

ON THE SOBOLEV QUOTIENT OF THREE-DIMENSIONAL CR MANIFOLDS

JIH-HSIN CHENG⁽¹⁾, ANDREA MALCHIODI⁽²⁾, PAUL YANG⁽³⁾

⁽¹⁾ Institute of Mathematics, Academia Sinica and NCTS
6F, Astronomy-Mathematics Building No. 1, Sec. 4
Roosevelt Road, Taipei 10617, TAIWAN

⁽²⁾ Scuola Normale Superiore, Piazza dei Cavalieri 7, 50126 Pisa, ITALY

⁽³⁾ Princeton University, Department of Mathematics
Fine Hall, Washington Road, Princeton NJ 08544-1000 USA

ABSTRACT. We exhibit examples of compact three-dimensional CR manifolds of positive Webster class, *Rossi spheres*, for which the pseudo-hermitian mass as defined in [CMY17] is negative, and for which the infimum of the CR-Sobolev quotient is not attained. To our knowledge, this is the first geometric context on smooth closed manifolds where this phenomenon arises, in striking contrast to the Riemannian case.

1. INTRODUCTION

The Yamabe problem consists in deforming conformally the metric of a manifold of dimension $n \geq 3$ so that its scalar curvature becomes a constant. Apart from being a natural conformal extension of the Uniformization Problem in two dimensions, the question was introduced in [Yam60] for trying to attack Poincaré's conjecture. Yamabe metrics have also been applied to other contexts, such as the study of degeneration of conformal structures. For example, in [TV05] it is shown that the set of Yamabe Bach-flat metrics on a four-manifold is compact up to orbifold degeneration.

Writing on (M, g) a conformal metric as $\tilde{g} = u^{\frac{4}{n-2}}g$, the scalar curvature transforms as

$$(1) \quad -\frac{4(n-1)}{n-2}\Delta_g u + S_g u = S_{\tilde{g}} u^{\frac{n+2}{n-2}}.$$

Therefore, if one wishes to have $S_{\tilde{g}}$ constant, the following elliptic problem must be solved

$$(Y) \quad -\frac{4(n-1)}{n-2}\Delta_g u + S_g u = \bar{S} u^{\frac{n+2}{n-2}} \quad \text{on } M; \quad \bar{S} \in \mathbb{R}.$$

Notice that the exponent on the right-hand side of the equation is critical with respect to the Sobolev embeddings. In [Yam60] an attempt was made to solve (Y) by lowering the exponent by a small amount, in order to obtain compactness, and then by letting it approach the critical one studying the limit of the corresponding solutions. The problem with this strategy though is that the weak limit of such solutions might be zero. Another way to attack (Y) was to view \bar{S} as a Lagrange multiplier, considering the Sobolev quotient

$$(2) \quad Q_{(M,g)}(u) := \frac{\int_M (c_n |\nabla_g u|^2 + S_g u^2) dV_g}{\left(\int_M |u|^{2^*} dV_g\right)^{\frac{2}{2^*}}} = \frac{\int_M S_{\tilde{g}} dV_{\tilde{g}}}{(Vol_{\tilde{g}}(M))^{\frac{2}{2^*}}},$$

where $c_n = \frac{4(n-1)}{(n-2)}$ and $2^* = \frac{2n}{n-2}$. If one could realise the minimum of $Q_{(M,g)}(u)$ over all non-zero u 's of class $W^{1,2}(M, g)$, this would give rise to a solution of (Y): notice that it is sufficient to consider functions in $W^{1,2}(M, g)$ that are non-negative, therefore by regularity theory one would obtain a positive smooth solution. Defining then

$$Y(M, g) := \inf_{u \in W^{1,2}(M, g), u \neq 0} Q_{(M,g)}(u),$$

2010 *Mathematics Subject Classification.* 32V20, 35J75, 35J20, 53C17, 32V30.

Key words and phrases. CR manifold, Rossi sphere, pseudo-hermitian mass, CR-Sobolev quotient.

¹E-mail addresses: cheng@math.sinica.edu.tw, andrea.malchiodi@sns.it, yang@math.princeton.edu

it can be proved using (1) that this quantity is independent of the conformal representative of g , and will therefore be denoted from now on by $Y(M, [g])$. Depending on the sign of the latter quantity, $(M, [g])$ is said to be of *negative*, *null* or of *positive Yamabe class*.

It was proved in [Tru68] that there exists a dimensional constant $\varepsilon_n > 0$ such that $Y(M, [g])$ is attained (and hence (Y) is solvable) provided $Y(M, [g]) \leq \varepsilon_n$. The result applies in particular to all manifolds with conformal classes of metrics of negative or null Yamabe class.

Consider the (normalized) Sobolev quotient in \mathbb{R}^n

$$(3) \quad S_n := \inf_{u \in C_c^\infty(\mathbb{R}^n), u \neq 0} \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

Using the stereographic projection from S^n to \mathbb{R}^n it can be proved that the above quantity coincides with the Yamabe quotient of the round sphere, i.e. for all $n \geq 3$ one has

$$S_n = Y(S^n, [g_{S^n}]).$$

It was shown in [Aub76] that one always has $Y(M, [g]) \leq S_n$, and that (Y) is solvable provided the strict inequality holds. It was also shown in [Aub76] that $Y(M, [g]) < S_n$ provided $n \geq 6$ and M is not locally conformally flat, i.e. when the Weyl tensor of (M, g) is not identically zero. It was proved then in [Sch84] that $Y(M, [g]) < S_n$ in all complementary cases (provided (M, g) is not conformally equivalent to the round sphere), i.e. when (M, g) has dimension less or equal to 5 or when it is locally conformally flat. While the argument in [Aub76] was based on a local energy expansion, the one in [Sch84] relied on the *Positive Mass Theorem* in general relativity, see [SY79b], [SY81], [SY79a], [SY17], which is in turn related to the expansion of the Green's function of the *conformal Laplacian* L_g near its pole, where

$$L_g u := -\frac{4(n-1)}{n-2} \Delta_g u + S_g u.$$

In both [Aub76] and [Sch84] the strict inequality was proved by evaluating the Yamabe-Sobolev quotient on (suitable perturbations of) highly concentrated extremals of (3) (classified in [Aub76], [Tal76]), suitably glued to (M, g) . Such extremals, parametrized using the Möbius group of S^n , can be chosen arbitrarily peaked near any point: these decay faster at infinity in higher dimensions and therefore the correction to the quotient due to the geometry of M is more *localized* in space for n large. In any case, we always have

$$Y(M, [g]) < S_n \quad \text{provided} \quad (M, g) \stackrel{\text{conf.}}{\not\cong} (S^n, g_{S^n}).$$

We consider in this paper compact three dimensional pseudo-hermitian manifolds (M, J, θ) : these are CR manifolds, i.e. endowed with a contact structure ξ and a CR structure $J : \xi \rightarrow \xi$ such that $J^2 = -1$. We assume (M, J) to be *pseudo-convex*, namely that it is globally defined a contact form θ which annihilates ξ and for which $\theta \wedge d\theta$ is always non-zero (see [BFG83]). We define the *Reeb vector field* as the unique T for which $\theta(T) \equiv 1$ and $T \lrcorner d\theta = 0$. Given J as above, we can define locally a vector field Z_1 such that

$$(4) \quad JZ_1 = iZ_1; \quad JZ_{\bar{1}} = -iZ_{\bar{1}} \quad \text{where} \quad Z_{\bar{1}} = \overline{(Z_1)}.$$

We also let $(\theta, \theta^1, \theta^{\bar{1}})$ be the dual triple to $(T, Z_1, Z_{\bar{1}})$, so that

$$d\theta = ih_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}} \quad \text{for some } h_{1\bar{1}} > 0 \quad (\text{possibly replacing } \theta \text{ by } -\theta).$$

In the following we will always assume that $h_{1\bar{1}} \equiv 1$.

The *connection* 1-form ω_1^1 and the *torsion* A_1^1 are uniquely determined by the structure equations

$$(5) \quad \begin{cases} d\theta^1 = \theta^1 \wedge \omega_1^1 + A_1^1 \theta \wedge \theta^{\bar{1}}; \\ \omega_1^1 + \omega_{\bar{1}}^{\bar{1}} = 0. \end{cases}$$

The *Tanaka-Webster curvature* (or *Webster curvature*) R_θ (or, simply, R) is then defined by the formula

$$d\omega_1^1 = R_\theta \theta^1 \wedge \theta^{\bar{1}} \quad (\text{mod } \theta).$$

A model with positive curvature is the standard sphere $(S^3, J_{S^3}, \hat{\theta})$, with $S^3 \subseteq \mathbb{C}^2 = \{(z_1, z_2)\}$, and

$$(6) \quad \hat{\theta} = \frac{1}{2}i(\bar{\partial} - \partial)(|z^1|^2 + |z^2|^2) = \frac{1}{2}i \sum_{k=1}^2 (z^k dz^{\bar{k}} - z^{\bar{k}} dz^k); \quad Z_1 = Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

Similarly to what happens with the classical stereographic projection, the CR three-sphere is CR equivalent to the *Heisenberg group* $\mathbb{H}^1 = \{(z, t), z \in \mathbb{C}, t \in \mathbb{R}\}$, see e.g. [CMY17].

The Tanaka-Webster curvature enjoys conformal properties similar to the scalar curvature on Riemannian manifolds. More precisely, scaling the contact form θ by a positive function, one has the following law for the transformation of the Webster curvature, similar to (1)

$$(7) \quad L_b u := -4\Delta_b u + R_\theta u = R_{\tilde{\theta}} u^3; \quad \tilde{\theta} = u^2 \theta.$$

Here $R_{\tilde{\theta}}$ is the Tanaka-Webster curvature corresponding to the pseudo-hermitian structure $(J, \tilde{\theta})$. Δ_b stands for the operator defined as follows

$$\Delta_b f = f_{,1}{}^1 + f_{,\bar{1}}{}^{\bar{1}} = f_{,1\bar{1}} + f_{,\bar{1}1},$$

where we have used $h^{1\bar{1}} = h_{\bar{1}1} = 1$ to raise or lower the indices, and where we set

$$(8) \quad f_1 = f_{,1} := Z_1 f; \quad f_{,1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1^1(Z_{\bar{1}}) Z_1 f; \quad f_{,0} = T f.$$

The CR-invariant sub-Laplacian transforms covariantly as follows

$$\hat{L}_b(\varphi) = u^{-\frac{Q+2}{Q-2}} L_b(u\varphi); \quad \hat{\theta} = u^2 \theta,$$

where $Q = 4$ is the *homogeneous dimension* of the manifold. By (7), finding $\tilde{\theta}$ with constant Webster curvature corresponds to solving the following analogous problem to (Y)

$$(W) \quad L_b u = \bar{R} u^{\frac{Q+2}{Q-2}} \quad \text{on } M; \quad \bar{R} \in \mathbb{R}, \quad u > 0.$$

In [JL87] the counterpart of the result in [Aub76] was obtained, i.e. if the infimum of the CR-Sobolev quotient satisfies

$$(9) \quad \mathcal{Y}(M, J) := \inf_{\hat{\theta}} \frac{\int_M R_{J, \hat{\theta}} \hat{\theta} \wedge d\hat{\theta}}{\left(\int_M \hat{\theta} \wedge d\hat{\theta}\right)^{\frac{1}{2}}} < \mathcal{Y}(S^3, J_{S^3}),$$

then it is attained and a solution of (W) exists (indeed, this holds true in any dimension). The same authors verified this condition when the dimension is greater or equal to five and (M, J) is not *spherical*, see [JL89] and [JL88].

However, in the CR setting new phenomena appear, related to the fact that most three-dimensional structures are non-embeddable, differently from the higher-dimensional case, see [BdM75], [BE90]. In [CMY17] some results in the above directions were proved, assuming embeddability of the structure.

More precisely, a notion of *pseudo-hermitian mass* was defined for three-dimensional *asymptotically-Heisenberg* manifolds (we refer to the latter paper for precise definitions and details) by setting

$$m(J, \theta) := i \oint_{S_\Lambda} \omega_1^1 \wedge \theta := \lim_{\Lambda \rightarrow +\infty} i \oint_{S_\Lambda} \omega_1^1 \wedge \theta,$$

where $S_\Lambda = \{\rho = \Lambda\}$, $\rho^4 = |z|^4 + t^2$ (with (z, t) coordinates on the Heisenberg group), and where ω_1^1 stands for the connection form of the structure. The above definition was introduced considering an analogue of the *Einstein-Hilbert action*.

As it happens in the Riemannian case, this mass is related to the expansion of the Green's function of the conformal sub-Laplacian L_b on a compact manifold M . When $\mathcal{Y}(M, J) > 0$ the latter operator is invertible, so for any $p \in M$ there exists a Green's function G_p verifying distributionally

$$(-4\Delta_b + R) G_p = 64 \pi^2 \delta_p,$$

where δ_p in the the right-hand side stands for the Dirac delta w.r.t. the volume measure $\theta \wedge d\theta$. In *CR normal coordinates* (z, t) (introduced in [JL89] and discussed in Section 2) G_p writes as

$$(10) \quad G_p = 2\rho^{-2} + A + O(\rho),$$

for some $A \in \mathbb{R}$ and where $\rho^4(z, t)$ is as above. For the latter expansion, we refer to Proposition 5.2 in [CMY17] (here we use an extra factor 4π in the definition of G_p), and to Subsection 2.1 for our notation $O(\rho)$. Given (M, J, θ) compact and $p \in M$, consider a blow-up of contact form as follows

$$(11) \quad N = (M \setminus \{p\}, J, G_p^2 \theta).$$

As it is shown in [CMY17], via an inversion of coordinates, the manifold N turns out to have asymptotically the geometry of the Heisenberg group, and its pseudo-hermitian mass satisfies

$$(12) \quad m = 12\pi A$$

(see Lemma 2.5 there, and recall the difference of 4π in our current notation), where A is as above. Using crucially a result in [HY15], in the same paper it was also proved that the pseudo-hermitian mass is non-negative (and zero only when (M, J, θ) is CR equivalent to S^3), provided that the *CR Paneitz operator* P on (M, J) is non-negative definite. The latter operator is

$$(13) \quad P\varphi := 4(\varphi_{\bar{1}1} + iA_{11}\varphi^1)^1,$$

it has a relation to the log-term coefficient in the Szegő kernel expansion, and it is pseudo-hermitian-covariant, namely $P_{\hat{\theta}}\varphi = e^{4f}P_{\theta}\varphi$ for the conformal change $\theta = e^{2f}\hat{\theta}$ ([Hir93]). By a result in [CCY12], manifolds for which P is non-negative and $W > 0$ can be embedded into some \mathbb{C}^N (see also [CCY16]).

The assumption on the positivity on the Paneitz operator is not technical, as in [CMY17] some counterexamples for the positivity of the pseudo-hermitian mass were also given for structures (arbitrarily) close to the spherical one, and hence with positive Webster curvature.

In this paper we are concerned with *Rossi spheres*: these are a one-parameter-family of CR structures on the 3-sphere of the form $S_s^3 := (S^3, J_{(s)}, \hat{\theta})$, where $\hat{\theta}$ is as in (6), and where $J_{(s)}$ is characterized by

$$(14) \quad J_{(s)}Z_{1(s)} = iZ_{1(s)}; \quad Z_{1(s)} = Z_1 + \frac{s}{\sqrt{1+s^2}}Z_{\bar{1}}, \quad Z_{\bar{1}(s)} = Z_{\bar{1}} + \frac{s}{\sqrt{1+s^2}}Z_1.$$

Rossi spheres are interesting because they are simple examples of CR structures on the three-sphere that cannot be embedded in \mathbb{C}^N . In [Bur79] it was shown that all the holomorphic functions on such structures are even functions if $s \neq 0$. On the other hand, there are explicit embeddings in \mathbb{C}^3 of the quotient of the Rossi spheres by the antipodal map, see [CS01]. By the above discussion, it follows that the Paneitz operator cannot be non-negative here. In addition, this family of CR structures are homogeneous and if we take the standard contact form, it is *pseudo-Einstein*, i.e. $R_{,1} - iA_{11,\bar{1}} = 0$, see [CY13] as well as our notation for covariant derivatives in Section 2.1.

Our first main result in this paper is the following theorem.

Theorem 1.1. *For $|s|$ small, $s \neq 0$, the pseudo-hermitian mass of the Rossi spheres S_s^3 is negative. More precisely, one has the expansion*

$$m_s = -18\pi s^2 + o(s^2) \quad \text{for } s \simeq 0.$$

We saw before (in both low-dimensional Riemannian and CR cases) that positivity of the mass implies attainment of the Sobolev quotient. We also strengthen the relation between mass and quotient by means of the following result.

Theorem 1.2. *For $|s|$ small, $s \neq 0$, the infimum of the CR-Sobolev quotient of S_s^3 coincides with $\mathcal{Y}(S^3, J_{S^3})$ and is not attained.*

Remark 1.3. (a) *The phenomenon in Theorem 1.2 is typical of some critical problems in a PDE context, like the Yamabe equation on Euclidean domains with Dirichlet boundary conditions or the case of some general elliptic operators on manifolds. However, to our knowledge this is the first time this is displayed in a purely geometric smooth context.*

(b) *We recall that in [Gam01] and [GY01] the CR-Yamabe problem was solved for every three dimensional CR manifolds, but there solutions were found via variational arguments and they are not of minimal type. Theorem 1.2 shows that the use of such methods is in some cases somehow necessary.*

Determining or estimating the *mass* of a manifold is in general a hard problem, since this is deeply related to the Green's function of the conformal (sub-)Laplacian, which is a *global* object. After recalling some preliminary facts in Section 2 on CR normal coordinates (introduced in [JL89]) and on Rossi spheres, we specialize in Section 3 to the latter manifolds, deriving first a suitable conformal factor and then expressing pseudo-hermitian coordinates depending on s . By the special expression of the Green's function in these coordinates, we are able to determine it quite precisely near the north pole, up to the constant term A appearing in (10).

By a formal expansion in s , worked-out at the beginning of Section 4, it is possible to characterize formally the Green's function for the conformal sub-Laplacian on Rossi spheres up to an order $O(s^3)$. However this expansion generates singular terms, with a particularly bad behavior near the pole, if expressed with respect to the standard complex coordinates of \mathbb{C}^2 , where S^3 embeds. However we verify in the second part of the section that the global singular expansion on S^3 matches with the one done in CR normal coordinates up to an order $O(s^3)$, allowing us to prove Theorem 1.1.

In Section 5, arguing by contradiction, we analyse the possible behaviours of minimizers for the CR Sobolev quotient. Due to a non-degeneracy result from [MU02], the analysis of minimizers can be reduced to a finite-dimensional one, and we show that the CR-Sobolev quotient of all candidate minimizers is strictly above the spherical one, i.e. $\mathcal{Y}(S^3, J_{S^3})$. With negative mass, this is expected for highly concentrated profiles, reversing the expansion in [Sch84]: however such a property has to be obtained in all cases, i.e. even for non-concentrated profiles. In Proposition 5.5 this is proved for s small in a fixed compact set of the CR maps of S^3 . One needs then to analyze the quotient in a regime with loss of compactness, which is particularly delicate due to the following reason. It is known from [Sch84] that the mass of a (given) manifold plays a role in the expansion for Sobolev quotients of highly concentrated functions. In our case this must be done uniformly in s , and the problem could be that the *principal term* coming from the mass could become negligible as $s \rightarrow 0$. To solve this issue we exploit a symmetry $s \rightarrow -s$ for Rossi spheres, discussed in Section 2, which implies that all variational expansions are indeed even in s and hence the mass, which vanishes with s , gives still a dominant sign to the asymptotic expansion of the CR-Sobolev quotient. Two appendices are devoted to the estimates of the latter quantity in two different scaling regimes.

Acknowledgements J.-H.C. (P.Y., resp.) are grateful to Scuola Normale Superiore and Princeton University (Academia Sinica in Taiwan, resp.) for the kind hospitality. J.-H.C. is supported by the project MOST 107-2115-M-001-011 of Ministry of Science and Technology and NCTS of Taiwan. A.M. is supported by the project *Geometric Variational Problems* from Scuola Normale Superiore and by MIUR Bando PRIN 2015 2015KB9WPT₀₀₁. He is also a member of GNAMPA as part of INdAM. He also would like to thank Academia Sinica in Taiwan and Princeton University for the kind hospitality. P. Y. acknowledges support from the NSF for the grant DMS 1509505.

2. BACKGROUND MATERIAL

In this section we recall some useful facts about CR manifolds and the properties of CR normal coordinates, constructed in [JL89]. We then describe some general features of Rossi spheres.

2.1. Preliminary facts on CR manifolds. Let us begin by recalling the following commutation relations on tensors, see Lemma 2.3 in [Lee88] (we also refer to this paper for our tensorial notation)

$$(15) \quad \begin{cases} c_{,1\bar{1}} - c_{,\bar{1}1} = ic_{,0} + kcR; \\ c_{,01} - c_{,10} = c_{,\bar{1}} A_{11} - kcA_{11,\bar{1}}; \\ c_{,0\bar{1}} - c_{,\bar{1}0} = c_{,1} A_{\bar{1}\bar{1}} + kcA_{\bar{1}\bar{1},1}. \end{cases}$$

Here c is a tensor with 1 or $\bar{1}$ as sub-indices, k is the number of 1-sub-indices of c minus the number of $\bar{1}$ -sub-indices of c and where, we recall, we are assuming that $h_{1\bar{1}} = 1$ (so $A_{\bar{1}\bar{1}} = A_{\bar{1}}^1$ and A_{11} is the complex conjugate of $A_{\bar{1}\bar{1}}$).

