \tag{1.4}$$

$$[u](x,0) = S(x)\,, \qquad \text{on } \Gamma; \tag{1.5}$$

$$u(x,t) = 0\,, \qquad \text{on } \partial\Omega; \tag{1.6}$$

where the operators div and ∇ act only with respect to the space variable x. The notation in (1.1)–(1.4), (1.6), means that the indicated equations are in force in the relevant spatial domain for $0 < t < T$, and $T > 0$ is a given time.

Here, Ω is an open connected bounded subset of $\mathbf{R}^N$ such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_1 and Ω_2 are two disjoint open subsets of Ω, $\Gamma = \overline{\partial\Omega_1 \cap \Omega} = \overline{\partial\Omega_2 \cap \Omega}$ is a compact regular set, and $|\Gamma \cap \partial\Omega|_{N-1} = 0$. We assume that Ω, Ω_1 and Ω_2 have Lipschitz boundaries. Moreover, σ_1, σ_2 and α are positive constants, and ν is the normal unit vector to Γ pointing into Ω_2. Since u_ε is not in general continuos across Γ we have set

$$u^{\text{(int)}} := \text{ trace of } u_{|\Omega_1} \text{on } \Gamma; \quad u^{\text{(out)}} := \text{ trace of } u_{|\Omega_2} \text{ on } \Gamma.$$

Indeed we refer conventionally to Ω_1 as to the *interior domain*, and to Ω_2 as to the *outer domain*. We also denote

$$[u] := u^{\text{(out)}} - u^{\text{(int)}} .$$

Similar conventions are employed for other quantities; for example (1.3) can be rewritten as

$$[\sigma \nabla u \cdot \nu] = 0 \,, \qquad \text{on } \Gamma,$$

where

$$\sigma = \sigma_1 \quad \text{in } \Omega_1, \qquad \sigma = \sigma_2 \quad \text{in } \Omega_2.$$

Suitable assumptions on the data g, f and S will be given in next section.

We point out that our results can be reproduced also in a periodic setting (see Remarks 3.3 and 3.5), a situation usually occurring in homogenization problems [1], [2].

The mathematical interest of this problem is due to the presence of the dynamic condition (1.4), and to the non linear terms f and g. Moreover, in the periodic setting, our approach yields estimates for the solution which are independent on the spatial period, a result which is of interest in homogenization [2].

Due to the presence of the nonlinear dynamic boundary condition (1.4), our problem can be compared to the problems studied in [4], [5] and [9] (in the last paper the partial differential equations are of parabolic type). However, equation (1.4) is set on the interface Γ and involves the jump $[u]$ of the unknown u across Γ, while the dynamic condition in [4], [5] and [9] is set on the boundary of the domain and involves the trace of the unknown. Moreover, our method, which relies on a fixed point theorem, is different from the semigroup theory used in [4], [5] and the potential theory used in [9].

Our mathematical scheme models the electrical conduction in biological tissues [3], where one of the phases, Ω_2, is the extracellular space, the other one, Ω_1, is the intracellular space, the interface Γ is the cell membrane and the unknown u is the electrical potential. We note that the dependence of the electrical potential on time is not merely parametrical, due to the equation (1.4), taking into account the nonlinear conductive/capacitive behaviour of the cell membrane. This model permits to investigate the Maxwell–Wagner interfacial polarization effect [6], that is the response of biological tissues to the injection of electrical currents in the radiofrequency range, which is relevant in clinical applications.

An alternative approach to the electrical conduction in biological tissues can be obtained by homogenization of the present model [1], [2], since the extracellular and intracellular spaces are finely mixed phases. Of course, the well-posedness results of this paper are needed even in that context.

The paper is organized as follows: in Section 2 we set our notations and assumptions on the data. Moreover, we state two crucial tools in the development of the paper: the Poincaré-like inequality (2.10), applying to functions that jumps on the interface Γ, and the energy estimate (2.17). In Section 3 we prove existence and uniqueness of weak solutions to (1.1)–(1.6). The main idea is to prove firstly the well-posedness of a suitable linear version of our problem, by means of a classical result in abstract parabolic theory (see, e.g., [10]), properly adapted to this situation. Then, the nonlinear case is studied using a fixed point technique.

2 Preliminary results

2.1 Notations

Given an open bounded and regular set A, we denote by $L^p(A)$, $1 \leq p \leq \infty$, and by $W^{k,p}(A)$, $k \in \mathbf{N}$, the standard Lebesgue and Sobolev spaces. In particular, $H^k(A)$ stands for $W^{k,2}(A)$. We denote also by $H^1_o(A)$ the subset of $H^1(A)$ of those functions which vanish on ∂A. Moreover, we recall that for any function in $H^k(A)$, its trace on the boundary belongs to the fractionary Sobolev space $H^{k-1/2}(\partial A)$.
Finally, $BV(A)$ indicates the space of functions in $L^1(A)$, whose distributional derivatives are measures of bounded total variation.

