Sobolev estimates for solutions of the transport equation and ODE flows associated to non-Lipschitz drifts

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Abstract

It is known, after [J16] and [ACM18], that ODE flows and solutions of the transport equation associated to Sobolev vector fields do not propagate Sobolev regularity, even of fractional order. In this paper, we show that some propagation of Sobolev regularity happens as soon as the gradient of the drift is exponentially integrable. We provide sharp Sobolev estimates and new examples. As an application of our main theorem, we generalize a regularity result for the 2D Euler equation obtained by Bahouri and Chemin in [BC94].

Key words: Ordinary differential equations with non smooth vector fields; transport equation; 2D Euler equation; Log-Lipschitz regularity.

MSC (2010): 34A12, 35F25, 35F10

Contents

Introduction

We consider the Cauchy problem for the transport equation associated to a vector field $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ on the flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$

$$\begin{cases} \partial_t u + b \cdot \nabla_x u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{Tr}$$

where $u_0 : \mathbb{T}^d \to \mathbb{R}$ is a given initial data and $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is the unknown to the problem.

The theory of characteristics establishes a link between solutions of (Tr) and the flow $X : [0,T] \times \mathbb{T}^d \to \mathbb{T}^d$ of b, i.e. the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)) \qquad x \in \mathbb{T}^d, t \in [0,T], \\ X(0,x) = x. \end{cases} \tag{ODE}$$

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Thanks to the classical Cauchy-Lipschitz theory both problems are well-posed when the drift b is regular enough, i.e. Lipschitz in the spatial variable uniformly in time. Unfortunately the Lipschitz regularity is a too strong assumption for applications. Indeed in various physical models of the mechanics of fluids it is essential to deal with non regular velocity, and this is not just a technical fact but corresponds to effective physical situations. For this reason in the last thirty years a big interest has grown on the study of (ODE) and (Tr) under weaker assumptions on the vector field.

In the present paper we study sharp regularity properties, in the scale of Sobolev spaces, of solutions of (Tr) and (ODE) in a setting that is in between the classical setting of the Cauchy-Lipschitz theory and the Sobolev setting considered in the DiPerna-Lions-Ambrosio theory [DPL89, A04]. More precisely, we assume that b admits a spatial distributional derivative satisfying

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\exp\{\,\beta\,|\,\nabla_x b(t,x)|\,\}\,\mathrm{d}x<\infty \quad \text{for some } \beta>0 \text{ and } \operatorname{div}_x b\in L^\infty([0,T]\times\mathbb{T}^d). \tag{HP}$$

We have chosen the ambient space $\mathbb{T}^d$ instead of $\mathbb{R}^d$ just because compactness allows to avoid integrability problems at infinity and to obtain global estimates. This makes statements shorter and more elegant. It is worth stressing, however, that any result we are going to present holds true also in the Euclidean space $\mathbb{R}^d$ provided one suitably localizes the estimates.

The study of (Tr) and (ODE) under (HP) is meaningful for applications to nonlinear partial differential equations. The 2D Euler equation in vorticity form (see [BM02, L96] for an overview) provides an important example of PDE where a vector field satisfying (HP) is involved. In particular, as an application of the main result in this work (Theorem 1.2 and Corollary 1.4) we obtain a propagation of regularity result (Theorem 3.1) for solutions of the Euler equation with bounded initial vorticity enjoying a fractional order regularity. This theorem is a non trivial improvement of [BC94, Corollary 1.1] stated in the periodic setting. See section 3 for details on this.

Let us now present the main regularity result of this manuscript underlying, by mean of examples, its sharpness in the Sobolev scale. We refer to section 1 for more details on our main theorems Theorem 1.2, Theorem 1.7 and related corollaries, while the examples Theorem 2.1, Theorem 2.2 and Theorem 2.4 are presented in section 2.

First of all it is worth mentioning that, under the assumptions (HP), it is well-known that (ODE) admits a unique flow in the classical sense. Indeed the velocity field satisfies the *log-Lipschitz* property that, identifying the drift with a periodic function from $\mathbb{R}^d$ to $\mathbb{R}^d$, reads

$$|b(t,x)-b(t,y)|\leq C|x-y|\max\{|\log(|x-y|)|,1\} \qquad \forall x,y\in\mathbb{R}^d\ t\in[0,T]. \tag{0.1}$$

This property implies in turn the existence and uniqueness of the curve $t\mapsto X_t(x)$ satisfying (ODE) (look at Lemma A.1 and the discussion in subsection 1.1). Moreover, $X_t:\mathbb{T}^d\to\mathbb{T}^d$ is invertible for any fixed time $t\in[0,T]$ and

$$u(t,x):=u_0((X_t)^{-1}(x)) \quad t\in[0,T], \quad \text{with } u_0\in L^p(\mathbb{T}^d), \tag{0.2}$$

provides the unique distributional solution of the Cauchy problem (Tr) in $L^\infty([0,T];L^p(\mathbb{T}^d))$, for $p\in[1,\infty]$ (see Remark 1.3 for more explanations). For distributional solutions we mean weakly continuous and bounded curves $t\mapsto u_t\in L^p(\mathbb{T}^d)$ satisfying

$$\int_{\mathbb{T}^d}u_t(x)\varphi(x)\,\mathrm{d}x-\int_{\mathbb{T}^d}u_0(x)\varphi(x)\,\mathrm{d}x=\int_0^t\int_{\mathbb{T}^d}u_s(x)(b_s(x)\cdot\nabla\varphi(x)-\operatorname{div}b_s(x)\varphi(x))\,\mathrm{d}x\,\mathrm{d}s,$$

for any $t\in[0,T]$ and $\varphi\in C^\infty(\mathbb{T}^d)$.

In order to make this introduction as clear as possible we do not illustrate here our main result Theorem 1.2 for (Tr), since it needs the introduction of a suitable functional class, we refer to section 1 for this. We prefer instead focusing the attention on the *Lagrangian* side of the problem (i.e. the study of (ODE)) that is really the core of our analysis. Indeed any regularity estimate for the flow X_t gives in turn results for the transport equation (Tr) as a consequence of the *Lagrangian identity* (0.2). In what follows d denotes the intrinsic distance on the flat torus $\mathbb{T}^d$.

Theorem 0.1. *Let $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ satisfy*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \exp\{\, \beta \mid \nabla b(t,x)| \,\}\, \mathrm{d}x = K < \infty \quad \text{for some } \beta > 0 \text{ and } \|\mathrm{div}\, b\|_{L^\infty([0,T]\times\mathbb{T}^d)} = L < \infty.$$

Then, for any $x, y \in \mathbb{T}^d$ and $t \in [0,T]$, we have

$$\mathsf{d}(X_t(x), X_t(y)) \le g_t(x)\mathsf{d}(x,y) \quad \text{and} \quad \mathsf{d}((X_t)^{-1}(x), (X_t)^{-1}(y)) \le g_t(x)\mathsf{d}(x,y), \tag{0.3}$$

for some nonnegative function g_t that fulfills

$$\|g_t\|_{L^{q_t}} \le e^{\frac{t^2 L C_1}{\beta}} (C_2 K)^{\frac{tC_1}{\beta}} \qquad \forall t \in [0,T], \quad \text{with} \quad q_t := \frac{\beta}{C_1 t}, \tag{0.4}$$

where $C_1 > 0$ and $C_2 > 0$ depend only on d.

Theorem 0.1 has to be understood as a quantitative approximation result in the spirit of Lusin's theorem for Sobolev functions (see [L77]). Indeed, (0.3) implies that, for any $\lambda > 0$, the flow map X_t and its inverse are λ-Lipschitz if restricted to the set $\{\, g_t < \lambda \,\}$. Moreover, since $g_t \in L^{q_t}(\mathbb{T}^d)$, by means of the Chebyschev inequality we can estimate the Lebesgue measure of the "bad" set

$$\mathscr{L}^d(\{\, g_t \ge \lambda \,\}) \le \frac{\|g_t\|_{L^{q_t}}^{q_t}}{\lambda^{q_t}} \le \frac{C_2 K e^{tL}}{\lambda^{q_t}},$$

where we do not control the oscillation of X_t.

It is well-known since the work [H96] that quantitative approximation properties à la Lusin are related (and actually characterize) Sobolev spaces for suitable choices of the exponents. This allows to deduce from Theorem 0.1 that

$$X_t \in W^{1,q_t}(\mathbb{T}^d; \mathbb{R}^d) \qquad \text{for any } 0 \le t < \frac{\beta}{C_1}, \quad \text{where} \quad q_t := \frac{\beta}{C_1 t}, \tag{0.5}$$

together with the quantitative bound

$$\|\nabla X_t\|_{L^{q_t}} \le C_d \|g\|_{L^{q_t}} \le C_d e^{\frac{t^2 L C_1}{\beta}} (C_2 K)^{\frac{tC_1}{\beta}}. \tag{0.6}$$

In other words X_t enjoys a definite Sobolev regularity until a critical time that depends only on β. The very same conclusion holds also for $(X_t)^{-1}$ (note that Theorem 0.1 gives a symmetric result in X_t and $(X_t)^{-1}$) but, for sake of simplicity, here and in the rest of the introduction we consider just the flow map X_t.

What at the first instance could sound surprising is that (0.5) is *sharp*: it can really happen that the flow associated to a vector field satisfying (HP) ceases to be $W^{1,1}$ regular after a time of order $\sim \beta$. In Theorem 2.2 we build a vector field with such a property. However, instead of explain this example, that is presented in detail in section 2, we want to present a formal computation to convey the idea that, if we are in a situation in which the Sobolev regularity of X_t is neither instantaneously lost (as in the DiPerna-Lions setting [ACM18]) nor fully preserved (as in the Cauchy-Lipschitz case), then it reasonably decreases according to (0.5) and (0.6) for structural reasons.