In the system of coordinates we will describe below, for $(z, t) \in \mathbb{H}^1$ near zero we will set

$$(16) \quad \rho^4 = |z|^4 + t^2.$$

For $k \in \mathbb{Z}$ we denote by $\tilde{O}(\rho^k)$ a function $f(z, \bar{z}, t)$ for which $|f| \leq C\rho^k$ for some $C > 0$; we use instead the symbol $\tilde{O}'(\rho^k)$ for a function $f(z, \bar{z}, t)$ such that

$$|f| \leq C\rho^k, \quad |\partial_z f| \leq C\rho^{k-1} |\partial_z \rho|, \quad |\partial_{\bar{z}} f| \leq C\rho^{k-1} |\partial_{\bar{z}} \rho|, \quad |\partial_t f| \leq C\rho^{k-1} |\partial_t \rho|.$$

One can define similarly the symbols $\tilde{O}''(\rho^k)$, $\tilde{O}'''(\rho^k)$, etc. We will use $O(\rho^k)$ for a function which is of the form $\tilde{O}^{(j)}(\rho^k)$ for every integer j , or for j large enough for our purposes.

Large positive constants are always denoted by C , and the value of C is allowed to vary from one formula to another and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C , as C_δ , etc.. Also constants with this kind of subscripts are allowed to vary.

Let us recall the notions of *pseudo-hermitian geometry* from [Web78] and [Lee86]. We would need the following result in [JL89] on page 313, Proposition 2.5. For a differential form η , let us denote by $\eta_{(m)}$ the part of its Taylor series that is homogeneous of degree m in terms of parabolic dilations (see [JL89] for more details).

Proposition 2.1. *Let \tilde{Z}_1 be a special frame dual to $\tilde{\theta}^1$ (with $\tilde{h}_{1\bar{1}} = 2$) and let $\theta^1 = \sqrt{2}\tilde{\theta}^1$ be a unitary coframe ($h_{1\bar{1}} = 1$). Then in pseudo-hermitian normal coordinates (z, t) with respect to $\tilde{Z}_1, \tilde{\theta}^1$, we have*

$$(a) \quad \theta_{(2)} = \overset{\circ}{\theta}; \quad \theta_{(3)} = 0; \quad \theta_{(m)} = \frac{1}{m} \sqrt{2} \left(iz\theta^{\bar{1}} - i\bar{z}\theta^1 \right)_{(m)}, \quad m \geq 4;$$

$$(b) \quad \theta_{(1)}^1 = \sqrt{2}dz; \quad \theta_{(2)}^1 = 0; \quad \theta_{(m)}^1 = \frac{1}{m} \left(\sqrt{2}z\omega_1^1 + 2tA_{1\bar{1}}\theta^{\bar{1}} - \sqrt{2}\bar{z}A_{1\bar{1}}\theta \right)_{(m)}, \quad m \geq 3$$

$$(c) \quad (\omega_1^1)_{(1)} = 0; \quad (\omega_1^1)_{(m)} = \frac{1}{m} \left(\sqrt{2}R(z\theta^{\bar{1}} - \bar{z}\theta^1) + A_{11,1}(\sqrt{2}z\theta - 2t\theta^1) - A_{1\bar{1},1}(\sqrt{2}\bar{z}\theta - 2t\theta^{\bar{1}}) \right)_{(m)},$$

$m \geq 2$, where $\overset{\circ}{\theta} = dt + izd\bar{z} - i\bar{z}dz$.

Definition 2.2. *Given a three dimensional pseudo-hermitian manifold (M, θ) we define a real symmetric tensor Q as*

$$Q = Q_{jk}\theta^j \otimes \theta^k, \quad j, k \in \{0, 1, \bar{1}\}$$

with $\theta^0 := \theta$, whose components with respect to any admissible coframe are given by

$$Q_{11} = \overline{Q_{\bar{1}\bar{1}}} = 3iA_{11}; \quad Q_{1\bar{1}} = Q_{\bar{1}1} = h_{1\bar{1}}R;$$

$$Q_{01} = Q_{10} = \overline{Q_{0\bar{1}}} = \overline{Q_{\bar{1}0}} = 4A_{11,1} + iR_{,1}; \quad Q_{00} = 16ImA_{11,1} - 2\Delta_b R.$$

We have then the following result, see page 315 in [JL89], Theorem 3.1.

Proposition 2.3. *Suppose M is a strictly pseudo-convex pseudo-hermitian manifold of dimension three, and let $q \in M$. Then for any integer $N \geq 2$ there exists a choice of contact form θ such that all symmetrized covariant derivatives of Q with total order less or equal than N vanish at q , that is*

$$(17) \quad Q_{\langle jk, l \rangle} = 0 \quad \text{at } q \text{ if } \mathbb{O}(jkl) \leq N.$$

By *CR normal coordinates* of order N we mean the pseudo-hermitian normal coordinates with θ chosen as in Proposition 2.3. We recall ([JL89]) that for a multi index $l = (l_1, \dots, l_s)$ we count its order as

$$\mathbb{O}(l) = \mathbb{O}(l_1) + \dots + \mathbb{O}(l_s),$$

where $\mathbb{O}(1) = \mathbb{O}(\bar{1}) = 1$ and where $\mathbb{O}(0) = 2$. The symmetrized covariant derivatives are defined by

$$Q_{(l)} = \frac{1}{s!} \sum_{\sigma \in \mathbb{S}_s} Q_{\sigma l}; \quad \sigma l = (l_{\sigma(1)}, \dots, l_{\sigma(s)}).$$

In [CMY17], Proposition A.5, the following result was proved.

Proposition 2.4. *In CR normal coordinates of order $N = 4$, we have a contact form θ such that*

$$\theta = (1 + O(\rho^4)) \overset{\circ}{\theta} + O(\rho^5)dz + O(\rho^5)d\bar{z}; \quad \theta^1 = (1 + O(\rho^4)) \sqrt{2}dz + O(\rho^4)d\bar{z} + O(\rho^3)\overset{\circ}{\theta};$$

$$\omega_1^1 = O(\rho^3)dz + O(\rho^3)d\bar{z} + O(\rho^2)\overset{\circ}{\theta};$$

$$Z_1 = (1 + O(\rho^4)) \overset{\circ}{Z}_1 + O(\rho^4)\overset{\circ}{Z}_{\bar{1}} + O(\rho^5)\frac{\partial}{\partial t}; \quad T = (1 + O(\rho^4))\frac{\partial}{\partial t} + O(\rho^3)\overset{\circ}{Z}_1 + O(\rho^3)\overset{\circ}{Z}_{\bar{1}},$$

where we recall

$$(18) \quad \overset{\circ}{\theta} = dt + izd\bar{z} - i\bar{z}dz; \quad \overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t} \right); \quad \rho^4 = t^2 + |z|^4.$$

2.2. Rossi spheres. We recall here some properties of *Rossi spheres*, introduced in [Ros65] as a non-embeddable example of CR manifold (see also [Bur79]). These are families of CR structures on S^3 , containing the standard one, obtained in the following way.

Considering the complex vector field Z_1 as in (6) and its conjugate $Z_{\bar{1}}$, one defines the CR structure $J_{(s)}$ by setting $J_{(s)}Z_{1(s)} = iZ_{1(s)}$, where

$$Z_{1(s)} = Z_1 + \frac{s}{\sqrt{1+s^2}}Z_{\bar{1}}, \quad Z_{\bar{1}(s)} = Z_{\bar{1}} + \frac{s}{\sqrt{1+s^2}}Z_1.$$

Corresponding to these vector fields, we have the dual forms

$$\theta_{(s)}^1 = (1 + s^2)\theta^1 - s\sqrt{1+s^2}\theta^{\bar{1}}, \quad \theta_{(s)}^{\bar{1}} = (1 + s^2)\theta^{\bar{1}} - s\sqrt{1+s^2}\theta^1,$$

where $\theta^1 = z^2dz^1 - z^1dz^2$. Compute

$$i\theta_{(s)}^1 \wedge \theta_{(s)}^{\bar{1}} = (1 + s^2)i\theta^1 \wedge \theta^{\bar{1}} = (1 + s^2)d\hat{\theta},$$

where $d\hat{\theta} = i\theta^1 \wedge \theta^{\bar{1}}$, i.e., $h_{1\bar{1}} = 1$. Hence, from (19) we get

$$h_{1\bar{1}}^{(s)} = \frac{1}{1+s^2} \text{ and } h_{(s)}^{1\bar{1}} := (h_{1\bar{1}}^{(s)})^{-1} = 1 + s^2.$$

By taking

$$\tilde{\theta}_{(s)}^1 = \frac{1}{\sqrt{2(1+s^2)}}\theta_{(s)}^1,$$

we have $\tilde{h}_{1\bar{1}}^{(s)} = 2$. The Webster curvature R of $(J, \hat{\theta})$ is identically equal to 2. Then we should take $\omega_1^1 = -2i\hat{\theta}$ in the structure equation (5), such that $d\omega_1^1 = 2\theta^1 \wedge \theta^{\bar{1}}$. We can then determine, from the structure equation for $(J_{(s)}, \hat{\theta})$, that

$$\omega_{1(s)}^1 = -2i(1 + 2s^2)\hat{\theta}, \quad h_{(s)}^{1\bar{1}}A_{1\bar{1}(s)} = 4is\sqrt{1+s^2}, \quad R_{(s)} = 2(1 + 2s^2).$$

Dual to $\theta^1 = z^2dz^1 - z^1dz^2$, we have

$$(19) \quad Z_1 = Z_1^{S^3} = z^{\bar{2}}\frac{\partial}{\partial z^1} - z^{\bar{1}}\frac{\partial}{\partial z^2}.$$

The sub-Laplacian associated to $(J_{(s)}, \hat{\theta})$ reads

$$(20) \quad \Delta_b^{(s)} = h_{(s)}^{1\bar{1}}(Z_{1(s)}Z_{\bar{1}(s)} + Z_{\bar{1}(s)}Z_{1(s)}) = (1 + 2s^2)\Delta_b^{(0)} + 2s\sqrt{1+s^2}(Z_1^2 + Z_{\bar{1}}^2).$$

It follows that, at $s = 0$, the first- and second-order derivatives of $\Delta_b^{(s)}$ w.r.t. s are given by

$$(21) \quad -\dot{\Delta}_b = 2Z_{\bar{1}}Z_{\bar{1}} + \text{conj.}; \quad -\ddot{\Delta}_b = -4\Delta_b.$$

Moreover since $R_s = 2(1 + 2s^2)$ it follows that, still at $s = 0$

$$(22) \quad \dot{R} = 0; \quad \ddot{R} = 8.$$

We next analyze a symmetry property of Rossi spheres, that will imply in particular the symmetry of the mass in s . Consider the diffeomorphism $\iota : S^3 \rightarrow S^3$ defined by

$$(23) \quad \iota(z^1, z^2) = (iz^1, z^2),$$

which fixes the point $(0,1)$. A direct computation shows that $\iota_* Z_1^{S^3} = i Z_1^{S^3}$ and hence $\iota_* Z_{\bar{1}}^{S^3} = (-i) Z_{\bar{1}}^{S^3}$. By (19), we compute

$$(24) \quad \iota_* Z_{1(s)} = \iota_* Z_1 + \frac{s}{\sqrt{1+s^2}} \iota_* Z_{\bar{1}} = i Z_1 + \frac{s}{\sqrt{1+s^2}} (-i) Z_{\bar{1}} = i Z_{1(-s)}.$$

It follows that

$$\begin{aligned} (\iota^* J_{(-s)}) Z_{1(s)} &= \iota_*^{-1} J_{(-s)} (\iota_* Z_{1(s)}) = \iota_*^{-1} J_{(-s)} (i Z_{1(-s)}) \text{ (by (24))} = \iota_*^{-1} (-Z_{1(-s)}) \\ &= (-1)(-i) Z_{1(s)} \text{ (by the inverse of (24))} = i Z_{1(s)} = J_{(s)} Z_{1(s)}. \end{aligned}$$

Hence we have shown

$$(25) \quad J_{(s)} = \iota^* J_{(-s)}.$$

Let $v_{(s)}$ denote the conformal factor in $\check{\theta}_{(s)} = e^{2v_{(s)}} \hat{\theta}$, yielding CR normal coordinates with respect to $J_{(s)}$. It then follows that

$$v_{(s)} = \iota^* v_{(-s)}, \quad \check{\theta}_{(s)} = \iota^* \check{\theta}_{(-s)},$$

and hence $\check{G}_s = \iota^* \check{G}_{-s}$ by observing

$$(26) \quad \iota^* \hat{\theta} = \hat{\theta}.$$

Write

$$\check{G}_s = 2\rho_s^{-2} + A_s + O(\rho_s)$$

in s -CR normal coordinates near $(0,1)$. Then $\rho_s = \iota^* \rho_{-s} = \rho_{-s} \circ \iota$ and

$$A_s = \iota^* A_{-s} = A_{-s} \circ \iota = A_{-s}$$

near the point $(0,1)$. So, we have obtained

$$m(J_{(s)}, \theta_{(s)}) = 12\pi A_s = 12\pi A_{-s} = m(J_{(-s)}, \theta_{(-s)}),$$

where $\theta_{(s)} = \check{G}_{(s)}^2 \check{\theta}_{(s)}$. This property (and other related ones) will be crucial in the last section of the paper.

3. CR NORMAL COORDINATES ON ROSSI SPHERES

In this section we will find the main-order terms of CR normal coordinates on Rossi spheres. We first determine the principal term in the required conformal factor, then discuss pseudo-hermitian coordinates and finally CR normal coordinates. This will allow us to express with a good precision the Green's function of the conformal sub-Laplacian near its pole.

3.1. Conformal factor in normalized contact form on Rossi spheres. Fix $p = (0,1) \in S^3 \subseteq \mathbb{C}^2$ and consider a contact form $\check{\theta}'_{(s)} = e^{2v_{(s)}} \hat{\theta}'$, where $\hat{\theta}' = 2\hat{\theta} = i(\bar{\partial} - \partial)(|z^1|^2 + |z^2|^2)$ yielding CR normal coordinates (see Proposition 2.3) with respect to $J_{(s)}$ for $N = 4$. We are going to solve an equation for $v_{(s)}$ as in Lemma 3.11 of Jerison-Lee's paper ([JL89]). Write

$$(27) \quad v_{(s)} = v_2 + v_3 + \dots,$$

where $v_2 \in \mathcal{R}_2 \subset \mathcal{P}_2$, $v_3 \in \mathcal{P}_3$. Recall that, in the notation of [JL89], \mathcal{P}_m denotes the vector space of polynomials in (z, t) that are homogeneous of degree m in terms of parabolic dilations (for which t has homogeneity 2), and $\mathcal{Y}_m \subseteq \mathcal{P}_m$ denotes the subspace of polynomials independent of t .

First, write $v_2 \in \mathcal{R}_2$ as $v_2 = az^2 + bz\bar{z} + c\bar{z}^2$ ((z, t) being pseudo-hermitian normal coordinates for $\hat{\theta}'$ at p) satisfying

$$(28) \quad L_2 v_2 = -z^2 Q_{11} - \bar{z}^2 Q_{\bar{1}\bar{1}} - z\bar{z} Q_{1\bar{1}} - \bar{z}z Q_{\bar{1}1}; \quad L_2 = -2|z|^2(\partial_z \partial_{\bar{z}} + \partial_{\bar{z}} \partial_z) - 12,$$

where $Q_{11} = 3iA_{11}^{JL} = Q_{\bar{1}\bar{1}}$ and $Q_{1\bar{1}} = R_{1\bar{1}}^{JL} = Q_{\bar{1}1}$ are w.r.t. the Jerison-Lee coframe $\theta_{JL}^1 = \theta_{(s)}^1 / \sqrt{1+s^2}$ with $h_{1\bar{1}}^{JL} = 2$ w.r.t. $\hat{\theta}'$ by the formulas for Q_{jk} on page 315 in [JL89] and (19). We compute

$$(29) \quad \tilde{Q}_{11} = 3iA_{11(s)} = \frac{12s}{\sqrt{1+s^2}} = \tilde{Q}_{\bar{1}\bar{1}}; \quad \tilde{Q}_{1\bar{1}} = R_{1\bar{1}(s)} = h_{\bar{1}\bar{1}}^{(s)} R_{(s)} = 2\frac{1+2s^2}{1+s^2},$$

with respect to the co-frame $\theta_{(s)}^1$. A direct computation shows that

$$L_2 v_2 = -12az^2 - 12c\bar{z}^2 - 16b|z|^2,$$

where $Q_{11} = 12a$, $Q_{\bar{1}\bar{1}} = 12c$, $Q_{1\bar{1}} = Q_{\bar{1}1} = 8b$ and

$$(30) \quad a = c = s\sqrt{1+s^2}, \quad b = \frac{1}{4}(1+2s^2).$$

For v_3 , we observe that all $Q_{jk,l}$'s for j, k, l being 1 or $\bar{1}$ vanish since the space derivatives of the constant R_0 is zero. On the other hand, Q_{0k} and Q_{k0} for $k = 1$ or $\bar{1}$ also vanish since they involve space derivatives by formulas on page 315 in [JL89]. Altogether, the right hand side of the equation in Lemma 3.11 in [JL89] for $m = 3$ equals zero, so we have

$$L_3 v_3 = 0.$$

By Lemma 3.9 in [JL89], we learn that L_3 is invertible on \mathcal{P}_3 . It follows that

$$(31) \quad v_3 = 0.$$

Therefore, from (30) and (31) we get the following result.

Lemma 3.1. *In pseudo-hermitian coordinates, the conformal factor expands in homogeneous powers as*

$$(32) \quad v_{(s)} = s\sqrt{1+s^2}(z^2 + \bar{z}^2) + \frac{1}{4}(1+2s^2)|z|^2 + v_4 + \dots$$

3.2. Pseudo-hermitian coordinates on Rossi spheres. Recall that on Rossi spheres we have

$$\theta_{(s)}^1 = (1+s^2)\theta^1 - s\sqrt{1+s^2}\theta^{\bar{1}}; \quad \omega_{1(s)}^1 = -i(1+2s^2)2\theta,$$

and that pseudo-hermitian coordinates near $(0, 1)$ are defined by the equation

$$(33) \quad \nabla_{\hat{\sigma}} \hat{\sigma} = 2c\hat{T}'; \quad \sigma(0) = (0, 1),$$

where \hat{T}' is the unique vector field such that $\hat{\theta}'(\hat{T}') = 1$ and $d\hat{\theta}'(\hat{T}', \cdot) = 0$. Recall also that

$$\hat{\theta}' = i \sum_{i=1}^2 (z^i d\bar{z}^i - \bar{z}^i dz^i); \quad \hat{T}' = -\text{Im} \left(z^1 \frac{\partial}{\partial z_1} + z^2 \frac{\partial}{\partial z_2} \right) = \frac{1}{2}i \sum_{i=1}^2 \left(z^i \frac{\partial}{\partial z_i} - \bar{z}^i \frac{\partial}{\partial \bar{z}_i} \right).$$

Setting

$$\dot{\sigma} = \alpha Z_{1(s)}^{JL} + \beta Z_{\bar{1}(s)}^{JL} + \gamma \hat{T}',$$

(33) becomes

$$(34) \quad \begin{aligned} 2c\hat{T}' &= \nabla_{\dot{\sigma}} \dot{\sigma} = \left(\dot{\alpha} + \alpha \omega_{1(s)}^1(\dot{\sigma}) \right) Z_{1(s)}^{JL} + \left(\dot{\beta} + \beta \omega_{\bar{1}(s)}^{\bar{1}}(\dot{\sigma}) \right) Z_{\bar{1}(s)}^{JL} + \dot{\gamma} \hat{T}' \\ &= (\dot{\alpha} - i\alpha(1+2s^2)\gamma) Z_{1(s)}^{JL} + (\dot{\beta} + i\beta(1+2s^2)\gamma) Z_{\bar{1}(s)}^{JL} + \dot{\gamma} \hat{T}'. \end{aligned}$$

If τ parametrizes the curve σ , the above formulas imply that

$$\gamma = 2c\tau; \quad \frac{\dot{\alpha}}{\alpha} = i(1+2s^2)\gamma; \quad \frac{\dot{\beta}}{\beta} = -i(1+2s^2)\gamma,$$

which in turn yields

$$\alpha(t) = \alpha(0)e^{ic(1+2s^2)\tau^2}; \quad \beta(t) = \beta(0)e^{-ic(1+2s^2)\tau^2}.$$

Therefore we obtained

$$\dot{\sigma} = \alpha(0)e^{ic(1+2s^2)\tau^2} Z_{1(s)}^{JL} + \beta(0)e^{-ic(1+2s^2)\tau^2} Z_{\bar{1}(s)}^{JL} + 2c\tau \hat{T}'.$$

Recall also that $Z_{1(s)} = Z_{1(0)} + \frac{s}{\sqrt{1+s^2}} Z_{\bar{1}(0)}$. Hence we need to solve for

$$\begin{aligned}\dot{z}_1(\tau) &= dz_1(\dot{\sigma}(\tau)) = \alpha(0)e^{ic(1+2s^2)\tau^2} dz_1(Z_{1(s)}^{JL}) + \beta(0)e^{-ic(1+2s^2)\tau^2} dz_1(Z_{\bar{1}(s)}^{JL}) + 2c\tau dz_1(\hat{T}') \\ &= \sqrt{1+s^2} \left(\alpha(0)e^{i\delta\tau^2} \bar{z}_2(\tau) + \beta(0)e^{-i\delta\tau^2} \frac{s}{\sqrt{1+s^2}} \bar{z}_2(\tau) \right) + c\tau iz_1(\tau),\end{aligned}$$

where $\delta = c(1+2s^2)$. Similarly, we obtain

$$\dot{z}_2(\tau) = \sqrt{1+s^2} \left(-\alpha(0)e^{i\delta\tau^2} \bar{z}_1(\tau) - \beta(0)e^{-i\delta\tau^2} \frac{s}{\sqrt{1+s^2}} \bar{z}_1(\tau) \right) + c\tau iz_2(\tau).$$