Let I be a real interval and X a Banach space. We denote by $L^p(I;X)$, $1 \leq p \leq \infty$, and by $H^k(I;X)$, $k \in \mathbf{N}$, the spaces of measurable functions $h : I \to X$ such that

$$\|h\|^p_{L^p(I;X)} = \int_I \|h(t)\|^p \, dt < +\infty \quad \text{if } 1 \leq p < +\infty \; ;$$

$$\|h\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t \in I} |h(t)| < +\infty \quad \text{if } p = +\infty \; ;$$

$$\|h\|^2_{H^k(I;X)} = \int_I \|h(t)\|^2 \, dt + \sum_{j=1}^{k} \int_I \|\frac{d^j h}{dt^j}(t)\|^2 \, dt < +\infty \; .$$

Moreover, we denote by $C^k(I;X)$, $0 \leq k \leq \infty$, the spaces of the continuous functions $h : I \to X$, having continuous derivatives, up to order k (continuous derivatives of any order, in the case $k = \infty$).

Finally, $L^p_{\rm loc}(A)$ is the space of those functions belonging to $L^p(K)$, for every open set K compactly contained in A, and analogously for the other functional spaces considered in this paper.

Let $Y = \Pi(a_j, b_j) := (a_1, b_1)\times(a_2, b_2)\times\cdots\times(a_N, b_N)$, with $a_j < b_j$, for every $j = 1,\dots,N$, be a cell in $\mathbf{R}^N$. A function defined on $\mathbf{R}^N$ is said to be Y-periodic if it is periodic of period $b_j - a_j$ with respect to each variable x_j, with $1 \le j \le N$. In such a case, Y will be said the periodicity cell.

2.2 Assumptions on the data

The function f in (1.4) fulfils

$$f \in W^{1,\infty}(\mathbf{R}) , \qquad f(0) = 0 . \tag{2.1}$$

The function g in equations (1.1) and (1.2) is a Caratheodory function such that

$$\begin{aligned} & g(\cdot,\cdot,0) \in L^\infty(\Omega\times(0,T)) \\ & |g(x,t,s) - g(x,t,s')| \le L|s-s'| , \qquad \text{for all } x\in\Omega,\ t>0,\ s,\ s'\in\mathbf{R}, \end{aligned} \tag{2.2}$$

for a constant L independent of x, t, s, s', and such that

$$LC < \gamma_0 , \tag{2.3}$$

where γ_0 is a suitable positive constant depending on the parameters in (1.1)–(1.6) and on $\|f'\|_\infty$, and $C > 0$ is the constant appearing in Poincaré's inequality in Proposition 2.2 or Corollary 2.3.
Moreover, we assume that $S \in H^{1/2}_o(\mathit{\Gamma},\Omega)$, where $H^{1/2}_o(\mathit{\Gamma},\Omega)$ is the space of the traces on $\mathit{\Gamma}$ of the functions belonging to $H^1_o(\Omega)$. Clearly, $H^{1/2}_o(\mathit{\Gamma},\Omega)$ is a Hilbert space. Note that $H^{1/2}_o(\mathit{\Gamma},\Omega) = H^{1/2}(\mathit{\Gamma})$, if $\mathit{\Gamma}\cap\partial\Omega = \emptyset$.

In the following we denote by γ a generic positive constant, taking in principle different values in different occurrences. These constants depend only on the geometrical properties of Ω, Ω_1 and Ω_2, and on α, σ_1, σ_2, T, L, and on the global bounds for g and f'. We also denote by ∇F the pointwise spatial gradient of a given function F, while DF denotes its variation measure (in the BV sense). In general ∇F and DF differ, as we often consider functions with jumps.
Throughout the paper, the dependence on the spatial variables of the involved functions is understood, even if omitted.

REMARK 2.1 - By means of minor changes in our approach, we may consider cases with non vanishing sources appearing on the right hand sides of (1.3), (1.4).

Of special interest in applications is the case of nonvanishing Dirichlet data, where (1.6) is replaced with

$$u(x,t) = \hat{u}(x,t) , \quad \text{on } \partial\Omega, \text{ where } \hat{u}\in L^2(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)). \tag{2.4}$$

In this case we look at the homogeneous Dirichlet problem for $v = u - \hat{u}$, i.e.,

$$-\mathrm{div}(\sigma\nabla v) = \sigma\Delta\hat{u} + g(x,t,v+\hat{u})\,, \qquad \text{in } \Omega_1, \Omega_2; \tag{2.5}$$

$$[\sigma\nabla v\cdot\nu] = -[\sigma]\nabla\hat{u}\cdot\nu\,, \qquad \text{on } \mathit{\Gamma}; \tag{2.6}$$

$$\alpha\frac{\partial}{\partial t}[v] + f([v]) = \sigma_2\nabla v^{(\mathrm{out})}\cdot\nu + \sigma_2\nabla\hat{u}\cdot\nu\,, \qquad \text{on } \mathit{\Gamma}; \tag{2.7}$$

$$[v](x,0) = S(x)\,, \qquad \text{on } \mathit{\Gamma}; \tag{2.8}$$

$$v(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{2.9}$$

2.3 Poincaré inequalities

In this section we will state a generalized version of the Poincaré's inequality, which applies to functions admitting jumps. For a similar result, see also Proposition 3.8 of [8].