Let us consider a drift b that does not depend on time, so its flow satisfies the semigroup property $X_{t+h} = X_t \circ X_h$. If $X_\delta \in W^{1,p}$ for some small time δ and some exponent $1 < p < \infty$ then the Hölder inequality suggests that, reasonably, $\nabla X_{n\delta} \in L^{p/n}$ for any integer $n \le p$. Indeed we can use the semigroup property and the chain rule to write

$$|\nabla X_{n\delta}|(x) \le |\nabla X_\delta|(X_{\delta(n-1)}) \cdot |\nabla X_\delta|(X_{\delta(n-2)}) \cdot \ldots \cdot |\nabla X_\delta|(X_\delta) \cdot |\nabla X_\delta|(x),$$

and observe that the right hand side is a product of n functions belonging to L^p. More precisely we have

$$\|\nabla X_\delta(X_{k\delta})\|_{L^p} \le e^{k\delta L} \|\nabla X_\delta\|_{L^p} \qquad \text{for } k = 0, \ldots, n-1,$$

where $L := \|\mathrm{div}\, b\|_{L^\infty}$. This immediately leads to $\|\nabla X_{n\delta}\|_{L^{p/n}} \le e^{L\delta n^2} \|\nabla X_\delta\|_{L^p}^n$. Eventually we set $t := n\delta$ and rewrite

$$\|\nabla X_t\|_{L^{\frac{p\delta^{-1}}{t}}} \le e^{L\delta^{-1}t^2} \|\nabla X_\delta\|_{L^p}^{\delta^{-1}t}, \qquad \text{for } 0 < t \le \frac{p\delta^{-1}}{t}. \tag{0.7}$$

Note that (0.7) is perfectly coherent with (0.5) and (0.6).

Theorem 0.1 allows also to describe the Sobolev regularity of X_t after the critical time $\frac{\beta}{C_1}$. In this case we can measure the regularity in the scale of *fractional* Sobolev spaces: what happens, roughly, is that X_t admits a derivative of order $\frac{\beta}{Ct} \wedge 1$ in L^1 for any $t \in [0,T]$ and again the conclusion is sharp in the scale of Sobolev spaces. Look at Corollary 1.4 for the rigorous statement written in terms of solution of the transport equation and to Theorem 2.2 for the example that underlines its sharpness.

Another simple outcome of Theorem 0.1 is the following: if the gradient of the drift satisfies an integrability condition slightly stronger than (HP), for instance

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \exp\{\, \beta \mid \nabla b_t(x)|\,\}\, \mathrm{d}x < \infty \qquad \text{for } \textit{any } \beta > 0,$$

then X_t belongs to $W^{1,p}$ for any $1 \le p < \infty$. Basically it follows from the explicit expressions of q_t and the critical time in (0.5), look at Corollary 1.8 for more details. On the other hand, we have an example (see Theorem 2.1) ensuring the existence of a drift satisfying a relaxed version of (HP), i.e.

$$\sup_{t>0} \int_{B_2} \exp\left\{ \frac{|\nabla b(t,x)|}{\log(1+|\nabla b_t(x)|)^a} \right\} \mathrm{d}x < \infty \quad \forall a > 0,$$

whose flow does not belong to any Sobolev space, even of fractional order, for any $t > 0$. Roughly, it amounts to say that the exponential integrability condition for ∇b_t, that we assume in (HP), is a threshold condition in order to hope for a Sobolev regularity of the flow map.

The examples we have been mentioning in this introduction are the content of section 2; they are all based on a technique introduced recently in [ACM18] by Alberti, Crippa and Mazzucato.

Let us finally spend a few words on the main idea behind the proof of Theorem 0.1. Our strategy builds upon the technique introduced by Crippa and De Lellis in [CDL08] for the quantitative study of generalized flows in the DiPerna-Lions-Ambrosio theory. The authors of the present paper have already used similar ideas in [BN18a] to obtain sharp regularity estimates for solutions of the continuity equation in the scale of *log-Sobolev* spaces assuming a Sobolev regularity on the drift. In order to explain a main technical point of the strategy let us recall the standard argument to prove that flow maps inherit the Lipschitz regularity of velocity fields. When X_t is associated to a uniformly K-Lipschitz vector field b, using the very definition of flow map, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|X_t(x) - X_t(y)| \le |b_t(X_t(x)) - b_t(X_t(y))| \le K|X_t(x) - X_t(y)| \qquad \forall x, y \in \mathbb{R}^d,$$

that together with a Grönwall lemma gives

$$|X_t(x) - X_t(y)| \le |x-y|e^{tK} \qquad \forall x, y \in \mathbb{R}^d,\ t \in [0,T].$$

Note that we have identified both b and X with periodic functions in $\mathbb{R}^d$.

In order to make a variant of this strategy work in our context we need to consider a weak version of the Lipschitz inequality

$$|b_t(x) - b_t(y)| \le K|x-y| \qquad \forall x, y \in \mathbb{R}^d,\ t \in [0,T], \tag{0.8}$$

that is not anymore available assuming just (HP).

In our setting a natural replacement of (0.8) is the log-Lipschitz property (0.1) that, if plugged in the Grönwall argument above, gives

$$|X_t(x) - X_t(y)| \le C|x-y|^{e^{-Ct/\beta}}, \tag{0.9}$$

see subsection 1.1 and the discussion therein for more details. Even though (0.9) is sharp in the scale of Hölder spaces (see [BC94]) it is not suitable for our purposes, indeed it cannot give either integer Sobolev regularity or approximation results by means Lipschitz functions. Moreover (0.9) cannot even implies our result in the case of fractional Sobolev spaces Corollary 1.4 since in our case the regularity dissipates in time with rate $\sim \frac{\beta}{t}$ (that is the sharp rate) while in (0.9) the rate is $\sim e^{-Ct/\beta}$. Let us point out that the use of the log-Lipschitz property (0.1) for the study of (ODE), (Tr) and related problems coming from PDE nowadays is consider standard, see for instance [BC94, CL95, Z02].

In this paper we adopt a change of prospective. We forget about the log-Lipschitz property and we take into account a different ingredient that has been already used by Crippa and De Lellis in the Sobolev setting. They have replaced (0.8) with the well-known inequality

$$|b_t(x) - b_t(y)| \le C_d|x-y|(M|\nabla b_t|(x) + M|\nabla b_t|(y)), \qquad \forall x,y \in \mathbb{R}^d,\ t\in[0,T], \tag{0.10}$$

available for any Sobolev map (see [ST70] for its proof), where M denotes the Hardy-Littlewood maximal operator. Assuming $\nabla b_t \in L^p$ for $p>1$ one has in turn $M|\nabla b_t| \in L^p$ (it is a general property of the maximal function, see [ST93, Theorem 1]) and it leads to a quantitative weak version of (0.8) that is suitable for the study of the regularity of X_t.

Under the assumption (HP) we can write a version of (0.10) as follows: there exists a nonnegative function h_t such that

$$|b_t(x) - b_t(y)| \le |x-y|\frac{C_d}{\beta}h_t(x) \quad \forall x,y\in\mathbb{T}^d,\ t\ge 0 \quad \text{and} \quad \sup_{t\in[0,T]}\int_{\mathbb{T}^d}\exp\{h_t(x)\}\,\mathrm{d}x < \infty \tag{0.11}$$

where C_d depends only on d, see Lemma A.2. This technical ingredient is the correct one to replace (0.8) in the Grönwall argument. We refer to section 1 for more details.

Notations. We denote by $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the flat torus of dimension $d\ge 1$ endowed with its geodesic distance d and its Haar measure $\mathscr{L}^d$. We denote by $B_r(x)$ the geodesic ball of radius $r>0$ centered at $x\in\mathbb{T}^d$.

We often identify $\mathbb{T}^d$ with $[0,1)^d$, in this way we can write

$$\mathsf{d}(x,y) := \min\{\ |\ x-y-k|\ :\ k\in\mathbb{Z}^d\ |k|\le 2\,\}, \tag{0.12}$$

where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^d$. Under this identification the Haar measure in $\mathbb{T}^d$ coincides with the Lebesgue measure on the square, while scalar functions can be identified $f:\mathbb{T}^d\to\mathbb{R}$ with 1-periodic functions on $\mathbb{R}^d$. We often use the double notation $f(t,x) = f_t(x)$ for functions depending both in the space and in the time variable.

We write

$$⨍_E f\,\mathrm{d}\mu = \frac{1}{\mu(E)}\int_E f\,\mathrm{d}x,$$

to denoted the average integral and

$$Mf(x) := \sup_{r>0}⨍_{B_r(x)}|f(y)|\,\mathrm{d}y, \qquad \forall\ x\in\mathbb{R}^d,$$

to denote the Hardy-Littlewood maximal function.

We often use the expression $a\lesssim_c b$ to mean that there exists a universal constant C depending only on c such that $a\le Cb$. The same convention is adopted for $\gtrsim_c$ and $\simeq_c$.

1 Regularity results

In this section we present regularity results for flows and solutions of the transport equation associated to drifts satisfying (HP). Let us begin by introducing a functional class.

Definition 1.1. Let $0<\alpha\le 1$ and $0<p\le\infty$ be fixed. We say that $f\in L^p(\mathbb{T}^d)$ belongs to F^α_p if

$$[f]_{F^\alpha_p} := \inf\left\{\|g\|\in L^p(\mathbb{T}^d):\ |f(x)-f(y)|\le \mathsf{d}(x,y)^\alpha(g(x)+g(y)) \quad \text{for every } x,y\in\mathbb{T}^d\right\} < \infty.$$

We set $\|f\|_{F^\alpha_p} := \|f\|_{L^p} + [f]_{F^\alpha_p}$.

