PAIRINGS BETWEEN BOUNDED DIVERGENCE–MEASURE VECTOR FIELDS AND BV FUNCTIONS

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Abstract. We introduce a family of pairings between a bounded divergence-measure vector field $\boldsymbol{A}$ and a function u of bounded variation, depending on the choice of the pointwise representative of u. We prove that these pairings inherit from the standard one, introduced in [6, 10], all the main properties and features (e.g. coarea, Leibniz and Gauss–Green formulas). We also characterize the pairings making the corresponding functionals semicontinuous with respect to the strict convergence in BV. We remark that the standard pairing in general does not share this property.

Contents

1. Introduction

In the seminal papers [6, 10], the product rule

$$\operatorname{div}(u\boldsymbol{A}) = u \operatorname{div} \boldsymbol{A} + \boldsymbol{A} \cdot \nabla u\,, \tag{1}$$

for smooth functions u and regular vector fields $\boldsymbol{A}$ in $\mathbb{R}^N$, has been suitably extended to BV functions and bounded divergence-measure vector fields. In particular, Chen and Frid [10] showed, using a regularization argument, that there exists a finite Radon measure $(\boldsymbol{A}, Du)_*$, which coincides to $\boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N$ in the smooth case, such that the relation

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)_* \tag{2}$$

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holds in the sense of measures. The measure $(\boldsymbol{A}, Du)_*$, usually called Anzellotti's pairing and that we call in the sequel the *standard pairing* between $\boldsymbol{A}$ and Du, is then defined in terms of the precise representative u^* of u, which is the pointwise value of u obtained as limit of regularizations by convolutions.

The standard pairing turns out to be a basic tool in many applications. We mention here, among others: extensions of the Gauss–Green formula [6, 8, 9, 13–17, 19, 31]; the setting of the Euler–Lagrange equations associated with integral functionals defined in BV [4, 32, 33]; Dirichlet problems for equations involving the 1–Laplace operator [5, 8, 21, 22, 26, 27]; conservation laws [10–14, 18]; the Prescribed Mean Curvature problem and capillarity [30, 31]; continuum mechanics [9, 23, 37, 38].

On the other hand, the standard pairing is not adequate when dealing with obstacle problems in BV (see [34–36]) or with semicontinuity properties, as we will explain below. The aim of this paper is to introduce a new family of pairings, depending on the choice of the pointwise representative of u, suitable to treat this kind of problems.

The main ingredients to build this family of pairings are the absolute continuity of the measure $\operatorname{div} \boldsymbol{A}$ with respect to the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$, and the fact that the pointwise value of a BV function can be specified up to a $\mathcal{H}^{N-1}$-negligible set. Indeed, a BV function u is approximately continuous outside a singular set S_u and its approximate upper and lower limits u^+ and u^- coincide with the traces of u on the countably $\mathcal{H}^{N-1}$-rectifiable jump set $J_u \subset S_u$, with $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ (see Section 2.2). Hence, a representative of u can be defined by its approximate limit $\tilde{u}$ outside S_u and through its traces $u^\pm$ on J_u. We remark again that the presence of $u^* := (u^+ + u^-)/2$ in (2) as the pointwise representative of u is due to the regularization argument used in [10] in order to define the standard pairing.

Recently, Scheven and Schmidt [34–36] have been in need to introduce the pairing

$$(\boldsymbol{A}, Du)_1 := -u^+ \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) \tag{3}$$

in order to study weakly 1-superharmonic functions and minimization problems for the total variation with an obstacle. Indeed, in this case, the presence of the representative u^+ comes out from (1) using the one-sided approximation procedure of u introduced in [7].

In this paper we prove that, for every Borel function $\lambda \colon \mathbb{R}^N \to [0, 1]$, there exists a measure $(\boldsymbol{A}, Du)_\lambda$ such that

$$\operatorname{div}(u\boldsymbol{A}) = u^\lambda \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)_\lambda\,, \tag{4}$$

where $u^\lambda := (1-\lambda)u^- + \lambda u^+$ is a selection of the multifunction $x \mapsto [u^-(x), u^+(x)]$. We show that, if the jump part $\operatorname{div}^j \boldsymbol{A}$ of $\operatorname{div} \boldsymbol{A}$ vanishes (see Proposition 2.3 for the definition), then $(\boldsymbol{A}, Du)_\lambda$ is independent of λ.

We show that this freedom in the choice of u^λ is necessary in order to obtain semicontinuity results in BV for the functionals

$$F_\varphi(u) := \langle (\boldsymbol{A}, Du)_\lambda\,, \varphi \rangle\,, \qquad \varphi \in C_c(\mathbb{R}^N), \quad \varphi \geq 0\,. \tag{5}$$

We characterize the selections λ such that these functionals are lower (resp. upper) semicontinuous with respect to the strict convergence in BV. More precisely, denoting by $(\operatorname{div} \boldsymbol{A})^\pm$ the positive and the negative part of the measure $\operatorname{div} \boldsymbol{A}$, the choices of λ which guarantee the lower semicontinuity of the functionals in (5) satisfy

$$(\boldsymbol{A}, Du)_\lambda = -u^+ (\operatorname{div} \boldsymbol{A})^+ + u^- (\operatorname{div} \boldsymbol{A})^- + \operatorname{div}(u\boldsymbol{A}), \tag{6}$$

whereas the upper semicontinuity is characterized by

$$(\boldsymbol{A}, Du)_\lambda = -u^- \, (\operatorname{div} \boldsymbol{A})^+ + u^+ \, (\operatorname{div} \boldsymbol{A})^- + \operatorname{div}(u\boldsymbol{A})\,. \tag{7}$$

As a consequence, it is a matter of fact that, in general, the standard pairing does not share these semicontinuity properties. On the other hand, if $\operatorname{div} \boldsymbol{A} \le 0$, as in [34–36], from the above result follows that the pairing (3) is upper semicontinuous with respect to the strict convergence in BV.

The plan of the paper is the following. In Section 2 we recall some known results on BV functions, divergence-measure vector fields and their weak normal traces. In Section 3 we focus our attention on the summability of u^λ with respect to the measure $|\operatorname{div} \boldsymbol{A}|$ and on some related properties of the truncated functions. In Sections 4, 5 and 6 we introduce the generalized pairing and we prove that it inherits from the standard one all the main properties and features. More precisely, $(\boldsymbol{A}, Du)_\lambda$ is a Radon measure, absolutely continuous with respect to $|Du|$, it satisfies the coarea, the chain rule and the Leibniz formulas, and it is consistent with the Gauss–Green formula.

The proofs of these results are based on the analogous properties valid for the standard pairing (see [19]), the fact that the generalized pairing differs from the standard one only by a term concentrated on J_u (see (20)), and some representation results of the normal traces of $\boldsymbol{A}$ on J_u (see [1]).

Our main application of the above theory is proposed in Section 7, where we consider the semicontinuity properties of the functionals F_φ defined in (5), with respect to the strict convergence in BV. In Theorem 7.6 we prove the characterizations (6)–(7) of the semicontinuous pairings. The proof is based on a recent result of Lahti (see [28]), which assures the lower (upper) semicontinuity of the lower u^- (upper u^+) limit under the strict convergence in BV, combined with the one-sided approximation result in [7], and a very careful treatment of the jump part of the measure $\operatorname{div} \boldsymbol{A}$. We show by easy examples that no semicontinuity property has to be expected with respect to the weak* convergence in BV.

2. Notation and preliminary results

In the following Ω will always denote a nonempty open subset of $\mathbb{R}^N$. For every $E \subset \Omega$, χ_E denotes its characteristic function. We say that E_h converges to E if χ_{E_h} converges to χ_E in $L^1(\Omega)$.

We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the Lebesgue measure and the $(N-1)$–dimensional Hausdorff measure in $\mathbb{R}^N$, respectively.

If $E \subset \mathbb{R}^N$ is an open set, the notation $\varphi \nearrow \chi_E$ denotes any family (φ_j) of smooth functions with support in E, such that $0 \le \varphi_j \le 1$, and $\lim_j \varphi_j(x) = 1$ for every $x \in E$.

Given an $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, For every $t \in [0,1]$ we denote by E^t the set

$$E^t := \left\{ x \in \mathbb{R}^N : \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} = t \right\}$$

of all points where E has density t. The sets E^0, E^1, $\partial^e E := \mathbb{R}^N \setminus (E^0 \cup E^1)$ are called respectively the *measure theoretic exterior*, the *measure theoretic interior* and the *essential boundary* of E.

A $\mathcal{H}^{N-1}$-measurable set $E \subset \mathbb{R}^N$ is *countably $\mathcal{H}^{N-1}$-rectifiable* if there exist countably many C^1 graphs $(\Sigma_i)_{i \in \mathbb{N}}$ such that $\mathcal{H}^{N-1}(E \setminus \bigcup_i \Sigma_i) = 0$.

2.1. **Measures.** The space of all Radon measures on Ω will be denoted by $\mathcal{M}(\Omega)$.

Given $\mu \in \mathcal{M}(\Omega)$, its *total variation* $|\mu|$ is the nonnegative Radon measure defined by

$$|\mu|(E) := \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)|:\ E_h\ \mu\text{-measurable sets, pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h\right\},$$

for every μ-measurable set E and its *positive* and *negative parts* are defined, respectively, by

$$\mu^+ := \frac{|\mu| + \mu}{2}, \qquad \mu^- := \frac{|\mu| - \mu}{2}.$$

Given $\mu \in \mathcal{M}(\Omega)$ and a μ-measurable set E, the restriction $\mu \llcorner E$ is the Radon measure defined by

$$\mu \llcorner E(B) = \mu(E \cap B), \qquad \forall\ B\ \mu\text{-measurable, } B \subset \Omega.$$

We recall the following property (see [3], Proposition 2.56 and formula (2.41)):

$$E \subset \Omega,\ |\mu|(E) = 0 \quad \Longrightarrow \quad |\mu|(B_r(x)) = o(r^{N-1}) \text{ for } \mathcal{H}^{N-1}\text{–a.e. } x \in E. \tag{8}$$

Given a nonnegative Radon measure ν, we say that $\mu \in \mathcal{M}(\Omega)$ is *absolutely continuous* with respect to ν (and we write $\mu \ll \nu$), if $|\mu|(B) = 0$ for every set B such that $\nu(B) = 0$.

We say that two positive measures ν_1, $\nu_2 \in \mathcal{M}(\Omega)$ are *mutually singular* (and we write $\nu_1 \perp \nu_2$) if there exists a Borel set E such that $|\nu_1|(E) = 0$ and $|\nu_2|(\Omega \setminus E) = 0$.

By the Radon–Nikodým theorem, given a nonnegative Radon measure ν, every $\mu \in \mathcal{M}(\Omega)$ can be uniquely decomposed as $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, and there exists a unique function (called the density of μ with respect to ν) $\psi_\nu \in L^1(\Omega, \nu)$ such that $\mu_1 = \psi_\nu \nu$. In particular, since $\mu \ll |\mu|$, then there exists $\psi \in L^1(\Omega, |\mu|)$, with $|\psi| = 1$ $|\mu|$–a.e. in Ω, and such that $\mu = \psi|\mu|$. This is usually called the *polar decomposition* of μ.

The following lemma shows the relation between the densities of μ and $|\mu|$, where μ is a Radon measure absolutely continuous with respect to $\mathcal{H}^{N-1}$.

Lemma 2.1. *Let $\mu \ll \mathcal{H}^{N-1}$ be a Radon measure in Ω, and let $\mu = \psi|\mu|$ be its polar decomposition. Then there exists a Borel set $Z \subset \Omega$, with $|\mu|(Z) = 0$, such that, for every $x \in \Omega \setminus Z$,*

$$\exists \lim_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^{N-1}} = L \in \mathbb{R} \quad \Longleftrightarrow \quad \exists \lim_{r \searrow 0} \frac{\mu(B_r(x))}{r^{N-1}} = \psi(x)\, L. \tag{9}$$

Proof. Let $A \subset \Omega$ be the set of Lebesgue points of ψ with respect to $|\mu|$ where $|\psi| = 1$. Clearly, we have that $|\mu|(\Omega \setminus A) = 0$. Moreover, from [3, Theorem 2.56 and (2.40)], the set

$$Z_1 := \left\{x \in \Omega :\ \limsup_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^{N-1}} = +\infty\right\}$$

has zero $\mathcal{H}^{N-1}$-measure, hence also $|\mu|(Z_1) = 0$.

If we set $Z := (\Omega \setminus A) \cup Z_1$, then $|\mu|(Z) = 0$ and (9) holds in $\Omega \setminus Z$. Namely, given $x \in \Omega \setminus Z$, $B_r(x) \subset \Omega$ and $\varphi \in C_c(\Omega)$ with support in $B_r(x)$, since $|\psi(x)| = 1$, we have that $|1 - \psi(y)\psi(x)| = |\psi(y) - \psi(x)|$, and hence

$$\begin{aligned}\left|\int_\Omega \varphi\, d|\mu| - \psi(x) \int_\Omega \varphi\, d\mu\right| &= \left|\int_{B_r(x)} \varphi(y)[1 - \psi(y)\psi(x)]\, d|\mu|(y)\right| \\ &\leq \|\varphi\|_\infty |\mu|(B_r(x)) \ -\!\!\!\!\!\!\int_{B_r(x)} |\psi(y) - \psi(x)|\, d|\mu|(y)\,.\end{aligned}$$

Taking $\varphi \nearrow \chi_{B_r(x)}$ and dividing by r^{N-1} we finally get

$$\left|\frac{|\mu|(B_r(x))}{r^{N-1}} - \psi(x)\,\frac{\mu(B_r(x))}{r^{N-1}}\right| \leq \frac{|\mu|(B_r(x))}{r^{N-1}} \fint_{B_r(x)} |\psi(y)-\psi(x)|\, d|\mu|(y)\,,$$

hence (9) follows because $x \notin Z_1$ and x is a Lebesgue point of ψ. □

Given $\mu \in \mathcal{M}(\Omega)$, we denote by $\mu = \mu^a + \mu^s$ its Lebesgue decomposition in the absolutely continuous part $\mu^a \ll \mathcal{L}^N$ and the singular part $\mu^s \perp \mathcal{L}^N$. We recall a relevant decomposition result for μ^s (see [2], Proposition 5).

