A FORMULA FOR THE ANISOTROPIC TOTAL VARIATION OF SBV FUNCTIONS

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Abstract. The purpose of this paper is to present the relation between certain BMO–type seminorms and the total variation of SBV functions. Following some ideas of [2], we give a representation formula of the total variation of SBV functions which does not make use of the distributional derivatives. We consider an anisotropic variant of the BMO–type seminorm introduced in [4], by using, instead of cubes, covering families made by translations of a given open bounded set with Lipschitz boundary.

1. Introduction

Inspired by the celebrated BMO (bounded mean oscillation) space of John and Nirenberg, recently Brezis, Bourgain and Mironescu introduced in [4] a new function space B which contains in particular BMO, BV, $W^{\frac{1}{p},p}$ for $1 \leq p < \infty$. This space is defined as the set of functions from the unit cube $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ such that the following seminorm is finite

$$[f]_B := \sup_{0<\varepsilon<1} [f]_\varepsilon, \quad \text{where } [f]_\varepsilon = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx\,. \tag{1.1}$$

Here $\mathcal{G}_\varepsilon$ denotes a collection of mutually disjoint ε-cubes Q' of the type $Q' = x + \varepsilon Q$ whose cardinality does not exceed ε^{1-n}. In particular in [4] it is proved that if $f \in B$ is an integer valued function such that $\lim_{\varepsilon\to 0}[f]_\varepsilon = 0$, then f is constant.

A variant of definition (1.1) involving ε-cubes of general orientation was introduced in [1] in order to characterize sets of finite perimeter. Precisely, setting for any measurable set $A \subset \mathbb{R}^n$ and $0 < \varepsilon < 1$

$$\mathrm{I}_\varepsilon(A) := \varepsilon^{n-1} \sup_{\mathcal{F}_\varepsilon} \sum_{Q' \in \mathcal{F}_\varepsilon} \fint_{Q'} \left| \chi_A(x) - \fint_{Q'} \chi_A \right| dx\,,$$

where $\mathcal{F}_\varepsilon$ denotes a collection of pairwise disjoint ε-cubes $Q' \subset \mathbb{R}^n$ with arbitrary orientation and cardinality not exceeding ε^{1-n}, in [1] it is proved that

$$\lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon(A) = \frac{1}{2} \min\{1, P(A)\}\,. \tag{1.2}$$

This result was generalized in [6] (see also [5] and [7]). In particular in [5, Sect. 6] it is proved that given an open set $\Omega \subset \mathbb{R}^n$ and a function $f \in SBV(\Omega)$, the space of special BV functions in Ω, see Section 2, then

$$\lim_{\varepsilon\to 0} \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{Q' \in \mathcal{H}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx = \frac{1}{4} \int_\Omega |\nabla f|\, dx + \frac{1}{2} |D^s f|(\Omega)\,. \tag{1.3}$$

Here $\mathcal{H}_\varepsilon$ is a family of pairwise disjoint cubes of side length ε contained in Ω.

A crucial fact for the validity of the representation formulas (1.2) and (1.3) is that the cubes in the families $\mathcal{F}_\varepsilon$ and $\mathcal{H}_\varepsilon$ can be chosen with any orientation. Things go very differently when rotations are not allowed. Indeed in [2] it is proved that if D is a bounded open set satisfying some mild regularity assumptions, there

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exists a bounded and lower semicontinuous function $\varphi^D : \mathbb{S}^{n-1} \to (0,+\infty)$ such that for any set $A \subset \mathbb{R}^n$ of finite perimeter

$$\lim_{\varepsilon\to 0} \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'} \left|\chi_A(x) - \fint_{D'} \chi_A\right| dx = \int_{\mathcal{F}A} \varphi^D(\nu_A(x))\, d\mathcal{H}^{n-1}(x)\,, \tag{1.4}$$

where $\mathcal{H}_\varepsilon$ is a family of pairwise disjoint sets of the form $D' = x + \varepsilon D$, $\mathcal{F}A$ is the reduced boundary of A and ν_A is the generalized outer normal to A.

Here we extend formula (1.4) to the case of a general function $f \in SBV(\Omega)$. More precisely, we consider a bounded open set $D \subset \mathbb{R}^n$, $n \geq 2$, such that there exists a Lipschitz function $\varrho : \mathbb{S}^{n-1} \to (0,+\infty)$ with the property that

$$D = \{x = tz : \ z \in \mathbb{S}^{n-1},\ 0 \leq t < \varrho(z)\}\,. \tag{H}$$

Given a function $f \in L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set, for any $\varepsilon > 0$ we consider the following quantity

$$H^D_\varepsilon(f,\Omega) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'} \left|f(x) - \fint_{D'} f\right| dx\,, \tag{1.5}$$

where $\mathcal{H}_\varepsilon$ is a family of pairwise disjoint translations D' of εD contained in Ω.

Then our main result reads as follows.

Theorem 1.1. *Let $D \subset \mathbb{R}^n$ be a bounded open set satisfying the assumption (H). There exist two Lipschitz continuous 1-homogeneous functions φ, $\psi : \mathbb{R}^n \to (0,+\infty)$, $\psi \leq \varphi$, strictly positive on $\mathbb{R}^n \setminus \{0\}$, ψ convex, such that if Ω is an open subset of $\mathbb{R}^n$ and $f \in SBV(\Omega)$ then*

$$\lim_{\varepsilon\to 0} H^D_\varepsilon(f,\Omega) = \int_\Omega \psi(\nabla f(x))\, dx + \int_{J_f} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x)\,. \tag{1.6}$$

Note that in (1.6) ∇f stands for the absolutely continuous part of the gradient measure Df, J_f is the jump set of f, $f^+ > f^-$ are the traces of f on both sides of J_f and ν_f is the generalized normal to J_f oriented in the direction going from f^- to f^+.

Note that every bounded open convex set containing the origin satisfies condition (H). However the convexity of D does not guarantee that φ is convex (see the examples in [5, Sect. 4.3]). On the other hand, ψ is always convex, see Proposition 3.6, and in general it does not coincide with the greatest convex function smaller than or equal to φ (see the example in Section 6). Note that if $f \in W^{1,1}(\Omega)$ then (1.6) holds without the assumption (H), see Theorem 4.1. On the contrary this assumption seems to be necessary for the Lipschitz continuity of φ. Moreover it is also needed in order to prove a crucial property in the argument leading to representation formula (1.6) for SBV functions. Namely, if D satisfies (H) and $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a bi-Lipschitz map such that $\mathrm{Lip}(I - \Phi)$, $\mathrm{Lip}(I - \Phi^{-1})$ are small, then (see Lemma 3.7) one can prove that $\Phi(D)$ is contained and contains two suitable dilations λD of the set D with λ close to 1. Indeed, simple examples show that this property fails to be true if D is only assumed to have Lipschitz boundary (see Remark 3.8).

2. Preliminaries

2.1. **Notation.** For a measurable set $E \subset \mathbb{R}^n$ we denote by $|E|$ its Lebesgue measure. The Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$ while $\mathcal{H}^{n-1}$ will stand for the Hausdorff $(n-1)$-dimensional measure. If $E \subset \mathbb{R}^n$ is a measurable set with strictly positive and finite measure and $g : E \to \mathbb{R}$ is measurable, we shall denote by

$$g_E := \fint_E g(x)\, dx := \frac{1}{|E|} \int_E g(x)\, dx$$

the average of g on E.

For $\delta > 0$ and $E \subset \mathbb{R}^n$, the δ–neighborhood of E is denoted by $I_\delta(E)$ and is defined as follows

$$I_\delta(E) := \{x \in \mathbb{R}^n : \ \mathrm{dist}\,(x, E) < \delta\}\,.$$

Given $z \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$and $\rho > 0$, we denote by $B_\rho(z)$ the ball of radius ρ centered in z and by $Q_\nu(z,\rho)$ a cube of center z and side ρ with two faces orthogonal to ν. If the center is at the origin and $\rho = 1$ we shall simply write Q_ν instead of $Q_\nu(0,1)$. Note that if $n = 2$, given $\nu \in \mathbb{S}^1$, there exists only one such cube Q_ν. On the other hand, if $n \geq 3$, the cubes Q_ν with two faces orthogonal to $\nu \in \mathbb{S}^{n-1}$, centered in 0 and of side length 1, form a family with $n-2$ degrees of freedom. This family will be denoted by $\mathbf{Q}_\nu$.

Throughout the paper C will denote a positive constant whose value may change from line to line.

2.2. **Functions of bounded variation.** We recall here some preliminary notions that will be used in the paper. We shall refer to [3] for a detailed account on BV functions and sets of finite perimeter. In what follows Ω will always denote an open subset of $\mathbb{R}^n$.

A function $h \in L^1(\Omega)$ is of *bounded variation* (and we write $h \in \mathrm{BV}(\Omega)$) if the distributional partial derivatives of h are measures with finite total variation in Ω. Namely, there exist Radon signed measures $D_1 h, \ldots, D_n h$ in Ω such that for $i = 1, \ldots, n$ we have $|D_i h|(\Omega) < \infty$ and

$$\int_\Omega h D_i \alpha dx = -\int_\Omega \alpha d D_i h(x)\,,$$

for all $\alpha \in C^1_0(\Omega)$. The gradient of Dh of h is a vector–valued measure with finite total variation, that is

$$|Dh|(\Omega) := \sup\left\{\int_\Omega h \operatorname{div}\alpha\, dx :\ \alpha \in C^1_c(\Omega,\mathbb{R}^n),\ \|\alpha\|_\infty \leq 1\right\},$$

is finite. We say that $h \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ if $h \in \mathrm{BV}(\Omega')$ for every $\Omega' \subset\subset \Omega$, i.e. every open set Ω' with $\bar{\Omega}'$ compact and contained in Ω.

It is well known, see Remark 3.50 in [3], that if Ω is a bounded connected open set with Lipschitz boundary, the following Poincaré–Wirtinger inequality holds for any $h \in \mathrm{BV}(\Omega)$, namely

$$\|h - h_\Omega\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C|Dh|(\Omega)\,, \tag{2.1}$$

where C depends only on Ω.