Once we will solve for this system, the pseudo-hermitian coordinates will be given by the map

$$(35) \quad (z, \bar{z}, t) = (\alpha(0)\tau, \beta(0)\tau, c\tau^2) \longmapsto (0, 1) + \int_0^t \dot{\sigma}(\eta) d\eta.$$

Setting for simplicity

$$\begin{aligned}A_0 &= \alpha(0); & B_0 &= \beta(0) \frac{s}{\sqrt{1+s^2}}; & C_0 &= 2c; \\ F_0(\tau) &:= \sqrt{1+s^2}(A_0 e^{i\delta\tau^2} + B_0 e^{-i\delta\tau^2}) = f_0(\tau) + ig_0(\tau),\end{aligned}$$

we have then the system of ODEs

$$\dot{z}_1(\tau) = F_0(\tau) \bar{z}_2(\tau) + iC_0\tau z_1(\tau); \quad \dot{z}_2(\tau) = -F_0(\tau) \bar{z}_1(\tau) + iC_0\tau z_2(\tau),$$

which in real form becomes

$$\begin{cases} \dot{x}_1(\tau) = f_0(\tau)x_2(\tau) + g_0(\tau)y_2(\tau) - C_0\tau y_1(\tau); \\ \dot{y}_1(\tau) = g_0(\tau)x_2(\tau) - f_0(\tau)y_2(\tau) + C_0\tau x_1(\tau); \\ \dot{x}_2(\tau) = -f_0(\tau)x_1(\tau) - g_0(\tau)y_1(\tau) - C_0\tau y_2(\tau); \\ \dot{y}_2(\tau) = f_0(\tau)y_1(\tau) - g_0(\tau)x_1(\tau) + C_0\tau x_2(\tau). \end{cases}$$

We rewrite this system as

$$\dot{\mathfrak{X}}(\tau) = \mathfrak{A}(\tau)\mathfrak{X}(\tau),$$

where

$$\mathfrak{A}(\tau) = \begin{pmatrix} 0 & -C_0\tau & f_0(\tau) & g_0(\tau) \\ C_0\tau & 0 & g_0(\tau) & -f_0(\tau) \\ -f_0(\tau) & -g_0(\tau) & 0 & -C_0\tau \\ -g_0(\tau) & f_0(\tau) & C_0\tau & 0 \end{pmatrix}.$$

We can Taylor-expand the solution to an arbitrary order in τ . Differentiating the above ODE we obtain

$$\ddot{\mathfrak{X}}(\tau) = \dot{\mathfrak{A}}(\tau)\mathfrak{X}(\tau) + \mathfrak{A}(\tau)^2\mathfrak{X}(\tau); \quad \ddot{\mathfrak{X}}(\tau) = \ddot{\mathfrak{A}}(\tau)\mathfrak{X}(\tau) + (\mathfrak{A}(\tau)\dot{\mathfrak{A}}(\tau) + 2\dot{\mathfrak{A}}(\tau)\mathfrak{A}(\tau))\mathfrak{X}(\tau) + \mathfrak{A}(\tau)^3\mathfrak{X}(\tau).$$

We have that

$$\begin{aligned}\mathfrak{A}(0) &= \begin{pmatrix} 0 & 0 & \operatorname{Re}A_0 + \operatorname{Re}B_0 & \operatorname{Im}A_0 + \operatorname{Im}B_0 \\ 0 & 0 & \operatorname{Im}A_0 + \operatorname{Im}B_0 & -\operatorname{Re}A_0 - \operatorname{Re}B_0 \\ -\operatorname{Re}A_0 - \operatorname{Re}B_0 & -\operatorname{Im}A_0 - \operatorname{Im}B_0 & 0 & 0 \\ -\operatorname{Im}A_0 - \operatorname{Im}B_0 & \operatorname{Re}A_0 + \operatorname{Re}B_0 & 0 & 0 \end{pmatrix} \\ \dot{\mathfrak{A}}(0) &= \begin{pmatrix} 0 & -C_0 & 0 & 0 \\ C_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_0 \\ 0 & 0 & C_0 & 0 \end{pmatrix}; \\ \ddot{\mathfrak{A}}(0) &= \begin{pmatrix} 0 & 0 & 2d(\operatorname{Im}B_0 - \operatorname{Im}A_0) & -2d(\operatorname{Re}B_0 - \operatorname{Re}A_0) \\ 0 & 0 & -2d(\operatorname{Re}B_0 - \operatorname{Re}A_0) & 2d(\operatorname{Im}A_0 - \operatorname{Im}B_0) \\ 2d(\operatorname{Im}A_0 - \operatorname{Im}B_0) & 2d(\operatorname{Re}B_0 - \operatorname{Re}A_0) & 0 & 0 \\ 2d(\operatorname{Re}B_0 - \operatorname{Re}A_0) & 2d(\operatorname{Im}B_0 - \operatorname{Im}A_0) & 0 & 0 \end{pmatrix}.\end{aligned}$$

In conclusion, looking at the first three terms in the Taylor expansion of $\mathfrak{X}(\tau)$ near $(0, 1)$ we find that

$$\mathfrak{X}(\tau) = \left(\begin{array}{c} \tilde{\tau}(\operatorname{Re}A_0 + \operatorname{Re}B_0) \\ \tilde{\tau}(\operatorname{Im}A_0 + \operatorname{Im}B_0) \\ 1 - \frac{1}{2}\tilde{\tau}^2 \left((\operatorname{Im}A_0 + \operatorname{Im}B_0)^2 + (\operatorname{Re}A_0 + \operatorname{Re}B_0)^2 \right) \\ \frac{C_0\tilde{\tau}^2}{2} \frac{1}{2(1+s^2)} \end{array} \right) + o(\tau^2); \quad \tilde{\tau} = \sqrt{1+s^2}\tau.$$

Recalling (35), we then obtain the following result.

Lemma 3.2. *Pseudo-hermitian coordinates near $(0, 1)$ on Rossi spheres w.r.t. $\hat{\theta}' = 2\hat{\theta}$ are given by the following map*

$$(z, \bar{z}, t) \mapsto \left(\begin{array}{c} \sqrt{1+s^2} \left(z + \frac{s}{\sqrt{1+s^2}} \bar{z} \right) \\ 1 - \frac{1}{2}(1+s^2) \left| z + \frac{s}{\sqrt{1+s^2}} \bar{z} \right|^2 + i \frac{t}{2} \end{array} \right) + o(\rho^2).$$

Inverting in the first component, we have in particular that

$$(36) \quad z = \frac{1}{\sqrt{1+s^2}}(1+s^2) \left(z_1 - \frac{s}{\sqrt{1+s^2}} \bar{z}_1 \right) + o(\rho^2).$$

3.3. CR normal coordinates. Recalling (27), Lemma 3.1 and using (36), we get

$$v_2 = \frac{1}{4} \left(3(z_1^2 + \bar{z}_1^2)s\sqrt{s^2+1}(2s^2+1) - |z_1|^2(12s^4+12s^2-1) \right) = A_1(z_1^2 + \bar{z}_1^2) + B_1|z_1|^2,$$

where

$$(37) \quad A_1 = \frac{1}{8}2(s^2+1)^{1/2}3s(2s^2+1); \quad B_1 = -\frac{1}{8}2(12s^4+12s^2-1).$$

Recall that also

$$\hat{\theta}_{(0)}^1 = z^2 dz^1 - z^1 dz^2; \quad \hat{\theta}' = i \sum_{i=1}^2 (z^i d\bar{z}^i - \bar{z}^i dz^i),$$

and that

$$\hat{\theta}_{(s)}^1 = (1+s^2)\hat{\theta}_{(0)}^1 - s\sqrt{1+s^2}\hat{\theta}_{(0)}^{\bar{1}}.$$

Conformally changing the contact form and recalling Appendix 1.1.1 in [CMY17], we have that $\hat{\theta}_{(s)}^1$ transforms as

$$(38) \quad \hat{\theta}_{(s)}^1 \mapsto e^v(\hat{\theta}_{(s)}^1 + 2iv^1\hat{\theta}'),$$

where $(d\hat{\theta}' = 2i\hat{\theta}_{(0)}^1 \wedge \hat{\theta}_{(0)}^{\bar{1}})$

$$v^1 = h_{(s)}^{\bar{1}\bar{1}} \hat{Z}_{\bar{1}(s)} v = \frac{(1+s^2)}{2} \hat{Z}_{\bar{1}(s)} v.$$

By computing explicitly, it turns out that

$$(v_2)^1 = \left(\frac{(s^2+1)}{2} \frac{\bar{z}_2 s(2A_1 z_1 + B_1 \bar{z}_1)}{\sqrt{s^2+1}} + \frac{(s^2+1)}{2} z_2 (2A_1 \bar{z}_1 + B_1 z_1) \right),$$

which can be written as

$$(39) \quad (v_2)^1 = A_2 z_1 z_2 + B_2 z_1 \bar{z}_2 + C_2 \bar{z}_1 z_2 + D_2 \bar{z}_1 \bar{z}_2,$$

with

(40)

$$A_2 = \frac{1}{2}(s^2+1)B_1, \quad B_2 = sA_1\sqrt{s^2+1}, \quad C_2 = A_1(s^2+1), \quad D_2 = \frac{1}{2}sB_1\sqrt{s^2+1}.$$

Up to higher order terms, we have that

$$(v_2)^1 = (A_2 + B_2)z_1 + (C_2 + D_2)\bar{z}_1.$$

Taylor expanding (38), up to higher-order terms $\hat{\theta}_{(s)}^1$ transforms into

$$\hat{\theta}_{(s)}^1 + \left[v_2 \hat{\theta}_{(s)}^1 + 2i(v_2)^1 \hat{\theta}' \right].$$

We now multiply by a complex unit factor $e^{i\psi}$, and impose a closeness condition on $e^{i\psi}$ multiplied by the latter form, up to higher-order terms, since by Proposition 2.4 it should be approximately a constant multiple of $d\tilde{z}_{CR}$, up to h.o.t.. We then find

$$\begin{aligned}
(41) \quad 0 &= d \left\{ e^{i\psi} \left(\hat{\theta}_{(s)}^1 + \left[v_2 \hat{\theta}_{(s)}^1 + 2i(v_2)^1 \hat{\theta}' \right] \right) \right\} \\
&= e^{i\psi} \left\{ id\psi \wedge \hat{\theta}_{(s)}^1 + i d\psi \wedge \left[v_2 \hat{\theta}_{(s)}^1 + 2i(v_2)^1 \hat{\theta}' \right] \right\} \\
&+ e^{i\psi} \left\{ d\hat{\theta}_{(s)}^1 + dv_2 \wedge \hat{\theta}_{(s)}^1 + v_2 d\hat{\theta}_{(s)}^1 + 2id((v_2)^1) \wedge \hat{\theta}' + 2i(v_2)^1 d\hat{\theta}' \right\} + h.o.t..
\end{aligned}$$

We also have

$$d\hat{\theta}_{(s)}^1 = 2(1+s^2)dz^2 \wedge dz^1 - 2s\sqrt{1+s^2}d\bar{z}^2 \wedge d\bar{z}^1.$$

We expand $d\psi$ and $\hat{\theta}_{(s)}^1$ in homogeneous powers of the (z, t) coordinates (w.r.t. parabolic scaling, including differentials) as follows

$$d\psi = (d\psi)_0 + (d\psi)_1 + (d\psi)_2 + \dots$$

Taylor-expanding the above system up to order one we obtain the relations

$$(42) \quad \begin{cases} (d\psi)_0 \wedge \hat{\theta}_{(s)}^1 = 0; \\ i(d\psi)_1 \wedge \hat{\theta}_{(s)}^1 + d\hat{\theta}_{(s)}^1 + (dv_2) \wedge \hat{\theta}_{(s)}^1 + 2i(v_2)^1 d\hat{\theta}' + 2i(d(v_2)^1) \wedge \hat{\theta}' = 0. \end{cases}$$

The first component is easy to solve setting $(d\psi)_0 = \mu \hat{\theta}_{(s)}^1$ for some $\mu \in \mathbb{R}$.

For the second component, recall that we have

$$\hat{\omega}_{1(s)}^1 = -i(1+2s^2)\hat{\theta}'; \quad \hat{A}_{1(s)}^1 = 2is\sqrt{1+s^2}.$$

It then follows

$$d\hat{\theta}_{(s)}^1 = \hat{\theta}_{(s)}^1 \wedge \hat{\omega}_{1(s)}^1 + \hat{A}_{1(s)}^1 \hat{\theta}' \wedge \hat{\theta}_{(s)}^{\bar{1}} = -i(1+2s^2)\hat{\theta}_{(s)}^1 \wedge \hat{\theta}' + 2is\sqrt{1+s^2} \hat{\theta}' \wedge \hat{\theta}_{(s)}^{\bar{1}}.$$

Moreover we have

$$d\hat{\theta}' = 2ih_{1\bar{1}(s)} \hat{\theta}_{(s)}^1 \wedge \hat{\theta}_{(s)}^{\bar{1}} = \frac{2i}{1+s^2} \hat{\theta}_{(s)}^1 \wedge \hat{\theta}_{(s)}^{\bar{1}},$$

and that (up to $\hat{\theta}'$)

$$d(v_2)^1 = \hat{Z}_{1(s)}(v_2)^1 \hat{\theta}_{(s)}^1 + \hat{Z}_{\bar{1}(s)}(v_2)^1 \hat{\theta}_{(s)}^{\bar{1}}.$$

By the above expression of $(v_2)^1$ and (39), this becomes

$$\begin{aligned}
d(v_2)^1 &= \left[(s^2 + 1/2)B_1(|z_2|^2 - |z_1|^2) + B_2\bar{z}_2^2 - C_2\bar{z}_1^2 + s\sqrt{1+s^2}A_1z_2^2 - s^2A_1z_1^2 \right] \hat{\theta}_{(s)}^1 \\
&+ \left[s\sqrt{1+s^2}B_1(|z_2|^2 - |z_1|^2) - B_2z_1^2 + s^2A_1\bar{z}_2^2 - s\sqrt{1+s^2}A_1\bar{z}_1^2 + C_2z_2^2 \right] \hat{\theta}_{(s)}^{\bar{1}} \quad \text{mod } \hat{\theta}'.
\end{aligned}$$

We next write

$$(d\psi)_1 = (A_3z_1 + B_3\bar{z}_1)\hat{\theta}_{(s)}^1 + (\bar{A}_3\bar{z}_1 + \bar{B}_3z_1)\hat{\theta}_{(s)}^{\bar{1}} + C_3\hat{\theta}'; \quad \bar{A}_3 = -A_3, \quad \bar{B}_3 = -B_3.$$

The $\hat{\theta}_{(s)}^{\bar{1}} \wedge \hat{\theta}_{(s)}^1$ -component of the second equation in (42) is given by

$$i(\bar{A}_3\bar{z}_1 + \bar{B}_3z_1) + \frac{6}{1+s^2}(v_2)^1 = 0.$$

This determines A_3 and B_3 by

$$i\bar{A}_3 + \frac{6}{1+s^2}(C_2 + D_2) = 0; \quad i\bar{B}_3 + \frac{6}{1+s^2}(A_2 + B_2) = 0,$$

giving

$$(43) \quad A_3 = -\frac{3}{8} \frac{2is}{\sqrt{1+s^2}}(7+6s^2); \quad B_3 = -\frac{3}{8}2i(1-6s^2).$$

Next, the $\hat{\theta}' \wedge \hat{\theta}_{(s)}^1$ component gives

$$iC_3 + i(1+2s^2) + \hat{T}v_2 - 2i\hat{Z}_{1(s)}(v_2)^1 = 0.$$

Finally, the $\hat{\theta}' \wedge \hat{\theta}_s^\top$ -component of the second equation in (42) is given by

$$2is\sqrt{1+s^2} - 2i\hat{Z}_{\overline{1}(s)}(v_2)^1 = 0.$$

This is true because, as one can check

$$\hat{Z}_{\overline{1}(s)}(v_2)^1 = (C_2 + D_2) + \frac{s}{\sqrt{1+s^2}}(A_2 + B_2); \quad \hat{Z}_{1(s)}(v_2)^1 = (A_2 + B_2) + \frac{s}{\sqrt{1+s^2}}(C_2 + D_2).$$

By a direct computation it follows that

$$\hat{Z}_{\overline{1}(s)}(v_2)^1 = s(s^2 + 1)^{1/2}; \quad \hat{Z}_{1(s)}(v_2)^1 = \frac{1}{8}(2s^2 + 1).$$

These also imply

$$C_3 = -\frac{3}{4}(1 + 2s^2) = -\frac{3}{2}s^2 - \frac{3}{4}.$$

Let us now try to integrate for the phase ψ . There holds

$$\hat{\theta}_{(s)}^1 = (1 + s^2)dz_1 - s\sqrt{1 + s^2}d\bar{z}_1 + h.o.t.; \quad \hat{\theta}' = i[(z_1d\bar{z}_1 - \bar{z}_1dz_1) + (d\bar{z}_2 - dz_2)] + h.o.t..$$

In this way, we have that $(d\psi)_1$ becomes

$$\begin{aligned} & \left[(A_3z_1 + B_3\bar{z}_1)(1 + s^2) + (A_3\bar{z}_1 + B_3z_1)s\sqrt{1 + s^2} - iC_3\bar{z}_1 \right] dz_1 + conj. - iC_3(dz_2 - d\bar{z}_2) \\ &= \left\{ \left[(1 + s^2)A_3 + s\sqrt{1 + s^2}B_3 \right] z_1 + \left[(1 + s^2)B_3 + s\sqrt{1 + s^2}A_3 - iC_3 \right] \bar{z}_1 \right\} dz_1 + conj. \\ &- iC_3(dz_2 - d\bar{z}_2). \end{aligned}$$

Since $(1 + s^2)B_3 + s\sqrt{1 + s^2}A_3 - iC_3 = 0$, we get

$$(d\psi)_1 = A_4z_1dz_1 + B_4dz_2 + conj.,$$

with

$$(44) \quad A_4 = -6is\sqrt{1 + s^2}; \quad B_4 = -iC_3 = \frac{3}{4}i(1 + 2s^2).$$

Integrating, we find

$$(\psi)_2 = \frac{1}{2}A_4z_1^2 + B_4z_2 + conj..$$

Taylor-expanding, we then get

$$(45) \quad dz'_{CR} = \hat{\theta}_{(s)}^1(1 + v_2 + i\psi_2) + 2i(v_2)^1\hat{\theta}' + h.o.t..$$

Writing the 0-th and 2nd order terms of the right hand side, we obtain

$$\begin{aligned} & \left[1 + A_1(z_1^2 + \bar{z}_1^2) + \frac{1}{2}iA_4(z_1^2 - \bar{z}_1^2) + B_1|z_1|^2 + iB_4(z_2 - \bar{z}_2) \right] \\ & \left\{ (1 + s^2)z_2dz_1 - s\sqrt{1 + s^2}z_2d\bar{z}_1 - (1 + s^2)z_1dz_2 + s\sqrt{1 + s^2}\bar{z}_1d\bar{z}_2 \right\} \\ & - 2[(A_2 + B_2)z_1 + (C_2 + D_2)\bar{z}_1][z_1d\bar{z}_1 - \bar{z}_1dz_1 + d\bar{z}_2 - dz_2]. \end{aligned}$$

Further expanding this, gives

$$(46) \quad \begin{aligned} & (1 + s^2)z_2dz_1 - s\sqrt{1 + s^2}z_2d\bar{z}_1 - (1 + s^2)z_1dz_2 + s\sqrt{1 + s^2}\bar{z}_1d\bar{z}_2 \\ & + [A_1(z_1^2 + \bar{z}_1^2) + \frac{1}{2}iA_4(z_1^2 - \bar{z}_1^2) + B_1|z_1|^2 + iB_4(z_2 - \bar{z}_2)] \\ & [(1 + s^2)dz_1 - s\sqrt{1 + s^2}d\bar{z}_1] - 2[(A_2 + B_2)z_1 + (C_2 + D_2)\bar{z}_1][z_1d\bar{z}_1 - \bar{z}_1dz_1 + d\bar{z}_2 - dz_2]. \end{aligned}$$

We next set

$$w = z_2 - \bar{z}_2,$$

and rewrite the terms involving z_2 as

$$(47) \quad \begin{aligned} z_2 &= 1 + \frac{1}{2}w - \frac{1}{2}|z_1|^2; & \bar{z}_2 &= 1 - \frac{1}{2}w - \frac{1}{2}|z_1|^2; \\ dz_2 &= \frac{1}{2}dw - \frac{1}{2}(\bar{z}_1dz_1 + z_1d\bar{z}_1); & d\bar{z}_2 &= -\frac{1}{2}dw - \frac{1}{2}(\bar{z}_1dz_1 + z_1d\bar{z}_1). \end{aligned}$$

