Characterization of BV functions on open domains: the Gaussian case and the general case

Davide Addona*

Department of Mathematics and applications
University of Milano Bicocca
via Cozzi 55, 20125 Milano, Italy

Giorgio Menegatti† and Michele Miranda jr.‡

Department of Mathematics and Computer Science
University of Ferrara
via Machiavelli 30, I-44121 Ferrara, Italy

March 12, 2019

Abstract

We provide three different characterizations of the space $BV(O,\gamma)$ of the functions of bounded variation with respect to a centred non-degenerate Gaussian measure γ on open domains O in Wiener spaces. Throughout these different characterizations we deduce a sufficient condition for belonging to $BV(O,\gamma)$ by means of the Ornstein-Uhlenbeck semigroup and we provide an explicit formula for one-dimensional sections of functions of bounded variation. Finally, we apply our technique to Fomin differentiable probability measures ν on a Hilbert space X, inferring a characterization of the space $BV(O,\nu)$ of the functions of bounded variation with respect to ν on open domains $O \subseteq X$.

Keywords: Infinite dimensional analysis; functions of bounded variation; open domains in Wiener spaces; geometric measure theory

Mathematics Subject Classification [2010] Primary: 58E, 28C20; Secondary: 26B30, 60H07

1 Introduction

Functions of bounded variation (BV functions in the sequel) have had an important role in several classical problems of the Calculus of Variations (see [2] for a complete and an in-depth dissertation). In one dimension they have been introduced in 1881 in [21] by Jordan who also pointed out the canonical decomposition of BV functions as the difference of two increasing functions.

A correct generalization to higher dimensions required over than 50 years and it is due to Fichera and De Giorgi, who related BV functions to distributions. In [15] Fichera defined BV functions

*email: davide.addona@unimib.it
†email: giorgio.menegatti@unife.it
‡email: michele.miranda@unife.it

as those functions whose partial derivatives, in the sense of distributions, are measures with finite total variation, i.e., given a continuous function u and open set $\Omega \in \mathbb{R}^n$, u is a function of bounded variation if the values

$$T_i u(Q) := \int_{\partial Q} u\nu_i d\mathcal{L}^{n-1},$$

are finite, where $Q \subseteq \Omega$ is a cube with sides parallel to the coordinate axes, $i \in \{1, \cdots, n\}$ and ν_i is the i-th component of the outward pointing unit normal to ∂Q.

In [11] De Giorgi showed that functions whose distributional derivatives are measures with finite total variation can be characterized by means of the behaviour near 0 of the heat semigroup T_t. To be more precise, he proved that $u \in L^\infty(\mathbb{R}^n)$ is a BV function if

$$I(u) := \lim_{t \downarrow 0} \int_{\mathbb{R}^n} |\nabla T_t u| dx < +\infty.$$

Further, in [25, 26] Mario Miranda provided an alternative definition of BV functions introducing the functional

$$V(u, \Omega) := \sup \left\{ \sum_{i=1}^m \int_\Omega u^i \mathrm{div}(v^i) dx : v \in [C_c(\Omega)]^{mN},\ \|v\|_\infty \le 1 \right\},$$

for any $u = (u^1, \ldots, u^m) \in \left[L^1_{\mathrm{loc}}(\Omega)\right]^n$ and any open set $\Omega \subseteq \mathbb{R}^N$, and showing that u is a function of bounded variation if and only if $V(u, \Omega) < +\infty$.

The last important characterization of BV functions is in terms of smooth functions. Indeed, a function $u \in [L^1(\Omega)]^m$ is a BV function if and only if there exists a sequence $(u_n) \subseteq [C^\infty(\Omega)]^m$ which converges to u in L^1 and whose gradients are uniformly bounded in L^1.

In infinite dimension, BV functions have been introduced by Fukushima and Hino in [16, 17]. The first problem which arises in infinite dimension is that there does not exist an analogous of the Lebesgue measure. Therefore, in [16, 17] the authors deal with a Wiener space, i.e., a Banach space endowed with a Gaussian measure γ and a related differential structure characterized by the Cameron–Martin space H, and they define the space of BV functions with respect to γ relying upon the theory of Dirichlet forms.

The first attempt to study BV functions in Wiener spaces with tools which are closer to those of Geometric Measure Theory in Euclidean setting is [3]. Here, the authors consider the Wiener space (X, γ, H) and analyse the connection among the distributional notion of vector-valued measures, approximations by means of smooth functions and the properties of the Ornstein–Uhlenbeck semigroup, which in Wiener spaces plays the role of the heat semigroup. One of the main problem in infinite dimension is the loss of local compactness of X which does not allow to apply the Riesz Theorem on the dual of $C_c(X)$ functions. Further, it is known that the dual space of $C_b(X)$ is strictly larger to the space of signed measures on X.

The main result of [3] is a characterization of BV functions in Wiener spaces which is completely analogous to the finite dimensional situation. Namely, [3, Theorem 4.1] states that a function $u \in L(\mathrm{Log}L)^{1/2}(X, \gamma)$ is a function of bounded variation, i.e., its distributional derivative along the directions of H are finite measures if and only if one of the following conditions holds true:

(i) the functional

$$V(u) := \sup \left\{ \int_X u \,\mathrm{div}_\gamma G d\gamma : G \in \mathcal{F}C^1_c(X, H) : |G(x)|_H \le 1\ \forall x \in X \right\},$$

is finite, where $\mathcal{F}C^1_c(X, H)$ is the space of H-valued cylindrical functions with "compact support" (in the sense that its support is an infinite cylinder with compact basis).

(ii) The functional

$$L(u) := \inf \left\{ \liminf_{n \to +\infty} \|\nabla_H u_n\|_{L^1(X,\gamma)} : u_n \in \mathbb{D}^{1,1}(X, \gamma),\ u_n \xrightarrow{L^1} u \right\},$$

is finite, where $\mathbb{D}^{1,1}(X, \gamma)$ is the Sobolev space defined in [16].

(ii) The limit

$$\mathcal{I}(u) := \lim_{t\downarrow 0}\int_X |\nabla_H T_t u|_H d\gamma,$$

is finite, where T_t is the Ornstein-Uhlenbeck on the whole space X.

The aim of this paper is trying to generalize as more as possible the results of [3] when an arbitrary open domain $O \subseteq X$ is considered. We say that $f \in L(\log L)^{1/2}(O,\gamma)$ belongs to $BV(O,\gamma)$ if there exists a vector measure $\mu \in \mathscr{M}(O,H)$ such that

$$\int_O f\partial_h^*\varphi d\gamma = -\int_O \varphi d\mu_h,$$

for any h in the Cameron–Martin space Here, ∂_h^* denotes, up to the sign, the adjoint in L^2 of the partial derivative along $h \in H$, $\mu_h = [\mu, h]_H$ and $\varphi \in \mathrm{Lip}_c(O,H)$ (set of bounded Lipschitz functions with bounded support with positive distance from ∂O). This definition coincides with [3, Definition 3.1] when $O = X$, and, in Theorem 3.1 we show that $f \in L(\mathrm{Log}L)^{1/2}(O,\gamma)$ is a function of bounded variation if and only if one of the following conditions is satisfied:

(a) $V_\gamma(f,O) < +\infty$, where

$$V_\gamma(f,O) := \sup\left\{\int_O f\mathrm{div}_\gamma^F G d\gamma : F \subseteq QX^* \text{ fin. dim.},\ G \in \mathrm{Lip}_c(O,F),\ |G(x)|_F \leq 1\ \forall x \in O\right\}.$$

(b) $L_\gamma(u,O) < +\infty$, where

$$L_\gamma(f,O) := \inf\left\{\liminf_{n\to+\infty}\|\nabla_H f_n\|_{L^1(O,\gamma)} :\ f_n \in \mathbb{D}^{1,1}(X,\gamma),\ f_n \stackrel{L^1(O,\gamma)}{\longrightarrow} f\right\}.$$

We point out that the equivalence of possible definitions of BV functions has been considered in the literature also in more general metric spaces, see for instance [4]. This characterization can also be considered in metric measure spaces with the construction given in [18]. Nevertheless, in such setting when one wants to define BV functions using the functional $V_\gamma(f,O)$, one usually requires the existence of a positive measure that realizes the total variation. In our setting we do not require a priori the existence of such a measure, but we prove its existence relying on the fact that in Wiener spaces when $O = X$ this was proved in [3].

In finite dimension the definition of the space of BV functions naturally extends to general open domains $\Omega \subseteq \mathbb{R}^n$; we refer for instance to [7] where the question was addressed on the problem of the existence of the extension operator. On the other hand, in infinite dimension the situation is quite more complicated; the first issue we deal with is that in Banach spaces the distance is, in general, not locally smooth. Moreover, compact sets in Banach spaces are not enough to approximate open sets (for instance, the closed unit ball is not compact in Hilbert spaces). Therefore, it is not obvious to find a good space of test functions. However, as often happens also in finite dimension, the space of Lipschitz functions with bounded support is a good compromise, since these functions are compatible with the differential structure of X related to the Cameron- Martin space H and the distance is a Lipschitz function. This choice, unfortunately, makes useless the finite dimensional approximations by means of conditional expectations (see [5, Corollary 3.5.2]), which also in [3] are crucial to get the main results.

To show that $V_\gamma(f,O) < +\infty$ we take advantage of [3, Theorem 4.1]. Indeed, we prove that if $V_\gamma(f,O) < +\infty$, then $V(\overline{f}) < +\infty$, where $\overline{f}$ is the null extension of f on O^c. This means that $\overline{f} \in BV(X,\gamma)$ and that its distributional derivatives along the directions of H are finite measure. We conclude by proving that these distributional derivatives satisfy the integration by parts formula states above. The fact that $L_\gamma(f,O) < +\infty$ implies that $f \in BV(O,\gamma)$ follows from an argument inspired by [24].

Finally, we prove that our techniques can also be applied in more general situations. We consider the results in [9], where X is a Hilbert space, $R \in \mathcal{L}(X)$ and BV functions on X with respect to a

Fomin differentiable measure ν along the directions $R^*(X)$ are considered. The authors prove that a function $u \in L^1(X, \nu)$ belongs to $BV(X, \nu)$ if for any $z \in X$ it holds that

$$V_z(u) := \sup \left\{ \int_X u(\langle R\nabla\varphi, z\rangle - v_z\varphi) d\nu : \ \varphi \in C_b^1(X), \ \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

Our arguments can be adapted to this setting and we obtain the characterization of BV functions on $O \subseteq X$ with respect to ν by means of the variation

$$V_z(f, O) := \sup \left\{ \int_X f(\langle RD\varphi, z\rangle - \varphi v_z) d\nu : \varphi \in \mathrm{Lip}_c(O), \ \|\varphi\|_\infty \leq 1 \right\}.$$

The unique additional hypothesis that we need is that Lipschitz functions are compatible with ν, in a sense that we make explicit later.

The paper is organized as follows. In Section 2 we give definitions and preliminary results which will be useful in the sequel of the paper. We begin with infinite dimensional measure theory, and we prove some properties of vector valued measures. Then, we introduce the Wiener space (X, γ, H), where X is a separable Banach space, γ is a centred non–degenerate Gaussian measure and H is the Cameron–Martin space associated to γ. Later, we present the standard construction of Sobolev spaces in Wiener setting and some features of the Ornstein–Uhlenbeck semigroup on X. We conclude by listing the main properties of the Orlicz space $L(\mathrm{Log}L)^{1/2}(O, \gamma)$ and by defining the space of functions of bounded variation $BV(O, \gamma)$, the variation $V_\gamma(f, O)$ and $L_\gamma(f, O)$, where $O \subseteq X$ is an open domain.

Section 3 is devoted to prove the equivalent definitions of $BV(O, \gamma)$, i.e., we show that a function $f \in L(\mathrm{Log}L)^{1/2}(O, \gamma)$ belongs to $BV(O, \gamma)$ if and only if either $V_\gamma(f, O) < +\infty$ or $L_\gamma(f, O) < +\infty$.

In Section 4 we collect some important consequences of the results in Section 3. To be more precise, let T_t denote the Ornstein–Uhlenbeck semigroup on the whole space. We prove that if for $f \in L(\mathrm{Log}L)^{1/2}(O, \gamma)$ the quantity

$$\mathscr{J}(f, O) := \liminf_{t \downarrow 0} \int_O |\nabla_H T_t(\bar{f})|_H d\gamma,$$

where $\bar{f}$ is the null extension of f outside O, is finite, then $f \in BV(O, \gamma)$. We are not able to prove that this condition is also necessary and indeed this in general is not the case. The main problem is that, at the best of our knowledges, the study of the Ornstein–Uhlenbeck semigroup on open domains is much more complicated. For example, in the whole space T_t has an explicit integral representation which allows direct computations. In this direction, in [22] the authors study BV functions on X restricted to an open convex set $\Omega \subseteq X$ in terms of the Ornstein–Uhlenbeck semigroup on Ω (see [8] for a first analysis of the Ornstein–Uhlenbeck and its properties on convex domains). However, In this case the convexity of Ω plays an essential role and it is not possible to generalize the techniques in [8] for a general open domain. Further, as in [2, Proposition 3.103] and [3, Proposition 3.9] we describe the connections between the one dimensional section of BV functions and directional derivatives.

Finally, in Section 5, X is a Hilbert space and, given $R \in \mathcal{L}(X)$, we consider a probability measure ν which is Fomin differentiable along the directions of $R^*(X)$. Starting from the results in [9], we provide a characterization of BV functions on open domains $O \subseteq X$ by means of the variation of a function on O with respect to ν.

1.1 Notations

Let X be a separable Banach space. We denote by $\|\cdot\|_X$ its norm and by X^* its topological dual, i.e., the set of bounded linear functionals on X. We denote by $\langle\cdot,\cdot\rangle$ the duality between X and X^*. Given $x_1^*, \ldots, x_m^* \in X^*$, we denote by $\pi_{x_1^*,\ldots,x_m^*} : X \longrightarrow \mathbb{R}^m$ the bounded linear map $\pi_{x_1^*,\ldots,x_m^*} x := (\langle x, x_1^*\rangle, \ldots, \langle x, x_m^*\rangle)$. If $F = \mathrm{Span}\{x_1^*, \ldots, x_m^*\} \subseteq X^*$, we also write π_F instead of $\pi_{x_1^*,\ldots,x_m^*}$. The symbol $\mathcal{F}\mathrm{C}_b^k(X)$ denotes the space of k-times Fréchet differentiable cylindrical functions with bounded derivatives up to the order k, that is $u \in \mathcal{F}\mathrm{C}_b^k(X)$ if there exists $v \in C_b^k(\mathbb{R}^m)$ (k-times continuously differentiable functions with bounded derivatives) such that $u(x) = v(\pi_{x_1^*,\ldots,x_m^*} x)$ for some $m \in \mathbb{N}$ and $x_1^*, \ldots, x_m^* \in X^*$.

The symbol $C_b^1(X)$ denotes the space of bounded functions from X to $\mathbb{R}$ which are Fréchet differentiable, with bounded Frechét derivative.

In the same spirit, we say that $E \subseteq X$ is a cylindrical set if there exists $m \in \mathbb{N}$, $x_1^*, \dots, x_m^* \in X^*$ and $B(\mathbb{R}^m)$ such that $E = \pi^{-1}_{x_1^*,\dots,x_m^*}(B)$.