Note that the quantity $H^D_\varepsilon(h,\Omega)$ defined in (1.5) is strictly related to the total variation of h. In fact, if D is a bounded connected open set with Lipschitz boundary, from (2.1), using Hölder inequality, we get that there exists a constant $C > 0$ depending only on D such that if $D' = \varepsilon D + x_0 \subset \Omega$, then

$$\varepsilon^{n-1} \fint_{D'} \left| h(x) - \fint_{D'} h \right| dx \leq C\,|Dh|(D')\,,$$

hence

$$\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon(\Omega)} \fint_{D'} \left| h(x) - \fint_{D'} h \right| dx \leq C\,|Dh|(\Omega)\,.$$

Thus,

$$H^D_\varepsilon(h,\Omega) \leq C\,|Dh|(\Omega)\,. \tag{2.2}$$

Let Ω be an open set in $\mathbb{R}^n$ and let $h \in \mathrm{BV}(\Omega)$. Then, see [3, Th. 3.78], at $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ there exist two real numbers $h^+(x)$ and $h^-(x)$, with $h^+(x) \geq h^-(x)$, and a unit vector $\nu_h(x)$ such that

$$\lim_{\rho\to 0} \fint_{B^+_\rho(x,\nu_h(x))} |h(y) - h^+(x)|\, dy = 0\,, \quad \lim_{\rho\to 0} \fint_{B^-_\rho(x,\nu_h(x))} |h(y) - h^-(x)|\, dy = 0\,, \tag{2.3}$$

where $B^\pm_\rho(x,\nu_h(x)) = \{y \in B_\rho(x)\ :\ \langle y - x, \nu_h(x)\rangle \gtrless 0\}$. We say that h is approximately continuous at a point $x \in \Omega$ if $h^+(x) = h^-(x) \in \mathbb{R}$ and we denote this common value by $\tilde{h}(x)$, the approximate limit of h at x. The set of points in Ω where h is approximately continuous is a Borel set that will be denoted by C_h. The set $S_h = \Omega \setminus C_h$ is called the approximate discontinuity set of h. Moreover, the set of points where (2.3) holds true and $h^+(x) > h^-(x)$ is denoted by J_h and it is called the jump set of h. We recall that the approximate discontinuity set coincides essentially with the jump set, that is

$$\mathcal{H}^{n-1}(S_h \setminus J_h) = 0\,.$$

For a function $h \in \mathrm{BV}(\Omega)$, we denote by $D^a h = \nabla h \mathcal{L}^n$ the absolutely continuous part of Dh with respect to the Lebesgue measure and by $D^s h$ the singular part. In turn $D^s h$ can be split in the jump part $D^j h = D^s h \lfloor J_h$ and in the Cantor part $D^c h = D^s h - D^j h$. By Proposition 3.92 of [3] we have

$$D^a h = \nabla h \mathcal{L}^n \lfloor \Omega\,, \quad D^c h = D^s h \lfloor (\Omega \setminus S_h)\,, \quad D^j h = (h^+ - h^-)\nu_h \mathcal{H}^{n-1} \lfloor J_h\,.$$

Let E be a measurable subset of $\mathbb{R}^n$. For any open set $\Omega \subset \mathbb{R}^n$, the perimeter of E in Ω, denoted by $P(E;\Omega)$, is the total variation of χ_E in Ω. We say that E is a set of finite perimeter in Ω if $P(E;\Omega) < \infty$.

We say that E has locally finite perimeter in Ω if $P(E;\Omega') < \infty$ for every bounded open set $\Omega' \subset\subset \Omega$. If E has locally finite perimeter in Ω, then the reduced boundary $\mathcal{F}E$ is the collection of all points $x \in \operatorname{supp}|D\chi_E| \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \mathcal{F}E \mapsto \mathbb{S}^{n-1}$ is called the generalized inner normal to E. From De Giorgi's structure theorem on sets of finite perimeter, see [3, Th. 3.59], we have that if E has finite perimeter in Ω then $P(E;\Omega) = \mathcal{H}^{n-1}(\mathcal{F}E \cap \Omega)$.

For every $t \in [0,1]$ and every $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, we denote by $E^{(t)}$ the set of points where E has density t, that is

$$E^{(t)} = \left\{ x \in \mathbb{R}^n \,:\, \lim_{\rho \downarrow 0} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}.$$

The essential boundary of E, denoted by $\partial^* E$, is then defined by setting $\partial^* E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$. We recall that for a set of locally finite perimeter the following inclusions hold $\mathcal{F}E \subset E^{(1/2)} \subset \partial^* E$ and that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, see [3, Th. 3.61].

We also recall that a sequence of functions $u_j \in \mathrm{BV}(\Omega)$ is said to converge strictly to $u \in \mathrm{BV}(\Omega)$ if the following conditions

$$u_j \to u \quad \text{in } L^1(\Omega), \qquad |Du_j|(\Omega) \to |Du|(\Omega)$$

are satisfied as $j \to \infty$. The space $\mathrm{SBV}(\Omega)$ of special functions of bounded variation is the set of all functions $h \in \mathrm{BV}(\Omega)$ for which the Cantor part $D^c h$ of the derivative vanishes. Thus

$$Dh = \nabla h \mathcal{L}^n \lfloor \Omega + D^j h\,.$$

We will use a recent result by De Philippis, Fusco and Pratelli (see [5, Th. A]) concerning the approximation of SBV functions. The precise statement goes as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathrm{SBV}(\Omega)$. Then, there exists a sequence of functions $u_j \in \mathrm{SBV}(\Omega)$ and of compact, C^1 manifolds $M_j \subset\subset \Omega$ with (possibly empty) C^1 boundary such that $J_{u_j} \subset M_j \cap J_u$, $\mathcal{H}^{n-1}(\overline{J_{u_j}} \setminus J_{u_j}) = 0$, and*

$$\|u_j - u\|_{\mathrm{BV}(\Omega)} \to 0\,, \qquad u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}})\,.$$

3. The functionals $H_\pm$

From now on we shall assume that D is a bounded and connected open set with Lipschitz boundary. Later on we shall explicitly mention assumption (H) when needed.

Given a function $f \in L^1(\Omega)$ we define the following quantities

$$H_+^D(f,\Omega) = \limsup_{\varepsilon \to 0} H_\varepsilon^D(f,\Omega)\,,$$

$$H_-^D(f,\Omega) = \liminf_{\varepsilon \to 0} H_\varepsilon^D(f,\Omega)\,.$$

Clearly, we have $H_-^D(f,\Omega) \le H_+^D(f,\Omega)$. The following scaling properties hold true:

$$H_\varepsilon^{\lambda D}(f,\Omega) = \lambda^{1-n} H_{\varepsilon\lambda}^D(f,\Omega) \qquad H_\pm^{\lambda D}(f,\Omega) = \lambda^{1-n} H_\pm^D(f,\Omega)\,. \tag{3.1}$$

Throughout the whole paper we shall assume without loss of generality that $\mathrm{diam}(D) = 1$. Indeed, setting $\tilde D := D/\mathrm{diam}(D)$, (3.1) gives that

$$H_\varepsilon^D(f,\Omega) = (\mathrm{diam}(D))^{1-n}\, H^{\tilde D}_{\varepsilon\,\mathrm{diam}\,D}(f,\Omega)\,, \qquad H_\pm^D(f,\Omega) = (\mathrm{diam}(D))^{1-n}\, H_\pm^{\tilde D}(f,\Omega)\,.$$

In the following, since the set D is fixed, we drop the superscript D and we only write H_ε and $H_\pm$.

3.1. **Properties of the functionals H_ε and $H_\pm$.** We list some properties of $H_\varepsilon(f,\Omega)$ and $H_\pm(f,\Omega)$, omitting the elementary proofs. To this end we denote by $\mathcal{A}_\Omega$ the family of all open subsets of Ω.

- *Translation invariance*: for any $\tau \in \mathbb{R}^n$, we have
$$H_\varepsilon(f(\cdot-\tau),\Omega+\tau) = H_\varepsilon(f,\Omega) \quad \text{and} \quad H_\pm(f(\cdot-\tau),\Omega+\tau) = H_\pm(f,\Omega)\,;$$
- *Monotonicity*: $H_\varepsilon(f,\cdot)$ and $H_\pm(f,\cdot)$ are increasing with respect to set inclusion;
- *Superadditivity of H_ε*: if $A_1,\ A_2 \in \mathcal{A}_\Omega$ and $A_1 \cap A_2 = \emptyset$, we have
$$H_\varepsilon(f,A_1\cup A_2) \ge H_\varepsilon(f,A_1) + H_\varepsilon(f,A_2)\,;$$
- *Homogeneity*: for any $t > 0$, we have
$$H_{t\varepsilon}(f(\cdot/t),t\Omega) = t^{n-1} H_\varepsilon(f(\cdot),\Omega)\,;$$
- *Superadditivity of H_-*: if $A_1,\ A_2 \in \mathcal{A}_\Omega$ and $A_1 \cap A_2 = \emptyset$, we have
$$H_-(f,A_1\cup A_2) \ge H_-(f,A_1) + H_-(f,A_2)\,.$$

The subadditivity of H_+ is less immediate and will be proved in Proposition 3.1 below. Given $A_1, A_2 \in \mathcal{A}_\Omega$ and a function $f \in L^1(\Omega)$, from the definition of H_+ it is plain to see that for any $\delta > 0$

$$H_+(f,A_1\cup A_2) \le H_+(f,W_1) + H_+(f,W_2)\,. \tag{3.2}$$

where $W_i = I_\delta(A_i) \cap (A_1 \cup A_2)$, for $i = 1,2$.

Proposition 3.1. *Let Ω be an open set and $f \in \mathrm{BV}(\Omega)$. Then for all $A \in \mathcal{A}_\Omega$*

$$H_+(f,A) = \sup\{H_+(f,A') \,:\, A' \subset\subset A,\ A' \in \mathcal{A}_\Omega\}\,. \tag{3.3}$$

Moreover $H_+(f,\cdot)$ is σ-subadditive on $\mathcal{A}_\Omega$.

Proof. Denote by $H_+^*(f,A)$ the right hand side of (3.3) and observe that $H_+^*(f,A) \le H_+(f,A)$. To prove the opposite inequality, fix $A \in \mathcal{A}_\Omega$ and denote by V and W two open sets such that $V \subset\subset W \subset\subset A$. Take a positive number $\delta < \min\{\mathrm{dist}(V,\partial W), \mathrm{dist}(W,\partial A)\}$. Since $A = W \cup (A \setminus \overline{I_\delta(V)})$, from (3.2) and (2.2) we have

$$H_+(f,A) \le H_+(f,I_\delta(W)) + H_+(f,I_\delta(A\setminus\overline{I_\delta(V)})\cap A) \le H_+^*(f,A) + C|Df|(A\setminus\overline{V})\,.$$

Then (3.3) follows letting $V \uparrow A$.

Let A_1, A_2 be two open sets in $\mathcal{A}_\Omega$ such that $H_+(f,A_1\cup A_2) > 0$ and let $0 < t < H_+(f,A_1\cup A_2)$. Thanks to (3.3) there exists $A \subset\subset A_1 \cup A_2$ such that $H_+(f,A) \ge t$. Since $\overline{A} \subset \Omega$ is compact there exist two open sets A_i', $i = 1,2$ such that $A \subset\subset A_1' \cup A_2'$ and $A_i' \subset\subset A_i$, $i = 1,2$. Thus from (3.2) we have

$$t \le H_+(f,A) \le H_+(f,A_1'\cup A_2') \le H_+(f,A_1) + H_+(f,A_2)\,.$$

Letting $t \to H_+(f,A_1\cup A_2)$, we have that H_+ is subadditive. Since H_+ is subadditive, using again (3.3), one gets at once that H_+ is σ-subadditive too. □

In the particular case that f is the linear function $f_\nu(x) := x\cdot\nu$ with $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$, some of the elementary properties of the functionals H_ε and $H_\pm$ read as

- *Translation invariance*: for any $\tau \in \mathbb{R}^n$, we have
$$H_\varepsilon(f_\nu,\Omega+\tau) = H_\varepsilon(f_\nu,\Omega) \quad \text{and} \quad H_\pm(f_\nu,\Omega+\tau) = H_\pm(f_\nu,\Omega)\,;$$

- *Homogeneity*: for any $t > 0$

$$H_{t\varepsilon}(f_\nu, t\Omega) = t^n H_\varepsilon(f_\nu, \Omega) \qquad \text{and} \qquad H_\pm(f_\nu, t\Omega) = t^n H_\pm(f_\nu, \Omega)\,. \tag{3.4}$$

3.2. **Definition of ψ.** Before we give the definition of $\psi : \mathbb{S}^{n-1} \to (0, +\infty)$ we observe that the functionals H_- and H_+ coincide if they act on a linear function $f_\nu(x) = x \cdot \nu$ and on any unitary cube centered in the origin.

Proposition 3.2. *Let $\nu \in \mathbb{S}^{n-1}$ and $f_\nu(x) = x \cdot \nu$. For any unitary cube $\tilde{Q}$ centered in the origin, we have*

$$H_+(f_\nu, \tilde{Q}) = H_-(f_\nu, \tilde{Q}) = \sup_{0<s\le 1} H_s(f_\nu, \tilde{Q})\,.$$

Moreover $H_+(f_\nu, \tilde{Q})$ is bounded from above.