Write (46) as

$$(48) \quad \begin{aligned} & C_6(z_2 dz_1 - z_1 dz_2) + D_6(\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2) \\ & + [A_1(z_1^2 + \bar{z}_1^2) + \frac{1}{2}iA_4(z_1^2 - \bar{z}_1^2) + B_1|z_1|^2 + iB_4w][C_8 dz_1 + D_8 d\bar{z}_1] \\ & - 2[(A_2 + B_2)z_1 + (C_2 + D_2)\bar{z}_1][z_1 d\bar{z}_1 - \bar{z}_1 dz_1 - dw], \end{aligned}$$

where

$$(49) \quad C_6 = C_8 = 1 + s^2; \quad D_6 = D_8 = -s\sqrt{1 + s^2}.$$

We now substitute $C_6(z_2 dz_1 - z_1 dz_2) + D_6(\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2) = C_6[(1 + \frac{1}{2}w)dz_1 + \frac{1}{2}z_1^2 d\bar{z}_1 - \frac{1}{2}z_1 dw] + D_6[(1 - \frac{1}{2}w)d\bar{z}_1 + \frac{1}{2}\bar{z}_1^2 dz_1 + \frac{1}{2}\bar{z}_1 dw]$ into (48) and collect terms involving w as follows:

$$(50) \quad \begin{aligned} & (iB_4C_8 + \frac{1}{2}C_6)w dz_1 + [2(A_2 + B_2) - \frac{1}{2}C_6]z_1 dw \\ & + (iB_4D_8 - \frac{1}{2}C_6)w d\bar{z}_1 + [2(C_2 + D_2) + \frac{1}{2}D_6]\bar{z}_1 dw. \end{aligned}$$

A direct computation shows that

$$(51) \quad iB_4C_8 + \frac{1}{2}C_6 = 2(A_2 + B_2) - \frac{1}{2}C_6 = (1 + s^2)(-\frac{3}{2}s^2 - \frac{1}{4}).$$

Similarly, we have

$$(52) \quad iB_4D_8 - \frac{1}{2}C_6 = 2(C_2 + D_2) + \frac{1}{2}D_6 = s\sqrt{1 + s^2}(\frac{3}{2}s^2 + \frac{5}{4}).$$

In view of (51) and (52) we can write (50) as

$$d[(1 + s^2)(-\frac{3}{2}s^2 - \frac{1}{4})z_1 w + s\sqrt{1 + s^2}(\frac{3}{2}s^2 + \frac{5}{4})\bar{z}_1 w].$$

On the other hand, we can write terms only involving z_1 and \bar{z}_1 in (48) as

$$(K_{11}z_1^2 + K_{\bar{1}\bar{1}}\bar{z}_1^2 + K_{1\bar{1}}|z_1|^2)dz_1 + (N_{11}z_1^2 + N_{\bar{1}\bar{1}}\bar{z}_1^2 + N_{1\bar{1}}|z_1|^2)d\bar{z}_1,$$

where

$$\begin{aligned} K_{11} &= (A_1 + \frac{1}{2}iA_4)C_8; & K_{\bar{1}\bar{1}} &= \frac{1}{2}D_6 + (A_1 - \frac{1}{2}iA_4)C_8 + 2(C_2 + D_2); \\ K_{1\bar{1}} &= B_1C_8 + 2(A_2 + B_2); & N_{11} &= \frac{1}{2}C_6 + (A_1 + \frac{1}{2}iA_4)D_8 - 2(A_2 + B_2); \\ N_{\bar{1}\bar{1}} &= (A_1 - \frac{1}{2}iA_4)D_8; & N_{1\bar{1}} &= B_1D_8 - 2(C_2 + D_2). \end{aligned}$$

Observe that

$$(53) \quad K_{\bar{1}\bar{1}} = \frac{1}{2}N_{\bar{1}\bar{1}} = s\sqrt{1 + s^2}(\frac{3}{2}s^4 + \frac{3}{4}s^2 - 1), \quad N_{11} = \frac{1}{2}K_{11} = (1 + s^2)(-\frac{3}{2}s^4 - \frac{9}{4}s^2 + \frac{1}{4}),$$

and

$$(54) \quad K_{11} = s(1 + s^2)^{3/2}(\frac{3}{2}s^2 + \frac{15}{4}), \quad N_{\bar{1}\bar{1}} = -s^2(1 + s^2)(\frac{3}{2}s^2 - \frac{9}{4}).$$

In view of (53) and (54), we can express (53) as

$$(55) \quad d\{\frac{1}{3}K_{11}z_1^3 + \frac{1}{3}N_{\bar{1}\bar{1}}\bar{z}_1^3 + K_{1\bar{1}}\bar{z}_1^2 z_1 + N_{11}z_1^2 \bar{z}_1\}.$$

Altogether, from (53) and (55) we obtain \tilde{z}_{CR} (see (45)) as follows:

$$(56) \quad \begin{aligned} \tilde{z}_{CR} &= (1 + s^2)z_1 - s\sqrt{1 + s^2}\bar{z}_1 + (1 + s^2)(-\frac{3}{2}s^2 - \frac{1}{4})z_1 w + s\sqrt{1 + s^2}(\frac{3}{2}s^2 + \frac{5}{4})\bar{z}_1 w \\ & + s(1 + s^2)^{3/2}(\frac{1}{2}s^2 + \frac{5}{4})z_1^3 - s^2(1 + s^2)(\frac{1}{2}s^2 - \frac{3}{4})\bar{z}_1^3 + s\sqrt{1 + s^2}(\frac{3}{2}s^4 + \frac{3}{4}s^2 - 1)z_1^2 \bar{z}_1 \\ & + (1 + s^2)(-\frac{3}{2}s^4 - \frac{9}{4}s^2 + \frac{1}{4})z_1^2 \bar{z}_1 + h.o.t.. \end{aligned}$$

The CR normal coordinate z'_{CR} w.r.t. Jerison-Lee's frame reads

$$z'_{CR} = \frac{\tilde{z}_{CR}}{\sqrt{1+s^2}}.$$

We want next to determine the t -component of CR normal coordinates. Recall the definition of $\hat{\theta}$ and (47): after some cancellations one can check that

$$\hat{\theta}' = i \left\{ z_1 d\bar{z}_1 - \bar{z}_1 dz_1 - dw + \frac{1}{2}|z_1|^2 dw - \frac{1}{2}w(\bar{z}_1 dz_1 + z_1 d\bar{z}_1) \right\}.$$

We now need to consider the conformal change of contact form

$$\check{\theta}' = e^{2v}\hat{\theta}' = (1 + 2v_2 + \dots)\hat{\theta}'.$$

Recalling that $v_2 = A_1(z_1^2 + \bar{z}_1^2) + B_1|z_1|^2$ we obtain that

$$\check{\theta}' = (1 + 2A_1(z_1^2 + \bar{z}_1^2) + 2B_1|z_1|^2) i \left\{ \left(z_1 - \frac{1}{2}wz_1 \right) d\bar{z}_1 - \left(\bar{z}_1 + \frac{1}{2}w\bar{z}_1 \right) dz_1 - \left(1 - \frac{1}{2}|z_1|^2 \right) dw \right\} + h.o.t..$$

From straightforward computations one finds

$$\begin{aligned} \check{\theta}' &= i \left(z_1 + 2A_1(z_1^2 + \bar{z}_1^2)z_1 + 2B_1|z_1|^2z_1 - \frac{1}{2}wz_1 \right) d\bar{z}_1 \\ &\quad - i \left(\bar{z}_1 + 2A_1(z_1^2 + \bar{z}_1^2)\bar{z}_1 + 2B_1|z_1|^2\bar{z}_1 + \frac{1}{2}w\bar{z}_1 \right) dz_1 \\ &\quad - i \left(1 + 2A_1(z_1^2 + \bar{z}_1^2) + 2B_1|z_1|^2 - \frac{1}{2}|z_1|^2 \right) dw + h.o.t.. \end{aligned}$$

Therefore, from (56) we deduce

$$\begin{aligned} d\tilde{z}_{CR} &= (1+s^2)dz_1 - s\sqrt{1+s^2}d\bar{z}_1 + (1+s^2) \left(-\frac{3}{2}s^2 - \frac{1}{4} \right) (wz_1 + z_1dw) \\ &\quad + s\sqrt{1+s^2} \left(\frac{3}{2}s^2 + \frac{5}{4} \right) (w\bar{z}_1 + \bar{z}_1dw) \\ &\quad + K_{11}z_1^2dz_1 + N_{\bar{1}\bar{1}}\bar{z}_1^2d\bar{z}_1 + K_{\bar{1}\bar{1}}(2\bar{z}_1z_1d\bar{z}_1 + \bar{z}_1^2dz_1) + N_{11}(2z_1\bar{z}_1dz_1 + z_1^2d\bar{z}_1) + h.o.t.. \end{aligned}$$

One can then expand $\check{\theta}' + i\tilde{z}'_{CR}d\tilde{z}'_{CR} - i\tilde{z}'_{CR}d\bar{\tilde{z}}'_{CR}$ to find that

$$(57) \quad t'_{CR} = -iw(1 + 1/2|z_1|^2) + is|z_1|^2(z_1^2 - \bar{z}_1^2) + is^2(\bar{z}_1^4 - z_1^4) + h.o.t..$$

We can summarize the above discussion into the following result.

Proposition 3.3. *The CR normal coordinates on Rossi spheres w.r.t. $\check{\theta}' = e^{2v}\hat{\theta}'$ are given by the formulas $z'_{CR} = \frac{\tilde{z}_{CR}}{\sqrt{1+s^2}}$, with \tilde{z}_{CR} as in (56) and t'_{CR} as in (57).*

We next collect some useful formulas derived from the latter proposition. Taylor-expanding z'_{CR} one finds

$$|z'_{CR}|^2 = |z_1|^2 \left(1 + \frac{1}{2}|z_1|^2 \right) - s(z_1^2 + \bar{z}_1^2 + w(z_1^2 - \bar{z}_1^2)) + \frac{1}{2}s^2(4|z_1|^2 - 4|z_1|^4 - z_1^4 - \bar{z}_1^4) + h.o.t.,$$

while taking its square we obtain

$$(58) \quad \begin{aligned} |z'_{CR}|^4 &= |z_1|^4(1 + |z_1|^2) - s|z_1|^2((z_1^2 + \bar{z}_1^2)(2 + |z_1|^2) + 2w(z_1^2 - \bar{z}_1^2)) \\ &\quad + s^2((z_1^4 + \bar{z}_1^4)(1 - |z_1|^2) + 2|z_1|^4(3|z_1|^2 - 1) + 2w(z_1^4 - \bar{z}_1^4)) + h.o.t.. \end{aligned}$$

The square of t'_{CR} is given by

$$(t'_{CR})^2 = -w^2(1 + |z_1|^2) + 2sw|z_1|^2(z_1^2 - \bar{z}_1^2) + 2s^2w(\bar{z}_1^4 - z_1^4) + h.o.t..$$

Summing the latter formula and (58) we obtain that, up to higher-order terms

$$\begin{aligned} (\rho'_{CR})^4 &= (1 + |z_1|^2)(|z_1|^4 - w^2) - s|z_1|^2(z_1^2 + \bar{z}_1^2)(2 + |z_1|^2) \\ &\quad + s^2[(z_1^4 + \bar{z}_1^4)(1 - |z_1|^2) + 2|z_1|^4(3 - |z_1|^2)]. \end{aligned}$$

It is also useful to expand the quantity $e^{v_2}(\rho'_{CR})^{-2}$, related to the conformal covariance for the Green's function, which up to higher-order terms is given by

$$(59) \quad e^{v_2}(\rho'_{CR})^{-2} = \frac{|z_1|^2 + 4}{4\sqrt{(|z_1|^2 + 1)(|z_1|^4 - w^2)}} + s \frac{(z_1^2 + \bar{z}_1^2)(12|z_1|^4 + 8|z_1|^2 - 6w^2)}{8(|z_1|^2 + 1)(|z_1|^4 - w^2)^{3/2}} \\ + s^2 \left[\frac{(z_1^4 + \bar{z}_1^4)(20|z_1|^6 + 8|z_1|^4 + 4w^2 - 5|z_1|^2 w^2) - 4|z_1|^{10} + 58|z_1|^6 w^2 + 24|z_1|^4 w^2 - 24|z_1|^2 w^4}{8(|z_1|^2 + 1)(|z_1|^4 - w^2)^{5/2}} \right].$$

Note that w.r.t. the contact from $\hat{\theta} = \frac{1}{2}\hat{\theta}'$ the CR normal coordinates and the *Heisenberg distance* would be $(z_{CR}, t_{CR}) = \left(\frac{z_{CR}}{\sqrt{2}}, \frac{t_{CR}}{2}\right)$ and $\rho_{CR} = \frac{\rho_{CR}}{\sqrt{2}}$ respectively.

4. PROOF OF THEOREM 1.1

In this section we determine the Green's function for the conformal sub-Laplacian on Rossi spheres, up to an error of order s^3 . This allows to estimate the mass of Rossi spheres, which turns out to be negative for $s \neq 0$ small. This is done by comparing the expression of the Green's function in CR-normal coordinates, locally near the pole, and by deriving a formal expansion in s globally away from the pole with respect to the standard coordinates (z_1, z_2) of S^3 .

4.1. Formal expansion of the Green's function in powers of s . Let L_s denote the conformal sub-Laplacian for the $J_{(s)}$ -structure on S^3 . For $s = 0$, the fundamental solution of $L_0 G_0 = 64\pi^2 \delta_p$ with pole at $p = (0, 1)$ is given by

$$(60) \quad G_0 = 2((1 - z_2)(1 - \bar{z}_2))^{-\frac{1}{2}}.$$

We next solve formally, up to an error $O(s^3)$, $L_s G_s = 0$ away from p in power series of s in the form

$$(61) \quad G_s = G_0 + sG_1 + \frac{1}{2}s^2(G_2 + \alpha G_0 - G_3),$$

where G_1, G_2 are suitable explicit singular functions near p , $\alpha \in \mathbb{R}$ and G_3 is a Hölder continuous function near p for which we would need to determine only $G_3(p)$. We chose to expand the second-order term including separately αG_0 : this will be useful later in order to fix the distributional component of the solution at the pole p . In principle this should be done also for the first-order term, but by our choice of G_1 this further correction will not be necessary.

For the above expansion, the following formulas will be used

$$(62) \quad Z_1 Z_1 z_1^a (1 - z_2)^b (1 - \bar{z}_2)^c = z_1^{a-2} (1 - z_2)^{b-2} (1 - \bar{z}_2)^c (b(|z_2|^2 - 1)(2a\bar{z}_2(z_2 - 1) - |z_2|^2 + 1) \\ + (a - 1)a\bar{z}_2^2(z_2 - 1)^2 + b^2(|z_2|^2 - 1)^2);$$

$$(63) \quad Z_{\bar{1}} Z_{\bar{1}} z_1^a (1 - z_2)^b (1 - \bar{z}_2)^c = (c - 1)c z_1^{a+2} (1 - z_2)^b (1 - \bar{z}_2)^{c-2};$$

$$(64) \quad Z_{\bar{1}} Z_1 z_1^a (1 - z_2)^b (1 - \bar{z}_2)^c \\ = z_1^a (1 - z_2)^{b-1} (-(1 - \bar{z}_2)^{c-1}) (a(z_2 - 1)((c + 1)\bar{z}_2 - 1) + b(c(|z_2|^2 - 1) + (\bar{z}_2 - 1)z_2));$$

$$(65) \quad Z_1 Z_{\bar{1}} z_1^a (1 - z_2)^b (1 - \bar{z}_2)^c = -c z_1^a (1 - z_2)^{b-1} (1 - \bar{z}_2)^{c-1} ((a + 1)\bar{z}_2(z_2 - 1) + b(|z_2|^2 - 1)),$$

with similar ones for $\bar{z}_1^a (1 - z_2)^b (1 - \bar{z}_2)^c$, passing to conjugates.

To find the first-order correction G_1 , we differentiate the relation $L_s G_s = 0$ with respect to s , evaluating it for $s = 0$. Using (21) and (22), this yields

$$L_0 G_1 = -\dot{L} G_0 = 8Z_1 Z_1 G_0 + 8Z_{\bar{1}} Z_{\bar{1}} G_0 \quad \text{on } S^3 \setminus \{p\},$$

where $\dot{L} = \frac{d}{ds}|_{s=0} L_s$. The right-hand side is given by

$$\frac{12((\bar{z}_2 - 1)^2 \bar{z}_1^2 + z_1^2 (z_2 - 1)^2)}{((\bar{z}_2 - 1)(z_2 - 1))^{5/2}}.$$

By formulas (62)-(65), the first-order correction G_1 to G_s can be chosen as

$$(66) \quad G_1 = \frac{1}{2}(z_1^2 + \bar{z}_1^2) \left[\frac{1}{1-z_2} + \frac{1}{1-\bar{z}_2} + 2 \right] \frac{1}{((1-z_2)(1-\bar{z}_2))^{\frac{1}{2}}}.$$

We pass next to the second order expansion for G_s : we will find it up to a smooth function that can be determined at p , which is enough for our purposes. Differentiating the relation $L_s G_s = 0$ twice with respect to s and evaluating at $s = 0$ we obtain (with analogous notation to above for the s -derivatives)

$$L_0 \ddot{G} = -2\dot{L}\dot{G} - \ddot{L}G_0.$$

Recalling from (21), (22) that $\ddot{L} = 4L_0$, we have

$$L_0(\ddot{G} + 4G_0) = -2\dot{L}\dot{G} = 16Z_1 Z_1 G_1 + 16Z_{\bar{1}} Z_{\bar{1}} G_1.$$

It is possible to show by direct computation, again from (62)-(65), that $16Z_1 Z_1 G_1 + 16Z_{\bar{1}} Z_{\bar{1}} G_1$ equals

$$(67) \quad \begin{aligned} & \frac{-1}{((1-z_2)(1-\bar{z}_2))^{\frac{7}{2}}} [z_1^4 (30(z_2-1)^3 + 6(z_2-1)^2(\bar{z}_2-1) - 12(z_2-1)^3(\bar{z}_2-1)) \\ & + \bar{z}_1^4 (30(\bar{z}_2-1)^3 + 6(\bar{z}_2-1)^2(z_2-1) - 12(\bar{z}_2-1)^3(z_2-1)) \\ & + 30(\bar{z}_2-1)^5 + 30(z_2-1)^5 + 18(\bar{z}_2-1)^4(z_2-1) + 18(z_2-1)^4(\bar{z}_2-1) \\ & + 6(\bar{z}_2-1)^5(z_2-1)^2 + 6(z_2-1)^5(\bar{z}_2-1)^2 - 18(\bar{z}_2-1)^4(z_2-1)^3 - 18(\bar{z}_2-1)^3(z_2-1)^4 \\ & - 12(z_2-1)^3(\bar{z}_2-1)^5 - 12(\bar{z}_2-1)^3(z_2-1)^5], \end{aligned}$$

where we grouped the terms by homogeneity in $z_2 - 1$ and $\bar{z}_2 - 1$.

We can invert L_0 explicitly for the terms with factors z_1^4 and \bar{z}_1^4 . The solution is given by

$$G_{2,1} := \frac{(z_1^4 + \bar{z}_1^4) g_{2,1}}{((1-z_2)(1+\bar{z}_2))^{\frac{5}{2}}},$$

where

$$\begin{aligned} g_{2,1} & := \frac{3}{8}(\bar{z}_2-1)^2 + \frac{3}{8}(z_2-1)^2 + \frac{1}{4}(z_2-1)(\bar{z}_2-1) + \frac{3}{2}(z_2-1)^2(\bar{z}_2-1)^2 \\ & - \frac{3}{4}(z_2-1)^2(\bar{z}_2-1) - \frac{3}{4}(z_2-1)(\bar{z}_2-1)^2. \end{aligned}$$

For the other terms, we can only find an explicit approximate solution. We set

$$\begin{aligned} g_{2,2} & = (z_2-1)^4 + (\bar{z}_2-1)^4 - \frac{4}{3}(z_2-1)^4(\bar{z}_2-1) - \frac{4}{3}(\bar{z}_2-1)^4(z_2-1) \\ & + 4(\bar{z}_2-1)^3(z_2-1)^2 + 4(z_2-1)^3(\bar{z}_2-1)^2 \\ & + \frac{11}{3}(z_2-1)^4(\bar{z}_2-1)^2 + \frac{11}{3}(\bar{z}_2-1)^4(z_2-1)^2 + 6(z_2-1)^3(\bar{z}_2-1)^3, \end{aligned}$$

and

$$G_{2,2} := \frac{3}{4} \frac{g_{2,2}}{((1-z_2)(1-\bar{z}_2))^{\frac{5}{2}}}.$$

Defining

$$(68) \quad G_2 = G_{2,1} + G_{2,2},$$

still by (62)-(65) one finds that

$$(69) \quad L_0 G_2 - 16Z_1 Z_1 G_1 - 16Z_{\bar{1}} Z_{\bar{1}} G_1 = -12 \frac{(z_2-1)^2 + (\bar{z}_2-1)^2 - 3(z_2-1)(\bar{z}_2-1)}{((1-z_2)(1-\bar{z}_2))^{\frac{1}{2}}} =: \Xi(z_2, \bar{z}_2),$$

with the right-hand side now bounded on S^3 .