These spaces have already appeared in the literature (see for instance [BC94]) and they coincide with the Triebel-Lizorkin class $F^{\alpha}_{p,\infty}$ when $\alpha \in (0,1)$ and $p>1$ (see [BC94, Proposition 3.2]). The Hajlasz characterization of Sobolev spaces [H96] gives

$$F^1_p = W^{1,p}(\mathbb{T}^d) \quad \text{for any } p>1, \quad \text{and} \quad F^1_1 \subset W^{1,1}(\mathbb{T}^d). \tag{1.1}$$

While, for $0<\alpha<1$, the class F^{α}_p is related to fractional Sobolev spaces (see [AF75])

$$W^{\alpha,p}(\mathbb{T}^d) := \{\, f\in L^p(\mathbb{T}^d):\ [f]_{W^{\alpha,p}} < \infty \,\} \qquad \alpha\in(0,1),\ p\geq 1$$

where

$$[f]_{W^{\alpha,p}} := \left(\int_{(0,1]^d} \int_{\mathbb{T}^d} \frac{|f(x+h)-f(x)|^p}{|h|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}h \right)^{1/p}, \tag{1.2}$$

is the socalled Gagliardo's seminorm. Precisely we have

$$W^{\alpha,p}(\mathbb{T}^d) \subset F^{\alpha}_p \subset W^{\alpha',p}(\mathbb{T}^d) \qquad \text{for any } 0<\alpha'<\alpha<1,\ p>1, \tag{1.3}$$

the proof of the first inclusion follows form [BN18b, Proposition 1.13] while the latter can be easily checked using the definition of F^{α}_p and Gagliardo's seminorm.

Let us finally mention that, the inequality

$$|f(x)-f(y)| = |f(x)-f(y)|^{\theta}|f(x)-f(y)|^{1-\theta} \lesssim \mathsf{d}(x,y)^{\alpha\theta}(g(x)^{\theta}+g(y)^{\theta})\,\|f\|^{1-\theta}_{L^\infty},$$

for any $\theta\in(0,1)$, $f\in F^{\alpha}_p$ and g competitors in the definition on $[f]_{F^{\alpha}_p}$ (see Definition 1.1), implies the interpolation estimate

$$[f]_{F^{\theta\alpha}_p} \lesssim \|f\|^{1-\theta}_{L^\infty}\,[f]^{\theta}_{F^{\alpha}_{p\theta}} \qquad \text{for any } \theta\in(0,1), \alpha\in(0,1),\ p>0, \tag{1.4}$$

that will play a role in the sequel.

This being said we are ready to state our main result.

Theorem 1.2. *Let $b:[0,T]\times\mathbb{T}^d\to\mathbb{R}^d$ satisfy*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \exp\{\,\beta\mid\nabla b_t(x)|\,\}\,\mathrm{d}x = K<\infty \quad \textit{for some } \beta>0 \textit{ and } \|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{T}^d)} = L<\infty.$$

where the derivatives are understood in the sense of distributions. Then, there exist constants $C_1>0$ and $C_2>2$ depending only on d, such that for any $0<\alpha\leq 1$, $p\geq 1$ and $u_0\in F^{\alpha}_p$ the unique solution $u\in L^\infty([0,T];L^p(\mathbb{T}^d))$ of (Tr) satisfies

$$[u_t]_{F^{\alpha}_{p_t}} \leq [u_0]_{F^{\alpha}_p}(e^{Lt}C_2K)^{1/p_t} \quad \forall t\in[0,T] \quad \textit{with} \quad p_t := \frac{p}{1+\beta^{-1}\alpha p C_1 t}. \tag{1.5}$$

Before proving Theorem 1.2 we present a remark and two important corollaries.

Remark 1.3. Let us explain why under the assumption (HP) the Cauchy problem (Tr) admits

$$u(t,x) := u_0((X_t)^{-1}(x)) \quad t\in[0,T], \tag{1.6}$$

where X is the flow map of b (see the discussion in subsection 1.1 for what concerns X), as a unique solution in $L^\infty([0,T];L^p(\mathbb{T}^d))$, for any $p\geq 1$.

First of all notice that $u(t,x):=u_0((X_t)^{-1}(x))$ is a weak solution of the transport equation in $L^\infty([0,T];L^p(\mathbb{T}^d))$ when $u_0\in L^p(\mathbb{T}^d)$ (look at the introduction for the definition of weak solution). Therefore to prove the sought claim it suffices to show the uniqueness property for (Tr) in the class $L^\infty([0,T];L^1(\mathbb{T}^d))$.

Using Lemma A.1 we deduce that b is Log-Lipschitz continuous and, if we further assume that $\operatorname{div} b=0$, then [BC94, Theorem 1.2] grants the uniqueness result we are looking for. It actually implies uniqueness in the larger class of signed measure, but we are not interested in this general case. In order to get rid of the assumption $\operatorname{div} b=0$ we can consider the recent result [CC18, Theorem 1.1 and Remark 1.5] together with the simple observation that in our case forward-backward curves are always trivial due to the pointwise uniqueness of trajectories in (ODE).

Corollary 1.4. *Let b, L, C_1 and C_2 be as in Theorem 1.2. Then for any $0 < \alpha \le 1$, $p \ge 1$ and $u_0 \in L^\infty(\mathbb{T}^d) \cap W^{\alpha,p}(\mathbb{T}^d)$ the unique solution $u \in L^\infty([0,T] \times \mathbb{T}^d)$ of (Tr) satisfies*

$$u_t \in W^{\alpha_t,p}(\mathbb{T}^d) \qquad \textit{with} \quad \alpha_t := \alpha \, \frac{1}{1 + 2\beta^{-1}\alpha p C_1 t}. \tag{1.7}$$

Moreover, if $p > 1$, for any $1 \le p' < p$ and $0 < \alpha' < \alpha$ it holds

$$u_t \in W^{\alpha',p'}(\mathbb{T}^d) \qquad t < \left(\frac{1}{p'} - \frac{1}{p}\right) \frac{\beta}{\alpha C_1}. \tag{1.8}$$

Finally, if we assume $\alpha = 1$ the conclusion (1.8) can be strengthen as follows

$$\|\nabla u_t\|_{L^{p'}} \lesssim_{d,p} \|\nabla u_0\|_{L^p} \left(e^{Lt} C_2 K\right)^{\frac{1+\beta^{-1}pC_1 t}{p}} \qquad t < \left(\frac{1}{p'} - \frac{1}{p}\right) \frac{\beta}{C_1}, \tag{1.9}$$

for any $1 \le p' < p$.

Proof. Using (1.4) with $\theta_t := \frac{1}{1+\beta^{-1}\alpha p C_1 t}$ and Theorem 1.2 we get

$$[u_t]_{F_p^{\theta_t \alpha}} \lesssim \|u_0\|_\infty^{1-\theta_t} \, [f]_{F_{p_t}^\alpha}^{\theta_t} \le \|u_0\|_\infty^{1-\theta_t} \, [u_0]_{F_p^\alpha}^{\theta_t} (e^{Lt} C_2 K)^{1/p},$$

since $\theta_t \alpha > \alpha_t$ (1.7) follows from (1.3). Let us address (1.8). If $t < \left(\frac{1}{p'} - \frac{1}{p}\right) \frac{\beta}{\alpha C_1}$ then $p_t > p'$, thus Theorem 1.2 and (1.3) gives

$$[u_t]_{W^{\alpha',p'}} \overset{(1.3)}{\lesssim}_{\alpha,\alpha',p,p'} [u_t]_{F_{p'}^\alpha} \lesssim [u_t]_{F_{p_t}^\alpha} \overset{(1.5)}{\lesssim} [u_0]_{F_p^\alpha} (e^{Lt} C_2 K)^{1/p_t} \qquad \forall \, 0 < \alpha' < \alpha.$$

Repeating the same argument with $\alpha = 1$ and taking into account (1.1) we get (1.9). ☐

Corollary 1.5. *Let $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ satisfy*

$$\sup_{t \in [0,T]} \int_{\mathbb{T}^d} \exp\{\, \beta \mid \nabla b_t(x) | \,\} \, \mathrm{d}x < \infty \quad \textit{for every } \beta > 0 \textit{ and } \|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{T}^d)} < \infty.$$

where derivatives are understood in the sense of distributions. Then for any $0 < \alpha \le 1$, $p > 1$ and $u_0 \in L^\infty(\mathbb{T}^d) \cap W^{s,p}(\mathbb{T}^d)$ the unique solution $u \in L^\infty([0,T] \times \mathbb{T}^d)$ of (Tr) satisfies

$$u_t \in W^{\alpha',p}(\mathbb{T}^d) \qquad \textit{for any} \quad 0 < \alpha' < \alpha, \quad t \in [0,T]. \tag{1.10}$$

In the case $\alpha = 1$ we also have

$$u_t \in W^{1,p'}(\mathbb{T}^d) \qquad \textit{for any} \quad 1 \le p' < p, \quad t \in [0,T]. \tag{1.11}$$

Proof. Let us first assume $p > 1$. An immediate application of Theorem 1.2 gives $u_t \in F_{p'}^\alpha$ for any $1 \le p' < \infty$ thus the sought conclusions follow from (1.1) and (1.3). In order to extend (1.10) to the case $p = 1$ it is enough to apply the Sobolev embedding theorem (1.12) stated below. ☐

Remark 1.6. The conclusions (1.8) can be extended to the case $p = 1$ as follows: for any $\alpha' < \alpha$ there exists $\bar{T} := \bar{T}(\alpha', \alpha, p, \beta, C_1) > 0$ such that

$$u_t \in W^{\alpha',p} \qquad \forall t < \bar{T}.$$

In order to do so it is enough to use the Sobolev embedding theorem:

$$\|f\|_{W^{s,r}} \lesssim_{s,s',r,r'} \|f\|_{W^{s',r'}}, \quad \text{with} \quad r' = \frac{dr}{d - (s - s')r}, \quad s, s' \in (0,1], \ r, r' \in (0,\infty), \tag{1.12}$$

where $\|f\|_{W^{\alpha,p}} := \|f\|_{L^p} + [f]_{W^{\alpha,p}}$. We refer to [AF75] and [DDN18] for more details.