Proof. By (3.4), we have

$$H_-(f_\nu, \tilde{Q}) = \liminf_{\varepsilon\to 0} H_\varepsilon(f_\nu, \tilde{Q}) = \liminf_{\varepsilon\to 0} \varepsilon^n H_1(f_\nu, (1/\varepsilon)\tilde{Q})\,.$$

Fixed $\varepsilon < s \le 1$, the cube $(1/\varepsilon)\tilde{Q}$ contains the union of at least $\lfloor (s/\varepsilon)\rfloor^n$ open disjoint cubes of side $1/s$ and $H_1(f_\nu, (1/s)\tilde{Q}) = H_1(f_\nu, z + (1/s)\tilde{Q})$ for any $z \in \mathbb{R}^n$. By the monotonicity in the second argument, the superadditivity of H_1 and the homogeneity, it holds

$$H_-(f_\nu, \tilde{Q}) \ge \liminf_{\varepsilon\to 0} \varepsilon^n \lfloor (s/\varepsilon)\rfloor^n H_1(f_\nu, (1/s)\tilde{Q}) = s^n H_1(f_\nu, (1/s)\tilde{Q}) = H_s(f_\nu, \tilde{Q})\,.$$

Hence,

$$H_-(f_\nu, \tilde{Q}) \ge \sup_{0<s\le 1} H_s(f_\nu, \tilde{Q})\,.$$

Moreover,

$$H_-(f_\nu, \tilde{Q}) \le H_+(f_\nu, \tilde{Q}) = \limsup_{\varepsilon\to 0} H_\varepsilon(f_\nu, \tilde{Q}) = \lim_{\varepsilon\to 0} \sup_{0<s<\varepsilon} H_s(f_\nu, \tilde{Q}) \le \sup_{0<s\le 1} H_s(f_\nu, \tilde{Q})\,.$$

So we have $H(f_\nu, \tilde{Q}) := H_+(f_\nu, \tilde{Q}) = H_-(f_\nu, \tilde{Q}) = \sup\limits_{0<s\le 1} H_s(f_\nu, \tilde{Q})$. The upper bound on H_+ is an immediate consequence of (2.2). □

We set now

$$\psi(\nu) := H(f_\nu, Q)\,, \tag{3.5}$$

where $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the canonical unit cube with edges parallel to the coordinate axes. Our next result shows that the values of the function ψ do not depend on the choice of the cube Q at the right hand side of (3.5).

Proposition 3.3. *Let $\nu \in \mathbb{S}^{n-1}$ and $f_\nu(x) = x \cdot \nu$. For any unitary cube $\tilde{Q}$ centered in the origin, we have*

$$\psi(\nu) = H(f_\nu, \tilde{Q})\,.$$

Proof. We can cover the cube Q with m disjoint open cubes $x_i + r\tilde{Q}$ plus an open set A_r with $|A_r| \to 0$ as $r \to 0$. By the subadditivity of $H_+(f_\nu, \cdot)$, the translation invariance and the homogeneity and by (2.2)

$$H_+(f_\nu, Q) \le \sum_{i=1}^m H_+(f_\nu, x_i + r\tilde{Q}) + H_+(f_\nu, A_r) \le H_+(f_\nu, \tilde{Q}) m r^n + C|A_r|\,.$$

Since $mr^n \le 1$, $H_+(f_\nu, Q) \le H_+(f_\nu, \tilde{Q})$. Interchanging the role of Q and $\tilde{Q}$, by Proposition 3.2 we have $H(f_\nu, Q) = H(f_\nu, \tilde{Q})$. □

We claim that ψ is Lipschitz continuous.

Proposition 3.4. *The function ψ is Lipschitz continuous and bounded away from zero.*

Proof. Let us first prove that ψ is Lipschitz continuous on $\mathbb{S}^{n-1}$. Fix $\nu, \tau \in \mathbb{S}^{n-1}$ and $\delta > 0$. There exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$

$$\psi(\nu) - \psi(\tau) \leq H_\varepsilon(f_\nu, Q) - H_\varepsilon(f_\tau, Q) + \delta .$$

There exists a family $\mathcal{H}_\varepsilon$ of translated copies D' of εD in Q such that

$$\psi(\nu) - \psi(\tau) \leq \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \left[\fint_{D'} \left| f_\nu - \fint_{D'} f_\nu \right| dx - \fint_{D'} \left| f_\tau - \fint_{D'} f_\tau \right| dx \right] + 2\delta \leq C|\nu - \tau| + 2\delta ,$$

where the last estimate follows by the triangular and Poincaré inequalities. The Lipschitz continuity of ψ then follows by letting $\delta \to 0$ and then interchanging the role of ν and τ.

To conclude the proof observe that $\frac{1}{2}D \subset Q$ since $\operatorname{diam}(D) = 1$. Therefore, from Proposition 3.2 we have

$$\min_{\nu \in \mathbb{S}^{n-1}} \psi(\nu) = \psi(\nu_0) \geq H_{\frac{1}{2}}(f_{\nu_0}, Q) \geq 2^{1-n} \fint_{\frac{1}{2}D} \left| f_{\nu_0} - \fint_{\frac{1}{2}D} f_{\nu_0} \right| dx .$$

Observe that the integral above cannot be zero, since otherwise $x - \fint_{\frac{1}{2}D} y\, dy \in \{x \in \mathbb{R}^n : x \cdot \nu_0 = 0\}$ for $\mathcal{L}^n$-a.e. $x \in \frac{1}{2}D$, which is impossible. Hence $\psi(\nu_0) > 0$. □

Let us consider the 1-homogeneous extension $\tilde{\psi}$ of ψ to $\mathbb{R}^n$, which is defined by setting $\tilde{\psi}(0) = 0$ and

$$\tilde{\psi}(\tau) = |\tau| \psi\left(\frac{\tau}{|\tau|}\right), \qquad \text{for all } \tau \in \mathbb{R}^n \setminus \{0\} .$$

Remark 3.5. Observe that $\tilde{\psi}$ is Lipschitz too. In fact, given $\xi, \zeta \in \mathbb{R}^n \setminus \{0\}$ we have

$$\begin{aligned} |\tilde{\psi}(\xi) - \tilde{\psi}(\zeta)| &= \left| |\xi| \psi\Big(\frac{\xi}{|\xi|}\Big) - |\zeta| \psi\Big(\frac{\zeta}{|\zeta|}\Big) \right| \leq \psi\Big(\frac{\xi}{|\xi|}\Big) \big||\xi| - |\zeta|\big| + |\zeta| \left| \psi\Big(\frac{\xi}{|\xi|}\Big) - \psi\Big(\frac{\zeta}{|\zeta|}\Big) \right| \\ &\leq C|\xi - \zeta| + C|\zeta| \left| \frac{\xi}{|\xi|} - \frac{\zeta}{|\zeta|} \right| \leq C|\xi - \zeta| . \end{aligned}$$

In the following, with a slight abuse of notation we shall still denote by ψ the 1-homogeneous extension $\tilde{\psi}$.

Proposition 3.6. *The function ψ is convex on $\mathbb{R}^n$.*

Proof. Since ψ is continuous and 1-homogeneous it is enough to show that for any $\nu_1, \nu_2 \in \mathbb{R}^n$ then $\psi(\nu_1 + \nu_2) \leq \psi(\nu_1) + \psi(\nu_2)$. To this end, given $\varepsilon > 0$ and a family $\mathcal{H}_\varepsilon$ of pairwise disjoint translated copies D' of εD in Q we have for any $D' \in \mathcal{H}_\varepsilon$

$$\begin{aligned} \fint_{D'} \left| f_{\nu_1+\nu_2} - \fint_{D'} f_{\nu_1+\nu_2} \right| dx &= \fint_{D'} \left| \Big(x - \fint_{D'} y\, dy\Big) \cdot (\nu_1 + \nu_2) \right| dx \\ &\leq \fint_{D'} \left| \Big(x - \fint_{D'} y\, dy\Big) \cdot \nu_1 \right| dx + \fint_{D'} \left| \Big(x - \fint_{D'} y\, dy\Big) \cdot \nu_2 \right| dx \\ &= \fint_{D'} \left| f_{\nu_1} - \fint_{D'} f_{\nu_1} \right| dx + \fint_{D'} \left| f_{\nu_2} - \fint_{D'} f_{\nu_2} \right| dx . \end{aligned}$$

Multiplying the above inequalities by ε^{n-1}, summing up over all $D' \in \mathcal{H}_\varepsilon$, passing to the supremum over all families $\mathcal{H}_\varepsilon$ and letting $\varepsilon \to 0$ we immediately get

$$H(f_{\nu_1+\nu_2}, Q) \leq H(f_{\nu_1}, Q) + H(f_{\nu_2}, Q) ,$$

that is, $\psi(\nu_1 + \nu_2) \leq \psi(\nu_1) + \psi(\nu_2)$. □

3.3. **Definition of φ.** In this section D will be a bounded open set satisfying assumption (H).

In order to define the function φ for any $\nu \in \mathbb{S}^{n-1}$ we consider the the characteristic function of the half space $S_\nu := \{x \in \mathbb{R}^n : x \cdot \nu > 0\}$. To this aim we set

$$H_+(S_\nu, Q_\nu) := H_+(\chi_{S_\nu}, Q_\nu)$$

and

$$H_-(S_\nu, Q_\nu) := H_+(\chi_{S_\nu}, Q_\nu)\,.$$

For a given $Q_\nu \in \mathbf{Q}_\nu$ in [2, Prop. 3.1] it is proved that

$$H_+(S_\nu, Q_\nu) = H_-(S_\nu, Q_\nu) = \sup_{0<s\le 1} H_s(S_\nu, Q_\nu) := H(S_\nu, Q_\nu)$$

and that $H(S_\nu, Q_\nu)$ is bounded from above and away from zero independently of ν.

We define now the function φ setting

$$\varphi(\nu) := \sup_{Q_\nu \in \mathbf{Q}_\nu} H(S_\nu, Q_\nu). \tag{3.6}$$

Note that as an immediate consequence of Theorem 1.1 we have that for every $Q_\nu \in \mathbf{Q}_\nu$

$$H(\chi_{S_\nu}, Q_\nu) = \int_{Q_\nu} \varphi(\nu)|\chi^+_{S_\nu} - \chi^-_{S_\nu}|\, d\mathcal{H}^{n-1} = \varphi(\nu)\,. \tag{3.7}$$

The equality above shows that in the definition (3.6) we may delete the supremum since $H(S_\nu, Q_\nu)$ does not depend on the choice of Q_ν in $\mathbf{Q}_\nu$. Although this property could be directly proved without the representation formula (1.6), since it will not be used in the following, we will omit its proof.

As before, we denote still by φ the 1-homogeneous extension of φ. In order to prove that φ is Lipschitz continuous we need a preliminary lemma which will be used also later. To this end, given a Lipschitz map $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$, we denote by $\mathrm{Lip}(\Phi)$ the Lipschitz constant of Φ.