Proposition 2.2. *If $\mu \in \mathcal{M}(\Omega)$ is such that $\mu^s \ll \mathcal{H}^{N-1}$, then μ^s can be uniquely decomposed as the sum $\mu^j + \mu^c$, where μ^j, $\mu^c \in \mathcal{M}(\Omega)$ are two mutually singular measures having the following properties:*

(i) *$\mu^c(B) = 0$ for every B such that $\mathcal{H}^{N-1}(B) < +\infty$;*

(ii) *the set*

$$\Theta_\mu := \left\{x \in \Omega \colon \limsup_{r\to 0+} \frac{|\mu|(B_r(x))}{r^{N-1}} > 0\right\}$$

is a Borel set, σ–finite with respect to $\mathcal{H}^{N-1}$;

(iii) *there exists $f \in L^1(\Theta_\mu, \mathcal{H}^{N-1} \llcorner \Theta_\mu)$ such that $\mu^j = f\,\mathcal{H}^{N-1} \llcorner \Theta_\mu$.*

The measures μ^j, μ^c are called jump part and Cantor part of the measure μ, while Θ_μ is called jump set of μ.

2.2. **Functions of bounded variation.** Let $u \colon \Omega \to \mathbb{R}$ be a Borel function. We denote by u^- and u^+ the *approximate lower limit* and the *approximate upper limit* of u, defined respectively by

$$u^+(x) := \inf\{t \in \mathbb{R} : \ \{u > t\} \text{ has density 0 at } x\},$$
$$u^-(x) := \sup\{t \in \mathbb{R} : \ \{u > t\} \text{ has density 1 at } x\}.$$

The function u is *approximately continuous* at $x \in \Omega$ if $u^+(x) = u^-(x)$ and, in this case, we denote by $\widetilde{u}(x)$ the common value.

Given $u \in L^1_{\text{loc}}(\Omega)$, $x \in \Omega$ is a *Lebesgue point* of u (with respect to $\mathcal{L}^N$) if there exists $z \in \mathbb{R}$ such that

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N\,(B_r(x))} \int_{B_r(x)} |u(y) - z|\; dy = 0.$$

In this case, x is a point of approximate continuity, and $z = \widetilde{u}(x)$ (see [24, Proposition 1.163]). We denote by $S_u \subset \Omega$ the set of points where this property does not hold.

We say that $x \in \Omega$ is an *approximate jump point* of u if there exist $a, b \in \mathbb{R}$ and a unit vector $\nu \in \mathbb{R}^n$ such that $a \neq b$ and

$$\begin{aligned} &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^i(x))} \int_{B_r^i(x)} |u(y) - a|\, dy = 0, \\ &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^e(x))} \int_{B_r^e(x)} |u(y) - b|\, dy = 0, \end{aligned} \tag{10}$$

where $B_r^i(x) := \{y \in B_r(x) : \ (y-x)\cdot\nu > 0\}$, and $B_r^e(x) := \{y \in B_r(x) : \ (y-x)\cdot\nu < 0\}$. The triplet (a, b, ν), uniquely determined by (10) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^i(x), u^e(x), \nu_u(x))$. The set of approximate jump points of u will be denoted by J_u.

We say that $u \in L^1(\Omega)$ is a *function of bounded variation* in Ω if the distributional derivative Du of u is a finite Radon measure in Ω. The vector space of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$. Moreover, we will denote by $BV_{\rm loc}(\Omega)$ the set of functions $u \in L^1_{\rm loc}(\Omega)$ that belongs to $BV(A)$ for every open set $A \Subset \Omega$ (i.e., the closure $\overline{A}$ of A is a compact subset of Ω).

If $u \in BV(\Omega)$, then Du can be decomposed as the sum of the absolutely continuous and the singular part with respect to the Lebesgue measure, i.e.

$$Du = D^a u + D^s u, \qquad D^a u = \nabla u\, \mathcal{L}^N,$$

where ∇u is the *approximate gradient* of u, defined $\mathcal{L}^N$-a.e. in Ω (see [3, Section 3.9]). The jump set J_u has the following properties: it is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ (see [3, Definition 2.57 and Theorem 3.78]); it is contained in the set Θ_{Du} defined in Proposition 2.2(ii) with $\mu = Du$, and $\mathcal{H}^{N-1}(\Theta_{Du} \setminus J_u) = 0$ (see [3, Proposition 3.92(b)]). By Proposition 2.2, the singular part $D^s u$ can be further decomposed as the sum of its Cantor and jump part, i.e. $D^s u = D^c u + D^j u$, $D^c u := D^s u \llcorner (\Omega \setminus S_u)$, and

$$D^j u := D^s u \llcorner J_u = (u^i - u^e)\, \nu_u\, \mathcal{H}^{N-1} \llcorner J_u.$$

We denote by $D^d u := D^a u + D^c u$ the diffuse part of the measure Du.

At every point $x \in J_u$ we have that $-\infty < u^-(x) < u^+(x) < +\infty$ and

$$u^-(x) = \min\{u^i(x), u^e(x)\}, \qquad u^+(x) = \max\{u^i(x), u^e(x)\}, \qquad x \in J_u.$$

Moreover, we can always choose an orientation on J_u such that $u^i = u^+$ on J_u (see [25, §4.1.4, Theorem 2]). In the following we shall always extend the functions u^i, u^e to $\Omega \setminus (S_u \setminus J_u)$ by setting

$$u^i = u^e = \widetilde{u} \quad \text{in } \Omega \setminus S_u.$$

Given a Borel function $\lambda \colon \Omega \to [0,1]$, the *$\lambda$–representative* of $u \in BV_{\rm loc}(\Omega)$ is defined by

$$u^\lambda(x) := \begin{cases} \tilde{u}(x), & x \in \Omega \setminus S_u, \\ (1-\lambda(x))u^-(x) + \lambda(x)u^+(x), & x \in J_u. \end{cases} \tag{11}$$

When $\lambda(x) = 1/2$ for every $x \in \Omega$, the λ–representative coincides with the *precise representative* $u^* := (u^+ + u^-)/2$ of u.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every open set $\Omega \subset \mathbb{R}^N$ the *perimeter* $P(E,\Omega)$ is defined by

$$P(E,\Omega) := \sup\left\{ \int_E \operatorname{div} \varphi\, dx : \ \varphi \in C^1_c(\Omega, \mathbb{R}^N),\ \|\varphi\|_\infty \le 1 \right\}.$$

We say that E is of *finite perimeter* in Ω if $P(E,\Omega) < +\infty$.

Denoting by χ_E the characteristic function of E, if E is a set of finite perimeter in Ω, then $D\chi_E$ is a finite Radon measure in Ω and $P(E,\Omega) = |D\chi_E|(\Omega)$.

If $\Omega \subset \mathbb{R}^N$ is the largest open set such that E is locally of finite perimeter in Ω, we call *reduced boundary* $\partial^* E$ of E the set of all points $x \in \Omega$ in the support of $|D\chi_E|$ such that the limit

$$\widetilde{\nu}_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^N$ and satisfies $|\widetilde{\nu}_E(x)| = 1$. The function $\widetilde{\nu}_E \colon \partial^* E \to S^{N-1}$ is called the *measure theoretic unit interior normal* to E.

A fundamental result of De Giorgi (see [3, Theorem 3.59]) states that $\partial^* E$ is countably $(N-1)$-rectifiable and $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$. If E has finite perimeter in Ω, Federer's structure theorem states that $\partial^* E \cap \Omega \subset E^{1/2} \subset \partial^e E$ and $\mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^e E \cup E^1)) = 0$ (see [3, Theorem 3.61]).

2.3. **Divergence–measure fields.** We will denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distributions is a finite Radon measure in Ω, acting as

$$\int_\Omega \varphi \, d \operatorname{div} \boldsymbol{A} = -\int_\Omega \boldsymbol{A} \cdot \nabla \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Similarly, $\mathcal{DM}^\infty_{\text{loc}}(\Omega)$ will denote the space of all vector fields $\boldsymbol{A} \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distributions is a Radon measure in Ω.

The basic properties of these vector fields are collected in the following proposition.

Proposition 2.3. *Let $\boldsymbol{A}$ be a vector field belonging to $\mathcal{DM}^\infty(\Omega)$, and let $\Theta_{\boldsymbol{A}}$ be the jump set of the measure $\mu = |\operatorname{div} \boldsymbol{A}|$, defined in Proposition 2.2(ii). Then the following hold.*

(i) $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$;
(ii) $\Theta_{\boldsymbol{A}}$ *is a Borel set, σ-finite with respect to* $\mathcal{H}^{N-1}$;
(iii) $\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} + \operatorname{div}^c \boldsymbol{A} + \operatorname{div}^j \boldsymbol{A}$, *where* $\operatorname{div}^a \boldsymbol{A}$ *is absolutely continuous with respect to* $\mathcal{L}^N$, $\operatorname{div}^c \boldsymbol{A}(B) = 0$ *for every set B with* $\mathcal{H}^{N-1}(B) < +\infty$, *and there exists* $f \in L^1(\Theta_{\boldsymbol{A}}, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}})$ *such that* $\operatorname{div}^j \boldsymbol{A} = f \, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}}$.

Proof. The main property (i) is proved in [10, Proposition 3.1]. The decomposition then follows from Proposition 2.2. □

2.4. **Weak normal traces.** In what follows, we will deal with the traces of the normal component of a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ on a countably $\mathcal{H}^{N-1}$–rectifiable set $\Sigma \subset \Omega$. In order to fix the notation, we briefly recall the construction given in [1] (see Propositions 3.2, 3.4 and Definition 3.3).

Given a domain $\Omega' \Subset \Omega$ of class C^1, the trace of the normal component of $\boldsymbol{A}$ on $\partial\Omega'$ is the distribution defined by

$$\langle \operatorname{Tr}(\boldsymbol{A}, \partial\Omega'), \varphi \rangle := \int_{\Omega'} \boldsymbol{A} \cdot \nabla \varphi \, dx + \int_{\Omega'} \varphi \, d \operatorname{div} \boldsymbol{A}, \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{12}$$

It turns out that this distribution is induced by an L^∞ function on $\partial\Omega'$, still denoted by $\operatorname{Tr}(\boldsymbol{A}, \partial\Omega')$, and

$$\| \operatorname{Tr}(\boldsymbol{A}, \partial\Omega') \|_{L^\infty(\partial\Omega', \mathcal{H}^{N-1} \llcorner \partial\Omega')} \leq \|\boldsymbol{A}\|_{L^\infty(\Omega')}. \tag{13}$$

Given a countably $\mathcal{H}^{N-1}$–rectifiable set Σ, there exist a covering $(\Sigma_i)_{i \in \mathbb{N}}$ of Σ and Borel sets $N_i \subseteq \Sigma_i$ with the following properties:

(R1) Σ_i is an *oriented* C^1 hypersurface, with (classical) normal vector field ν_{Σ_i};
(R2) $N_i \subseteq \Sigma_i$ are pairwise disjoint Borel sets such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$;
(R3) for every $i \in \mathbb{N}$, there exist two open bounded sets Ω_i, Ω_i' with C^1 boundary and exterior normal vectors ν_{Ω_i} and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial\Omega_i \cap \partial\Omega_i'$, and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \qquad \forall x \in N_i.$$

By a deep localization property proved in [1, Proposition 3.2], we can fix an orientation on Σ, given by

$$\nu_\Sigma(x) := \nu_{\Sigma_i}(x), \qquad \mathcal{H}^{N-1} - \text{a.e. on } N_i$$

and the normal traces of $\boldsymbol{A}$ on Σ are defined by

$$\operatorname{Tr}^e(\boldsymbol{A},\Sigma) := \operatorname{Tr}(\boldsymbol{A},\partial\Omega_i),\quad \operatorname{Tr}^i(\boldsymbol{A},\Sigma) := -\operatorname{Tr}(\boldsymbol{A},\partial\Omega_i'),\qquad \mathcal{H}^{N-1}-\text{a.e. on } N_i.$$

Moreover, the normal traces belong to $L^\infty(\Sigma,\mathcal{H}^{N-1}\llcorner\Sigma)$ and

$$\operatorname{div}\boldsymbol{A}\llcorner\Sigma = \left[\operatorname{Tr}^i(\boldsymbol{A},\Sigma) - \operatorname{Tr}^e(\boldsymbol{A},\Sigma)\right]\mathcal{H}^{N-1}\llcorner\Sigma \tag{14}$$

(see [1, Proposition 3.4]). In particular, by (13), $|\operatorname{div}\boldsymbol{A}|(\Sigma)\le 2\|\boldsymbol{A}\|_\infty\mathcal{H}^{N-1}(\Sigma)$.

Remark 2.4. We observe that, if Σ is oriented by a normal vector field ν and Σ' is the same set oriented by $\nu' := -\nu$, then

$$\operatorname{Tr}^e(\boldsymbol{A},\Sigma') = -\operatorname{Tr}^i(\boldsymbol{A},\Sigma),\quad \operatorname{Tr}^i(\boldsymbol{A},\Sigma') := -\operatorname{Tr}^e(\boldsymbol{A},\Sigma),$$

so that the difference $\operatorname{Tr}^i(\boldsymbol{A},\Sigma)-\operatorname{Tr}^e(\boldsymbol{A},\Sigma)$ is independent of the choice of the orientation on Σ.

