On the Hölder regularity of the landscape function

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Abstract

In this paper we study the Hölder regularity of the landscape function introduced by Santambrogio in [S]. We develop a new technique which both extends Santambrogio's result to lower Ahlfors regular measures in general dimension h and simplifies its proof.

1 Introduction

In the last decade, a huge attention has been given to optimal transportation problems. We briefly review here the main definitions and concepts that are a premiss to the subject of the paper.

1.1 Monge-Kantorovich transportation problem

This is the original formulation of transportation problems. Given a *macroscopic displacement* $(\mu^+, \mu^-) \in \mathcal{P}(\mathbf{R}^N) \times \mathcal{P}(\mathbf{R}^N)$, the problem consists (in the Monge's version) to find the best *transport map*, i.e. a measurable map $t : \mathbf{R}^N \to \mathbf{R}^N$ such that for all Borel sets B we have $\mu^-(B) = \mu^+(t^{-1}(B))$, which minimizes the cost functional

$$M(t) := \int_{\mathbf{R}^N} c(x, t(x)) \mathrm{d}\mu^+(x). \tag{1.1}$$

Usually c is the p-th power of the Euclidean distance. The set of transport maps between μ^+, μ^- will be denoted by $\mathcal{M}(\mu^+, \mu^-)$.

In the Kantorovich's version of the problem, transport maps are replaced by *transport plans*, i.e. probability measures $\pi \in \mathcal{P}(\mathbf{R}^N \times \mathbf{R}^N)$ such that for all Borel sets $A, B \subseteq \mathbf{R}^N$ we have $\pi(A \times \mathbf{R}^N) = \mu^+(A), \pi(\mathbf{R}^N \times B) = \mu^-(B)$, while the functional becomes

$$K(\pi) := \int_{\mathbf{R}^N \times \mathbf{R}^N} c(x, y) \mathrm{d}\pi(x, y). \tag{1.2}$$

The set of transport plans between μ^+, μ^- will be denoted by $\mathcal{P}(\mu^+, \mu^-)$.

As it is pointed out in [MS-DIST], both the transport maps and the transport plans represent a *microscopic displacement*, that is a transport map or plan tells only the initial and final position of a single particle.

We recall that given a transport map t for the macroscopic displacement (μ^+, μ^-), the measure π_t given by $\pi_t(C) := \mu^+(\{x \in \mathbf{R}^N \ : \ (x, t(x)) \in C\})$ is a transport plan for the same macroscopic displacement and $M(t) = K(\pi_t)$. Kantorovich's problem is then a generalization of Monge's one. Actually, it can be seen (see, for example, [ACBBV-CIME] or [V]) that K is the lower semicontinuous envelope of M w.r.t. the weak convergence of measures.

Wasserstein spaces. In the sequel, we will make use of Wasserstein distances and spaces. Set $c(x, y) = |x - y|^p$, $p \geq 1$. The *Wasserstein distance* of order p between the measures μ^+, μ^- is defined by

$$w_p(\mu^+, \mu^-) := \left(\min_{\pi \in \mathcal{P}(\mu^+, \mu^-)} K(\pi) \right)^{\frac{1}{p}}.$$

If $p = +\infty$, $w_\infty(\mu^+, \mu^-)$ is defined as

$$w_\infty(\mu^+, \mu^-) := \min_{\pi \in \mathcal{P}(\mu^+, \mu^-)} \pi\text{- esssup}\, |x - y|.$$

Given an open set $X \subseteq \mathbf{R}^N$ and $x_0 \in \overline{X}$, the *Wasserstein space* of order p over $\overline{X}$ is the set of measures

$$W_p(\overline{X}) := \{\mu \in \mathcal{P}(\mathbf{R}^N) \ : \ \operatorname{spt} \mu \subseteq \overline{X}, \ w_p(\mu, \delta_{x_0}) < +\infty\}.$$

It can be seen that:

- the definition does not depend on x_0;
- $W_p(\overline{X})$ endowed with the Wasserstein distance w_p is a complete metric space;
- in the case X is bounded, the convergence w.r.t. w_p is equivalent to the weak convergence of measures.

For the details related to the Wasserstein distance see, for example, [V].

1.2 Irrigation models

In order to introduce the concept of *macroscopic/microscopic motion* and of *ramified transportation* several attempts have been done. The first two are the papers by Maddalena, Morel and Solimini ([MMS]) and by Xia ([X]). The model proposed by Xia is the relaxation on vector measures of an appropriate functional defined on weighted directed graphs.

In their approach Maddalena, Morel and Solimini consider paths starting from a source and representing the trajectory in $\mathbf{R}^N$ of a fluid particle or the fiber of a tree. All the paths start from a common source S and the irrigated measure is defined counting how many fibers stop in a given volume.

In the original formulation, given a probability space $(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$ (the *reference space*), a *set of fibers with source point* $S \in \mathbf{R}^N$ is a mapping $\chi : \Omega \times \mathbf{R}_+ \to \mathbf{R}^N$ satisfying the following assumptions:

- for μ_Ω-a.e. $p \in \Omega$, the map $\chi_p(\cdot) := \chi(p, \cdot)$ is 1-Lipschitz;
- for μ_Ω-a.e. $p \in \Omega$, $\chi_p(0) = S$.

Recall that if Ω is a complete separable metric space with uncountable cardinality and μ_Ω has no atoms, then $(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$ is isomorphic to the standard space $([0,1], \mathcal{B}([0,1]), \mathcal{L}^1_{|[0,1]})$ (see, for example, Theorem 16 in Chapter 15, Section 5 of [R]). In the following, we will then always assume that we are in the hypothesis of that result.

In this model the *irrigating measure* is the Dirac mass in the point S, i.e. $\mu^+ = \delta_S$, while the *irrigated measure* is the the image of μ_Ω via the map $p \mapsto i_\chi(p) := \chi(p, \sigma_\chi(p))$, where $\sigma_\chi(p)$ is the *stopping time*, i.e. the infimum of t such that $\chi_p(\cdot)$ is constant on $[t, +\infty]$. This measure is μ^- (but, we will simply write μ in the following). Then, Maddalena, Morel and Solimini consider the functional

$$\chi \mapsto J^0_\alpha(\chi) := \int_{\mathbf{R}_+} \int_{M_t(\chi)} [\mu_\Omega([p]^0_t)]^{\alpha-1} \mathrm{d}p \mathrm{d}t, \tag{1.3}$$

where $M_t(\chi)$ is the set of fibers not stopped at time t, and $[p]^0_t$ is the set of fibers which moves together up to time t, i.e. if $q \in [p]^0_t$, then $\chi_q(s) = \chi_p(s)$ for all $s \in [0, t]$.

Here, and in all models considered in this paper, the irrigated measure μ will be supposed compactly supported. The parameter α belongs to the interval $[0, 1[$; in such case the concavity of the model function $|x|^\alpha$ gives rise to the branched transportation since moving the mass together as much as possible becomes cheaper.

Let us mention the approach by Bernot, Caselles and Morel in [BCM]. They consider the functional given by

$$\chi \mapsto J_\alpha^2(\chi) := \int_\Omega \int_{\mathbf{R}_+} [\mu_\Omega([p]_t^2)]^{\alpha-1} |\dot\chi(p,t)| \mathrm{d}t\mathrm{d}p, \tag{1.4}$$

where $[p]_t^2 := \{q \in \Omega \ : \ \chi(p,t) \in \chi(\{q\} \times I)\}$. In this setting (as in the extended one which will be introduced in the next section), the irrigating and irrigated measure can be chosen arbitrarily among probability measures.

1.3 Extended setting

In this paper we will consider the general framework introduced by Maddalena and Solimini in [MS-DIST] and further developed in [MS-SYNCH] which includes and extends all the formulations from [MMS, BCM].

Definition 1.1 (Irrigation pattern)**.** Let $I \subseteq \mathbf{R}$ be a generic interval. By *irrigation pattern* we will mean a measurable function $\chi : \Omega \times I \to \mathbf{R}^N$ such that $\chi_p \in \mathrm{AC}(I)$ for almost all p. The pattern $\tilde\chi$ will be *equivalent* to χ if the images of μ_Ω through the maps $p \mapsto \chi_p, p \mapsto \tilde\chi_p$ are the same.

Notation. In the whole paper we will always denote by a (respectively, b) the infimum (respectively, supremum) of I.

Definition 1.2 (Solidarity classes)**.** For every $(p,t) \in \Omega \times I$ we consider the sets

$$[p]_t^0 := \{q \in \Omega \ : \ \chi(q,s) = \chi(p,s), \ \ \forall s \in [0,t]\}, \tag{1.5}$$
$$[p]_t^1 := \{q \in \Omega \ : \ \chi(q,t) = \chi(p,t)\}, \tag{1.6}$$
$$[p]_t^2 := \{q \in \Omega \ : \ \chi(p,t) \in \chi(\{q\} \times I)\}. \tag{1.7}$$

For every $i \in \{0,1\}$ and every $t \in I$, $\{[p]_t^i \ : \ p \in \Omega\}$ is a partition of Ω.

The *masses* m_χ^i are given by:

$$m_\chi^i(p,t) := \mu_\Omega([p]_t^i). \tag{1.8}$$

Definition 1.3 (Cost densities, cost functionals)**.** For $i \in \{0,1,2\}$ we consider the following *cost densities*:

$$s_\alpha^i(p,t) := [m_\chi^i(p,t)]^{\alpha-1} \tag{1.9}$$

The *cost functionals* we are interested in will be:

$$J_\alpha^i(\chi) := \int_{\Omega \times I} s_\alpha^i(p,t) |\dot\chi(p,t)| \mathrm{d}p\mathrm{d}t. \tag{1.10}$$

The variational problem considered in this paper will then be the minimization of J^i_α, given the irrigating and irrigated measure. The irrigating measure will always be the Dirac mass δ_S, while the irrigated measure will be denoted by μ (we will write μ_χ to stress that it is the measure irrigated by the pattern χ). Finally, we will denote by $d_\alpha(\delta_S, \mu)$ the least irrigation cost (which is the same for all the functionals as proved in [MS-SYNCH]).

For $i = 0, 1$ the functional is *synchronous*, i.e. if the trajectories of two particles given by an optimal pattern are the same, then they will move together. For $i = 2$, the functional is *asynchronous*, since each particle can move independently on its trajectory, i.e. for every p the function χ_p can be reparameterized (independently) without losing the optimality of the reparameterized pattern. $J^0_\alpha, J^1_\alpha, J^2_\alpha$ are respectively the functionals originally introduced in [MMS], [MS-DIST] and [BCM]. We refer to [MS-SYNCH] for proof of the next theorem, which is the fundamental tool to present the unified theory of the irrigation functionals.

Theorem 1.4 (Synchronization Theorem)**.** *The following statements hold:*

- $J^2_\alpha \le J^1_\alpha \le J^0_\alpha$.
- J^0_α, J^1_α *share the same minima, if the initial mass is a Dirac mass;*
- *every optimal pattern for* J^2_α *can be reparameterized to be a minimum for* J^1_α*, i.e. every optimal pattern for* J^2_α *is synchronizable (see also [BF] for a proof);*
- *every optimal pattern for* J^1_α *is optimal for* J^2_α.

When we will say that χ is *optimal*, we will always mean that χ is a *minimum* for J^1_α (hence, a minimum of J^0_α, too). Notice that by Theorem 1.4 if a result which involves quantities which are invariant under scaling (as, for instance, Santambrogio landscape function introduced in next section) holds for optimal patterns must also hold for minima of J^2_α.

1.4 The landscape function

The proper subject of this paper is the *landscape function* introduced by Santambrogio in [S]. The landscape function is, actually, a natural mathematical object to be introduced in the context of the branched transportation models. Also, it is connected to the shaping of river basins as many works of geophysicists pointed out (we refer to the introduction of [S] for a detailed discussion).

Suppose χ is a optimal pattern for the functional J_α^2 irrigating a measure μ from a Dirac mass and $\alpha > 1 - 1/N$. In a point $x = \chi(p,t)$, the landscape function, as introduced by Santambrogio, is defined by

$$z(x) := \int_0^t [\mu_\Omega([p]_t^2)]^{\alpha-1} |\dot{\chi}_p(s)| \mathrm{d}s. \tag{1.11}$$

It is the transportation cost of the mass from the initial source S to the point x. The main result contained in [S] is the Hölder regularity of the landscape function of exponent $1 + N(1-\alpha)$ if the irrigated measure μ has a density (w.r.t. the Lebesgue measure) bounded from below by a positive constant and its support is of type A, i.e. for every $x_0 \in \operatorname{spt}\mu$ and every $r \in [0, \operatorname{diam}\operatorname{spt}\mu]$ we have $\mathcal{L}^N(\operatorname{spt}\mu \cap B_r(x_0)) \geq Ar^N$. As a consequence, the irrigated measure is Ahlfors from below in dimension N.

