Quantitative analysis of a singularly perturbed shape optimization problem in a polygon

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Abstract

We carry on our study of the connection between two shape optimization problems with spectral cost. On the one hand, we consider the optimal design problem for the survival threshold of a population living in a heterogenous habitat Ω; this problem arises when searching for the optimal shape and location of a shelter zone in order to prevent extinction of the species. On the other hand, we deal with the spectral drop problem, which consists in minimizing a mixed Dirichlet-Neumann eigenvalue in a box Ω. In a previous paper [12] we proved that the latter one can be obtained as a singular perturbation of the former, when the region outside the refuge is more and more hostile. In this paper we sharpen our analysis in case Ω is a planar polygon, providing quantitative estimates of the optimal level convergence, as well as of the involved eigenvalues.

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1 Introduction

In this note we investigate some relations between the two following shape optimization problems, settled in a box $\Omega \subset \mathbb{R}^N$, that is, a bounded, Lipschitz domain (open and connected).

Definition 1.1. Let $0 < \delta < |\Omega|$ and $\beta > \dfrac{\delta}{|\Omega| - \delta}$. For any measurable $D \subset \Omega$ such that $|D| = \delta$, we define the *weighted eigenvalue*

$$\lambda(\beta, D) := \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_D u^2 \, dx - \beta \int_{\Omega \setminus D} u^2 \, dx} : u \in H^1(\Omega),\ \int_D u^2 \, dx > \beta \int_{\Omega \setminus D} u^2 \, dx \right\},$$

and the *optimal design problem for the survival threshold as*

$$\Lambda(\beta, \delta) = \min \left\{ \lambda(\beta, D) : D \subset \Omega,\ |D| = \delta \right\}.$$

Definition 1.2. Let $0 < \delta < |\Omega|$. Introducing the space $H_0^1(D, \Omega) := \left\{ u \in H^1(\Omega) : u = 0 \text{ q.e. on } \Omega \setminus D \right\}$ (where q.e. stands for quasi-everywhere, i.e. up to sets of zero capacity), we can define, for any quasi-open $D \subset \Omega$ such that $|D| = \delta$, the *mixed Dirichlet-Neumann eigenvalue* as

$$\mu(D, \Omega) := \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} : u \in H_0^1(D, \Omega) \setminus \{0\} \right\},$$

and the *spectral drop problem* as

$$\mathrm{M}(\delta) = \min\Big\{\mu(D,\Omega) : D \subset \Omega,\ \text{quasi-open},\ |D| = \delta\Big\}.$$

The two problems above have been the subject of many investigations in the literature. The interest in the study of the eigenvalue $\lambda(\beta, D)$ goes back to the analysis of the optimization of the survival threshold of a species living in a heterogenous habitat Ω, with the boundary $\partial\Omega$ acting as a reflecting barrier. As explained by Cantrell and Cosner in a series of paper [3, 4, 5] (see also [11, 9, 12]), the heterogeneity of Ω makes the intrinsic growth rate of the population, represented by a $L^\infty(\Omega)$ function $m(x)$, be positive in favourable sites and negative in the hostile ones. Then, if $m^+ \not\equiv 0$ and $\int m < 0$, it turns out that the positive principal eigenvalue $\lambda = \lambda(m)$ of the problem

$$\begin{cases} -\Delta u = \lambda m u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e.

$$\lambda(m) = \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega m u^2 dx} : u \in H^1(\Omega), \int_\Omega m u^2 dx > 0 \right\},$$

acts a survival threshold, namely the smaller $\lambda(m)$ is, the greater the chances of survival become. Moreover, by [11], the minimum of $\lambda(m)$ w.r.t. m varying in a suitable class is achieved when m is of bang-bang type, i.e. $m = \mathbb{1}_D - \beta\mathbb{1}_{\Omega\setminus D}$, being $D \subset \Omega$ with fixed measure. As a consequence, one is naturally led to the shape optimization problem introduced in Definition 1.1.

On the other hand, the spectral drop problem has been introduced in [2] as a class of shape optimization problems where one minimizes the first eigenvalue $\mu = \mu(D, \Omega)$ of the Laplace operator with homogeneous Dirichlet conditions on $\partial D \cap \Omega$ and homogeneous Neumann ones on $\partial D \cap \partial\Omega$:

$$\begin{cases} -\Delta u = \mu u & \text{in } D \\ u = 0 & \text{on } \partial D \cap \Omega \\ \partial_\nu u = 0 & \text{on } \partial D \cap \partial\Omega. \end{cases}$$

In our paper [12], we analyzed the relations between the above problems, showing in particular that $\mathrm{M}(\delta)$ arises from $\Lambda(\beta, \delta)$ in the singularly perturbed limit $\beta \to +\infty$, as stated in the following result.