The remaining part of this section is dedicated to the proof of Theorem 1.2. As we have anticipated in the introduction, we carry out a Lagrangian approach, meaning that the core of our argument is a regularity result for flows, whose proof is based on a technique introduced by Crippa and De Lellis [CDL08] in the context of DiPerna-Lions-Ambrosio's theory [DPL89, A04].

1.1 Regularity of flows

In this subsection we prove Theorem 0.1. It is restated below for reader's convenience.

Theorem 1.7 (Regularity of the flow). *Let $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ satisfy*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \exp\{\,\beta \mid \nabla b(t,x)|\,\}\,\mathrm{d}x = K < \infty \quad \textit{for some } \beta > 0 \textit{ and } \|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{T}^d)} = L < \infty.$$

Then, for any $x, y \in \mathbb{T}^d$ and $t \in [0,T]$, we have

$$\mathsf{d}(X_t(x), X_t(y)) \le g_t(x)\mathsf{d}(x,y) \quad \textit{and} \quad \mathsf{d}((X_t)^{-1}(x), (X_t)^{-1}(y)) \le g_t(x)\mathsf{d}(x,y), \tag{1.13}$$

for some nonnegative function g_t that fulfills

$$\|g_t\|_{L^{q_t}} \le e^{\frac{t^2 L C_1}{\beta}} (C_2 K)^{\frac{tC_1}{\beta}} \qquad \forall t \in [0,T], \quad \textit{with} \quad q_t := \frac{\beta}{C_1 t}, \tag{1.14}$$

where $C_1 > 0$ and $C_2 > 0$ depend only on d.

Before proving Theorem 1.7, let us recall that, under the assumption

$$\sup_{t>0} \int_{\mathbb{T}^d} \exp\{\,\beta \mid \nabla b_t(x)|\,\}\,\mathrm{d}x =: K(\beta) < \infty \qquad \text{for some } \beta > 0,$$

there exists a unique classical solution of the problem (ODE). Indeed thanks to Lemma (A.1) we know that b is Log-Lipschitz, namely

$$|b_t(x) - b_t(y)| \le \frac{C}{\beta}\mathsf{d}(x,y)\log\left(\frac{CK}{\mathsf{d}(x,y)^d}\right) \qquad \forall x, y \in \mathbb{T}^d \ \forall t \ge 0. \tag{1.15}$$

In particular b satisfies the Osgood condition, so it admits a unique solution for any initial data $x \in \mathbb{T}^d$.

Moreover, by mean of (1.15), it is possible to show that X_t is Hölder continuous:

$$(CK)^{-1-e^{Ct/\beta}}\,\mathsf{d}(x,y)^{e^{Ct/\beta}} \le \mathsf{d}(X_t(x), X_t(y)) \le (CK)^{1+e^{-Ct/\beta}}\,\mathsf{d}(x,y)^{e^{-Ct/\beta}}, \tag{1.16}$$

for some $C > 2$ depending only on d. To see this we use again (1.15) obtaining

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}(X_t(x), X_t(y))\right| \le |b_t(X_t(x)) - b_t(X_t(y))| \le \frac{C}{\beta}\mathsf{d}(X_t(x), X_t(y))\log\left(\frac{CK}{\mathsf{d}(X_t(x), X_t(y))^d}\right),$$

that amounts to

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\log\log\left(\frac{CK}{\mathsf{d}(X_t(x), X_t(y))^d}\right)\right| \le \frac{C}{\beta}, \tag{1.17}$$

where the constant in the left hand side may be bigger then the one in the previous line but still depends only on d. Thus (1.17) immediately implies (1.16).

If we further assume $\|\operatorname{div} b\|_{L^\infty} := L < \infty$ we also deduce

$$e^{-tL}\mathscr{L}^d \le (X_t)_\#\mathscr{L}^d \le e^{tL}\mathscr{L}^d \qquad \text{for any } t \ge 0. \tag{1.18}$$

In particular when b is divergence-free, X_t is a measure preserving map for any $t \ge 0$. The property (1.18) can be checked observing that X coincides with the unique *Regular Lagrangian flow* associated to b according to Ambrosio's axiomatization (see [A04]).

Let us refer to [BC94], [CL95] and [Z02] for further details on well-posedness results for flows and solutions of the continuity and transport equation associated to Log-Lipschitz drifts.

We conclude this subsection by proving Theorem 1.7 and stating a simple corollary.

Proof of Theorem 1.7. Fix $x, y \in \mathbb{T}^d$, recalling (0.12) we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}(X_t(x), X_t(y))\right| \leq & |b_t(X_t(x)) - b_t(X_t(y))| \\ \leq & \mathsf{d}(X_t(x), X_t(y))\frac{2C_1}{\beta}\left(1 + \log\left(c_d M\left(\exp\left\{2^{-1}\beta|\nabla b_t|\right\}\right)(X_t(x))\right)\right),\end{aligned}$$

where in the second line we used (A.2) with $\beta' = \beta/2$. Setting

$$g_t(x) := \exp\left\{\frac{2C_1}{\beta}\int_0^t \left(1 + \log\left(c_d M\left(\exp\left\{2^{-1}\beta|\nabla b_s|\right\}\right)(X_s(x))\right)\right)\mathrm{d}s\right\},$$

the Grönwall inequality gives (1.13). It remains to prove (1.14). Let us fix $t > 0$ and set $q_t := \frac{\beta}{C_1 t}$, using Jensen's inequality and (1.18) we deduce

$$\begin{aligned}\int_{\mathbb{T}^d} g_t(x)^{q_t}\,\mathrm{d}x = & \int_{\mathbb{T}^d} \exp\left\{2 + \fint_0^t 2\log\left(c_d M\left(\exp\left\{2^{-1}\beta|\nabla b_s|\right\}\right)(X_s(x))\right)\mathrm{d}s\right\}\mathrm{d}x \\ \lesssim_d & \int_{\mathbb{T}^d}\fint_0^t \left[M\left(\exp\left\{2^{-1}\beta|\nabla b_s|\right\}\right)(X_s(x))\right]^2\mathrm{d}s\,\mathrm{d}x \\ \lesssim_d & e^{tL}\fint_0^t\int_{\mathbb{T}^d}\left[M\left(\exp\left\{2^{-1}\beta|\nabla b_s|\right\}\right)(x)\right]^2\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Exploiting the boundness of the maximal function between L^2 spaces (see [ST70]) we get the sought conclusion:

$$\int_{\mathbb{T}^d} g_t(x)^{q_t}\,\mathrm{d}x \lesssim_d e^{tL}\fint_0^t\int_{\mathbb{T}^d}\exp\left\{\beta|\nabla b_s(x)|\right\}\mathrm{d}x\,\mathrm{d}s \leq e^{tL}K.$$

□

An immediate consequence of Theorem 1.7 is the following.

Corollary 1.8. *Let $b : [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ satisfy*

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\exp\left\{\beta\mid\nabla b_t(x)|\right\}\mathrm{d}x < \infty \quad \textit{for every } \beta > 0 \textit{ and } \|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{T}^d)} < \infty.$$

Then, for any $t \in [0,T]$, X_t and its inverse belong to $W^{1,p}(\mathbb{T}^d;\mathbb{R}^d)$ for every $1 \leq p < \infty$.

Proof. Let us fix $t > 0$. Using (1.14) with $\beta = pC_1 t$ we deduce $g_t \in L^p(\mathbb{T}^d)$. The sought conclusion follows from (1.1). □

We conclude the section with the proof of Theorem 1.2.

1.2 Proof of Theorem 1.2

We assume without loss of generality that $[u_0]_{F_p^\alpha} = 1$. It is enough to prove that, for any $t \in [0,T]$, there exists a positive function $\bar{g}_t$ such that

$$|u_t(x) - u_t(y)| \lesssim \mathsf{d}(x,y)^\alpha(\bar{g}_t(x) + \bar{g}_t(y)) \qquad \forall x, y \in \mathbb{T}^d, \tag{1.19}$$

and

$$\|\bar{g}_t\|_{L^{p_t}} \lesssim_{d,p,\alpha} (e^{tL}C_2K)^{1/p_t} \qquad \text{with} \quad p_t = \frac{p}{1 + \beta^{-1}\alpha p C_1 t}, \tag{1.20}$$

where C_1 and C_2 are as in Theorem 1.7.