The object of this paper is to provide a more general definition for the landscape function which will work for a generic pattern and agrees with Santambrogio's definition for optimal ones. In Section 2 we develop a general theory on the landscape function, where it is introduced in a sufficiently general setting to suit the functionals $J_\alpha^0, J_\alpha^1, J_\alpha^2$. For a general pattern (i.e., one that is not optimal) the landscape cannot be introduced directly with an explicit expression like (1.11). We then introduce this function as a sort of relaxation of (1.11).

In Section 3 we develop various type of "gain formulas". These formulas establish how the value of the functional $J_\alpha^0(\chi)$ varies, when we consider certain variation of the pattern χ. The variation is expressed in terms of a difference of the landscape function values. Combining gain formulas with the results in Section 5 we give estimates on the variation of the landscape between two points of an optimal pattern.

In Section 4 we prove that the Hölder continuity of the landscape function and the decay of the mass on a fiber are related. If the landscape function is Hölder continuous of exponent β, then mass $m_\chi(x)$ can be estimated from below (up to a constant) by a power $(1-\beta)/(1-\alpha)$ of the residual length of the fiber, and vice versa. Moreover, we prove that Hölder regularity of the landscape function w.r.t. the distance on fibers is equivalent to Hölder regularity w.r.t. the Euclidean distance.

In Section 5 we provide an upper estimate for the irrigation cost of a probability measure which satisfies an lower Ahlfors regularity condition (see Definition 5.1).

In Section 6 we extend the result by Santambrogio to all Ahlfors regular *from below* measures in a general dimension $h \geq 0$. In Theorem 6.2 we prove that if the irrigated measure μ is in such hypothesis, then the landscape function is Hölder continuous with exponent $1 + h(\alpha - 1)$. Even though our main result is a generalization of the one in [S], the proof is completely different and a bit more elementary.

Finally, in Section 8 we provide several examples and counter-examples to show that our results are in a certain sense as sharp as possible. For example, if μ is Ahlfors regular, then the best Hölder exponent is actually the one provided by Theorem 6.2.

For the reader's convenience, Hausdorff, Minkowski and resolution dimensions main properties and definitions are provided in Appendix A. Some technical measurability results are provided in Appendix B. A list of the main symbols involved in the paper can be found in Appendix C.

2 Landscape function

In view of Theorem 1.4, in this section and in the following ones we will consider only the functional J^0_α. We will then drop the superscript and if we write J_α we will always mean J^0_α.

Consider the J_α cost in the extended setting. By Fubini Theorem, it is the integral on Ω of

$$p \mapsto c(p) := \int_I s_\alpha(p,t)|\dot{\chi}_p(t)|\mathrm{d}t. \tag{2.1}$$

$c(p)$ is finite μ_Ω-a.e. $p \in \Omega$ whenever $J_\alpha(\chi) < +\infty$. We are then driven to consider the following definition.

In analogy with the former setting of the problem, we define $i_\chi(p) := \chi(p,b)$. Before we go on, we introduce the definition of the domain of a pattern, which we will often encounter in the rest of the paper.

Definition 2.1 (Domain of a pattern)**.** Let χ be a pattern. The *domain of the pattern* χ denoted by D_χ is the set defined by

$$D_\chi = \{x \ : \ \exists A \subseteq \Omega, \mu_\Omega(A) > 0, \exists t \in I \text{ s.t. } \forall p \in A, \chi(p,t) = x\}.$$

We now define the *landscape function.* We remark that in the following we are *not* supposing that χ is an optimal pattern, but only a finite cost one, i.e. $J_\alpha(\chi) < +\infty$. We will implicitly assume that the pattern χ has finite cost, whenever its landscape function will be considered.

Definition 2.2 (Landscape function)**.** For μ_Ω-a.e. p and all $t \in I$, we define the function $Z_\chi : \Omega \times I \to \mathbf{R}^N$ as

$$Z_\chi(p,t) := \int_a^t s_\alpha(p,s)|\dot{\chi}_p(s)|\mathrm{d}s.$$

A lower semicontinuous function φ is *admissible* for χ if

$$\varphi(\chi(p,t)) \leq Z_\chi(p,t) \tag{2.2}$$

holds for μ_Ω-a.e. p and for all $t \in I$. The *landscape function* $\overline{Z}_\chi$ of the pattern χ is then defined by:

$$\overline{Z}_\chi := \sup\{\varphi \ : \ \varphi \text{ admissible for } \chi\}. \tag{2.3}$$

Remark 2.3. Some remarks:

1. $\overline{Z}_\chi$ is lower semicontinuous;

2. it is equivalent to require equation (2.2) for μ_Ω-a.e. p and all $t \in I$ or $(\mu_\Omega \otimes \mathcal{L}^1)$-a.e. (p,t) by the continuity of the fibers. Indeed, for any given $p \in \Omega$ consider set

   $$S_p = \{t \in I \ : \ (\varphi \circ \chi)(p,t) - Z_\chi(p,t) > 0\}.$$

   Let

   $$S = \{(p,t) \in \Omega \times I \ : \ (\varphi \circ \chi)(p,t) - Z_\chi(p,t) > 0\}.$$

   By Fubini Theorem,

   $$(\mu_\Omega \otimes \mathcal{L}^1)(S) = \int_\Omega \mathcal{L}^1(S_p)\mathrm{d}p.$$

   Then, S is negligible w.r.t. $(\mu_\Omega \otimes \mathcal{L}^1)$, if and only if S_p is negligible for a.e. p w.r.t. $\mathcal{L}^1$;

3. $\overline{Z}_\chi$ is the maximal l.s.c. extension on $\overline{D}_\chi$ of its restriction to D_χ;

4. if there is no misunderstanding, we will simply write $\overline{Z}$ instead of $\overline{Z}_\chi$.

The next proposition shows that $\overline{Z}$ is the "right" relaxation of Z follows directly from the definition.

Proposition 2.4. *$\overline{Z}$ satisfies the inequality*

$$\overline{Z}(\chi(p,t)) \leq Z(p,t),$$

for a.e. $p \in \Omega$ and for all $t \in I$.

Proof. The key point to note is that for a.e. p the quantity $m_\chi(p,\cdot)$ is monotone decreasing, so it is sufficient to prove the inequality for a fiber p such that $m_\chi(p,t) > 0$ for all t in the interior of I. Given φ admissible, there exists $q \in [p]_t$ such that

$$\varphi(\chi(p,t)) = \varphi(\chi(q,t)) \le Z(q,t) = Z(p,t).$$

Taking the supremum, we finally get:

$$\overline{Z}(\chi(p,t)) \le Z(p,t). \qquad \square$$

Proposition 2.5 (Alternative definition of the landscape function)**.** *The landscape function* $\overline{Z}$ *can be characterized as*

$$\overline{Z}(x) = \sup_{r>0} \operatorname{essinf}\{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_r(x))\}.$$

Proof. The map

$$x \mapsto \tilde{Z}(x) := \sup_{r>0} \operatorname{essinf}\{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_r(x))\}$$

is lower semicontinuous, since the set $\{\tilde{Z} > a\}$ is open. Indeed, if for some x we have $\tilde{Z}(x) > a$, then for some $r > 0$

$$\operatorname{essinf}\{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_r(x))\} > a.$$

We now prove that $B_r(x) \subseteq \{\tilde{Z} > a\}$. For any $x_1 \in B_r(x)$, we have

$$\operatorname{essinf}\{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_{r_1}(x_1))\} > a, \tag{2.4}$$

if $B_{r_1}(x_1) \subseteq B_r(x)$, since

$$\{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_{r_1}(x_1))\} \subseteq \{Z(p,t) \ : \ (p,t) \in \chi^{-1}(B_r(x))\},$$

and essinf is a monotone decreasing function. By (2.4), $\tilde{Z}(x_1) > a$, so that $B_r(x) \subseteq \{\tilde{Z} > a\}$. Since $\tilde{Z}$ is clearly admissible for χ, we must then have $\tilde{Z} \le \overline{Z}$.

Let φ any admissible function. Then, by the lower semicontinuity of φ, for all $x \in \mathbf{R}$

$$\begin{aligned}\varphi(x) &= \sup_{r>0} \inf\{\varphi(y) \ : \ y \in B_r(x)\} \le \\ &\le \sup_{r>0} \operatorname{essinf}\{\varphi(\chi(p,t)) \ : \ (p,t) \in \chi^{-1}(B_r(x))\} \le \tilde{Z}(x).\end{aligned}$$

Then, $\varphi \le \tilde{Z}$, which implies $\overline{Z} \le \tilde{Z}$. $\square$

Remark 2.6. If $\overline{Z}(x) < v$, then there exists a sequence (p_n, t_n) such that $\chi(p_n, t_n) \to x$ and $Z(p_n, t_n) < v$. This is an immediate consequence of Proposition 2.5.

We end this section showing (Theorem 2.8) that $\overline{Z} \circ \chi = Z$ almost everywhere w.r.t. the product measure on $\Omega \times I$. To this aim, we shall need to employ some estimates which are going to be proved in next section, obviously without the use of such a property.

Lemma 2.7. *Let* $\chi : \Omega \times I \to X$ *be an* optimal *pattern. For a.e.* $p \in \Omega$ *we have*

$$\overline{Z}(\chi(p, b)) = Z(p, b).$$

Proof. First of all, note that by Proposition 2.4 for a.e. $p \in \Omega$

$$\overline{Z}(\chi(p, b)) \leq Z(p, b).$$

Suppose moreover that the set

$$\Omega' = \{p \in \Omega \ : \ \overline{Z}(\chi(p, b)) < Z(p, b)\}$$

has strictly positive measure. This means that there exist $r, s \in \mathbf{Q}$ such that the set

$$\Omega'' = \{p \in \Omega \ : \ \overline{Z}(\chi(p, b)) < r < s < Z(p, b)\} \tag{2.5}$$

has strictly positive measure. Let X_j^n be a partition of $\mathbf{R}^N$ with equal squares of $\operatorname{diam} X_j^n < 1/n$. Since the support of the irrigated measure is compact (and, hence, bounded), only a finite number of X_j^n is needed to cover it. Let

$$\Omega_j^n := \Omega'' \cap i_\chi^{-1}(X_j^n).$$

If Ω_j^n has positive measure, take $x_j^n = \chi(p_j^n, t_j^n) \in \mathbf{R}^N$ such that (just apply Remark 2.6 to any point $\chi(p, b)$ with $p \in \Omega_j^n$)

$$d(x_j^n, X_j^n) < \frac{1}{n}, \quad Z(p_j^n, t_j^n) < r.$$

The set $\{\chi(p, b) \ : \ p \in \Omega_j^n\}$ (on such set we have $Z > s$) is moved on x_j^n. For such a pattern, using the estimates developed in the next section (apply Theorem 3.16 below a finite number of times),

$$J_\alpha(\chi_n) - J_\alpha(\chi) \leq \alpha \mu_\Omega(\Omega'')(r - s).$$

Let μ, μ_n be the measures irrigated by χ, χ_n respectively. Since $w_1(\mu_n, \mu) < 1/n$, by compactness $\chi_n \to \overline{\chi}$ and $\overline{\chi}$ irrigates μ. By lower semicontinuity of the J_α cost function (see Lemma 4.6 of [MS-DIST])

$$J_\alpha(\overline{\chi}) \leq J_\alpha(\chi) + \alpha \mu_\Omega(\Omega'')(r - s) < J_\alpha(\chi),$$

which is not possible because of the minimality of χ. □

The next theorem shows the equivalence between $\overline{Z}$ and z (defined in the Introduction). The theorem provides it for a minimum of J^0 and, consequently, for a minimum of J^1. Since every mimimum of J^2 can be synchronized in a minimum of J^0 without changing z, the proof of the equivalence is complete.

Theorem 2.8. *Let* $\chi : \Omega \times I \to X$ *be an* optimal *pattern. For a.e.* $p \in \Omega$ *and all* $t \in I$ *we have*

$$\overline{Z}(\chi(p,t)) = Z(p,t).$$

Proof. By Proposition 2.4 we need to prove only that $\overline{Z}(\chi(p,t)) \geq Z(p,t)$. Given $T \geq 0$, consider the pattern stopped at time $T < b$ given by

$$\hat{\chi}(p,t) := \begin{cases} \chi(p,t) & 0 \leq t \leq T \\ \chi(p,T) & t > T. \end{cases}$$

We have for every $p \in \Omega$ and $t \leq T$,

$$Z_{\hat{\chi}}(p,t) = Z_{\chi}(p,t).$$

By Lemma 2.7, for a.e. $p \in \Omega$,

$$\overline{Z}_{\chi}(\chi(p,T)) = \overline{Z}_{\hat{\chi}}(\hat{\chi}(p,T)) = Z_{\hat{\chi}}(p,T) = Z_{\chi}(p,T). \tag{2.6}$$

Equation (2.6) is not sufficient to conclude the proof, since the full measure set Ω_T provided by Lemma 2.7 depends on T. Suppose that $c(p) < +\infty$. We can fix $\overline{t} > T$ such that $[p]_{\overline{t}}$ has positive measure. If we had

$$\overline{Z}_{\chi}(\chi(p,T)) < Z_{\chi}(p,T),$$

it would be

$$\overline{Z}_{\chi}(\chi(q,b)) < Z_{\chi}(q,b), \tag{2.7}$$

for $q \in [p]_{\overline{t}}$. Since $[p]_{\overline{t}}$ is a set of positive measure, equation (2.7) contradicts Lemma 2.7. □

We recall the following definition from [DS-ELE].