Given a Borel set $E \in \mathcal{B}(X)$ and an open set $O \subseteq X$, O^c is the complementary set of O, the writing $E \Subset O$ means that E is a bounded subset of O with $\mathrm{dist}(E, O^c) > 0$, where $\mathrm{dist}(E, O^c) := \inf\{\|x - y\|_E : x \in E,\ y \in O^c\}$. Further, for any open set $A \subseteq X$ and any $\eta > 0$, we define $A_{-\eta} := \{x \in A : d(x, A^c) > \eta\}$. For any $x \in X$ and $r > 0$ we denote by $B(x, r) := \{y \in X : \|y - x\|_X < r\}$. If $x = 0$ we simply write $B(r)$ instead of $B(0, r)$.

If f is a function defined on O, we denote by $\bar{f}$ its null extension.

2 Definitions and preliminary results

2.1 Infinite-dimensional measure theory

Let $\mathcal{B}(X)$ be the Borel σ-field on X, Y be a Hilbert space with inner product $[\cdot,\cdot]_Y$. Since X is separable, $\mathcal{B}(X)$ is generated by the family of the cylindrical sets (see [29, Fernique Corollary in I.1.2]). For any open set $O \subseteq X$, we denote by $\mathscr{M}(O, Y)$ the set of Borel countably additive measures on O which take values in a Hilbert space Y. We stress that $\mathscr{M}(O, Y) \subseteq \mathscr{M}(X, Y)$. Indeed, if $\mu \in \mathscr{M}(O, Y)$ we extend μ on X by introducing the measure $\widetilde{\mu}(E) = \mu(E \cap O)$ for any $E \in \mathcal{B}(X)$. For any $\mu \in \mathscr{M}(O, Y)$ and $y \in Y$ we denote by μ_y the scalar measure $[\mu, y]_Y$. When $Y = \mathbb{R}$ we simply write $\mathscr{M}(O)$ instead of $\mathscr{M}(O, Y)$. We recall that the total variation of a measure $\mu \in \mathscr{M}(O, Y)$ is a positive finire measure defined for any Borel set $B \in \mathcal{B}(O)$ by

$$|\mu|(B) := \sup\left\{\sum_{n=1}^{\infty} |\mu(B_n)|_Y :\ B = \bigcup_{n\in\mathbb{N}} B_n \text{ and the Borel sets } B_n \text{ are pairwise disjoint}\right\},$$

We remind that since we are working in a separable Banach space, any finite measure is a Radon measure.

For a given $E \in \mathcal{B}(O)$ and $\mu \in \mathscr{M}(O, Y)$, we denote by $\mu \llcorner E \in \mathscr{M}(E, Y)$ the restriction of μ on E defined by

$$\mu \llcorner E(B) = \mu(E \cap B), \qquad \forall B \in \mathcal{B}(E).$$

By using the polar decomposition $\mu = \sigma_\mu |\mu|$, where $\sigma_\mu : O \to Y$ is a $|\mu|$-measurable map which satisfies $|\sigma_\mu|_Y = 1$ for $|\mu|$-a.e. $x \in O$, by the inclusion $\mathscr{M}(O, Y) \subseteq (C_b(O, Y))^*$ we get the equalities

$$\begin{aligned}
|\mu|(O) = &\sup\left\{\int_O [\sigma_\mu, \Phi]_Y d|\mu| :\ \Phi \in C_b(O, Y),\ |\Phi(x)|_Y \le 1\ \forall x \in O\right\} \\
= &\sup\left\{\int_O [\sigma_\mu, \Phi]_Y d|\mu| :\ \Phi \in \mathcal{F}C_b^1(O, Y),\ |\Phi(x)|_Y \le 1\ \forall x \in O\right\}
\end{aligned}$$

where

$$\begin{aligned}
C_b(O, Y) := &\{u : O \to Y \text{ continuous and bounded}\}, \\
\mathcal{F}C_b^1(O, Y) := &\Big\{u : O \to Y : \\
&u(x) = \sum_{i=1}^{n} v_i(\pi_{x_1^*,\dots,x_m^*} x) y_i,\ x \in O,\ n, m \in \mathbb{N}, v_i \in C_b^1(\mathbb{R}^m),\ \ y_i \in Y\ i = 1, \dots, n\Big\}.
\end{aligned}$$

If $Y = \mathbb{R}$, we simply write $C_b(O)$ and $\mathcal{F}C_b^1(O)$.

We denote by $\mathrm{Lip}_c(O, Y)$ the set of bounded Lipschitz Y-valued functions $G : X \to Y$ such that $\mathrm{supp}(G) \Subset O$. If $Y = \mathbb{R}$ we simply write $\mathrm{Lip}_c(O)$. It is clear that if $O_1 \subseteq O_2$ then $Lip_c(O_1, Y) \subseteq Lip_c(O_2, Y) \subseteq Lip_c(X, Y)$. Finally, $\mathrm{Lip}_b(X, Y)$ denotes the space of Y-valued bounded Lipschitz continuous functions on X.

The space $C_c^1(O)$ and $C_c^1(O, Y)$ are defined in a similar way.

Lemma 2.1 *Let $O \subseteq X$ be an open set, and let $\mu \in \mathscr{M}(O, Y)$. Then, for any open set $A \subseteq O$ we have*

$$|\mu|(A) = \sup\left\{\int_A [\sigma, G]_Y d|\mu| : G \in \mathrm{Lip}_c(A, Y),\ |G(x)|_Y \leq 1\ \forall x \in O\right\}, \tag{2.1}$$

where $\mu = \sigma|\mu|$ is the polar decomposition of μ. In particular, for any $y \in Y$

$$|\mu_y|(A) = \sup\left\{\int_A G d\mu_y : G \in \mathrm{Lip}_c(A),\ |G(x)| \leq 1\ \forall x \in A\right\}. \tag{2.2}$$

Proof. We limit ourselves to show (2.1), since from it we easily deduce (2.2). Clearly, for any $G \in \mathrm{Lip}_c(A, Y)$ with $|G(x)|_Y \leq 1$ for any $x \in X$, we have $[\sigma, G]_Y \leq 1$, hence

$$\int_A [\sigma, G]_Y \ d|\mu| \leq |\mu|(A).$$

We have to prove the converse inequality. Let $\varepsilon > 0$. Since $|\mu|$ is a Radon measure, there exists a compact set $K \subseteq A$ such that $|\mu|(A \setminus K) < \varepsilon$. By the properties of vector measures, there exists $\varphi_\varepsilon \in C_b(A, Y)$ with $\|\varphi_\varepsilon\|_{L^\infty(O,\gamma)} \leq 1$ and such that

$$|\mu|(A) \leq \int_A [\sigma, \varphi_\varepsilon]_Y d|\mu| + \varepsilon \leq \int_K [\sigma, \varphi_\varepsilon]_Y d|\mu| + 2\varepsilon.$$

Let $\{y_n : n \in \mathbb{N}\}$ be an orthonormal basis of Y. For $n \in \mathbb{N}$ we consider $\sigma_n := \sum_{i=1}^n [\sigma, y_i]_Y y_i$. From the dominated convergence theorem there exists $n_\varepsilon \in \mathbb{N}$ such that $\sigma_{n_\varepsilon} =: \sigma_\varepsilon$ is a Borel function with range contained in $\mathrm{span}\{y_1, \ldots, y_{n_\varepsilon}\}$, $|\sigma_\varepsilon|_Y \leq 1$ everywhere and

$$\|\sigma - \sigma_\varepsilon\|_{L^1(K,|\mu|)} \leq \varepsilon, \quad |\mu|(A) \leq \int_K [\sigma_\varepsilon, \varphi_\varepsilon]_Y d|\mu| + 3\varepsilon.$$

Since σ_ε has finite dimensional range, only a finite number of components of φ_ε is involved in the above integral. From the Stone-Weierstrass Theorem, there exists $g_\varepsilon \in \mathcal{F}C_b^1(X, Y)$ with finite dimensional range such that $\|g_\varepsilon - \varphi_\varepsilon\|_{L^\infty(K)} < \varepsilon$. To conclude, let $\delta := d(K, A^c) > 0$ and let us consider a Lipschitz function ψ such that $\psi \equiv 1$ in K and $\psi \equiv 0$ in $(\cup_{x\in K} B(x, \delta/2))^c$. Hence, setting

$$F(h) := \begin{cases} h, & |h|_Y \leq 1, \\ h/|h|_Y, & |h|_Y \geq 1, \end{cases}$$

for any $h \in Y$, the function $G_\varepsilon := \psi \cdot (F \circ g_\varepsilon)$ belongs to $\mathrm{Lip}_c(A, Y)$, $|G_\varepsilon(x)|_Y \leq 1$ for any $x \in Y$ and $\|G_\varepsilon - \varphi_\varepsilon\|_{L^\infty(K)} \leq 2\varepsilon$, from which it follows that

$$\begin{aligned} |\mu|(A) &\leq \int_K [\sigma_\varepsilon, G_\varepsilon]_Y d|\mu| + 5\varepsilon \leq \int_A [\sigma, G_\varepsilon]_Y d|\mu| + 6\varepsilon \\ &\leq \sup\left\{\int_A [\sigma, G]_Y d|\mu| : G \in \mathrm{Lip}_c(A, Y),\ |G(x)|_Y \leq 1\ \forall x \in X\right\} + 6\varepsilon. \end{aligned}$$

The arbitrariness of ε gives the thesis. QED

From the proof of Lemma 2.1 we immediately deduce the following result.

Corollary 2.2 *Let $\mu \in \mathscr{M}(O, H)$. Then, for any open set $A \subseteq O$ we have*

$$|\mu|(A) = \sup\left\{\int_A [\sigma, G]_H d|\mu| : F \subseteq QX^* \textit{ fin. dim.},\ G \in \mathrm{Lip}_c(A, F),\ |G(x)|_F \leq 1\ \forall x \in A\right\}, \tag{2.3}$$

where $\mu = \sigma|\mu|$ is the polar decomposition of μ.

Remark 2.3 We point out that the space $C_c^1(A, Y)$ is not in general a good set of test functions. Indeed, in the previous proof we have used Lipschitz cut-off and the existence of C^1 cut-off functions is strictly related to the separability of X^* (see for instance [12, Theorem 5.3]).

Remark 2.4 Let μ be a signed Radon measure on X and let $O \subseteq X$ be an open set. By the additivity of μ, and since the sets ∂O_{-t} are pairwise disjoint, there exist at most countably many $t \in (0, 1)$ such that $|\mu|(\partial O_{-t}) \neq 0$.

2.2 The abstract Wiener space

We consider a nondegenerate centred Gaussian measure γ on X, i.e., γ is a probability measure such that, for any $x^* \in X^*$ the image measure $\gamma \circ (x^*)^{-1}$ is a centred Gaussian measure on $\mathbb{R}$ and its Fourier transform satisfies

$$\widehat{\gamma}(x^*) := \int_X e^{i\langle x, x^* \rangle}\gamma(dx) = \exp\left(-\frac{1}{2}\langle Qx^*, x^*\rangle\right),$$

for some nonnegative and symmetric operator $Q \in \mathcal{L}(X^*, X)$, said the covariance operator. The nondegeneracy hypothesis on γ means that Q is a positive definite operator, that is, $\langle Qx^*, x^*\rangle > 0$ for any $x^* \neq 0$.

Moreover, Q is uniquely determined by

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle\langle x, y^*\rangle\gamma(dx).$$

The boundedness of Q follows from Fernique's Theorem (see [5, Theorem 2.8.5]), which states that there exists $\alpha > 0$ such that

$$\int_X e^{\alpha\|x\|_X^2}\gamma(dx) < +\infty.$$

Further, it is easy to prove that the function $x \mapsto \langle x, x^*\rangle$ belongs to $L^p(X, \gamma)$, for any $p \in [1, +\infty)$. Let us denote by $R^* : X^* \longrightarrow L^2(X, \gamma)$ the embedding of X^* in $L^2(X, \gamma)$, and by $\mathscr{H}$ the closure of R^*X^* in $L^2(X, \gamma)$. $\mathscr{H}$ is called the *reproducing kernel* of γ and clearly R^*X^* is dense in it. By putting together Fernique's Theorem and [5, Theorem 2.10.9] for any $\widehat{h} \in \mathscr{H}$ there exists an $\alpha > 0$ such that

$$\int_X e^{\alpha(\widehat{h}(x))^2}\gamma(dx) < +\infty. \tag{2.4}$$

It is also possible to prove that $Q = RR^*$, where $R : \mathscr{H} \longrightarrow X$ is the operator defined by the Bochner integral

$$R\widehat{h} := \int_X \widehat{h}(x)x\gamma(dx), \quad \widehat{h} \in \mathscr{H},$$

and that R is a injective and compact operator. The space $H := R\mathscr{H} \subseteq X$ is called the Cameron–Martin space, and it plays a crucial role in infinite dimensional analysis. H enjoys nice properties: indeed, H is dense subspace of X and it is a Hilbert space endowed with the scalar product $[h_1, h_2]_H := \langle \widehat{h}_1, \widehat{h}_2\rangle_{L^2(X,\gamma)}$, where $\widehat{h}_1, \widehat{h}_2 \in \mathscr{H}$ and $h_i := R\widehat{h}_i$, $i = 1, 2$. In particular, if $x^*, y^* \in X^*$, then $[Qx^*, Qy^*]_H = \langle Qx^*, y^*\rangle$. We denote by $|\cdot|_H$ the norm in H induced by $[\cdot,\cdot]_H$. When no confusion is possible, we simply write $|\cdot|$ instead of $|\cdot|_H$. Moreover, the embedding $H \hookrightarrow X$ is compact and $\gamma(H) = 0$ if and only if X is infinite dimensional. The importance of H follows from the Cameron-Martin Theorem (see [5, Theorem 2.4.5]). For any $h \in X$ let us consider the shifted measure $\gamma^h := \gamma(\cdot - h)$. Then, γ^h is absolutely continuous with respect to γ if and only if $h \in H$. In this case, if we write $h = R\widehat{h}$, we have

$$\gamma^h(dx) := \exp\left(\widehat{h}(x) - \frac{1}{2}|h|_H^2\right)\gamma(dx),$$

for any $h \in H$.

In the sequel, when $h \in H$, we denote by $\widehat{h} \in \mathscr{H}$ the corresponding element in the reproducing kernel such that $R\widehat{h} = h$. The non degeneracy of the measure implies that for $h \in QX^*$ there exists a unique element $x^* \in X^*$ such that $h = Qx^*$.

For $h \in QX^*$, $h = Qx^*$, we denote by $X_h^\perp := \mathrm{Ker}(\pi_{x^*})$, by γ_h the image of γ under π_{x^*} (i.e., $\gamma_h = \gamma \circ \pi_{x^*}^{-1}$) and by $\gamma_h^\perp$ the Gaussian measure on X concentrated on $X_h^\perp$ which is the image of γ under $\mathrm{Id}_X - \pi_{x^*}$. With this construction we get the decomposition of $\gamma = \gamma_h \otimes \gamma_h^\perp$.