As we mentioned before we exploit the Lagrangian representation formula

$$u_t(x) := u_0((X_t)^{-1}(x)) \qquad \text{for any } x \in \mathbb{T}^d, \text{ and } t \in [0,T], \tag{1.21}$$

where X_t is the solution of (ODE). Note that the inverse of X_t is well-defined thanks to Theorem 1.7. By Definition 1.1 we know that there exists $h : \mathbb{T}^d \to [0,\infty]$ satisfying

$$|u_0(x) - u_0(y)| \le \mathsf{d}(x,y)^\alpha (h(x) + h(y)) \qquad \forall x, y \in \mathbb{T}^d \quad \text{and} \quad \|h\|_{L^p} \le 1 + \varepsilon, \tag{1.22}$$

where $0 < \varepsilon < 1$ is fixed. Building upon (1.21), (1.22) and Theorem 1.7(ii) we get

$$\begin{aligned} |u_t(x) - u_t(y)| =& |u_0((X_t)^{-1}(x)) - u_0((X_t)^{-1}(y))| \\ \le& \mathsf{d}((X_t)^{-1}(x), (X_t)^{-1}(y))^\alpha (h((X_t)^{-1}(x)) + h((X_t)^{-1}(y))) \\ \le& \mathsf{d}(x,y)^\alpha g_t(x)^\alpha (h((X_t)^{-1}(x)) + h((X_t)^{-1}(y))). \end{aligned}$$

Setting $q_t = \frac{\beta}{C_1 t}$, where C_1 is as in Theorem 1.7, and using the Young inequality with exponents $\left(\frac{q_t}{\alpha p_t}, \frac{p}{p_t}\right)$ we deduce

$$\begin{aligned} & g_t(x)^\alpha (h((X_t)^{-1}(x)) + h((X_t)^{-1}(y))) \\ & \lesssim \left(\frac{\alpha p_t}{q_t} g_t(x)^{q_t/p_t} + \frac{p_t}{p} (h((X_t)^{-1}(x))^{p/p_t} \right) + \left(\frac{\alpha p_t}{q_t} g_t(y)^{q_t/p_t} + \frac{p_t}{p} (h((X_t)^{-1}(y))^{p/p_t} \right) \\ & =: \bar{g}_t(x) + \bar{g}_t(y). \end{aligned}$$

Thanks to (1.18) and (1.14) we get

$$\begin{aligned} \|\bar{g}_t\|_{L^{p_t}} \le& \frac{\alpha p_t}{q_t} \|g_t\|_{L^{q_t}}^{q_t/p_t} + \frac{p_t}{p} e^{\frac{tL}{p_t}} \|h\|_{L^p}^{p/p_t} \le e^{\frac{tL}{p_t}} (1+\varepsilon)^{p/p_t} \left(\frac{\alpha p_t}{q_t} (C_2 K)^{1/p_t} + \frac{p_t}{p} \right) \\ \le& (1+\varepsilon)^{p/p_t} (e^{tL} C_2 K)^{1/p_t}, \end{aligned}$$

letting $\varepsilon \to 0$ we conclude the proof.

2 Counterexamples

In this section we prove that Corollary 1.4 and Theorem 1.7 are optimal in the scale of Sobolev spaces by mean of three different examples. The first one tries to answer the question whether the integrability condition

$$\sup_{t>0} \int_{\mathbb{T}^d} \exp\{ \beta \mid \nabla b_t(x) | \} \, \mathrm{d}x < \infty \qquad \text{for some } \beta > 0,$$

assumed in Theorem 1.2 and Theorem 1.7 can be relaxed.

Theorem 2.1. *There exist a divergence free velocity field b satisfying*

$$\operatorname{supp} b_t \subset B_{1/2} \quad \forall t \ge 0; \qquad \sup_{t>0} \int_{B_{1/2}} \exp\left\{ \frac{|\nabla b_t(x)|}{\log(1 + |\nabla b_t(x)|)^a} \right\} \mathrm{d}x < \infty \quad \forall a > 0, \tag{2.1}$$

and $u_0 \in C_c^\infty(B_{1/2})$ such that $\operatorname{supp} u_t \subset B_{1/2}$ $\forall t \ge 0$ and

$$[u_t]_{W^{s,p}} = \infty \qquad \forall t > 0, \ \forall s > 0, \ \forall p \ge 1,$$

where u_t is the solution in $L^\infty([0,\infty) \times \mathbb{R}^d)$ of (Tr) with initial data u_0.

The second example shows that the conclusions in Corollary 1.4 are sharp in the scale of fractional Sobolev spaces.

Theorem 2.2. *For any $m \in \mathbb{N}$ and $\lambda > 0$ there exist a divergence free velocity field b satisfying*

$$\operatorname{supp} b_t \subset B_{1/2} \quad \forall t \ge 0; \qquad \sup_{t>0} \int_{B_{1/2}} \exp\{\lambda |\nabla b_t(x)|\} \, \mathrm{d}x < \infty \tag{2.2}$$

and $u_0 \in C_c^m(B_{1/2})$ such that the unique solution $u \in L^\infty([0,\infty) \times \mathbb{R}^d)$ of (Tr) with $\operatorname{supp} u_t \subset B_{1/2}$ for any $t \ge 0$ fulfills *(i)* $\|u_t\|_{W^{1,1}} = \infty$ *for any* $t > c_1\lambda$*;*

(ii) $[u_t]_{W^{\alpha_t,1}} = \infty$ *for any* $t > 0$*, where* $\alpha_t := 1 \wedge \frac{c_2\lambda}{t}$ *for any* $t > 0$*;*

where $c_1 > 0$ *and* $c_2 > 0$ *depend only on* m *and* d*.*

Remark 2.3. An immediate consequence of Theorem 2.2 is that the flow map X_t associated to b satisfies $\|\nabla X_t\|_{W^{1,1}} = \infty$ for any $t > c\lambda$, where $c = c(d) > 0$.

The last example shows that, in general, we cannot hope for the Lipschitz regularity of the flow associated to b when $\|\nabla b\|_{L^\infty} = \infty$, even under a very strong integrability assumption (in the Orlicz sense) on ∇b. In particular we cannot extend Corollary 1.8 and (1.11) to the case $p = \infty$.

Theorem 2.4. *Let us fix an increasing* $\Phi : [0,\infty) \to [0,\infty)$ *satisfying* $\Phi(0) = 0$ *and* $\lim_{r\to\infty} \Phi(r) = \infty$*. Then there exist a divergence free velocity field* b *satisfying*

$$\operatorname{supp} b_t \subset B_{1/2} \quad \forall t \geq 0; \qquad \sup_{t>0} \int_{B_{1/2}} \Phi(|\nabla b_t(x)|)\,\mathrm{d}x < \infty, \tag{2.3}$$

and $u_0 \in C^1_c(B_{1/2})$ *such that* $\operatorname{supp} u_t \subset B_{1/2}$ $\forall t \geq 0$ *and*

$$\|\nabla u_t\|_{L^\infty} = \infty \qquad \forall t > 0,$$

where $u_t \in L^\infty([0,T) \times \mathbb{R}^d)$ *is the solution of* (Tr) *with initial data* u_0*.*

Let us spend a few words explaining the idea behind the construction of the examples in Theorem 2.1, Theorem 2.2 and Theorem 2.4. Basically, they are built following a common strategy that has been introduced for a first time in [ACM14], [ACM18] and recently adopted in [BN18a]. Following this scheme the construction of the vector field b and the solution u_t of (Tr) is achieved by patching together a countable number of pairs v_n and ρ_n of velocity fields and solutions to (Tr) with disjoint supports. They are obtained by rescaling in space, time and size v and ρ, that are the fundamental building block provided by Proposition 2.5. Choosing properly the scaling parameters we get the three different examples.

Proposition 2.5. *Assume* $d \geq 2$ *and let* Q *be the open cube with unit side centered at the origin of* $\mathbb{R}^d$*. There exist a velocity field* $v \in C^\infty([0,\infty) \times \mathbb{R}^d)$ *and a solution* $\rho \in L^\infty([0,\infty) \times \mathbb{R}^d)$ *of* (Tr) *such that*

(i) v_t *is bounded, divergence-free and compactly supported in* Q *for any* $t \geq 0$*;*

(ii) ρ_t *has zero average and it is bounded and compactly supported in* Q *for any* $t \geq 0$*;*

(iii) $\sup_{t\geq 0} \|v_t\|_{W^{1,\infty}(\mathbb{R}^d)} < \infty$ *for any* $t \geq 0$*, for any* $1 \leq p \leq \infty$*;*

(iv) there exists a constant $c > 0$ *such that*

$$\|\rho_t\|_{\dot W^{s,p}(\mathbb{R}^d)} \gtrsim \exp(cst), \qquad \forall t \geq 0, \quad s > 0, \quad 1 \leq p \leq \infty. \tag{2.4}$$

Proof. As remarked in [ACM18, Remark 10] we can assume $d = 2$. In [ACM16] the authors proved the existence of a velocity field $v \in C^\infty([0,\infty) \times \mathbb{R}^d)$ and a solution $\rho \in L^\infty([0,\infty) \times \mathbb{R}^d)$ of (Tr) satisfying (i), (ii), (iii) and

$$\|\rho_t\|_{H^{-s}(\mathbb{R}^2)} \leq C_s \exp(-sct), \qquad \text{for any } s \in (0,1).$$

However, from [ACM16, page 33, proof of 6.4], we also have

$$\|\rho_t\|_{W^{-s,q}(\mathbb{R}^2)} \lesssim_{s,q} \exp(-cst) \quad \text{for any } 1 \leq q \leq 2, \text{ and } 0 < s < 2.$$

Therefore, thanks to Gagliardo–Nirenberg interpolation inequality (see [AF75]) we obtain (2.4). □

Before going into details with the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.4 we present a technical ingredient.

Lemma 2.6. *Let $\gamma \in (-\infty, 1)$ be fixed. For every $n \in \mathbb{N}$ consider an open set Ω_n, a function $f_n \in L^p(\mathbb{R}^d)$ and a parameter $0 < \lambda_n < 1/4$. Assume that the family $\{\Omega_n\}_{n\in\mathbb{N}}$ is disjoint and that the distance between $\operatorname{supp} f_n$ and $\mathbb{R}^d \setminus \Omega_n$ is bigger than λ_n for every $n \in \mathbb{N}$. Then it holds*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \\
&\quad\geq \limsup_{N\to\infty} \sum_{n=1}^{N} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h - \frac{c(d)2^p}{sp}\left(\frac{2}{\lambda_n}\right)^{sp} \|f_n\|_{L^p}^p \right). \qquad (2.5)
\end{aligned}$$

Proof. Let us call $\bar{\Omega}_n \subset \Omega_n$ the set of $x \in \mathbb{R}^d$ whose distance from $\operatorname{supp} f_n$ is smaller than $\lambda_n/2$. Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \\
&\geq \limsup_{N\to\infty} \sum_{n=1}^{N} \int_{B_{\lambda_n/2}}\int_{\bar{\Omega}_n} \frac{|f_n(x+h) - f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \\
&= \limsup_{N\to\infty} \sum_{n=1}^{N} \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \\
&\qquad\qquad - \int_{\mathbb{R}^d\setminus B_{\lambda_n/2}}\int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \Big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus B_{\lambda_n/2}}\int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^p}{|h|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}h \\
&\leq 2^p \|f_n\|_{L^p}^p \int_{\mathbb{R}^d\setminus B_{\lambda_n/2}} \frac{1}{|h|^{d+sp}}\, \mathrm{d}h \leq \frac{c(d)2^p}{sp}\left(\frac{2}{\lambda_n}\right)^{sp} \|f_n\|_{L^p}^p .
\end{aligned}$$

Combining these inequalities we get the sought conclusion. □

Remark 2.7. The inequality (2.5) in the case $p = 2$ was proven in [ACM18].