It will be now sufficient to add a more regular correction (which is Hölder continuous by standard regularity theory) to solve the equation for G_2 pointwise, away from p . From (69), setting $G_3 = L_0^{-1}\Xi(z, w)$ we then find that

$$L_0(G_2 - G_3) - 16Z_1 Z_1 G_1 - 16Z_{\bar{1}} Z_{\bar{1}} G_1 = 0 \quad \text{on } S^3 \setminus \{p\},$$

which corresponds to (61) up to the term $s^2 \alpha G_0$, which will be determined later. To obtain $G_3(p)$, we use the Green's representation formula, convoluting $\Xi(z_2, \bar{z}_2)$ with G_0 :

$$G_3(p) = \frac{1}{64\pi^2} \int_{S^3} -24 \frac{(z_2 - 1)^2 + (\bar{z}_2 - 1)^2 - 3(z_2 - 1)(\bar{z}_2 - 1)}{((1 - z_2)(1 + \bar{z}_2))} \hat{\theta} \wedge d\hat{\theta}.$$

The Taylor expansion of the integrand in z_2, \bar{z}_2 is

$$(24\bar{z}_2^5 + 24\bar{z}_2^4 + 24\bar{z}_2^3 + 24\bar{z}_2^2 - 24) + (-24\bar{z}_2^5 - 24\bar{z}_2^4 - 24\bar{z}_2^3 - 24\bar{z}_2^2 - 48\bar{z}_2) z_2 \\ + (24 - 24\bar{z}_2) z_2^2 + (24 - 24\bar{z}_2) z_2^3 + (24 - 24\bar{z}_2) z_2^4 + (24 - 24\bar{z}_2) z_2^5 + \dots$$

Integrated, this gives

$$\int_{S^3} (24 + 48|z_2|^2) \hat{\theta} \wedge d\hat{\theta} = 48 \cdot 2\pi^2 + 96 \pi^2 = 192\pi^2,$$

which implies that

$$(70) \quad G_3(p) = 3.$$

In conclusion, we found that

$$\ddot{G} = G_2 - G_3 + \alpha G_0,$$

i.e. (61), where α is a real number to be determined later. We proved therefore the following result.

Proposition 4.1. *For every compact set K in $S^3 \setminus \{p\}$, $p = (0, 1)$, there exists a constant $C_K > 0$ such that the function G_s in (61) satisfies*

$$|L_s G_s| \leq C_K s^3 \quad \text{on } K.$$

4.2. Rigorous estimates. We prove next that the function G_s in Lemma 4.1 well matches with the expression of the Green's function of L_s in CR normal coordinates. Recall from the end of Section 3 that $\rho_{CR}^2 = \frac{1}{2}(\rho'_{CR})^2$: then from (59) we obtain that

$$(71) \quad 2e^{v^2} \rho_{CR}^{-2} = \frac{|z_1|^2 + 4}{\sqrt{(|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2)}} + s \frac{(z_1^2 + \bar{z}_1^2)(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{2((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{3/2}} \\ + s^2 \left[\frac{(z_1^4 + \bar{z}_1^4)(20|z_1|^6 + 8|z_1|^4 + 4\mathbf{w}^2 - 5|z_1|^2\mathbf{w}^2) - 4|z_1|^{10} + 58|z_1|^6\mathbf{w}^2 + 24|z_1|^4\mathbf{w}^2 - 24|z_1|^2\mathbf{w}^4}{2((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{5/2}} \right] \\ + O(s^3 \rho^{-2}),$$

where $\mathbf{w} = z_2 - \bar{z}_2$. Given the covariance property of the Green's function ($G_{\tilde{\theta}} = e^u G_{\theta}$ if $\tilde{\theta} = e^{2u}\theta$), we aim to compare this expression to the function G_s in Lemma 4.1 on a suitable small annulus centered around p . We do it term by term for the Taylor series in s , and for this purpose the following formulas will be useful. Since $z_2 - \bar{z}_2$ is purely imaginary, we can write

$$|z_1|^4 + |z_2 - \bar{z}_2|^2 = |z_1|^4 - (z_2 - \bar{z}_2)^2 = (|z_1|^2 + (z_2 - \bar{z}_2)) (|z_1|^2 - (z_2 - \bar{z}_2)).$$

As $|z_1|^2 + |z_2|^2 = 1$, we get

$$(72) \quad |z_1|^4 - \mathbf{w}^2 = |z_1|^4 + |z_2 - \bar{z}_2|^2 = (1 + z_2)(1 + \bar{z}_2)(1 - z_2)(1 - \bar{z}_2).$$

Setting $\mathbf{v} = z_2 + \bar{z}_2 - 2$ (which is real), we have that $z_2 = 1 + \frac{\mathbf{v}}{2} + \frac{\mathbf{w}}{2}$, which implies

$$|z_2|^2 = 1 + \mathbf{v} + \frac{\mathbf{v}^2}{4} - \frac{\mathbf{w}^2}{4} + o(\rho^4).$$

Squaring this relation, we obtain

$$(73) \quad |z_1|^4 = \mathbf{v}^2 + \frac{\mathbf{v}^3}{2} - \frac{\mathbf{v}\mathbf{w}^2}{2} + o(\rho^6).$$

We also have that $|z_2|^2 = 1 + \mathbf{v}$ up to an error $O(\rho^4)$, so $\mathbf{v}^2 = -|z_1|^4 + o(\rho^4)$. These imply that

$$(74) \quad |z_2|^2 + 1 - (z_2 + \bar{z}_2) = \frac{1}{4}|z_1|^2 - \frac{1}{4}\mathbf{w}^2 = \frac{1}{4}|z_1|^2 + \frac{1}{4}|\mathbf{w}|^2 + o(\rho^4).$$

Furthermore, there holds

$$(75) \quad 1 + z_2 + \bar{z}_2 + |z_2|^2 \simeq 3 + \mathbf{v} + |z_2|^2 \simeq 2 + 2|z_2|^2 = 4 - 2|z_1|^2 + o(\rho^2).$$

Recalling our notation from Section 2, we have then the following result.

Lemma 4.2. *For $\alpha = -\frac{3}{4}$, the following estimate holds*

$$(76) \quad 2e^{v_2} \rho_{CR}^{-2} = \left(G_0 + sG_1 + \frac{1}{2}s^2(G_2 + \alpha G_0) \right) + o(s^2)O''(\rho^{-2}) + o_\rho(1),$$

where $o_\rho(1) \rightarrow 0$ as $\rho \rightarrow 0$.

PROOF. We analyse separately different orders in s for the left-hand side and the first term in the right-hand side of (76).

Zero-th order in s . Recalling that $G_0 = 2((1 - z_2)(1 - \bar{z}_2))^{-\frac{1}{2}}$, we need to compare the two quantities

$$(77) \quad \frac{|z_1|^2 + 4}{\sqrt{(|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2)}} \quad \text{and} \quad \frac{2}{((1 - z_2)(1 - \bar{z}_2))^{\frac{1}{2}}}.$$

Taylor-expanding the terms involving $|z_1|^2$ in the left-hand side we are left with comparing

$$\frac{4(1 - \frac{1}{4}|z_1|^2)}{\sqrt{(|z_1|^4 - \mathbf{w}^2)}} \quad \text{and} \quad \frac{2}{((1 - z_2)(1 - \bar{z}_2))^{\frac{1}{2}}}.$$

Using (72) and multiplying by $((1 - z_2)(1 - \bar{z}_2))^{\frac{1}{2}}$, we are left with the comparison of

$$\frac{4(1 - \frac{1}{4}|z_1|^2)}{((1 + z_2)(1 + \bar{z}_2))^{\frac{1}{2}}} \quad \text{and} \quad 2.$$

From (75) we are left with comparing

$$\frac{4(1 - \frac{1}{4}|z_1|^2)}{(4 - 2|z_1|^2)^{\frac{1}{2}}} \quad \text{and} \quad 2,$$

which holds true up to an error of order $O(\rho^4)$. Therefore the two quantities in (77) coincide up to an error of order $O(\rho^2)$.

First order in s . Recalling (66), we have that

$$G_1 = \frac{1}{4}(z_1^2 + \bar{z}_1^2) \left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{(1 - z_2)(1 - \bar{z}_2)} \right] G_0.$$

Considering the first-order term in s of (71), we need to compare the two quantities

$$\frac{(z_1^2 + \bar{z}_1^2)(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{2((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{3/2}} \quad \text{and} \quad \frac{1}{4}(z_1^2 + \bar{z}_1^2) \left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{(1 - z_2)(1 - \bar{z}_2)} \right] G_0.$$

Using the expression of G_0 , dividing by $(z_1^2 + \bar{z}_1^2)$ and multiplying by 2 we need to compare

$$\frac{(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{3/2}} \quad \text{and} \quad \left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{((1 - z_2)(1 - \bar{z}_2))^{\frac{3}{2}}} \right].$$

Using (58), this is equivalent to the comparison of

$$\frac{(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{((|z_1|^2 + 1)(1 + z_2)(1 + \bar{z}_2))^{3/2}} \quad \text{and} \quad 4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2.$$

Using (75) and Taylor-expanding the left-hand side we arrive to comparing

$$\frac{(1 - 3/4|z_1|^2)(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{8} \quad \text{and} \quad 4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2.$$

Using instead (74) we transform the right-hand side, arriving to the comparison of

$$\frac{(1 - 3/4|z_1|^2)(12|z_1|^4 + 8|z_1|^2 - 6\mathbf{w}^2)}{8} \quad \text{and} \quad |z_1|^2 + \frac{3}{4}|z_1|^4 - \frac{3}{4}\mathbf{w}^2,$$

which is again true up to an error of order $O(\rho^6)$. Therefore, we get matching of the first-order terms in s in both sides of (76) up to an error $O(\rho^2)$.

Second order in s . Recalling again (71) and the fact that G_2 comes with a factor $\frac{1}{2}$, let us first compare

$$\frac{(z_1^4 + \bar{z}_1^4)(20|z_1|^6 + 8|z_1|^4 + 4\mathbf{w}^2 - 5|z_1|^2\mathbf{w}^2)}{2((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{5/2}} \quad \text{and} \quad \frac{1}{2}G_{2,1} := \frac{1}{2} \frac{(z_1^4 + \bar{z}_1^4)g_{2,1}}{((1 - z_2)(1 + \bar{z}_2))^{\frac{5}{2}}},$$

where, up to order $O(\rho^8)$

$$g_{2,1} := \frac{3}{8}(\bar{z}_2 - 1)^2 + \frac{3}{8}(z_2 - 1)^2 + \frac{1}{4}(z_2 - 1)(\bar{z}_2 - 1) - \frac{3}{4}(z_2 - 1)^2(\bar{z}_2 - 1) - \frac{3}{4}(z_2 - 1)(\bar{z}_2 - 1)^2.$$

Factoring out $(z_1^4 + \bar{z}_1^4)$ and using (72), (75) we need to compare

$$\left(1 - \frac{5}{4}|z_1|^2\right) \frac{(20|z_1|^6 + 8|z_1|^4 + 4\mathbf{w}^2 - 5|z_1|^2\mathbf{w}^2)}{64} \quad \text{and} \quad \frac{1}{2}g_{2,1}.$$

Expanding $g_{2,1}$ and using (73) we arrive to the comparison of

$$\left(1 - \frac{5}{4}|z_1|^2\right) \frac{(20|z_1|^6 + 8|z_1|^4 + 4\mathbf{w}^2 - 5|z_1|^2\mathbf{w}^2)}{64} \quad \text{and} \quad \frac{1}{2} \frac{1}{16} (-3\mathbf{v}^3 + 4\mathbf{v}^2 + 3\mathbf{v}\mathbf{w}^2 + 2\mathbf{w}^2),$$

which is correct, up to an error of order $O(\rho^{12})$.

We need next to compare

$$\left[\frac{-4|z_1|^{10} + 58|z_1|^6\mathbf{w}^2 + 24|z_1|^4\mathbf{w}^2 - 24|z_1|^2\mathbf{w}^4}{2((|z_1|^2 + 1)(|z_1|^4 - \mathbf{w}^2))^{5/2}} \right] \quad \text{and} \quad \frac{1}{2} \frac{3}{4} \frac{g_{2,2}}{((1 - z_2)(1 - \bar{z}_2))^{\frac{5}{2}}},$$

where, up to higher order terms

$$\begin{aligned} g_{2,2} &= (z_2 - 1)^4 + (\bar{z}_2 - 1)^4 - \frac{4}{3}(z_2 - 1)^4(\bar{z}_2 - 1) - \frac{4}{3}(\bar{z}_2 - 1)^4(z_2 - 1) \\ &\quad + 4(\bar{z}_2 - 1)^3(z_2 - 1)^2 + 4(z_2 - 1)^3(\bar{z}_2 - 1)^2. \end{aligned}$$

Using again (72), we then need to compare

$$\left[\frac{-4|z_1|^{10} + 58|z_1|^6\mathbf{w}^2 + 24|z_1|^4\mathbf{w}^2 - 24|z_1|^2\mathbf{w}^4}{2((|z_1|^2 + 1)(1 - z_2)(1 - \bar{z}_2))^{5/2}} \right] \quad \text{and} \quad \frac{3}{8}g_{2,2}.$$

As before, we are then comparing

$$\left(1 - \frac{5}{4}|z_1|^2\right) \frac{-4|z_1|^{10} + 58|z_1|^6\mathbf{w}^2 + 24|z_1|^4\mathbf{w}^2 - 24|z_1|^2\mathbf{w}^4}{64} \quad \text{and} \quad \frac{3}{8}g_{2,2}.$$

In fact, we can add to G_2 any multiple of G_0 . In the latter formula, we can then replace $g_{2,2}$ with $\tilde{g}_{2,2}$, where

$$\tilde{g}_{2,2} = g_{2,2} - 2(z_2 - 1)^2(\bar{z}_2 - 1)^2.$$

It turns out that

$$\frac{3}{8}\tilde{g}_{2,2} = \frac{1}{16}\mathbf{v}(\mathbf{v}^4 - 4\mathbf{v}^2\mathbf{w}^2 + 6\mathbf{v}\mathbf{w}^2 + 3\mathbf{w}^4).$$

Using (73) and the previous formula to expand $|z_1|^2$ as $|z_1|^2 = -\mathbf{v} - \frac{1}{4}\mathbf{v}^2 + \frac{1}{4}\mathbf{w}^2$, the left-hand side in the above formula becomes

$$\mathbf{v}^5/16 + (3\mathbf{v}^2\mathbf{w}^2)/8 - (\mathbf{v}^3\mathbf{w}^2)/4 + (3\mathbf{v}\mathbf{w}^4)/16 + O(\rho^{12}),$$

so it coincides with the right-hand side, i.e. with $\frac{3}{8}\tilde{g}_{2,2}$ up to error terms of order $O(\rho^{12})$. Therefore, also the second-order terms in s of both sides of (76) coincide up to an error of order $O(\rho^{-2})$.

It is standard to check that the above matching also holds up to computing first- and second-order derivatives, which then implies the conclusion. ■

PROOF OF THEOREM 1.1. Consider a small annulus of the form

$$A_r := \{r \leq \rho \leq 2r\},$$

and a smooth cut-off function χ_r satisfying

$$\begin{cases} \chi_r = 1 & \text{on } \{\rho \leq r\}; \\ \chi_r = 1 & \text{on } \{\rho \geq 2r\}; \\ |\nabla_b \chi_r| \leq \frac{C}{r}; & |\nabla_b^2 \chi_r| + |\nabla_T \chi_r| \leq \frac{C}{r^2}. \end{cases}$$

If v is the conformal factor as in Proposition 2.3 then, with obvious notation, the Green's function conformally transforms as $G_\theta = e^{-v} G_{\hat{\theta}}$. Consider then the function

$$\check{G}_s = \chi_r \left(2\rho_{CR}^{-2} - \frac{1}{2} G_3(p) s^2 \right) + (1 - \chi_r) e^{-v} G_{\hat{\theta}}.$$

From the conformal covariance of L_s , Lemma 4.1 and Lemma 4.2 it follows that, applying the conformal sub-Laplacian with respect to the contact form θ :

$$|L_s^v \check{G}_s| \leq C_r o(s^2) \quad \text{pointwise on } S^3.$$

It then follows from standard regularity theory that the Green's function G_θ of the conformal sub-Laplacian satisfies $\|G_\theta - \check{G}_s\|_{L^\infty(S^3)} = o(s^2)$. Sending s to zero and recalling that $G_3(p) = 3$, we deduce

$$(78) \quad A = -\frac{3}{2} s^2 + o(s^2).$$

Therefore, given that $m = 12\pi A$ (see (70) and (12)), we obtain the conclusion. ■

5. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2 by an implicit function argument and some asymptotic expansions, which crucially use also Theorem 1.1.

We start by analysing the relation of the CR Sobolev quotient on Rossi spheres with the minimizers on standard spheres found in [JL88]. Recall that in [JL87] it was proved that for any three-dimensional CR manifold one has $\mathcal{Y}(M, J) \leq \mathcal{Y}(S^3, J_{S^3})$, which in particular implies

$$(79) \quad \mathcal{Y}(S^3, J_{(s)}) \leq \mathcal{Y}(S^3, J_{S^3} = J_{(0)}).$$

In [JL88] it was proven that $\mathcal{Y}(S^3, J_{S^3})$ is precisely attained by the following functions, up to composing (z_1, z_2) with elements of $SU(2)$

$$(80) \quad \varphi_\lambda = \lambda \left(\frac{(|z_1|^2 + |z_2 + 1|^2)^2 - (z_2 - \bar{z}_2)^2}{(\lambda^2 |z_1|^2 + |z_2 + 1|^2)^2 - \lambda^4 (z_2 - \bar{z}_2)^2} \right)^{\frac{1}{2}}; \quad \lambda > 0.$$

Recalling that $\hat{\theta} \wedge d\hat{\theta}$ is a volume form double w.r.t. the Euclidean one, the φ_λ 's satisfy the following normalization condition

$$(81) \quad \int_{S^3} \varphi_\lambda^4 \hat{\theta} \wedge d\hat{\theta} = 4\pi^2 \quad \text{for all } \lambda > 0.$$

On the standard S^3 , see [FS74], the *Folland-Stein space* $\mathfrak{S}^{1,2}(S^3)$ is defined as the completion of the (complex-valued) C^∞ functions on S^3 with respect to the norm

$$\|u\|_{\mathfrak{S}^{1,2}} := \left(\int_{S^3} (u_{,1} \bar{u}_{,1} + u_{,\bar{1}} \bar{u}_{,\bar{1}}) \hat{\theta} \wedge d\hat{\theta} \right)^{\frac{1}{2}} + \left(\int_{S^3} |u|^2 \hat{\theta} \wedge d\hat{\theta} \right)^{\frac{1}{2}}.$$

Notice that, for $|s|$ small, this defines an equivalent norm on Rossi spheres too: from now on, this will be assumed understood.

We show next that, if a minimizer for the CR-Sobolev quotient on Rossi spheres exists for $|s|$ small, it must be close in $\mathfrak{S}^{1,2}(S^3)$ to some function φ_λ as in (80). We have indeed the following result.

Lemma 5.1. *Fix $s \in \mathbb{R}$, $|s|$ small. Assume $u_s > 0$ attains $\inf Q_{(s)} = \mathcal{Y}(S^3, J_{(s)})$. Then, if u_s is normalized so that $\int_{S^3} u_s^4 \hat{\theta} \wedge d\hat{\theta} = 4\pi^2$, up to a homogeneous action on S^3 there exists $\lambda > 0$ such that*

$$\|u_s - \varphi_\lambda\|_{\mathfrak{S}^{1,2}(S^3)} = o_s(1),$$

where $o_s(1) \rightarrow 0$ as $s \rightarrow 0$.

PROOF. It is sufficient to notice that, if $Z_{1(s)}$ is as in (14), then for all smooth u 's one has

$$\int_{S^3} (Z_{1(s)}u Z_{\bar{1}(s)}\bar{u} + Z_{\bar{1}(s)}u Z_{1(s)}\bar{u}) \hat{\theta} \wedge d\hat{\theta} = (1 + o_s(1)) \int_{S^3} (Z_{1(s)}u Z_{\bar{1}(s)}\bar{u} + Z_{\bar{1}(s)}u Z_{1(s)}\bar{u}) \hat{\theta} \wedge d\hat{\theta}.$$

Since we are assuming u_s to be normalized in $L^4(S^3)$ as in the statement, its $\mathfrak{S}^{1,2}(S^3)$ -norm is uniformly bounded from above, and therefore

$$\int_{S^3} (Z_{1(s)}u_s Z_{\bar{1}(s)}\bar{u}_s + Z_{\bar{1}(s)}u_s Z_{1(s)}\bar{u}_s) \hat{\theta} \wedge d\hat{\theta} = \int_{S^3} (Z_{1(s)}u_s Z_{\bar{1}(s)}\bar{u}_s + Z_{\bar{1}(s)}u_s Z_{1(s)}\bar{u}_s) \hat{\theta} \wedge d\hat{\theta} + o_s(1).$$

This relation implies that u_s is nearly a minimizer also for $\mathcal{Y}(S^3, J_{S^3} = J_{(0)})$ and therefore, since the minimizers of the latter quantity must be of the form (80), the conclusion follows. ■

5.1. Finite-dimensional reduction. Let φ_λ be as in (80), and define the following family of functions

$$(82) \quad \mathcal{M} = \{\varphi_\lambda(U(\cdot)) : |\lambda| > 0, U \in SU(2)\}.$$

Even though $SU(2)$ is a four-dimensional Lie group, since φ_λ is invariant by a complex rotation in z_1 , the result of these compositions is a set of three dimensions. We previously saw that the functions in \mathcal{M} are global minimizers of the CR-Sobolev quotient $Q_{(s)}$ on the standard S^3 when $s = 0$, where

$$(83) \quad Q_{(s)}(u) = \frac{\int_{S^3} u L_s u \hat{\theta} \wedge d\hat{\theta}}{\left(\int_{S^3} u^4 \hat{\theta} \wedge d\hat{\theta}\right)^{\frac{1}{2}}}.$$

In [MU02], Lemma 5, it was proved that the linearization of the Yamabe equation (with $s = 0$) at \mathcal{M} is minimally degenerate, in the sense that its kernel coincides with the tangent space to \mathcal{M} .