Theorem 1.3 ([12, Thm. 1.4, Lemma 3.3])**.** *If* $0 < \delta < |\Omega|$, $\beta > \dfrac{\delta}{|\Omega| - \delta}$ *and* $\dfrac{\delta}{\beta} < \varepsilon < |\Omega| - \delta$ *then*

$$\mathrm{M}(\delta + \varepsilon)\left(1 - \sqrt{\frac{\delta}{\varepsilon\beta}}\right)^2 \le \Lambda(\beta, \delta) \le \mathrm{M}(\delta).$$

As a consequence, for every $0 < \delta < |\Omega|$,

$$\lim_{\beta\to+\infty} \Lambda(\beta, \delta) = \mathrm{M}(\delta).$$

In respect of this asymptotic result, let us also mention [8], where the relation between the above eigenvalue problems has been recently investigated for $D \subset \Omega$ fixed and regular.

In [12], we used the theorem above to transfer information from the spectral drop problem to the optimal design one. In particular, we could give a contribution in the comprehension of the shape of an optimal set D^* for $\Lambda(\beta, \delta)$. This topic includes several open questions starting from the analysis performed in [4] (see also [9, 11]) when $\Omega = (0, 1)$: in this case it is shown that any optimal set D^* is either $(0, \delta)$ or $(1 - \delta, 1)$. The knowledge of analogous features in the higher dimensional case is far from being well understood, but it has been recently proved in [9] that when Ω is an N-dimensional rectangle, then ∂D^* does not contain any portion of sphere, contradicting previous conjectures and numerical studies [1, 15, 7]. This result prevents the existence of optimal *spherical shapes*, namely optimal D^* of the form $D^* = \Omega \cap B_{r(\delta)}(x_0)$ for suitable x_0 and $r(\delta)$ such that $|D^*| = \delta$.

On the other hand, we have shown that spherical shapes are optimal for $\mathrm{M}(\delta)$, for small δ, when Ω is an N-dimensional polytope. This, together with Theorem 1.3, yields the following result.

Theorem 1.4 ([12, Thm. 1.7])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded, convex polytope. There exists $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$:*

- *D^* is a minimizer of the spectral drop problem in Ω, with volume constraint δ, if and only if $D^* = B_{r(\delta)}(x_0) \cap \Omega$, where x_0 is a vertex of Ω with the smallest solid angle;*
- *if $|D| = \delta$ and D is not a spherical shape as above, then, for β sufficiently large,*

$$\lambda(\beta, D) > \lambda(\beta, B_{r(\delta)}(x_0) \cap \Omega).$$

In particular, in case $\Omega = (0, L_1) \times (0, L_2)$, with $L_1 \leq L_2$, and $0 < \delta < L_1^2/\pi$, then any minimizing spectral drop is a quarter of a disk centered at a vertex of Ω.

Then, even though the optimal shapes for $\Lambda(\beta, \delta)$ can not be spherical for any fixed β, they are asymptotically spherical as $\beta \to +\infty$, at least in the qualitative sense described in Theorem 1.4.

The main aim of the present note is to somehow revert the above point of view: we will show that, in case $\mathrm{M}(\delta)$ is explicit as a function of δ, one can use Theorem 1.3 in order to obtain quantitative bounds on the ratio

$$\frac{\Lambda(\beta, \delta)}{\mathrm{M}(\delta)}.$$

In particular, we will pursue this program in case Ω is a planar polygon: indeed, on the one hand, in such case the threshold $\bar{\delta}$ in Theorem 1.4 can be estimated explicitly; on the other hand, such theorem implies that the optimal shapes for $\mathrm{M}(\delta)$ are spherical, so that $\mathrm{M}(\delta)$ can be explicitly computed. This will lead to quantitative estimates about the convergence of $\Lambda(\beta, \delta)$ to $\mathrm{M}(\delta)$.

As a byproduct of this analysis, we will also obtain some quantitative information on the ratio

$$\frac{\lambda(\beta, B_{r(\delta)}(p) \cap \Omega)}{\Lambda(\beta, \delta)},$$

thus providing a quantitative version of the second part of Theorem 1.4.