The following result is a consequence of (14) and will be used in the study of the semicontinuity of the generalized pairing (see Theorem 7.6).

Theorem 2.5. *Let $\boldsymbol{A}\in\mathcal{DM}^\infty(\Omega)$, let $\operatorname{div}\boldsymbol{A} = \psi_{\boldsymbol{A}}|\operatorname{div}\boldsymbol{A}|$ be the polar decomposition of the measure $\operatorname{div}\boldsymbol{A}$, and let $\Sigma\subset\Omega$ be a countably $\mathcal{H}^{N-1}$-rectifiable set. Then*

$$\operatorname{Tr}^i(\boldsymbol{A},\Sigma)(x) - \operatorname{Tr}^e(\boldsymbol{A},\Sigma)(x) = \lim_{r\searrow 0}\frac{\operatorname{div}\boldsymbol{A}\,(B_r(x))}{\omega_{N-1}r^{N-1}}\,,\qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\Sigma\,, \tag{15}$$

$$\operatorname{Tr}^i(\boldsymbol{A},\Sigma)(x) - \operatorname{Tr}^e(\boldsymbol{A},\Sigma)(x) = \psi_{\boldsymbol{A}}(x)\lim_{r\searrow 0}\frac{|\operatorname{div}\boldsymbol{A}|(B_r(x))}{\omega_{N-1}r^{N-1}}\,,\quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\Sigma\,. \tag{16}$$

Proof. From (8) with $\mu := |\operatorname{div}\boldsymbol{A}|\llcorner\Sigma$ and $E := \Omega\setminus\Sigma$, we have that

$$\lim_{r\searrow 0}\frac{|\operatorname{div}\boldsymbol{A}|\llcorner(\mathbb{R}^N\setminus\Sigma)\,(B_r(x))}{\omega_{N-1}r^{N-1}} = 0\,,\qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\Sigma.$$

On the other hand, by (14)

$$\lim_{r\searrow 0}\frac{\operatorname{div}\boldsymbol{A}\llcorner\Sigma\,(B_r(x))}{\omega_{N-1}r^{N-1}} = \operatorname{Tr}^i(\boldsymbol{A},\Sigma)(x) - \operatorname{Tr}^e(\boldsymbol{A},\Sigma)(x)\,,\qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\Sigma\,,$$

and hence (15) holds. Finally, (16) follows from (15) and Lemma 2.1. □

For later use, we recall here a result proved in [19, Proposition 3.1].

Proposition 2.6. *Let $\boldsymbol{A}\in\mathcal{DM}^\infty(\Omega)$, $u\in BV(\Omega)\cap L^\infty(\Omega)$ and let $\Sigma\subset\Omega$ be an oriented countably $\mathcal{H}^{N-1}$–rectifiable set. Then $u\boldsymbol{A}\in\mathcal{DM}^\infty(\Omega)$ and the normal traces of $u\boldsymbol{A}$ on Σ are given by*

$$\operatorname{Tr}^e(u\boldsymbol{A},\Sigma) = \begin{cases} u^e\operatorname{Tr}^e(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-\text{a.e. in } J_u\cap\Sigma,\\ \widetilde{u}\operatorname{Tr}^e(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-\text{a.e. in } \Sigma\setminus J_u.\end{cases}$$

$$\operatorname{Tr}^i(u\boldsymbol{A},\Sigma) = \begin{cases} u^i\operatorname{Tr}^i(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-\text{a.e. in } J_u\cap\Sigma,\\ \widetilde{u}\operatorname{Tr}^i(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-\text{a.e. in } \Sigma\setminus J_u.\end{cases}$$

3. Some remarks on $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$

In this section we analyze the properties of the functional spaces needed to define the pairing $(\boldsymbol{A}, Du)_\lambda$ introduced in (4).

Definition 3.1. *Given $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, let us define the spaces:*

$$BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|) := \left\{u \in BV(\Omega) : \ u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)\right\} ,$$

$$BV_{\mathrm{loc}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|) := \left\{u \in BV_{\mathrm{loc}}(\Omega) : \ u^* \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)\right\} .$$

Notice that $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$ and u^* is defined $\mathcal{H}^{N-1}$-a.e. in Ω, hence the definitions are well-posed.

The following lemma shows that if $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ then any representative u^λ of u defined in (11) (in particular u^+, u^-) is summable with respect to the measure $|\operatorname{div} \boldsymbol{A}|$, hence the definitions of the spaces $BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ and $BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ are independent of the choice of the pointwise representative.

Lemma 3.2. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV_{\mathrm{loc}}(\Omega)$. Given two Borel selections $\lambda, \mu \colon \Omega \to [0, 1]$, then it holds:*

(i) *$u^\lambda \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$ if and only if $u^\mu \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$;*

(ii) *for every countably $\mathcal{H}^{N-1}$-rectifiable set $\Sigma \subset \Omega$, $u^\lambda \in L^1_{\mathrm{loc}}(\Sigma, \mathcal{H}^{N-1} \llcorner \Sigma)$ if and only if $u^\mu \in L^1_{\mathrm{loc}}(\Sigma, \mathcal{H}^{N-1} \llcorner \Sigma)$.*

Proof. We prove only (i), being the proof of (ii) entirely similar. By the representation (14) of $\operatorname{div} \boldsymbol{A} \llcorner J_u$ and the estimate (13), for every compact set $K \Subset \Omega$ we have

$$\begin{aligned}\int_{J_u \cap K} (u^+ - u^-)\, d|\operatorname{div} \boldsymbol{A}| &= \int_{J_u \cap K} (u^+ - u^-) |\operatorname{Tr}^i(\boldsymbol{A}, J_u) - \operatorname{Tr}^e(\boldsymbol{A}, J_u)|\, d\mathcal{H}^{N-1} \\ &\leq 2\|\boldsymbol{A}\|_{L^\infty(K)} |D^j u|(K).\end{aligned}$$

Recalling that $u^+ - u^- = 0$ in $\Omega \setminus S_u$, i.e. $\mathcal{H}^{N-1}$-a.e. in $\Omega \setminus J_u$, it follows that $u^+ - u^- \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$. The result now follows by observing that $u^\lambda = u^\mu + (\lambda - \mu)(u^+ - u^-)$. □

We underline that, for every $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and every $u \in BV(\Omega)$, it holds

$$\int_{J_u} |\operatorname{Tr}^{i,e}(\boldsymbol{A}, J_u)|\, (u^+ - u^-)\, d\mathcal{H}^{N-1} \leq \|\boldsymbol{A}\|_\infty |D^j u|(\Omega) < +\infty.$$

Nevertheless, in general the functions $|\operatorname{Tr}^{i,e}(\boldsymbol{A}, J_u)|\, u^\pm$ are not summable with respect to $\mathcal{H}^{N-1} \llcorner J_u$, even under the additional assumption $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, as it is shown in the following example.

Example 3.3. Let $\Omega = B_1(0) \subset \mathbb{R}^2$. Let us show that there exist a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and a function $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ such that

$$\int_{J_u} |\operatorname{Tr}^{i,e}(\boldsymbol{A}, J_u)|\, u^\pm\, d\mathcal{H}^1 = +\infty.$$

Let $1 = r_0 > r_1 > \cdots > r_n > \cdots$ be a decreasing sequence converging to 0, such that

$$\sum_j r_j < +\infty, \qquad \sum_j j\, r_j = +\infty,$$

and let $u \colon \Omega \to \mathbb{R}$ be defined by $u(x) = j$, if $r_j \leq |x| < r_{j-1}$, $j \in \mathbb{N}$. Since

$$\int_\Omega u\, dx = \pi \sum_{j=1}^\infty r_j^2 < \infty, \quad Du = \sum_{j=1}^\infty \mathcal{H}^1 \llcorner \partial B_{r_j}(0)\,, \quad |Du|(\Omega) = 2\pi \sum_{j=1}^\infty r_j < \infty,$$

then $u \in BV(\Omega)$. We choose on the jump set $J_u = \bigcup_{j=1}^{\infty} \partial B_{r_j}(0)$ the orientation such that $u^i = u^+ = j+1$ and $u^e = u^- = j$ on $\partial B_{r_j}(0)$.

Let $(a_j) \subset \mathbb{R}$ be a bounded sequence, and let

$$\boldsymbol{A}(x) := a(|x|)\,\frac{x}{|x|}\,, \qquad \text{with} \quad a(\rho) := \sum_{j=1}^{\infty} a_j\,\chi_{[r_j, r_{j-1})}(\rho), \ \ \rho \in (0,1).$$

We have that $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^2)$, $\operatorname{Tr}^i(\boldsymbol{A}, J_u) = a_{j+1}$, $\operatorname{Tr}^e(\boldsymbol{A}, J_u) = a_j$ on ∂B_{r_j}, and

$$\operatorname{div}\boldsymbol{A} = \frac{a(|x|)}{|x|}\,\mathcal{L}^2 + \sum_{j=1}^{\infty}(a_{j+1} - a_j)\mathcal{H}^1 \llcorner \partial B_{r_j}\,,$$

$$|\operatorname{div}\boldsymbol{A}|(\Omega) \leq \|\boldsymbol{A}\|_\infty \int_\Omega \frac{1}{|x|}\,dx + 2\pi \sum_{j=1}^{\infty} |a_{j+1} - a_j|\, r_j < +\infty,$$

so that $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$. On the other hand, if we choose a sequence (a_j) such that $|a_j| \geq c > 0$ for every $j \in \mathbb{N}$, we have that

$$\int_{J_u} u^- \,|\operatorname{Tr}^{i,e}(\boldsymbol{A}, J_u)|\, d\mathcal{H}^1 \geq 2\pi c \sum_{j=1}^{\infty} j\, r_j = +\infty.$$

We collect here the main features of the truncation operator that will be useful to generalize to $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$ properties valid in $BV(\Omega) \cap L^\infty(\Omega)$.

Proposition 3.4 (Properties of the truncated functions)**.** *For every $k > 0$, let*

$$T_k(s) := \max\{\min\{s, k\}, -k\}\,, \qquad s \in \mathbb{R}. \tag{17}$$

Let $u \in BV(\Omega)$ and let $\lambda\colon \Omega \to [0,1]$ be a Borel function. Then the following hold.

(i) *$T_k(u^\pm) = [T_k(u)]^\pm \to u^\pm$, $[T_k(u)]^\lambda \to u^\lambda$, $\mathcal{H}^{N-1}$-a.e. in Ω;*

(ii) *$|DT_k(u)| \leq |Du|$ in the sense of measures, for every $k > 0$;*

(iii) *$|[T_k(u)]^\pm| \leq |u^\pm|$ for every $k > 0$, hence*

$$|T_k(u^\lambda)| \leq (1-\lambda)|u^-| + \lambda\,|u^+| \qquad \forall k > 0;$$

(iv) *if $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$, then $T_k(u^\lambda) \to u^\lambda$ in $L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$.*

Proof. The proof of (i) can be found in [3, Theorem 4.34(a)].

The inequality in (ii) is a consequence of the fact that T_k is a 1-Lipschitz function (see the first part of the proof of Theorem 3.96 in [3]).

The inequalities in (iii) follow from $|T_k(s)| \leq |s|$ and the equalities in (i), whereas (iv) follows from (iii), Lemma 3.2, and Lebesgue's Dominated Convergence Theorem. □

4. Definition and basic properties of pairings

Definition 4.1 (Generalized pairing)**.** *Given a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and a Borel function $\lambda\colon \Omega \to [0,1]$, for every $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$ the λ–pairing between $\boldsymbol{A}$ and Du is the distribution $(\boldsymbol{A}, Du)_\lambda : C_c^\infty(\Omega) \to \mathbb{R}$ acting as*

$$\langle (\boldsymbol{A}, Du)_\lambda\,, \varphi\rangle := -\int_\Omega u^\lambda\,\varphi\, d(\operatorname{div}\boldsymbol{A}) - \int_\Omega u\boldsymbol{A}\cdot\nabla\varphi\, dx\,, \qquad \varphi \in C_c^\infty(\Omega)\,. \tag{18}$$

Remark 4.2. The standard pairing

$$\langle (\boldsymbol{A}, Du)_* \,, \varphi \rangle := -\int_\Omega u^* \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u \, \boldsymbol{A} \cdot \nabla \varphi \, dx,$$

introduced in [6], and deeply studied in recent years (see e.g. [10], [19] and the references therein), is the λ-pairing corresponding to the constant selection $\lambda(x) = \frac{1}{2}$ for every $x \in \Omega$.

Remark 4.3. The definition of generalized pairing and the properties proved in the rest of the paper can be extended straightforwardly to vector fields $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and functions $u \in BV_{\mathrm{loc}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$.

Clearly, the change of pointwise values of u may just affect the behavior of the pairing on the jump set J_u of u. More precisely, the following basic properties hold.

Proposition 4.4. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, and $\lambda \colon \Omega \to [0,1]$ be a Borel function. Then $(\boldsymbol{A}, Du)_\lambda$ is a Radon measure in Ω, and the equations*

$$\operatorname{div}(u\boldsymbol{A}) = u^\lambda \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)_\lambda \,, \tag{19}$$

$$(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du)_* + \left(\frac{1}{2} - \lambda\right) (u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u \,, \tag{20}$$

hold in the sense of measures in Ω. Moreover, $(\boldsymbol{A}, Du)_\lambda$ is absolutely continuous with respect to $|Du|$, and

$$|\,(\boldsymbol{A}, Du)_\lambda\,| \leq (1 + |1 - 2\lambda|) \|\boldsymbol{A}\|_\infty |Du|. \tag{21}$$

In what follows we will write

$$(\boldsymbol{A}, Du)_\lambda = \theta_\lambda(\boldsymbol{A}, Du, x) |Du|, \tag{22}$$

where $\theta_\lambda(\boldsymbol{A}, Du, \cdot)$ denotes the Radon–Nikodým derivative of $(\boldsymbol{A}, Du)_\lambda$ with respect to $|Du|$.