Lemma 3.7. *Let D a bounded open set satisfying (H) for some Lipschitz function $\varrho : \mathbb{S}^{n-1} \to (0, +\infty)$. Then there exist $\gamma > 0$, $\sigma_0 \in (0, 1/\gamma)$, with the property that if $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a bi-Lipschitz map such that* $\mathrm{Lip}(I-\Phi)$, $\mathrm{Lip}(I - \Phi^{-1}) \le \sigma$, *with* $0 < \sigma < \sigma_0$, $\Phi(0) = 0$*, then*

$$(1 - \gamma\sigma)D \subset \Phi(D) \subset (1 + \gamma\sigma)D\,. \tag{3.8}$$

Proof. We show only the first inclusion, since the proof of the second one is completely similar. Set

$$r_0 = \min \varrho, \quad R_0 = \max \varrho\,.$$

Take $0 < \sigma < \sigma_0 < 1/\gamma$ with σ_0 and γ to be chosen. Proving the first inclusion in (3.8) is equivalent to showing that if $x = t\varrho(z)z$, with $0 \le t < 1 - \gamma\sigma$ for some $z \in \mathbb{S}^{n-1}$ then $y := \Phi^{-1}(x) = s\varrho(w)w$ for some $0 \le s < 1$ and $w \in \mathbb{S}^{n-1}$. Observe that there exists $\sigma_0 > 0$ such that $I_{R_0\sigma_0}(\frac{1}{2}D) \subset D$. Thus if $t < \frac{1}{2}$ then the ball centered in x with radius $\sigma_0 R_0$ is contained in D, hence $y \in D$ since $|x - y| = |(I - \Phi^{-1})(x)| \le \sigma|x| < \sigma_0 R_0$.

Therefore, in order to prove that $y \in D$ it is enough to assume $t \ge \frac{1}{2}$. Observe that

$$\begin{aligned}|z - w| &= \frac{|t\varrho(z)z - t\varrho(z)w|}{t\varrho(z)} \le \frac{|t\varrho(z)z - s\varrho(w)w|}{t\varrho(z)} + \frac{|(s\varrho(w) - t\varrho(z))w|}{t\varrho(z)} \\ &= \frac{|x - y|}{t\varrho(z)} + \frac{||x| - |y||}{t\varrho(z)} \le \frac{4R_0\sigma}{r_0}\,.\end{aligned}$$

Therefore, $|\varrho(z) - \varrho(w)| \le C_1\sigma$ for some positive constant C_1 depending only on r_0, R_0, $\mathrm{Lip}(\varrho)$, and we have

$$s \le \left|s - t\frac{\varrho(z)}{\varrho(w)}\right| + \left|t\frac{\varrho(z)}{\varrho(w)} - t\right| + t \le \frac{||x| - |y||}{r_0} + t\frac{C_1\sigma}{r_0} + t \le \frac{R_0\sigma}{r_0} + t\frac{C_1\sigma}{r_0} + t < 1\,,$$

provided

$$t < \frac{r_0 - R_0\sigma}{r_0 + C_1\sigma}\,.$$

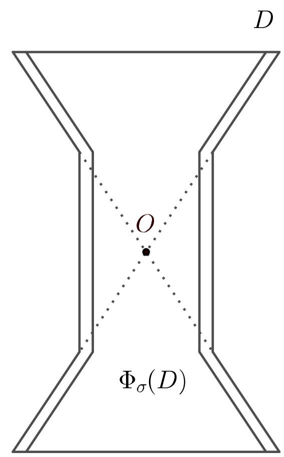

FIGURE 1. Hourglass-shaped domain

Thus the first inclusion in (3.8) follows if we choose $\gamma > (C_1 + R_0)/r_0$ and σ_0 such that $I_{R_0\sigma_0}(\frac{1}{2}D) \subset D$ and $\sigma_0 < 1/\gamma$. □

Remark 3.8. Let us consider a set D as in Figure 1. Observe that D has Lipschitz boundary, it is star-shaped with respect to the origin but the assumption (H) is not satisfied since

$$D = \{x = tz : \ z \in \mathbb{S}^1,\ 0 \le t < \varrho(z)\}$$

with a discontinuous $\varrho \in BV(\mathbb{S}^1)$. Observe that the sets $\Phi_\sigma(D)$ in Figure 1, with σ small, are images of D under a map $\Phi_\sigma : \mathbb{R}^2 \mapsto \mathbb{R}^2$ satisfying the assumptions of Lemma 3.7. Nevertheless, when σ goes to 0 it is not possible to find a set of the type $x_\sigma + \lambda_\sigma D$ contained in $\Phi_\sigma(D)$ and such that $\lambda_\sigma \to 1$.

We conclude this section proving the Lipschitz continuity of φ.

Proposition 3.9. *The function φ is Lipschitz continuous.*

Proof. Observe first that given two unit vectors ν, τ and a cube $Q_\nu \in \mathbf{Q}_\tau$ there exists a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$R(Q_\nu) \in \mathbf{Q}_\tau, \quad R(S_\nu) = S_\tau, \quad \mathrm{Lip}(I - R),\ \mathrm{Lip}(I - R^{-1}) \le |\nu - \tau|\,.$$

To see this choose an orthonormal frame $\{\varepsilon_1, \dots, \varepsilon_n\}$ in $\mathbb{R}^n$ such that

$$\nu = \varepsilon_1, \qquad \tau = \tau_1\varepsilon_1 + \tau_2\varepsilon_2$$

and define R by setting $R(\varepsilon_1) = \tau$, $R(\varepsilon_2) = -\tau_2\varepsilon_1 + \tau_1\varepsilon_2$ and $R(\varepsilon_i) = \varepsilon_i$ for all $i = 3, \dots, n$.

Fix now $\nu, \tau \in \mathbb{S}^{n-1}$ such that $|\nu - \tau| < \sigma_0$, where σ_0 is the constant provided by Lemma 3.7 and a cube $Q_\nu \in \mathbf{Q}_\nu$. Given $\varepsilon > 0$, take a family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets $D' \subset Q_\nu$ of the form $x + \varepsilon D$. We may assume without loss of generality that each $D' \in \mathcal{H}_\varepsilon$ intersects ∂S_ν, since otherwise the corresponding term in the sum

$$\sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| \chi_{S_\nu}(x) - \fint_{D'} \chi_{S_\nu} \right| dx$$

will be zero. Therefore, since the diameter of each D' is equal to ε and each D' intersects ∂S_ν, we have that each D' is contained in a slab of measure 2ε and we get that $\#(\mathcal{H}_\varepsilon)|D'| \le 2\varepsilon$.

Thanks to Lemma 3.7 one can find two families of sets $D'' \subset R(D') \subset D'''$ such that D'' is a translation of $(1-\gamma|\nu-\tau|)D'$ and D''' is a translation of $(1+\gamma|\nu-\tau|)D'$. Thus, for every D' as above we have, since the rotation R is measure preserving,

$$\fint_{D'}\left|\chi_{S_\nu}-\fint_{D'}\chi_{S_\nu}\right|dx=\frac{2|D'\cap S_\nu||D'\setminus S_\nu|}{|D'|^2}=\frac{2|R(D')\cap S_\tau||R(D')\setminus S_\tau|}{|D'|^2}\,. \tag{3.9}$$

Observing that

$$|R(D')\cap S_\tau|\le|D''\cap S_\tau|+|(D'''\setminus D'')\cap S_\tau|\le|D''\cap S_\tau|+C|\nu-\tau||D'|$$

and that similarly $|R(D')\setminus S_\tau|\le|D''\setminus S_\tau|+C|\nu-\tau||D'|$, it is easily checked that

$$|R(D')\cap S_\tau||R(D')\setminus S_\tau|\le|D''\cap S_\tau||D''\setminus S_\tau|+C|D'|^2|\nu-\tau|\,.$$

Therefore, recalling that $D''\subset R(D')$, we get that

$$\begin{aligned}
&\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|\chi_{S_\nu}-\fint_{D'}\chi_{S_\nu}\right|dx\le\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\frac{2|D''\cap S_\tau||D''\setminus S_\tau|}{|D'|^2}+C\varepsilon^{n-1}|\nu-\tau|\#(\mathcal{H}_\varepsilon)\\
&=\varepsilon^{n-1}(1-\gamma|\nu-\tau|)^{2n}\sum_{D'\in\mathcal{H}_\varepsilon}\frac{2|D''\cap S_\tau||D''\setminus S_\tau|}{|D''|^2}+C\varepsilon^{n-1}|\nu-\tau|\#(\mathcal{H}_\varepsilon)\\
&\le H_{\varepsilon(1-\gamma|\nu-\tau|)}(\chi_{S_\tau},R(Q_\nu))+C|\nu-\tau|\,.
\end{aligned}\tag{3.10}$$

From the previous inequality, passing to the supremum over all families $\mathcal{H}_\varepsilon$ and letting $\varepsilon\to0$ we conclude that

$$H(S_\nu,Q_\nu)\le H(S_\tau,R(Q_\nu))+C|\nu-\tau|\le\varphi(\tau)+C|\nu-\tau|\,,$$

where the last inequality follows from the fact that $R(Q_\nu)\in\mathbf{Q}_\tau$. Taking the supremum with respect to all possible cubes Q_ν in $\mathbf{Q}_\nu$, we have

$$\varphi(\nu)-\varphi(\tau)\le C|\nu-\tau|\,.$$

Interchanging ν with τ we have thus proved that $|\varphi(\nu)-\varphi(\tau)|\le C|\nu-\tau|$ whenever $|\nu-\tau|\le\sigma_0$. Therefore φ is Lipschitz continuous on $\mathbb{S}^{n-1}$, hence on $\mathbb{R}^n$ by Remark 3.5. □

4. $W^{1,1}$ functions

In this section we prove Theorem 1.1 for a function $f\in W^{1,1}(\Omega)$ where Ω is an open set in $\mathbb{R}^n$. Beside being of interest in itself, this special case will be also needed in the next section to prove (1.6) in the general case.

Theorem 4.1. *Let D be a bounded connected open set with Lipschitz boundary. If $f\in W^{1,1}(\Omega)$, then*

$$H_+(f,\Omega)=H_-(f,\Omega)=\int_\Omega\psi(\nabla f)\,dx\,.$$

Proof. **Step 1.** *For $f\in W^{1,1}(\Omega)$, we have*

$$H_-(f,\Omega)\ge\int_\Omega\psi(\nabla f)\,dx\,. \tag{4.1}$$

To prove this inequality assume first that Ω is a bounded open set and that $f\in C^1(\overline{\Omega})$. Fix $t>0$ and $\sigma>0$ and observe that there exist $r>0$ and a finite family of pairwise disjoint open cubes $Q(x_i,r)$ with edges parallel to the coordinate axes, contained in $U_t:=\{x\in\Omega:\ |\nabla f(x)|>t\}$, $i=1,\dots,m$, such that

$$\left|U_t\setminus\bigcup_{i=1}^m Q(x_i,r)\right|<\sigma, \tag{4.2}$$

$$|\nabla f(x)-\nabla f(y)|<\sigma\qquad\text{for all }\ x,y\in Q(x_i,r)\,, \tag{4.3}$$