Definition 2.9 (Simple patterns)**.** We say that a pattern χ is *simple* if all the fibers which share a common point coincide as functions of the time parameter. In other words, if $\chi(p,t) = \chi(p',t')$, then $t = t'$ and $\chi(p,s) = \chi(p',s)$ for all $s \in [0,t]$. See Definition 6.1 in [DS-ELE].

Recall that any optimal pattern χ is simple.

If χ is a simple pattern, the function $Z(p,t)$ does not actually depend on (p,t), meaning that if $x = \chi(p,t)$ then Z depends actually on x (and not on the particular couple (p,t) which realizes x). This is the content of the following proposition.

Proposition 2.10. *Suppose that χ is a simple pattern (see Definition 2.9). Then, given $x \in D_\chi$, the function Z is constant on the set $\chi^{-1}(x)$.*

Proof. By the hypothesis of the proposition, for every given $x \in D_\chi$ and a.e. $(p,t) \in \chi^{-1}(x)$, the function $s_\alpha(p,\cdot)$ does not depend on p on the interval $[0,t]$. □

Notation. In view of Proposition 2.10, if $x = \chi(p,t)$, we will write $Z(x)$ instead of $Z(\chi(p,t))$ if χ is simple.

Under this notation, Theorem 2.8 implies the following corollary.

Corollary 2.11. *Suppose that χ is an optimal pattern. Then $Z = \overline{Z}$ on D_χ.*

Proof. Let $x \in D_\chi$, then $x = \chi(p,t)$ for p in a set of positive measure. For a.e. p we then have:

$$\overline{Z}(x) = Z(\chi(p,t)) = Z(x).$$

□

3 Gain formula

3.1 Moving a mass m from a point x to a point y

Recall (see [DS-ELE]) that if χ is a simple pattern, then the following formula holds:

$$J_\alpha(\chi) = \int_{\mathbf{R}^N} |P_\chi(x)|^\alpha \mathrm{d}\mathcal{H}^1(x), \tag{3.1}$$

where $P_\chi(x) = [p]_t$ whenever (p,t) satisfies $x = \chi(p,t)$. As far as the validity of (3.1) and the good definition of $P_\chi(x)$ we refer to Theorem 9.2 of [DS-ELE].

Consider the following modification of an irrigation pattern χ. In the next part of the section, we will always suppose that χ is a simple pattern.

Definition 3.1 (Mass function)**.** Suppose χ is a simple pattern. The *mass function* is defined as

$$m_\chi(x) = |P_\chi(x)|.$$

If $x = \chi(p,t)$, we clearly have $m_\chi(x)^{\alpha-1} = s_\alpha(p,t)$.

We now come to a key definition. We will refer to Figure 1. Suppose that χ is a simple pattern. Let x and y be points on distinct fibers. These two fibers coincide up to a certain bifurcation point. Let C_1 (respectively, C_2) be the curve between the bifurcation of the fibers containing x and y and x (respectively, y). First we remove a mass $m \leq m_\chi(x)$ from the branch passing through x, add it to the branch passing through y up to y (on C_2) through a deviation of the fibers in a set $M \subset \Omega$, such that $\mu_\Omega(M) = m$. This pattern will be named "mass deviation" of the pattern χ and denoted by $\overline{\chi}_{x,y,M}$.

Definition 3.2 (Mass deviation of a pattern χ)**.** Suppose that χ is a simple pattern. Let x and y be points on distinct fibers. Let $x = \chi(p_1, t_1)$ and $y = \chi(p_2, t_2)$. Let $M \subseteq [p_1]_{t_1}$ such that $\mu_\Omega(M) = m$. Define:

$$\overline{\chi}_{x,y,M}(p,t) = \begin{cases} \chi(p,t) & \text{if } p \in \Omega \setminus M, \\ \chi(p_2,t) & \text{if } p \in M, t \leq t_2, \\ \chi(p_2,t_2) & \text{if } p \in M, t \geq t_2. \end{cases} \tag{3.2}$$

We call this new pattern a *mass deviation* of χ. If there is not ambiguity on x, y, M we will simply write $\overline{\chi}$.

Remark 3.3. The pattern $\overline{\chi}_{x,y,M}$ does not irrigate the same measure as χ. The irrigated measures are related by

$$\mu_{\overline{\chi}} = \mu_{\chi|(\Omega \setminus M)} + m\delta_y.$$

Suppose that χ is a simple pattern. Consider the pattern $\overline{\chi}$ of Definition 3.2. If we move the mass deviated in y from y to x with on a straight line and recovering the irrigated measure we get a new pattern which we call a "mass by-pass" of the pattern χ and denote by $\tilde{\chi}_{x,y,M}$. In this way, the original irrigated measure is recovered, i.e. $\mu_{\tilde{\chi}} = \mu_\chi$. The way described here, nevertheless, is not the only way of recovering the original irrigated measure.

Definition 3.4 (Mass by-pass of a pattern χ)**.** Suppose that χ is a simple pattern. Let x and y be points on distinct fibers. Let $x = \chi(p_1, t_1)$ and $y = \chi(p_2, t_2)$. Let $M \subseteq [p_1]_t$ such that $\mu_\Omega(M) = m$. Consider the pattern $\overline{\chi}$ of Definition 3.2 and consider the composition the $\overline{\chi}$ with a pattern between $m\delta_y$ and $\nu_{x,y,M} = \mu_\chi - \mu_{\overline{\chi}} + m\delta_y = \mu_{\chi|M}$. We call the new pattern a *mass by-pass* of χ. If there is not ambiguity on x, y, M we will simply write $\tilde{\chi}$.

Remark 3.5. The pattern involved in the composition may be the optimal one between $m\delta_y$ and $\nu_{x,y,M}$ or, as it is sometimes useful, the pattern built from a straight line between x and y and the optimal one between $m\delta_x$ and $\nu_{x,y,M}$.

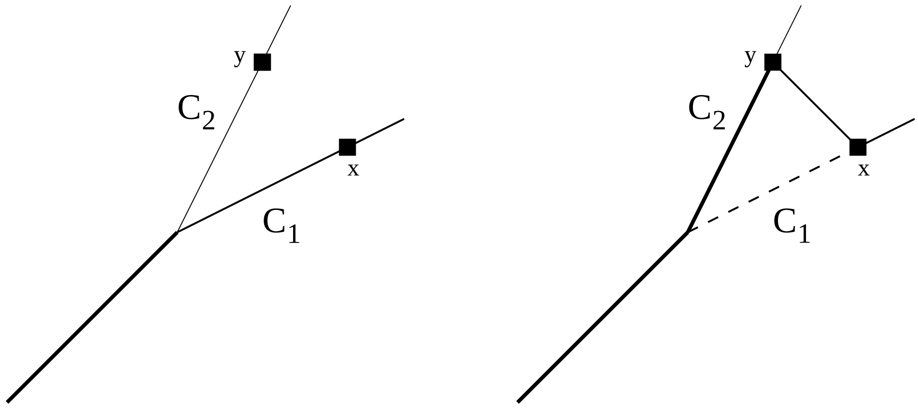

Figure 1: From the pattern χ to $\overline{\chi}_{x,y,M}$ and $\tilde{\chi}_{x,y,M}$.

Remark 3.6. The pattern $\tilde{\chi}_{x,y,M}$ irrigates the same measure as χ.

Theorem 3.7 (First order gain formula, deviation case)**.** *Suppose that χ is a simple pattern. Then, the pattern $\overline{\chi}$ satisfies*

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \le \alpha m(Z(y) - Z(x)).$$

Proof. Recall that by Proposition 2.10 the function Z can be regarded as a function of x instead of (p,t) so that we will write $Z(x)$ meaning $Z(\chi^{-1}(x))$ without confusion.

Let C_1 and C_2 be the curves in Definition 3.2. We have:

$$\begin{aligned} J_\alpha(\overline{\chi}) - J_\alpha(\chi) = &\int_{C_1} (m_\chi(w) - m)^\alpha \mathrm{d}\mathcal{H}^1(w) + \int_{C_2} (m_\chi(w) + m)^\alpha \mathrm{d}\mathcal{H}^1(w) + \\ &- \left(\int_{C_1} m_\chi(w)^\alpha \mathrm{d}\mathcal{H}^1(w) + \int_{C_2} m_\chi(w)^\alpha \mathrm{d}\mathcal{H}^1(w) \right). \end{aligned}$$

The concavity of $u \mapsto u^\alpha$ gives

$$(-m+u)^\alpha - u^\alpha \le -\alpha m u^{\alpha-1}, \quad (m+u)^\alpha - u^\alpha \le \alpha m u^{\alpha-1}.$$

Then,

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \le \alpha m(Z(y) - Z(x)).$$

Indeed, we have:

$$Z(y) - Z(x) = \int_{C_2} m_\chi(w)^{\alpha-1} \mathrm{d}\mathcal{H}^1(w) - \int_{C_1} m_\chi(w)^{\alpha-1} \mathrm{d}\mathcal{H}^1(w). \qquad \square$$

We now consider the mass by-pass case. In the following we will denote by $d_\alpha(\mu^+, \mu^-)$ the least irrigation cost where the initial mass μ^+ is moved on μ^-, i.e.,

$$d_\alpha(\mu^+, \mu^-) := \min_\chi J_\alpha(\chi), \tag{3.3}$$

where χ ranges among the patterns moving μ^+ on μ^-. Recall that the convergence w.r.t. d_α is equivalent to the weak one (see Section 4.2 of [MS-DIST] for the details).

Corollary 3.8 (First order gain formula, mass by-pass case)**.** *Suppose that χ is a simple pattern. Then, the pattern $\tilde{\chi}$ satisfies*

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y) - Z(x)) + d_\alpha(m\delta_y, \nu_{x,y,M}).$$

Corollary 3.9. *Suppose that χ is optimal for J_α. Then,*

$$\alpha m(\overline{Z}(x) - \overline{Z}(y)) \leq d_\alpha(m\delta_y, \nu_{x,y,M}). \tag{3.4}$$

Proof. Since χ is optimal and $\tilde{\chi}$ irrigates the same measure, we must obviously have:

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) \geq 0.$$

The conclusion then follows by Corollary 3.8 and Theorem 2.8. □

Corollary 3.10. *Suppose that χ is optimal for J_α. Then,*

$$\alpha m(\overline{Z}(x) - \overline{Z}(y)) \leq m^\alpha |x - y|. \tag{3.5}$$

Proof. Use as recovery pattern the second one of Remark 3.5. □

3.2 From a set X to a point y

In this part we want to generalize Corollary 3.8 where we replace the point x by a set X.

Let us begin with a generalization of Definition 3.2.

Definition 3.11 (Mass deviation in the discrete case)**.** Suppose that χ is a simple pattern. Let $x_1, x_2, \ldots, x_n$ points with masses $m_\chi(x_i) \geq m_i$ for $i = 1, 2, \ldots, n$. A *mass deviation* of χ from the set $X = \{x_1, x_2, \ldots, x_n\}$ to the point y is a pattern given by Definition 3.2 applied iteratively on each point of X.

The following lemma is a close generalization of the main theorem of the preceding section.

Lemma 3.12 (Discrete first order gain formula)**.** *Suppose that χ is a simple pattern. Let $x_1, x_2, \ldots, x_n$ points with masses $m_\chi(x_i) \geq m_i$ for $i = 1, 2, \ldots, n$. Suppose that for $i = 1, 2, \ldots, n$ we move the masses m_i from x_i on a given point y. The new pattern $\overline{\chi}$ satisfies*

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y) - \inf_{x \in X} Z(x)),$$

where $m = m_1 + \ldots + m_n$ and $X = \{x_1, x_2, \ldots, x_n\}$.