Proof. Assume, in addition, that $u \in BV(\Omega) \cap L^\infty(\Omega)$. In this case the fact that $(\boldsymbol{A}, Du)_\lambda$ is a Radon measure, and the validity of (19) are straightforward consequences of the fact that the distribution

$$\langle \operatorname{div}(u\boldsymbol{A}) \,, \varphi \rangle = \int_\Omega u \, \boldsymbol{A} \cdot \nabla \varphi \, dx, \qquad \varphi \in C^\infty_c(\Omega)$$

is a Radon measure in Ω (see [10]). Moreover, we have that

$$\begin{aligned}(\boldsymbol{A}, Du)_\lambda &= -u^* \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) + \left(\frac{1}{2} - \lambda\right) (u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u \\ &= (\boldsymbol{A}, Du)_* + \left(\frac{1}{2} - \lambda\right) (u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u \,,\end{aligned}$$

Using the representation (14) of $\operatorname{div} \boldsymbol{A} \llcorner J_u$, with $\Sigma = J_u$ oriented in such a way that $u^i = u^+$, we have that

$$\begin{aligned}(u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u &= (u^+ - u^-) \left[\operatorname{Tr}^i(\boldsymbol{A}, J_u) - \operatorname{Tr}^e(\boldsymbol{A}, J_u)\right] \mathcal{H}^{N-1} \llcorner J_u \\ &= \left[\operatorname{Tr}^i(\boldsymbol{A}, J_u) - \operatorname{Tr}^e(\boldsymbol{A}, J_u)\right] |D^j u|\end{aligned}$$

and hence

$$|(u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u| \leq 2 \|\boldsymbol{A}\|_\infty |D^j u|.$$

Since $|\,(\boldsymbol{A}, Du)_*\,| \leq \|\boldsymbol{A}\|_\infty |Du|$ (see [19, Theorem 3.3]), then (21) holds true.

Consider now the general case $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Let $u_k := T_k(u)$ be the sequence of truncated functions. By Proposition 3.4(i) and (iv), we have that $(u_k)^\lambda \to u^\lambda$ $\mathcal{H}^{N-1}$-a.e. in Ω and in $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Hence, we can pass to the limit in

$$\langle (\boldsymbol{A}, Du_k)_\lambda \,, \varphi \rangle = -\int_\Omega (u_k)^\lambda \, \varphi \, d(\operatorname{div} \boldsymbol{A}) - \int_\Omega u_k \boldsymbol{A} \cdot \nabla \varphi \, dx$$

and obtain the validity of (19) in the sense of distributions. Since, by the estimate (21) and Proposition 3.4(ii), we have that

$$|\, (\boldsymbol{A}, Du_k)_\lambda \,|(\Omega) \leq 2\|\boldsymbol{A}\|_\infty |Du_k|(\Omega) \leq 2\|\boldsymbol{A}\|_\infty |Du|(\Omega), \qquad \forall k \in \mathbb{N}.$$

Hence, the distribution $(\boldsymbol{A}, Du)_\lambda$ is in fact a measure, and (19), (20) and (21) hold in the sense of measures. □

Remark 4.5. In the last part of the proof of Proposition 4.4 we have shown that, for every $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and every $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, the pairing $(\boldsymbol{A}, Du)_\lambda$ is the weak* limit, in the sense of measures, of the sequence $(\boldsymbol{A}, DT_k(u))_\lambda$.

Remark 4.6. Since $(u+v)^+ \leq u^+ + v^+$ and $(u+v)^- \geq u^- + v^-$, with possibly strict inequalities, the map $u \mapsto (\boldsymbol{A}, Du)_\lambda$ is not linear, in general. On the other hand, the map $u \mapsto u^*$ is linear, hence the standard pairing is linear with respect to u. More precisely, the λ-pairing is linear if and only if $(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du)_*$ for every $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Indeed, for every $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we have that

$$\begin{aligned}(\boldsymbol{A}, Du)_\lambda + (\boldsymbol{A}, D(-u))_\lambda &= \left(\frac{1}{2} - \lambda\right) [u^+ - u^- + (-u)^+ - (-u)^-] \operatorname{div} \boldsymbol{A} \llcorner J_u \\ &= 2\left(\frac{1}{2} - \lambda\right) (u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner J_u.\end{aligned}$$

Hence, if there exists $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $(\boldsymbol{A}, Du)_\lambda \neq (\boldsymbol{A}, Du)_*$, then the claim follows from (20).

Using (20), and the results of Theorem 3.3 in [19], we are able to compute explicitly the diffuse part $(\boldsymbol{A}, Du)^d_\lambda$, the absolutely continuous part $(\boldsymbol{A}, Du)^a_\lambda$, and the jump part $(\boldsymbol{A}, Du)^j_\lambda$ of the generalized pairing.

Proposition 4.7. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Then the diffuse, the absolutely continuous and the jump part of the measure $(\boldsymbol{A}, Du)_\lambda$ are respectively*

$$(\boldsymbol{A}, Du)^d_\lambda = (\boldsymbol{A}, Du)^d_* \,, \qquad (\boldsymbol{A}, Du)^a_\lambda = \boldsymbol{A} \cdot \nabla u \, \mathcal{L}^N,$$
$$(\boldsymbol{A}, Du)^j_\lambda = \left[(1-\lambda) \operatorname{Tr}^e(\boldsymbol{A}, J_u) + \lambda \operatorname{Tr}^i(\boldsymbol{A}, J_u)\right] \, (u^+ - u^-) \, \mathcal{H}^{N-1} \llcorner J_u \,,$$

where $\operatorname{Tr}^i(\boldsymbol{A}, J_u)$ and $\operatorname{Tr}^e(\boldsymbol{A}, J_u)$ are the normal traces corresponding to the orientation of J_u such that $u^+ = u^i$.

Proof. The computation of the diffuse part involves the pointwise values of u up to sets of Lebesgue measure zero, hence $(\boldsymbol{A}, Du)^d_\lambda = (\boldsymbol{A}, Du)^d_*$. Moreover, by Theorem 3.2 in [10], $(\boldsymbol{A}, Du)^a_\lambda = (\boldsymbol{A}, Du)^a_* = \boldsymbol{A} \cdot \nabla u \, \mathcal{L}^N$.

Concerning the jump part $(\boldsymbol{A}, Du)^j_\lambda$, by (21), we already know that it is concentrated on J_u. Denoting by $\alpha^i := \operatorname{Tr}^i(\boldsymbol{A}, J_u)$ and $\alpha^e := \operatorname{Tr}^e(\boldsymbol{A}, J_u)$, by Theorem 3.3 in [19] we already know that

$$(\boldsymbol{A}, Du)^j_* = \frac{\alpha^i + \alpha^e}{2} \, (u^+ - u^-) \, \mathcal{H}^{N-1} \llcorner J_u \,.$$

Finally, by (20) and (14), we conclude that

$$\begin{aligned}(\boldsymbol{A}, Du)^j_\lambda &= (\boldsymbol{A}, Du)^j_* + \left(\frac{1}{2} - \lambda\right)(u^+ - u^-)\operatorname{div}\boldsymbol{A} \llcorner J_u \\ &= \frac{\alpha^i + \alpha^e}{2}(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u + \left(\frac{1}{2} - \lambda\right)(u^+ - u^-)(\alpha^i - \alpha^e)\,\mathcal{H}^{N-1} \llcorner J_u \\ &= [(1-\lambda)\alpha^e + \lambda\alpha^i]\,(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u. \end{aligned}$$

□

Remark 4.8 (The pairing trivializes on $W^{1,1}$). From Proposition 4.7, we have that

$$(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du)_* = \boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N, \qquad \forall u \in W^{1,1}(\Omega) \cap L^\infty(\Omega).$$

Remark 4.9 (*BV* vector fields). If $\boldsymbol{A} \in BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$, then clearly $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and

$$\operatorname{Tr}^{i,e}(\boldsymbol{A}, J_u) = \boldsymbol{A}^{i,e}_{J_u} \cdot \nu_u\,, \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_u,$$

where $\boldsymbol{A}^{i,e}_{J_u}$ are the traces of $\boldsymbol{A}$ on J_u in the sense of BV (see [3, Theorem 3.77]). Hence, the jump part of $(\boldsymbol{A}, Du)_\lambda$ can be written as

$$(\boldsymbol{A}, Du)^j_\lambda = \left[(1-\lambda)\boldsymbol{A}^i_{J_u} + \lambda\, \boldsymbol{A}^e_{J_u}\right] \cdot D^j u.$$

The following result is an improvement of Proposition 4.15 in [19], Theorem 1.2 in [10] and Lemma 2.2 in [6].

Proposition 4.10 (Approximation by C^∞ fields)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$. Then there exists a sequence $(\boldsymbol{A}_k)_k$ in $C^\infty(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ satisfying the following properties.*

(i) *$\boldsymbol{A}_k \to \boldsymbol{A}$ in $L^1(\Omega, \mathbb{R}^N)$ and $\int_\Omega |\operatorname{div}\boldsymbol{A}_k|\,dx \to |\operatorname{div}\boldsymbol{A}|(\Omega)$.*

(ii) *$\operatorname{div}\boldsymbol{A}_k \stackrel{*}{\rightharpoonup} \operatorname{div}\boldsymbol{A}$ in the weak* sense of measures in Ω.*

(iii) *For every oriented countably $\mathcal{H}^{N-1}$-rectifiable set $\Sigma \subset \Omega$ it holds*

$$\lim_{k\to+\infty} \left\langle \operatorname{Tr}^{i,e}(\boldsymbol{A}_k, \Sigma)\,,\, \varphi\right\rangle = \left\langle \operatorname{Tr}^*(\boldsymbol{A}, \Sigma)\,,\, \varphi\right\rangle \qquad \forall \varphi \in C_c(\Omega),$$

where $\operatorname{Tr}^*(\boldsymbol{A}, \Sigma) := [\operatorname{Tr}^i(\boldsymbol{A}, \Sigma) + \operatorname{Tr}^e(\boldsymbol{A}, \Sigma)]/2$.

Moreover, for every $u \in BV(\Omega) \cap L^\infty(\Omega)$, it holds

(iv) *$(\boldsymbol{A}_k, Du)_* \stackrel{*}{\rightharpoonup} (\boldsymbol{A}, Du)_*$ locally in the weak* sense of measures in Ω;*

(v) *$\theta(\boldsymbol{A}_k, Du, x) \to \theta(\boldsymbol{A}, Du, x)$ for $|Du|$-a.e. $x \in \Omega$, where $\theta(\boldsymbol{A}, Du; \cdot)$ is the Radon–Nikodým derivative of $(\boldsymbol{A}, Du)_*$ with respect to $|Du|$.*

Remark 4.11. It is not difficult to show that a similar approximation result holds also for $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ with a sequence $(\boldsymbol{A}_k)$ in $C^\infty(\Omega, \mathbb{R}^N)$.

Proof. (i) This part is proved in [10, Theorem 1.2]. We just recall, for later use, that for every k the vector field $\boldsymbol{A}_k$ is of the form

$$\boldsymbol{A}_k = \sum_{i=1}^\infty \rho_{\varepsilon_i} * (\boldsymbol{A}\varphi_i), \tag{23}$$

where (φ_i) is a partition of unity subordinate to a locally finite covering of Ω depending on k and, for every i, $\varepsilon_i \in (0, 1/k)$ is chosen in such a way that

$$\int_\Omega |\rho_{\varepsilon_i} * (\boldsymbol{A} \cdot \nabla\varphi_i) - \boldsymbol{A} \cdot \nabla\varphi_i|\,dx \le \frac{1}{k\,2^i} \tag{24}$$

(see [10], formula (1.8)).

(ii) From (i) we have that

$$\lim_{k\to+\infty}\int_\Omega \boldsymbol{A}_k\cdot\nabla\varphi\,dx=\int_\Omega \boldsymbol{A}\cdot\nabla\varphi\,dx\qquad\forall\varphi\in C^1_c(\Omega),$$

hence (ii) follows from the density of $C^1_c(\Omega)$ in $C_0(\Omega)$ in the norm of $L^\infty(\Omega)$ and the bound $\sup_k\int_\Omega|\operatorname{div}\boldsymbol{A}_k|\,dx<+\infty$.