Let us start with the construction of our examples. Let $p \geq 1$ be fixed. We consider v and ρ as in Proposition 2.5, and a family of disjoint open cubes $\{Q_n\}_{n\in\mathbb{N}}$ contained in $B_{1/4}$. Assuming that the cube Q_n has side of length $3\lambda_n$ and center at $x_n \in B_{1/4}$, we set

$$v_n(t,x) := \frac{\lambda_n}{\tau_n} v\left(\frac{t}{\tau_n}, \frac{x - x_n}{\lambda_n}\right), \qquad \rho_n(t,x) := \gamma_n \rho\left(\frac{t}{\tau_n}, \frac{x - x_n}{\lambda_n}\right),$$

for every $x \in \mathbb{R}^d$, $t \geq 0$ and $n \in \mathbb{N}$ where $\gamma_n, \tau_n, \lambda_n \in (0, 1/10)$ convergence to 0 and

$$\sup_n \lambda_n/\tau_n < \infty.$$

Observe that u_n is supported in Q_n and $\operatorname{dist}(\operatorname{supp} u_n, \mathbb{R}^d \setminus Q_n) \geq \lambda_n$ for every $n \in \mathbb{N}$. Let us set

$$b(t,x) := \sum_n v_n(t,x), \qquad u(t,x) := \sum_n \rho_n(t,x) \qquad \forall x \in \mathbb{R}^d, \quad \forall t > 0.$$

Note that $\operatorname{supp} b_t \subset B_{1/2}$ and $\operatorname{supp} u_t \subset B_{1/2}$. It remains to choose properly the parameters λ_n, γ_n, τ_n in order to get our three examples.

Proof of Theorem 2.1. We choose

$$\gamma_n = \exp\left\{-\log(n)\log\log\log\log(n)\right\}, \ \tau_n = \frac{1}{\log(n)\log\log\log(n)}, \ \lambda_n = \frac{1}{n^{1/d}\exp\left\{\frac{\log(n)}{\log\log\log(n)}\right\}}$$

for $n \in \mathbb{N}$ big enough[1]. It is easily seen that $\sup_n \gamma_n \lambda_n^{-m} + \gamma_n < \infty$ for any m, it implies $u_0 \in C_c^\infty(B_{1/2})$ and $u \in L^\infty((0,\infty) \times \mathbb{R}^d)$. Let us check (2.1).

$$
\begin{aligned}
&\int_{B_1(0)} \left(\exp\left\{ \frac{|\nabla b(t,x)|}{\log(1+|\nabla b(t,x)|)^a} \right\} - 1 \right) \mathrm{d}x \\
&= \sum_n \int_{B_1(0)} \left(\exp\left\{ \frac{|\nabla v_n(t,x)|}{\log(1+|\nabla v_n(t,x)|)^a} \right\} - 1 \right) \mathrm{d}x \\
&= \sum_n \int_{B_1(0)} \left(\exp\left\{ \frac{\frac{1}{\tau_n}|\nabla v(t/\tau_n,x)|}{\log(1+\frac{1}{\tau_n}|\nabla v(t/\tau_n,x)|)^a} \right\} - 1 \right\} \lambda_n^d \, \mathrm{d}x \\
&\le C \sum_n \left(\exp\left\{ \frac{\frac{1}{\tau_n} C}{\log(1+\frac{C}{\tau_n})^a} \right\} - 1 \right) \lambda_n^d \\
&\le \sum_n C \exp\left\{ C' |\log(\tau_n)|^{-a} \tau_n^{-1} \right\} \lambda_n^d \\
&\le C \sum_n \exp\left\{ C'' \frac{\log(n) \log\log\log(n)}{(\log\log(n))^a} \right\} \frac{1}{n \exp\left\{ \frac{d \log(n)}{\log\log\log(n)} \right\}},
\end{aligned}
$$

where C, C' and C'' are positive constants. Observe that for $n_0 \in \mathbb{N}$ large enough, we have

$$
C'' \frac{\log\log\log(n)}{(\log\log(n))^a} \le \frac{d}{2} \frac{1}{\log\log\log(n)} \qquad \forall n \ge n_0.
$$

Thus,

$$
\int_{B_1(0)} \left(\exp\left\{ \frac{|\nabla b(t,x)|}{\log(1+|\nabla b(t,x)|)^a} \right\} - 1 \right) dx \le C \sum_n \frac{1}{n \exp\left\{ \frac{d \log(n)}{2 \log\log\log(n)} \right\}} \lesssim \sum_n \frac{1}{n^2} < \infty.
$$

Let us eventually verify

$$
[u_t]_{W^{s,p}} = \infty \qquad \forall t > 0, \ \forall s > 0, \ \forall p \ge 1.
$$

An application of Lemma 2.6 leads to

$$
\begin{aligned}
[u_t]^p_{W^{s,p}} &\ge \limsup_{N\to\infty} \sum_{n=1}^N \left([\rho_n(t,\cdot)]^p_{W^{s,p}} - \frac{c(d)2^p}{sp} \left(\frac{2}{\lambda_n} \right)^{sp} \|\rho_n(t,\cdot)\|^p_{L^p} \right) \\
&\ge \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^p \lambda_n^{d-sp} \left([\rho(t/\tau_n,\cdot)]^p_{W^{s,p}} - C(s,p,d) \|\rho(t/\tau_n,\cdot)\|^p_{L^p} \right) \\
&\ge \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^p \lambda_n^{d-sp} \left(C \exp\{cpst/\tau_n\} - C(s,p,d) \|\rho_0\|^p_{L^p} \right),
\end{aligned}
$$

where in the last passage we used Proposition 2.5(iv). Now observe that

$$
\begin{aligned}
&\gamma_n^p \lambda_n^{d-sp} \exp\left\{ \frac{cpst}{2\tau_n} \right\} \\
&= \frac{1}{n^{1-sp/d}} \exp\left\{ \log(n) \left(\frac{cpst}{2} \log\log\log(n) - p \log\log\log\log(n) - \frac{(d-sp)}{\log\log\log(n)} \right) \right\} \\
&\ge \frac{1}{n^{1-sp/d}} \exp\{2\log(n)\} = n^{1+sp/d} \ge 1,
\end{aligned}
$$

for any n large enough. Moreover $\sum_{n=1}^\infty \gamma_n^p \lambda_n^{d-sp} < \infty$, so

$$
[u_t]^p_{W^{s,p}} \ge C' \sum_{n \ge n_0}^\infty \exp\left\{ \frac{cpst}{2\tau_n} \right\} - C'' = \infty.
$$

The proof is complete. $\square$

[1] It suffices that $\log\log\log\log(n) \ge 1$, namely $n \ge \exp\exp\exp\exp(1)$

Let us now pass to the second example.

Proof of Theorem 2.2. For any $m \in \mathbb{N}$ positive and any $\lambda > 0$, we choose

$$\tau_n = \frac{2c_0\lambda}{\log(n)\log\log\log(n)}, \qquad \lambda_n = \frac{1}{\exp\{\log(n)\log\log\log(n)\}}, \qquad \gamma_n = \lambda_n^m,$$

for $n \geq 10^{100}$, where $c_0 := \|v\|_{L^\infty}$. Since $\sup_n \gamma_n\lambda_n^{-m} + \gamma_n < \infty$, we have $u_0 \in C_c^m(B_{1/2})$ and $u \in L^\infty((0,\infty)\times\mathbb{R}^d)$. Let us check (2.2). Using the identity $\lambda_n = \exp(-2c_0\lambda/\tau_n)$ we get

$$\begin{aligned}\int_{B_1}(\exp\{\lambda|\nabla b_t(x)|\} - 1)\,\mathrm{d}x &= \sum_n \int_{B_1}(\exp\{\lambda|\nabla v_n(t,x)|\} - 1)\,\mathrm{d}x \\ &\leq C\sum_n \exp\{c_0\lambda/\tau_n\}\lambda_n^d \\ &= C\sum_n \lambda_n^{d-\frac{1}{2}} < \infty.\end{aligned}$$

Let us now prove (i).