These new quantitative estimates are the main results of this note, and they are contained in Theorems 2.2 and 2.3, respectively. The next section is devoted to their statements and proofs, together with further details of our analysis.

2 Setting of the problem and main results.

Let $\Omega \subset \mathbb{R}^2$ denote a convex n-gon, $n \geq 3$. We introduce the following quantities and objects, all depending on Ω:

- $\alpha_{\min}$ is the smallest interior angle;
- $\mathcal{V}_{\min}$ is the set of vertices having angle $\alpha_{\min}$;

- $e_1, \dots, e_n$ are the (closed) edges;
- d denotes the following quantity:

$$d = \min\{\mathrm{dist}(e_i \cap e_j, e_k) : i \neq j,\ i \neq k,\ j \neq k\}.$$

Under the above notation, we define the threshold

$$\bar{\delta} := \frac{d^2}{2\alpha_{\min}}. \tag{1}$$

Remark 2.1. Notice that, as far as $n \geq 4$, d corresponds to the shortest distance between two non-consecutive edges:

$$d = \min\{\mathrm{dist}(x_i, x_j) : x_i \in e_i,\ x_j \in e_j,\ e_i \cap e_j = \emptyset\}.$$

Moreover, for any n,

$$0 < \bar{\delta} < |\Omega|.$$

Indeed, let $e_i \cap e_j \in \mathcal{V}_{\min}$, with $|e_i| \leq |e_j|$. Then

$$d \leq |e_i| \sin\alpha_{\min} \qquad \text{and} \qquad |\Omega| \geq \frac{1}{2}|e_i||e_j| \sin\alpha_{\min},$$

and the claim follows since $\sin\alpha_{\min} < \alpha_{\min}$.

Our main results are the following.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ denote a convex n-gon, let $\bar{\delta}$ be defined in (1), and let us assume that*

$$0 < \delta < \bar{\delta}.$$

Then $\mathrm{M}(\delta)$ is achieved by D^* if and only if $D^* = B_{r(\delta)}(p) \cap \Omega$, where $p \in \mathcal{V}_{\min}$. Moreover

$$\beta > \max\left\{\left(\frac{\delta}{\bar{\delta} - \delta}\right)^3, 1\right\} \qquad \Longrightarrow \qquad (1 + \beta^{-1/3})^{-1}\left(1 - \beta^{-1/3}\right)^2 < \frac{\Lambda(\beta, \delta)}{\mathrm{M}(\delta)} < 1.$$

By taking advantage of the asymptotic information on $\Lambda(\beta, \delta)/\,\mathrm{M}(\delta)$, we can deduce the corresponding relation between the eigenvalue of a spherical shape and the minimum $\Lambda(\beta, \delta)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^2$ denote a convex n-gon, $\beta > 1$, and let us assume that*

$$\delta < \frac{\beta^{1/3}}{\beta^{1/3} + 1}\,\bar{\delta},$$

where $\bar{\delta}$ is defined in (1). Then, taking $p \in \mathcal{V}_{\min}$ and $r(\delta)$ such that $|B_{r(\delta)}(p) \cap \Omega| = \delta$,

$$1 < \frac{\lambda(\beta, B_{r(\delta)}(p) \cap \Omega)}{\Lambda(\beta, \delta)} < \left(1 + \beta^{-\frac{1}{3}}\right)\left(1 - \beta^{-\frac{1}{3}}\right)^{-2}.$$

To prove our results, we will use the analysis we developed in [12, Section 4] to estimate $\mathrm{M}(\delta)$ by means of α-symmetrizations on cones [13, 10]. To this aim we will first evaluate a suitable isoperimetric constant.