As a consequence, one has that the CR-Sobolev quotient on the standard sphere is *non-degenerate in the sense of Bott* on \mathcal{M} . Thanks to this fact and to Lemma 5.1, for s small we can characterize with particular precision all the solutions of the CR-Yamabe equation lying in a fixed neighborhood (in $\mathfrak{S}^{1,2}$) of the manifold \mathcal{M} , and in particular the (hypothetical) minimal ones. We first show that the CR-Yamabe equation is always solvable, in a fixed neighborhood of \mathcal{M} , up to a Lagrange multiplier: see [AM06] for a general reference on this method.

Proposition 5.2. *For φ_λ as in (80) there exists a unique $w_\lambda \in \mathfrak{S}^{1,2}(S^3)$, depending smoothly on λ , such that $\|w_\lambda\|_{\mathfrak{S}^{1,2}(S^3)} \leq C s$ and which satisfies*

$$(84) \quad \int_{S^3} \varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda} w_\lambda \hat{\theta} \wedge d\hat{\theta} = 0; \quad L_s(\varphi_\lambda + w_\lambda) - 2(\varphi_\lambda + w_\lambda)^3 = \ell \varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda}$$

for some $\ell \in \mathbb{R}$. Moreover, there exists $\delta > 0$ with the following property: if there exists a critical point of $Q_{(s)}$ in a δ -neighborhood of \mathcal{M} (in $\mathfrak{S}^{1,2}$ norm), then it must be of the form $\varphi_\lambda + w_\lambda$ up to a homogeneous action on S^3 and up to a scalar multiple, with w_λ as above.

PROOF. For $\lambda > 1$, φ_λ has a global maximum at $(z_1, z_2) = (0, 1)$. Locally near these functions, all other extremals can be obtained composing on the right with elements of $SU(2)$. When also λ varies, the extremals can be described locally near the φ_λ 's by

$$\Sigma_{\Lambda, \gamma} = \{\varphi_{\mathbf{a}, \lambda}(z_1, z_2) := \varphi_\lambda(U_{\mathbf{a}}(z_1, z_2)) : \mathbf{a} \in (-\gamma, \gamma)^3, \lambda \in [1/2, 2\Lambda]\} \subseteq \mathcal{M},$$

where

$$(85) \quad U_{\mathbf{a}}(z_1, z_2) = \left(\exp \begin{pmatrix} 0 & a_1 + i a_2 \\ -a_1 + i a_2 & i a_3 \end{pmatrix} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \mathbf{a} = (a_1, a_2, a_3).$$

Consider next the CR-Yamabe equation on the standard sphere

$$L_0 u = 2u^3 \quad \text{on } S^3.$$

It was proved in [MU02] (see Lemma 5 there) that solutions of the linearized equation at φ_λ

$$L_0 v = 6\varphi_\lambda^2 v \quad \text{on } S^3$$

are of the form

$$v = l_0 \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial \lambda} + \sum_{i=1}^3 l_i \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial a_i},$$

where $l_i \in \mathbb{R}$ and where the latter derivatives are evaluated at $\mathbf{a} = 0$.

Define $\widetilde{W} = \widetilde{W}_\lambda$ to be the space of functions \tilde{w} satisfying the four constraints

$$(86) \quad \int_{S^3} \varphi_{\mathbf{a}, \lambda}^2 \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial \lambda} \tilde{w} \hat{\theta} \wedge d\hat{\theta} = 0; \quad \int_{S^3} \varphi_{\mathbf{a}, \lambda}^2 \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial a_i} \tilde{w} \hat{\theta} \wedge d\hat{\theta} = 0; \quad i = 1, 2, 3,$$

where, again, the derivatives are evaluated at $\mathbf{a} = 0$.

It follows from the classification result in [MU02] and Fredholm's theory that the operator

$$A_{\widetilde{W}} : \tilde{w} \mapsto P_{\widetilde{W}} [L_s \tilde{w} - 6\varphi_\lambda^2 \tilde{w}],$$

where $P_{\widetilde{W}}$ denotes the projection onto \widetilde{W} , is invertible from \widetilde{W} in itself.

Setting $S_{\mathbf{a}, \lambda}(\tilde{w}) := L_s(\varphi_\lambda + \tilde{w}) - 2(\varphi_{\mathbf{a}, \lambda} + \tilde{w})^3$, equation (84) becomes $P_{\widetilde{W}} S_{\mathbf{a}, \lambda}(\tilde{w}) = 0$. Since $A_{\widetilde{W}}$ is invertible (with the norm of the inverse uniformly bounded), we have that

$$P_{\widetilde{W}} S_{\mathbf{a}, \lambda}(\tilde{w}) = 0 \quad \iff \quad \tilde{w} = T_{\mathbf{a}, \lambda}(\tilde{w}),$$

where

$$T_{\mathbf{a}, \lambda}(\tilde{w}) = -(A_{\widetilde{W}})^{-1} \{S_{\mathbf{a}, \lambda}(0) - 2[(\varphi_{\mathbf{a}, \lambda} + \tilde{w})^3 - \varphi_{\mathbf{a}, \lambda}^3 - 3\varphi_{\mathbf{a}, \lambda}^2 \tilde{w}]\}.$$

From the smoothness in s of the $J_{(s)}$ structures it follows that $\|T_{\mathbf{a}, \lambda}(0)\| = O(s)$. Moreover, it is quite standard that for s and δ small

$$\|T_{\mathbf{a}, \lambda}(\tilde{w}_1) - T_{\mathbf{a}, \lambda}(\tilde{w}_2)\| = o(1)\|\tilde{w}_1 - \tilde{w}_2\|, \quad \|\tilde{w}_1\|, \|\tilde{w}_2\| \leq \delta.$$

It follows that for s small $T_{\mathbf{a}, \lambda}$ is a contraction in a normed ball of radius Cs for $C > 0$ large and fixed, so in such a ball there exists a unique fixed point w_λ of $T_{\mathbf{a}, \lambda}$.

In this way we found a (unique) solution to the problem

$$L_s(\varphi_\lambda + w_\lambda) - 2(\varphi_\lambda + w_\lambda)^3 = \ell \varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda} + \sum_{i=1}^3 \ell_i \varphi_\lambda^2 \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial a_i} \Big|_{\mathbf{a}=0}$$

for some Lagrange multipliers ℓ, ℓ_i . However the last three vanish by *Palais' criticality principle*. In fact, let us recall that, being $(S^3, J_{(s)})$ a homogeneous space, $Q_{(s)}$ is invariant under the maps $U_{\mathbf{a}}$ as in (85). Therefore, with obvious notation, we have with the same Lagrange multipliers that

$$L_s(\varphi_{\lambda, \mathbf{a}} + w_{\lambda, \mathbf{a}}) - 2(\varphi_{\lambda, \mathbf{a}} + w_{\lambda, \mathbf{a}})^3 = \ell \varphi_{\lambda, \mathbf{a}}^2 \frac{\partial \varphi_{\lambda, \mathbf{a}}}{\partial \lambda} + \sum_{i=1}^3 \ell_i \varphi_{\lambda, \mathbf{a}}^2 \frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial a_i},$$

for \mathbf{a} in a neighborhood of zero. Differentiating with respect to a_i and then scalar-multiplying by $\frac{\partial \varphi_{\mathbf{a}, \lambda}}{\partial a_j}$ one obtains an invertible system for $(\ell_i)_i$, yielding that $\ell_i = 0$ for $i = 1, 2, 3$, as desired.

Let now u be a critical point of $Q_{(s)}$ in a δ -neighborhood of \mathcal{M} for s small. Then it satisfies $L_s u = \mu u^3$ for some Lagrange multiplier μ . Since u is close to the family of φ_λ 's, satisfying $L_0 \varphi_\lambda = 2\varphi_\lambda^3$, the multiplier μ must be δ -close to 2.

Defining $\tilde{u} = \mu^{-\frac{1}{2}} u$, this is still close of order δ to \mathcal{M} , and it satisfies $L_s \tilde{u} = 2\tilde{u}^3$, i.e. the second equation in (84) with $\ell = 0$. By uniqueness of the fixed point, we must then have $\tilde{u} = \varphi_\lambda + w_\lambda$, up to a homogeneous action on S^3 . This concludes the proof. ■

Remark 5.3. *In Proposition 5.2 it is possible to replace the φ_λ 's with other approximate solutions to the CR-Yamabe equation on Rossi spheres. With a better approximate solution, for example, one would then require a correction as in (84) of smaller norm, yielding a more precise expansion for the quotient $Q_{(s)}$. This observation will be crucially used in the next two sections.*

5.2. Expansion of the CR Sobolev quotient. Recalling the latter statement in Proposition 5.2, we analyze the CR Sobolev quotient on functions of the form $\varphi_\lambda + w_\lambda$, showing that it is strictly higher than the standard spherical one. We first show that the latter expansion is always even in s .

Lemma 5.4. *Let $s > 0$ be small, and let $w_\lambda^{(s)}$ and $w_\lambda^{(-s)}$ denote the counterparts of w_λ in Proposition 5.2 for s and $-s$ respectively. Then one has that*

$$Q_{(s)}(\varphi_\lambda + w_\lambda^{(s)}) = Q_{(-s)}(\varphi_\lambda + w_\lambda^{(-s)}).$$

PROOF. Let $\iota : S^3 \rightarrow S^3$ be the diffeomorphism given in (23). We notice that φ_λ is invariant under ι and that, due to (24), (25) and (26), for any $u \in \mathfrak{S}^{1,2}(S^3)$ one has

$$Q_{(s)}(\iota^*u) = Q_{(-s)}(u).$$

From this covariance property and the uniqueness in Proposition 5.2 it follows that $w_\lambda^{(s)} = \iota^*w_\lambda^{(-s)}$, and therefore we get

$$Q_{(s)}(\varphi_\lambda + w_\lambda^{(s)}) = Q_{(s)}(\iota^*(\varphi_\lambda + w_\lambda^{(-s)})) = Q_{(-s)}(\varphi_\lambda + w_\lambda^{(-s)}),$$

which is the desired conclusion. ■

We analyse next two situations. The first is when the parameter λ in the previous lemma tends to infinity or to zero, and the second when $\log \lambda$ remains bounded. In the latter case we will show that the CR Sobolev quotient would be strictly higher than $\mathcal{Y}(S^3, J_{S^3})$, which would give a contradiction to (79). On the other hand, we can also rule out the former case using the estimates on the Green's function in Section 4, and in particular the negativity of the mass of $(S^3, J_{(s)})$ for s small and non zero. The proofs of the next two results, beginning from the latter case, are given in the next two appendices.

Proposition 5.5. *Let $\Lambda > 1$ be a fixed number. Then there exist $C_\Lambda > 0$ such that, for $\lambda \in [1/\Lambda, \Lambda]$ and for s small one has $Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi + s^2\mathcal{A}_\lambda + \mathcal{B}_{\lambda,s}$, where*

$$\mathcal{A}_\lambda = \frac{16\pi\lambda^2(3 + 12\lambda^2 + 2\lambda^4 + 12\lambda^6 + 3\lambda^8)}{(1 + \lambda^2)^6},$$

and where $|\mathcal{B}_{\lambda,s}| \leq C_\Lambda s^3$.

Proposition 5.6. *The following expansion holds true, uniformly in s (small)*

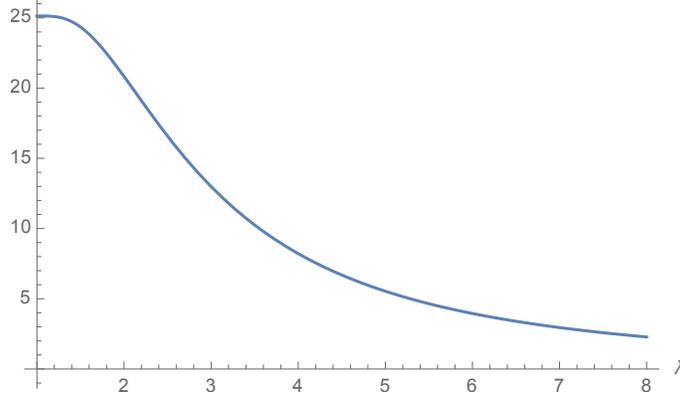
$$Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi - \frac{8}{3} \frac{m_s}{\lambda^2} + O\left(\frac{s^2}{\lambda^3}\right) = 4\pi + 48\pi \frac{s^2}{\lambda^2} (1 + o_s(1)) + O\left(\frac{s^2}{\lambda^3}\right),$$

for λ large.

Remark 5.7. *The above function $\lambda \mapsto \mathcal{A}_\lambda$ is positive and strictly decreasing for $\lambda > 1$, see the picture below. Notice that the matching of the first-order correction terms for λ large in the above two propositions: the expansions are indeed obtained with two completely different approaches. However, while the mass does not appear in the expansions of Section 6, it is somehow hidden in the fact that there we are using standard coordinates on S^3 , and not CR normal coordinates.*

We can finally prove our second main result.

PROOF OF THEOREM 1.2. Assume by contradiction that u is a minimizer of the CR-Sobolev quotient $Q_{(s)}$ for $s \neq 0$ small. By Lemma 5.1, u must then lie in a δ -neighborhood of the manifold \mathcal{M} defined in (82). From the second part of Proposition 5.2 we have also that $u = \varphi_\lambda + w_\lambda$ up to a homogeneous action on S^3 , where w_λ is as in the first part of the Proposition. The conclusion then follows from Proposition 5.5 and Proposition 5.6, which cover all ranges of λ for s small enough. ■



6. APPENDIX A: PROOF OF PROPOSITION 5.5

We consider the *Cayley map* from S^3 into \mathbb{H}^1 given by

$$(87) \quad \mathcal{F}(z_1, z_2) = \left(\frac{z_1}{1+z_2}, \operatorname{Re} \left(i \frac{1-z_2}{1+z_2} \right) \right),$$

with inverse

$$\mathcal{F}^{-1}(z, t) = \left(\frac{2iz}{t+i(1+|z|^2)}, \frac{-t+i(1-|z|^2)}{t+i(1+|z|^2)} \right).$$

Using \mathcal{F} , we can derive explicit expressions for the CR maps on S^3 . Letting \mathfrak{d}_λ denote the natural dilation in the Heisenberg group

$$\mathfrak{d}_\lambda(z, t) = (\lambda z, \lambda^2 t); \quad \lambda > 0,$$

consider the map $\Phi_\lambda : S^3 \rightarrow S^3$ defined by

$$\Phi_\lambda(p) = (\mathcal{F}^{-1} \circ \mathfrak{d}_\lambda \circ \mathcal{F})(p).$$

By explicit computations one finds that the inverse is given by

$$(88) \quad \Phi_\lambda^{-1}(z_1, z_2) = \left(\frac{2\lambda(\bar{z}_2 + 1)z_1}{\lambda^2|z_2 + 1|^2 + \bar{z}_2 + |z_1|^2 - z_2}, \frac{\lambda^2|z_2 + 1|^2 - \bar{z}_2 - |z_1|^2 + z_2}{\lambda^2|z_2 + 1|^2 + \bar{z}_2 + |z_1|^2 - z_2} \right).$$

For later purposes the following formula will be useful

$$(89) \quad \varphi_\lambda(\Phi_\lambda^{-1}(z_1, z_2))^{-3} = \frac{1}{2}\lambda^{-1} \left(\frac{|1+z_2|^2}{(\lambda^2|1+z_2|^2 + |z_1|^2)^2 - (z_2 - \bar{z}_2)^2} \right)^{\frac{1}{2}}.$$

Notice also that $\varphi_{\lambda=1} \equiv 1$ on S^3 .

6.1. Approximate solutions. We construct next, on every compact interval in the range of λ , approximate solutions to the CR-Yamabe equation with $s \neq 0$ up an order $O(s^2)$, improving the accuracy of the φ_λ 's (approximate up to order $O(s)$) for $s \neq 0$.

Lemma 6.1. *Let $\Lambda > 1$ be a fixed number. Then there exist $C_\Lambda > 0$ and regular functions \hat{w}_λ , depending smoothly on λ such that for $\lambda \in [1/\Lambda, \Lambda]$ and for s small one has*

$$L_s(\varphi_\lambda + s\hat{w}_\lambda) - 2(\varphi_\lambda + s\hat{w}_\lambda)^3 = f_\lambda,$$

with $\|f_\lambda\|_{L^\infty(S^3)} \leq C_\Lambda s^2$.

PROOF. Recall that the extremals of the CR-Sobolev inequality (up to a homogeneous CR-action of S^3) have the expression in (80), namely

$$\varphi_\lambda = \lambda \left(\frac{(|z_1|^2 + |z_2 + 1|^2)^2 - (z_2 - \bar{z}_2)^2}{(\lambda^2|z_1|^2 + |z_2 + 1|^2)^2 - \lambda^4(z_2 - \bar{z}_2)^2} \right)^{\frac{1}{2}} = 2\lambda \left(\frac{|1+z_2|^2}{(\lambda^2|z_1|^2 + |z_2 + 1|^2)^2 - \lambda^4(z_2 - \bar{z}_2)^2} \right)^{\frac{1}{2}},$$

and for all $\lambda > 0$ they satisfy the equation

$$(90) \quad L_0 \varphi_\lambda = -4\Delta_b \varphi_\lambda + 2\varphi_\lambda = 2\varphi_\lambda^3 \quad \text{on } S^3.$$

Our goal is to find a correction $s\hat{w}_\lambda$ such that $\varphi_\lambda + s\hat{w}_\lambda$ satisfies the CR-Yamabe equation on $(S^3, J_{(s)})$ up to an order s^2 . Recalling (21) and (22), it is sufficient to solve for

$$-4\Delta_b \hat{w}_\lambda + 2\hat{w}_\lambda - 6\varphi_\lambda^2 \hat{w}_\lambda = \mathcal{G}_\lambda := 8Z_1 Z_1 \varphi_\lambda + \text{conj.}$$

From a straightforward computation one has that

$$\begin{aligned} \mathcal{G}_\lambda(z_1, z_2) &= \frac{192\lambda(\lambda^2 - 1)^2 |1 + z_2|^8 \operatorname{Re} \left[z_1^2 (1 + z_2 + \lambda^2(z_2 - 1))^2 \right]}{\left[(\lambda^2 |z_1|^2 + |z_2 + 1|^2)^2 - \lambda^4 (z_2 - \bar{z}_2)^2 \right]^{\frac{5}{2}} \left[(|z_1|^2 + |z_2 + 1|^2)^2 - (z_2 - \bar{z}_2)^2 \right]^{\frac{3}{2}}} \\ &= \frac{3}{4} \lambda^{-4} \frac{(\lambda^2 - 1)^2 \operatorname{Re} \left[z_1^2 (1 + z_2 + \lambda^2(z_2 - 1))^2 \right]}{\left[(\lambda^2 |z_1|^2 + |z_2 + 1|^2)^2 - \lambda^4 (z_2 - \bar{z}_2)^2 \right]^{\frac{5}{2}}} \varphi_\lambda(z_1, z_2)^5. \end{aligned}$$

It is useful to evaluate this expression after composing with the inverse CR map defined in (88): by direct computation, using also (88), one finds that

$$(91) \quad \begin{aligned} \mathcal{G}_\lambda(\Phi_\lambda^{-1}(z_1, z_2)) &= \frac{3}{4} \lambda^{-3} (\lambda^2 - 1)^2 \left(\frac{|1 + z_2|^2}{(\lambda^2 |1 + z_2|^2 + |z_1|^2)^2 - (z_2 - \bar{z}_2)^2} \right)^{\frac{3}{2}} \\ &\times \left[z_1^2 (1 - \bar{z}_2 + \lambda^2(1 + \bar{z}_2))^4 + \bar{z}_1^2 (1 - z_2 + \lambda^2(1 + z_2))^4 \right]. \end{aligned}$$

Let us recall the covariance of the conformal sub-Laplacian L_θ : for a conformal contact form $\tilde{\theta} = u^{\frac{4}{Q-2}} \theta$ one has

$$L_{\tilde{\theta}} \varphi = u^{-\frac{Q+2}{Q-2}} L_\theta(u \cdot).$$

Let $\mathfrak{L}_{\varphi_\lambda}$ be the linearized CR-Yamabe operator at φ_λ on $(S^3, J_{(0)})$, i.e.

$$(92) \quad \mathfrak{L}_{\varphi_\lambda} v = -4\Delta_b v + 2v - 6\varphi_\lambda^2 v,$$

and let \mathfrak{w}_λ denote the pull-back of \hat{w}_λ via Φ_λ , namely

$$(93) \quad \mathfrak{w}_\lambda(z) = \varphi_\lambda^{-1}(\Phi_\lambda^{-1}(z)) \hat{w}_\lambda(\Phi_\lambda^{-1}(z)).$$

Then the covariance of L_θ implies that

$$(94) \quad (\mathfrak{L}_{\varphi_1 \equiv 1} \mathfrak{w}_\lambda)(x) = \varphi_\lambda(\Phi_\lambda^{-1}(x))^{-3} (\mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda)(\Phi_\lambda^{-1}(x)).$$