Fix i and $\varepsilon > 0$ and consider a family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets D_j of the form $z_j + \varepsilon D \subset Q(x_i, r)$, for $j = 1, \dots, k$. For every $x \in z_j + \varepsilon D$ we may write

$$f(x) = f(z_j) + \nabla f(z_j) \cdot (x - z_j) + R_j(x)\,,$$

where $R_j(x) = (\nabla f(\overline{x}) - \nabla f(z_j)) \cdot (x - z_j)$ for some $\overline{x} \in Q(x_i, r)$. Thus, using the estimate (4.3), we have that $|R_j(x)| \leq \sigma\varepsilon$. Thus, using again (4.3),

$$\begin{aligned}\varepsilon^{n-1} \sum_{j=1}^{k} \fint_{D_j} \left| f(x) - \fint_{D_j} f \right| dx &\geq \varepsilon^{n-1} \sum_{j=1}^{k} \fint_{D_j} \left| \nabla f(z_j) \cdot (x - z_j) - \fint_{D_j} \nabla f(z_j) \cdot (y - z_j)\, dy \right| dx - 2k\sigma\varepsilon^n \\ &\geq \varepsilon^{n-1} \sum_{j=1}^{k} \fint_{D_j} \left| \nabla f(x_i) \cdot (x - z_j) - \fint_{D_j} \nabla f(x_i) \cdot (y - z_j)\, dy \right| dx - Ck\sigma\varepsilon^n \\ &\geq \varepsilon^{n-1} |\nabla f(x_i)| \sum_{j=1}^{k} \fint_{D_j} \left| \frac{\nabla f(x_i)}{|\nabla f(x_i)|} \cdot x - \fint_{D_j} \frac{\nabla f(x_i)}{|\nabla f(x_i)|} \cdot y\, dy \right| dx - C\sigma r^n\,,\end{aligned}$$

where in the last inequality we used the fact that $k\varepsilon^n = |\cup_{j=1}^k D_j|/|D| \leq r^n/|D|$. Thus, from the inequality above, taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the lim inf with respect to ε, we get

$$H_-(f, Q(x_i, r)) \geq r^n \psi(\nabla f(x_i)) - C\sigma r^n \geq \int_{Q(x_i, r)} \psi(\nabla f(x))\, dx - C\sigma r^n\,, \tag{4.4}$$

where again the last inequality follows by a simple use of the estimates (4.3) and from the Lipschitz continuity of ψ, see Remark 3.5. Thus, summing up with respect to i, by the superadditivity of $H_-(f, \cdot)$, we get

$$H_-(f, \Omega) \geq \sum_{i=1}^{m} H_-(f, Q(x_i, r) \geq \sum_{i=1}^{m} \int_{Q(x_i, r)} \psi(\nabla f(x))\, dx - C\sigma|\Omega| \geq \int_{U_t} \psi(\nabla f(x))\, dx - C\sigma\,,$$

where the last inequality follows from (4.2) and the constant C depends on $|\Omega|, |D|$ and on $\|\nabla f\|_{L^\infty(\Omega)}$. Then (4.1) follows at once letting first $\sigma \to 0$ and then $t \to 0$ in the previous inequality.

Take now any open set Ω and $f \in W^{1,1}(\Omega)$. Given $\sigma > 0$, take $f_\sigma \in C^1(\Omega)$ such that $\|f - f_\sigma\|_{W^{1,1}(\Omega)} < \sigma$. Given an open set $A \subset\subset \Omega$, for any family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets $D' \subset A$ of the type $z + \varepsilon D$ we have

$$\begin{aligned}\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right| dx &\geq \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f_\sigma(x) - \fint_{D'} f_\sigma \right| dx \\ &\qquad - \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| (f - f_\sigma)(x) - \fint_{D'} (f - f_\sigma) \right| dx \\ &\geq \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f_\sigma(x) - \fint_{D'} f_\sigma \right| dx - C(D)\varepsilon^n \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} |\nabla f(x) - \nabla f_\sigma(x)|\, dx \\ &\geq \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f_\sigma(x) - \fint_{D'} f_\sigma \right| dx - \frac{C(D)}{|D|} \int_A |\nabla f(x) - \nabla f_\sigma(x)|\, dx\,,\end{aligned}$$

where $C(D)$ is the Poincaré constant of D. Thus, passing to the supremum with respect to the family $\mathcal{H}_\varepsilon$ and letting ε tend to 0, from (4.1) applied to f_σ we get

$$H_-(f; \Omega) \geq H_-(f_\sigma, A) - C\|f - f_\sigma\|_{W^{1,1}(A)} \geq \int_A \psi(\nabla f_\sigma(x))\, dx - C\sigma\,,$$

where the constant C depends only on D. Recalling Proposition 3.4, (4.1) then follows letting first $\sigma \to 0$ and then $A \uparrow \Omega$.

Step 2. *If $f \in W^{1,1}(\Omega)$, then*

$$H_+(f,\Omega) \le \int_\Omega \psi(\nabla f)\,dx\,. \tag{4.5}$$

Using (3.3) we may always assume that Ω is a bounded open set with Lipschitz boundary. Moreover, by an approximation argument similar to the one used in the final part of the previous step, we may assume without loss of generality that $f \in C^1(\overline{\Omega})$.

Recall that $|\partial U_t| = 0$ for all but countably many $t > 0$. Then fix t so that $|\partial U_t| = 0$ and $\sigma > 0$ and consider the same cubes $Q(x_i, r)$, $i = 1,\dots m$, as before. Using the subadditivity of $H_+(f,\cdot)$ we have

$$H_+(f,\Omega) \le \sum_{i=1}^m H_+(f, Q(x_i,r)) + H_+(f, \Omega \setminus \overline{U}_t) + H_+(f, W_t)\,,$$

where $W_t \subset \Omega$ is an open set such that $\overline{U}_t \setminus \cup_{i=1}^m Q(x_i,r) \subset W_t$ and $|W_t| < \sigma$. Note that this choice of W_t is possible thanks to (4.2) and to the fact that $|\partial U_t| = 0$. Then, arguing as in the proof of (4.4) we get that for every $i = 1,\dots,m$

$$H_+(f, Q(x_i,r)) \le \int_{Q(x_i,r)} \psi(\nabla f(x))\,dx + C\sigma r^n\,,$$

for some positive constant C depending only on $|D|$ and $\|\nabla f\|_{L^\infty(\Omega)}$. Thus, from the two previous inequalities, recalling (2.2), we have

$$H_+(f,\Omega) \le \int_\Omega \psi(\nabla f(x))\,dx + C\sigma|\Omega| + C\int_{\{|\nabla f|\le t\}} |\nabla f(x)|\,dx + C\sigma\|\nabla f\|_{L^\infty(\Omega)}\,.$$

Then (4.5) follows letting first $\sigma \to 0$ and then $t \to 0$. □

The following corollary is an almost immediate consequence of Theorem 4.1.

Corollary 4.2. *Let D be a bounded connected open set with Lipschitz boundary and let $f \in L^1(\Omega)$. If $H_-(f,\Omega)$ is finite then $f \in BV(\Omega)$. Conversely, if $f \in BV(\Omega)$ then $H_+(f,\Omega)$ is finite.*

Proof. We prove only that if $H_-(f,\Omega) < \infty$ then f is in $BV(\Omega)$, since the other implication follows at once from (2.2). Fix an open set $A \subset\subset \Omega$ and $0 < \sigma < \operatorname{dist}(A, \partial\Omega)$. For all $x \in A$ set $f_\sigma(x) = (\varrho_\sigma * f)(x)$, where ϱ is a standard mollifier with compact support in the unit ball B and $\varrho_\sigma(x) = \sigma^{-n}\varrho(x/\sigma)$. Then, given any family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets D' of the form $z + \varepsilon D' \subset A$, using the definition of f_σ and Fubini's theorem, we get, recalling that $\int_B \varrho\,dx = 1$,

$$\begin{aligned}
\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'} \Big| f_\sigma(x) - \fint_{D'} f_\sigma \Big|\,dx &= \varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'} \Big| \int_B \varrho(y) f(x-\sigma y)\,dy - \fint_{D'}\int_B \varrho(y) f(z-\sigma y)\,dydz \Big|\,dx \\
&\le \varepsilon^{n-1}\int_B \varrho(y)\Big(\sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'} \Big| f(x-\sigma y) - \fint_{D'} f(z-\sigma y)\,dz \Big|\,dx \Big)\,dy \\
&= \varepsilon^{n-1}\int_B \varrho(y)\Big(\sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'-\sigma y} \Big| f(x) - \fint_{D'-\sigma y} f \Big|\,dx \Big)\,dy \le H_\varepsilon(f,\Omega)\,.
\end{aligned}$$

Therefore, taking the supremum over all families $\mathcal{H}_\varepsilon$ and letting $\varepsilon \to 0$ we get for all $\sigma > 0$ sufficiently small

$$\int_A |\nabla f_\sigma|\,dx \le C\int_A \psi(\nabla f_\sigma)\,dx = CH_-(f_\sigma, A) \le CH_-(f,\Omega)\,.$$

Hence the conclusion follows by letting first $\sigma \to 0$ and then $A \uparrow \Omega$. □

5. SBV functions

In this section D will be a bounded open set satisfying assumption (H).

The following lemma is a simple consequence of Lemma 3.7. In order to state it, given $\nu \in \mathbb{S}^{n-1}$ we denote by π_ν the $(n-1)$-dimensional subspace orthogonal to ν and by $\pi_\nu^\perp$ the orthogonal subspace to π_ν. Given $x \in \mathbb{R}^n$, with a slight abuse of notation, we denote by $\pi_\nu(x)$ and $\pi_\nu^\perp(x)$ the projections of x on π_ν and $\pi_\nu^\perp$, respectively.

Lemma 5.1. *There exist $C_1, \sigma_1 > 0$ with the following property. If $\nu \in \mathbb{S}^{n-1}$, $\eta : \pi_\nu \to \pi_\nu^\perp$ is a Lipschitz continuous function such that $\eta(0) = 0$, $\mathrm{Lip}(\eta) \le \sigma < \sigma_1$, and $S^+ := \{x \in \mathbb{R}^n : \pi_\nu^\perp(x) > \eta(\pi_\nu(x))\}$, then for any Q_ν in $\mathbf{Q}_\nu$ and any $0 < r \le 1$*

$$H(S_\nu, rQ_\nu) - C_1\sigma r^{n-1} \le H_-(\chi_{S^+}, rQ_\nu) \le H_+(\chi_{S^+}, rQ_\nu) \le H(S_\nu, rQ_\nu) + C_1\sigma r^{n-1}\,. \tag{5.1}$$

Proof. Fix $\nu \in \mathbb{S}^{n-1}$, $Q_\nu \in \mathbf{Q}_\nu$ and $0 < r \le 1$. Assume that $\sigma_1 \le 1/4$ and observe that there exists a bi-Lipschitz map $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $\Phi(x) = x$ if $x \in Q_\nu$ and $|\pi_\nu^\perp(x)| \ge r/2$, $\mathrm{Lip}(I - \Phi)$, $\mathrm{Lip}(I - \Phi^{-1}) \le C_0\sigma$, for a constant C_0 depending only on the dimension, and such that $\Phi(S^+ \cap rQ_\nu) = S_\nu \cap rQ_\nu$, $\Phi(rQ_\nu) = rQ_\nu$.

The proof now is very similar to the one of Proposition 3.9. Given $0 < \varepsilon < r$, we take a family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets $D' \subset Q_\nu$ of the form $x + \varepsilon D$. As in the proof of Proposition 3.9 we may assume without loss of generality that each D' intersects S^+ and thus that it is contained in $I_\varepsilon(\Gamma_\eta) \cap rQ_\nu$, where Γ_η stands for the graph of η. Thus, since $|I_\varepsilon(\Gamma_\eta) \cap rQ_\nu| \le C\varepsilon r^{n-1}$, for some constant depending only on n, we have that $\#(\mathcal{H}_\varepsilon)|D'| \le C\varepsilon r^{n-1}$ with the same constant C.