For $D \subset \Omega$, we write

$$\mathcal{R}(D, \Omega) := \frac{P(D, \Omega)}{2|D \cap \Omega|^{1/2}},$$

where P denotes the relative De Giorgi perimeter. For $0 < \delta < |\Omega|$ we consider the isoperimetric problem

$$I(\Omega, \delta) := \inf \{\mathcal{R}(D, \Omega) : D \subset \Omega, \ |D| = \delta\},$$

and we call

$$K(\Omega, \delta) = \inf_{0<\delta' \leq \delta} I(\Omega, \delta').$$

Given the unbounded cone with angle α,

$$\Sigma_\alpha := \{(r \cos \vartheta, r \sin \vartheta) \in \mathbb{R}^2 : 0 < \vartheta < \alpha, \ r > 0\},$$

it is well known that

$$I(\Sigma_\alpha, \alpha r^2/2) = \mathcal{R}(B_r(0) \cap \Sigma_\alpha, \Sigma_\alpha) = \frac{\alpha r}{2|\alpha r^2/2|^{1/2}} = \sqrt{\frac{\alpha}{2}}, \tag{2}$$

is independent on r, and hence on $\delta = |B_r(0) \cap \Sigma_\alpha|$. As a consequence, also

$$K(\Sigma_\alpha, \delta) = \sqrt{\frac{\alpha}{2}},$$

for every δ.

Lemma 2.4. *If $\Omega \subset \mathbb{R}^2$ is a convex n-gon and $\delta < \bar{\delta}$, then $I(\Omega, \delta)$ is achieved by D^* if and only if $D^* = B_{r(\delta)}(p) \cap \Omega$, where $p \in \mathcal{V}_{\min}$. Moreover $K(\Omega, \bar{\delta})$ is achieved by the same D^* too.*

Proof. Notice that, by assumption, for any $p \in \mathcal{V}_{\min}$ the set $D = B_{r(\delta)}(p) \cap \Omega$ is a circular sector of measure δ, with $\partial D \cap \Omega$ a circular arc. Then (2) implies

$$I(\Omega, \delta) \leq I(\Sigma_{\alpha_{\min}}, \delta) = \sqrt{\frac{\alpha_{\min}}{2}}, \tag{3}$$

and we are left to show the opposite inequality (strict, in case D is not of the above kind). Applying Theorems 4.6 and 5.12 in [14], and Theorems 2 and 3 in [6], we deduce that I is achieved by $D^*_\delta \subset \Omega$, which is an open, connected set, such that $\Gamma := \partial D^*_\delta \cap \Omega$ is either a (connected) arc of circle or a straight line segment. Moreover, $\partial D^*_\delta \cap \partial\Omega$ consists in exactly two points (the endpoints of Γ), and $\partial D^*_\delta \cap \Omega$ reaches the boundary of Ω orthogonally at flat points (i.e. not at a vertex). Hence, there are three possible configurations (see Fig. 1).

A. The endpoints of Γ belong to the interior of two consecutive edges e_i and e_{i+1}. In this case Γ is orthogonal to both e_i and e_{i+1}, and D^*_δ is a portion of a disk centered at $e_i \cap e_{i+1}$. Recalling (2), we deduce that $e_i \cap e_{i+1} \in \mathcal{V}_{\min}$, and the lemma follows.

B. The endpoints of Γ belong to the same edge e_i.

C. The endpoints of Γ belong to two non-consecutive edges.

The rest of the proof will be devoted to show that cases B and C can not occur.

In case B, assume w.l.o.g. that $e_i \subset \{(x, 0) \in \mathbb{R}^2\}$ and $\Omega \subset \{(x, y) \in \mathbb{R}^2 : y \geq 0\} = \Sigma_\pi$. Then $D^*_\delta \cap \Omega = D^*_\delta \cap \Sigma_\pi$, $P(D^*_\delta, \Omega) = P(D^*_\delta, \Sigma_\pi)$, and

$$\mathcal{R}(D^*_\delta, \Omega) \geq I(D^*_\delta, \Sigma_\pi) = \sqrt{\frac{\pi}{2}} > \sqrt{\frac{\alpha_{\min}}{2}},$$

in contradiction with (3).

Finally, in order to rule out configuration C, by definition of d we have

$$\mathcal{R}(D^*_\delta, \Omega) \geq \frac{d}{2\sqrt{|D^*_\delta|}} = \frac{d}{2\sqrt{\delta}} > \sqrt{\frac{\alpha_{\min}}{2}}$$

whenever $\delta < \bar{\delta}$, which is fixed as $d/2\alpha_{\min}$. So that we get again a contradiction concluding the proof.