Proof. The proof is by induction. For $k \in \mathbf{N}$, set

$$\Delta_k := Z(y) - \inf\{Z(x_1), Z(x_2), \ldots, Z(x_k)\}.$$

The statement obtained for $n = 1$ is true by Corollary 3.8. Suppose that the statement is true for the $n-1$ points $x_1, \ldots, x_{n-1}$. Let $\overline{\chi}_{n-1}$ the pattern where the masses $x_1, x_2, \ldots, x_{n-1}$ have been moved on y. By the inductive hypothesis $\overline{\chi}_{n-1}$ satisfies

$$J_\alpha(\overline{\chi}_{n-1}) - J_\alpha(\chi) \leq \alpha(m_1 + m_2 + \ldots + m_{n-1})\Delta_{n-1}.$$

Let Z_{n-1} be the landscape function of $\overline{\chi}_{n-1}$. Moving the mass in x_n, by Corollary 3.8 we get a new pattern $\overline{\chi}$ such that

$$J_\alpha(\overline{\chi}) - J_\alpha(\overline{\chi}_{n-1}) \leq \alpha m_n(Z_{n-1}(y) - Z_{n-1}(x_n)).$$

Clearly, we have $\Delta_{n-1} \leq \Delta_n$. We just need to prove that

$$Z_{n-1}(y) - Z_{n-1}(x_n) \leq Z(y) - Z(x_n) \leq \Delta_n \tag{3.6}$$

The proof of inequality (3.6) follows from the fact that, in this and any previous step, moving some mass to the branch containing y does not increase the landscape gap $Z(y) - Z(x_n)$ between y and x_n. □

Definition 3.13 (Fibers passing through a set X)**.** Given a closed set $X \subseteq \mathbf{R}^N$, consider the set of fibers which transit on X, i.e. the set

$$T(X) := \{p \in \Omega \ : \ \chi_p(I) \cap X \neq \emptyset\}.$$

Lemma 3.14 (Discrete first order gain formula, countable case)**.** *Suppose that χ is a simple pattern. Let x_i, $i \in \mathbf{N}$, be points with masses $m_\chi(x_i) \geq m_i$ for $i = 1, 2, \ldots, n$. Suppose that we move the masses m_i from x_i on a given point y. The new pattern $\overline{\chi}$ satisfies*

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y) - \inf_{x \in X} Z(x)),$$

where $m = \sum_i m_i$ and $X = \{x_i \ : \ i \in \mathbf{N}\}$.

Proof. Let $\overline{\chi}_n$ the pattern obtained moving the n points $x_1, \dots, x_n$. $\overline{\chi}_n \to \overline{\chi}$ fiberwise and by the lower semicontinuity of the cost functional J_α:

$$J_\alpha(\overline{\chi}) \le \liminf_{n\to+\infty} J_\alpha(\overline{\chi_n}) \le \liminf_{n\to+\infty}\left(Z(y) - \inf_{1\le i\le n} Z(x_i)\right) \le Z(y) - \inf_X Z(x),$$

since, for every n

$$Z(y) - \inf_{1\le i\le n} Z(x_i) \le Z(y) - \inf_X Z(x). \qquad \square$$

Remark 3.15. In Lemma 3.12 and 3.14, if we are given a subset M of fibers passing through X, i.e. for every $p \in M$ we have $\chi(p, I) \cap X \ne \emptyset$, by Proposition B.3, M can be split in n measurable subsets of fibers passing through $x_1, x_2, \dots, x_n$ respectively.

We now go on with the last generalization of Theorem 3.7, which we will deduce by a pruning argument from the previous statement.

By Proposition B.4, $T(X)$ is measurable.

Given a subset $M \subseteq T(X)$ such that $\mu_\Omega(M) = m$. Set

$$X_M := \{x \in X \;:\; x \in \chi_p(I),\ p \in S\}.$$

Given a pattern χ, let $\overline{\chi}_M$ be the pattern obtained moving the mass carried by the set of fibers in M on a given $y \in \mathbf{R}^N$.

We have then the following theorem.

Theorem 3.16 (Continuous first order gain formula)**.** *Let χ be a simple pattern. Given a measurable subset X and a subset $M \subseteq X_M$ of the fibers passing through X such that $\mu_\Omega(M) = m$. We then have:*

$$J_\alpha(\overline{\chi}_M) - J_\alpha(\chi) \le \alpha m(Z(y) - \inf_X Z). \tag{3.7}$$

Proof. The case $S \in X$ can be easily handled as in the proof of Theorem 3.7. Indeed, using the same notation where C_1 is no longer the fiber of the point x, but the rectifiable set where the mass diminishes, we have

$$J_\alpha(\overline{\chi}_M) - J_\alpha(\chi) = \int_{C_1} [(m(x) - c(x))^\alpha - m(x)^\alpha] \mathrm{d}\mathcal{H}^1(x) + \\ + \int_{C_2} [(m(x) + m)^\alpha - m(x)^\alpha] \mathrm{d}\mathcal{H}^1(x).$$

The first term in the r.h.s. of the previous formula is negative, so thatx

$$J_\alpha(\overline{\chi}_M) - J_\alpha(\chi) \le \\ \le \int_{C_2} [(m(x) + m)^\alpha - m(x)^\alpha] \mathrm{d}\mathcal{H}^1(x) \le \alpha m \int_{C_2} m(x)^{\alpha-1} \le \alpha m Z(y).$$

The last formula proves the statement of the theorem in the case $S \in X$ since, in such case, $\inf_X Z = 0$.

Suppose now that $S \notin X$. For every $\varepsilon > 0$, let $X \subseteq V_\varepsilon$ an open subset contained in the ε-neighborhood of X such that

$$\inf_{V_\varepsilon} Z \geq \inf_X Z - \varepsilon.$$

We can also suppose that $S \notin V_\varepsilon$. Except for a null subset of fibers of infinite cost, for every $\overline{p} \in T(X)$ by continuity there exists $\tilde{t} \in \mathbf{Q}$ such that $\chi(\overline{p}, \tilde{t}) \in V_\varepsilon$ and $m_\chi(\chi(\overline{p}, \tilde{t})) > 0$.

Let

$$F_t = \{x \in \mathbf{R}^N \;:\; \exists p \in \Omega,\; m_\chi([p]_t) > 0,\; \chi(p,t) = x\}.$$

Recall that given two fibers p, q, we have $[p]_t = [q]_t$ or $[p]_t \cap [q]_t = \emptyset$. Given $x_1, x_2 \in F_t$, we then have $x_1 = \chi(p_1, t), x_2 = \chi(p_2, t)$ with $[p_1]_t \cap [p_2]_t = \emptyset$. Since the total measure of such disjoint solidarity classes must not exceed one, they must be at most countable. Hence, the set F_t is countable.

Consider now the sets

$$F = \bigcap_{t \in \mathbf{Q}} F_t, \quad X_\varepsilon = F \cap V_\varepsilon.$$

This proves that $T(X) \subseteq T(X_\varepsilon)$. Applying Lemma 3.14, we have:

$$J_\alpha(\overline{\chi}_M) - J_\alpha(\chi) \leq \alpha m(Z(y) - \inf_{X_\varepsilon} Z) \leq \alpha m(Z(y) - \inf_X Z) + \alpha m \varepsilon.$$

Since ε can be chosen arbitrarily, the proof is concluded. □

Given a pattern χ, let $\tilde{\chi}_M$ be the pattern obtained moving the mass carried by the set of fibers in M on a given $y \in \mathbf{R}^N$.

Corollary 3.17. *Suppose that χ is a simple pattern. Let $X \subseteq \mathbf{R}^N$ be measurable and $y \in \mathbf{R}^N$. Suppose that*

$$Z(y) < \inf_{x \in X} Z(x).$$

Let m as in Theorem 3.16. Then, we have

$$J_\alpha(\tilde{\chi}_M) - J_\alpha(\chi) \leq \alpha m(Z(y) - \inf_{x \in X} Z(x)) + d_\alpha(m\delta_y, \nu),$$

where $\nu = i_{\chi\#}\mu_\Omega - i_{\overline{\chi}\#}\mu_\Omega - m\delta_y$.

Corollary 3.18. *In the same assumptions of Corollary 3.17, if χ is optimal we have:*

$$\alpha m(\inf_{x \in X} Z(x) - Z(y)) \leq d_\alpha(y, \nu), \tag{3.8}$$

where $\nu = i_{\chi\#}\mu_\Omega - i_{\overline{\chi}\#}\mu_\Omega - m\delta_y$.

Proof. The proof is the same as that of Corollary 3.9. □

4 Decay of the multiplicity on a fiber

Definition 4.1 (Fiber distance)**.** Suppose that χ is a simple pattern and let p be a fiber. Given two points $x_1 = \chi_p(t_1), x_2 = \chi_p(t_2)$ with $t_1 < t_2$, their *fiber distance* $d(x_1, x_2)$ is given by:

$$d(x_1, x_2) := \int_{t_1}^{t_2} |\dot{\chi}_p(t)| \mathrm{d}t. \tag{4.1}$$

Since χ is a simple pattern, there is a unique fiber between x and y and the integral defining $d(x_1, x_2)$ does not depend on p. So $d(x_1, x_2)$ is well-defined.

Definition 4.2. Let p be a fiber of the pattern $\chi : \Omega \times I \to \mathbf{R}^N$. The function $Z : \mathbf{R}^N \to \mathbf{R}$ is *Hölder continuous* with exponent β *w.r.t. the fiber distance* if, for some constant C,

$$|Z(x_2) - Z(x_1)| \leq C d(x_1, x_2)^\beta, \tag{4.2}$$

for all $x_1 = \chi_p(t_1), x_2 = \chi_p(t_2)$.

Theorem 4.3. *Let χ be a* simple pattern*, let Z be as in Definition 2.2, and let $p \in \Omega$ be given. Then, the following conditions are equivalent:*

1. *Z is Hölder continuous with exponent β on the fiber p (formula* (4.2)*);*

2. *the Hölder continuity condition holds when one of the points is a terminal point, i.e. if $x = \chi(p, t)$, we have*

$$Z(\chi(p, b)) - Z(x) \leq C d(x, i_\chi(p))^\beta, \tag{4.3}$$

for some constant $C > 0$;

3. *if $x = \chi(p, t)$,*

$$m(x) \geq C d(x, i_\chi(p))^{\frac{1-\beta}{1-\alpha}}, \tag{4.4}$$

for some constant $C > 0$.

Proof. The proof is divided in three steps.

- 1. $\Rightarrow$ 2. Obvious;

- 2. $\Rightarrow$ 3. Recall that the function $s \mapsto m(\chi(p, s))$ is non-increasing. Thus,

$$C d(x, i_\chi(p))^\beta \geq Z(i_\chi(p)) - Z(x) = \int_t^b m(\chi(p, s))^{\alpha-1} |\dot{\chi}_p(s)| \mathrm{d}s \geq$$
$$\geq m(\chi(p, t))^{\alpha-1} \int_t^b |\dot{\chi}_p(s)| \mathrm{d}s = m(x)^{\alpha-1} d(x, i_\chi(p)).$$

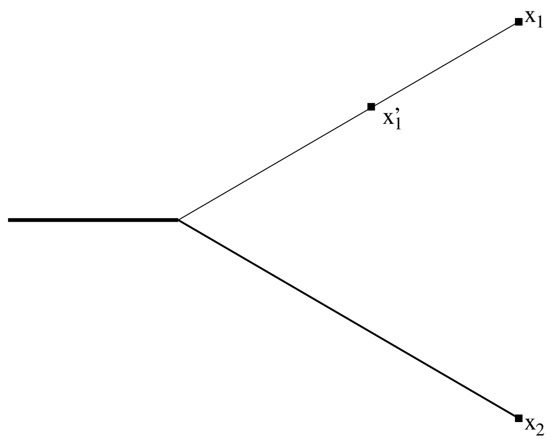

Figure 2: Proof of Theorem 4.5

- 3. ⇒ 1. Consider the reparameterized fiber, so that $|\dot{\chi}_p(t)| = 1$. Let l be its length. From equation (4.4)

$$m(\chi_p(s))^{\alpha-1} \leq C|l-s|^{\beta-1}.$$

Integrating between $t_1 < t_2$, we get

$$Z(\chi_p(t_2)) - Z(\chi_p(t_1)) = \int_{t_1}^{t_2} m(\chi(p,s))^{\alpha-1}\mathrm{d}s \leq$$

$$\leq \int_{t_1}^{t_2} C(l-s)^{\beta-1}\mathrm{d}s = C\left[\frac{(l-t_1)^\beta}{\beta} - \frac{(l-t_2)^\beta}{\beta}\right]. \quad (4.5)$$

By subadditivity $(l-t_1)^\beta - (l-t_2)^\beta \leq (t_2-t_1)^\beta$, so

$$Z(x_2) - Z(x_1) \leq \frac{C}{\beta}(t_2-t_1)^\beta = \frac{C}{\beta}d(x_1,x_2)^\beta. \qquad \square$$

Remark 4.4. If C is the constant appearing in inequality (4.2), then our computation provides $C^{1/(\alpha-1)}$ as the constant in inequality (4.4).