Finally, there exists an orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of H such that $h_n = Qx_n^*$ with $x_n^* \in X^*$ for any $n \in \mathbb{N}$ (see [5, Corollary 3.2.8]). In the following sections, we denote by π_m the projection $\pi_{x_1^*,\ldots,x_m^*}$.

2.3 Sobolev spaces and Ornstein-Uhlenbeck semigroup

Due to the Cameron-Martin Theorem, the derivatives along the directions of H will be of crucial importance. Therefore, for any $f \in \mathcal{F}C_b^1(X)$ and any $h \in H$ we define

$$\partial_h f(x) := \lim_{t\to 0} \frac{f(x+th)-f(x)}{t},$$

and

$$\partial_h^* f(x) := \partial_h f(x) - f(x)\widehat{h}(x).$$

For any $f \in \mathcal{F}C_b^1(X)$ with $f(x) = \varphi(\pi_{x_1^*,\ldots,x_m^*}x)$, $\varphi \in C_b^k(\mathbb{R}^m)$ and $x_1^*, \ldots, x_m^* \in X^*$, we set

$$\nabla_H f(x) := \sum_{i=1}^m \partial_i \varphi(\pi_{x_1^*,\ldots,x_m^*}x) Q x_i^*.$$

Clearly, it follows that $\partial_h f(x) = [\nabla_H f(x), h]_H$ for any $h \in H$. For a function $\Phi \in \mathcal{F}C_b^1(X, H)$, if we write

$$\Phi(x) = \sum_{i=1}^m \varphi_i(x) k_i$$

with $m \in \mathbb{N}$, $k_1, \ldots, k_m \in H$ and $\varphi_1, \ldots, \varphi_m \in \mathcal{F}C_b^1(X)$, then the γ-divergence Φ is defined by

$$\mathrm{div}_\gamma \Phi(x) := \sum_{i=1}^m \partial_{k_i}^* \varphi_i(x).$$

The operators ∂_h^* and div_γ are, up to the sign, the adjoint operators of ∂_h and ∇_H in L^2, respectively, namely,

$$\int_X \partial_h f g d\gamma = -\int_X f \partial_h^* g d\gamma, \qquad \int_X [\nabla_H f, \Phi]_H d\gamma = -\int_X f \mathrm{div}_\gamma \Phi d\gamma, \tag{2.5}$$

for any $f, g \in \mathcal{F}C_b^1(X)$ and any $\Phi \in \mathcal{F}C_b^1(X, H)$. Integration by parts formulae (2.5) imply that $\nabla_H : \mathcal{F}C_b^1(X) \to L^p(X, \gamma; H)$ is a closable operator for any $p \in [1, +\infty)$ (see [5, Section 5.2]) and we still denote by ∇_H its closure. In the next definition we follow the notations of [16].

Definition 2.5 *For $p \in [1, +\infty)$ we define the* Sobolev space $\mathbb{D}^{1,p}(X, \gamma)$ *as the domain of the closure of ∇_H in $L^p(X, \gamma; H)$. We denote the closure as ∇_H and $\mathbb{D}^{1,p}(X, \gamma)$ is a Banach space endowed with the norm*

$$\|f\|_{\mathbb{D}^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \|\nabla_H f\|_{L^p(X,\gamma;H)}. \tag{2.6}$$

Notice that the same space is denoted by $W^{p,1}(X, \gamma)$ in [5].

Remark 2.6 By approximation it is possible to prove that the first equality in (2.5) holds true for any $f \in \mathbb{D}^{1,p}(X, \gamma)$, any $g \in \mathbb{D}^{1,q}(X, \gamma)$, with $1 < p < +\infty$ and $q = p'$ being its conjugate exponent. Further, the second equality in (2.5) holds true for any $f \in \mathbb{D}^{1,p}(X, \gamma)$ and any $\Phi \in \mathrm{Lip}_b(X, H)$, with $1 \le p < +\infty$ and $\mathrm{Lip}_b(X, H)$ has been introduced in Subsection 2.1 (see [5, Proposition 5.8.8]).

Let us introduce the Ornstein-Uhlenbeck semigroup $(T_t)_{t\ge 0}$ as follows: for any $f \in L^1(X, \gamma)$, we set

$$T_t f(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy), \quad t \ge 0. \tag{2.7}$$

Let us recall that $(T_t)_{t\ge 0}$ is a strongly continuous contraction semigroup on $L^p(X, \gamma)$, for any $p \in [1, +\infty]$ (see [28, Proposition 2.4]). Moreover, if $f \in L^p(X, \gamma)$ with $p > 1$, then $T_t f \in \mathbb{D}^{k,q}(X, \gamma)$ for any $k \in \mathbb{N}$ and any $q > 1$ (see [5, Proposition 5.4.8]). Further, from the definition of T_t and of ∇_H, if $f \in \mathbb{D}^{1,1}(X, \gamma)$ then $T_t f \in \mathbb{D}^{1,1}(X, \gamma)$ and

$$\nabla_H T_t f = e^{-t} T_t \nabla_H f,$$

where the above equality reads componentwise.

For every $m \in \mathbb{N}$ and $f \in L^1(X,\gamma)$ we introduce the canonical cylindrical approximation $\mathbb{E}_m f$ of f as the conditional expectation relative to the σ-algebra generated by $\{\langle x, x_1^*\rangle, \dots, \langle x, x_m^*\rangle\}$. [5, Corollary 3.5.2] show that

$$\mathbb{E}_m f(x) := \int_X f(\pi_m x + (\mathrm{Id}_X - \pi_m)y)\gamma(dy).$$

Definition 2.7 *Let $h \in H$, let $O \subseteq X$ be an open set and let $f : O \to \mathbb{R}$ be a γ-measurable function. For every $y \in X$ we set*

$$O_y := \{t \in \mathbb{R} | y + th \in O\}$$

and $f_y : O_y \to \mathbb{R}$ defined as

$$f_y(t) := f(y + th).$$

We denote by $D\mathscr{E}_h^O$ the set of $\gamma-$measurable functions f such that for γ-a.e. $y \in X$, the function f_y on O_y has representative $\widetilde{f}_y$ (i.e. $f_y(t) = \widetilde{f}_y(t)$ for a.e. $t \in \mathbb{R}$ with respect to the Lebesgue measure) which is locally absolutely continuous. It is clear that for γ-a.e. $y \in X$ and for a.e. $t \in O_y$, $f_y'(t)$ is well defined and for such a t we have

$$f_y'(t) = f_{y+rh}'(t - r), \quad \forall r > 0. \tag{2.8}$$

If for such a y and t we put

$$\partial_h f(y + th) := f_y'(t),$$

$\partial_h f$ is well defined γ-a.e. and, by (2.8), it does not depend on t.

From the definition of $D\mathscr{E}_h^O$, it follows that any $f \in D\mathscr{E}_h^O$ has a representative $\widetilde{f}$ such that for γ-a.e. $y \in X$ the function $\widetilde{f}_y$ is locally absolutely continuous.

We can now give the definition of Sobolev spaces on arbitrary open sets; this approach was used in the case $p = 2$ and on the whole space X by [19] and generalized for any p and on domains in [7]

Definition 2.8 *Given $p \in [1, +\infty]$, we say that $f \in W^{1,p}(O,\gamma)$ if $f \in L^p(O,\gamma)$, $f \in D\mathscr{E}_h^O$ for all $h \in H$, and there exists $\nabla_H f \in L^p(O,\gamma;H)$ such that $\partial_h f = [\nabla_H f, h]_H$.*

It is a standard argument to prove that $W^{1,p}(O,\gamma)$ is a Banach space with norm given by

$$\|f\|_{W^{1,p}(O,\gamma)} = \|f\|_{L^p(O,\gamma)} + \|\nabla_H f\|_{L^p(O,\gamma;H)}.$$

Remark 2.9 Let $f \in W^{1,p}(X,\gamma)$. By definition it follows that $f_{|O} \in W^{1,p}(O,\gamma)$.

The proof of the following result can be deduced by the result [5, Proposition 5.4.6]; we repeat the proof for reader's convenience.

Lemma 2.10 *$W^{1,p}(X,\gamma) = \mathbb{D}^{1,p}(X,\gamma)$ for any $p \in [1,\infty)$.*

Proof. If $f \in \mathcal{F}C_b^1(X)$, then f is an element of $W^{1,p}(X,\gamma)$; hence, by the definition of $\mathbb{D}^{1,p}(X,\gamma)$ and the fact that $W^{1,p}(X,\gamma)$ is complete, we have $\mathbb{D}^{1,p}(X,\gamma) \subseteq W^{1,p}(X,\gamma)$.
To prove the converse inclusion, we consider $f \in \mathbb{D}^{1,p}(X,\gamma)$ and we build a sequence in $\mathcal{F}C_b^1(X)$ which converges to f. For $n \in \mathbb{N}$ we define $f_n(x) = \mathbb{E}_n f(x)$. We consider $\gamma_n = \gamma \circ \pi_n^{-1}$, Borel measure on $\mathbb{R}^n$: clearly there exists $g_n \in W^{1,p}(\mathbb{R}^n,\gamma_n)$ such that $f_n(x) = g_n(\widehat{h_1}(x), \dots, \widehat{h_n}(x))$ with $\nabla g = (\mathbb{E}_n(\partial_{h_1} f), \dots, \mathbb{E}_n(\partial_{h_n} f))$. Each g_n can be approximated by a sequence $g_{n,m} \in C_b^1(\mathbb{R}^n)$. So we can find a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{F}C_b^1(X)$ where $f_k(x) = g_{n_k,m_k}(\widehat{h_1}(x), \dots, \widehat{h_{n_k}}(x))$, and f_k converges to f in $\mathbb{D}^{1,p}(X,\gamma)$. QED

We close this section with the following remark.

Remark 2.11 If $f \in W^{1,1}(O,\gamma)$ and $g \in \mathrm{Lip}_c(O)$, then the integration by parts formula holds

$$\int_O f\partial_h^* g d\gamma = -\int_O \partial_h f g d\gamma, \qquad \forall h \in H.$$

2.4 The Orlicz spaces

We use the concepts of Orlicz space (see [27]); in particular we recall two particular examples of Orlicz spaces, $L(\log L)^{1/2}(O,\gamma)$ and $L^\Psi(O,\gamma)$. Let $O\subseteq X$ be an open set. We introduce the spaces $L(\log L)^{1/2}(O,\gamma)$ and $L^\Psi(O,\gamma)$ as follows: let

$$A_{1/2}(x):=\int_0^x(\ln(1+t))^{1/2}dt,\quad x\geq 0,$$

and let Ψ be its complementary function, namely,

$$\Psi(y):=\int_0^y(A'_{1/2}(t))^{-1}dt=\int_0^y\left(\exp\left(t^2\right)-1\right)dt,\quad y\geq 0.$$

We define the spaces

$$L(\log L)^{1/2}(O,\gamma):=\{f\in L^1(O,\gamma):A_{1/2}(|f|)\in L^1(O,\gamma)\},$$
$$L^\Psi(O,\gamma):=\{g\in L^1(O,\gamma):\text{there exists } c>0 \text{ such that } \Psi(c|g|)\in L^1(O,\gamma)\}.$$

We stress that, with the notations of [27], $L(\log L)^{1/2}(O,\gamma)=\widetilde{\mathcal{L}}_\Phi(\gamma)$. Since the function $\Phi(t):=(\log(1+t))^{1/2}\in\Delta_2$ (see [27, Definition 1, Section 2.3]), from [27, Theorem 2(ii), Section 3.1] it follows that $L(\log L)^{1/2}(O,\gamma)$ is a vector space. Following the notations of [27, Section 3.2] we consider the Luxemburg norm

$$N_\Phi(f)=\|f\|_{L(\log L)^{1/2}(O,\gamma)}:=\inf\left\{t>0:\int_O A_{1/2}(|f|/t)d\gamma\leq 1\right\}$$

and the space

$$\widetilde{L}_\Phi(\gamma):=(L(\log L)^{1/2}(O,\gamma),\|\cdot\|_{L(\log L)^{1/2}(O,\gamma)})=(\widetilde{\mathcal{L}}_\Phi(\gamma),N_\Phi).$$

Further, [27, Corollary 4, Section 3.4] gives $\widetilde{L}_\Phi(\gamma)$ is equivalent to the spaces M_Φ and $L_\Phi(\gamma)$, defined in [27, Definition 2, Section 3.4] and [27, Definition 5, Section 3.1]. Hence, [27, Theorem 10, Section 3.3] implies that $L(\log L)^{1/2}(O,\gamma)$ is a Banach space. Moreover, similar arguments give that L^Ψ is a Banach space endowed with the norm

$$\|g\|_{L^\Psi(O,\gamma)}:=\inf\left\{t>0:\int_O\Psi(|g|/t)d\gamma\leq 1\right\}.$$

Remark 2.12 We notice that if two measurable functions $g_1,g_2:O\to\mathbb{R}$ have the same image measure, then

$$\|g_1\|_{L^\Psi(O,\gamma)}=\|g_2\|_{L^\Psi(O,\gamma)}.$$

This simply follows by the identity

$$\int_O\Psi(|g_1|/t)d\gamma=\int_{\mathbb{R}}\Psi(|s|/t)(\gamma\circ g_1^{-1})(ds)=\int_O\Psi(|g_2|/t)d\gamma.$$

We conclude this part with two important results on Orlicz spaces. The former, which is [27, Proposition 1, Section 3.3] and the Remark below therein, is a sort of Hölder inequality for complementary Orlicz spaces. The latter is a dominated convergence theorem in Orlicz spaces, and it is [27, Theorem 14, Section 3.4] rewritten in our situation and using our notations.

Proposition 2.13 *If $f\in L(\log L)^{1/2}(O,\gamma)$ and $g\in L^\Psi(O,\gamma)$ then $fg\in L^1(O,\gamma)$ and*

$$\|fg\|_{L^1(O,\gamma)}\leq 2\|f\|_{L(\log L)^{1/2}(O,\gamma)}\|g\|_{L^\Psi(O,\gamma)}.$$

Theorem 2.14 *Let $\varphi\in L(\log L)^{1/2}(O,\gamma)$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $\varphi_n(x)\to\widetilde{\varphi}(x)$ and $|\varphi_n(x)|\leq|\varphi(x)|$ for γ-a.e. $x\in O$. Then, $\varphi_n,\widetilde{\varphi}\in L(\mathrm{Log}L)^{1/2}(O,\gamma)$ and*

$$\|\varphi_n-\widetilde{\varphi}\|_{L(\log L)^{1/2}(O,\gamma)}\to 0,\quad n\to+\infty.$$

Finally, thanks to (2.4), it is easy to see that the function $x\mapsto|\ell(x)|\in L^\Psi(O,\gamma)$, and therefore the integral $\int_O f\ell d\gamma$ is well defined for any $f\in L(\log L)^{1/2}(O,\gamma)$ and $\ell\in\mathscr{H}$.

2.5 BV functions on open domains in infinite dimension

We begin this subsection by providing the definition of function of bounded variation on an open domain of X.