Proposition 2.2 *Let $u : \Omega \to \mathbf{R}$ be given by*

$$u_{|\Omega_1} = u_{1|\Omega_1}\,, \quad u_{|\Omega_2} = u_{2|\Omega_2}\,, \qquad u_1\,,\, u_2 \in H^1_o(\Omega)\,.$$

Then

$$\int_\Omega u^2\,\mathrm{d}x \le C\Big\{\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_{\mathit{\Gamma}} [u]^2\,\mathrm{d}\sigma\Big\}\,, \tag{2.10}$$

where C depends on Ω and $\mathit{\Gamma}$.

Proof - As u^2 is of class $W^{1,1}$ both in Ω_1 and in Ω_2, $u^2 \in BV(\Omega)$, and the usual contradiction argument, exploiting $u^2 = 0$ on $\partial\Omega$ in the sense of traces, shows that

$$\int_\Omega u^2\,\mathrm{d}x \le \gamma|Du^2(\Omega)| \le \gamma\int_\Omega |u||\nabla u|\,\mathrm{d}x + \gamma\int_{\mathit{\Gamma}} |[u^2]|\,\mathrm{d}\sigma\,, \qquad \gamma = \gamma(\Omega)\,. \tag{2.11}$$

Indeed the singular part of the variation of u (and therefore of u^2) is concentrated on $\mathit{\Gamma}$. We estimate above last integral by

$$\int_{\mathit{\Gamma}} |[u]|(|u^{(\mathrm{int})}| + |u^{(\mathrm{out})}|)\,\mathrm{d}\sigma \le \delta^{-1}\int_{\mathit{\Gamma}} [u]^2\,\mathrm{d}\sigma + \delta\int_{\mathit{\Gamma}} (|u^{(\mathrm{int})}|^2 + |u^{(\mathrm{out})}|^2)\,\mathrm{d}\sigma\,, \tag{2.12}$$

for a $\delta \in (0,1)$ to be chosen presently. Using standard trace inequalities, we check that

$$\int_{\mathit{\Gamma}} (|u^{(\mathrm{int})}|^2 + |u^{(\mathrm{out})}|^2)\,\mathrm{d}\sigma \le \gamma\int_\Omega \big(u^2 + |\nabla u|^2\big)\,\mathrm{d}x\,, \tag{2.13}$$

where $\gamma = \gamma(\mathit{\Gamma}, \Omega_1, \Omega_2)$. A further application of Cauchy-Schwartz inequality to (2.11) yields

$$\int_\Omega u^2\,\mathrm{d}x \le \gamma\delta^{-1}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \gamma\delta^{-1}\int_{\mathit{\Gamma}} [u]^2\,\mathrm{d}\sigma + \gamma\delta\int_\Omega u^2\,\mathrm{d}x\,,$$

whence (2.10) on selecting a small enough δ. $\square$

In many applications (see e.g. [1] and [2]), it is useful to obtain a precise dependence of the constant γ in the proof of previous lemma, and then of C in (2.10), on $\Omega_1,\ \Omega_2$ and $\varGamma$. To this purpose, set $E_k = k^{-N}E$, where E is a given regular Y-periodic set. Clearly, it follows that $E_k \cap k^{-N}Y$ has a measure of order k^{-N} and its boundary area is of order k^{1-N}, $k \in \mathbf{N}$. Moreover, assume that $\Omega_1 = E_k \cap \Omega$, $\Omega_2 = \Omega \setminus \Omega_1$ and $\varGamma = \overline{\partial\Omega_1 \cap \Omega}$. Note that $|\Omega_1| \sim 1$, while $|\varGamma| \sim k$. Then, the following result holds (see Lemma 7.1 in [2]).

Corollary 2.3 *Assume that Ω_1, $\varGamma$ and E are as stated before. Let $u : \Omega \to \mathbf{R}$ be as in Proposition 2.2. Then*

$$\int_\Omega u^2 \,\mathrm{d}x \le C\Big\{ \int_\Omega |\nabla u|^2 \,\mathrm{d}x + k \int_\varGamma [u]^2 \,\mathrm{d}\sigma \Big\} . \tag{2.14}$$

Here C depends only on Ω and E and not on k.