(iii) As a first step we prove that, for every $u\in BV(\Omega)\cap L^\infty(\Omega)$,

$$\lim_{k\to+\infty}\int_\Omega u\,\varphi\,\operatorname{div}\boldsymbol{A}_k\,dx=\int_\Omega u^*\,\varphi\,d\operatorname{div}\boldsymbol{A},\qquad\forall\varphi\in C_c(\Omega). \tag{25}$$

Namely, from the definition (23) of $\boldsymbol{A}_k$ and the identity $\sum_i\nabla\varphi_i=0$ we have that

$$\operatorname{div}\boldsymbol{A}_k=\sum_i\rho_{\varepsilon_i}*(\varphi_i\operatorname{div}\boldsymbol{A})+\sum_i\left[\rho_{\varepsilon_i}*(\boldsymbol{A}\cdot\nabla\varphi_i)-\boldsymbol{A}\cdot\nabla\varphi_i\right].$$

From the estimate (24) we have that

$$\left|\sum_i\int_\Omega u\,\varphi\left[\rho_{\varepsilon_i}*(\boldsymbol{A}\cdot\nabla\varphi_i)-\boldsymbol{A}\cdot\nabla\varphi_i\right]\,dx\right|<\frac1k\,\|\varphi\|_\infty\,\|u\|_\infty\,,$$

and hence, to prove (25), it is enough to show that

$$\lim_{k\to+\infty}\sum_i\int_\Omega u\,\varphi\,\rho_{\varepsilon_i}*(\varphi_i\operatorname{div}\boldsymbol{A})=\int_\Omega u^*\,\varphi\,d\operatorname{div}\boldsymbol{A}. \tag{26}$$

On the other hand,

$$\sum_i\int_\Omega u\,\varphi\,\rho_{\varepsilon_i}*(\varphi_i\operatorname{div}\boldsymbol{A})=\sum_i\int_\Omega\rho_{\varepsilon_i}*(u\,\varphi)\,\varphi_i\,d\operatorname{div}\boldsymbol{A}\,,$$

hence (26) follows by observing that the functions $\rho_{\varepsilon_i}*(u\,\varphi)$ converge pointwise $\mathcal{H}^{N-1}$–a.e. in Ω to $u^*\varphi$, so that

$$u^*\varphi-\sum_i\varphi_i\rho_{\varepsilon_i}*(u\,\varphi)=\sum_i\varphi_i\left[u^*\varphi-\rho_{\varepsilon_i}*(u\,\varphi)\right]\to0,\qquad|\operatorname{div}\boldsymbol{A}|\text{-a.e. in }\Omega.$$

We remark that, as a consequence of (25), if $E\Subset\Omega$ is a set of finite perimeter, then

$$\lim_{k\to+\infty}\int_\Omega\chi_E\,\varphi\,\operatorname{div}\boldsymbol{A}_k\,dx=\int_\Omega\chi_E^*\,\varphi\,d\operatorname{div}\boldsymbol{A},\qquad\forall\varphi\in C_c(\Omega). \tag{27}$$

Let us now prove (iii). Let $\omega\Subset\Omega$ be a set of class C^1. By the definition (12) of normal traces, by (i), (ii) and (27), for every $\varphi\in C^\infty_c(\Omega)$ we have that

$$\begin{aligned}\langle\operatorname{Tr}(\boldsymbol{A}_k,\partial\omega)\,,\,\varphi\rangle&=\int_\omega\boldsymbol{A}_k\cdot\nabla\varphi\,dx+\int_\omega\varphi\,\operatorname{div}\boldsymbol{A}_k\,dx\\&=\int_\Omega\chi_\omega\,\boldsymbol{A}_k\cdot\nabla\varphi\,dx+\int_\Omega\chi_\omega\,\varphi\,\operatorname{div}\boldsymbol{A}_k\,dx\,,\end{aligned}$$

so that

$$\begin{aligned}\lim_{k\to+\infty}\langle\operatorname{Tr}(\boldsymbol{A}_k,\partial\omega)\,,\,\varphi\rangle&=\int_\Omega\chi_\omega\,\boldsymbol{A}\cdot\nabla\varphi\,dx+\int_\Omega\chi^*_\omega\,\varphi\,d\operatorname{div}\boldsymbol{A}\\&=\int_\omega\boldsymbol{A}\cdot\nabla\varphi\,dx+\int_\omega\varphi\,d\operatorname{div}\boldsymbol{A}+\frac12\int_{\partial\omega}\varphi\,d\operatorname{div}\boldsymbol{A}\\&=\langle\operatorname{Tr}(\boldsymbol{A},\partial\omega)\,,\,\varphi\rangle+\frac12\,\langle\operatorname{div}\boldsymbol{A}\llcorner\partial\omega\,,\,\varphi\rangle\;.\end{aligned}$$

Hence, by (14), we have proved that

$$\lim_{k\to+\infty} \mathrm{Tr}^e(\boldsymbol{A}_k, \partial\omega) = \mathrm{Tr}^e(\boldsymbol{A}, \partial\omega) + \frac{1}{2}\left[\mathrm{Tr}^i(\boldsymbol{A}, \partial\omega) - \mathrm{Tr}^e(\boldsymbol{A}, \partial\omega)\right] = \mathrm{Tr}^*(\boldsymbol{A}, \partial\omega)\,,$$

in the sense of distributions. Using the arguments of Section 2.4, this relation can be extended to the countably $\mathcal{H}^{N-1}$-rectifiable set Σ. By a density argument as in (ii), this relation hold for every $\varphi \in C_c(\Omega)$, hence (iii) holds true for $\mathrm{Tr}^e(\boldsymbol{A}_k, \Sigma)$. Finally, a similar computation holds for $\mathrm{Tr}^i(\boldsymbol{A}_k, \Sigma)$.

(iv) Using the passage to the limit in (25) we obtain straightforwardly

$$\begin{aligned}\lim_{k\to+\infty} \langle (\boldsymbol{A}_k, Du)_*\ , \varphi\rangle &= \lim_{k\to+\infty}\left[-\int_\Omega u^*\,\varphi\,\mathrm{div}\,\boldsymbol{A}_k\,dx - \int_\Omega u\,\boldsymbol{A}_k\cdot\nabla\varphi\,dx\right]\\ &= -\int_\Omega u^*\,\varphi\,d\,\mathrm{div}\,\boldsymbol{A} - \int_\Omega u\,\boldsymbol{A}\cdot\nabla\varphi\,dx\\ &= \langle (\boldsymbol{A}, Du)_*\ , \varphi\rangle\end{aligned}$$

for every $\varphi \in C^1_c(\Omega)$, and hence (iv) holds.

(v) Using the definition (22) of the density θ, we have that, for every $\varphi \in C_c(\Omega)$,

$$\begin{aligned}\lim_{k\to+\infty}\int_\Omega \theta(\boldsymbol{A}_k, Du, x)\varphi(x)\,d|Du| &= \lim_{k\to+\infty}\langle (\boldsymbol{A}_k, Du)_*\ , \varphi\rangle\\ &= \langle (\boldsymbol{A}, Du)_*\ , \varphi\rangle = \int_\Omega \theta(\boldsymbol{A}, Du, x)\varphi(x)\,d|Du|,\end{aligned}$$

hence (v) follows. □

5. Coarea formula for generalized pairings

This section is devoted to the proof of the coarea formula for the λ-pairing, and a related slicing result for its density θ_λ.

Theorem 5.1 (Coarea formula)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega)\cap L^1(\Omega, |\,\mathrm{div}\,\boldsymbol{A}|)$. Then*

$$\langle (\boldsymbol{A}, Du)_\lambda\ , \varphi\rangle = \int_{\mathbb{R}} \left\langle \left(\boldsymbol{A}, D\chi_{\{u>t\}}\right)_\lambda\ , \varphi\right\rangle\,dt, \qquad \forall \varphi \in C_0(\Omega)\,. \tag{28}$$

Proof. Since $(\boldsymbol{A}, Du)_\lambda$ and $\left(\boldsymbol{A}, D\chi_{\{u>t\}}\right)_\lambda$, $t \in \mathbb{R}$, are measures in Ω, it is enough to prove (28) for $\varphi \in C^\infty_c(\Omega)$.

Let us first consider the case $u \in L^\infty(\Omega)$. By possibly replacing u with $u + \|u\|_\infty$, it is not restrictive to assume that $u \geq 0$. Given a test function $\varphi \in C^\infty_c(\Omega)$, we have that

$$\begin{aligned}&\int_{\mathbb{R}} \left\langle \left(\boldsymbol{A}, D\chi_{\{u>t\}}\right)_\lambda\ , \varphi\right\rangle\,dt\\ &\quad= -\int_0^{+\infty}\left(\int_\Omega \chi^\lambda_{\{u>t\}}\varphi\,d\,\mathrm{div}\,\boldsymbol{A}\right)dt - \int_0^{+\infty}\left(\int_\Omega \chi_{\{u>t\}}\boldsymbol{A}\cdot\nabla\varphi\,dx\right)dt\\ &\quad= -\int_0^{+\infty}\left(\int_\Omega \chi^\lambda_{\{u>t\}}\varphi\,d\,\mathrm{div}\,\boldsymbol{A}\right)dt - \int_\Omega u\,\boldsymbol{A}\cdot\nabla\varphi\,dx,\end{aligned} \tag{29}$$

where the last equality follows from the coarea formula for BV functions (see [3, Theorem 3.40]).

Moreover, by [20, Lemma 2.2], we have that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, there exists a Borel set $N_t \subset \Omega$, with $\mathcal{H}^{N-1}(N_t) = 0$, such that

$$\forall x \in \Omega \setminus N_t: \qquad \chi^-_{\{u>t\}}(x) = \chi_{\{u^->t\}}(x), \quad \chi^+_{\{u>t\}}(x) = \chi_{\{u^+>t\}}(x),$$

so that, since $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$, we obtain that

$$\chi^\lambda_{\{u>t\}}(x) = (1-\lambda(x))\chi_{\{u^->t\}}(x) + \lambda(x)\chi_{\{u^+>t\}}(x)\,, \qquad \text{for } |\operatorname{div} \boldsymbol{A}|\text{-a.e. } x \in \Omega.$$

Hence, we get

$$\begin{aligned}
&\int_0^{+\infty} \left(\int_\Omega \chi^\lambda_{\{u>t\}} \varphi \, d \operatorname{div} \boldsymbol{A} \right) dt \\
&= \int_0^{+\infty} \left(\int_\Omega [(1-\lambda)\chi_{\{u^->t\}} + \lambda \chi_{\{u^+>t\}}]\, \varphi \, d\operatorname{div} \boldsymbol{A} \right) dt \\
&= \int_\Omega (1-\lambda)\varphi \left(\int_0^{+\infty} \chi_{\{u^->t\}}\, dt \right) d\operatorname{div}\boldsymbol{A} + \int_\Omega \lambda\varphi \left(\int_0^{+\infty} \chi_{\{u^+>t\}}\, dt \right) d\operatorname{div}\boldsymbol{A} \\
&= \int_\Omega u^\lambda\, \varphi \, d\operatorname{div}\boldsymbol{A}\,.
\end{aligned} \tag{30}$$

As a consequence, from (29), (30) and the definition (18) of $(\boldsymbol{A}, Du)_\lambda$, we conclude that (28) holds for every test function $\varphi \in C^\infty_c(\Omega)$ and for every $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Finally, the general case $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ follows applying the previous step to the truncated functions $u_k := T_k(u)$. Namely, (28) gives, for every $k > 0$,

$$\langle (\boldsymbol{A}, Du_k)_\lambda\,,\, \varphi \rangle = \int_{\mathbb{R}} \left\langle \left(\boldsymbol{A}, D\chi_{\{u_k>t\}}\right)_\lambda\,,\, \varphi \right\rangle dt, \qquad \forall \varphi \in C^1_c(\Omega). \tag{31}$$

By Remark 4.5, the left-hand side of (31) converges to $\langle (\boldsymbol{A}, Du)_\lambda\,,\, \varphi\rangle$. On the other hand, since

$$\begin{gathered}
\{u>t\} = \{u_k > t\}, \quad D\chi_{\{u>t\}} = D\chi_{\{u_k>t\}}\,, \qquad \forall t \in [-k,k), \\
D\chi_{\{u_k>t\}} = 0\,, \qquad \forall t \in \mathbb{R} \setminus [-k,k),
\end{gathered}$$

the right-hand side in (31) is equal to

$$\int_{-k}^{k} \left\langle \left(\boldsymbol{A}, D\chi_{\{u>t\}} \right)_\lambda\,,\, \varphi \right\rangle dt\,. \tag{32}$$

By the estimate (21) we have that

$$\left| \left\langle \left(\boldsymbol{A}, D\chi_{\{u>t\}} \right)_\lambda\,,\, \varphi \right\rangle \right| \le 2\|\varphi\|_\infty \|\boldsymbol{A}\|_\infty |D\chi_{\{u>t\}}|(\Omega)\,,$$

and hence, by the coarea formula in BV and the Lebesgue Dominated Convergence Theorem, the integral in (32) converges to the right-hand side of (28) as $k \to +\infty$. □

Proposition 5.2. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then*

$$\text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}: \quad \theta_\lambda(\boldsymbol{A}, Du, x) = \theta_\lambda(\boldsymbol{A}, D\chi_{\{u>t\}}, x) \quad \text{for } |D\chi_{\{u>t\}}|\text{-a.e. } x \in \Omega\,. \tag{33}$$

Proof. Thanks to Proposition 4.10(iv), the proof can be done following the lines of [6, Proposition 2.7(iii)]. For the reader's convenience, we recall here the main points.