Definition 2.15 *Let $O \subseteq X$ be an open set. We say that $f \in L(\log L)^{1/2}(O,\gamma)$ is a function of bounded variation in O, and we write $f \in BV(O,\gamma)$, if there exists an H-valued measure $\mu_f \in \mathscr{M}(O,H)$ such that*

$$\int_O \partial_h^* G f d\gamma = -\int_O G d[\mu_f, h]_H, \tag{2.9}$$

for any $G \in \mathrm{Lip}_c(O)$ and any $h \in H$. We write $D_\gamma^O f := \mu_f$, we call it weak gradient of f and we set $D_\gamma^{O,h} f := [\mu_f, h]_H$. As a consequence of Corollary 2.2, if the weak gradient of f there exists then it is unique. Finally, if $E \in \mathcal{B}(O)$, $u = \chi_E$ and $u \in BV(O,\gamma)$, then we say that E has finite perimeter in O.

Remark 2.16 We point out that the requirement on the validity of (2.9) can be equivalently required only for $h \in QX^*$. Indeed if (2.9) holds for any $h \in QX^*$, we can pass to any $h \in H$ by considering a sequence $(h_j)_{j\in\mathbb{N}} \subseteq QX^*$ converging to h in H. Then if $\sigma_j = |h_j - h|_H$ and $k \in H$ is such that $|k|_H = 1$, the functions $\widehat{h}_j - \widehat{h}$ and $\sigma_j \widehat{k}$ have the same image measure (see e.g [5, Lemma 2.2.8]). By (2.4) $\widehat{h}_j - \widehat{h} \in L^\Psi(O,\gamma)$ and

$$\|\widehat{h}_j - \widehat{h}\|_{L^\Psi(O,\gamma)} = \sigma_j \|\widehat{k}\|_{L^\Psi(O,\gamma)}$$

so $\widehat{h}_j \to \widehat{h}$ in $L^\Psi(O,\gamma)$. Thanks to Proposition 2.13 we can then pass to the limit in (2.9).

In the next Lemma we state that the Definition 2.15 is equivalent to [3, Def. 3.1].

Lemma 2.17 *If $O = X$, then in Definition 2.15 the space $\mathrm{Lip}_c(X)$ can be replaced by $\mathcal{F}C_b^1(X)$.*

Remark 2.18 We stress that there is no inclusion between $\mathrm{Lip}_c(X)$ and $\mathcal{F}C_b^1(X)$. Indeed, functions in $\mathcal{F}C_b^1(X)$ should be smoother than those in $\mathrm{Lip}_c(X)$. Further, we have no condition on the support of functions in $\mathcal{F}C_b^1(X)$, and so $\mathcal{F}C_b^1(X) \not\subseteq \mathrm{Lip}_c(X)$.

Proof. [Proof of Lemma 2.17] Let $f \in L(\log L)^{1/2}(X,\gamma)$ satisfy (2.9) for every $\widetilde{G} \in \mathrm{Lip}_c(X)$, and let $G \in \mathcal{F}C_b^1(X)$. Clearly, G can be approximated by a sequence $(G_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{Lip}_c(X)$ such that $G_n \to G$ and $\nabla_H G_n \to \nabla_H G$ γ-a.e. in X and G_n, $\nabla_H G_n$ are uniformly bounded: it suffices to consider a sequence (θ_n) of Lipschitz functions such that $\theta_{n|B(n)} \equiv 1$ and $\theta_{n|X\setminus B(n+1)} \equiv 0$ for every $n \in \mathbb{N}$, and define $G_n = \theta_n G$. For any $h \in QX^*$ we have

$$\int_X f \partial_h^* G_n \, d\gamma = -\int_X G_n \, d[\mu_f, h]_H, \tag{2.10}$$

and by the dominated convergence theorem the right-hand side of (2.10) converges to

$$-\int_X G \, d[\mu_f, h]_H, \qquad \text{as } n \to +\infty.$$

From the definition of ∂_h^*, we can split the left-hand side of (2.10) as

$$\int_X f \partial_h^* G_n d\gamma = \int_X f \partial_h G_n d\gamma - \int_X f G_n \widehat{h} d\gamma =: I_1^n + I_2^n.$$

Again, by the dominated convergence theorem we infer that $I_1^n \to \int_X f \partial_h G d\gamma$ as $n \to +\infty$. As far as I_2^n is concerned, we can apply Theorem 2.14, with $\varphi := f\|G\|_\infty$, $\widetilde{\varphi} := fG$ and $\varphi_n := fG_n$ for any $n \in \mathbb{N}$, so, $f(G_n - G)$ goes to 0 in $L(\log L)^{1/2}(X,\gamma)$. Hence, by Proposition 2.13 and by $\widehat{h} \in L^\Psi(X,\gamma)$ we have $I_2^n \to \int_X fG\widehat{h} d\gamma$, and therefore (2.9) holds true for any $G \in \mathcal{F}C_b^1(X)$.

To prove the converse implication, let $f \in L(\log L)^{1/2}(X,\gamma)$, and assume that (2.9) is satisfied for every $\widetilde{G} \in \mathcal{F}C_b^1(X)$. We claim that every $G \in \mathrm{Lip}_c(X)$ can be approximated by a sequence G_n of functions in $\mathcal{F}C_b^1(X)$ such that $G_n \to G$ and $\nabla_H G_n \to \nabla_H G$ γ-a.e. in X and G_n, $\nabla_H G_n$ are uniformly bounded. If the claim is true, we can argue as above to conclude. Hence, it remains to prove the claim.

Let $G \in \mathrm{Lip}_c(X)$, and for any $n \in \mathbb{N}$ let $\widetilde{G}_n = \mathbb{E}_n G$. By [5, Proposition 5.4.5], $\widetilde{G}_n$ converges to G in $\mathbb{D}^{1,2}(X,\gamma)$ and $\widetilde{G}_n$ and $\nabla_H \widetilde{G}_n$ are uniformly bounded. Moreover, $\widetilde{G}_n$ is a cylindrical function and therefore there exists $v_n \in \mathrm{Lip}_b(\mathbb{R}^n)$ such that $\widetilde{G}_n = v_n \circ \pi_n$, by identifying $H_n := \mathrm{span}\{h_1, \ldots, h_n\}$ with $\mathbb{R}^n$. For every $n \in \mathbb{N}$, v_n can be approximated by a sequence $(v_{m,n})_{m \in \mathbb{N}}$ of convolutions of v_n with a sequence of standard mollifiers ϕ_m in $\mathbb{R}^n$, and we define $G_{m,n} := v_{m,n} \circ \pi_n$. Easy computations reveal that $G_{m,n} \to G_n$ in $\mathbb{D}^{1,2}(X,\gamma)$ (see e.g. [20, Lemma 3.2]). From the definition, $G_{m,n} \in \mathcal{F}\mathrm{C}^1_b(X)$ and $G_{m,n}$ and $\nabla_H G_{m,n}$ are uniformly bounded with respect to $m,n \in \mathbb{N}$. Now, with a diagonal argument, we find a sequence $G_n \in \mathcal{F}\mathrm{C}^1_b(X)$ which converges to G in $\mathbb{D}^{1,2}(X,\gamma)$ and such that G_n, $\nabla_H G_n$ are uniformly bounded. In particular, up to a subsequence, both G_n and $\nabla_H G_n$ converge to G and $\nabla_H G$ γ-a.e., respectively. QED

Remark 2.19 If $O_1 \subseteq O_2$ are open sets and $f \in BV(O_2,\gamma)$ then $f_{|O_1} \in BV(O_1,\gamma)$. If f_1, $f_2 \in BV(X,\gamma)$ and $f_{1|O} = f_{2|O}$ then clearly

$$D^O_\gamma(f_{1|O}) = D^O_\gamma(f_{2|O}) = D^X_\gamma f_1 \llcorner O = D^X_\gamma f_2 \llcorner O.$$

We stress that Lemma 2.17 implies that our definition of $BV(X,\gamma)$ is coherent with that used in literature.

Remark 2.20 As in [3, Section 3], we may slightly modify the requirement in Definition 2.15. Let $\{h_j : j \in \mathbb{N}\} \subseteq QX^*$ be an orthonormal basis of H and let $\partial^*_j := \partial^*_{h_j}$ for any $j \in \mathbb{N}$. We say that $f \in BV(O,\gamma)$ if there exists a family $\{\mu_j\}_{j\in\mathbb{N}}$ of real valued measures such that $\sigma := \sup_{j\in\mathbb{N}} |(\mu_1,\ldots,\mu_j)|(O) < +\infty$ and

$$\int_O u \partial^*_j G \, d\gamma = -\int_O G \, d\mu_j, \quad G \in \mathrm{Lip}_c(O), \quad j \in \mathbb{N}.$$

The measure $\mu := \sum_{j\in\mathbb{N}} \mu_j h_j$ is well defined and belongs to $\mathscr{M}(O,H)$. It is enough to consider the density f_j of μ_j with respect to $|\mu|$ for any $j \in \mathbb{N}$. Hence, $\sum_{j\in\mathbb{N}} f_j^2 \le 1$ for σ-a.e. and $\mu = \sum_{j\in\mathbb{N}} f_j h_j \sigma$.

Clearly, the restriction of a Sobolev function in X to O is a function of bounded variation.

Lemma 2.21 *Let O be an open subset of X. If $f \in \mathbb{D}^{1,1}(X,\gamma)$ then $f_{|O} \in BV(O,\gamma)$ and $D^O_\gamma f = \nabla_H f \gamma \llcorner O$.*

Proof. From [17, Proposition 3.2] it is well known that $\mathbb{D}^{1,1}(X,\gamma)$ is continuously embedded into $L(\log L)^{1/2}(X,\gamma)$, and so $f \in L(\log L)^{1/2}(O,\gamma)$. Since $\mathrm{Lip}_c(O) \subseteq \mathbb{D}^{1,q}(X,\gamma)$ for any $q \in [1,+\infty]$, applying Lemma 2.6 we have

$$\int_O f \partial^*_h G d\gamma = \int_X f \partial^*_h G d\gamma = -\int_X \partial_h f G d\gamma = -\int_O \partial_h f G d\gamma,$$

for any $G \in \mathrm{Lip}_c(O)$ and any $h \in H$. In particular, for any $h \in H$ we have

$$[D^O_\gamma f, h]_H = \partial_h f \gamma \llcorner O = [\nabla_H f, h]_H \gamma \llcorner O,$$

and equality $D^O_\gamma f = \nabla_H f \gamma \llcorner O$ follows. QED

If we consider the Ornstein-Uhlenbeck semigroup defined in (2.7), then

$$\int_O (T_t \bar{f})(\mathrm{div}_\gamma G) d\gamma = e^{-t} \int_O f \, (\mathrm{div}_\gamma (T_t G)) \, d\gamma,$$

for any $f \in L(\log L)^{1/2}(O,\gamma)$ and any $G \in \mathrm{Lip}_c(O,H)$ (where $T_t G$ is calculated componentwise). Indeed, it has been proved in [3, Section 2.4] that

$$\int_X (T_t u)(\mathrm{div}_\gamma \Phi) d\gamma = e^{-t} \int_X u \, (\mathrm{div}_\gamma (T_t \Phi)) \, d\gamma,$$

for any $u \in L(\log L)^{1/2}(X,\gamma)$ and $\Phi \in \mathcal{F}C_b^1(X,H)$ ($T_t\Phi$ again calculated componentwise), and by an approximation argument it also holds for any $\Phi \in \mathrm{Lip}_b(X,H)$. Therefore, if we consider $f \in L(\log L)^{1/2}(O,\gamma)$ we have

$$\int_O (T_t\bar{f})(\mathrm{div}_\gamma G)d\gamma = \int_X (T_t\bar{f})(\mathrm{div}_\gamma G)d\gamma = e^{-t}\int_X \bar{f}\mathrm{div}_\gamma(T_tG)d\gamma = e^{-t}\int_O f\mathrm{div}_\gamma(T_tG)d\gamma,$$

for every $G \in \mathrm{Lip}_c(O,H)$.

The variation of integrable functions plays a crucial role in the setting of BV functions. Indeed, both in finite dimension (see [2, Definition 3.4 & Proposition 3.6]) and in Wiener spaces, when the whole space is considered (see [3, Definition 3.8 & Theorem 4.1]), it is possible to characterize functions of bounded variation by means of their variation. We introduce this concept also in our context.

Definition 2.22 *For any open set $O \subseteq X$ and any $f \in L(\log L)^{1/2}(O,\gamma)$, we define the variation of f in O by*

$$V_\gamma(f,O) := \sup\left\{\int_O f\mathrm{div}_\gamma^F G d\gamma :\ F \subseteq QX^* \textit{ fin. dim.},\ G \in \mathrm{Lip}_c(O,F),\ |G(x)|_F \le 1\ \forall x \in O\right\}, \tag{2.11}$$

where for $F = \mathrm{span}\{k_1,\dots,k_m\}$ for some $k_1,\dots,k_m \in QX^*$ and $G \in \mathrm{Lip}_c(O,F)$, we define $\mathrm{div}_\gamma^F G := \sum_{i=1}^m \partial_{k_i}^* G_i(x)$ with $G(x) := \sum_{i=1}^m G_i(x)k_i$ and $|G(x)|_F^2 := \sum_{i=1}^m |G_i(x)|^2$. Since $f \in L(\log L)^{1/2}(O,\gamma)$, the integral term in (2.11) is well defined for any $F \subseteq QX^*$ of finite dimension.

If $O = X$, we denote $V_\gamma(f,X)$ by $V_\gamma(f)$; it is not hard to see that this definition coincides with [3, Definition 3.8], by arguing as in Lemma 2.17.

Under our assumptions, G has bounded support and $F \subseteq QX^*$, hence $\mathrm{div}_\gamma^F G$ is bounded. Therefore, it follows that $V_\gamma(f,O)$ is lower semicontinuous with respect to the L^1 convergence of f.

For any $f \in L(\log L)^{1/2}(O,\gamma)$ we introduce the functional

$$L_\gamma(f,O) := \inf\left\{\liminf_{n\to+\infty}\int_O |\nabla_H f_n|_H d\gamma :\ f_n \in W^{1,1}(O,\gamma),\ f_n \overset{L^1(O,\gamma)}{\longrightarrow} f\right\}. \tag{2.12}$$

The variation of a function f along a subspace F of H generated by a finite number of elements of QX^* deserves a particular attention. Let $h_i,\dots,h_k \in QX^*$ be orthonormal elements of H and let $F = \mathrm{span}\{h_1,\dots,h_k\}$. We define the variation of $f \in L(\log L)^{1/2}(O,\gamma)$ in O along F by

$$V_\gamma^F(f,O) := \sup\left\{\sum_{j=1}^k \int_O f\partial_{h_j}^* G_j d\gamma : G_j \in \mathrm{Lip}_c(O),\ j = 1,\dots,k,\ \sum_{j=1}^k |G_j(x)|^2 \le 1\ \forall x \in O\right\}, \tag{2.13}$$

where $\partial_h^*\psi = \partial_h\psi - \psi\widehat{h}$ for any smooth enough function ψ and any $h \in QX^*$. If $O = X$ we denote $V_\gamma^F(f,X)$ by $V_\gamma^F(f)$. Further, let $h \in QX^*$. When $F = \mathrm{span}\{h\}$, we denote $V_\gamma^F(f,O)$ by $V_\gamma^h(f,O)$. As $V_\gamma(f,O)$, also $V_\gamma^F(f,O)$ and $V_\gamma^h(f,O)$ are lower semicontinuous respect to f in the L^1 topology.