Proof - Arguing as in the proof of Proposition 2.2 and replacing δ with δ/k in (2.12), we obtain

$$\begin{aligned} \int_\Omega u^2 \,\mathrm{d}x \le \gamma \int_\Omega |u||\nabla u| \,\mathrm{d}x \quad &+ \gamma(\delta/k)^{-1} \int_\varGamma [u]^2 \,\mathrm{d}\sigma \\ &+ \gamma(\delta/k) \int_\varGamma (|u^{(\mathrm{int})}|^2 + |u^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma , \end{aligned} \tag{2.15}$$

for a $\delta \in (0,1)$ to be chosen presently. Setting $Q_i = k^{-N}Y + z_i$, $i = 1, \dots, k^N$, and using the periodic structure of Ω_1 together with standard trace inequalities, we check that for each $i \in \mathbf{N}$

$$\int_{\varGamma\cap Q_i} (|u^{(\mathrm{int})}|^2 + |u^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma \le \gamma k \int_{\Omega\cap Q_i} \big(u^2 + k^{-2}|\nabla u|^2\big) \,\mathrm{d}x , \tag{2.16}$$

where now $\gamma = \gamma(E, Y)$ does not depend on Q_i. Next we add (2.16) over all the cells covering Ω, and use the resulting inequality in (2.15). Then, applying again Cauchy-Schwartz inequality and selecting a small enough δ, we may conclude following the same argument as in the proof of Proposition 2.2. $\square$

REMARK 2.4 - If E^c is connected, one can prove an estimate similar to (2.14), but with the factor k formally replaced by k^{-1} (in this spirit, see Lemma 6 of [7]).
We note also that, as a consequence of (2.14), the Lipschitz constant L in (2.3) is independent of the scaling k.

2.4 Energy estimates

On multiplying (1.1), (1.2) by u and integrating formally by parts, using also (1.3)–(1.5), (2.2) and Proposition 2.2 or Corollary 2.3, we obtain, for $0 < t < T$,

$$
\begin{aligned}
&\int_0^t \int_\Omega \sigma |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{\alpha}{2} \int_\Gamma [u]^2(x,t)\,\mathrm{d}\sigma + \int_0^t \int_\Gamma [u] f([u])\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= \frac{\alpha}{2} \int_\Gamma S^2(x)\,\mathrm{d}\sigma + \int_0^t \int_\Omega g(x,t,u) u \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{\alpha}{2} \int_\Gamma S^2(x)\,\mathrm{d}\sigma + L \int_0^t \int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega g(x,t,0) u \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{\alpha}{2} \int_\Gamma S^2(x)\,\mathrm{d}\sigma + LC \int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}\tau + LC \int_0^t \int_\Gamma [u]^2(x,t)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad + \delta^{-1} \int_0^t \int_\Omega g^2(x,t,0)\,\mathrm{d}x\,\mathrm{d}\tau + \delta \int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \delta \int_0^t \int_\Gamma [u]^2(x,t)\,\mathrm{d}\sigma\,\mathrm{d}\tau \,,
\end{aligned}
$$

whence, by Gronwall's inequality, (2.3) and using the linear growth of f, it follows

$$
\begin{aligned}
&\int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Gamma [u]^2(x,t)\,\mathrm{d}\sigma \le \\
&\gamma \left(\int_\Gamma S^2(x)\,\mathrm{d}\sigma + \int_0^t \int_\Omega g^2(x,t,0)\,\mathrm{d}x\,\mathrm{d}\tau \right) \le \gamma \,,
\end{aligned}
\tag{2.17}
$$

where γ depends on Ω, T, L, σ, α, $\|S\|_{L^2(\Gamma)}$, $\|g(\cdot,\cdot,0)\|_{L^\infty(\Omega\times(0,T))}$ and the Poincaré constant C. This energy estimate will be instrumental in the following.

3 Existence and uniqueness of weak solutions

Given a space $X(\Omega)$ of real functions defined over Ω, we denote by $X_o(\Omega)$ the subspace of $X(\Omega)$ comprised of those functions which have zero trace on the boundary $\partial\Omega$. Then we set

$$
\begin{aligned}
\mathcal{X}_o(\Omega) = \{ u = (u_1, u_2) \mid u_1 := u_{|\Omega_1} = u'_{|\Omega_1} \,,\ u_2 := u_{|\Omega_2} = u''_{|\Omega_2} \,, \\
\text{with } u', u'' \in X_o(\Omega) \} \,, \\
\|u\|_{\mathcal{X}_o(\Omega)} = \|u_1\|_{X(\Omega_1)} + \|u_2\|_{X(\Omega_2)} \,.
\end{aligned}
$$

Let us firstly consider the linear case; i.e., $g(x,t,s) \equiv P(x,t)$ independent of $s \in \mathbf{R}$ and $f(s) = \beta s$, with $\beta \in \mathbf{R}$. We also assume more general conditions in (1.3) and