It follows from this formula and (91) that the pull-back \mathfrak{w}_λ satisfies the following equation on S^3 , which has constant coefficients on the left-hand side

$$(95) \quad -4\Delta_b \mathfrak{w}_\lambda - 4\mathfrak{w}_\lambda = 12(\lambda^2 - 1)^2 \operatorname{Re} \frac{(1 - \bar{z}_2 + \lambda^2(1 + \bar{z}_2)) z_1^2}{(1 - z_2 + \lambda^2(1 + z_2))^3}.$$

The latter equation can be solved explicitly in \mathfrak{w}_λ via Fourier decomposition: in fact, the right-hand side in (95) is given by

$$12 \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \operatorname{Re} \frac{z_1^2 (1 - \Gamma \bar{z}_2)}{(1 - \Gamma z_2)^3}; \quad \text{with } \Gamma = \frac{1 - \lambda^2}{1 + \lambda^2}.$$

Since we have the expansion

$$\frac{1}{(1 - \Gamma z_2)^3} = 1 + 3\Gamma z_2 + 6\Gamma^2 z_2^2 + 10\Gamma^3 z_2^3 + 15\Gamma^4 z_2^4 + \dots,$$

we obtain that

$$(96) \quad \begin{aligned} 12 \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \operatorname{Re} \frac{z_1^2 (1 - \Gamma \bar{z}_2)}{(1 - \Gamma z_2)^3} &= 12 \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \operatorname{Re} \left\{ z_1^2 \left[1 + 3\Gamma z_2 + 6\Gamma^2 z_2^2 + 10\Gamma^3 z_2^3 + 15\Gamma^4 z_2^4 + \dots \right. \right. \\ &\left. \left. - \Gamma \bar{z}_2 (1 + 3\Gamma z_2 + 6\Gamma^2 z_2^2 + 10\Gamma^3 z_2^3 + 15\Gamma^4 z_2^4 + \dots) \right] \right\}. \end{aligned}$$

While the set of functions of the right-hand side in the first line are spherical harmonics, i.e. satisfying

$$(97) \quad -\Delta_b(z_1^2 z_2^k) = (k+2)z_1^2 z_2^k,$$

the functions on the second line of the right-hand side are not. However, they can be easily modified in order to satisfy an eigenvalue equation. More precisely, one has that (see [JL89])

$$(98) \quad -\Delta_b \left(z_2^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) \right) = (10+3k) \left(z_2^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) \right).$$

Hence we rewrite the right-hand side in (95) in the following way

$$\begin{aligned} & 12 \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \operatorname{Re} \frac{z_1^2 (1 - \Gamma \bar{z}_2)}{(1 - \Gamma z_2)^3} \\ = & 12 \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \operatorname{Re} \left\{ z_1^2 \left[1 + 3\Gamma z_2 + 6\Gamma^2 z_2^2 + 10\Gamma^3 z_2^3 + 15\Gamma^4 z_2^4 + \dots \right. \right. \\ & - \Gamma \bar{z}_2 - 3\Gamma^2 \left(z_2 \bar{z}_2 - \frac{1}{4} \right) - 6\Gamma^3 z_2 \left(z_2 \bar{z}_2 - \frac{2}{5} \right) - 10\Gamma^4 z_2^2 \left(z_2 \bar{z}_2 - \frac{3}{6} \right) + \dots \\ & \left. \left. - 3\Gamma^2 \frac{1}{4} - 6\Gamma^3 z_2 \frac{2}{5} - 10\Gamma^4 z_2^2 \frac{3}{6} - \dots \right] \right\}. \end{aligned}$$

The latter expression can in turn be rewritten as

$$(99) \quad \begin{aligned} & 12\Gamma^2 \operatorname{Re} \left\{ z_1^2 \left[\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} (\Gamma z_2)^k \left(1 - \Gamma^2 \frac{k+3}{k+4} \right) \right. \right. \\ & \left. \left. - \Gamma^2 \sum_{k=-1}^{\infty} \frac{(k+2)(k+3)}{2} (\Gamma z_2)^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) \right] \right\}. \end{aligned}$$

Recall that by (95), to obtain \mathfrak{w}_λ , we need to invert the operator $-4\Delta_b - 4$ on the latter expression, so we have to divide the coefficients of the spherical harmonics respectively by (using (97) and (98)) $4(k+2) - 4 = 4(k+1)$ and by $4(3k+10) - 4 = 12(k+3)$. We then find

$$(100) \quad \mathfrak{w}_\lambda = \frac{3}{2} \Gamma^2 \operatorname{Re} \left\{ z_1^2 \left[\sum_{k=0}^{\infty} (k+2) (\Gamma z_2)^k \left(1 - \Gamma^2 \frac{k+3}{k+4} \right) - \frac{1}{3} \Gamma^2 \sum_{k=-1}^{\infty} (k+2) (\Gamma z_2)^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) \right] \right\},$$

with $\Gamma = \frac{1-\lambda^2}{1+\lambda^2}$. Notice that since $|\Gamma| < 1$ all the above series are absolutely converging on S^3 . Finally, the correction \hat{w}_λ to φ_λ for the CR-Yamabe equation can be obtained from (93). ■

6.2. Second order expansion of the CR Sobolev quotient. We want next to analyse the order s^2 in the expansion of the CR Sobolev quotient.

Lemma 6.2. *If $Q_{(s)}$ is as in (83), then we have that*

$$Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) = 4\pi + \frac{16\pi\lambda^2(3 + 12\lambda^2 + 2\lambda^4 + 12\lambda^6 + 3\lambda^8)}{(1 + \lambda^2)^6} s^2 + \mathcal{B}_{\lambda,s},$$

with $|\mathcal{B}_{\lambda,s}| \leq C_\Lambda s^3$.

PROOF. Recall that, at $s = 0$, from (22) one has $\frac{d}{ds} R_s = 0$ and $\frac{d^2}{ds^2} R_s = 8$. We use the choice of contact form

$$\hat{\theta} = \frac{1}{2} i \sum_{i=1}^2 (z_i d\bar{z}_i - \bar{z}_i dz_i); \quad \hat{\theta} \wedge d\hat{\theta} = 2 d\sigma_{Eucl}.$$

From the expression of $-\ddot{\Delta}_b$ (in (21)) and of \ddot{R} we have that the second derivative $\ddot{Q}(\varphi_\lambda)$ of $Q_{(s)}(\varphi_\lambda)$ at $s = 0$ is given by

$$\ddot{Q}(\varphi_\lambda) = \int_{S^3} \varphi_\lambda \left(-4\ddot{\Delta}_b \varphi_\lambda + \ddot{R} \varphi_\lambda \right) \hat{\theta} \wedge d\hat{\theta} = \int_{S^3} \varphi_\lambda \left(-16\Delta_b \varphi_\lambda + 8\varphi_\lambda \right) \hat{\theta} \wedge d\hat{\theta}.$$

Using (90), this also becomes

$$(101) \quad \ddot{Q}(\varphi_\lambda) = 8 \int_{S^3} \varphi_\lambda^4 \hat{\theta} \wedge d\hat{\theta} = 32\pi^2,$$

since the integral is independent of λ and since $\varphi_{\lambda=1} \equiv 1$.

Our next goal is to expand to second order in s the quantity $Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda)$. We claim that

$$(102) \quad \begin{aligned} Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) &= 4\pi + \frac{s^2}{2\pi} \left(\frac{1}{2} \ddot{Q}(\varphi_\lambda) - \int_{S^3} \hat{w}_\lambda \mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda \hat{\theta} \wedge d\hat{\theta} \right) + o(s^2) \\ &= 4\pi + \frac{s^2}{2\pi} \left(\frac{1}{2} \ddot{Q}(\varphi_\lambda) - \int_{S^3} \mathfrak{w}_\lambda \mathfrak{L}_{\varphi_1} \mathfrak{w}_\lambda \hat{\theta} \wedge d\hat{\theta} \right) + o(s^2). \end{aligned}$$

Here, \mathfrak{w}_λ is given in (100) (see also (93)) and $\mathfrak{L}_{\varphi_\lambda}$ is given in (92). The latter equality follows from the covariance property (94). To check this claim, we want to expand $Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda)$, which we write as

$$\frac{\int_{S^3} (\varphi_\lambda + s\hat{w}_\lambda) \left(L_0 + s\dot{L} + \frac{1}{2}s^2\ddot{L} \right) (\varphi_\lambda + s\hat{w}_\lambda) \hat{\theta} \wedge d\hat{\theta}}{\left(\int_{S^3} (\varphi_\lambda + s\hat{w}_\lambda)^4 \hat{\theta} \wedge d\hat{\theta} \right)^{\frac{1}{2}}}.$$

Expanding in s we find that this quantity is equal to

$$\begin{aligned} & Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) \\ &= \frac{\int_{S^3} \left[\varphi_\lambda L_0 \varphi_\lambda + s \left(\varphi_\lambda \dot{L} \varphi_\lambda + 2\hat{w}_\lambda L_0 \varphi_\lambda \right) + s^2 \left(\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda + 2\hat{w}_\lambda \dot{L} \varphi_\lambda + \hat{w}_\lambda L_0 \hat{w}_\lambda \right) \right] \hat{\theta} \wedge d\hat{\theta}}{\left(\int_{S^3} (\varphi_\lambda^4 + 4s\varphi_\lambda^3 \hat{w}_\lambda + 6s^2 \varphi_\lambda^2 \hat{w}_\lambda^2) \hat{\theta} \wedge d\hat{\theta} \right)^{\frac{1}{2}}} \\ &+ o(s^2). \end{aligned}$$

The first-order term in s vanishes, as one can see using the Euler equation for φ_λ , so we will just consider the second-order term. Since \mathfrak{w}_λ only consists of spherical harmonics of positive order, see (100), using (93) it also turns out that

$$\int_{S^3} \varphi_\lambda^3 \hat{w}_\lambda \hat{\theta} \wedge d\hat{\theta} = \int_{S^3} \mathfrak{w}_\lambda \hat{\theta} \wedge d\hat{\theta} = 0,$$

so there is no contribution to the expansion of the denominator from the first-order term (in s) in the denominator.

Since $\int_{S^3} \varphi_\lambda L_0 \varphi_\lambda \hat{\theta} \wedge d\hat{\theta} = 8\pi^2$ and $\int_{S^3} \varphi_\lambda^4 \hat{\theta} \wedge d\hat{\theta} = 4\pi^2$, we can collect these numbers in the numerator and denominator respectively to get that

$$Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) = \frac{8\pi^2}{(4\pi^2)^{1/2}} \frac{1 + \frac{s^2}{8\pi^2} \int_{S^3} (\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda + 2\hat{w}_\lambda \dot{L} \varphi_\lambda + \hat{w}_\lambda L_0 \hat{w}_\lambda) \hat{\theta} \wedge d\hat{\theta}}{\left(1 + \frac{s^2}{4\pi^2} \int_{S^3} 6\varphi_\lambda^2 \hat{w}_\lambda^2 \hat{\theta} \wedge d\hat{\theta} \right)^{\frac{1}{2}}} + o(s^2).$$

Taylor-expanding one finds

$$\begin{aligned} & Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) \\ &= \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} (\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda + 2\hat{w}_\lambda \dot{L} \varphi_\lambda + \hat{w}_\lambda L_0 \hat{w}_\lambda) \hat{\theta} \wedge d\hat{\theta} - \int_{S^3} 6\varphi_\lambda^2 \hat{w}_\lambda^2 \hat{\theta} \wedge d\hat{\theta} \right) \right] \\ &+ o(s^2). \end{aligned}$$

We now use the fact that \hat{w}_λ satisfies

$$\mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda := L_0 \hat{w}_\lambda - 6\varphi_\lambda^2 \hat{w}_\lambda = -\dot{L} \varphi_\lambda$$

to deduce that

$$(103) \quad \begin{aligned} Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) &= \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} (\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda - \hat{w}_\lambda \mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda) \hat{\theta} \wedge d\hat{\theta} \right) \right] + o(s^2) \\ &= \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} (\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda - \mathfrak{w}_\lambda \mathfrak{L}_{\varphi_1} \mathfrak{w}_\lambda) \hat{\theta} \wedge d\hat{\theta} \right) \right] + o(s^2). \end{aligned}$$

We next compute the latter integral. To explicitly integrate spherical harmonics, we need the following explicit formula (see Proposition 5.3 in [JL89])

$$(104) \quad \int_{S^3} |z_1|^4 |z_2|^{2k} \hat{\theta} \wedge d\hat{\theta} = \frac{8\pi^2}{(k+1)(k+2)(k+3)}.$$

Both \mathbf{w}_λ and $\mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda$ consist of two types of spherical harmonics, orthogonal to each-other. For the first series, taking real parts, we need to compute integrals of the form (notice that only products of conjugate terms contribute)

$$\frac{1}{4} \int_{S^3} (z_1^2 z_2^k + \bar{z}_1^2 \bar{z}_2^k)^2 \hat{\theta} \wedge d\hat{\theta} = \frac{4\pi^2}{(k+1)(k+2)(k+3)}.$$

For the second series, still taking real parts, we need to compute instead

$$\begin{aligned} & \frac{1}{4} \int_{S^3} \left[z_1^2 z_2^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) + \bar{z}_1^2 \bar{z}_2^k \left(z_2 \bar{z}_2 - \frac{k+1}{k+4} \right) \right]^2 \hat{\theta} \wedge d\hat{\theta} \\ &= \frac{1}{2} \int_{S^3} |z_1|^4 |z_2|^{2k} \left(|z_2|^4 - 2|z_2|^2 \frac{k+1}{k+4} + \left(\frac{k+1}{k+4} \right)^2 \right) \hat{\theta} \wedge d\hat{\theta}. \end{aligned}$$

Using (104), the expression becomes

$$\frac{12\pi^2}{(k+2)(k+3)(k+4)^2(k+5)}.$$

Therefore, from (99) and (100) we obtain

$$\begin{aligned} - \int_{S^3} \mathbf{w}_\lambda \mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda \hat{\theta} \wedge d\hat{\theta} &= -24 \frac{3}{2} \Gamma^4 \left\{ \sum_{k=0}^{\infty} \frac{(k+1)(k+2)^2}{2} \left(1 - \Gamma^2 \frac{k+3}{k+4} \right)^2 \Gamma^{2k} \frac{2\pi^2}{(k+1)(k+2)(k+3)} \right. \\ &\quad \left. + \sum_{k=-1}^{\infty} \frac{\Gamma^4}{6} (k+2)^2 (k+3) \Gamma^{2k} \frac{6\pi^2}{(k+2)(k+3)(k+4)^2(k+5)} \right\}. \end{aligned}$$

After some simplification, this gives

$$- \int_{S^3} \mathbf{w}_\lambda \mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda \hat{\theta} \wedge d\hat{\theta} = -36\Gamma^4 \pi^2 \left\{ \sum_{k=0}^{\infty} \frac{k+2}{k+3} \left(1 - \Gamma^2 \frac{k+3}{k+4} \right)^2 \Gamma^{2k} + \sum_{k=-1}^{\infty} \Gamma^4 \frac{\Gamma^{2k}(k+2)}{(k+4)^2(k+5)} \right\}.$$

Notice that the last series starts from $k = -1$, so after relabelling we get

$$- \int_{S^3} \mathbf{w}_\lambda \mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda \hat{\theta} \wedge d\hat{\theta} = -36\Gamma^4 \pi^2 \left\{ \sum_{k=0}^{\infty} \frac{k+2}{k+3} \left(1 - \Gamma^2 \frac{k+3}{k+4} \right)^2 \Gamma^{2k} + \sum_{k=0}^{\infty} \Gamma^2 \frac{\Gamma^{2k}(k+1)}{(k+3)^2(k+4)} \right\}.$$

After some manipulation, the series reduces to a finite one, and we find

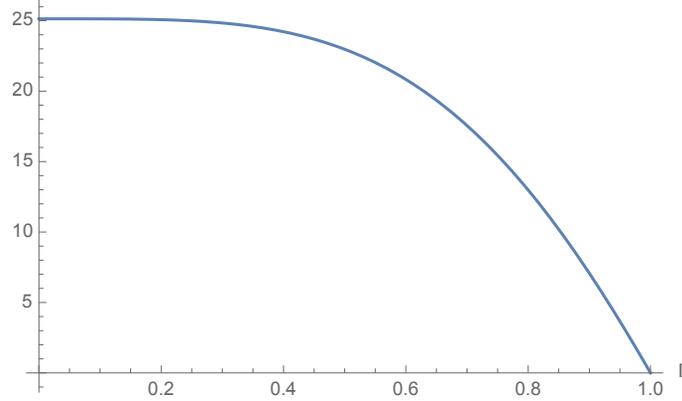
$$- \int_{S^3} \mathbf{w}_\lambda \mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda \hat{\theta} \wedge d\hat{\theta} = 8\pi^2 \Gamma^4 (\Gamma^2 - 3).$$

Collecting this formula and (103), from (101) and (102) we obtain the second order expansion

$$\begin{aligned} Q_{(s)}(\varphi_\lambda + s \hat{w}_\lambda) &= 4\pi + \frac{s^2}{2\pi} \left(\frac{1}{2} \ddot{Q}(\varphi_\lambda) - \int_{S^3} \mathbf{w}_\lambda \mathfrak{L}_{\varphi_1} \mathbf{w}_\lambda \hat{\theta} \wedge d\hat{\theta} \right) + o(s^2) \\ (105) \quad &= 4\pi + 4\pi s^2 (\Gamma^6 - 3\Gamma^4 + 2) + o(s^2), \quad \Gamma = \frac{1 - \lambda^2}{1 + \lambda^2}. \end{aligned}$$

This concludes the proof. ■

We display next the graph of the function $4\pi(\Gamma^6 - 3\Gamma^4 + 2)$ in Γ . This shows that the second-order correction of the Sobolev quotient is always positive in λ , and tends to zero as $\lambda \rightarrow \infty$.



6.3. Conclusion. We can use the observation in Remark 5.3, to work out the contraction argument in Proposition 5.2 starting from $\varphi_\lambda + s\hat{w}_\lambda$ instead of from φ_λ only. Given the improved accuracy in Lemma 6.1, the contraction can be performed in a ball of radius $O(s^2)$ in $\mathfrak{S}^{1,2}(S^3)$, yielding a corresponding correction \check{w}_λ of that order. By Lemma 6.1 and the smoothness of $Q_{(s)}$, we then have that

$$Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda + \check{w}_\lambda) = Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) + dQ_{(s)}(\varphi_\lambda + s\hat{w}_\lambda)[\check{w}_\lambda] + O(\|\check{w}_\lambda\|^2) = Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) + O(s^4).$$

By uniqueness in the fixed point of the contraction, it must be $\varphi_\lambda + s\hat{w}_\lambda + \check{w}_\lambda = \varphi_\lambda + w_\lambda$, so the conclusion follows from Lemma 6.2.

7. APPENDIX B: PROOF OF PROPOSITION 5.6

The goal of this section is to expand $Q_{(s)}$ on the functions $\varphi_\lambda + w_\lambda$ given by Proposition 5.2 for large values of λ . Since the estimates of the previous section deteriorate for λ in this range, we choose approximate solutions in terms of CR normal coordinates, better suited for highly-concentrated profiles.

Recall from the results in Section 5 of [CMY17] that, given $p \in M$, the Green's function of the conformal sub-Laplacian satisfies, in CR normal coordinates

$$(106) \quad G_p = 2\rho^{-2} + A + O(\rho).$$

7.1. Approximate solutions. For $p \in S^3$, fix a small number $r > 0$ and define in CR normal coordinates a function F such that

$$\begin{cases} F(z, t) = |z|^2 & \text{for } \rho \leq r; \\ F \equiv 0 & \text{for } \rho \geq 2r. \end{cases}$$

In this way, F can be extended via cut-offs to all of S^3 as the zero function away from p , so F can be written as

$$(107) \quad F(z, t) = |z|^2 + O(\rho^5).$$

For $\lambda > 0$ large, let us consider a test function in CR normal coordinates as follows

$$(108) \quad \check{\varphi}_\lambda = \frac{\lambda}{\left(1 + \lambda^2 F + \lambda^4 \tilde{G}\right)^{\frac{1}{2}}},$$

where $\tilde{G} = G_p^{-2}$.