We also define the weak gradient of $f \in BV(O,\gamma)$ along F. Let $D_\gamma^O f = \sigma|D_\gamma^O f|$ be the polar decomposition of $D_\gamma^O f$. We define the weak gradient of f along F by

$$D_\gamma^{O,F} f := \sum_{j=1}^k [\sigma, h_{i_j}]_H h_{i_j} |D_\gamma^O f|.$$

If $O = X$, we denote $V_\gamma^h(f,X)$ by $V_\gamma^h(f)$; it is not hard to see that this definition coincides with [3, Definition 3.8], by arguing as in Lemma 2.17.

3 Equivalent characterizations of BV on domains

Let $O \subseteq X$ be an open set. The aim of this section is to prove that, analogously to the case $O = X$ (see [3, Theorem 4.1]) it is possible to characterize the space $BV(O,\gamma)$ in terms of (2.11) and (2.12). To begin with, we state the main theorem of the paper.

Theorem 3.1 *Let $f \in L(\log L)^{1/2}(O,\gamma)$. The following are equivalent:*

1. *$f \in BV(O,\gamma)$;*
2. *$V_\gamma(f,O) < +\infty$;*
3. *$L_\gamma(f,O) < +\infty$.*

Moreover, if one (and then all) of the previous holds true, then $|D^O_\gamma f|(O) = V_\gamma(f,O) = L_\gamma(f,O)$.

Since the proof is rather long, for reader's convenience we split it into two different subsections. In the former we prove implication (2) $\Rightarrow$ (1), in the latter we show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

3.1 (2) $\Rightarrow$ (1)

In this subsection we show that, if $f \in L(\log L)^{1/2}(O,\gamma)$ has finite variation, then $f \in BV(O,\gamma)$ and $|D^O_\gamma f|(O) = V_f(f,O)$.

The following result is a part of [3, Theorem 4.1].

Proposition 3.2 *Let $u \in L(\log L)^{1/2}(X,\gamma)$. Then, the following are equivalent:*

(i) there exists $\mu \in \mathscr{M}(X,H)$ such that

$$\int_X u\partial_h^* G d\gamma = -\int_X G[\mu,h]_H, \qquad \forall G \in \mathrm{Lip}_c(X).$$

(ii) $V_\gamma(u) < +\infty$.

We remark that, arguing component by component in a way similar to the proof of [3, Theorem 4.1] and taking into account [3, Definition 3.1] and subsequent paragraphs we have the following result.

Remark 3.3 Let $h \in QX^*$ and let $u \in L(\log L)^{1/2}(X,\gamma)$. Then, the following are equivalent:

(i) there exists $\mu_h \in \mathscr{M}(X)$ such that

$$\int_X u\partial_h^* G d\gamma = -\int_X G d\mu_h, \qquad \forall G \in \mathrm{Lip}_c(X).$$

(ii) $V^h_\gamma(u) < +\infty$.

Further, if we consider $F \subseteq QX^*$ with $F = \mathrm{span}\{h_1,\dots,h_m\}$ such that $V^F_\gamma(u) < +\infty$, then there exists a measure $\mu_F := (\mu_1,\dots,\mu_m) \in \mathscr{M}(X,F)$ which satisfies

$$\int_X u\partial_h^* G d\gamma = -\int_X G d[\mu_F,h]_H, \quad G \in \mathrm{Lip}_c(X),$$

for any $h \in F$ and

$$|\mu_F|(X) = |(\mu_1,\dots,\mu_m)|(X) \le V^F_\gamma(u).$$

Finally, if $\sup_{F\subseteq QX^*} V^F_\gamma(u) < +\infty$, then $u \in BV(X,\gamma)$.

Remark 3.4 The results of [3] can be used because the definitions of BV and variations are equivalent to ours. Moreover, from [3, Theorem 4.1] the statements in Proposition 3.3 hold true for $G \in \mathcal{F}C^1_b(X)$. We can generalize to $G \in \mathrm{Lip}_c(X)$ by arguing as in Lemma 2.17.

Theorem 3.5 *Let $f \in L(\log L)^{1/2}(O,\gamma)$ be such that $V_\gamma(f,O) < +\infty$. Then, $f \in BV(O,\gamma)$, i.e., there exists a unique H-valued measure on $\mathcal{B}(O)$, denoted by $D_\gamma^O f$, which satisfies*

$$\int_O f \operatorname{div}_\gamma^F G d\gamma = -\int_O [G, dD_\gamma^O f]_H, \quad G \in \operatorname{Lip}_c(O,F), \tag{3.1}$$

where $F \subseteq QX^*$ finite dimensional subspace. Moreover, $|D_\gamma^O f|(O) = V_\gamma(f,O)$ and

$$|D_\gamma^O f|(A) = \sup\left\{\int_A f \operatorname{div}_\gamma^F G d\gamma : F \subseteq QX^* \text{ fin. dim.}, \ G \in \operatorname{Lip}_c(A,F), \ |G(x)|_F \le 1 \ \forall x \in A\right\},$$

for any open set $A \subseteq O$.

Proof.

Let g be a Lipschitz function such that $\operatorname{dist}(\operatorname{supp} g, O^c) > 0$ and $\|g\|_\infty \le 1$. Then, $g\overline{f} \in L(\log L)^{1/2}(X,\gamma)$ and $\sup_{F\subseteq QX^*} V_\gamma^F(g\bar{f}) < +\infty$. Recalling the concepts in Subsection 2.4 it is easy to see that

$$\int_X A_{1/2}(|g\overline{f}|)d\gamma \le \int_X A_{1/2}(|\overline{f}|)d\gamma = \int_O A_{1/2}(|f|)d\gamma < +\infty,$$

since $f \in L(\log L)^{1/2}(O,\gamma)$. Hence, $A_{1/2}(|g\overline{f}|) \in L^1(X,\gamma)$ and therefore $g\overline{f} \in L(\log L)^{1/2}(X,\gamma)$. Further, for any $t > 0$ we have

$$\int_X A_{1/2}(|(g\overline{f}|/t)d\gamma \le \int_O A_{1/2}(|f|/t)d\gamma,$$

and so $\|g\overline{f}\|_{L(\log L)^{1/2}(X,\gamma)} \le \|f\|_{L(\log L)^{1/2}(O,\gamma)}$. Finally, for any $F \subseteq QX^*$ finite dimensional and any $G \in \operatorname{Lip}_c(X,F)$ with $|G(x)|_F \le 1$ for any $x \in X$, the function gG belongs to $\operatorname{Lip}_c(O,F)$ and $\|gG\|_\infty \le 1$. Since $\operatorname{div}_\gamma^F(gG) = g\operatorname{div}_\gamma^F G + [G, \nabla_H g]_H$, it follows that

$$\begin{aligned}\int_X (g\bar{f})\operatorname{div}_\gamma^F G d\gamma &= \int_X \bar{f}\operatorname{div}_\gamma^F(gG)d\gamma - \int_X f[G, \pi_F(\nabla_H g)]_H d\gamma \\ &\le V_\gamma(f,O) + \|\nabla_H g\|_{L^\infty(O,\gamma;H)}\|f\|_{L^1(O,\gamma)},\end{aligned}$$

since g is Lipschitz. From Proposition 3.2, there exists a Borel H–valued measure $D_\gamma^X(g\bar{f})$ such that

$$\int_X (g\bar{f})\partial_h \psi d\gamma = -\int_X \psi d[h, D_\gamma^X(g\bar{f})]_H, \tag{3.2}$$

for any $\psi \in \operatorname{Lip}_c(X)$ and any $h \in QX^*$. Hence, $g\bar{f} \in BV(X,\gamma)$ by Definition 2.15.

Let $A \Subset O$ be an open set. For every $g \in \operatorname{Lip}_c(O)$ such that $g_{|A} \equiv 1$, we have seen that $g\bar{f} \in L(\log L)^{1/2}(X,\gamma)$ and $g\bar{f} \in BV(X,\gamma)$. We define

$$\nu_A(B) := D_\gamma^X(g\bar{f})(B \cap A)$$

for every $B \in \mathcal{B}(X)$. Clearly, ν_A is concentrated on A and by Remark 2.19 we deduce that ν_A does not depend on the choice of g. Further, Corollary 2.2 implies

$$|\nu_A|(O) = |\nu_A|(A) = \sup\left\{\int_A [G,\sigma]_H \, d|\nu_A| : G \in \operatorname{Lip}_c(A,F), \ F \subseteq QX^* \text{ fin. dim.}, |G|_H \le 1\right\}$$

where $\nu_A = \sigma|\nu_A|$. For any $G \in \operatorname{Lip}_c(A,F)$ we have

$$\int_A [G,\sigma]_H \ d|\nu_A| = -\int_X fg\operatorname{div}_\gamma G \ d\gamma = -\int_X f\operatorname{div}_\gamma G \ d\gamma.$$

Since $\operatorname{Lip}_c(A,F) \subseteq \operatorname{Lip}_c(O,F)$, we infer that $\int_X f\operatorname{div}_\gamma G \le V_\gamma(f,O)$. Then, $|\nu_A|(O) \le V_\gamma(f,O)$.

We consider an increasing sequence (A_n) of open sets such that $A_n \Subset O$ for any $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} A_n = O$ and for any $B \Subset O$ there exists $\overline{n} \in \mathbb{N}$ such that $B \Subset A_{\overline{n}}$. A possible choice is

$A_n := \{x \in O : \mathrm{dist}(x, O^c) > n^{-1}, \|x\| \le n\}$ for any $n \in \mathbb{N}$. Since $(|\nu_{A_n}|(O))_{n\in\mathbb{N}}$ is an increasing bounded sequence, it admits limit $\mathrm{L} \le V_\gamma(f, O)$.

By the definition of ν_A, it is clear that if $m < n$ then $\nu_{A_n|A_m} = \nu_{A_m|A_m}$. It follows that

$$|\nu_{A_n} - \nu_{A_m}|(O) = |\nu_{A_n}|(A_n \backslash A_m) = |\nu_{A_n}|(A_n) - |\nu_{A_m}|(A_m) = |\nu_{A_n}|(O) - |\nu_{A_m}|(O)$$

for $n > m$. Since $|\nu_{A_n}|(O)$ converges to L, the previous equation implies that (ν_{A_n}) is a Cauchy sequence in $\mathscr{M}(O, H)$, which is a Banach space with norm

$$\|\mu\| := |\mu|(O)$$

(see e.g. [13, Section I.5]) the discussion after Corollary 6). Therefore, $(\nu_{A_n})_{n\in\mathbb{N}}$ converges to a measure which we denote by $D^O_\gamma f = \sigma_1|D^O_\gamma f|$. Moreover, $|D^O_\gamma f|(O) = \mathrm{L}$ and $D^O_\gamma f \llcorner A_n = \nu_{A_n}$ for every $n \in \mathbb{N}$.

By Definition 2.22 there exists a sequence of functions G_n such that, for every $n \in \mathbb{N}$, $|G_n|_H \le 1$, $G_n \in \mathrm{Lip}_c(O, F_n)$ for some finite dimensional subspace $F_n \le H$ and

$$V_\gamma(f, O) = \lim_{n\to+\infty} \int_O f \mathrm{div}_\gamma G_n d\gamma.$$

The assumptions on $(A_n)_{n\in\mathbb{N}}$ imply that there exists an increasing sequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that for any $n \in \mathbb{N}$ we have $G_n \in \mathrm{Lip}_c(A_{m_n}, F_n)$. Therefore,

$$V_\gamma(f, O) = \lim_{n\to+\infty} \int_X f \mathrm{div}_\gamma G_n d\gamma \le \lim_{n\to+\infty} |\nu_{A_{m_n}}|(O) = |D^O_\gamma f|(O) \le V_\gamma(f, O),$$

hence $|D^O_\gamma f|(O) \le V_\gamma(f, O)$.

Finally, let $G \in \mathrm{Lip}_c(O)$ and $h \in QX^*$. There exists $n \in \mathbb{N}$ such that $G \in \mathrm{Lip}_c(A_n, F_n)$ and

$$\int_O \partial^*_h f G \, d\gamma = \int_{A_n} \partial^*_h f G \, d\gamma = \int_{A_n} G d \left[h, \nu_{A_n}\right]_H = \int_{A_n} G \left[h, \sigma_1\right]_H d|D^O_\gamma f| = \int_O G \left[h, \sigma_1\right]_H \, d|D^O_\gamma f|.$$

Therefore, by Definition 2.15 it follows that f is a function of bounded variation with weak gradient $D^O_\gamma f$ which does not depend on the choice of A_n.

The second part of the statement follows from Corollary 2.2. QED

Arguing as in Theorem 3.5 it is possible to prove that if f has finite variation $F \subseteq QX^*$ finite dimensional, then there exists a measure $D^{O,F}_\gamma f \in \mathscr{M}(O, F)$.

Corollary 3.6 *Let $f \in L(\log L)^{1/2}(O, \gamma)$, let $F = \mathrm{span}\{h_1, \ldots, h_k\}$, where $h_i \in QX^*$ are orthonormal, and let us consider $V^F_\gamma(f, O)$ defined in (2.13). If $V^F_\gamma(f, O) < +\infty$, then there exists a measure $D^{O,F}_\gamma f \in \mathscr{M}(O, F)$ which enjoys*

$$\int_O f \mathrm{div}^F_\gamma G d\gamma = -\int_O [G, dD^{O,F}_\gamma f]_H, \quad G \in \mathrm{Lip}_c(O, F),$$

$|D^{O,F}_\gamma f| = V^F_\gamma(f, O)$ and for any open set $A \subseteq O$ we have

$$|D^{O,F}_\gamma f|(A) = \sup\left\{\sum_{j=1}^k \int_A f \partial^*_{h_j} G_j d\gamma : G_j \in \mathrm{Lip}_c(A),\ j = 1, \ldots, k,\ \sum_{j=1}^k |G_j(x)|^2 \le 1\ \forall x \in A\right\}.$$

Moreover, let $h \in QX^*$: if $V^h_\gamma(O, f) < +\infty$, then there exists a finite measure $D^{O,h}_\gamma f \in \mathscr{M}(O)$ which satisfies

$$\int_O f \partial_h G d\gamma = -\int_O G dD^{O,h}_\gamma f, \quad G \in \mathrm{Lip}_c(O),$$

$|D^{O,h}_\gamma f| = V^h_\gamma(f, O)$ and for any open set $A \subseteq O$

$$|D^{O,h}_\gamma f|(A) = \sup\left\{\int_A f \partial^*_h G d\gamma : G \in \mathrm{Lip}_c(A),\ |G(x)| \le 1\ \forall x \in A\right\}.$$

3.2 The implications (1) ⇒ (3) ⇒ (2)

In this subsection we prove the remaining implications of Theorem 3.1. Before stating the result we are interested in, we provide a useful result involving the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ and the space $BV(X,\gamma)$.