(1.4):

$$
\begin{aligned}
-\sigma\Delta u &= P(t)\,, && \text{in } \Omega_1,\, \Omega_2; && (3.1)\\
[\sigma\nabla u\cdot\nu] &= Q(t)\,, && \text{on } \Gamma; && (3.2)\\
\alpha\frac{\partial}{\partial t}[u]+\beta[u] &= \sigma_2\nabla u^{(\text{out})}\cdot\nu + h(t)\,, && \text{on } \Gamma; && (3.3)\\
[u](x,0) &= S\,, && \text{on } \Gamma\,, && (3.4)\\
u(x,t) &= 0\,, && \text{on } \partial\Omega. && (3.5)
\end{aligned}
$$

Here $P \in L^2(\Omega\times(0,T))$, $Q \in L^2(0,T;L^2(\Gamma))$, $h \in L^2(0,T;L^2(\Gamma))$ and $S \in H_o^{1/2}(\Gamma,\Omega)$. The constant $\beta \in \mathbf{R}$ equals $f'(0)$ in many interesting applications (see [1] and [2]).

In order to state the existence of a solution of this problem, we first consider its stationary formulation, given by

$$
\begin{aligned}
-\sigma\Delta u &= P\,, && \text{in } \Omega_1,\, \Omega_2; && (3.6)\\
[\sigma\nabla u\cdot\nu] &= Q\,, && \text{on } \Gamma; && (3.7)\\
[u] &= S\,, && \text{on } \Gamma; && (3.8)\\
u &= 0\,, && \text{on } \partial\Omega; && (3.9)
\end{aligned}
$$

where $P \in L^2(\Omega)$, $Q \in L^2(\Gamma)$, and $S \in H_o^{1/2}(\Gamma,\Omega)$. The rigorous weak formulation of (3.6)–(3.9) is

$$
\int_\Omega \sigma\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_\Gamma Q\varphi\,\mathrm{d}\sigma = \int_\Omega P\varphi\,\mathrm{d}x\,, \tag{3.10}
$$

for all $\varphi \in H_o^1(\Omega)$, with $u \in \mathcal{H}_o^1(\Omega)$, such that $[u] = S$ in the sense of traces on Γ.

Lemma 3.1 *Let S, P, Q be as above. Then* (3.6)–(3.9) *(i.e.,* (3.8), (3.10)*), has a unique solution $u \in \mathcal{H}_o^1(\Omega)$, satisfying*

$$
\|u\|_{\mathcal{H}_o^1(\Omega)} \le \gamma\big(\|S\|_{H^{1/2}(\Gamma)} + \|P\|_{L^2(\Omega)} + \|Q\|_{L^2(\Gamma)}\big)\,. \tag{3.11}
$$

Proof - Let us begin with the auxiliary problems

$$
\begin{aligned}
&-\sigma_1\Delta u_1 = P \quad \text{in } \Omega_1, \qquad u_1 = 0 \quad \text{on } \Gamma, \qquad u_1 = 0 \quad \text{on } \partial\Omega,\\
&-\sigma_2\Delta u_2 = P \quad \text{in } \Omega_2, \qquad u_2 = S \quad \text{on } \Gamma, \qquad u_2 = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.12}
$$

Existence and uniqueness of the solutions to these Dirichlet problems are classical and we have that $\bar u = (u_1, u_2) \in \mathcal{H}_o^1(\Omega)$. Note that $[\bar u] = S$. Moreover

$$
\|\bar u\|_{\mathcal{H}_o^1(\Omega)} \le \gamma\big(\|S\|_{H^{1/2}(\Gamma)} + \|P\|_{L^2(\Omega)}\big)\,. \tag{3.13}
$$

Then we seek u in the form

$$
u = w + \bar u\,, \quad \text{where } w \in H_o^1(\Omega).
$$

More exactly, the problem for w is

$$\int_\Omega \sigma\nabla w\cdot\nabla\varphi\,\mathrm{d}x = -\int_\Omega \sigma\nabla\bar u\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega P\varphi\,\mathrm{d}x - \int_\Gamma Q\varphi\,\mathrm{d}\sigma\,, \tag{3.14}$$

for all $\varphi\in H^1_o(\Omega)$. The right hand side of this equation defines a continuous linear operator in $H^1_o(\Omega)$, whose norm is bounded exactly by the right hand side of (3.11), as a consequence of (3.13). The left hand side defines a continuous bilinear form, which is easily seen to be coercive in the space $H^1_o(\Omega)$. Thus, an application of Lax-Milgram theorem yields existence and uniqueness of a solution w as above, and therefore existence of u. Now (3.11) follows from (3.13) and from standard continuous dependence of w on the data in (3.14). Uniqueness of solutions to (3.10) is immediate. $\square$

Next let us come back to the evolution problem (3.1)–(3.5). Rigorously, the problem can be formulated as: find a function $u\in L^2(0,T;\mathcal{H}^1_o(\Omega))$, such that $[u]\in C(0,T;L^2(\Gamma))$, satisfying (3.4) and

$$\int_0^T\int_\Omega \sigma\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Gamma \varphi^{(\mathrm{int})}Q\,\mathrm{d}\sigma\,\mathrm{d}t - \alpha\int_0^T\int_\Gamma [u]\frac{\partial}{\partial t}[\varphi]\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+\beta\int_0^T\int_\Gamma [u][\varphi]\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T\int_\Gamma [\varphi]h\,\mathrm{d}\sigma\,\mathrm{d}t = \int_0^T\int_\Omega P\varphi\,\mathrm{d}x\,\mathrm{d}t\,, \tag{3.15}$$

for all $\varphi\in L^2(0,T;\mathcal{H}^1_o(\Omega))$ such that $[\varphi]\in H^1(0,T;L^2(\Gamma))$, and $\varphi=0$ at $t=0$, $t=T$.