Theorem 4.5. *Let χ be an* optimal *pattern. Then, the* uniform Hölder continuity *on the fibers of the landscape function (Definition 4.2) implies its Hölder continuity w.r.t. the Euclidean distance on $\overline{D}_\chi$.*

Proof. Fix x_1, x_2, let $d = |x_1 - x_2|$. Suppose first you can take backward x_1' on the fiber of x_1 at a fiber distance d (see Figure 2). Thanks to the decay inequality (4.4) of Theorem 4.3, we have

$$m(x_1') \geq Cd^{\frac{1-\beta}{1-\alpha}},$$

that is

$$m(x_1')^{\alpha-1} \leq Cd^{\beta-1}$$

By Corollary 3.10

$$\alpha(\overline{Z}(x_1') - \overline{Z}(x_2)) \leq m(x_1')^{\alpha-1}|x_1' - x_2| \leq (Cd^{\beta-1})(2d) = 2Cd^{\beta}.$$

Finally, since $\overline{Z}(x_1) - \overline{Z}(x_2) = (\overline{Z}(x_1) - \overline{Z}(x_1')) + (\overline{Z}(x_1') - \overline{Z}(x_2))$ and each of the two terms is bounded by some constant times d^{β} (the first thanks to the Hölder continuity on the fiber and to the equality $Z = \overline{Z} \circ \chi$ implied by the minimality of χ), the first part of the proof is complete on D_χ and, by a continuity argument, on $\overline{D_\chi}$ (see item 3 of Remark 2.3).

Suppose finally that x_1' cannot be taken at a fiber distance d. Then, we can take $x_1' = S$ and the term $\overline{Z}(x_1') - \overline{Z}(x_2)$ can be estimated by zero. □

Remark 4.6. Note that if the Hölder continuity constant w.r.t. fibers is given by C, then the constant w.r.t. the Euclidean distance is at most $C(1+2/\alpha)$.

5 Irrigation cost of a measure μ with a given Ahlfors dimension

Before entering in the last part of the proof of the main result, we prove an estimate from above of the irrigation cost between a Dirac mass and an Ahlfors regular from below measure. The irrigation cost is bounded by the α-power of the mass of the irrigated measure times the diameter of its support times a *universal* constant. We recall that the definition and a complete treatment of the main tool used here, the *hierarchy of collectors*, can be found in [DS-DIM] (see, in particular, Definition 3.1 and Corollary 3.1).

Definition 5.1 (Lower Ahlfors regular measure)**.** A measure μ is *Ahlfors regular from below* in dimension h, if there exists $C_A > 0$ such that

$$\mu(B(x,r)) \geq C_A r^h,$$

for all $r \in [0,1]$ and for all $x \in \operatorname{spt}\mu$.

A *dyadic cube of order* n in $\mathbf{R}^N$ is a Cartesian product of N intervals of the kind $[k2^{-n}, (k+1)2^{-n}[$ for $k \in \mathbf{Z}, n \in \mathbf{N}$:

$$Q^n_{k_1,\dots,k_N} := \prod_{i=1}^{N} [k2^{-n}, (k+1)2^{-n}[;$$

the center of $Q^n_{k_1,\dots,k_N}$ is the point

$$c(Q^n_{k_1,\dots,k_N}) := \left(2^{-n}\left(k_1+\frac{1}{2}\right),\dots,2^{-n}\left(k_N+\frac{1}{2}\right)\right).$$

Lemma 5.2. *Let B be a ball of unitary radius and let $\mu \in \mathcal{P}(\mathbf{R}^N)$ such that* $\operatorname{spt}\mu \subseteq \overline{B}$. *Suppose also that μ is Ahlfors regular from below of dimension h. Let P_n the set of centers of the dyadic squares of order n which meet* $\operatorname{spt}\mu$. *Then,*

$$\#P_n \le C_A^{-1} 2^{h(n+1)}.$$

Proof. Consider the balls whose center is the center of some dyadic square of order n and whose radius is $r = 2^{-(n+1)}$. Their mass (w.r.t. the measure μ) is at least (by the Ahlfors regularity from below of μ) $C_A 2^{-h(n+1)}$. Moreover, such balls are disjoint. We then have:

$$(\#P_n)C_A 2^{-h(n+1)} \le \sum_{x\in P_n} \mu(B_{2^{-(n+1)}}(x)) \le \mu\left(\bigcup_{x\in P_n} B_{2^{-(n+1)}}(x)\right) \le 1,$$

since the total mass is unitary. The previous inequality finally gives

$$\#P_n \le C_A^{-1} 2^{h(n+1)}.$$

□

Theorem 5.3. *Let B be a ball of unitary radius and let $\mu \in \mathcal{P}(\mathbf{R}^N)$ such that* $\operatorname{spt}\mu \subseteq \overline{B}$. *Suppose also that μ is Ahlfors regular from below of dimension h. Then, given $\alpha > 1 - 1/h$ and $S \in B$, we have*

$$d_\alpha(\mu, \delta_S) \le C(C_A, h, N, \alpha) < +\infty.$$

Proof. Without loss of generality, we can suppose that S is the center of B. A different source can be managed using a pattern which initially moves the mass from the source to the center of B. $C(C_A, h, N, \alpha)$ will be increased at most by 1, the cost of transportation of a Dirac mass from the boundary of B to its center.

Since $\operatorname{spt}\mu$ is compact, for every n we can P_n as in Lemma 5.2. Set $P_0 = \{S\}$. Consider any dependence map $\gamma_n : P_n \to P_{n-1}$ which maps a point in P_n to a point in P_{n-1} whose distance does not exceed 2^{-n+1} and the correspondent hierarchy of collectors $(P_n, \gamma_n)_{0\le n\le N_{\max}}$.

The cost of the pattern connecting δ_S to μ is then estimated by (we refer to Corollary 3.3 of [DS-DIM] and to Lemma 5.2):

$$\sum_{n=1}^{+\infty} 2^{-n+1}\sqrt{N}(\#P_n)^{1-\alpha}.$$

So $C(C_A, h, N, \alpha)$ can be chosen as

$$C(C_A, h, N, \alpha) = \frac{4\sqrt{N} C_A^{\alpha-1}}{1 - 2^{-1+h(1-\alpha)}}. \qquad \square$$

Corollary 5.4. *Let $\mu \in \mathcal{P}(\mathbf{R}^N)$ such that $\mu(\mathbf{R}^N) = m$. Suppose that* $\operatorname{spt}\mu$ *is contained in a ball of radius d. Suppose also that μ is Ahlfors regular from below in dimension h. Then, given $\alpha > 1 - 1/h$ and $S \in \mathbf{R}^N$,*

$$d_\alpha(\mu, \delta_S) \leq C(C_A, h, N, \alpha) m^\alpha d.$$

Proof. The proof is the same as of Theorem 5.3. Note that scaling the length by a factor d, the cost is multiplied by a factor d. A mass scaling of a factor m implies that the cost is multiplied by a factor m^α. $\square$

6 Hölder continuity of the landscape function

In this section we are going to prove that inequality (4.3) holds for a.e. $p \in \Omega$ with

$$\frac{1-\beta}{1-\alpha} = h,$$

that is

$$\beta = 1 + h(\alpha - 1),$$

under the hypothesis that the irrigated measure is Ahlfors regular from below in dimension h.

Lemma 6.1. *Let μ be the irrigated measure, which is supposed to be Ahlfors regular* from below *in dimension h. Let $\overline{Z}$ be the landscape function associated to the* optimal *pattern χ. Let $\beta := 1 + h(\alpha - 1)$. Then, for some $c > 0$*

$$\overline{Z}(x_0) - \overline{Z}(x) \leq c|x_0 - x|^\beta, \tag{6.1}$$

for x_0 in the support of the irrigated measure μ and $x \in \mathbf{R}^N$.

Proof. By contradiction, fix x, x_0 with $x_0 \in \operatorname{spt}\mu$ such that, for a suitably large constant c (see Figure 3),

$$Z(x_0) - Z(x) > c|x_0 - x|^\beta.$$

Let $r = |x_0 - x|$, so $Z(x_0) - Z(x) > cr^\beta$. Since we are assuming, $\operatorname{spt}\mu$ bounded, we can suppose that $r \leq 1$. Indeed, if (6.1) would hold for all couples with $r \leq 1$, it would also hold for all the remaining couples with $r > 1$.

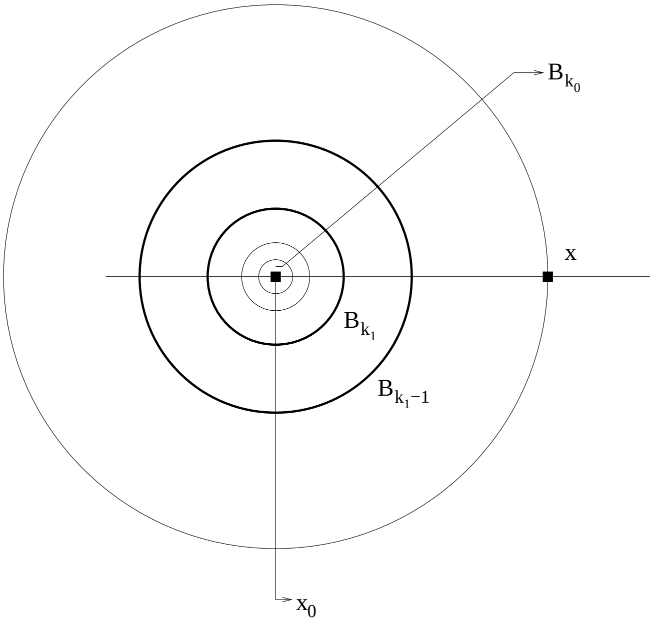

Figure 3: Proof of Lemma 6.1

Let $B_k := \overline{B}_{r_k}(x_0)$, with $r_k = r2^{-k}$. Since Z is lower semicontinuous, we have that

$$Z(x_0) = \sup_{k\geq 0} \inf_{B_k} Z.$$

Then, given $\varepsilon > 0$ we must have $Z(x_0) - \varepsilon < \inf_{B_k} Z$ for every $k \geq k_0$ for a suitable integer k_0. Since $x \in B_0$, we clearly have $\inf_{B_0} Z \leq Z(x)$. For $\varepsilon = cr^\beta/2$, we get

$$\inf_{B_{k_0}} Z - \inf_{B_0} Z \geq \frac{1}{2} cr^\beta.$$

Choose

$$c' = 2^{\beta-1}(1 - 2^{-\beta})c. \tag{6.2}$$

Since

$$\inf_{B_{k_0}} Z - \inf_{B_0} Z = \sum_{k=1}^{k_0} \left(\inf_{B_k} Z - \inf_{B_{k-1}} Z \right),$$

we cannot have for all $k = 1, \ldots, k_0$

$$\inf_{B_k} Z - \inf_{B_{k-1}} Z < c' r_k^\beta,$$

otherwise

$$c < \frac{2^{1-\beta} c'}{1 - 2^{-\beta}},$$

in contradiction to (6.2). For a certain value of $k = k_1$ we must then have:

$$\inf_{B_{k_1}} Z - \inf_{B_{k_1 - 1}} Z \geq c' r_{k_1}^{\beta}.$$

Let x_{k_1-1} a point which realizes the infimum of Z on B_{k_1-1}. We now apply Corollary 3.18 with $X = B_{k_1}$,$y = x_{k_1-1}$, and the moved mass is that carried by fibers stopping on $X = B_{k_1}$. The difference of the values of the landscape function between $X = B_{k_1}$ and $y = x_{k_1-1}$ is then given by $c' r_{k_1}^{\beta}$. Taking into account Corollary 5.4, since the irrigated measure is Ahlfors from below and $x_0 \in \operatorname{spt} \mu$, inequality (3.8) of Corollary 3.18 becomes:

$$\alpha c' r_k^{\beta} \leq C (r_k^h)^{\alpha - 1} r_k = C r_k^{\beta}$$

Here C is the constant of Corollary 5.4. This means that if we choose c, and therefore c', sufficiently large we get in a contradiction. □

Now, we reach the main theorem of the paper.

Theorem 6.2. *Let μ be the irrigated measure, which is supposed to be Ahlfors regular* from below *in dimension h. Let Z be the landscape function associated to the* optimal *pattern χ. Let $\beta := 1 + h(\alpha - 1)$. Then,*

$$|Z(x) - Z(y)| \leq c|x - y|^{\beta},$$

on $\overline{D}_{\chi}$ of the pattern χ.

Proof. The result simply follows from Lemma 6.1 and the equivalence stated in Theorem 4.5. □

7 Necessary conditions for the Hölder continuity of the landscape function

The main result of this section is the following theorem. In Appendix A the reader will find the definitions of the various notion of measure dimension which play a role in this and the following section.