Lemma 3.7 *Let $f \in BV(X,\gamma)$, let $h \in H$ and let $O \subseteq X$ be an open set such that $|D^h_\gamma f|(\partial O) = 0$. Then,*

$$\lim_{t\to 0}\int_O |\partial_h T_t f| d\gamma = |D^{X,h}_\gamma f|(O).$$

Proof. We can apply [3, Theorem 4.4] because the definitions of variation are equivalent, therefore we know that $\int_X |\partial_h T_t f| d\gamma \to |D^{X,h}_\gamma f|(X)$ as $t \to 0^+$. Further, [6, Theorem 8.2.3] gives

$$\limsup_{t\to 0}\int_C |\partial_h T_t f| d\gamma \leq |D^{X,h}_\gamma f|(C),$$

for any closed set $C \subseteq X$. Therefore,

$$|D^{X,h}_\gamma f|(O) \leq \liminf_{t\to 0}\int_O |\partial_h T_t f| d\gamma \leq \limsup_{t\to 0}\int_{\overline{O}} |\partial_h T_t f| d\gamma \leq |D^{X,h}_\gamma f|(\overline{O}) = |D^{X,h}_\gamma f|(O),$$

since $|D^{X,h}_\gamma f|(\partial O) = 0$. QED

Remark 3.8 We easily deduce that (1) ⇒ (2) i.e. that $f \in BV(O,\gamma)$ implies $V_\gamma(f,O) = |D^O_\gamma f|(O)$. Indeed, let $f \in BV(O,\gamma)$ and let F be a finite dimensional subspace of QX^*. Then for any $G \in \mathrm{Lip}_c(O,F)$, the integration by parts formula (2.9) gives

$$\int_O f(\mathrm{div}^F_\gamma G) d\gamma = -\int_O [G, dD^O_\gamma f]_H = -\int_O [G,\sigma]_H d|D^O_\gamma f| \leq |D^O_\gamma f|(O), \tag{3.3}$$

where $D^O_\gamma f = \sigma |D^O_\gamma f|$ is the polar decomposition of $D^O_\gamma f$.
Since $D^O_\gamma f$ is a finite Radon measure, taking the supremum with respect to F and $G \in \mathrm{Lip}_c(O,F)$ with $|G(x)|_F \leq 1$ for any $x \in O$, in both sides of (3.3) and taking into account (2.3) in Corollary 2.2 we infer that $V_\gamma(f,O) \leq |D^O_\gamma f|(O)$. In particular, from Theorem 3.5 we conclude that $V_\gamma(f,O) = |D^O_\gamma f|(O)$.

Implication (1) ⇒ (3) is the content of the following proposition.

Proposition 3.9 *If $f \in BV(O,\gamma)$ then $L_\gamma(f,O) \leq |D^O_\gamma f|(O)$.*

Proof. We adapt the proofs in [2, Theorem 3.9] and in [23, Proposition 7.5.9]. Our aim is proving that there exists a sequence $(f_\varepsilon) \subseteq \mathbb{D}^{1,1}(X,\gamma)$ such that $f_\varepsilon \to f$ in $L^1(O,\gamma)$ and $\int_O |\nabla_H f_\varepsilon|_H d\gamma \to |D^O_\gamma f|(O)$ as $\varepsilon \to 0$. Then by definition $L_\gamma(f,O) \leq |D^O_\gamma f|(O)$.

Assume that $f \in BV(O,\gamma)$. Since $|D^O_\gamma f|(O) < +\infty$, from Remark 2.4 for all but at most countable $r \in (1,+\infty)$ we have $|D^O_\gamma f|(\partial(O_{(-1/r)})) = 0$, where $O_{-\alpha} := \{x \in O : \mathrm{dist}(x,O^c) > \alpha\}$ for any positive α. Thus, for any $i \in \mathbb{N}$ we set $O_i := O_{-(r_i)^{-1}}$ such that (r_i) is an increasing sequence of positive numbers, $r_i \to +\infty$ as $i \to +\infty$ and $|D^O_\gamma f|(\partial(O_{(-1/r_i)})) = 0$ for any $i \in \mathbb{N}$. We introduce a sequence of Lipschitz functions $(\varphi_i)_{i\in\mathbb{N}} \subseteq \mathrm{Lip}_c(O)$ such that $\varphi_i \equiv 1$ on O_i for any $i \in \mathbb{N}$. Further, we define

$$f_i := \begin{cases} f\varphi_i, & \text{in } O,\\ 0, & \text{in } O^c.\end{cases}$$

Arguing as in the proof of Theorem 3.5, it follows that $f_i \in BV(X,\gamma)$ and

$$D^X_\gamma f_i = \varphi_i D^O_\gamma f + f\nabla_H\varphi_i\gamma. \tag{3.4}$$

Since $f_{i|O_i} \equiv f$, it follows that $D^X_\gamma f_i \llcorner O_i = D^O_\gamma f \llcorner O_i$. Let us set

$$f_t := T_t\overline{f}, \quad f_{i,t} := T_t(f_i), \quad i \in \mathbb{N},\ t \geq 0,$$

where $(T_t)_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup given by (2.7). Since $\overline{f}, f_i \in L(\log L)^{1/2}(X,\gamma)$, it follows that $f_{i,t} \in \mathbb{D}^{1,1}(X,\gamma)$ for any $i \in \mathbb{N}$ and any $t > 0$ and $f_{i,t} \to f_i$ (resp. $f_t \to f$) in $L(\log L)^{1/2}(O_i,\gamma)$ (resp. $L(\log L)^{1/2}(O,\gamma)$) as $t \to 0^+$ (see [17, Proposition 3.6(i)-(ii)]). Therefore $f_t \to f$ (resp. $f_{i,t} \to f_i$) in $L^1(O,\gamma)$ (resp. $L^1(O_i,\gamma)$) as $t \to 0^+$. Hence, from Lemma 3.7 and (3.4) we deduce that

$$\lim_{t\to 0}\int_O |\nabla_H f_{i,t}|_H d\gamma = |D^X_\gamma f_i|(O), \quad \lim_{t\to 0}\int_{O_i} |\nabla_H f_{i,t}|_H d\gamma = |D^O_\gamma f|(O_i), \quad i \in \mathbb{N}. \tag{3.5}$$

We define $U_1 := O_1$, $U_2 := O_2$ and $U_i := O_i \backslash \overline{O_{i-2}}$ for any $i > 2$. Further, (U_i) is an open covering of O and $U_i \cap U_j = \emptyset$ for any $|i-j| > 1$. Therefore, from [1, Corollary 1.4] with $W = \mathrm{Lip}_c(O)$, there exists a partition of unity $(\psi_i) \subseteq \mathrm{Lip}_c(O)$ subordinated to (U_i), i.e.,

$$\psi_i \geq 0, \quad \sum_{i\in\mathbb{N}} \psi_i \equiv 1, \qquad \mathrm{supp}(\psi_i) \subseteq U_i, \ i \in \mathbb{N}.$$

Let us fix $\varepsilon > 0$. Since $D^O_\gamma f$ is a finite Radon measure and thanks to Lemma 3.7 there exists $i_\varepsilon > 0$ such that

$$|D^O_\gamma f|(O \setminus O_i)| \leq \varepsilon, \qquad i \geq i_\varepsilon - 1. \tag{3.6}$$

Moreover, from the convergence of $f_{i,t}$ to f in $L(\log L)^{1/2}(O_i,\gamma)$ and (3.5) there exists $t_\varepsilon > 0$ such that for any $i \in \{1,\dots,i_\varepsilon\}$ we have

$$\|f_{i_\varepsilon,t_\varepsilon} - f\|_{L(\log L)^{1/2}(O_{i_\varepsilon},\gamma)} \leq \varepsilon, \tag{3.7}$$

$$\|f_{i_\varepsilon,t_\varepsilon} - f\|_{L^1(O_{i_\varepsilon},\gamma)} \leq 2^{-i}(1 + \|\nabla_H\psi_i\|_{L^\infty(X;H)})^{-1}\varepsilon, \tag{3.8}$$

$$\left|\int_{O_{(i_\varepsilon-1)}} |\nabla_H f_{i_\varepsilon,t_\varepsilon}|_H d\gamma - |D^O_\gamma f|(O_{(i_\varepsilon-1)})\right| \leq \varepsilon. \tag{3.9}$$

Finally, again from the convergence of $f_{i,t}$ to f in $L(\log L)^{1/2}(O_i,\gamma)$ and (3.5), for any $i > i_\varepsilon$ there exists $t_{\varepsilon,i} > 0$ such that

$$\|f_{i,t_{\varepsilon,i}} - f\|_{L(\log L)^{1/2}(O_i,\gamma)} \leq 2^{-i}\varepsilon, \tag{3.10}$$

$$\|f_{i,t_{\varepsilon,i}} - f\|_{L^1(O_i,\gamma)} \leq 2^{-i}(1 + \|\nabla_H\psi_i\|_{L^\infty(X;H)})^{-1}\varepsilon, \tag{3.11}$$

$$\left|\int_O \psi_i|\nabla_H f_{i,t_{\varepsilon,i}}|_H d\gamma - \int_O \psi_i d|D^O_\gamma f|\right| \leq 2^{-i}\varepsilon. \tag{3.12}$$

We set

$$f_{\varepsilon,i} := \begin{cases} f_{i_\varepsilon,t_\varepsilon}, & i \leq i_\varepsilon, \\ f_{i,t_{\varepsilon,i}}, & i > i_\varepsilon, \end{cases} \qquad f_\varepsilon := \sum_{i\in\mathbb{N}} \psi_i f_{\varepsilon,i}. \tag{3.13}$$

f_ε is well defined since for any $x \in O$ the series is indeed a finite sum (the support of ψ_i is contained in U_i and therefore the series in $f_\varepsilon(x)$ involves at most two terms).

By Lemma 2.10, $W^{1,1}(X,\gamma) = \mathbb{D}^{1,1}(X,\gamma)$. Since $f_{\varepsilon,i} \in \mathbb{D}^{1,1}(X,\gamma)$ we get $f_{\varepsilon,i} \in W^{1,1}(X,\gamma)$. It is clear that $f_\varepsilon \in D\mathscr{E}^O_h$ for all $h \in QX^*$ and there exists $\nabla_H f_\varepsilon = \sum_{i\in\mathbb{N}} \psi_i \nabla_H f_{\varepsilon,i}$ which satisfies $\partial_h f_\varepsilon = [\nabla_H f_\varepsilon, h]_H$. So, to prove that $f_\varepsilon \in W^{1,1}(O,\gamma)$, it suffices to prove that $f_\varepsilon \in L^1(O,\gamma)$ and $\nabla_H f_\varepsilon \in L^1(O,\gamma;H)$.

We stress that $f_{\varepsilon|O^c} \equiv 0$ and $\nabla_H f_\varepsilon$ is well defined γ-a.e. $x \in X$. It is also worth noticing that $f_{\varepsilon|O_{(i_\varepsilon-1)}} = f_{i_\varepsilon,t_\varepsilon|O_{(i_\varepsilon-1)}}$. Then,

$$\begin{aligned}\int_O |f_\varepsilon| d\gamma \leq & \int_O \left|\sum_{i\in\mathbb{N}} \psi_i(f_{\varepsilon,i} - f)\right| d\gamma + \int_O |f| d\gamma \\ \leq & \int_O \left|\sum_{i=1}^{i_\varepsilon} \psi_i(f_{\varepsilon,i} - f)\right| d\gamma + \int_O \left|\sum_{i=i_\varepsilon+1}^{+\infty} \psi_i(f_{\varepsilon,i} - f)\right| d\gamma + \|f\|_{L(\log L)^{1/2}(O,\gamma)} \\ \leq & 2\varepsilon + \|f\|_{L(\log L)^{1/2}(O,\gamma)},\end{aligned} \tag{3.14}$$

thanks to (3.8), (3.11), (3.13) and the fact that $\mathrm{supp}(\psi_i)\subseteq U_i\subseteq O_i$ for any $i\in\mathbb{N}$. Let us consider $\nabla_H f_\varepsilon$. We have

$$\int_O |\nabla_H f_\varepsilon|_H d\gamma \le \int_O \left|\sum_{i\in\mathbb{N}} \nabla_H\psi_i f_{\varepsilon,i}\right|_H d\gamma + \int_O \left|\sum_{i\in\mathbb{N}} \psi_i \nabla_H f_{\varepsilon,i}\right|_H d\gamma =: I_1+I_2.$$

Let us deal with I_1. Since $\sum_{i\in\mathbb{N}}\nabla_H\psi_i\equiv 0$, it follows that

$$I_1=\int_O \left|\sum_{i\in\mathbb{N}} \nabla_H\psi_i (f_{\varepsilon,i}-f)\right|_H d\gamma \le \varepsilon, \tag{3.15}$$

from the definition of ψ_i and of $f_{\varepsilon,i}$, and by applying (3.8) and (3.11). As far as I_2 is concerned, we get

$$I_2\le \int_O \sum_{i=1}^{i_\varepsilon} \psi_i|\nabla_H f_{\varepsilon,i}|_H d\gamma + \int_O \sum_{i=i_\varepsilon+1}^{+\infty} \psi_i|\nabla_H f_{\varepsilon,i}|_H d\gamma =: J_1+J_2.$$

We recall that from the definition $\sum_{i=1}^{i_\varepsilon}\psi_i\equiv 1$ on $O_{(i_\varepsilon-1)}$ and from (3.13) we have $f_{\varepsilon,i}=f_{i_\varepsilon,t_\varepsilon}$ for $i=1,\ldots,i_\varepsilon$. Hence, (3.9) gives

$$J_1=\int_{O_{(i_\varepsilon-1)}} |\nabla_H f_{i_\varepsilon,t_\varepsilon}|_H d\gamma \le |D_\gamma^O f|(O_{(i_\varepsilon-1)})+\varepsilon. \tag{3.16}$$

As far as J_2 is concerned, we stress that $\psi_i\equiv 0$ on $O_{(i_\varepsilon-1)}$ for any $i\ge i_\varepsilon+1$. Therefore, from (3.12) we deduce that

$$J_2\le \sum_{i\in\mathbb{N}}\int_{O\setminus O_{(i_\varepsilon-1)}} \psi_i d|D_\gamma^O f|+\varepsilon = |D_\gamma^O f|(O\setminus O_{(i_\varepsilon-1)})+\varepsilon, \tag{3.17}$$

and the claim is so proved.

Finally, above computations reveal that $f_\varepsilon\to f$ in $L^1(O,\gamma)$ and $\int_O |\nabla_H f_\varepsilon|_H d\gamma\to |D_\gamma^O f|(O)$ as $\varepsilon\to 0$. Indeed, (3.14) shows that

$$\int_O |f_\varepsilon - f| d\gamma\le\varepsilon.$$

Moreover, we have

$$\left|\int_O |\nabla_H f_\varepsilon|_H d\gamma - |D_\gamma^O f|(O)\right| \le I_1 + \left|J_1-|D_\gamma^O f|(O)\right| + \left||D_\gamma^O f|(O_{(i_\varepsilon-1)})-|D_\gamma^O f|(O)\right| + J_2.$$

Therefore, from (3.6), (3.15), (3.16) and (3.17) we conclude that

$$\left|\int_O |\nabla_H f_\varepsilon|_H d\gamma - |D_\gamma^O f|(O)\right|\le 5\varepsilon.$$

QED

The following proposition, which shows that (3) ⇒ (2), concludes the proof of Theorem 3.1.