$$\|u_t\|_{W^{1,1}} = \sum_n \|\rho_n(t,\cdot)\|_{W^{1,1}} = \sum_n \gamma_n\lambda_n^{d-1}\|\rho(t/\tau_n,\cdot)\|_{W^{1,1}} \geq C\sum_n \gamma_n\lambda_n^{d-1}\exp\{ct/\tau_n\}$$

where in the last line we used Proposition 2.5(iv). Now observe that

$$C\sum_n \gamma_n\lambda_n^{d-1}\exp\{ct/\tau_n\} \geq C\sum_n \exp\{(-2c_0\lambda(d+m-1)+ct)/\tau_n\} = +\infty$$

provided $-2c_0\lambda(d+m-1)+ct \geq 0$, that is to say

$$t \geq c_1\lambda \qquad \text{with} \quad c_1 = 2c_0(d+m-1)/c.$$

Let us finally prove (ii). For any $0 < s < 1$ we have

$$\begin{aligned}[u_t]_{W^{s,1}} &\geq \limsup_{N\to\infty}\sum_{n=1}^N \left(\|\rho_n(t,\cdot)\|_{W^{s,1}} - \frac{c(d)2}{s}\left(\frac{2}{\lambda_n}\right)^s \|\rho_n(t,\cdot)\|_{L^1}\right) \\ &\geq \limsup_{N\to\infty}\sum_{n=1}^N \gamma_n\lambda_n^{d-s}\left(\|\rho(t/\tau_n,\cdot)\|_{W^{s,1}} - C(s,d)\|\rho(t/\tau_n,\cdot)\|_{L^1}\right) \\ &\geq C\sum_{n=1}^\infty \gamma_n\lambda_n^{d-s}\exp\{cst/\tau_n\} - C' \\ &= C\sum_n \exp\{(-2c_0\lambda(d+m-s)+cst)/\tau_n\} - C',\end{aligned}$$

where we used $\sum_{n=1}^\infty \gamma_n\lambda_n^{d-s} < \infty$. From the previous estimate we deduce

$$[u_t]_{W^{s,1}} = \infty \qquad \text{provided} \qquad -2c_0\lambda(d+m-s)+cst \geq 0,$$

that implies our conclusion with $c_2 = 2c_0(d-1+m)/c$. □

Proof of Theorem 2.4. Let us choose

$$\tau_n = \frac{1}{\log\log(n)}, \qquad \lambda_n = \frac{1}{n^2\Phi(c_0\log\log(n))^{1/d}}, \qquad \gamma_n = \frac{\lambda_n}{\log\log(n)} \qquad \text{for } n \geq 10^{100}, \tag{2.6}$$

where $c_0 = \|\nabla v\|_{L^\infty}$. First of all let us observe that $u_0 \in C_c^1(B_{1/2})$, since $\gamma_n/\lambda_n = 1/\log\log(n)$. In order to check (2.3), we estimate

$$\int_{\mathbb{R}^d}\Phi(|\nabla b_t(x)|)\,\mathrm{d}x = \sum_n \lambda_n^d\int_{B_2}\Phi(|\nabla v(t/\tau_n,x)|/\tau_n)\,\mathrm{d}x = C\sum_n \frac{1}{n^{2d}} < \infty.$$

Moreover, using Proposition 2.5(ii) we have

$$\begin{aligned}\|\nabla u_t\|_{L^\infty} &\geq \sup_n \frac{\gamma_n}{\lambda_n} \|\nabla \rho(t/\tau_n, \cdot)\|_{L^\infty} \\ &\geq C \sup_n \frac{\gamma_n}{\lambda_n} \exp\{ct/\tau_n\} \\ &= \sup_n \frac{C}{\log\log n} \exp\{ct \log\log(n)\} = \infty.\end{aligned}$$

The proof is complete. □

3 Application to the 2D Euler equation

In this section we present an application of Corollary 1.4 to the study of the 2D Euler equation with bounded initial vorticity in the class $L^\infty \cap W^{\alpha,p}$. We prove a propagation of regularity result that generalizes [BC94, Corollary 1.1].

Let us start by introducing the Cauchy problem associated to the 2D Euler equation in vorticity formulation. Here we set the problem in the 2 dimensional torus:

$$\begin{cases} \partial_t \omega_t + \operatorname{div}(b_t \omega_t) = 0, \\ b_t = K * \omega_t, \\ \omega_0 = \bar{\omega}, \end{cases} \tag{E}$$

where $\bar{\omega}$ is the initial data and K is the Biot-Savart kernel.

In this section, we consider only solutions of class $L^\infty([0,T] \times \mathbb{T}^2) \cap C([0,T]; L^1(\mathbb{T}^2))$ satisfying the weak formulation

$$\int_{\mathbb{T}^2} \omega_t \varphi \,\mathrm{d}x - \int_{\mathbb{T}^2} \omega_0 \varphi \,\mathrm{d}x = \int_0^t \int \omega_s \, b_s \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}s \qquad \forall \varphi \in C^\infty(\mathbb{T}^2). \tag{3.1}$$

It is well-known since the work [YU63] that in this class (E) admits a unique solution (in the sense of (3.1)). We refer to [BM02, CH98, L96, MP94] for a detailed description of the classical theory for the 2D Euler equation.

Let us state the main result of this section.

Theorem 3.1. *Let $0 < \alpha \leq 1$ and $p \geq 1$ be fixed. Consider a weak solution of (E) $\omega \in L^\infty([0,T] \times \mathbb{T}^2) \cap C([0,T]; L^1(\mathbb{T}^2))$. The following hold true:*

(i) if $\omega_0 \in W^{\alpha,p}(\mathbb{T}^2)$ then

$$\omega_t \in W^{\alpha_t,p}(\mathbb{T}^2) \qquad \text{with} \quad \alpha_t := \alpha \, \frac{1}{1 + C\|\omega_0\|_{L^\infty} \alpha p t} \qquad \forall t \in [0,T], \tag{3.2}$$

where $C > 0$ is a universal constant;

(ii) if $\omega_0 \in C(\mathbb{T}^2) \cap W^{\alpha,p}(\mathbb{T}^2)$ with $p > 1$ then

$$\omega_t \in W^{\alpha',p}(\mathbb{T}^2) \qquad \text{for any } 0 < \alpha' < \alpha, \quad \forall t \in [0,T]. \tag{3.3}$$

When $\alpha = 1$ we also have

$$\omega_t \in W^{1,p'}(\mathbb{T}^2) \qquad \text{for any } 1 \leq p' < p, \quad \forall\, t \in [0,T]; \tag{3.4}$$

(iii) If $\omega_0 \in W^{\alpha,p}(\mathbb{T}^2)$ with $p > 2/\alpha$ it holds $\omega_t \in W^{\alpha,p}(\mathbb{T}^2)$ for any $t \in [0,T]$.

Remark 3.2. The conclusion (iii) in Theorem 3.1 can be also obtained using the Hölder theory for the Euler equation (see for instance [BM02]). Indeed, the Sobolev embedding gives $W^{\alpha,p}(\mathbb{T}^2) \subset C^{1-\frac{2}{p\alpha}}(\mathbb{T}^2)$ for $\alpha p > 2$ that implies in turn $b \in L^1([0,T]; C^{2-\frac{2}{p\alpha}}(\mathbb{T}^2))$ and the classical Cauchy-Lipschitz theory can be applied.

Before proving Theorem 3.1 let us recall the main properties of the Biot-Savart kernel K in (E). In the whole space $\mathbb{R}^2$ it can be explicitly written as

$$K(y) := \frac{1}{2\pi}\frac{y^{\perp}}{|y|^2},$$

while in our periodic setting has a more complicated form [2] but still satisfies the following properties:

(i) $K \in L^1(\mathbb{T}^2;\mathbb{R}^2)$;

(ii) $\nabla K : \mathbb{T}^2 \to \mathbb{R}^{2\times 2}$ is a vector valued Calderon-Zygmund kernel, in particular there exists a constant $C > 1$ such that

$$\int_{\mathbb{T}^2} \exp\left\{ \frac{|\nabla (K * f)|}{C\,\|f\|_{L^\infty}} \right\} \mathrm{d}x \le C \qquad \forall f \in L^\infty(\mathbb{T}^2). \tag{3.5}$$

(3.5) follows from the fact that Calderon-Zygmund operators map L^∞ in BMO (see [ST70]) and from the exponential integrability of BMO functions (see [N61]). We refer to [S96] for a detailed analysis of the Biot-Savart Kernel K in the periodic setting.

The next lemma shows that (3.5) can be slightly improved when $f \in C(\mathbb{T}^2)$. It is basically a consequence of the fact that Calderon-Zygmund operator map $C(\mathbb{T}^2)$ to VMO$(\mathbb{T}^2)$ (the space of vanishing mean oscillation functions), see [ST93, page 180].

Lemma 3.3. *Let K as above. Then for any $f \in C(\mathbb{T}^2)$ and every $\beta > 0$ it holds*

$$\int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla (K * f)| \,\} \,\mathrm{d}x < C(\beta, \|K\|_{L^1}, \|f\|_{L^\infty}, \rho_f^{-1}(C/\beta)),^{3} \tag{3.6}$$

where $\rho_f(r) := \sup\{\, \mid f(x) - f(y)| : \; x, y \in \mathbb{T}^2 \text{ with } \mathsf{d}(x,y) \le r \,\}$ is the modulus of continuity of f.

Proof. Let us fix $\beta > 1$ and $f \in C(\mathbb{T}^2)$. There exists $\bar f \in C^\infty(\mathbb{T}^2)$ such that $\|f - \bar f\|_{L^\infty} \le \frac{C}{\beta}$ and $\|\nabla \bar f\|_{L^\infty} \le \frac{2\|f\|_\infty}{\rho_f^{-1}(C/\beta)}$ where C is as in (3.5). Observe that, since $K \in L^1(\mathbb{T}^2;\mathbb{R}^2)$, we have

$$\|\nabla (K * \bar f)\|_{L^\infty} = \|K * \nabla \bar f\|_{L^\infty} \le \frac{2\,\|f\|_{L^\infty}\|K\|_{L^1}}{\rho_f^{-1}(C/\beta)}. \tag{3.7}$$

Therefore we can estimate

$$\begin{aligned}
\int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla (K * f)| \,\} \,\mathrm{d}x &\le \int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla (K * \bar f)| + \beta |\nabla (K * (f - \bar f))| \,\} \,\mathrm{d}x \\
&\le \exp\left\{ \frac{2\beta\,\|f\|_{L^\infty}\|K\|_{L^1}}{\rho_f^{-1}(C/\beta)} \right\} \int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla (K * (f - \bar f))| \,\} \,\mathrm{d}x \\
&\le \exp\left\{ \frac{2\beta\,\|f\|_{L^\infty}\|K\|_{L^1}}{\rho_f^{-1}(C/\beta)} \right\} \int_{\mathbb{T}^2} \exp\left\{ \frac{|\nabla (K * (f - \bar f))|}{C\,\|f - \bar f\|_{L^\infty}} \right\} \mathrm{d}x \\
&\le C \exp\left\{ \frac{2\beta\,\|f\|_{L^\infty}\|K\|_{L^1}}{\rho_f^{-1}(C/\beta)} \right\} < \infty,
\end{aligned}$$

where in the last line we used (3.5). $\square$

[2] For any function f in $\mathbb{T}^2$, it holds

$$K * f(x) = \sum_{n\in\mathbb{Z}} \int_{[0,1]^2} K(x - y + n) f(y) \,\mathrm{d}y.$$

[3] This constant blows up when $\beta \to \infty$

Proof of Theorem 3.1. Any weak solution ω_t of (E) is also a distributional solution of (Tr) with drift $b_t := K * \omega_t$. Observe that $\operatorname{div} b_t = 0$ and it holds

$$\sup_{t>0} \int_{\mathbb{T}^2} \exp\left\{ \frac{|\nabla b_t|}{C\,\|\omega_0\|_{L^\infty}} \right\} \mathrm{d}x \le C,$$

thanks to (3.5) and the identity $\|\omega_t\|_{L^\infty} = \|\omega_0\|_{L^\infty}$. Applying Corollary 1.4(i) the first conclusion follows.