Theorem 7.1. *Let $\mu \in \mathcal{P}(\Omega)$. Let χ be an* optimal pattern *between δ_S and μ. Suppose that the landscape function is Hölder continuous of exponent $\beta \leq \alpha$ (i.e. the decay exponent $h = (1 - \beta)/(1 - \alpha)$ satisfies $h \geq 1$). Then, we have*

$$\dim_{\mathrm{M}}(\mu) \leq h.$$

As a consequence of Theorem 7.1, we deduce a corollary which matches Theorem 6.2.

Definition 7.2 (Upper Ahlfors regular measure)**.** A measure μ is *Ahlfors regular from above* in dimension h, if there exists $C_A > 0$ such that

$$\mu(B(x,r)) \leq C_A r^h.$$

Corollary 7.3. *In the same assumptions of Theorem 7.1, suppose also that μ is* upper *Ahlfors regular in dimension h'. Then, the following inequality must hold:*

$$h' \leq h = \frac{1-\beta}{1-\alpha},$$

i.e. $\beta \leq 1 + h'(\alpha - 1)$.

Proof of Corollary 7.3. Recall the following results from [DS-DIM]:

- Theorem 1.1: $\dim_c(\mu) \leq \max\{\dim_M(\mu), 1\}$; recall that $\dim_c(\mu)$ is the least Hausdorff dimension of a set on which μ is concentrated;
- Corollary 1.4: if μ is upper Ahlfors regular in dimension h', then $\dim_c(\mu) \geq h'$.

We then have:

$$h' \leq \dim_c(\mu) \leq \max\{\dim_M(\mu), 1\} \leq h,$$

which completes the proof of the corollary. □

Corollary 7.4. *If the irrigated measure is Ahlfors regular in dimension $h \geq 1$, the exponent given by Theorem 6.2 is the highest one and is independent on the source of irrigation.*

Proof. Suppose that μ is Ahlfors regular from below in dimension h_1 and from above in dimension h_2. The exponents must satisfy $h_2 \leq h_1$. By Theorem 6.2 the landscape function is Hölder with exponent $\beta_1 = 1+h_1(\alpha-1)$ (or less). If we add the hypothesis that $h_2 \geq 1$, and apply Corollary 7.3 we find that the Hölder continuity exponent must not exceed $\beta_2 = 1 + h_2(\alpha - 1) \geq \beta_1$. This shows that when μ is Ahlfors regular in dimension $h \geq 1$ (i.e., $h_1 = h_2 = h$, and $\beta_1 = \beta_2 = 1 + h(\alpha - 1)$), the Hölder regularity given by Theorem 6.2 is the best possible and the regularity does not depend on the position of the source. □

To be able to prove Theorem 7.1 we have to show some preliminary results and notation.

Remark 7.5. If the landscape function is Hölder continuous of exponent β, then the length of the fibers is bounded by $\|\overline{Z}\|_{C^\beta}(\operatorname{diam}(\{S\} \cup \operatorname{spt}\mu))^\beta$ on the support of the pattern D_χ.

Indeed, If p is a fiber, then its length is given by $\int_I |\dot{\chi}(p,t)|\mathrm{d}t$. Thanks to the Hölder continuity of the landscape function we can estimate the length as:

$$\int_I |\dot{\chi}(p,t)|\mathrm{d}t \leq \overline{Z}(\chi(p,b)) - \overline{Z}(S) \leq \|\overline{Z}\|_{C^\beta}|S - \chi(p,b)|^\beta \leq \\ \leq C(\operatorname{diam}(\{S\} \cup \operatorname{spt}\mu))^\beta.$$

Since the left hand-side is the length of the fiber, the statement is proved.

Given $p \in \Omega$ and a point x define $t_p(x)$ as

$$t_p(x) := \inf\{t \in I \ : \ \chi(p,t) = x\}.$$

Coherently, $t_p(x) = b := \sup I$ if the fiber p does not pass through x. Consider the function l defined on the image of an optimal pattern χ which associates to every x the supremum of the distance along the fiber χ_p from x to the terminal point of the fiber, given by

$$l(x) = \operatorname{esssup} l_p(x), \tag{7.1}$$

where

$$l_p(x) := \int_{t_p(x)}^b |\dot{\chi}_p(s)| \ \mathrm{d}s \tag{7.2}$$

(if Z is Hölder continuous, $l_p(x) < +\infty$ for a.e. $p \in \Omega$ by Remark 7.5), and the essential supremum is taken among the particles p such that its equivalence class at the time instant where it passes through x are of positive measure, i.e. $|[p]_{t_p(x)}| > 0$.

With this notation, the implication between item 1 and 3 in Theorem 4.3 can be restated as: if Z is Hölder continuous of exponent β, then

$$m_\chi(x) \geq Cl(x)^h, \tag{7.3}$$

for some constant C and $h = (1-\beta)/(1-\alpha)$.

Let us recall a definition from Paragraph 4.4 in [MS-SYNCH].

Definition 7.6 (Flow ordering)**.** Consider an optimal pattern χ. Let $x, y \in \mathbf{R}^N$. We say that x precedes y in the *flow order* if there exists $A \subseteq \Omega$, with $\mu_\Omega(A) > 0$, such that for all $p \in A$ we have that $c(p) < +\infty$ and $t_x \leq t_y$, where $\chi_p(t_x) = x, \chi_p(t_y) = y$. In this case we write $x \leq y$. Note this is a partial ordering.

Lemma 7.7. *Consider an optimal pattern χ. Suppose that $x \le y$. For a.e. $p \in \Omega$:*

$$l(y) - l(x) \le -d(x, y).$$

As a consequence, l is decreasing w.r.t. $\le$.

Proof. If $x, y \in \chi_p(I)$, we must have by definition:

$$l_p(x) - l_p(y) = d(x, y).$$

Formula 7.1 implies that for every $\varepsilon > 0$, there exists a set P of fibers with a positive measure passing through y such that if $q_\varepsilon \in P$ then

$$l(y) \le l_{q_\varepsilon}(y) + \varepsilon.$$

Since χ is a simple pattern, $x \in \chi_{q_\varepsilon}$. We have then for some q_ε:

$$l(x) \ge l_{q_\varepsilon}(x) = l_{q_\varepsilon}(y) + d_{q_\varepsilon}(x, y) = l_{q_\varepsilon}(y) + d(x, y) \ge l(y) - \varepsilon + d(x, y).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the lemma is proved. □

Lemma 7.8. *Consider an* optimal *pattern χ. Suppose that the landscape function is Hölder continuous of exponent β. Then, $l \circ \chi_p$ is upper semi-continuous for a.e. $p \in \Omega$.*

Proof. $l \circ \chi_p$ is decreasing by Lemma 7.7, then we just need to prove the left-continuity. Given $t_0 \in I$ and any increasing sequence $t_n \to t_0$ we will prove that

$$c < \lim_{n \to +\infty} l(\chi_p(t_n)) \Longrightarrow c \le l(\chi_p(t_0)).$$

This will imply $\lim_n l(\chi_p(t_n)) = l(\chi_p(t_0))$. Note that, in principle, we could have $c > l(\chi_p(t_0))$, if the fibers passing through $\chi_p(t_n)$ and almost realizing the value of $l(\chi_p(t_n))$ would not also pass through $\chi_p(t_0)$. In the following we will rule out this occurrence, showing that there must be a set of fibers of positive measure passing through $\chi_p(t_0)$ and of "residual length" at least c. Let

$$A_n = \{q \in [p]_{t_n} \ : \ l_q(\chi_p(t_n)) \ge c\}.$$

We have $A_{n+1} \subseteq A_n$, since $[p]_{t_{n+1}} \subseteq [p]_{t_n}$ and $l_q(\chi_p(t_n)) \ge l_q(\chi_p(t_{n+1}))$. Let c' any number such that:

$$c < c' < \lim_{n \to +\infty} l(\chi_p(t_n)).$$

Since $l(\chi_p(t_n)) > c'$, we can find $p_n \in [p]_{t_n}$ (with $[p]_{t_n}$ of positive measure) such that $l_{p_n}(\chi_p(t_n)) > c'$. Then, if we take $\tau_n > t_n$ such that $d(\chi_{p_n}(t_n), \chi_{p_n}(\tau_n)) = c$, we have

$$l_{p_n}(\chi_{p_n}(\tau_n)) > c' - c.$$

Therefore, from inequality (7.3),

$$|[p_n]_{\tau_n}| \geq C(c' - c)^h,$$

where, as usual, $h = (1-\beta)/(1-\alpha)$. Obviously, $[p_n]_{\tau_n} \subseteq A_n$, so,

$$\mu_\Omega(A_n) \geq C(c' - c)^h.$$

Let

$$A = \bigcap_{n>0} A_n.$$

Then,

$$\mu_\Omega(A) \geq C(c' - c)^h > 0.$$

Every fiber $q \in A$ passes through $\chi_p(t_0)$ (thanks to the continuity of χ_p) and from $l_q(\chi_p(t_n)) \geq c$ it follows that $l_q(\chi_p(t_0)) \geq c$. Consequently $l(\chi_p(t_0)) \geq c$. □

We now go on with a key proposition in the proof of Theorem 7.1. This theorem is a kind of "intermediate values theorem" for l (in spite of the fact it is not continuous).

Proposition 7.9. *Suppose that l is the function associated to an* optimal *pattern χ. Let $l_1 \leq l(x_1)$. Then, there exists $x_2 \geq x_1$, such that $l(x_2) = l_1$.*

Proof. Let $l_1 > 0$, otherwise we just choose a terminal point. Consider the following minimization problem:

$$m = \inf\{l(y) \ : \ y \geq x_1, l(y) \geq l_1\}. \tag{7.4}$$

Let y_n be a minimizing sequence and suppose $l(y_n) \geq l(y_{n+1}), l(y_n) \to m \geq l_1$. Let

$$P_n = \{p \in \Omega \ : \ y_n \in \chi_p(I)\}.$$

Because of our hypothesis on the Hölder continuity of the landscape function, we deduce from inequality (7.3),

$$\mu_\Omega(P_n) \geq C l_1^h.$$

Let

$$P = \limsup_{n\to+\infty} P_n = \bigcap_{n\geq 0} \bigcup_{k\geq n} P_k.$$

By Fatou Lemma,

$$\mu_\Omega(P) \geq \limsup_{n\to+\infty} \mu_\Omega(P_n) \geq C l_1^h.$$

By definition of P, if $p \in P$ for some increasing sequences n_k and t_k we must have $\chi(p, t_k) = y_{n_k}$. Finally, $l(y_{n_k}) \geq l_1$ and $t_k \to t_\infty$ (since y_n is a minimizing sequence on which l is decreasing).

Let $y_{\min} := \chi(p, t_\infty)$. Then, $y_{n_k} \to y_{\min}$ and by Lemma 7.8

$$l(y_{\min}) \geq \limsup_{k \to +\infty} l(y_{n_k}) = \lim_{k \to +\infty} l(y_{n_k}) \geq l_1,$$

$y_{\min}$ then solves (7.4), since for all k we have that $l(y_{\min}) \leq l(y_{n_k})$ (y_{n_k} precedes $y_{\min}$ and l is monotone on the fiber as stated in Lemma 7.7).

If now we set $x_2 = y_{\min}$, we just need to prove that $l(y_{\min}) = l_1$. Suppose, on the contrary, that $l(y_{\min}) > l_1$. Then, for some fiber p_1, $l_{p_1}(y_{\min}) > l_1$. Then for some ε_1 we would find a point $\overline{y} = \chi(p_1, t_\infty + \varepsilon_1) \geq x_1$, $l_{p_1}(\overline{y}) > l_1$ and, by Lemma 7.7, $l(\overline{y}) < l(y_{\min})$. Then, $y_{\min}$ would not be the minimizer of the problem stated in (7.4), since $\overline{y}$ would satisfy the constraints and $l(\overline{y}) < l(y_{\min})$. □

Proof of Theorem 7.1. Consider the following construction. We start from a terminal point x_0 and go back along its fiber p by a length equal to δ. We reach a point x_1 where $l(x_1) \geq l_p(x_1) = \delta$. Let $x_2 \geq x_1$ be the point in Proposition 7.9 such that $l(x_2) = k\delta$ for the maximum positive integer k possible. Since k is maximum, the Euclidean distance between x_0, x_2 is at most 2δ by Lemma 7.7. Note that the maximum integer $k(\delta)$ obtained in this way does not exceed $\delta^{-1} C d^\beta$ (where C is the Hölder constant of the landscape function and $d = \operatorname{diam}(\{S\} \cup \operatorname{spt} \mu)$). Indeed, by Remark 7.5 we have

$$k(\delta)\delta = l(x_2) \leq C d^\beta.$$

Let $S(\delta)$ the set of points obtained like x_2 from a terminal point x_0 and ν the measure which realizes the minimum of the ∞-Wasserstein distance between the irrigated measure μ and the set of probability measures supported on $S(\delta)$. The distance of each of the selected points to the support of the final measure is at most 2δ, so that $w_\infty(\mu, \nu) \leq 2\delta$.