Thanks to Lemma 3.7 if $0 < \sigma < \sigma_1 := \min\{1/4, \sigma_0/C_0\}$ we can find two families of sets $D'' \subset \Phi(D') \subset D'''$ such that D'' is a translation of $(1 - \gamma C_0\sigma)D'$ and D''' is a translate of $(1 + \gamma C_0\sigma)D'$. Thus, for every D' as above, arguing as in the proof of (3.9) we have

$$\begin{aligned}\fint_{D'}\Big|\chi_{S^+} - \fint_{D'}\chi_{S^+}\Big|\,dx &= \frac{2|D'\cap S^+||D'\setminus S^+|}{|D'|^2}\\ &\le (1 + C_0\sigma)^n\frac{2|\Phi(D')\cap S_\nu||\Phi(D')\setminus S_\nu|}{|D'|^2} \le (1 + C\sigma)\frac{2|\Phi(D')\cap S_\nu||\Phi(D')\setminus S_\nu|}{|D'|^2}\,.\end{aligned}$$

Since

$$|\Phi(D')\cap S_\nu| \le |D''\cap S_\nu| + C\sigma|D'|,\quad |\Phi(D')\setminus S_\nu| \le |D''\setminus S_\nu| + C\sigma|D'|\,,$$

arguing exactly as in the proof of (3.10), we have

$$\begin{aligned}\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\Big|\chi_{S^+} - \fint_{D'}\chi_{S^+}\Big|\,dx &\le \varepsilon^{n-1}(1 + C\sigma)(1-\gamma\sigma)^{2n}\sum_{D'\in\mathcal{H}_\varepsilon}\frac{2|D''\cap S_\nu||D''\setminus S_\nu|}{|D''|^2} + C\varepsilon^{n-1}\sigma\#(\mathcal{H}_\varepsilon)\\ &\le (1 + C\sigma)H_{\varepsilon(1-\gamma\sigma)}(\chi_{S_\nu}, rQ_\nu) + C\sigma r^{n-1}\,.\end{aligned}$$

Therefore, passing first to the supremum over all families $\mathcal{H}_\varepsilon$ and then taking the lim sup as $\varepsilon \to 0$ we conclude that

$$H_+(\chi_{S^+}, rQ_\nu) \le (1 + C\sigma)H(S_\nu, rQ_\nu) + C\sigma r^{n-1} \le H(S_\nu, rQ_\nu) + C_1\sigma r^{n-1}\,.$$

This proves the second inequality in (5.1). The first one is proved in a similar way. □

Theorem 5.2. *Let D an open set satisfying (H) for some Lipschitz function $\varrho : \mathbb{S}^{n-1} \to (0, +\infty)$ and let $f \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$. Given $\delta > 0$, we have*

$$H_-(f, I_\delta(J_f)\cap\Omega) \ge \int_{J_f}(f^+(x) - f^-(x))\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)\,. \tag{5.2}$$

Proof. We divide the proof into several steps.

Step 1: *Construction of a family of cubes covering the jump set up to a small error.*

We start by fixing $\eta > 0$. Since $\mathcal{H}^{n-1}\llcorner J_f$ is a σ–finite measure, we find a compact set $H \subset J_f$ such that $\mathcal{H}^{n-1}(H) < \infty$ and

$$|Df|(J_f\setminus H) < \eta\,, \tag{5.3}$$

$$\nu_f|_H, f^{\pm}|_H \qquad \text{are continuous.}$$

Moreover, H can be chosen with the property that

$$\lim_{r\to 0^+} \fint_{Q^{\pm}_{\nu_f(x)}(x,r)} |f(y) - f^{\pm}(x)|\,dy = 0 \quad \text{uniformly with respect to } x \in H \text{ and to any } Q^{\pm}_{\nu_f(x)}(x,r) \in \mathbf{Q}_{\nu_f(x)}\,, \tag{5.4}$$

where $Q^{\pm}_{\nu}(x,r) := \{y \in Q_\nu(x,r) \,:\, (y-x)\cdot\nu(x) \gtrless 0\}$.

Let us fix $0 < \sigma < \sigma_1 < 1$, where σ_1 is as in Lemma 5.1. Since J_f is $(n-1)$–countably rectifiable, we may find a compact set $K \subset H$ such that

$$|Df|(H \setminus K) < \eta \tag{5.5}$$

with the property that K is the union of finitely disjoint many compact sets $K_1, \dots, K_m$ where each K_i is contained in the graph Γ_i of a C^1 function γ_i with compact support defined on some $(n-1)$–dimensional hyperplane π_i with $\|\nabla\gamma_i\|_{L^\infty(\pi_i)} < \sigma$. We start a construction of a suitable family of cubes for the compact K_1. Up to a translation and a rotation of the orthonormal frame, we may assume without loss of generality that $\pi_1 = \{x_n = 0\}$. Then we may extend $\nu_f|_{K_1}, f^{\pm}|_{K_1}$ to the whole Γ_1 in such a way that the corresponding extensions $\tilde f^{\pm}$ and $\tilde\nu_f$ are continuous functions with compact support satisfying $\|\tilde f^{\pm}\|_\infty \le \|f\|_\infty$, $\|\tilde\nu_f\|_\infty \le 1$. Denoting by $\pi_1(K_1)$ the projection of K_1 on $\{x_n = 0\}$, we may find an open set $U \subset \mathbb{R}^{n-1}$ such that $\pi_1(K_1) \subset U$ and $\mathcal{H}^{n-1}(U \setminus \pi_1(K_1)) < \sigma\mathcal{H}^{n-1}(K_1)$.

Let us now fix a cube $\tilde Q \in \mathbf{Q}_{e_n}$, $\tilde Q = \tilde Q^{n-1} \times (-1/2, 1/2)$ such that

$$H(S_{e_n}, \tilde Q) \ge \varphi(e_n) - \sigma. \tag{5.6}$$

For $r > 0$ we consider the standard covering of $\mathbb{R}^{n-1}$ obtained by translating the cube $r\tilde Q^{n-1}$. Taking r sufficiently small, since $\mathcal{H}^{n-1}(U \setminus \pi_1(K_1)) < \sigma\mathcal{H}^{n-1}(K_1)$ we may find finitely many pairwise disjoint $(n-1)$–dimensional open cubes $Q_1^{n-1}, \dots, Q_k^{n-1}$ belonging to this covering with the property that each of them intersects $\pi_1(K_1)$ and

$$\mathcal{H}^{n-1}\Big(U \setminus \bigcup_{i=1}^{k} Q_i^{n-1}\Big) < 2\sigma\mathcal{H}^{n-1}(K_1)\,. \tag{5.7}$$

Choosing r smaller if needed, we may also assume that

$$|\tilde f^{\pm}(x) - \tilde f^{\pm}(y)| \le \sigma \qquad \text{if } x, y \in \Gamma_1 \text{ with } |x-y| < \sqrt{n}r/2 \tag{5.8}$$

and that there exists a constant C depending only on n and $\|f\|_\infty$ such that

$$\fint_{Q^{\pm}_{e_n}(x,\rho)} |f(y) - f^{\pm}(x)|\,dy < C\sigma \qquad \text{for all } 0 < \rho < 2r, \text{ for all } x \in K_1 \text{ and for any } Q^{\pm}_{e_n}(x,\rho) \in \mathbf{Q}_{e_n}. \tag{5.9}$$

Indeed, given a cube $Q^{\pm}_{e_n}(x,\rho) \in \mathbf{Q}_{e_n}$, since $|\nu_f(x) - e_n| \le C\sigma$ we may construct a rotation $R : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $\mathrm{Lip}(I - R), \mathrm{Lip}(I - R^{-1}) \le C\sigma$ and $R(Q^{\pm}_{e_n}(x,\rho)) \in \mathbf{Q}_{\nu_f(x)}$. Thus from (5.4) we have

$$\begin{aligned}
\fint_{Q^{\pm}_{e_n}(x,\rho)} |f(y) - f^{\pm}(x)|\,dy &\le \fint_{R(Q^{\pm}_{e_n}(x,\rho))} |f(y) - f^{\pm}(x)|\,dy + 2\|f\|_\infty \frac{\big|Q^{\pm}_{e_n}(x,\rho) \setminus R(Q^{\pm}_{e_n}(x,\rho))\big|}{\rho^n} \\
&\le \fint_{R(Q^{\pm}_{e_n}(x,\rho))} |f(y) - f^{\pm}(x)|\,dy + C\sigma\|f\|_\infty\,.
\end{aligned}$$

For all $i = 1, \dots, k$ we consider the cube $Q_i := y_i + r\tilde Q \subset U \times \mathbb{R}$ whose projection on the horizontal plane is Q_i^{n-1} and whose center y_i belongs to Γ_1. Taking r smaller if necessary, we may assume that all cubes Q_i are contained in $I_\delta(J_f) \cap \Omega$.

Step 2: *Estimate of $H_-(f, Q_i)$ for $i = 1, \dots, k$.*

We concentrate our attention on the first cube $Q_1 = Q_{e_n}(y_1, r)$ and we let $\mathcal{H}_\varepsilon$ be a family of pairwise disjoint translations D' of εD, with $0 < \varepsilon < r$, contained in Q_1. For the sake of simplicity we denote by $D'_1, \dots, D'_s$ the elements of $\mathcal{H}_\varepsilon$ such that $D'_h \cap K_1 \neq \emptyset$, by $D'_{s+1}, \dots, D'_t$ the elements of $\mathcal{H}_\varepsilon$ intersecting Γ_1 but not K_1 and by $D'_{t+1}, \dots, D'_T$ the remaining ones. Then for any $h = 1, \dots, s$ we choose a point $x_h \in D'_h \cap K_1$. Let us introduce the sets

$$S^\pm := \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon\ x_n \gtrless \gamma_1(x')\right\}$$

and, for $h = 1, \dots, s$, the functions

$$w_h(x) := \begin{cases} \tilde{f}^+(x_h) & \text{if } x \in S^+\,, \\ \tilde{f}^-(x_h) & \text{if } x \in S^-\,. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right| dx &\geq \sum_{h=1}^s \fint_{D'_h} \left| w_h(x) - \fint_{D'_h} w_h \right| dx - \sum_{h=1}^s \fint_{D'_h} \left| f(x) - w_h(x) - \fint_{D'_h} (f - w_h) \right| dx \\ &\geq \sum_{h=1}^s 2\left(\tilde{f}^+(x_h) - \tilde{f}^-(x_h)\right) \frac{|D'_h \cap S^+||D'_h \setminus S^+|}{|D'_h|^2} - 2\sum_{h=1}^s \fint_{D_h} |f - w_h|\, dx\,. \end{aligned} \tag{5.10}$$

Since $D'_h \subset \tilde{Q}(x_h, 2\varepsilon) := x_h + 2\varepsilon\tilde{Q}$, we have

$$\begin{aligned} \fint_{D'_h} |f - w_h|\, dx &\leq \frac{1}{\varepsilon^n |D|}\left[\int_{D'_h \cap S^+} \left|f - \tilde{f}^+(x_h)\right| dx + \int_{D'_h \cap S^-} \left|f - \tilde{f}^-(x_h)\right| dx\right] \\ &\leq \frac{1}{\varepsilon^n |D|}\left[\int_{\tilde{Q}(x_h, 2\varepsilon) \cap S^+} \left|f - \tilde{f}^+(x_h)\right| dx + \int_{\tilde{Q}(x_h, 2\varepsilon)) \cap S^-} \left|f - \tilde{f}^-(x_h)\right| dx\right]. \end{aligned} \tag{5.11}$$