Define, for almost every $t\in(0,T)$, $w(t)\in H^1_o(\Omega)$ as the solution to (3.6)–(3.9) with $P=P(t)$, $Q=Q(t)$, $S=0$. Then $u=v+w$, where v solves in the sense above

$$-\sigma\Delta v = 0\,,\qquad \text{in } \Omega_1,\Omega_2; \tag{3.16}$$
$$[\sigma\nabla v\cdot\nu] = 0\,,\qquad \text{on } \Gamma; \tag{3.17}$$
$$\alpha\frac{\partial}{\partial t}[v] + \beta[v] = \sigma_2\nabla v^{(\mathrm{out})}\cdot\nu + q(t)\,,\qquad \text{on } \Gamma; \tag{3.18}$$
$$[v](x,0) = S\,,\qquad \text{on } \Gamma; \tag{3.19}$$
$$v(x,t) = 0\,,\qquad \text{on } \partial\Omega; \tag{3.20}$$

with

$$q(t) = h(t) + \sigma_2\nabla w(t)^{(\mathrm{out})}\cdot\nu\,,\qquad 0<t<T\,. \tag{3.21}$$

Clearly, $q\in L^2(0,T;H^{-1/2}(\Gamma,\Omega))$, where $H^{-1/2}(\Gamma,\Omega)$ denotes the topological dual space of $H^{1/2}_o(\Gamma,\Omega)$. Indeed, for a.e. $t\in(0,T)$ and every $r\in H^{1/2}_o(\Gamma,\Omega)$, we have

$$\langle q(t),r\rangle = \int_\Gamma h(t)r\;\mathrm{d}\sigma - \langle\langle\sigma_2\Delta w(t),\tilde r\rangle\rangle - \int_{\Omega_2}\sigma_2\nabla w(t)\nabla\tilde r\;\mathrm{d}x$$

where $\tilde{r}$ is an extension to $H^1_o(\Omega)$ of r, such that $\|\tilde{r}\|_{H^1_o(\Omega)} \le \gamma \|r\|_{H^{1/2}(\Gamma)}$, and $\langle\cdot,\cdot\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle$ denote the duality pairing between $H^{-1/2}(\Gamma,\Omega)$, $H^{1/2}_o(\Gamma,\Omega)$ and $H^{-1}(\Omega)$, $H^1_o(\Omega)$, respectively.

Theorem 3.2 *For any given $q \in L^2(0,T;H^{-1/2}(\Gamma,\Omega))$, problem* (3.16)–(3.20) *admits a unique solution $v \in L^2(0,T;\mathcal{H}^1_o(\Omega))$ with $[v] \in C(0,T;L^2(\Gamma))$. Therefore, problem* (3.1)–(3.5) *admits a unique solution.*

Proof - The proof follows from an application of abstract parabolic theory, as summarized for example in [10], chapter 23. We consider the three Hilbert spaces $H^{1/2}_o(\Gamma,\Omega) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma,\Omega)$, and a bilinear form on the first one

$$a(\rho,r) = \int_\Omega \sigma \nabla V_{(\rho)} \cdot \nabla V_{(r)} \,\mathrm{d}x\,, \qquad \rho\,, r \in H^{1/2}_o(\Gamma,\Omega)\,, \tag{3.22}$$

where $V_{(r)}$ is the solution to (3.6)–(3.9) with $P=0$, $Q=0$, $S=r$, for each $r \in H^{1/2}_o(\Gamma,\Omega)$. Clearly a is bilinear, symmetric and continuous (owing to estimate (3.11)). Moreover we have the coercivity estimate

$$\begin{aligned} a(r,r) + \int_\Gamma r^2 \,\mathrm{d}\sigma &= \int_\Omega \sigma |\nabla V_{(r)}|^2 \,\mathrm{d}x + \int_\Gamma r^2 \,\mathrm{d}\sigma \\ &\ge \gamma \|V_{(r)}\|^2_{\mathcal{H}^1_o(\Omega)} \ge \gamma \|r\|^2_{H^{1/2}(\Gamma)}\,, \end{aligned} \tag{3.23}$$

where we have made use, in this order, of Poincaré's inequality (Proposition 2.2), and of classical trace inequalities. Thus there exists a unique solution, which we denote as $[v]$, to

$$\begin{aligned} &[v] \in L^2(0,T;H^{1/2}_o(\Gamma,\Omega)) \cap C(0,T;L^2(\Gamma))\,, \quad [v](0) = S\,, \\ &\alpha \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma [v](t) r \,\mathrm{d}\sigma + a([v](t),r) = \langle q(t), r\rangle\,, \qquad \text{for all } r \in H^{1/2}_o(\Gamma,\Omega), \end{aligned}$$

in the sense of distributions (Theorem 23.A [10]). A standard density argument proves that $v = V_{([v])}$ is the sought after solution.