Given k, the number of points of $S(\delta)$ such that $l(x) = k\delta$ can be estimated by $C^{1/(1-\alpha)} \delta^{-h} k^{-h}$: indeed, these points cannot be on a flow line by the way they are chosen, then their number N_k must satisfy

$$C^{1/(\alpha-1)} N_k (k\delta)^h \leq 1,$$

since by (7.3) each point carries a mass at least given by $C^{1/(\alpha-1)}(k\delta)^h$ and their total mass must be less than 1.

Hence, for the total number of points $N(\delta)$ we have the following estimate:

$$N(\delta) = \sum_{k=1}^{k(\delta)} N_k \leq C^{1/(1-\alpha)} \delta^{-h} \left[1 + \int_1^{\delta^{-1} C d^\beta} x^{-h} \mathrm{d}x \right]. \tag{7.5}$$

In the case $h > 1$,

$$\int_1^{\delta^{-1}Cd^\beta} x^{-h}\mathrm{d}x = -\frac{[\delta C^{-1}d^{-\beta}]^{h-1} - 1}{h-1} \leq -\log(\delta C^{-1}d^{-\beta}).$$

The last inequality follows from the convexity of the exponential function. In the case $h = 1$,

$$\int_1^{\delta^{-1}Cd^\beta} x^{-1}\mathrm{d}x = -\log(\delta C^{-1}d^{-\beta}).$$

In any case, we can estimate

$$N(\delta) \leq C^{1/(1-\alpha)}\delta^{-h}(1 - \log(\delta C^{-1}d^{-\beta})) =: \delta^{-h}f(\delta),$$

where $f(\delta)$ grows as $-\log\delta$ as $\delta \to 0^+$. Now, given n choose the unique δ (the uniqueness is true only for δ small) such that $n = \delta^{-h}f(\delta)$. For such a choice of δ,

$$w_\infty(\mu, D_n) \leq w_\infty(\mu, D_{N(\delta)}) \leq 2\delta.$$

We then easily have:

$$\dim_{\mathrm{res}}^\infty(\mu) = \limsup_{n\to+\infty}\left(\frac{\log w_\infty(\mu, D_n)}{-\log n}\right)^{-1} \leq \limsup_{\delta\to 0^+}\left(\frac{\log(2\delta)}{-\log(\delta^{-h}f(\delta))}\right)^{-1} = h.$$

The statement of the theorem then follows from the following results of [DS-DIM]: $\dim_{\mathrm{M}}(\mu) = \dim_{\mathrm{res}}^\infty(\mu)$ (Proposition 5.4). □

Remark 7.10. Note that in the previous proof, in the case $h > 1$, $N(\delta)$ can be estimated by

$$N(\delta) = \sum_{k=1}^{k(\delta)} N_k \leq C^{1/(1-\alpha)}\delta^{-h}\left[1 + \int_1^{+\infty} x^{-h}\mathrm{d}x\right].$$

Since it does not depend on $k(\delta)$ and, consequently, on the size d of the diameter of the convex hull of the supports of the initial and final measure (so that the irrigated measure does not need to be compactly supported). Of course, this is not the case when $h = 1$.

8 Counter-examples

We now provide some counter-examples which show the opportunity of the hypothesis assumed in this paper.

Consider a probability measure $\mu \in \mathcal{P}(\mathbf{R})$. Its *distribution function* is $F(x) = \mu((-\infty, x))$ (this definition is slightly different from the usual one). Recall that F is a non-decreasing, left-continuous (hence, lower semicontinuous) function. Moreover, $0 \le F(x) \le 1$ and

$$\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to+\infty} F(x) = 1.$$

Set

$$J(x_0) := \lim_{x\to x_0^+} F(x) - F(x_0) = \mu(\{x_0\}).$$

Given $y \in [0,1]$, the set

$$F^{-1}(]-\infty, y]) = \{x \in \mathbf{R} \ : \ F(x) \le y\}$$

is a closed interval of the form $]-\infty, \alpha]$. We then set for $y \in [0,1[$, taking advantage of the lower semicontinuity of F,

$$G(y) := \max\{x \in \mathbf{R} \ : \ F(x) \le y\}.$$

G is the so-called *quantile function* of F. Since

$$F^{-1}((-\infty, y]) = (-\infty, G(y)],$$

we have by construction

$$F(x) \le y \Longleftrightarrow x \le G(y).$$

G is a non-decreasing and right-continuous (hence, upper semicontinuous) function. Of course, if F is one-to-one, then G is just F^{-1}. Note also that in general this may not happen because μ can be null on some interval.

Suppose now that we are given a measure μ such that $\operatorname{spt}\mu \subseteq [0,1]$ and consider the problem of the irrigation of μ from $S = 0$ (in the following we will always consider the irrigation from $S = 0$, unless differently stated). The optimal pattern is given on $\Omega = [0,1]$ by

$$\chi(p,t) = \min\{t, G(p)\},$$

since the support must be convex and the no-loop condition must hold, and the multiplicity is given by

$$m(x) = 1 - F(x) = \mu([x,1]).$$

The landscape function is then given by

$$Z(x) = \int_0^x m(t)^{\alpha-1}\mathrm{d}t = \int_0^x (1 - F(t))^{\alpha-1}\mathrm{d}t. \tag{8.1}$$

We have seen in Corollary 7.4 that when μ is Ahlfors regular in dimension h, the best Hölder of exponent of the landscape function is $1 + h(\alpha - 1)$ and it does not depend on how we choose the source S. The same fact is not true if the irrigated measure is only Ahlfors regular from below, as the following example shows.

Example 8.1. *When the measure is Ahlfors regular from below in dimension h, the regularity of the landscape function may depend on the location of the source S and may assume both the lowest best possible value* $1 + h(\alpha - 1)$ *(given by Theorem 6.2) and the highest best possible value* 1. Consider the measure $\mu \in \mathcal{P}([0, 1])$ given by

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2} f \mathcal{L}_{|[0,1]}$$

where $f(x) = h(1 - x)^{h-1}$ with $h > 1$. It is easy to check that μ is Ahlfors regular from below in dimension h (but not from above). The distribution function of μ is given by

$$F(x) = 1 - \frac{1}{2}(1 - x)^h \tag{8.2}$$

Suppose now that $\alpha > 1 - 1/h$. By formula (8.1) we have

$$Z(x) = \int_0^x \frac{1}{2^{\alpha-1}}(1 - t)^{h(\alpha-1)}\mathrm{d}t = \frac{1}{2^{\alpha-1}\beta}(1 - (1 - x)^\beta),$$

where $\beta = 1 + h(\alpha - 1)$. Z is then Hölder continuous with the exponent given by Theorem 6.2. Note that in this case $x = 1$ is a terminal point. The same regularity holds for $S < 1$.

On the other side, if we irrigate the same measure from a point $S \geq 1$, the mass function in $0 \leq x \leq 1$ is given by

$$m(x) = 1 - \frac{1}{2}(1 - x)^h \geq \frac{1}{2},$$

while $m(x) = 0$ if $x \geq 1$. The landscape function is given by

$$Z_1(x) = \int_0^x [m(x)]^{\alpha-1}\mathrm{d}t + S - 1,$$

which is Lipschitz continuous, the best possible regularity of the landscape function.

An analogous construction leads to the following remark.

Example 8.2. *A counterexample to the thesis of Corollary 7.3 if we drop the hypothesis* $h \geq 1$. An estimate analogous to inequality (7.5) in the case $h < 1$ actually provides only $\dim_{\mathrm{M}}(\mu) \leq 1$, which gives $h' \leq \dim_{\mathrm{c}}(\mu) \leq \min\{\dim_{\mathrm{M}}(\mu), 1\} = 1$ since $\dim_{\mathrm{M}}(\mu) \leq h < 1$. Actually, this is in the nature of things. Indeed, consider the irrigation from $S = 0$ of the measure

$$\mu = \frac{1}{2}\mathcal{L}^1_{|[0,1]} + \frac{1}{2}\delta_1.$$

Since the multiplicity is bounded from below, the landscape function is Lipschitz continuous ($h = 0$), but the measure has Minkowski dimension $h' = 1$.

In general, without assumptions on the irrigated measure, the landscape function may be no more than lower semicontinuous. This may happen in dimension greater than one (as the next example shows). In the 1-dimensional case, the continuity is guaranteed by the following proposition.

Proposition 8.3. *In the 1-dimensional case, the landscape function is continuous. Moreover, it is locally Lipschitz continuous in the set where it is finite.*

Proof. Let $[a, b]$ be the convex hull of the support of the irrigated measure and the source of irrigation. Without any restriction, let $S = 0$ be the source of irrigation. The landscape function is then defined on $[a, b]$. Suppose that $a \geq 0$. Since the function $(1 - F(x))^{\alpha-1}$ is non-decreasing, it is bounded in any interval of the type $[0, b - \varepsilon]$, so that $Z \in \mathrm{Lip}_{\mathrm{loc}}([0, b - \varepsilon])$ and the proposition is proved.

If $a < 0$, the optimal pattern is built up merging the optimal pattern irrigating $\mu_{|[0,b]}$ and $\mu_{|[a,0]}$. This proves the continuity and Lipschitz continuity separately, which ultimately given continuity of the landscape function on $[a, b]$ and its locally Lipschitz continuity on $]a, b[$. □

The landscape function remains continuous even if it takes value $+\infty$. It is easy to see that this case may happen for an irrigable measure as shown, for instance, in the following example.

Remark 8.4. Note that we may have $Z(b) = +\infty$ (but μ is nevertheless irrigable). This is the case when we consider a measure μ whose density is given by

$$f(x) = h(1 - x)^{h-1}\mathcal{L}^1_{|[0,1]}.$$

In this case $b = 1, 1 - F(x) = (1 - x)^h$. If we take $h(1 - \alpha) > 1$, for $x < 1$, we have

$$Z(x) = \int_0^x (1 - F(t))^{\alpha-1} \mathrm{d}t = \frac{1}{1 + h(\alpha - 1)} - \frac{(1 - x)^{1+h(\alpha-1)}}{1 + h(\alpha - 1)},$$

while $Z(1) = +\infty$. Nevertheless, μ is irrigable since in the 1-dimensional case every measure is irrigable for $0 < \alpha \leq 1$.

Example 8.5. *The landscape function is, in general, not continuous in dimension $N \geq 2$.* Let $\alpha \in [0, 1[$ and let the source be $S = (0, 0)$.

1. Consider any sequence $\{x_n\}_{n\geq 1}$ of points in $\mathbf{R}^2$ such that:

   - $x_n \to S$ as $n \to +\infty$;
   - for every $n \geq 1$, the point x_{n+1} is a positive distance apart from the (closed) convex envelope C_n of $S, x_1, x_2, \ldots, x_n$.

   For example, the sequence given, for $n \geq 1$, by

   $$x_n = \frac{1}{n}\left(\cos\left(\frac{\pi}{2n}\right), \sin\left(\frac{\pi}{2n}\right)\right),$$

   would fit these first requirements.

2. Consider now the following construction given by recurrence. Suppose that, given $n \geq 1$, we have defined a measure μ_n of this type:

   $$\mu_n = \sum_{k=1}^{n} a_k \delta_{x_k},$$

   with

   $$\sum_{k=1}^{n} a_k = 1.$$

   Note that this conditions determine μ_1 (which turns out to be δ_{x_1}). Define

   $$\mu_n^a = a\nu + (a_n - a)\delta_{x_n} + \sum_{k=1}^{n-1} a_k \delta_{x_k},$$

   where ν is any probability measure supported in the closure of the set $\{x_k \ : \ k > n\}$. Since $\mu_n^a \rightharpoonup \mu_n$ as $a \to 0^+$, $\chi_n^a(p, \cdot) \to \chi_n(p, \cdot)$ uniformly for a.e. $p \in \Omega$ by the Skorohod Theorem.

3. Fix a radius r_{n+1} such that the $B_{r_{n+1}}(x_{n+1})$ is a positive distance apart from C_n. We now prove that for every $\varepsilon > 0$ there exists $\delta_n(\varepsilon) > 0$ such that, whenever $a < \delta_n(\varepsilon)$, the measure of the set

   $$P_n^a = \{p \in \Omega \ : \ \chi_n(p, I) \cap B_{r_{n+1}}(x_{n+1}) \neq \emptyset\}$$

is less that ε. Indeed, suppose on the contrary that for some $\varepsilon > 0$ there exists a sequence $a_k \to 0$ such that $\mu_\Omega(P_n^{a_k}) \geq \varepsilon$. Then,

$$\mu_\Omega\left(\limsup_{k\to+\infty} P_n^{a_k}\right) \geq \limsup_{k\to+\infty} \mu_\Omega(P_n^{a_k}) \geq \varepsilon,$$

which is in contradiction to the a.e. uniform convergence given by the Skorohod Theorem (the previous step).