Proposition 3.10 *Let $f\in L(\log L)^{1/2}(O,\gamma)$. If $L_\gamma(f,O)<+\infty$ then $V_\gamma(f,O)\le L_\gamma(f,O)$.*

Proof. Let $f\in L(\log L)^{1/2}(O,\gamma)$ and let us show that $L_\gamma(f,O)<+\infty$ implies $V_\gamma(f,O)\le L_\gamma(f,O)$. Since $L_\gamma(f,O)<+\infty$, there exists a sequence $(f_n)\subseteq W^{1,1}(O,\gamma)$ such that $f_n\to f$ in $L^1(O,\gamma)$ as $n\to+\infty$ and

$$\lim_{n\to+\infty}\int_O |\nabla_H f_n|_H d\gamma = L_\gamma(f,O).$$

Further, for any $G\in \mathrm{Lip}_c(O,F)$ where F is a finite dimensional subspace QX^* and such that $|G(x)|_H\le 1$ for any $x\in O$, we have $\mathrm{div}_\gamma^F G\in L^\infty(X)$. Then thanks to Remark 2.11, we get

$$\int_O f(\mathrm{div}_\gamma^F G)d\gamma = \lim_{n\to+\infty}\int_O f_n(\mathrm{div}_\gamma^F G)d\gamma = \lim_{n\to+\infty}\left(-\int_O [\nabla_H f_n, G]_H d\gamma\right)\le L_\gamma(f,O).$$

Taking the supremum over G, we get $V_\gamma(f,O)\le L_\gamma(f,O)$. QED

4 Further results

In this section we collect some consequences of the results of Section 3. At first, we give a sufficient condition (related to the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ introduced in (2.7)) which ensures that $f \in L(\log L)^{1/2}(O,\gamma)$ belongs to $BV(O,\gamma)$. We stress that, differently from [3, Theorem 4.1], we don't have the equivalence of this condition with those in Theorem 3.1 since we are concerning with the semigroup $(T_t)_{t\geq 0}$ defined on the whole space X. Unfortunately, at the best of our knowledges there is no good definition of the Ornstein–Uhlenbeck semigroup on open domains in Wiener spaces and therefore we don't recover the same result of [3]. For any $f \in L(\log L)^{1/2}(O,\gamma)$ we define the (possible infinite) limit

$$\mathscr{J}(f,O) := \liminf_{t\downarrow 0} \int_O |\nabla_H T_t(\bar{f})|_H d\gamma, \tag{4.1}$$

and we show that if $\mathscr{J}(f,O) < +\infty$, then $f \in BV(O,\gamma)$ and $|D^O_\gamma f|(O) \leq \mathscr{J}(f,O)$.

Proposition 4.1 *Let $f \in L(\log L)^{1/2}(O,\gamma)$ and let $\mathscr{J}(f,O) < +\infty$. Then, $f \in BV(O,\gamma)$ and $|D^O_\gamma f|(O) \leq \mathscr{J}(f,O)$.*

Proof. Arguing as in the proof of Theorem 3.5 it follows that $\overline{f} \in L(\log L)^{1/2}(X,\gamma)$ and therefore from [16, Proposition 3.6(i)] we infer that $T_t\overline{f} \in \mathbb{D}^{1,1}(X,\gamma)$ for any $t > 0$. Then, it is enough to apply Theorem 3.1(3) with $f_n := T_{t_n}(\bar{f})$, being (t_n) be any sequence of positive real numbers which satisfies $t_n \downarrow 0^+$ as $n \to +\infty$. QED

Corollary 3.6 allows us to prove a connection between the variation $V^h_\gamma(f,O)$ along h and the one dimensional sections of f (see [2, Chapter 3.11, Theorem 3.103] for the finite dimensional case and [3, Theorem 3.10] for the Wiener setting). To this aim, let us fix $h \in QX^*$ (so $\widehat{h} \in X^*$) and let us set $K := \ker(\widehat{h})$. For any open set $\Omega \subseteq X$ and for γ-a.e. $y \in K$ we introduce the set $\Omega^h_y := \{t \in \mathbb{R} : y + th \in \Omega\}$. Further, for any function $u : \Omega \to \mathbb{R}$ we set $u_y : \Omega^h_y \to \mathbb{R}$ as $u_y(t) = u(y+th)$. Moreover, we consider the decomposition of $\gamma = \gamma_1 \otimes \gamma^\perp_h$, where $\gamma_1 = \gamma \circ \pi_h^{-1}$ and $\gamma^\perp_h = \gamma \circ (I - \pi_h)^{-1}$ and $\pi_h : X \to \mathrm{span}\{h\}$ is the projection on $\mathrm{span}\{h\}$. Finally, for any $f \in L(\mathrm{Log}L)^{1/2}(\Omega,\gamma)$ we define

$$V^h_{\gamma_1}(f_y,\Omega^h_y) := \sup\left\{\int_{\Omega^h_y} f_y(t)(\psi'(t) - t\psi(t))d\gamma_1(t) : \psi \in \mathrm{Lip}_c(\Omega^h_y),\ |\psi(t)| \leq 1\ \forall t \in \Omega^h_y\right\}.$$

We stress that $\psi'(t) - t\psi(t) = \mathrm{div}_{\gamma_1}\psi(t)$ is the Gaussian divergence in dimension 1.

Proposition 4.2 *Let $O \subseteq X$ be an open set, let $f \in L(\log L)^{1/2}(O,\gamma)$ and let $h \in QX^*$. Then,*

$$V^h_\gamma(f,O) = \int_K V^h_{\gamma_1}(f_y,O^h_y)\gamma^\perp_h(dy). \tag{4.2}$$

Proof. At first, we remark that the inequality $\leq$ in (4.2) easily follows from Fubini's theorem. Indeed,

$$\int_O f(x)\partial^*_h G d\gamma = \int_K \int_{O^h_y} f_y(t)(G'(t) - tG(t))d\gamma_1(t)\gamma^\perp_h(dy) \leq \int_K V^h_{\gamma_1}(f_y,O^h_y)\gamma^\perp_h(dy),$$

for any admissible function G. Taking the supremum over G we get the desired inequality.

In order to prove the converse one, we use both an approximation and a smoothing argument. Since $|D^{O,h}_\gamma f|(O) < +\infty$, from Remark 2.4 there exists at most countably many indexes $\varepsilon \in (0,1)$ such that $|D^{O,h}_\gamma f|(\partial O_{-\varepsilon}) \neq 0$.

Let $\varepsilon \in (0,1)$ be such that $|D^{O,h}_\gamma f|(\partial O_{-\varepsilon}) = 0$, we introduce g_ε such that $g_\varepsilon \in \mathrm{Lip}_c(O)$ and $g_{\varepsilon|O_{-\varepsilon}} \equiv 1$. For $t > 0$ let use define $f_t := T_t(g_\varepsilon \bar{f})$ where T_t is the Ornstein-Uhlenbeck semigroup in $L^1(X,\gamma)$, which is a strongly continuous semigroup, therefore $f_{t|O_{-\varepsilon}}$ converges to $fg_{\varepsilon|O_{-\varepsilon}}$ in L^1 for $t \to 0$. f_t is in $\mathbb{D}^{1,p}(X,\gamma) = W^{1,p}(X,\gamma)$, hence it is in $BV(X,\gamma)$. Arguing as in the first part of the proof of Theorem 3.5, we have $g_\varepsilon\bar{f} \in BV(X,\gamma)$, and from Lemma 3.7 it follows that $|D_\gamma^{X,h} f_t|(O_{-\varepsilon}) \to |D_\gamma^{X,h}(g_\varepsilon \bar{f})|(O_{-\varepsilon})$ as $t \to 0$. From Corollaries 2.2 and 3.6 we deduce that $|D_\gamma^{X,h} f_t|(O_{-\varepsilon})$ converges to $|D_\gamma^{O,h} f|(O_{-\varepsilon}) = |D_\gamma^{X,h}(g_\varepsilon \bar{f})|(O_{-\varepsilon})$ as $t \to 0$. Moreover, since

$$\int_K \|(f_t)_y - (f)_y\|_{L^1((O_{-\varepsilon})^h_y, \gamma_1)} \gamma_h^\perp(dy) = \int_{O_{-\varepsilon}} |f_t - f| d\gamma \to 0, \qquad t \to 0,$$

there exists a sequence (t_n) decreasing to 0 as $n \to +\infty$ such that

$$\int_{(O_{-\varepsilon})^h_y} |(f_n)_y - (f)_y| d\gamma_1 \to 0, \qquad n \to +\infty,$$

for $\gamma_h^\perp$-a.e. $y \in K$, where $f_n := f_{t_n}$ for any $n \in \mathbb{N}$. Hence, the lower semicontinuity of $V^h_{\gamma_1}$, Fatou's Lemma, the convergence of $(|D_\gamma^{X,h} f_n|(O_{-\varepsilon}))$ and Corollary 3.6 imply that

$$\begin{aligned}\int_K V^h_{\gamma_1}(f_y, (O_{-\varepsilon})^h_y)\gamma_h^\perp(dy) &\leq \liminf_{n\to+\infty} \int_K V^h_{\gamma_1}((f_n)_y, (O_{-\varepsilon})^h_y)\gamma_h^\perp(dy) = \\ &= \liminf_{n\to+\infty} |D_\gamma^{X,h} f_n|(O_{-\varepsilon}) = |D_\gamma^{O,h} f|(O_{-\varepsilon}) \leq V^h_\gamma(f, O). \qquad (4.3)\end{aligned}$$

Letting $\varepsilon \to 0$ in (4.3) we conclude. QED

5 BV functions on domains in Hilbert spaces

In this section we show that the arguments in the proof of Theorem 3.5 allow us to prove a different characterization of BV functions on open domains in Hilbert spaces with respect to more general probability measures. In particular, we consider the setting of [9], and we recall the main definitions and results. Let X be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, let ν be a Borel probability measure on X and let $R \in \mathcal{L}(X)$ be such that the following hypothesis is satisfied.

Hypothesis 5.1 *For any $z \in X$ there exists $v_z \in \bigcap_{1\leq p<\infty} L^p(X, \nu)$ such that*

$$\int_X \langle RD\varphi, z\rangle d\nu = \int_X \varphi v_z d\nu, \quad \varphi \in C^1_b(X). \qquad (5.1)$$

Here, D denotes the Fréchet derivative of $\varphi \in C^1_b(X)$.

In particular, it follows that the map $z \mapsto v_z$ is closed from X to $L^p(X, \nu)$ for any $p \geq 1$, and therefore it is continuous. Hence, there exists a positive constant C_p such that

$$\|v_z\|_{L^p(X,\nu)} \leq C_p \|z\|, \quad z \in X, \ p \geq 1.$$

Further, since a crucial tool of our investigation is the space of Lipschitz functions, we need an additional hypothesis.

Hypothesis 5.2 *For any $G \in \mathrm{Lip}(X)$ there exists a subset $N = N_G \subseteq X$ such that $\nu(N) = 0$ and G is Gâteaux differentiable on $X \setminus N$.*

From [9, Proposition 2.3] we inherit the following result.

Proposition 5.3 *The operator $RD : D(RD) := C^1_b(X) \to L^p(X, \nu; X)$ is closable in $L^p(X, \nu)$ and we denote by M_p and $\mathbb{D}^{1,p}(X, \nu)$ its closure and the domain of its closure, respectively. In particular, if $p > 1$ for any $f \in \mathbb{D}^{1,p}(X, \nu)$ and $z \in X$ it holds that*

$$\int_X \langle M_p f, z\rangle d\nu = \int_X f v_z d\nu. \qquad (5.2)$$

For any $p < \infty$ we denote by $M_p^* : D(M_p^*) \subseteq L^{p'}(X, \nu; X) \to L^{p'}(X, \nu)$ the adjoint operator of M_p. Hence, for any $f \in \mathbb{D}^{1,p}(X, \nu)$ and any $F \in D(M_p^*)$ we have

$$\int_X M_p^* F f d\nu = \int_X \langle M_p f, F\rangle d\nu.$$

To simplify the notations, for functions $f\in\cup_{p\geq1}\mathbb{D}^{1,p}(X,\nu)$ we set $Mf:=M_pf$ for any $p\geq1$, and for vector fields $F\in\cup_{p>1}D(M_p^*)$ we set $M^*F:=M_p^*F$ for any $p>1$.

If $f\in\mathbb{D}^{1,p}(X,\nu)$ and $\varphi\in\mathrm{Lip}_b(X)$, then $(f\varphi)\in\mathbb{D}^{1,p}(X,\nu)$ and $M_p(f\varphi)=\varphi M_pf+fRD\varphi$, where D is the Gâteaux derivative of φ. Hence, if $p>1$ from (5.2) it follows that

$$\int_X\varphi\langle M_pf,z\rangle d\nu=-\int_X f(\langle RD\varphi,z\rangle-\varphi v_z)d\nu,\quad z\in X. \tag{5.3}$$

This yields that $M_p^*(\varphi)(z)=-\langle RD\varphi,z\rangle+\varphi v_z$. We stress that for $p=1$ the right-hand side of (5.3) is not meaningful, in general, since nothing ensures that $fv_z\in L^1(X,\nu)$. However, formula (5.3) is the starting point for the definition of BV functions in (X,ν) given in [9, Definition 3.1].

Definition 5.4 *Let $f\in L^1(X,\nu)$ be such that $fv_z\in L^1(X,\nu)$ for any $z\in X$. We say that $u\in BV(X,\nu)$ if there exists an X-valued Borel measure $m\in\mathscr{M}(X,X)$ such that, setting $m_z(B):=\langle m(B),z\rangle$ for any $z\in X$ and any $B\in\mathcal{B}(X)$, we have*

$$\int_X f(\langle RD\varphi,z\rangle-\varphi v_z)d\nu=-\int_X\varphi dm_z,\quad\varphi\in C_b^1(X). \tag{5.4}$$

We also denote m by $D_\nu f$ to stress its dependence on ν and f.

Further, we introduce the variation along $z\in X$ of $f\in L^1(X,\nu)$.

Definition 5.5 *Let $z\in X$ and let $f\in L^1(X,\nu)$ such that $fv_z\in L^1(X,\nu)$. Then, we define the variation of f along z by*

$$V_\nu^z(f):=\sup\left\{\int_X f(\langle RD\varphi,z\rangle-\varphi v_z)d\nu:\varphi\in C_b^1(X),\|\varphi\|_\infty\leq1\right\}.$$

Assume that $fv_z\in L^1(X,\nu)$ for any $z\in X$. Then, we define the variation of f by

$$V_\nu(f):=\sup\left\{\int_X fM^*\varphi d\nu:\varphi\in C_b^1(X,F),\ F\subseteq X\ \textit{fin. dim.},\ \|\varphi\|_\infty\leq1\right\}.$$

It is easy to see that, if $\{e_1,\ldots,e_k\}$ is an orthonormal basis of F, then

$$M^*\varphi=-\sum_{i=1}^k\langle RD\varphi,e_i\rangle-\varphi v_{e_i}$$

for any $\varphi\in C_b^1(X,F)$.

Let $O\subseteq X$ be an open set. We provide the definition of BV functions and of the variation of a function on O.