Uniqueness of solutions of (3.1)–(3.5) follows from the observation that the difference of two solutions solves (3.1)–(3.5) with zero data P, Q, h, S. □

REMARK 3.3 - Assume $\Omega = Y$, then Theorem 3.2 holds also if we replace the boundary condition (3.5) with the requirement that $u \in L^2(0,T;H^1_\times(Y))$, where

$$\begin{aligned} H^1_\times(Y) = \{u \mid u_1 := u_{|\Omega_1} = u'_{|\Omega_1}\,,\ u_2 := u_{|\Omega_2} = u''_{|\Omega_2}\,, \\ \text{with } u', u'' \in H^1_{\mathrm{per}}(Y)\,; \int_Y u = 0\}\,, \end{aligned}$$

and

$$H^1_{\mathrm{per}}(Y) = \{u \in H^1_{\mathrm{loc}}(\mathbf{R}^N) \ :\ u \text{ is } Y\text{-periodic}\}\,.$$

Clearly, in this case, we must assume that the data P, Q, h, S of the problem are Y-periodic with respect to the x variable, that the compatibility condition

$$\int_Y P \, \mathrm{d}x = \int_\Gamma Q \, \mathrm{d}\sigma \tag{3.24}$$

is satisfied and that in (3.23) it is used Poicaré's inequality in the version of [8].

Next we extend previous results to the nonlinear problem relevant to us. Let us begin with

$$\begin{aligned}
-\sigma \Delta u &= g(t,u)\,, && \text{in } \Omega_1, \Omega_2; && (3.25)\\
[\sigma \nabla u \cdot \nu] &= Q(t)\,, && \text{on } \Gamma; && (3.26)\\
\alpha \frac{\partial}{\partial t}[u] + f([u]) &= \sigma_2 \nabla u^{(\mathrm{out})} \cdot \nu + h(t)\,, && \text{on } \Gamma; && (3.27)\\
[u](x,0) &= S\,, && \text{on } \Gamma; && (3.28)\\
u(x,t) &= 0\,, && \text{on } \partial\Omega. && (3.29)
\end{aligned}$$

Theorem 3.4 *Assume that g satisfies* (2.2)*, with L as in* (2.3)*, and that f satisfies* (2.1)*. Moreover let us stipulate that Q, $h \in L^2(0,T;L^2(\Gamma))$. Then there exists a unique solution of problem* (3.25)–(3.29).

Proof - Consider the Banach space $X = L^2(0,T_1;\mathcal{H}^1_o(\Omega))$, endowed with the natural norm

$$\|u\|_{L^2(0,T_1;\mathcal{H}^1_o(\Omega))} := \left(\int_0^{T_1} \|u(t)\|^2_{\mathcal{H}^1_o(\Omega)} \, dt \right)^{1/2} ,$$

where T_1 will be choosen later. Let us introduce an operator H acting on X by means of $H(u) = w$, where w is the solution of

$$\begin{aligned}
-\sigma \Delta w &= g(t,u)\,, && \text{in } \Omega_1, \Omega_2; && (3.30)\\
[\sigma \nabla w \cdot \nu] &= Q(t)\,, && \text{on } \Gamma; && (3.31)\\
\alpha \frac{\partial}{\partial t}[w] + f([u]) &= \sigma_2 \nabla w^{(\mathrm{out})} \cdot \nu + h(t)\,, && \text{on } \Gamma; && (3.32)\\
[w](x,0) &= S\,, && \text{on } \Gamma; && (3.33)\\
w(x,t) &= 0\,, && \text{on } \partial\Omega. && (3.34)
\end{aligned}$$

Clearly, the operator H is well defined; moreover, multiplying previous equation by w and integrating by parts, we obtain that $H(X) \subset X$.