Finally, the assertion concerning $K(\Omega, \bar{\delta})$ follows by its definition and from the fact that for all $\delta \leq \bar{\delta}$ (see also [12, Corollary 4.3]), we have just showed that $I(\Omega, \delta) = \sqrt{\frac{\alpha}{2}}$ is a constant independent of δ. □

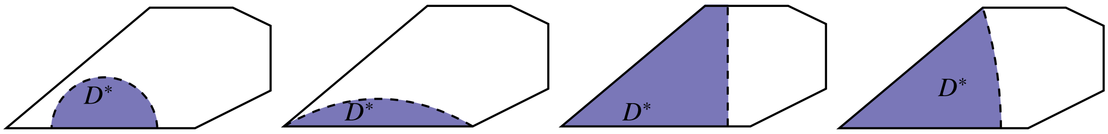

Figure 1: some possibilities for cases B (on the left) and C (on the right) in the proof of Lemma 2.4. The Dirichlet boundary $\partial D^* \cap \Omega$ is dashed.

Remark 2.5. Notice that the threshold $\bar{\delta}$ in Lemma 2.4 has no reason to be optimal. On the other hand, one can easily check that in the case of a rectangle, as treated in Theorem 1.4 it is actually optimal, since, for $\delta > \bar{\delta}$, $I(\Omega, \delta)$ is achieved by a rectangle (see e.g. [12, Remark 4.5]).

We are now in position to prove our main results.

Proof of Theorem 2.2. First of all, we take $\varepsilon \in (\delta/\beta, \bar{\delta} - \delta) \neq \emptyset$ by the assumption on δ and we apply [12, Corollary 4.3] and Lemma 2.4 to deduce that

$$\begin{aligned} \mathrm{M}(\delta) &= K^2(\Omega, \delta)\delta^{-1}\lambda_1^{\mathrm{Dir}} = \alpha_{\min}(2\delta)^{-1}\lambda_1^{\mathrm{Dir}} \\ \mathrm{M}(\delta+\varepsilon) &= K^2(\Omega, \delta+\varepsilon)(\delta+\varepsilon)^{-1}\lambda_1^{\mathrm{Dir}} = \alpha_{\min}[2(\delta+\varepsilon)]^{-1}\lambda_1^{\mathrm{Dir}}, \end{aligned}$$

where λ_1^{Dir} stands for the first eigenvalue of the Dirichlet-Laplacian in the ball of unit radius. By Theorem 1.3 we obtain

$$1 \geq \frac{\Lambda(\beta,\delta)}{\mathrm{M}(\delta)} \geq \frac{\mathrm{M}(\delta+\varepsilon)}{\mathrm{M}(\delta)}\left(1 - \sqrt{\frac{\delta}{\varepsilon\beta}}\right)^2 = \frac{\delta}{\delta+\varepsilon}\left(1 - \sqrt{\frac{\delta}{\varepsilon\beta}}\right)^2,$$

for all $\varepsilon \in (\delta/\beta, \bar{\delta} - \delta)$. Then we make the choice of $\varepsilon = \delta/\beta^{1/3}$, which is admissible since $\beta > 1$ and $\delta < \beta^{1/3}\bar{\delta}/(1+\beta^{1/3})$, and obtain

$$1 \geq \frac{\Lambda(\beta,\delta)}{\mathrm{M}(\delta)} \geq \frac{1}{1+\beta^{-1/3}}\left(1 - \beta^{-1/3}\right)^2,$$

yielding the conclusion. □

Proof of Theorem 2.3. Calling $D^* = B_{r(\delta)}(p) \cap \Omega$, for some $p \in \mathcal{V}_{\min}$ and using conclusion 2 of [12, Lemma 3.1], we infer that $\lambda(\beta, D^*) \leq \mu(D^*, \Omega)$. As a consequence we can use Theorem 2.2 to write

$$1 \leq \frac{\lambda(\beta, D^*)}{\Lambda(\beta,\delta)} \leq \frac{M(\delta)}{\Lambda(\beta,\delta)} \leq (1+\beta^{-1/3})\left(1 - \beta^{-1/3}\right)^{-2}. \quad \square$$

Remark 2.6. The estimate of Theorem 2.3 can be read as,

$$1 \leq \frac{\lambda(\beta, D^*)}{\Lambda(\beta, \delta)} \leq 1 + 3\beta^{-1/3} + o(\beta^{-1/3}), \qquad \text{as } \beta \to \infty.$$

On the other hand, even without using asymptotic expansions, as β increases, the estimate becomes more precise. As an example, for all $\beta > 8$, one has the explicit estimate

$$1 \leq \frac{\lambda(\beta, D^*)}{\Lambda(\beta, \delta)} \leq 1 + 15\beta^{-1/3} + 14\beta^{-2/3}.$$

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