Lemma 7.1. *In CR normal coordinates one has the expansion*

$$L_b \check{\varphi}_\lambda = \check{\varphi}_\lambda^3 (2 + O(\rho^3) + \lambda^{-2} O(\rho^2)) + \check{\varphi}_\lambda^5 \left[-\frac{3}{2} |z|^2 \rho^2 (4 + \lambda^2 |z|^2) A + O(\rho^5) + O(\lambda^2 \rho^7) \right].$$

PROOF. By direct computation we have that

$$\check{\varphi}_{\lambda,1} = -\frac{1}{2} \frac{\lambda^3}{\left(1 + \lambda^2 F(z) + \lambda^4 \tilde{G}\right)^{\frac{3}{2}}} \left[F_{,1} + \lambda^2 \tilde{G}_{,1}\right] = -\frac{1}{2} \check{\varphi}_{\lambda}^3 \left[F_{,1} + \lambda^2 \tilde{G}_{,1}\right],$$

and similarly for its conjugate. As a consequence, we have that

$$\check{\varphi}_{\lambda,1\bar{1}} = -\frac{1}{2} \check{\varphi}_{\lambda}^3 \left[F_{,1\bar{1}} + \lambda^2 \tilde{G}_{,1\bar{1}}\right] + \frac{3}{4} \check{\varphi}_{\lambda}^5 \left|F_{,1} + \lambda^2 \tilde{G}_{,1}\right|^2,$$

which implies

$$\Delta_b \check{\varphi}_{\lambda} = -\frac{1}{2} \check{\varphi}_{\lambda}^3 \left[\Delta_b F + \lambda^2 \Delta_b \tilde{G}\right] + \frac{3}{2} \check{\varphi}_{\lambda}^5 \left|F_{,1} + \lambda^2 \tilde{G}_{,1}\right|^2.$$

By direct computation one finds (with $G = G_p$)

$$\Delta_b \tilde{G} = -2G^{-3} \Delta_b G + 12G^{-4} G_{,1} G_{,\bar{1}}.$$

We then deduce

$$L_b \check{\varphi}_{\lambda} = 2 \check{\varphi}_{\lambda}^3 \left[\Delta_b F + \lambda^2 (12G^{-4} G_{,1} G_{,\bar{1}} - 2G^{-3} \Delta_b G)\right] - 6 \check{\varphi}_{\lambda}^5 \left|F_{,1} + \lambda^2 \tilde{G}_{,1}\right|^2 + R \check{\varphi}_{\lambda}.$$

We can next write

$$R \check{\varphi}_{\lambda} = R \check{\varphi}_{\lambda}^3 (\lambda^{-2} + F + \lambda^2 G^{-3} G).$$

Since G satisfies $L_b G = 0$, we get some cancellation and find that

$$L_b \check{\varphi}_{\lambda} = \check{\varphi}_{\lambda}^3 (2\Delta_b F + \lambda^{-2} R + R F) + 6 \check{\varphi}_{\lambda}^5 \left[4G^{-4} G_{,1} G_{,\bar{1}} \left(1 + \lambda^2 F(z) + \lambda^4 \tilde{G}\right) - \left|F_{,1} + \lambda^2 \tilde{G}_{,1}\right|^2\right].$$

Using some further cancellation we then obtain

$$L_b \check{\varphi}_{\lambda} = \check{\varphi}_{\lambda}^3 (2\Delta_b F + \lambda^{-2} R + R F) + 6 \check{\varphi}_{\lambda}^5 \left[4G^{-4} G_{,1} G_{,\bar{1}} (1 + \lambda^2 F(z)) - F_{,1} F_{,\bar{1}} - \lambda^2 (F_{,1} \tilde{G}_{,\bar{1}} + \tilde{G}_{,1} F_{,\bar{1}})\right].$$

From Proposition A.5 in [CMY17] (where a different but analogous notation is used) one has that, in CR normal coordinates

$$\begin{aligned} Z_1 &= (1 + O(\rho^4)) \overset{\circ}{Z}_1 + O(\rho^4) \overset{\circ}{Z}_{\bar{1}} + O(\rho^5) \frac{\partial}{\partial t}; \\ \omega_1^1 &= O(\rho^3) dz + O(\rho^3) d\bar{z} + O(\rho^2) \overset{\circ}{\theta}, \end{aligned}$$

see (18). By direct computation, one then has

$$\begin{aligned} G_{,1} &= -\frac{i\sqrt{2}\bar{z}}{(t + i|z|^2)\rho^2} + O(1); & F_{,1} &= \frac{\bar{z}}{\sqrt{2}} + O(\rho^4); \\ \tilde{G}_{,1} &= \frac{2\sqrt{2}\bar{z}(|z|^2 + it)}{(A\rho^2 + 2)^3} + O(\rho^6); & \Delta_b F &= 1 + O(\rho^3). \end{aligned}$$

Using these expressions in the above formula for $L_b \check{\varphi}_{\lambda}$ one finally finds

$$L_b \check{\varphi}_{\lambda} = \check{\varphi}_{\lambda}^3 (2 + \lambda^{-2} O(\rho^2) + O(\rho^3)) + 6 \check{\varphi}_{\lambda}^5 \left[-\frac{1}{4} |z|^2 \rho^2 (4 + \lambda^2 |z|^2) A + O(\rho^5) + O(\lambda^2 \rho^7)\right],$$

which is the desired result. ■

If the contact form θ involved in the definition of CR normal coordinates writes as $\theta = e^{2v} \hat{\theta}$, setting

$$(109) \quad \bar{\varphi}_{\lambda} = e^{-v} \check{\varphi}_{\lambda},$$

by the covariance property of the conformal sub-Laplacian one has that

$$(110) \quad \check{\varphi}_{\lambda}^4 \theta \wedge d\theta = \bar{\varphi}_{\lambda}^4 \hat{\theta} \wedge d\hat{\theta}; \quad \check{\varphi}_{\lambda} L_b^{(\theta)} \check{\varphi}_{\lambda} \theta \wedge d\theta = \bar{\varphi}_{\lambda} L_s \bar{\varphi}_{\lambda} \hat{\theta} \wedge d\hat{\theta}, \quad L_s = L_b^{(\hat{\theta})}.$$

These imply the invariance

$$Q_{(s)}(\bar{\varphi}_{\lambda}) = \frac{\int_{S^3} \bar{\varphi}_{\lambda} L_s \bar{\varphi}_{\lambda} \hat{\theta} \wedge d\hat{\theta}}{\left(\int_{S^3} \bar{\varphi}_{\lambda}^4 \hat{\theta} \wedge d\hat{\theta}\right)^{\frac{1}{2}}} = \frac{\int_{S^3} \check{\varphi}_{\lambda} L_b^{(\theta)} \check{\varphi}_{\lambda} \theta \wedge d\theta}{\left(\int_{S^3} \check{\varphi}_{\lambda}^4 \theta \wedge d\theta\right)^{\frac{1}{2}}}.$$

We then get the following consequence of Lemma 7.1, concerning the differential of $Q_{(s)}$ at $\bar{\varphi}_\lambda$.

Corollary 7.2. *There exists a constant $C > 0$ such that for all s small and λ large one has the inequality $|dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \leq \frac{C}{\lambda^2} \|v\|_{\mathfrak{S}^{1,2}}$ for every $v \in \mathfrak{S}^{1,2}(S^3)$.*

PROOF. By direct computation, for $v \in \mathfrak{S}^{1,2}(S^3)$, one has

$$(111) \quad dQ_{(s)}(\bar{\varphi}_\lambda)[v] = \frac{2}{\left(\int_{S^3} \bar{\varphi}_\lambda^4 \hat{\theta} \wedge d\hat{\theta}\right)^{\frac{3}{2}}} \int_{S^3} \left[\left(\int_{S^3} \bar{\varphi}_\lambda^4 \hat{\theta} \wedge d\hat{\theta} \right) L_s \bar{\varphi}_\lambda - \left(\int_{S^3} \bar{\varphi}_\lambda L_s \bar{\varphi}_\lambda \hat{\theta} \wedge d\hat{\theta} \right) \bar{\varphi}_\lambda^3 \right] v \hat{\theta} \wedge d\hat{\theta}.$$

From (110) and Lemma 7.1 it follows that

$$\int_{S^3} \bar{\varphi}_\lambda L_s \bar{\varphi}_\lambda \hat{\theta} \wedge d\hat{\theta} = 2 \int_{S^3} \check{\varphi}_\lambda^4 \theta \wedge d\theta + \int_{S^3} [\check{\varphi}_\lambda^4 (O(\rho^3) + \lambda^2 O(\rho^2)) + \check{\varphi}_\lambda^6 (O(\rho^4) + \lambda^2 O(\rho^6))] \theta \wedge d\theta.$$

Using a change of variable it is possible then to show

$$\int_{S^3} \check{\varphi}_\lambda^4 \theta \wedge d\theta = - \int_{S^3} \check{\varphi}_\lambda L_b^\theta \check{\varphi}_\lambda \theta \wedge d\theta + O(\lambda^{-2}).$$

Therefore, inserting the latter estimate and the result of Lemma 7.1 into (111) we find that

$$|dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \leq \int_{S^3} \check{\varphi}_\lambda^3 [O(\rho^2) + O(\lambda^{-2}) + \check{\varphi}_\lambda^5 (O(\rho^4) + \lambda^2 O(\rho^6))] |v| \theta \wedge d\theta.$$

Applying Hölder's inequality we get that

$$\begin{aligned} & |dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \\ & \leq \left[O(\lambda^{-2}) + \left(\int_{S^3} \check{\varphi}_\lambda^4 O(\rho^{\frac{8}{3}}) \right)^{\frac{3}{4}} + \left(\int_{S^3} \check{\varphi}_\lambda^{20/3} O(\rho^{\frac{16}{3}}) \right)^{\frac{3}{4}} + \lambda^2 \left(\int_{S^3} \check{\varphi}_\lambda^{20/3} O(\rho^8) \right)^{\frac{3}{4}} \right] \|v\|_{\mathfrak{S}^{1,2}}, \end{aligned}$$

where all integrals are computed w.r.t. the volume form $\theta \wedge d\theta$. By the expression of $\check{\varphi}_\lambda$, all terms are integrable and of order λ^{-2} , which concludes the proof. ■

7.2. Expansion of the CR Sobolev quotient. We expand next the CR Sobolev quotient $Q_{(s)}$ on the approximate solutions $\bar{\varphi}_\lambda$ in (109), obtaining the following result.

Lemma 7.3. *Let $\check{\varphi}_\lambda$ be defined in (108). Then for λ large one has the expansion*

$$Q_{(s)}(\bar{\varphi}_\lambda) = 4\pi + 48\pi \frac{s^2}{\lambda^2} (1 + o_s(1)) + O\left(\frac{1}{\lambda^3}\right).$$

PROOF. We use (110), Lemma 7.1 and integrate: expanding the numerator in $Q_{(s)}$ we find that

$$\begin{aligned} \int_{S^3} \check{\varphi}_\lambda L_b^{(\theta)} \check{\varphi}_\lambda \theta \wedge d\theta &= 2 \int_{S^3} \check{\varphi}_\lambda^4 \theta \wedge d\theta - \frac{3}{2} A \int_{\mathbb{H}^1} |z|^2 (4 + \lambda^2 |z|^2) \rho^2 \check{\varphi}_\lambda^6 \hat{\theta} \wedge d\hat{\theta} \\ &+ \int_{S^3} \varphi^4 (O(\lambda^2 \rho^2) + O(\rho^3)) \theta \wedge d\theta + \int_{S^3} \varphi^6 (O(\rho^5) + O(\lambda^2 \rho^7)) \theta \wedge d\theta, \end{aligned}$$

where

$$\check{\varphi}_\lambda = \frac{\lambda}{\left(1 + \lambda^2 |z|^2 + \frac{1}{4} \lambda^4 (|z|^4 + t^2)\right)^{\frac{1}{2}}}; \quad (z, t) \in \mathbb{H}^1.$$

For the first term, which also appears in the above expression, we Taylor-expand \tilde{G} as

$$\tilde{G} = \left(\frac{2 + A\rho^2}{\rho^2}\right)^{-2} = \frac{1}{4} \rho^4 (1 - A\rho^2) + O(\rho^8).$$

Therefore, $\check{\varphi}_\lambda$ expands as

$$\begin{aligned}\check{\varphi}_\lambda &= \frac{\lambda}{(1 + \lambda^2(|z|^2 + O(\rho^5)) + \frac{1}{4}\lambda^4[\rho^4(1 - A\rho^2) + O(\rho^8)])^{\frac{1}{2}}} \\ &= \left(1 + \frac{1}{8} \frac{A\rho^6\lambda^4}{1 + \lambda^2|z|^2 + \frac{1}{4}\lambda^4\rho^4} + O\left(\frac{\rho^{12}\lambda^8}{(1 + \lambda^4\rho^4)^2}\right)\right) \mathring{\varphi}_\lambda \\ &= \mathring{\varphi}_\lambda + \frac{1}{8}A\rho^6\lambda^2\mathring{\varphi}_\lambda^3 + O\left(\frac{\rho^{12}\lambda^8}{(1 + \lambda^4\rho^4)^2}\right) \mathring{\varphi}_\lambda.\end{aligned}$$

Taylor-expanding the integral of the fourth power of $\check{\varphi}_\lambda$ and using a change of variable we get that

$$\int_{S^3} \check{\varphi}_\lambda^4 \theta \wedge d\theta = \int_{\mathbb{H}^1} \mathring{\varphi}_\lambda^4 \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} + O(1/\lambda^3).$$

Hence, using the fact that $\int_{\mathbb{H}^1} \mathring{\varphi}_\lambda^4 \mathring{\theta} \wedge d\mathring{\theta}$ is independent of λ , $Q_{(s)}(\bar{\varphi}_\lambda)$ becomes

$$\frac{2 \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} \right) - \frac{3A}{2} \int_{\mathbb{H}^1} |z|^2 \rho^2 (4 + \lambda^2|z|^2) \mathring{\varphi}_1^6 \mathring{\theta} \wedge d\mathring{\theta}}{\left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} \right)^{\frac{1}{2}}} + O(1/\lambda^3).$$

We can expand the denominator in the latter expression as

$$\begin{aligned}& \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} \right)^{-\frac{1}{2}} \\ &= \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} \right)^{-\frac{1}{2}} \left(1 + \frac{\frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta}}{\int_{\mathbb{H}^1} \mathring{\varphi}_1^4 \mathring{\theta} \wedge d\mathring{\theta}} \right)^{-\frac{1}{2}} \\ &= \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} \right)^{-\frac{1}{2}} \left(1 - \frac{1}{4}A\lambda^2 \frac{\int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta}}{\int_{\mathbb{H}^1} \mathring{\varphi}_1^4 \mathring{\theta} \wedge d\mathring{\theta}} \right) + O(1/\lambda^3),\end{aligned}$$

which gives

$$\begin{aligned}& Q_{(s)}(\bar{\varphi}_\lambda) \\ &= \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} \right)^{-\frac{1}{2}} \left[2 \int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} - \frac{3}{2}A \int_{\mathbb{H}^1} |z|^2 \rho^2 (4 + \lambda^2|z|^2) \mathring{\varphi}_1^6 \mathring{\theta} \wedge d\mathring{\theta} \right] \\ &+ O\left(\frac{1}{\lambda^3}\right),\end{aligned}$$

equivalent to

$$\begin{aligned}Q_{(s)}(\bar{\varphi}_\lambda) &= \left(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} \right)^{-\frac{1}{2}} \left[2 \int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} - \frac{1}{2}A \int_{\mathbb{H}^1} (3|z|^2(4 + \lambda^2|z|^2) - \lambda^2\rho^4) \rho^2 \mathring{\varphi}_\lambda^6 \mathring{\theta} \wedge d\mathring{\theta} \right] \\ (112) \quad &+ O\left(\frac{1}{\lambda^3}\right).\end{aligned}$$

The computation on page 177 in [JL87] (where θ_1 in their notation equals $2\hat{\theta}$) shows that $\mathring{\varphi}_{\sqrt{2}}^4$ is the scaling factor for the volume of the Cayley map. Recalling that $\hat{\theta} \wedge d\hat{\theta}$ is twice the (induced) Euclidean volume on S^3 , this implies

$$(113) \quad \int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \mathring{\theta} \wedge d\mathring{\theta} = \int_{S^3} \hat{\theta} \wedge d\hat{\theta} = 4\pi^2.$$

We now make the following change of variables $\lambda z \mapsto \sqrt{2}z'$, $\lambda^2 t \mapsto 2t'$, and notice that

$$\mathring{\varphi}_\lambda(z, t) = \frac{\lambda}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2} \left((1 + |z'|^2)^2 + (t')^2 \right)^{\frac{1}{2}}} = \frac{\lambda}{\sqrt{2}} \mathring{\varphi}_{\sqrt{2}}(z', t').$$

In this way we have

$$\int_{\mathbb{H}^1} (3|z|^2(4 + \lambda^2|z|^2) - \lambda^2\rho^4) \rho^2 \mathring{\varphi}_\lambda \mathring{\theta} \wedge d\mathring{\theta}(z, t) = \frac{4}{\lambda^2} \int_{\mathbb{H}^1} (3|z'|^2(2 + |z'|^2) - (\rho')^4) (\rho')^2 \mathring{\varphi}_{\sqrt{2}} \mathring{\theta} \wedge d\mathring{\theta}(z', t').$$

As one can check by direct computations, the primitive w.r.t. t of the integrand is

$$\frac{3(|z'|^6 + 8|z'|^4 + 19|z'|^2 + 8)|z'|^6 \log \left(\frac{t^2(|z'|^4 + 4|z'|^2 + 2) - 2t(|z'|^2 + 1)\sqrt{2|z'|^2 + 1}\sqrt{t^2 + |z'|^4} + (|z'|^2 + 1)^2|z'|^4}{|z'|^4(t^2 + (|z'|^2 + 1)^2)} \right)}{2(|z'|^2 + 1)^5(2|z'|^2 + 1)^{3/2}} \\ - \frac{t\sqrt{t^2 + |z'|^4} \left(t^2(3|z'|^8 + 4|z'|^6 - 17|z'|^4 - 4|z'|^2 + 2) + (|z'|^3 + |z'|)^2(3|z'|^6 - 8|z'|^4 - 55|z'|^2 - 24) \right)}{(|z'|^2 + 1)^4(2|z'|^2 + 1)(t^2 + (|z'|^2 + 1)^2)^2}.$$

As a consequence, we deduce that

$$\int_{\mathbb{R}} (3|z'|^2(2 + |z'|^2) - (\rho')^4) (\rho')^2 \mathring{\varphi}_{\sqrt{2}} dt \\ = \frac{-3(|z'|^6 + 8|z'|^4 + 19|z'|^2 + 8)|z'|^6 \log \left(\frac{|z'|^4 - 2(\sqrt{2|z'|^2 + 1} - 2)|z'|^2 - 2\sqrt{2|z'|^2 + 1} + 2}{|z'|^4 + 2(\sqrt{2|z'|^2 + 1} + 2)|z'|^2 + 2(\sqrt{2|z'|^2 + 1} + 1)} \right)}{2(|z'|^2 + 1)^5(2|z'|^2 + 1)^{3/2}} \\ - \frac{4(|z'|^2 + 1)\sqrt{2|z'|^2 + 1}(3|z'|^8 + 4|z'|^6 - 17|z'|^4 - 4|z'|^2 + 2)}{2(|z'|^2 + 1)^5(2|z'|^2 + 1)^{3/2}}.$$

Multiplying this quantity by $2\pi|z'|$, its primitive w.r.t. $|z'|$ is

$$\frac{\pi \left(3(|z'|^2 + 2)|z'|^8 \log \left(\frac{|z'|^4 - 2(\sqrt{2|z'|^2 + 1} - 2)|z'|^2 - 2\sqrt{2|z'|^2 + 1} + 2}{|z'|^4 + 2(\sqrt{2|z'|^2 + 1} + 2)|z'|^2 + 2(\sqrt{2|z'|^2 + 1} + 1)} \right) + 4\sqrt{2|z'|^2 + 1}(|z'|^6 + 6|z'|^4 + 6|z'|^2 + 1) \right)}{2(|z'|^2 + 1)^4\sqrt{2|z'|^2 + 1}},$$

whose difference between the values $|z'| \rightarrow +\infty$ and $|z'| = 0$ is 8π . Therefore, recalling that the volume form $\mathring{\theta} \wedge d\mathring{\theta}$ is four times the Euclidean one, we obtain that

$$\int_{\mathbb{H}^1} (3|z|^2(4 + \lambda^2|z|^2) - \lambda^2\rho^4) \rho^2 \mathring{\varphi}_\lambda \mathring{\theta} \wedge d\mathring{\theta} = 32\pi.$$

Recalling (113) and the fact that $A = -\frac{3}{2}s^2(1 + o_s(1))$, from (78) and (112) we deduce that

$$Q_{(s)}(\mathring{\varphi}_\lambda) = 4\pi - \frac{32\pi A}{\lambda^2} s^2(1 + o_s(1)) + O\left(\frac{1}{\lambda^3}\right) = 4\pi + \frac{48\pi}{\lambda^2} s^2(1 + o_s(1)) + O\left(\frac{1}{\lambda^3}\right).$$

This concludes the proof. \blacksquare

7.3. Conclusion. We can use the observation in Remark 5.3, to perform the contraction argument in Proposition 5.2 starting from $\mathring{\varphi}_\lambda$ instead of from φ_λ only. Given the improved accuracy in Lemma 5.5, the contraction can be performed in a ball of radius $O(\frac{1}{\lambda^2})$ in $\mathfrak{S}^{1,2}(S^3)$, yielding a corresponding correction \mathring{w}_λ of that order. By Lemma 5.5 and the smoothness of $Q_{(s)}$, we then have similarly to Subsection 6.3

$$Q_{(s)}(\mathring{\varphi}_\lambda + \mathring{w}_\lambda) = Q_{(s)}(\mathring{\varphi}_\lambda) + O(\|\mathring{w}_\lambda\|^2) = Q_{(s)}(\mathring{\varphi}_\lambda) + O\left(\frac{1}{\lambda^4}\right).$$

By uniqueness of the fixed point, it must be $\mathring{\varphi}_\lambda + \mathring{w}_\lambda = \varphi_\lambda + w_\lambda$, so from Lemma 7.3 we get that

$$(114) \quad Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi + 48\pi \frac{s^2}{\lambda^2} (1 + o_s(1)) + O\left(\frac{1}{\lambda^3}\right).$$

Notice that

$$Q_{(0)}(\varphi_\lambda + w_\lambda) = Q_{(0)}(\varphi_\lambda) \equiv 4\pi,$$

and therefore the term $O(\frac{1}{\lambda^3})$ appearing in (114) is identically zero for $s = 0$, even and smooth in s . It therefore must be of the form $O(\frac{s^2}{\lambda^3})$. Hence the statement of the proposition holds true.

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