Given $u_1, u_2 \in X$ and the corresponding solutions $w_1 = H(u_1)$ and $w_2 = H(u_2)$, subtracting from each other the two formulations of (3.30)–(3.34) written for u_1 and u_2 respectively, multiplying against $w_1 - w_2$ and integrating by parts, we obtain

$$
\begin{aligned}
& \int_{\Omega_t} |\nabla(w_1 - w_2)|^2 \, dx \, dt + \int_{\Gamma} [w_1 - w_2]^2(t) \, d\sigma \\
\leq \; & \gamma_1 \left[\int_{\Omega_t} |g(u_1) - g(u_2)| \, |w_1 - w_2| \, dx \, dt \right. \\
+ \; & \left. \int_0^t \int_{\Gamma} |f([u_1]) - f([u_2])| \, |[w_1 - w_2]| \, d\sigma \, dt \right] \\
\leq \; & \gamma_1 \left[L \int_{\Omega_t} |u_1 - u_2| \, |w_1 - w_2| \, dx \, dt + \int_0^t \int_{\Gamma} |[u_1 - u_2]| \, |[w_1 - w_2]| \, d\sigma \, dt \right] \\
\leq \; & \gamma_1 \left[\frac{L}{2} \int_{\Omega_{T_1}} |u_1 - u_2|^2 \, dx \, dt + \frac{LC}{2} \int_{\Omega_t} |\nabla(w_1 - w_2)|^2 \, dx \, dt \right. \\
+ \; & \left. \delta \int_0^{T_1} \int_{\Gamma} [u_1 - u_2]^2 \, d\sigma \, dt + \frac{1}{\delta} \int_0^t \int_{\Gamma} [w_1 - w_2]^2 \, d\sigma \, dt \right]
\end{aligned} \tag{3.35}
$$

where γ_1 depends only on $\sigma, \alpha, \|f'\|_\infty$. Taking into account (2.3), the second term in the last inequality can be absorbed into the left hand side, while the fourth term will be treated using Gronwall's Lemma. If we set

$$
\begin{aligned}
F(t) \quad & = \int_{\Gamma} [w_1 - w_2]^2(t) \, d\sigma \\
G(T_1) \quad & = \gamma_1 \left[\frac{L}{2} \int_{\Omega_{T_1}} |u_1 - u_2|^2 \, dx \, dt + \delta \int_0^{T_1} \int_{\Gamma} [u_1 - u_2]^2 \, d\sigma \, dt \right]
\end{aligned}
$$

by (3.35), it follows

$$
F(t) \leq G(T_1) + \frac{\gamma_1}{\delta} \int_0^t F(\tau) \, d\tau \, ,
$$

which implies

$$
\int_0^{T_1} F(\tau) \, d\tau \leq G(T_1) T_1 \mathrm{e}^{\gamma_1 \frac{T_1}{\delta}} \, .
$$

Inserting this estimate in (3.35) and using Proposition 2.2, we obtain

$$
\begin{aligned}
\int_0^{T_1} \|w_1(t) - w_2(t)\|^2_{\mathcal{H}^1_o(\Omega)} \, dt \quad & \leq \gamma_2 \left[1 + (T_1 + T_1/\delta) \mathrm{e}^{\gamma_1 \frac{T_1}{\delta}} \right] \cdot \\
& \left(\frac{L}{2} \int_{\Omega_{T_1}} |u_1 - u_2|^2 \, dx \, dt + \delta \int_{\Omega_{T_1}} |\nabla(u_1 - u_2)|^2 \, dx \, dt \right) ,
\end{aligned}
$$

where now γ_2 depends also on the Poincare's constant and the trace embedding. Taking into account that, by (2.3), we may assume that $L < \min\left(\frac{1}{C\gamma_1}, \frac{1}{\gamma_2(1+2\mathrm{e})} \right)$

and choosing

$$\delta \le \frac{1}{2(1+2\mathrm{e})\gamma_2} \qquad \text{and} \qquad T_1 = \min\left(1, \delta, \frac{\delta}{\gamma_1}\right)$$

it follows

$$\int_0^{T_1} \|w_1(t) - w_2(t)\|^2_{\mathcal{H}^1_o(\Omega)} \, dt \le \frac{1}{2} \int_0^{T_1} \|u_1(t) - u_2(t)\|^2_{\mathcal{H}^1_o(\Omega)} \, dt$$

which implies that H is a contraction. So, it admits a unique fixed point, i.e., a solution of problem (3.25)–(3.29) exists in X. Noting that the width T_1 of the time interval is independent of the iteration step, we may conclude the proof by iterating this argument over $(0, T)$. □

REMARK 3.5 - As already pointed out in Remark 3.3, existence and uniqueness of solutions to (3.25)–(3.28) in the periodic case with null mean average can be proven by means of methods analogous to the ones employed above in Theorem 3.4. We omit the details. Moreover, we note that, if the assumption of null mean average is not required, then uniqueness of the solution does not hold. However, it can be easily proved that, for any two periodic solutions $u_1,\ u_2$ of (3.25)–(3.28) with $g \equiv 0$, there exists a function $\psi \in L^2(0, T)$, depending only on t such that $u_1(x, t) - u_2(x, t) = \psi(t)$ for a.e. $t \in (0, T)$. Clearly, $\psi(t) = \frac{1}{|Y|} \int_Y u_1(x, t) - u_2(x, t) \, \mathrm{d}x$.

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