4. Fix now ε_{n+1} such that $\varepsilon_{n+1}^{\alpha-1} r_{n+1} > n+1$ and $a_{n+1} < \min\{a_n, \delta_n(\varepsilon_{n+1})\}$ such that $P_n^{a_{n+1}}$ has measure less that ε_{n+1}.

5. Consider now an optimal pattern χ irrigating the measure μ built in this way. This pattern is irrigable if $\alpha > 1/2$ (note that we could also require $\sum_n a_n^\alpha |x_n| < +\infty$, so that the measure would be also irrigable for $\alpha \leq 1/2$). By construction, $Z(x_n) \geq \varepsilon_n^{\alpha-1} r_n > n$ since a mass at most given by ε_n has to cover a distance at least given by r_n to reach x_n. Then,
$$\lim_{n\to+\infty} Z(x_n) = +\infty > 0 = Z(S).$$
The landscape function is then lower semicontinuous, but not continuous in S.

A Dimensions

In this appendix we recall definitions and main properties of the dimensions used in the paper. For the details we refer to [DS-DIM] or [M].

A.1 Hausdorff dimension

Given a set A, its *Hausdorff dimension* is defined by:

$$\dim_{\mathcal{H}}(A) := \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(A) = 0\}.$$

Definition A.1 (Hausdorff concentration dimension)**.** Given a Borel measure μ, we define the *Hausdorff concentration dimension* of μ as

$$\dim_{\mathrm{c}}(\mu) = \inf\left\{\dim_{\mathcal{H}}(B) : \mu(B^c) = 0\right\}.$$

Proposition A.2. *If a Borel measure μ is upper Ahlfors regular in dimension h', then $\dim_{\mathrm{c}}(\mu) \geq h'$.*

Proof. See Corollary 1.4 of [DS-DIM]. □

A.2 Minkowski dimension

Minkowski dimension can be defined in several ways. Let A be a bounded subset of $\mathbf{R}^N$. Define $N(A,\varepsilon)$ as

$$N(A,\varepsilon) := \min\left\{k \in \mathbf{N} : A \subseteq \bigcup_{i=1}^{k} B_\varepsilon(x_i),\ x_i \in \mathbf{R}^N\right\}.$$

$N(A,\varepsilon)$ is the least number of balls of radius ε whose union covers A.

Definition A.3 (Minkowski dimension)**.** The *Minkowski dimension* of a set A is defined as:

$$\dim_{\mathrm{M}}(A) := \inf\left\{\alpha \geq 0 : \limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\alpha = 0\right\}.$$

Definition as a power of ε^{-1}. It is easy to see that Minkowski upper dimension is also given by:

$$\dim_{\mathrm{M}}(A) = \limsup_{\varepsilon\to 0^+} \frac{\log N(A,\varepsilon)}{-\log\varepsilon} = \limsup_{\varepsilon\to 0^+} \log_{1/\varepsilon} N(A,\varepsilon).$$

The Minkowski dimension measures how fast $N(A,\varepsilon)$ grows as $\varepsilon \to 0^+$ in terms of a power of ε^{-1}.

Definition via the Minkowski content. Another equivalent definition is:

$$\dim_{\mathrm{M}}(A) := N + \limsup_{\varepsilon\to 0^+} \frac{\log \mathcal{L}^N(A_\varepsilon)}{-\log\varepsilon},$$

where A_ε is the close ε-neighborhood given by

$$A_\varepsilon := \left\{x \in \mathbf{R}^N : d(x,A) \leq \varepsilon\right\}.$$

Definition as box counting dimension. There is a furher way to definite Minkowski dimension. Recall that a *dyadic cube of order* m in $\mathbf{R}^N$ is a Cartesian product of N intervals of the kind $[k2^{-m},(k+1)2^{-m}[$ for $k \in \mathbf{Z}, m \in \mathbf{N}$. For every given m, the dyadic cubes of order m are a disjoint cover of $\mathbf{R}^N$. Let $Q(A,m)$ be the cardinality of dyadic cubes of order m which meet A. Minkowski dimension is given by

$$\dim_{\mathrm{M}}(A) = \limsup_{\varepsilon\to 0^+} \frac{\log Q(A,m)}{m\log 2}$$

Proposition A.4. *For every set A, we have*

$$\dim_{\mathcal{H}}(A) \leq \dim_{\mathrm{M}}(A).$$

Definition A.5 (Minkowski dimension)**.** The *Minkowski dimension* of a measure μ is given by the infimum of Minkowski dimensions of the sets B on which μ is concentrated (or equivalently of the support of μ). It is denoted by $\dim_{\mathrm{M}}(\mu)$.

Proposition A.6. *Let $\mu \in \mathcal{P}(\mathbf{R}^N)$. Then,*

$$\dim_{\mathrm{c}}(\mu) \leq \max\{\dim_{\mathrm{M}}(\mu), 1\}.$$

Proof. See Theorem 1.1 of [DS-DIM]. □

A.3 Resolution dimension

The *resolution dimension* was introduced by Devillanova and Solimini in [DS-DIM], to which we refer for the proof of the statements stated. Let $\mu \in \mathcal{P}(\Omega)$. Consider the set D_n of discrete measures ν with $\#(\operatorname{spt}\nu) \leq n$ and the minimization problem

$$w_p(\mu, D_n) := \min_{\nu \in D_n} w_p(\mu, \nu).$$

Definition A.7 (Resolution dimension)**.** Let $\mu \in \mathcal{P}(\mathbf{R}^N)$ and $p \in [1, +\infty]$, then the *resolution dimension* of μ of index p is given by

$$\dim_{\mathrm{res}}^{\mathrm{p}}(\mu) := -\left(\limsup_{n\to+\infty} \frac{\log w_p(\mu, D_n)}{\log n}\right)^{-1}.$$

The next proposition is contained in [DS-DIM].

Proposition A.8. *For all probability measures μ we have*

$$\dim_{\mathcal{H}}(\mu) \leq \dim_{\mathrm{res}}^{\mathrm{p}}(\mu) \leq \dim_{\mathrm{M}}(\mu).$$

Moreover for $p = +\infty$ the resolution dimension coincides with the Minkowski dimension.

B Measurability facts

Recall that in the model considered a pattern $\chi : \Omega \times I \to \mathbf{R}^N$ is a Carathéodory function. This means that:

- for all $t \in I$ the function $p \mapsto \chi_t(p) := \chi(p,t)$ is measurable;
- for μ_Ω-a.e. $p \in \Omega$ the function $t \mapsto \chi_p(t) := \chi(p,t)$ is continuous.

Definition B.1 (Fibers passing through a point x)**.** Given $x \in \mathbf{R}^N$, consider the set $T(x)$ defined by

$$T(x) := \{p \in \Omega \ : \ x \in \chi(p, I)\}.$$

Equivalently, $T(x)$ is the set of the fibers $p \in \Omega$ such that for some $t \in I$ we have $\chi(p,t) = x$, i.e. the fibers passing through x. Hence,

$$T(x) = \bigcup_{t \in I} \chi_t^{-1}(\{x\}). \tag{B.1}$$

Remark B.2. $T(x)$ is nothing other than $T(\{x\})$ of Definition 3.13.

Proposition B.3. *$T(x)$ is a measurable set.*

Proof. The set $\chi_t^{-1}(\{x\})$ is measurable for all $t \in I$. Of course, this is not sufficient to prove the measurability since equation (B.1) does not define $T(x)$ as a countable union. Let I_m be an increasing sequence of compact intervals whose union is I (if I is itself compact, choose $I_m = I$ for all m). Define the sets $T_{m,n}(x), T_n(x)$ as

$$T_{m,n}(x) = \bigcup_{t \in I_m} \chi_t^{-1}(B_{1/n}(x)), \quad T_m(x) = \bigcup_{t \in I_m} \chi_t^{-1}(\{x\}).$$

We have that:

$$T_m(x) = \bigcap_{n>0} T_{m,n}(x), \quad T(x) = \bigcup_{m>0} T_m(x). \tag{B.2}$$

First, we prove that $T_{m,n}(x)$ is a measurable set. We have:

$$T_{m,n}(x) = \bigcup_{t \in I_m \cap \mathbf{Q}} \chi_t^{-1}(B_{1/n}(x)). \tag{B.3}$$

Indeed, if $\overline{p} \in T_{m,n}(x)$, then $\chi(\overline{p}, \overline{t}) \in B_{1/n}(x)$, then, by the continuity of χ w.r.t. the variable t, $\chi(\overline{p}, t) \in B_{1/n}(x)$ for t in a suitable open interval containing $\overline{t}$. In particular, $\chi(\overline{p}, \tilde{t}) \in B_{1/n}(x)$ for some $\tilde{t} \in \mathbf{Q}$. Since in equation (B.3), $T_{m,n}(x)$ is a countable union of measurable sets, it is measurable.

We now prove that the first equality in (B.2) holds, the second being straighforward. Since $T_m(x) \subseteq T_{m,n}(x)$ for every n,

$$T_m(x) \subseteq \bigcap_{n>0} T_{m,n}(x).$$

Now, suppose that $\overline{p} \in \cap_n T_{m,n}(x)$. For every $n > 0$ there must be $t_n \in I_m$ such that $\chi(\overline{p}, t_n) \in B_{1/n}(x)$. Since I_m is compact, up to a subsequence $t_n \to \overline{t} \in I_m$. Clearly, $\chi(\overline{p}, \overline{t}) = x$, that is $\overline{p} \in B_{1/n}(x)$. □

This result can be easily generalized to a general closed set X.

Proposition B.4. *If X is a closed subset of $\mathbf{R}^N$, $T(X)$ is a measurable set.*

Proof. Just replace $B_{1/n}(x)$ by the $1/n$-neighborhood of X in the proof of Proposition B.3. □

C Notation

$\mathbf{R}^N$:	the Euclidean N-dimensional space.
$\mathcal{P}(\mathbf{R}^N)$:	the set of Borel probability measures on $\mathbf{R}^N$.
$M(t)$:	the Monge functional (see equation (1.1)).
$\mathcal{M}(\mu^+, \mu^-)$:	the set of transport maps between μ^+, μ^-.
$K(\pi)$:	the Kantorovich functional (see equation (1.2)).
$\mathcal{P}(\mu^+, \mu^-)$:	the set of transport plans between μ^+, μ^-.
$w_p(\mu^+, \mu^-)$:	Wasserstein distance of order p between μ^+, μ^-.
$W_p(\overline{X})$:	Wasserstein space of order p.
$(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$:	the space of particles or reference space; μ_Ω is the reference measure.
$\sigma_\chi(p)$:	the stopping time: $\sigma_\chi(p) := \inf\{t \in \mathbf{R} \ : \ \chi_p \text{ const on } [t, +\infty]\}$.
$i_\chi(p)$:	the terminal point of the fiber p: $i_\chi(p) := \chi(p, \sigma_\chi(p)) = \chi(p, b)$.
μ:	the irrigated measure: $\mu = i_{\chi\#}\mu_\Omega$; $\operatorname{spt}\mu$ is compact, unless differently stated.
AC(I):	absolutely continuous functions on the interval I.
$[p]_t^i, i = 0, 1, 2$:	solidarity classes (see Definition 1.2).
$m_\chi^i(p, t), i = 0, 1, 2$:	mass function (see equation (1.8)).
$s_\alpha^i(p, t), i = 0, 1, 2$:	cost densities (see Definition 1.3).
$J_\alpha^i(p, t), i = 0, 1, 2$:	cost functionals (see Definition 1.3).
$c(p)$:	the cost of the fiber p (see equation (2.1)).
D_χ:	the domain of the pattern χ (see Definition 2.1).
Z_χ:	see Definition 2.2.
$\overline{\chi}$:	the mass deviation of the pattern χ.
$\tilde{\chi}$:	the mass by-pass of the pattern χ.
$d(x_1, x_2)$:	the distance of x_1, x_2 on the fiber. for a simple pattern (see equation 4.1).

$\beta = 1 + h(\alpha - 1)$:	the usual relation between β, h, α.
$l(x)$:	the maximal distance of x to a terminal point (see Definition 7.1).
$l_p(x)$:	the distance of x to the terminal point of the fiber p (see Definition 7.2).
$d_\alpha(\mu, \nu)$:	the least cost of the irrigation of ν from μ (see equation (3.3)).
$\mathcal{H}^\alpha$:	outer Hausdorff measure in dimension α.
$\dim_{\mathcal{H}}(A)$:	Hausdorff dimension of the set A.
$\dim_{\mathrm{c}}(\mu)$:	Concentration dimension of a measure μ (see Definition A.1).
$\dim_{\mathrm{M}}(\mu)$:	Minkowski dimension of a measure μ (see Definition A.3).
$\dim_{\mathrm{res}}^{\mathrm{p}}(\mu), \dim_{\mathrm{res}}^{\infty}(\mu)$:	Resolution dimension of a measure μ (see Definition A.7).

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