Given two real numbers $a < b$, the function $v := \max\{\min\{u, b\}, a\}$ satisfies

$$\begin{gathered}
\{u>t\} = \{v>t\}, \quad D\chi_{\{u>t\}} = D\chi_{\{v>t\}}\,, \qquad \forall t \in [a,b), \\
D\chi_{\{v>t\}} = 0\,, \qquad \forall t < a,\ t \ge b.
\end{gathered} \tag{34}$$

Since

$$\frac{dDu}{d|Du|} = \frac{dD\chi_{\{u>t\}}}{d|D\chi_{\{u>t\}}|}\,, \qquad |D\chi_{\{u>t\}}|\text{-a.e. in } \Omega$$

(see [25, §4.1.4, Theorem 2(i)]), we deduce that

$$\frac{dDu}{d|Du|} = \frac{dDv}{d|Dv|} \qquad |Dv|\text{-a.e. in } \Omega.$$

Let $(\boldsymbol{A}_k) \subset C^\infty(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be the sequence of smooth vector fields approximating $\boldsymbol{A}$ as in Proposition 4.10. Since, by [6, Proposition 2.3], we have

$$\theta(\boldsymbol{A}_k, Du, x) = \boldsymbol{A}_k(x) \cdot \frac{dDu}{d|Du|}(x) = \boldsymbol{A}_k(x) \cdot \frac{dDv}{d|Dv|}(x) = \theta(\boldsymbol{A}_k, Dv, x) \qquad |Dv|\text{-a.e. in } \Omega,$$

then, from Proposition 4.10(v), we obtain that

$$\theta(\boldsymbol{A}, Du, x) = \theta(\boldsymbol{A}, Dv, x) \qquad |Dv|\text{-a.e. in } \Omega.$$

Recalling the definition (22) of θ_λ and the relation (20), we conclude that

$$\theta_\lambda(\boldsymbol{A}, Du, x) = \theta_\lambda(\boldsymbol{A}, Dv, x) \qquad |Dv|\text{-a.e. in } \Omega. \tag{35}$$

Namely, $\theta_\lambda(\boldsymbol{A}, Du, x) = \theta(\boldsymbol{A}, Du, x) = \theta(\boldsymbol{A}, Dv, x) = \theta_\lambda(\boldsymbol{A}, Dv, x)$ for $|D^d v|$-a.e. $x \in \Omega$, whereas, by Proposition 4.7 (and using the notations therein) and the inclusion $J_v \subset J_u$, $\theta_\lambda(\boldsymbol{A}, Du, x) = (1-\lambda)\operatorname{Tr}^e(\boldsymbol{A}, J_u) + \lambda \operatorname{Tr}^i(\boldsymbol{A}, J_u) = \theta_\lambda(\boldsymbol{A}, Dv, x)$ for $|D^j v|$-a.e. $x \in \Omega$.

Given $\varphi \in C_c^\infty(\Omega)$, let us compute $\langle (\boldsymbol{A}, Dv)_\lambda\,,\, \varphi\rangle$. By the definition of $\theta_\lambda(\boldsymbol{A}, Dv, x)$, equality (35), the coarea formula in BV (see [3, Theorem 3.40]) and (34) it holds

$$\begin{aligned}\langle (\boldsymbol{A}, Dv)_\lambda\,,\, \varphi\rangle &= \int_\Omega \theta_\lambda(\boldsymbol{A}, Dv, x)\,\varphi(x)\,d|Dv| \\ &= \int_\Omega \theta_\lambda(\boldsymbol{A}, Du, x)\,\varphi(x)\,d|Dv| \\ &= \int_a^b dt \int_\Omega \theta_\lambda(\boldsymbol{A}, Du, x)\,\varphi(x)\,d|D\chi_{\{u>t\}}|\,.\end{aligned} \tag{36}$$

On the other hand, by the coarea formula (28) and (34), it holds

$$\begin{aligned}\langle (\boldsymbol{A}, Dv)_\lambda\,,\, \varphi\rangle &= \int_{\mathbb{R}} \Big\langle \big(\boldsymbol{A}, D\chi_{\{v>t\}}\big)_\lambda\,,\, \varphi\Big\rangle\, dt \\ &= \int_a^b \Big\langle \big(\boldsymbol{A}, D\chi_{\{u>t\}}\big)_\lambda\,,\, \varphi\Big\rangle\, dt \\ &= \int_a^b dt \int_\Omega \theta_\lambda(\boldsymbol{A}, D\chi_{\{u>t\}}, x)\,\varphi(x)\,d|D\chi_{\{u>t\}}|\,.\end{aligned} \tag{37}$$

Comparing (36) with (37), we finally conclude that, for every $a < b$,

$$\int_a^b dt \int_\Omega \theta_\lambda(\boldsymbol{A}, Du, x)\,\varphi(x)\,d|D\chi_{\{u>t\}}| = \int_a^b dt \int_\Omega \theta_\lambda(\boldsymbol{A}, D\chi_{\{u>t\}}, x)\,\varphi(x)\,d|D\chi_{\{u>t\}}|\,,$$

so that (33) follows. □

6. Chain rule, Leibniz and Gauss–Green formulas for generalized pairings

In this section we show that some relevant formulas, proved in [19] for the standard pairing, remain valid for general λ–pairings.

Proposition 6.1 (Chain Rule)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then it holds:*

(i) $(\boldsymbol{A}, Dh(u))^d_\lambda = (\boldsymbol{A}, Dh(u))^d_*$, *and* $(\boldsymbol{A}, Dh(u))^a_\lambda = h'(\widetilde{u})\, \boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N$.

Moreover, if h is non-decreasing, then

(ii) $(\boldsymbol{A}, Dh(u))^j_\lambda = \dfrac{h(u^+) - h(u^-)}{u^+ - u^-}\, (\boldsymbol{A}, Du)^j_\lambda$;

(iii) $\theta_\lambda(\boldsymbol{A}, Dh(u), x) = \theta_\lambda(\boldsymbol{A}, Du, x)$, *for $|Dh(u)|$-a.e. $x \in \Omega$.*

The same characterization holds if $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ and $h\colon I \to \mathbb{R}$ is a locally Lipschitz function such that $u(\Omega) \Subset I$.

Proof. Although the proof is essentially the same of [19, Proposition 4.5], for the sake of completeness we prefer to illustrate it in some detail.

One of the main ingredients is the Chain Rule Formula for BV functions (see [3, Theorem 3.99]):

$$D^d h(u) = h'(\widetilde{u}) D^d u, \quad D^a h(u) = h'(u) \nabla u\, \mathcal{L}^N, \quad D^j h(u) = [h(u^i) - h(u^e)]\, \nu_u\, \mathcal{H}^{N-1} \llcorner J_u.$$

Statement (i) easily follows from the first two relations above and Proposition 4.7.

Concerning (ii), we have that $[h(u)]^{i,e} = h(u^{i,e})$ (see [3, Proposition 3.69(c)]). Moreover, since h is non-decreasing, also the relations $[h(u)]^\pm = h(u^\pm)$ hold true, and hence (ii) follows again from Proposition 4.7.

Let us prove (iii). If h is strictly increasing, we can follow the proof of [6, Proposition 2.8]. Namely, $\{u > t\} = \{h(u) > h(t)\}$ for every $t \in \mathbb{R}$, hence

$$D\chi_{\{u>t\}} = D\chi_{\{h(u)>h(t)\}}, \qquad \forall t \in \mathbb{R}.$$

From Proposition 5.2, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ it holds

$$\theta_\lambda(\boldsymbol{A}, Du, x) = \theta_\lambda(\boldsymbol{A}, D\chi_{\{u>t\}}, x) = \theta_\lambda(\boldsymbol{A}, D\chi_{\{h(u)>h(t)\}}, x) = \theta_\lambda(\boldsymbol{A}, Dh(u), x)$$

for $|D\chi_{\{u>t\}}|$-a.e. $x \in \Omega$, and (iii) follows.

If h is non-decreasing, we can adapt the proof of [29, Proposition 2.7]. Namely, let $h_\varepsilon(t) := h(t) + \varepsilon\, t$, so that h_ε is strictly increasing for every $\varepsilon > 0$. Since

$$[h_\varepsilon(u)]^\lambda = (1-\lambda) h_\varepsilon(u^-) + \lambda\, h_\varepsilon(u^+) = [h(u)]^\lambda + \varepsilon\, u^\lambda,$$

by the previous step we deduce that

$$(\boldsymbol{A}, Dh(u))_\lambda + \varepsilon\, (\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Dh_\varepsilon(u))_\lambda = \theta_\lambda(\boldsymbol{A}, Du, x)\, |Dh_\varepsilon(u)|. \tag{38}$$

On the other hand, again by the Chain Rule Formula in BV,

$$Dh_\varepsilon(u) = [h'(\widetilde{u}) + \varepsilon]\, D^d u + [h(u^i) - h(u^e) + \varepsilon(u^i - u^e)]\, D^j u = Dh(u) + \varepsilon\, Du,$$

hence, passing to the limit in (38) as $\varepsilon \to 0$, we deduce that

$$(\boldsymbol{A}, Dh(u))_\lambda = \theta_\lambda(\boldsymbol{A}, Du, x)\, |Dh(u)| \qquad \text{as measures in } \Omega,$$

and (iii) follows. □

Proposition 6.2 (Leibniz formula)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u, v \in BV(\Omega) \cap L^\infty(\Omega)$. Then, choosing on J_u the orientation such that $u^+ = u^i$, it holds*

$$(vA, Du)^d_\lambda = v^\lambda\,(A, Du)^d_\lambda = v^*\,(A, Du)^d_*\,, \tag{39}$$

$$(vA, Du)^j_\lambda = [(1-\lambda)\operatorname{Tr}^i(A, J_u)v^i + \lambda\operatorname{Tr}^e(A, J_u)v^e]\,(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u\,. \tag{40}$$

Proof. By [19, Proposition 4.9], denoting $\alpha^i := \operatorname{Tr}^i(\boldsymbol{A}, J_u)$ and $\alpha^e := \operatorname{Tr}^e(\boldsymbol{A}, J_u)$ we have that

$$(vA, Du)^d_* = v^*\,(A, Du)^d_*\,, \tag{41}$$

$$(vA, Du)^j_* = \frac{\alpha^i v^i + \alpha^e v^e}{2}(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u\,, \tag{42}$$

hence (39) follows from (41) and Proposition 4.7.

From the representation formulas (14) and Proposition 2.6, we get

$$\operatorname{div}(vA) \llcorner J_u = \left[\operatorname{Tr}^i(vA, J_u) - \operatorname{Tr}^e(vA, J_u)\right]\mathcal{H}^{N-1} \llcorner J_u = (v^i\alpha^i - v^e\alpha^e)\mathcal{H}^{N-1} \llcorner J_u\,,$$

hence, from (42), we obtain

$$(vA, Du)^j_\lambda = \left[\frac{\alpha^i v^i + \alpha^e v^e}{2} + \left(\frac{1}{2} - \lambda\right)(v^i\alpha^i - v^e\alpha^e)\right](u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u,$$

that is (40) holds. □

In the last part of this section we will prove a generalized Gauss–Green formula for vector fields $\boldsymbol{A} \in \mathcal{DM}^\infty(\mathbb{R}^N)$ on a set $E \subset \mathbb{R}^N$ of finite perimeter, generalizing the analogous result for the standard pairing proved in [19, Theorem 5.1].

Using the conventions of Section 2.4, we will assume that the generalized normal vector on $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$ with the measure–theoretic *interior* unit normal vector to E.

Theorem 6.3 (Gauss-Green)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\mathbb{R}^N)$ and $u \in BV(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, |\operatorname{div}\boldsymbol{A}|)$. Let $E \subset \mathbb{R}^N$ be a bounded set with finite perimeter. Then the following Gauss–Green formulas hold:*

$$\int_{E^1} u^\lambda\, d\operatorname{div} A + \int_{E^1} (A, Du)_\lambda = -\int_{\partial^* E} \operatorname{Tr}^i(A, \partial^* E)\, u^i\; d\mathcal{H}^{N-1}\,, \tag{43}$$

$$\int_{E^1 \cup \partial^* E} u^\lambda\, d\operatorname{div} A + \int_{E^1 \cup \partial^* E} (A, Du)_\lambda = -\int_{\partial^* E} \operatorname{Tr}^e(A, \partial^* E)\, u^e\; d\mathcal{H}^{N-1}\,, \tag{44}$$

where E^1 is the measure theoretic interior of E and $\partial^* E$ is oriented with respect to the interior unit normal vector.

Proof. We recall that, by Lemma 3.2, $u^\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}^N, |\operatorname{div}\boldsymbol{A}|)$, whereas, from [3, Theorem 3.84], $u^i, u^e \in L^1(\partial^* E, \mathcal{H}^{N-1} \llcorner \partial^* E)$. Recalling (20), we have that

$$\int_{E^1} (A, Du)_\lambda = \int_{E^1} (A, Du)_* + \int_{E^1} \left(\frac{1}{2} - \lambda\right)(u^+ - u^-)\, d\operatorname{div} A\,.$$

On the other hand, by the definition (11) of u^λ, it holds

$$\int_{E^1} u^\lambda\, d\operatorname{div} A = \int_{E^1} u^* d\operatorname{div} A - \int_{E^1} \left(\frac{1}{2} - \lambda\right)(u^+ - u^-)\, d\operatorname{div} A$$

so that (43) follows from the Gauss–Green formula for the standard pairing proved in [19, Theorem 5.1]. The validity of (44) can be checked in a very similar way. □

7. Semicontinuity results

In this section we consider the pairing as a function in BV

$$BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|) \ni u \mapsto (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}_b(\Omega),$$

where $\mathcal{M}_b(\Omega)$ denotes the space of finite Borel measures on Ω (see (21)).

Our aim is to characterize the selections $\lambda\colon \Omega \to [0,1]$ such that the above map is lower (resp. upper) semicontinuous, meaning that, if $(u_n) \subset BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ is a sequence converging to a function $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ (in a suitable way), then

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \leq \liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0$$

$$\left(\text{resp.} \quad \langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \geq \limsup_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0\right).$$

Since $(\boldsymbol{A}, Du)_\lambda$ is affected by the pointwise value of u, the correct notion of convergence in BV seems to be the strict one (see e.g. [3, Definition 3.14]).

Definition 7.1. *The sequence $(u_n) \subset BV(\Omega)$ strictly converges to $u \in BV(\Omega)$ if (u_n) converges to u in $L^1(\Omega)$ and the total variations $|Du_n|(\Omega)$ converge to $|Du|(\Omega)$.*

We recall a recent result concerning the pointwise behavior of strictly converging sequences.