Since $\|\nabla\gamma_1\|_\infty < \sigma$, it is easily checked that $\left|\tilde{Q}^-(x_h, 2\varepsilon)) \cap S^+\right| \leq C(n)\sigma\varepsilon^n$. A similar estimate holds for the measure of $\tilde{Q}^+_{e_n}(x_h, 2\varepsilon) \cap S^-$, hence we have

$$\frac{1}{\varepsilon^n} \int_{\tilde{Q}^\mp(x_h, 2\varepsilon) \cap S^\pm} \left|f(x) - \tilde{f}^\pm(x_h)\right| dx \leq C(n)\|f\|_\infty \sigma$$

and thus, from (5.9) and (5.11), we may estimate

$$\fint_{D'_h} \left| f(x) - w_h(x) - \fint_{D'_h} (f - w_h) \right| dx \leq C\sigma \tag{5.12}$$

for some constant C depending only on n, $\|f\|_\infty$ and $|D|$. Note that

$$\bigcup_{h=1}^s D'_h \subset Q_1 \cap \{x = (x', x_n) \in \tilde{Q}_1^{n-1} \times \mathbb{R} :\ |x_n - \gamma_1(x')| \leq C\varepsilon\}\,,$$

for some constant C depending only on n. Therefore we may conclude that $s|D|\varepsilon^n \leq Cr^{n-1}\varepsilon$, that is

$$s|D|\varepsilon^{n-1} \leq Cr^{n-1}\,. \tag{5.13}$$

Thus from (5.12) we have

$$\varepsilon^{n-1} \sum_{h=1}^s \fint_{D'_h} \left| f(x) - w_h(x) - \fint_{D'_h} (f - w_h) \right| dx \leq Cr^{n-1}\sigma$$

for some constant C depending once more only on n, $\|f\|_\infty$ and D. Inserting the latter inequality in (5.10) yields

$$\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right| dx \geq \varepsilon^{n-1} \sum_{h=1}^s 2\left(\tilde{f}^+(x_h) - \tilde{f}^-(x_h)\right) \frac{|D'_h \cap S^+||D'_h \setminus S^+|}{|D'_h|^2} - Cr^{n-1}\sigma\,.$$

So, from (5.8) we see that

$$
\begin{aligned}
\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f(x)-\fint_{D'}f\right|dx &\geq \varepsilon^{n-1}\left(\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)-2\sigma\right)\sum_{h=1}^{s}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}-Cr^{n-1}\sigma \\
&\geq \varepsilon^{n-1}\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\sum_{h=1}^{s}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}-Cr^{n-1}\sigma\,,
\end{aligned}
\tag{5.14}
$$

where the last inequality follows since by (5.13) we have

$$\varepsilon^{n-1}\sum_{h=1}^{s}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}\leq 2\varepsilon^{n-1}s\leq Cr^{n-1}\,.$$

Note that, by Poincaré inequality, we get

$$\varepsilon^{n-1}\sum_{h=s+1}^{t}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}\leq C(D)\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)\,.$$

Therefore, recalling (5.14), we may estimate

$$
\begin{aligned}
&\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f(x)-\fint_{D'}f\right|dx \\
&\geq \varepsilon^{n-1}\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\sum_{h=1}^{t}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}-C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)-Cr^{n-1}\sigma \\
&\geq \varepsilon^{n-1}\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\sum_{h=1}^{T}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}-C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)-C\sigma\mathcal{H}^{n-1}(K_1\cap Q_1)\,,
\end{aligned}
$$

where the last inequality follows since when $h>t$ then $D'_h\cap\Gamma_1=\emptyset$ and since $r^{n-1}\leq\mathcal{H}^{n-1}(\Gamma_1\cap Q_1)$. Note that C depends now on D, n and $\|f\|_\infty$. Taking the supremum over all families $\mathcal{H}_\varepsilon$ in the previous inequality and letting ε tend to zero, we obtain

$$H_-(f,Q_1)\geq\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)H_-(\chi_{S^+},Q_1)-C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)-C\sigma\mathcal{H}^{n-1}(K_1\cap Q_1)\,. \tag{5.15}$$

Recalling that $\|\nabla\gamma\|_{L^\infty}<\sigma$ and using the first inequality in Lemma 5.1, the Lipschitz continuity of φ, (5.6) and (5.8) we have, using also the homogeneity property of H_-,

$$
\begin{aligned}
\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)H_-(\chi_{S^+},Q_1) &\geq \left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)H_-(S_{e_n},\tilde Q)r^{n-1}-C\sigma r^{n-1} \\
&\geq \left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\varphi(e_n)r^{n-1}-C\sigma r^{n-1}\geq\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\varphi(e_n)\mathcal{H}^{n-1}(\Gamma_1\cap Q_1)-C\sigma r^{n-1} \\
&\geq \left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\int_{K_1\cap Q_1}\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)-C\sigma\mathcal{H}^{n-1}(\Gamma_1\cap Q_1) \\
&\geq \int_{K_1\cap Q_1}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)-C\sigma\mathcal{H}^{n-1}(\Gamma_1\cap Q_1)\,,
\end{aligned}
$$

Inserting this estimate in (5.15) we then get

$$H_-(f,Q_1)\geq\int_{K_1\cap Q_1}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)-C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)-C\sigma\mathcal{H}^{n-1}(K_1\cap Q_1)\,.$$

Step 3. *Proof of* (5.2).

By repeating the same argument for all the other cubes Q_j, with $j = 2, \dots, k$, and summing up the corresponding inequalities we then get, by the superadditivity of $H_-(f, \cdot)$,

$$H_-\left(f, \cup_{i=1}^k Q_i\right) \geq \int_{\cup_{i=1}^k Q_i \cap K_1} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C\mathcal{H}^{n-1}((\Gamma_1 \setminus K_1) \cap (U \times \mathbb{R})) - C\sigma\mathcal{H}^{n-1}(K_1)$$
$$\geq \int_{K_1} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C\sigma\mathcal{H}^{n-1}(K_1)\,,$$

where the last inequality follows at once from (5.7). At this point, we repeat the same argument for all $j = 2, \dots, m$ by constructing finitely many pairwise disjoint cubes $Q_{j,i} \subset I_\delta(J_f)$, with $i = 1, \dots k_j$, so to get

$$H_-\left(f, \cup_{i=1}^{k_j} Q_{j,i}\right) \geq \int_{K_j} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C\sigma\mathcal{H}^{n-1}(K_j)\,.$$

Note that, since the sets $K_1, \dots, K_m$ are pairwise disjoint all the cubes $Q_{j,i}$ can be constructed in such a way that they are all pairwise disjoint for $j = 2, \dots, m$ and $i = 1, \dots, k_j$ and such that they do not intersect the cubes $Q_1, \dots, Q_k$ constructed for K_1. Therefore, from the previous inequalities, using again the subadditivity of $H_-(f, \cdot)$, we get

$$H_-\left(f, I_\delta(J_f) \cap \Omega\right) \geq \sum_{j=1}^m \int_{K_j} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C\sigma \sum_{j=1}^m \mathcal{H}^{n-1}(K_j)$$
$$\geq \int_{J_f} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C|Df|(J_f \setminus K) - C\sigma\mathcal{H}^{n-1}(H)$$
$$\geq \int_{J_f} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) - C\eta - C\sigma\mathcal{H}^{n-1}(H)\,,$$

where the last inequality follows from (5.3) and (5.5). Finally (5.2) follows by letting first $\sigma \to 0$ and then $\eta \to 0$ in the previous inequality. □

Theorem 5.3. *Let D an open set satisfying (H) for some Lipschitz function $\varrho : \mathbb{S}^{n-1} \to (0, +\infty)$ and let $f \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$. Given $\delta > 0$, we have*

$$H_+(f, I_\delta(J_f) \cap \Omega) \leq \int_{J_f} (f^+(x) - f^-(x))\, \varphi(\nu_f(x))\, d\mathcal{H}^{n-1}(x) + C|Df|((I_\delta(J_f) \cap \Omega) \setminus J_f)\,, \tag{5.16}$$

where C is a constant depending only on n, D and $\|f\|_\infty$.

Proof. The proof is similar to the one of Theorem 5.2. Therefore we shall only indicate a few changes.

Step 1: *Construction of a family of cubes covering the jump set up to a small error.*

We repeat the same construction of the Step 1 of Theorem 5.2. Indeed in this case the construction is even a bit simpler since we do not need to choose a cube $\tilde{Q}$ satisfying (5.6). Therefore, we can choose the cubes Q_i^{n-1} with all the edges parallel to the coordinate axes.

Step 2: *Estimate of $H_+(f, Q_i)$ for $i = 1, \dots, k$.*

We fix the cube Q_1 and using the same notation as in Step 2 of Theorem 5.2 we obtain, similarly to (5.10),

$$\sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right| dx$$
$$\leq \sum_{h=1}^s \fint_{D'_h} \left| w_h(x) - \fint_{D'_h} w_h \right| dx + \sum_{h=1}^s \fint_{D'_h} \left| f(x) - w_h(x) - \fint_{D'_h} (f - w_h) \right| dx + \sum_{h=s+1}^T \fint_{D'_h} \left| f(x) - \fint_{D'_h} f \right| dx$$
$$\leq \sum_{h=1}^s 2\left(\tilde{f}^+(x_h) - \tilde{f}^-(x_h)\right) \frac{|D'_h \cap S^+||D'_h \setminus S^+|}{|D'_h|^2} + 2\sum_{h=1}^s \int_{D_h} |f - w_h|\, dx + \sum_{h=s+1}^T \fint_{D'_h} \left| f(x) - \fint_{D'_h} f \right| dx\,.$$

Since

$$\varepsilon^{n-1}\sum_{h=s+1}^{t}\fint_{D'_h}\left|f(x)-\fint_{D'_h}f\right|dx\le C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)\,,$$

and

$$\varepsilon^{n-1}\sum_{h=t+1}^{T}\fint_{D'_h}\left|f(x)-\fint_{D'_h}f\right|dx\le C|Df|(I_\delta(J_f)\setminus\Gamma_1)\,,$$

with a similar estimate as in Step 2 of Theorem 5.2 we have

$$\begin{aligned}\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f(x)-\fint_{D'}f\right|dx\le&\,\varepsilon^{n-1}\left(\tilde f^+(y_1)-\tilde f^-(y_1)\right)\sum_{h=1}^{T}2\frac{|D'_h\cap S^+||D'_h\setminus S^+|}{|D'_h|^2}\\&+C\sigma\mathcal{H}^{n-1}(K_1\cap Q_1)+C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)+C|Df|(I_\delta(J_f)\setminus\Gamma_1)\,.\end{aligned}$$

Taking the supremum over all families $\mathcal{H}_\varepsilon$ in the previous inequality, letting ε tend to zero and using now the estimate of H_+ in Lemma 5.1 and the Lipschitz continuity of φ, we obtain

$$\begin{aligned}H_+(f,Q_1)\le&\int_{K_1\cap Q_1}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)\\&+C\sigma\mathcal{H}^{n-1}(K_1\cap Q_1)+C\mathcal{H}^{n-1}((\Gamma_1\setminus K_1)\cap Q_1)+C|Df|(I_\delta(J_f)\setminus\Gamma_1)\,.\end{aligned}$$

Step 3. *Proof of* (5.16).