Definition 5.6 *Let $f\in L^1(O,\nu)$ such that $fv_z\in L^1(O,\nu)$ for any $z\in X$. We say that $f\in BV(O,\nu)$ if there exists an X-valued Borel measure $\mu\in\mathscr{M}(O,X)$ such that, setting $\mu_z(B):=\langle\mu(B),z\rangle$ for any $z\in X$ and any $B\in\mathcal{B}(O)$, we have*

$$\int_O f(\langle R\nabla\varphi,z\rangle-\varphi v_z)d\nu=-\int_O\varphi d\mu_z,\quad\varphi\in\mathrm{Lip}_c(O), \tag{5.5}$$

where ∇ denotes the Gâteaux derivative. We also denote μ by D_ν^Of to stress its dependence on ν,O and f.

Definition 5.7 *Let $z\in X$ and let $f\in L^1(O,\nu)$ such that $fv_z\in L^1(O,\nu)$. Then, we define the variation of f along z by*

$$V_\nu^z(f,O):=\sup\left\{\int_O f(\langle R\nabla\varphi,z\rangle-\varphi v_z)d\nu:\varphi\in\mathrm{Lip}_c(O),\ \|\varphi\|_\infty\leq1\right\}.$$

assume that $fv_z\in L^1(O,\nu)$ for any $z\in X$. We define the variation of f in O by

$$V_\nu(f,O):=\sup\left\{\int_O fM^{*,F}\varphi d\nu:\varphi\in\mathrm{Lip}_c(O,F),\ F\subseteq X\ \textit{fin. dim.},\ \|\varphi\|_\infty\leq1\right\},$$

where

$$M^{*,F}\varphi := \sum_{j=1}^{m}\langle RD\varphi, e_j\rangle - \varphi v_{e_j}, \tag{5.6}$$

and $\{e_1, \ldots, e_m\}$ *is an orthonormal basis of* F.

Lemma 5.8 *If* $O = X$*, then:*

1. (5.5) *implies* (5.4) *for any* $z \in X$ *and* $f \in L^p(X, \nu)$ *with* $p \in [1, +\infty)$*;*
2. (5.4) *implies* (5.5) *for any* $z \in X$ *and* $f \in L^p(X, \nu)$ *with* $p \in (1, +\infty)$*.*

Proof.

To prove that (5.5) implies (5.4), it is enough to argue as in the first part of the proof of Lemma 2.17.

Let $f \in L^p(X, \nu)$ for some $p > 1$, we prove that (5.4) gives (5.5). Let $G \in \mathrm{Lip}_c(X)$, let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of X and let $G \in \mathrm{Lip}_c(X)$. Similarly to what is done in the proof of [9, Lemma 2.1(ii)], for any $n, k \in \mathbb{N}$ we set

$$G_{n,k}(x) := \int_{\mathbb{R}^n} G\left(P_n x + \frac{1}{k}\sum_{j=1}^{n}\xi_j e_j\right)\rho_n(\xi)d\xi,$$

where $P_n x := \sum_{j=1}^n \langle x, e_j\rangle e_j$ for any $x \in X$ and ρ_n is any nonnegative smooth function supported in the unit ball of $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\rho_n(\xi)d\xi = 1$. We infer that $G_{n,k} \in \mathcal{F}C^1_b(X)$ and

$$\lim_{k\to+\infty}\lim_{n\to+\infty} G_{n,k}(x) = G(x),\ \forall x \in X, \quad \|G_{n,k}\|_\infty \le \|G\|_\infty,$$
$$\|DG_{n,k}\|_{L^\infty(X,X)} \le \|\nabla G\|_{L^\infty(X,X)}, \quad n, k \in \mathbb{N}.$$

Then, $(G_{n,k})$ is bounded in $\mathbb{D}^{1,p'}(X, \nu)$, and it follows that, up to a subsequence, $RDG_{n,k}$ weakly converges to a function $\Psi \in L^{p'}(X, \nu; X)$. By [13, Chapter IV, Theorem 1.1] $L^{p'}(X, \nu; X)$ is the dual of $L^p(X, \nu; X)$, hence, for any $z \in X$ and any $g \in C^1_b(X)$ we have

$$\int_X g\langle R\nabla G, z\rangle d\nu = \int_X (\langle M_p g, z\rangle - g v_z)G d\nu = \lim_{k,n\to+\infty}\int_X (\langle M_p g, z\rangle - g v_z)G_{k,n})d\nu$$
$$= \lim_{k,n\to+\infty}\int_X g\langle RDG_{k,n}, z\rangle d\nu = \int_X g\langle \Psi, z\rangle d\nu.$$

Thanks to [9, Lemma 2.1(ii)] it follows that

$$\int_X g\langle R\nabla G, z\rangle d\nu = \int_X g\langle \Psi, z\rangle d\nu,$$

holds true for any $g \in C_b(X)$. If we define the measures $\nu_1 := \langle R\nabla G, z\rangle\nu$ and $\nu_2 := \langle \Psi, z\rangle\nu$, above arguments imply that

$$\int_X g d\nu_1 = \int_X g d\nu_2, \quad g \in C_b(X),$$

which means that $\nu_1 = \nu_2$. Therefore,

$$\int_X f d\nu_1 = \int_X f d\nu_2,$$

and we deduce that, up to a subsequence,

$$\int_X f(\langle RDG_{n,k}, z\rangle - G_{n,k}v_z)d\nu \to \int_X f(\langle \Psi, z\rangle - Gv_z)d\nu \int_X f(\langle R\nabla G, z\rangle - Gv_z)d\nu.$$

Repeating this arguments for any $z \in X$ we get the thesis. QED

We want to prove the equivalence of $V_\nu(f, O) < +\infty$ and $f \in BV(O, \nu)$. As usual, by standard arguments (see e.g. Remark 3.8) we infer that $f \in BV(O, \nu)$ gives $V_\nu(f, O) < +\infty$ and $V_\nu(f, O) \le |D^O_\nu f|(O)$. To show that if $V(f, O) < +\infty$ then $f \in BV(O, \nu)$ and $|D^O_\nu f|(O) \le V_\nu(f, O)$ we first prove that, if $V^z_\nu(f, O) < +\infty$, then there exists a Borel measure μ_z which satisfies (5.5).

Proposition 5.9 *Let $z \in X$ be such that $R^*z \neq 0$ and let $f \in L^1(O,\nu)$ be such that $fv_z \in L^1(O,\nu)$ and $V_\nu^z(f,O) < +\infty$. Then, there exists a Borel measure μ_z such that (5.5) is satisfied.*

Proof. Let $f \in L^1(O,\nu)$ be such that $fv_z \in L^1(O,\nu)$. Further, let $g \in \mathrm{Lip}_c(O)$ such that $\|g\|_\infty \leq 1$. Clearly, $\overline{f}g \in L^1(X,\nu)$ and $(\overline{f}g)v_z \in L^1(X,\nu)$. Further, for any $G \in C_b^1(X)$ with $\|G\|_\infty \leq 1$, we have $gG \in \mathrm{Lip}_c(O)$ and $\nabla(gG) = G\nabla g + gDG$. Hence,

$$\begin{aligned}\int_X (\overline{f}g)(\langle RDG, z\rangle - Gv_z)d\nu =& \int_X \overline{f}(\langle R\nabla(Gg), z\rangle - (Gg)v_z)d\nu - \int_X \overline{f}G\langle R\nabla g, z\rangle d\nu \\ \leq& V_\nu^z(f,O) + \|R\|_{\mathcal{L}(X)}\|\nabla g\|_\infty \|f\|_{L^1(X,\nu)}.\end{aligned}$$

Hence, $V_\nu^z(\overline{f}g) < +\infty$ and from [9, Theorem 3.3] there exists a measure m_z which satisfies (5.5) and $|m_z|(X) = V_z(\overline{f}g)$. Arguing as in Theorem 3.5, it is easy to build a measure μ_z as limit for $n \to +\infty$ of $m_z \llcorner A_n$, where (A_n) is a suitable increasing sequence of open subsets of O, such that (5.5) is fulfilled. QED

We are ready to state the main theorem of this section, which is a characterization of the space $BV(O,\nu)$ in terms of the variation V_ν.

Theorem 5.10 *Let $f \in L^p(O,\nu)$ for some $p > 1$. Then, $f \in BV(O,\nu)$ if and only if $V_\nu(f,O) < +\infty$. In this case, $|D_\nu^O f|(O) = V_\nu(f,O)$.*

Proof. The proof follows the arguments in [9, Theorem 3.5], hence we only give a brief sketch.

The fact that $f \in BV(O,\nu)$ implies $V_\nu(f,O) < +\infty$ is standard, hence we limit ourselves to prove the converse implication.

Let $f \in L^1(O,\nu)$ be such that $V_\nu(f,O) < +\infty$. Therefore, for any $z \in X$ we have $V_\nu^z(f,O) < +\infty$. Let us fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of X and for any $n \in \mathbb{N}$ let us set $\mu_n := \mu_{e_n}$, where μ_{e_n} is the measure constructed in Proposition 5.9. We claim that

$$m(B) := \sum_{n=1}^{+\infty} \mu_n(B)e_n, \quad B \in \mathcal{B}(O), \tag{5.7}$$

is a well defined vector measure belonging to $\mathscr{M}(O,X)$ which satisfies (5.3) and $|m|(X) \leq V_\nu(f,O)$. To this aim, for any $k \in \mathbb{N}$ we define the measure

$$M_k(B) := \sum_{n=1}^{k} \mu_n(B)e_n, \quad B \in \mathcal{B}(O).$$

From (2.1) we have

$$\begin{aligned}|M_k|(O) =& \sup\left\{\int_O \langle\varphi, dM_k\rangle : \varphi \in \mathrm{Lip}_c(O,X),\ |\varphi(x)|_X \leq 1\ \forall x \in O\right\} \\ =& \sup\left\{\int_O \langle\varphi, dM_k\rangle : \varphi \in \mathrm{Lip}_c(O,P_k(X)),\ |\varphi(x)|_X \leq 1\ \forall x \in O\right\},\end{aligned}$$

where P_k is the projection on the subspace generated by $\{e_1, \ldots, e_k\}$. Then, Proposition 5.3 gives

$$\int_O \langle\varphi, dM_k\rangle = \sum_{n=1}^{k} \int_O \varphi d\mu_n = \sum_{n=1}^{k} \int_O f(\langle R\nabla\varphi, e_n\rangle - \varphi v_{e_n})d\nu = \int_O f M^{*,P_k(X)}\varphi d\nu \leq V_\nu(f,O)\|\varphi\|_\infty,$$

for any $\varphi \in \mathrm{Lip}_c(O,P_k(X))$, where $M^{*,P_k(X)}$ has been defined in (5.6). This means that $|M_k|(O) \leq V_\nu(f,O)$ for any $k \in \mathbb{N}$. Classical results in measure theory imply that the series $\sum_{n=1}^{+\infty} \mu_n(B)e_n$ converges for any $B \in \mathcal{B}(O)$, that $m \in \mathscr{M}(O,X)$ and that $|m|(O) \leq V_\nu(f,O)$.

Finally, the validity of (5.5) for $\varphi \in \mathrm{Lip}_c(O)$ follows arguing as in [9]. QED

5.1 Examples

Here we provide some examples of measures which satisfy both Hypotheses 5.1 and 5.2.

Weighted Gaussian measure. We consider a weighted Gaussian measure

$$\nu(dx) = \frac{e^{-U(x)}}{\int_X e^{-U} d\gamma}\gamma(dx),$$

where γ is a Gaussian measure on X and U satisfies the following assumptions.

Hypothesis 5.11 *$U : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function such that $U \in \mathbb{D}^{1,q}(X, \gamma)$ for any $q > 1$.*

Here we take $R = Q^{1/2}$. Under Hypothesis 5.11 it easy to see that Hypothesis 5.1 is satisfied, for any $z \in X$ we have $v_z = \widehat{h} + \langle Q^{1/2}DU, z\rangle$, where $h = Q^{1/2}z$ and $v_z \in L^q(X, \nu)$ for any $z \in X$ and any $q \geq 1$ (for a deep study of Banach space endowed with weighted Gaussian measure see [14]). Further, Hypothesis 5.2 is fulfilled since ν is absolutely continuous with respect to the Gaussian measure γ and [5, Theorem 5.11.2(ii)] implies that any Lipschitz function is Gâteaux differentiable γ-a.e. in X. Hence, Theorem 5.10 reads as follows.

Theorem 5.12 *Let $O \subseteq X$ be an open subset of X and let $p > 1$. Then, $f \in L^p(X, \nu)$ belongs to $BV(O, \nu)$ if and only if $V_\nu(f, O) < +\infty$. In this case for any $\varphi \in \mathrm{Lip}_c(O)$ we have*

$$\int_O f(\langle Q^{1/2}D\varphi, z\rangle - \varphi\widehat{h} - \varphi\langle Q^{1/2}DU, z\rangle)d\nu = -\int_X \varphi dm_z,$$

for any $z \in X$, where $h = Q^{1/2}z$.

A non Gaussian product measure. From [9, Subsection 5.2] and [10] we can consider a non Gaussian example of measure ν which satisfies our assumptions. For $m \geq 1$ and $\mu > 0$ we consider the measure

$$\nu_\mu(d\xi) := a\mu^{-\frac{1}{2m}} e^{-\frac{|\xi|^{2m}}{2m}} d\xi, \quad \xi \in \mathbb{R},$$

where $a := (2m)^{1-1/(2m)}/\Gamma(1/(2m))$ is a normalizing factor such that $\nu_\mu(\mathbb{R}) = 1$. We consider a sequence of positive numbers μ_j, $j \in \mathbb{N}$, such that

$$\sum_{j\in\mathbb{N}} \mu_j^{\frac{1}{m}} < +\infty,$$

which gives that the product measure on $\mathbb{R}^{\mathbb{N}}$ defined by

$$\nu := \prod_{j\in\mathbb{N}} \nu_{\mu_j},$$

is well defined and concentrated on ℓ^2 (space of sequences with Euclidean norm). We set $X = L^2(0, 1)$, we fix an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of X consisting of equibounded functions and we consider the standard isomorphism from X to $\mathbb{R}^{\mathbb{N}}$, $x \mapsto (x_k)$, where $x_k = \langle x, e_k\rangle$ for any $k \in \mathbb{N}$. The induced measure is still called ν, and in [10] it has been proved that Hypothesis 5.1 is satisfied with $R = Q^{1/2}$, where Q is the covariance operator of ν, i.e.,

$$Qe_j = b_1\mu_j^{\frac{1}{m}} e_j, \quad j \in \mathbb{N}, \quad b_1 = (2m)^{\frac{1}{m}} \frac{\Gamma\left(\frac{3}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}.$$

Finally, we show that the measure ν enjoys the property in Hypothesis 5.2. Let $G \in \mathrm{Lip}(X)$ and, for any $n \in \mathbb{N}$ and let us consider the n-dimensional subspace X_n of X generated by $\{e_1, \ldots, e_n\}$ and its orthogonal complement $X_n^\perp$. For any $y \in X_n^\perp$, on the finite dimensional affine spaces $y + X_n$ we can choose the product measure

$$\nu^n := \prod_{i=1}^{n} \nu_{\mu_i},$$

which is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^n$. Therefore, thanks to the finite dimensional Rademacher Theorem we infer that $D_{e_k}G$ exists ν-a.e. for any $k = 1, \ldots, n$, where D_{e_k} denotes the Gâteaux derivative along e_k. We conclude by proceeding as in [5, Theorem 5.11.2(ii)].

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