Proposition 7.2. *Every sequence (u_n) strictly convergent in $BV(\Omega)$ to u admits a subsequence (u_{n_k}) such that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$*

$$u^-(x) \leq \liminf_k u^-_{n_k}(x) \leq \limsup_k u^+_{n_k}(x) \leq u^+(x). \tag{45}$$

In particular, $\lim_k \widetilde{u}_{n_k}(x) = \widetilde{u}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus J_u$.

Proof. See [28], Theorem 3.2, and Corollary 3.3. □

Combining Proposition 7.2 with Theorem 3.3 in [7], we obtain the following approximation result.

Proposition 7.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $u \in BV(\Omega)$. Then there exist two sequences $(u_n), (v_n) \subset W^{1,1}(\Omega)$ such that:*

(a) *for every $n \in \mathbb{N}$, $\widetilde{v}_n \leq u^-$ and $u^+ \leq \widetilde{u}_n$ $\mathcal{H}^{N-1}$-a.e. in Ω;*
(b) *$u_n \to u$, $v_n \to u$ strictly in BV;*
(c) *$\widetilde{u}_n(x) \to u^+(x)$ and $\widetilde{v}_n(x) \to u^-(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$.*

If, in addition, $u \in L^\infty(\Omega)$, then the above sequences are bounded in $L^\infty(\Omega)$.

Proof. From Theorem 3.3 in [7], there exists a sequence $(u_n) \subset W^{1,1}(\Omega)$, strictly convergent to u, and such that $\widetilde{u}_n \geq u^+$ $\mathcal{H}^{N-1}$-a.e. in Ω, for every $n \in \mathbb{N}$. Moreover, if u is bounded, then this sequence is bounded in $L^\infty(\Omega)$. By Proposition 7.2, we can extract a subsequence (not relabeled) such that

$$\limsup_n u_n^+(x) \leq u^+(x), \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega.$$

On the other hand, the inequality $\widetilde{u}_n \geq u^+$ gives

$$\liminf_n u_n^+(x) \geq u^+(x), \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega,$$

hence the assertion for (u_n) follows. The construction of (v_n) can be done in a similar way. □

In order to state the semicontinuity results, a more piece of notation is needed. Given a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, let us denote by $\Omega_{\boldsymbol{A}}$ the set of points $x \in \Omega$ such that x belongs to the support of $\operatorname{div} \boldsymbol{A}$ (i.e. $|\operatorname{div} \boldsymbol{A}|(B_r(x) \cap \Omega) > 0$ for every $r > 0$), and the limit

$$\psi_{\boldsymbol{A}}(x) := \lim_{r\to 0} \frac{\operatorname{div} \boldsymbol{A}(B_r(x))}{|\operatorname{div} \boldsymbol{A}|(B_r(x))}$$

exists in $\mathbb{R}$, with $|\psi_{\boldsymbol{A}}(x)| = 1$. If we extend $\psi_{\boldsymbol{A}} = 0$ in $\Omega \setminus \Omega_{\boldsymbol{A}}$, we have that $\psi_{\boldsymbol{A}} \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ and the polar decomposition $\operatorname{div} \boldsymbol{A} = \psi_{\boldsymbol{A}} |\operatorname{div} \boldsymbol{A}|$ holds. Moreover, if we define the sets

$$\Omega^+_{\boldsymbol{A}} := \{x \in \Omega_{\boldsymbol{A}} :\ \psi_{\boldsymbol{A}}(x) = 1\}, \quad \Omega^-_{\boldsymbol{A}} := \{x \in \Omega_{\boldsymbol{A}} :\ \psi_{\boldsymbol{A}}(x) = -1\}, \tag{46}$$

then $(\operatorname{div} \boldsymbol{A})^+ = \operatorname{div} \boldsymbol{A} \llcorner \Omega^+_{\boldsymbol{A}}$ and $(\operatorname{div} \boldsymbol{A})^- = \operatorname{div} \boldsymbol{A} \llcorner \Omega^-_{\boldsymbol{A}}$.

Let $\Theta_{\boldsymbol{A}}$ be the jump set of the measure $|\operatorname{div} \boldsymbol{A}|$ (see Proposition 2.3). Since $\Theta_{\boldsymbol{A}}$ is σ-finite with respect to $\mathcal{H}^{N-1}$, then there exists a countably $\mathcal{H}^{N-1}$-rectifiable Borel set $\Theta^r_{\boldsymbol{A}} \subseteq \Theta_{\boldsymbol{A}}$ such that $\Theta^u_{\boldsymbol{A}} := \Theta_{\boldsymbol{A}} \setminus \Theta^r_{\boldsymbol{A}}$ is purely $\mathcal{H}^{N-1}$-unrectifiable (i.e. $\mathcal{H}^{N-1}(\Theta^u_{\boldsymbol{A}} \cap \Sigma) = 0$ for every countably $\mathcal{H}^{N-1}$-rectifiable set Σ, see [3, Definition 2.64 and Proposition 2.76]).

Let us define the families of selections

$$\Lambda_{\text{lsc}} := \left\{\lambda\colon \Omega \to [0,1] \text{ Borel: } \lambda = 0\ \mathcal{H}^{N-1}\text{-a.e. in } \Theta^r_{\boldsymbol{A}} \cap \Omega^-_{\boldsymbol{A}}, \lambda = 1\ \mathcal{H}^{N-1}\text{-a.e. in } \Theta^r_{\boldsymbol{A}} \cap \Omega^+_{\boldsymbol{A}}\right\}$$
$$\Lambda_{\text{usc}} := \left\{\lambda\colon \Omega \to [0,1] \text{ Borel: } \lambda = 1\ \mathcal{H}^{N-1}\text{-a.e. in } \Theta^r_{\boldsymbol{A}} \cap \Omega^-_{\boldsymbol{A}}, \lambda = 0\ \mathcal{H}^{N-1}\text{-a.e. in } \Theta^r_{\boldsymbol{A}} \cap \Omega^+_{\boldsymbol{A}}\right\}.$$

These families satisfy the following extremality properties.

Lemma 7.4. *Given $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, $\varphi \in C_0(\Omega)$, $\varphi \geq 0$, then for every Borel function $\lambda\colon \Omega \to [0,1]$ it holds*

$$\int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} \leq \int_{\Omega^+_{\boldsymbol{A}}} u^+ \varphi \, d \operatorname{div} \boldsymbol{A}, \qquad \int_{\Omega^-_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} \leq \int_{\Omega^-_{\boldsymbol{A}}} u^- \varphi \, d \operatorname{div} \boldsymbol{A}, \tag{47}$$

with equality if $\lambda \in \Lambda_{\text{lsc}}$.

Similarly,

$$\int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} \geq \int_{\Omega^+_{\boldsymbol{A}}} u^- \varphi \, d \operatorname{div} \boldsymbol{A}, \qquad \int_{\Omega^-_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} \geq \int_{\Omega^-_{\boldsymbol{A}}} u^+ \varphi \, d \operatorname{div} \boldsymbol{A}, \tag{48}$$

with equality if $\lambda \in \Lambda_{\text{usc}}$.

Proof. Let us prove the claim only for the first inequality in (47), the other being similar.

Since, by the very definition of $\Omega^+_{\boldsymbol{A}}$,

$$\int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} = \int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d |\operatorname{div} \boldsymbol{A}|$$

and $u^\lambda \leq u^+$ $\mathcal{H}^{N-1}$-a.e. in Ω, the first inequality in (47) follows.

Let $\lambda \in \Lambda_{\text{lsc}}$ and let us prove that equality holds in the first inequality in (47). Let us decompose the set $\Omega^+_{\boldsymbol{A}}$, defined in (46), as the union of the disjoint sets

$$\Omega^+_{\boldsymbol{A}} \setminus J_u, \quad \Omega^+_{\boldsymbol{A}} \cap (J_u \cap \Theta_{\boldsymbol{A}}), \quad \Omega^+_{\boldsymbol{A}} \cap (J_u \setminus \Theta_{\boldsymbol{A}}),$$

that, in turn, coincide up to sets of $\mathcal{H}^{N-1}$-measure zero respectively with

$$\Omega^+_{\boldsymbol{A}} \setminus S_u, \quad \Omega^+_{\boldsymbol{A}} \cap \Theta^r_{\boldsymbol{A}} \cap J_u, \quad (\Omega^+_{\boldsymbol{A}} \setminus \Theta_{\boldsymbol{A}}) \cap J_u.$$

Observe that $u^\lambda = \widetilde{u}\ \mathcal{H}^{N-1}$-a.e. (hence $|\operatorname{div}\boldsymbol{A}|$-a.e.) in $\Omega^+_{\boldsymbol{A}} \setminus S_u$, $u^\lambda = u^+\ \mathcal{H}^{N-1}$-a.e. in $\Omega^+_{\boldsymbol{A}} \cap \Theta^r_{\boldsymbol{A}}$, and, by Proposition 2.3, $|\operatorname{div}\boldsymbol{A}|((\Omega^+_{\boldsymbol{A}} \setminus \Theta_{\boldsymbol{A}}) \cap J_u) = 0$ Hence,

$$\begin{aligned}\int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d \operatorname{div}\boldsymbol{A} &= \int_{\Omega^+_{\boldsymbol{A}}} u^\lambda \varphi \, d\,|\operatorname{div}\boldsymbol{A}| \\ &= \int_{\Omega^+_{\boldsymbol{A}}\setminus S_u} \widetilde{u}\,\varphi \, d\,|\operatorname{div}\boldsymbol{A}| + \int_{\Omega^+_{\boldsymbol{A}}\cap\Theta^r_{\boldsymbol{A}}\cap J_u} u^+\,\varphi \, d\,|\operatorname{div}\boldsymbol{A}| \\ &= \int_{\Omega^+_{\boldsymbol{A}}} u^+ \varphi \, d \operatorname{div}\boldsymbol{A}\,. \end{aligned}$$

□

Corollary 7.5. *Given $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$, it holds:*

$$(\boldsymbol{A}, Du)_\lambda = -u^+ (\operatorname{div}\boldsymbol{A})^+ + u^- (\operatorname{div}\boldsymbol{A})^- + \operatorname{div}(u\,\boldsymbol{A}), \qquad \forall \lambda \in \Lambda_{\mathrm{lsc}}, \tag{49}$$

$$(\boldsymbol{A}, Du)_\lambda = -u^- (\operatorname{div}\boldsymbol{A})^+ + u^+ (\operatorname{div}\boldsymbol{A})^- + \operatorname{div}(u\,\boldsymbol{A}), \qquad \forall \lambda \in \Lambda_{\mathrm{usc}}. \tag{50}$$

In particular,

$$(\boldsymbol{A}, Du)_\lambda = \min\{(\boldsymbol{A}, Du)_0\,, (\boldsymbol{A}, Du)_1\}, \qquad \forall \lambda \in \Lambda_{\mathrm{lsc}}, \tag{51}$$

$$(\boldsymbol{A}, Du)_\lambda = \max\{(\boldsymbol{A}, Du)_0\,, (\boldsymbol{A}, Du)_1\}, \qquad \forall \lambda \in \Lambda_{\mathrm{usc}}. \tag{52}$$

Moreover, if the orientation of J_u is chosen in such a way that $u^+ = u^i$, then,

$$(\boldsymbol{A}, Du)^j_\lambda = \min\{\mathrm{Tr}^i(\boldsymbol{A}, J_u),\ \mathrm{Tr}^e(\boldsymbol{A}, J_u)\}\,(u^+ - u^-)\mathcal{H}^{N-1} \llcorner J_u, \qquad \forall \lambda \in \Lambda_{\mathrm{lsc}}\,, \tag{53}$$

$$(\boldsymbol{A}, Du)^j_\lambda = \max\{\mathrm{Tr}^i(\boldsymbol{A}, J_u),\ \mathrm{Tr}^e(\boldsymbol{A}, J_u)\}\,(u^+ - u^-)\mathcal{H}^{N-1} \llcorner J_u, \qquad \forall \lambda \in \Lambda_{\mathrm{usc}}\,. \tag{54}$$

Proof. The first part is a direct consequence of the equality case in Lemma 7.4.

Let us prove (51). To simplify the notation, let

$$\mu := \operatorname{div}\boldsymbol{A}, \qquad \nu := \min\{(\boldsymbol{A}, Du)_0\,, (\boldsymbol{A}, Du)_1\}.$$

Since $(\boldsymbol{A}, Du)_0 = -u^-\mu + \operatorname{div}(u\boldsymbol{A})$ and $(\boldsymbol{A}, Du)_1 = -u^+\mu + \operatorname{div}(u\boldsymbol{A})$, by definition of minimum of two measures, for every Borel set $E \subset \Omega$ one has

$$\nu(E) = \operatorname{div}(u\boldsymbol{A})(E) + \inf\left\{-u^-\mu^+(E_0) - u^+\mu^+(E_1) + u^-\mu^-(E_0) + u^+\mu^-(E_1)\right\},$$

where the infimum is taken over the pairs E_0, E_1 of disjoint Borel sets such that $E = E_0 \cup E_1$. Setting $E^- := E \cap \Omega^-_{\boldsymbol{A}}$ and $E^+ := E \setminus E^-$, then $E \cap \Omega^+_{\boldsymbol{A}} \subset E^+$ and

$$\begin{aligned} -u^-\mu^+(E_0) - u^+\mu^+(E_1) &\geq -u^+\mu^+(E^+) = -u^+\mu^+(E), \\ u^-\mu^-(E_0) + u^+\mu^-(E_1) &\geq u^-\mu^-(E^-) = u^-\mu^-(E), \end{aligned}$$

for every partition $\{E_0, E_1\}$ of E. Hence,

$$\nu(E) = \operatorname{div}(u\boldsymbol{A})(E) - u^+\mu^+(E) + u^-\mu^-(E) = (\boldsymbol{A}, Du)_\lambda\,(E), \qquad \forall \lambda \in \Lambda_{\mathrm{lsc}}\,.$$

The proof of (52) is similar. Finally, (53) and (54) are consequences of (51) and (52), respectively, and Proposition 4.7. □

Theorem 7.6. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, and let $\lambda\colon \Omega \to [0,1]$ be a Borel function.*

Then $\lambda \in \Lambda_{\mathrm{lsc}}$ if and only if, for every $u_n, u \in BV(\Omega)$ satisfying

(a) *$u_n \to u$ strictly in BV,*
(b) *there exists $g \in L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$ such that, for every $n \in \mathbb{N}$, $|u_n| \leq g$ $|\operatorname{div}\boldsymbol{A}|$-a.e. in Ω,*

it holds

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \leq \liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi\in C_c^\infty(\Omega),\ \varphi\geq 0. \tag{55}$$

Analogously, $\lambda\in\Lambda_{\rm usc}$ if and only if, for every $u_n, u\in BV(\Omega)$ satisfying (a), (b), (c), it holds

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \geq \limsup_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi\in C_c^\infty(\Omega),\ \varphi\geq 0. \tag{56}$$

Proof. Let us prove only the statement concerning the lower semicontinuity, the other being similar.