We repeat the same argument for all the other cubes Q_j and as in Step 3 of Theorem 5.2, by the subadditivity of $H_+(f,\cdot)$ we have

$$\begin{aligned}H_+\left(f,I_\delta(J_f)\cap\Omega\right)\le&\,H_+\left(f,\cup_{j=1}^m\cup_{i=1}^{k_j}Q_{j,i}\right)+H_+\left(f,(I_\delta(J_f)\cap\Omega)\setminus\cup_{j=1}^m\cup_{i=1}^{k_j}Q_{j,i}\right)\\\le&\int_{J_f}\left(\tilde f^+(x)-\tilde f^-(x)\right)\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)+C\sigma\mathcal{H}^{n-1}(H)+C\eta+C|Df|\big((I_\delta(J_f)\cap\Omega)\setminus J_f\big)\,.\end{aligned}$$

Letting first $\sigma\to0$ and then $\eta\to0$ we obtain (5.16). □

Combining the estimates proved in the previous Theorems 5.2 and 5.3, it is now easy to give the

Proof of Theorem 1.1. We start by assuming that $f\in\mathrm{SBV}(\Omega)\cap L^\infty(\Omega)$ and that $|\overline{J_f}\cap\Omega|=0$. Note that in this case $f\in W^{1,1}(\Omega\setminus\overline{J_f})$. Therefore, given $\delta>0$, by applying Theorem 4.1 on $\Omega_\delta=\Omega\setminus\overline{I_\delta(J_f)}$ and Theorem 5.2 on $I_\delta(J_f)\cap\Omega$, we get by the superadditivity of $H_-(f,\cdot)$

$$H_-(f,\Omega)\ge H_-(f,\Omega_\delta)+H_-(f,I_\delta(J_f)\cap\Omega)\ge\int_{\Omega_\delta}\psi(\nabla f)\,dx+\int_{J_f}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)\,.$$

In turn, letting $\delta\to0$, we have

$$H_-(f,\Omega)\ge\int_\Omega\psi(\nabla f)\,dx+\int_{J_f}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)\,.\tag{5.17}$$

Conversely, by applying Theorem 4.1 again and Theorem 5.3 and using the subadditivity of $H_+(f,\cdot)$ we have, for $\delta>0$

$$\begin{aligned}H_+(f,\Omega)\le&\,H_+(f,\Omega\setminus\overline{J_f})+H_+(f,I_\delta(J_f))\\\le&\int_\Omega\psi(\nabla f)\,dx+\int_{J_f}(f^+(x)-f^-(x))\,\varphi(\nu_f(x))\,d\mathcal{H}^{n-1}(x)+C\int_{I_\delta(J_f)\cap\Omega}|\nabla f|\,dx\,.\end{aligned}$$

Letting $\delta\to0$ and recalling that $|\overline{J_f}\cap\Omega|=0$, we that $H_+(f,\Omega)$ is smaller than or equal to the right hand side of (5.17). Hence (1.6) follows.

Step 2: Observe that if f_h is a sequence of functions converging to f in $\mathrm{BV}(\Omega)$, then, by the Poincaré inequality we have for $\varepsilon \in (0,1)$

$$|H_\varepsilon(f_h,\Omega) - H_\varepsilon(f,\Omega)| \le C\|f_h - f\|_{\mathrm{BV}(\Omega)}\,.$$

Therefore $H_\varepsilon(f_h,\Omega) \to H_\varepsilon(f,\Omega)$ uniformly with respect to ε, hence $H_\pm(f_h,\Omega) \to H_\pm(f,\Omega)$. On the other hand, by Reschetnyak continuity theorem, see [3, Th. 2.39], we have also that if $f_h, f \in \mathrm{SBV}(\Omega)$ and $f_h \to f$ in $\mathrm{BV}(\Omega)$ then

$$\liminf_{h\to\infty}\left[\int_\Omega \psi(\nabla f_h)\,dx + \int_{J_{f_h}} \left(f_h^+ - f_h^-\right)\varphi(\nu_{f_h})\,d\mathcal{H}^{n-1}\right] = \int_\Omega \psi(\nabla f)\,dx + \int_{J_f} (f^+ - f^-)\,\varphi(\nu_f)\,d\mathcal{H}^{n-1}\,.$$

With this observation and a simple approximation argument one immediately gets that (1.6) still holds without the assumption $f \in L^\infty(\Omega)$. Then, the assumption $|\overline{J_f} \cap \Omega| = 0$ can be removed using the approximation Theorem 2.1.

Step 3: To prove that $\psi \le \varphi$ we fix $\nu \in \mathbb{S}^{n-1}$. From (3.7), given a cube $Q_\nu \in \mathbf{Q}_\nu$, we have

$$\varphi(\nu) = H(\chi_{S_\nu}, Q_\nu)\,.$$

Fix $\sigma > 0$ small and set $u_{\nu,\sigma} := (\varrho_\sigma * \chi_{S_\nu})$, where ϱ is a standard mollifier as in the proof of Corollary 4.2. By construction $u_{\nu,\sigma}$ strictly converges to χ_{S_ν} as $\sigma \to 0$. Then, arguing exactly as in the proof of Corollary 4.2 we have that $H(u_{\nu,\sigma}, Q_\nu) \le H(\chi_{S_\nu}, Q_\nu)$, that is

$$\int_{Q_\nu} \psi(\nabla u_{\nu,\sigma})\,dx \le \varphi(\nu)\,.$$

From this inequality, Reshetnyak continuity theorem yields

$$\psi(\nu) = \int_{Q_\nu} \psi\left(\frac{D\chi_{S_\nu}}{|D\chi_{S_\nu}|}\right) d|D\chi_{S_\nu}| = \lim_{\sigma\to 0}\int_{Q_\nu} |\nabla u_{\nu,\sigma}|\psi\left(\frac{\nabla u_{\nu,\sigma}}{|\nabla u_{\nu,\sigma}|}\right) dx \le \varphi(\nu)\,,$$

as required. □

6. An example of Ambrosio-Comi revisited

In this section we take $D = Q$, where $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ denotes the unit cube centered in the origin in $\mathbb{R}^n$ and give the explicit expression of the function ψ when $n = 2$. Note that in [2] the authors give the explicit expression for φ in the two-dimensional case. It turns out that φ is not convex, while ψ is convex, thanks to Proposition 3.6. In this section we calculate ψ and show that ψ is strictly smaller than the greatest convex function less than or equal to φ. We recall, see (3.5), that

$$\psi(\nu) = \lim_{\varepsilon\to 0} H_\varepsilon\,(f_\nu, Q)\ ,$$

where $f_\nu(x) := x\cdot\nu$ with $x \in \mathbb{R}^n$. For $\varepsilon > 0$ and $x_0 \in Q$ it is straightforward to see that

$$\fint_{Q(x_0,\varepsilon)} \left| x\cdot\nu - \fint_{Q(x_0,\varepsilon)} y\cdot\nu \right| dx = \varepsilon \fint_Q |x\cdot\nu|\,dx\,. \tag{6.1}$$

since $\fint_Q x\cdot\nu\,dx = 0$. Then $H_\varepsilon\,(f_\nu, Q_\nu)$ takes the form

$$H_\varepsilon\,(f_\nu, Q_\nu) = \varepsilon^{n-1}\sup_{\mathcal{H}_\varepsilon}\sum_{Q'\in\mathcal{H}_\varepsilon}\fint_{Q'}\left| x\cdot\nu - \fint_{Q'} y\cdot\nu\,dy\right| dx\,,$$

where the supremum runs over all possible families $\mathcal{H}_\varepsilon$ of translations Q' of the cube εQ contained in Q_ν. Thus, by (6.1) we have

$$H_\varepsilon\,(f_\nu, Q_\nu) = \varepsilon^n \sup_{\mathcal{H}_\varepsilon}\#(\mathcal{H}_\varepsilon)\int_Q |x\cdot\nu|\,dx = \varepsilon^n\,\lfloor(1/\varepsilon)\rfloor^n\int_Q |x\cdot\nu|\,dx\,.$$

Therefore

$$\psi(\nu) = \int_Q |x \cdot \nu| \, dx \, .$$

We consider now the case $n = 2$. From the above formula it is clear that for every $\nu = (\nu_1, \nu_2) \in \mathbb{S}^1$

$$\psi(-\nu_1, -\nu_2) = \psi(-\nu_2, \nu_1) = \psi(\nu_2, -\nu_1) = \psi(\nu_1, \nu_2) \, . \tag{6.2}$$

Let us compute $\psi(\nu)$ when $\nu_2 \neq 0$. If we set

$$m := -\frac{\nu_1}{\nu_2}$$

then we have

$$\int_Q |x \cdot \nu| \, dx = |\nu_2| \int_Q |t - ms| \, dsdt = |\nu_2|(I^+(m) + I^-(m)) \, ,$$

where

$$I^+(m) := \int_{Q \cap \{t \geq ms\}} (t - ms) \, dsdt, \qquad I^-(m) := \int_{Q \cap \{t \leq ms\}} (ms - t) \, dsdt \, .$$

We start with the case $m > 0$, namely ν_1 and ν_2 have opposite signs. We distinguish three cases.

Case 1: $\nu_2 > 0$ and $\nu_1 < -\nu_2$.

Note that in this case $m > 1$ and therefore the straight line $\{t = ms\}$ intersects the lines $\left\{t = \pm\frac{1}{2}\right\}$ at the points $\left(\frac{1}{2m}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2m}, -\frac{1}{2}\right)$. To calculate $I^+(m)$ observe that

$$I^+(m) = \int_{\left[-\frac{1}{2}, -\frac{1}{2m}\right] \times \left[-\frac{1}{2}, -\frac{1}{2}\right]} (t - ms) \; dsdt + \int_W (t - ms) \; dsdt$$

where $W := \left\{(s, t) \in \mathbb{R}^2 : \; -\frac{1}{2m} \leq x \leq \frac{1}{2m} \quad \text{and} \quad ms \leq t \leq \frac{1}{2}\right\}$. A direct computation gives that

$$I^+(m) = \frac{1}{8}\left(m + \frac{1}{3m}\right)$$

It can be easily seen that $I^+(m) = I^-(m)$. From the the definition of the parameter m we end up with

$$\psi(\nu_1, \nu_2) = -\frac{1}{4}\nu_1\left(1 + \frac{1}{3}\frac{\nu_2^2}{\nu_1^2}\right)$$

Case 2: $\nu_2 > 0$ and $\nu_1 \geq -\nu_2$.

Note that in this case $0 < m \leq 1$ and therefore the straight line $\{y = mx\}$ intersects the lines $\left\{x = \pm\frac{1}{2}\right\}$ in two points, whose coordinate are respectively given by $\left(\frac{1}{2}, \frac{m}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{m}{2}\right)$. Arguing as in the previous case, we immediately get

$$\psi(\nu_1, \nu_2) = \frac{1}{4}\nu_2\left(1 + \frac{1}{3}\frac{\nu_1^2}{\nu_2^2}\right)$$

Case 3: $\nu = (1, 0)$.

A direct computation which follows along the lines of previous cases gives that

$$\psi(1, 0) = \frac{1}{4}$$

Due to (6.2), the cases that we considered before allow us to conclude that for all $\nu \in \mathbb{R}^2$

$$\psi(\nu_1, \nu_2) = \begin{cases} \frac{1}{4}|\nu_2|\left(1 + \frac{1}{3}\frac{\nu_1^2}{\nu_2^2}\right) & \text{if } |\nu_1| \leq |\nu_2| \, , \\ \frac{1}{4}|\nu_1|\left(1 + \frac{1}{3}\frac{\nu_2^2}{\nu_1^2}\right) & \text{if } |\nu_2| \leq |\nu_1| \, . \end{cases} \tag{6.3}$$

Let us now compare this function with the function φ that in this case, see [2, Section 4.3], is given for any $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ by

$$\varphi(\nu_1, \nu_2) = \begin{cases} \frac{2}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \le |\nu_1| \le \frac{32}{27}|\nu_2|\,, \\ \frac{1}{2}\max\{|\nu_1|, |\nu_2|\} & \text{if } |\nu_1| \le \frac{27}{32}|\nu_2| \ \text{ or } \ |\nu_1| \ge \frac{32}{27}|\nu_2|\,. \end{cases}$$

Then a tedious, but elementary computation shows that $\frac{3}{2}\psi \le \varphi \le 2\psi$. Thus ψ is a convex function strictly smaller than the greatest convex function less than or equal to φ.

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