Let us now address (ii). First of all observe that if $\omega_0 \in C(\mathbb{T}^2)$ then $\omega_t \in C(\mathbb{T}^2)$ for any $t \ge 0$. It follows from the identity $\omega_t(x) = \omega_0((X_t)^{-1}(x))$ where X is the flow associated to b_t, that is continuous together with its inverse thanks to (1.16). Therefore Lemma 3.3 infers that, for any $\beta > 0$ and $t \ge 0$

$$\int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla b_t(x)|\,\}\, \mathrm{d}x < \infty. \tag{3.8}$$

Exploiting Lemma 3.3 and the fact that the modulus of continuity of ω_t fulfills $\sup_{0<t<T} \rho_{\omega_t} \le \rho$ for some nondecreasing $\rho : (0,\infty) \to (0,\infty)$ satisfying $\lim_{r\to 0} \rho(r) = 0$ we can strengthen (3.8) as

$$\sup_{0<t<T} \int_{\mathbb{T}^2} \exp\{\, \beta \mid \nabla b_t(x)|\,\}\, \mathrm{d}x < \infty \qquad \text{for any } \beta > 0.$$

We are in position to apply Corollary 1.5 and conclude the proof of (ii). We eventually prove (iii). Since $W^{\alpha,p}(\mathbb{T}^2) \subset C(\mathbb{T}^2)$ when $p > 2/\alpha$ we use (ii) to deduce that $\omega_t \in W^{\alpha',p}(\mathbb{T}^2)$ for any $0 < \alpha < \alpha'$ and $t \ge 0$. This infers that $\nabla b_t \in W^{\alpha',p}(\mathbb{T}^2; \mathbb{R}^{2\times 2})$ for any $0 < \alpha' < \alpha$. Sobolev's embedding theorem implies that b_t is a Lipschitz vector field with respect to the spatial variable. It is also clear that this estimate is locally uniform in time, therefore the standard Cauchy-Lipschitz theory applies and we conclude that X_t is actually biLipschitz, thus $\omega_t = \omega_0(X_t^{-1}) \in W^{\alpha,p}(\mathbb{T}^2)$. $\square$

A Appendix

In this appendix we collect two technical results concerning functions whose gradient is exponentially integrable. The first one is a consequence of [AF75, Theorem 8.40], we add its proof for the reader convenience.

Lemma A.1. *Let $f \in W^{1,1}(\mathbb{T}^d)$ satisfy*

$$\int_{\mathbb{T}^d} \exp\{\, \beta \mid \nabla f(x)|\,\}\, \mathrm{d}x =: K < \infty, \qquad \text{for some } \beta > 0.$$

Then f admits a continuous representative satisfying

$$|f(x) - f(y)| \le \frac{C}{\beta} \mathsf{d}(x,y) \log\left(\frac{CK}{\mathsf{d}(x,y)^d} \right) \qquad \forall x, y \in \mathbb{T}^d, \tag{A.1}$$

where $C = C_d > 2$.

Proof. Thanks to the Poincaré inequality there exists a constant $\bar{C} > 0$ such that

$$\fint_{B_{2r}(x)} \left| f(y) - \fint_{B_r(x)} f \right| \mathrm{d}y \le \bar{C} r \fint_{B_{2r}(x)} |\nabla f(y)|\, \mathrm{d}y \le \bar{C} r \left(\fint_{B_{2r}(x)} |\nabla f(y)|^p\, \mathrm{d}y \right)^{1/p}, \tag{A.2}$$

for any $x \in \mathbb{T}^d$, for any $r \in (0,2)$ and $p \ge 1$. Using the notation $f_{x,r} := \fint_{B_r(x)} f$ we deduce from (A.2)

$$\frac{\left(\frac{\beta}{r} \fint_{B_{2r}(x)} |f - f_{x,r}|\, \mathrm{d}y \right)^p}{p!} \le \frac{\bar{C}}{r^d \omega_d} \int_{\mathbb{T}^d} \frac{(\beta |\nabla f(y)|)^p}{p!}\, \mathrm{d}y, \qquad \forall p \in \mathbb{N},\ p \ge 1.$$

Taking the sum for $p \ge 1$ we conclude

$$\exp\left(\frac{\beta}{r} \fint_{B_{2r}(x)} |f - f_{x,r}|\, \mathrm{d}y \right) - 1 \le \frac{\bar{C}}{r^d \omega_d} (K - 1).$$

This gives

$$|f_{x,2r} - f_{x,r}| \le \fint_{B_{2r}(x)} |f - f_{x,r}| \,\mathrm{d}y \le \frac{r}{\beta} \log\left(\frac{K\bar{C}}{\omega_d r^d}\right), \tag{A.3}$$

up to increase $\bar{C}$. Thanks to Morrey's inequality (see [AF75]) we know that $f \in C(\mathbb{T}^d)$ and thus $f(x) = \lim_{r\to 0} \fint_{B_r(x)} f\,\mathrm{d}y$. Using this, (A.3) and a standard iteration procedure we get

$$|f_{x,r} - f(x)| \le \sum_{k=0}^{\infty} |f_{x,2^{-k}r} - f_{x,2^{-k-1}r}| \le \sum_{k=0}^{\infty} \frac{r2^{-k}}{\beta} \log\left(\frac{K\bar{C}}{\omega_d (2^{-k}r)^d}\right) \le \frac{C'r}{\beta} \log\left(\frac{K\bar{C}}{\omega_d r^d}\right),$$

for some $C' > 0$ depending only on d. Observe that when $r = \mathsf{d}(x,y)$ we have $|f_{x,r} - f_{y,r}| \le 2^d \fint_{B_{2r}(x)} |f - f_{x,r}| \,\mathrm{d}y$, therefore

$$\begin{aligned} |f(x) - f(y)| \le & |f(x) - f_{x,r}| + 2^d \fint_{B_{2r}(x)} |f - f_{x,r}| \,\mathrm{d}y + |f_{y,r} - f(y)| \\ \le & \frac{(2C' + 2^d)r}{\beta} \log\left(\frac{K\bar{C}}{\omega_d r^d}\right) \le \frac{C}{\beta} \mathsf{d}(x,y) \log\left(\frac{CK}{\mathsf{d}(x,y)^d}\right), \end{aligned}$$

where $C = C_d$. □

Lemma A.2. *Let $f \in W^{1,1}(\mathbb{T}^d)$ satisfy*

$$\int_{\mathbb{T}^d} \exp\{\, \beta \mid \nabla f(x) | \,\} \,\mathrm{d}x < \infty, \qquad \textit{for some } \beta > 0.$$

Then for any $0 < \beta' \le \beta$ we have

$$\frac{\beta'}{C_1} \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} \le 1 + \log\left(c_d M \left(\exp\{\beta' |\nabla f|\}\right)(x)\right) \qquad \forall x, y \in \mathbb{R}^d,$$

where $c_d \ge 1$ and $C_1 > 0$ depend only on d.

Proof. Let us fix $x, y \in \mathbb{R}^d$. Using Morrey's inequality (see [AF75]) we have

$$\frac{1}{p!} \left(\frac{\beta' |f(x) - f(y)|}{Cr}\right)^p \le \fint_{B_r(x)} \frac{(\beta' |\nabla f|)^p}{p!} \,\mathrm{d}z \qquad \forall p > d \text{ integer}, \tag{A.4}$$

where $C > 0$ depends only on d and $r := 2\mathsf{d}(x,y)$. In order to make notation short let us set

$$A := \frac{\beta'}{2C} \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)}.$$

Taking the sum in (A.4) for p between $d+1$ and ∞ we end up with

$$e^A - 1 - A - \frac{A^2}{2!} - \dots - \frac{A^d}{d!} \le \fint_{B_r(x)} \exp\{\beta' |\nabla f|\} \,\mathrm{d}z \le M\left(\exp\{\beta' |\nabla f|\}\right)(x).$$

Now observe that, when $A \ge 1$, we have

$$e^{A/2} \le c_d (e^A - 1 - A - \frac{A^2}{2!} - \dots - \frac{A^d}{d!}) \le c_d M \left(\exp\{\beta' |\nabla f|\}\right)(x),$$

for some constant $c_d \ge 1$, thus we deduce

$$\frac{A}{2} \le \log\left(c_d M \left(\exp\{\beta' |\nabla f|\} - 1\right)(x)\right) \qquad \text{when } A > 1,$$

that trivially gives

$$\frac{A}{2} \le \frac{1}{2} + \log\left(c_d M \exp\{\beta' |\nabla f|\}(x)\right),$$

without any restriction on A. Recalling the definition of A we conclude. □

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