Let $\lambda\in\Lambda_{\rm lsc}$, let $u_n, u\in BV(\Omega)$ satisfy (a), (b), and let us prove that the semicontinuity property in (55) holds. Let $\varphi\in C_c^\infty(\Omega)$, $\varphi\geq 0$, and let (u_{n_k}) be a subsequence such that

$$\liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle = \lim_k \left\langle (\boldsymbol{A}, Du_{n_k})_\lambda \,,\, \varphi\right\rangle,$$

and (45) holds true (here we use (a) and Proposition 7.2).

From Lemma 7.4, assumption (b), Fatou's Lemma and the pointwise estimates (45) we have that

$$\limsup_k \int_{\Omega_{\boldsymbol{A}}^+} u_{n_k}^\lambda\,\varphi\,d\operatorname{div}\boldsymbol{A} \leq \limsup_k \int_{\Omega_{\boldsymbol{A}}^+} u_{n_k}^+\,\varphi\,d\operatorname{div}\boldsymbol{A} \leq \int_{\Omega_{\boldsymbol{A}}^+} u^+\,\varphi\,d\operatorname{div}\boldsymbol{A}\,. \tag{57}$$

Recalling that

$$\int_{\Omega_{\boldsymbol{A}}^-} u_{n_k}^\lambda\,\varphi\,d\operatorname{div}\boldsymbol{A} = -\int_{\Omega_{\boldsymbol{A}}^-} u_{n_k}^\lambda\,\varphi\,d|\operatorname{div}\boldsymbol{A}|\,,$$

the same argument gives

$$\limsup_k \int_{\Omega_{\boldsymbol{A}}^-} u_{n_k}^\lambda\,\varphi\,d\operatorname{div}\boldsymbol{A} \leq \int_{\Omega_{\boldsymbol{A}}^-} u^-\,\varphi\,d\operatorname{div}\boldsymbol{A}\,. \tag{58}$$

Since $|\operatorname{div}\boldsymbol{A}|(\Omega\setminus(\Omega_{\boldsymbol{A}}^-\cup\Omega_{\boldsymbol{A}}^+))=0$, from (57), (58) and the equality case in (47) we get

$$\limsup_k \int_{\Omega} u_{n_k}^\lambda\,\varphi\,d\operatorname{div}\boldsymbol{A} \leq \int_{\Omega_{\boldsymbol{A}}^+} u^+\,\varphi\,d\operatorname{div}\boldsymbol{A} + \int_{\Omega_{\boldsymbol{A}}^-} u^-\,\varphi\,d\operatorname{div}\boldsymbol{A} = \int_{\Omega} u^\lambda\,\varphi\,d\operatorname{div}\boldsymbol{A}\,. \tag{59}$$

Finally, from (59) and (a) we conclude that

$$\begin{aligned}
&\liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle = -\limsup_k \left(\int_\Omega u_{n_k}^\lambda\,\varphi\,d\operatorname{div}A + \int_\Omega u_{n_k}\,\boldsymbol{A}\cdot\nabla\varphi\,dx\right)\\
&\geq -\int_{\Omega_{\boldsymbol{A}}} u^\lambda\,\varphi\,d\operatorname{div}A - \int_\Omega u\,\boldsymbol{A}\cdot\nabla\varphi\,dx\\
&= \langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle\,,
\end{aligned}$$

i.e., (55) holds true.

Assume now that (55) holds true for every $u_n, u\in BV(\Omega)$ satisfying (a), (b), and let us prove that $\lambda\in\Lambda_{\rm lsc}$. We claim that, under these assumptions,

$$(\boldsymbol{A}, Du)_\lambda \leq (\boldsymbol{A}, Du)_0\,,\quad (\boldsymbol{A}, Du)_\lambda \leq (\boldsymbol{A}, Du)_1\,,\qquad \forall u\in BV(\Omega)\cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|), \tag{60}$$

in the sense of measures. By a truncation argument and Remark 4.5, it is enough to show that the above inequality holds for every $u\in BV(\Omega)\cap L^\infty(\Omega)$. Let $u\in BV(\Omega)\cap L^\infty(\Omega)$ and let $(u_n), (v_n)\subset W^{1,1}(\Omega)\cap L^\infty(\Omega)$ be the approximating sequences given by Proposition 7.3.

Since $(\widetilde{u}_n)$ converges to u^+ $|\operatorname{div} \boldsymbol{A}|$-a.e. in Ω and, by (b), also in $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, for every test function $\varphi \in C_c^\infty(\Omega)$ we have that

$$\begin{aligned}\lim_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle &= \lim_n \left(-\int_\Omega \widetilde{u}_n \,\varphi \, d\operatorname{div} \boldsymbol{A} - \int_\Omega u_n \,\boldsymbol{A}\cdot\nabla\varphi\, dx\right) \\ &= -\int_\Omega u^+ \,\varphi\, d\operatorname{div}\boldsymbol{A} - \int_\Omega u\,\boldsymbol{A}\cdot\nabla\varphi\,dx = \langle (\boldsymbol{A}, Du)_1 \,,\, \varphi\rangle\,,\end{aligned}$$

hence, by the semicontinuity assumption, if $\varphi \geq 0$,

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \leq \liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle = \langle (\boldsymbol{A}, Du)_1 \,,\, \varphi\rangle\,.$$

The same argument, using the sequence (v_n), shows that

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle \leq \liminf_n \langle (\boldsymbol{A}, Dv_n)_\lambda \,,\, \varphi\rangle = \langle (\boldsymbol{A}, Du)_0 \,,\, \varphi\rangle\,,$$

so that (60) follows.

Let $\Omega' \Subset \Omega$ be an open domain with C^1 boundary. From Proposition 4.7 we have that

$$(\boldsymbol{A}, D\chi_{\Omega'})_\lambda = \left[(1-\lambda)\operatorname{Tr}^i(\boldsymbol{A}, \partial\Omega') + \lambda \operatorname{Tr}^e(\boldsymbol{A}, \partial\Omega')\right] \mathcal{H}^{N-1} \llcorner \partial\Omega',$$

hence, the inequalities (60) give

$$\begin{cases} (1-\lambda)\left[\operatorname{Tr}^i(\boldsymbol{A}, \partial\Omega') - \operatorname{Tr}^e(\boldsymbol{A}, \partial\Omega')\right] \leq 0, \\ -\lambda \left[\operatorname{Tr}^i(\boldsymbol{A}, \partial\Omega') - \operatorname{Tr}^e(\boldsymbol{A}, \partial\Omega')\right] \leq 0, \end{cases} \qquad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega'. \tag{61}$$

Let $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$-rectifiable set. Recalling the definition of normal traces given in Section 2.4, from (61) we deduce that

$$\begin{cases} (1-\lambda)\left[\operatorname{Tr}^i(\boldsymbol{A}, \Sigma) - \operatorname{Tr}^e(\boldsymbol{A}, \Sigma)\right] \leq 0, \\ -\lambda \left[\operatorname{Tr}^i(\boldsymbol{A}, \Sigma) - \operatorname{Tr}^e(\boldsymbol{A}, \Sigma)\right] \leq 0, \end{cases} \qquad \mathcal{H}^{N-1}\text{-a.e. on } \Sigma. \tag{62}$$

Let us choose an orientation for the countably $\mathcal{H}^{N-1}$-rectifiable set $\Sigma^+ := \Theta^r_{\boldsymbol{A}} \cap \Omega^+_{\boldsymbol{A}}$. Since $\Sigma^+ \subset \Omega_{\boldsymbol{A}}$ and $\psi_{\boldsymbol{A}}(x) = 1$ for $|\operatorname{div}\boldsymbol{A}|$-a.e. $x \in \Sigma^+$, from (16) we have that

$$\operatorname{div}\boldsymbol{A} \llcorner \Sigma^+ = \left[\operatorname{Tr}^i(\boldsymbol{A}, \Sigma^+) - \operatorname{Tr}^e(\boldsymbol{A}, \Sigma^+)\right] \mathcal{H}^{N-1} \llcorner \Sigma^+ > 0.$$

Hence, from the first inequality in (62), we deduce that $\lambda = 1$ $\mathcal{H}^{N-1}$-a.e. on Σ^+. A similar argument, using $\Sigma^- := \Theta^r_{\boldsymbol{A}} \cap \Omega^-_{\boldsymbol{A}}$, shows that $\lambda = 0$ $\mathcal{H}^{N-1}$-a.e. on Σ^-. □

Corollary 7.7. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and let $\lambda \colon \Omega \to [0,1]$ be a Borel function. Then the continuity property*

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle = \lim_n \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{63}$$

holds for every $u_n, u \in BV(\Omega)$ satisfying (a) and (b) in Theorem 7.6 if and only if $\mathcal{H}^{N-1}(\Theta^r_{\boldsymbol{A}}) = 0$.

Proof. We have that the stated property holds if and only if both (55) and (56) hold. From Theorem 7.6, these inequalities hold (for every (u_n), u) if and only if $\lambda \in \Lambda_{\mathrm{lsc}} \cap \Lambda_{\mathrm{usc}}$. Finally, from the very definition of Λ_{lsc} and Λ_{usc}, we have that $\Lambda_{\mathrm{lsc}} \cap \Lambda_{\mathrm{usc}} \neq \emptyset$ if and only if $\mathcal{H}^{N-1}(\Theta^r_{\boldsymbol{A}}) = 0$. □

Remark 7.8. The assumption $\mathcal{H}^{N-1}(\Theta^r_{\boldsymbol{A}}) = 0$ is trivially satisfied if $\operatorname{div}^j \boldsymbol{A} = 0$, e.g. if $\operatorname{div}\boldsymbol{A} \in L^1(\Omega)$.

Example 7.9. In view of Corollary 7.7 we have that, in general, the continuity property (63) does not hold with respect to the strict convergence in BV. Namely, let $\Omega = (-2,2) \subset \mathbb{R}$ and consider $\boldsymbol{A} := \chi_{(-1,1)}$, so that $\operatorname{div} \boldsymbol{A} = \delta_{-1} - \delta_1$, and $\Theta^r_{\boldsymbol{A}} = \Theta_{\boldsymbol{A}} = \{-1,+1\}$ is not empty. Let $\lambda\colon \Omega \to [0,1]$ be any Borel function. Let $u_n(x) := \max\{\min\{n+1-n|x|, 1\}, 0\}$. It is readily seen that (u_n) strictly converges to $u := \chi_{[-1,1]}$, so that $(-u_n)$ strictly converges to $-u$, and $\langle (\boldsymbol{A}, Du_n)_\lambda\,, \varphi\rangle = 0$ for every n. On the other hand, choosing φ such that $\varphi(-1) = 0$ and $\varphi(1) = 1$, one has

$$\langle (\boldsymbol{A}, Du)_\lambda\,, \varphi\rangle = [1-\lambda(-1)]\,\varphi(-1) + [\lambda(1)-1]\,\varphi(1) = \lambda(1) - 1,$$
$$\langle (\boldsymbol{A}, D(-u))_\lambda\,, \varphi\rangle = -\lambda(-1)\,\varphi(-1) + \lambda(1)\,\varphi(1) = \lambda(1),$$

and at least one of the right-hand sides must be different from 0.

Example 7.10. We remark that, in general, (55) does not hold if assumption (a) is replaced by the weak* convergence in BV. Namely, let us consider $\Omega = (-2,2) \subset \mathbb{R}$, $\boldsymbol{A} := \chi_{(0,1)}$ and $u_n(x) := \max\{1-n|x|, 0\}$. Since (u_n) converges to $u = 0$ in $L^1(\Omega)$ and $|Du_n|(\Omega) = 2$ for every n, then (u_n) converges weakly* to u in $BV(\Omega)$. (see [3, Proposition 3.13]): If $\varphi \in C^\infty_c(\Omega)$ is strictly positive in 0, one has

$$\liminf_n \langle (\boldsymbol{A}, Du_n)_\lambda\,, \varphi\rangle = \liminf_n \left(-u_n(0)\,\varphi(0) + u_n(1)\,\varphi(1) - \int_0^1 u_n\,\varphi'\right)$$
$$= -\varphi(0) < 0 = \langle (\boldsymbol{A}, Du)_\lambda\,, \varphi\rangle\,,$$

so that (55) does not hold.

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