Results and Questions on Minimum Problems for Eigenvalues

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Abstract

For every open subset Ω of $I\!R^N$ we consider the sequence $\lambda(\Omega) = (\lambda_j(\Omega))_j$ of eigenvalues of the Laplace operator $-\Delta$ on $H_0^1(\Omega)$ and we consider the minimization problem

$$\min\{\Phi(\lambda(\Omega)) \ : \ |\Omega| = m, \Omega \in \mathcal{A}\}$$

where the function Φ is given and $\mathcal{A}$ denotes the class of admissible domains. About this problem, we present some known results and we list some open questions.

1 Introduction

Minimization problems for eigenvalues are not new in the literature; since the first result by Krahn [20], [21] and Faber [16] concerning the minimality of a disk in $I\!R^2$ for the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet boundary conditions, among domains with equal area, many other results have been obtained.

We are concerned with problems of the form

$$\min\{\Phi(\lambda(\Omega)) \ : \ |\Omega| = m, \Omega \in \mathcal{A}\} \tag{1}$$

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where $\lambda(\Omega)$ denotes the sequence $(\lambda_j(\Omega))_j$ of all eigenvalues of the Laplace operator with Dirichlet boundary conditions, Φ is a given function, $m > 0$ is given and $\mathcal{A}$ is the class of admissible domains. A choice we can do is to take $\mathcal{A} = \{\Omega \subseteq D\}$ where the domain D (the so called "*design region*") is given, a bounded open subset of $I\!R^N$, or also $\mathcal{A} = \{\Omega \subseteq I\!R^N\}$ without any a priory imposed boundedness. We shall see that the two cases are quite different, and most of the results obtained for the first cannot be easily extended to the second.

The plan of the paper is the following.

- In section 2 we present a general result concerning shape optimization problems with monotone cost functionals; we introduce the notion of γ-convergence and we show that monotone decreasing costs which are γ-lower semicontinuous always admit a minimizer among domains with given measure contained in a bounded design region. This will imply, as a consequence, the existence of a solution for problem (1) under monotonicity assumptions on the function Φ.

- In section 3 the case of geometrical constraints on the class of admissible domains is considered. Starting from the very strict case of admissible domains with a Lipschitz constant a priori bounded, some weaker conditions are listed, all implying the existence of solutions of shape optimization problems, and in particular of (1) for any continuous function Φ.

- In section 4 we consider the very special case of a function Φ which depends only on the first two eigenvalues λ_1, λ_2. We show that without any geometrical constraint on the admissible domains, in this case we obtain the existence of minimizers for problem (1) for any continuous (even nonmonotone) function Φ.

- Finally, section 5 is devoted to some open questions.

2 Monotone cost functions

In this section we consider general shape optimization problems of the form

$$\min\{F(\Omega) : |\Omega| = m, \Omega \in \mathcal{A}\} \tag{2}$$

where the admissible class $\mathcal{A}$ is the family of all quasi-open subsets Ω of a given bounded design region D of $I\!R^N$. The choice of quasi-open sets in this framework seems to be very natural (see Buttazzo and Dal Maso [14]); we refer to the book [19] for any further detail about these domains which are very useful in potential theory.

To every $\Omega \in \mathcal{A}$ we can associate the function w_Ω which is defined as the unique solution of the problem formally written as

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \overline{D} \setminus \Omega. \end{cases} \tag{3}$$

More precisely, w_Ω can be seen as the unique minimizer of the functional defined on $H_0^1(D)$ by

$$F_\Omega(u) = \begin{cases} \int_D \frac{1}{2}|\nabla u|^2 dx - \int_D u dx & \text{if } u = 0 \text{ q.e. on } \overline{D} \setminus \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

By the help of the functions w_Ω we can introduce now a convergence on the class $\mathcal{A}$.

Definition 2.1 *We say that a sequence $(\Omega_n)_n$ in $\mathcal{A}$ γ-converges to a domain $\Omega \in \mathcal{A}$ if the solutions w_{Ω_n} converge weakly in $H_0^1(D)$ to the solution w_Ω.*

Remark 2.2 The convergence above turns out to be equivalent (see Sverak [23]) to the Γ-convergence of the functionals

$$F_{\Omega_n}(u) = \begin{cases} \int_D \frac{1}{2}|\nabla u|^2 dx & \text{if } u \in H_0^1(\Omega_n) \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

to the functional F_Ω defined in the similar way, with respect to the $L^2(D)$ strong convergence; this explains why we adopt the notation γ-convergence.

Remark 2.3 The γ-convergence on $\mathcal{A}$ is not compact; indeed several shape optimization problems of the form (2) do not admit any solution, and the introduction of relaxed formulation is needed in order to describe the behavior of minimizing sequences. We do not want to deal here with this topic, and we refer the interested reader to the paper by Buttazzo and Dal Maso [13] where all details about relaxed shape optimization problems with Dirichlet boundary conditions can be found.

Even if in general problem (2) does not admit a solution in $\mathcal{A}$, some particular cases of existence results are available. This is indeed the case when the cost F is monotone decreasing with respect to the set inclusion. More precisely, the following result holds.

Theorem 2.4 *Let $F : \mathcal{A} \to [0, +\infty]$ be a functional such that*
(i) F is γ-lower semicontinuous,
(ii) F is monotone decreasing with respect to the set inclusion.
Then for every $0 \leq m \leq |D|$ problem (2) admits at least a solution in $\mathcal{A}$.

The proof of theorem 2.4 was first given by Buttazzo and Dal Maso in [14]. Another simpler proof can be given by using the new concept of weak γ-convergence, introduced by Bucur, Buttazzo and Henrot in [8]. In this way, the existence proof follows by the direct methods of the calculus of variations.

We want to remark that assumption (i) in theorem 2.4 is verified for a large number of situations; for instance we have (see Buttazzo and Dal Maso [14])

$$\Omega_n \xrightarrow{\gamma} \Omega \qquad \Longrightarrow \qquad \lambda_j(\Omega_n) \to \lambda_j(\Omega) \quad \text{for all } \ j \in I\!N.$$

Therefore the cost functions

$$F(\Omega) = \Phi(\lambda(\Omega))$$

satisfies assumption (i) as soon as the function $\Phi : I\!R^{I\!N} \to [0, +\infty]$ is lower semicontinuous, in the sense that

$$z_j^n \to z_j \ \text{ for all } \ j \in I\!N \implies \Phi(z) \leq \liminf_{n\to\infty} \Phi(z^n). \tag{6}$$

Assumption (ii) is on the contrary much more restrictive. However, in the case of problems involving eigenvalues, of the form (1), theorem 2.4 includes the case of functions Φ which are monotone increasing, that is

$$z_j^1 \leq z_j^2 \ \text{ for all } \ j \in I\!N \implies \Phi(z^1) \leq \Phi(z^2). \tag{7}$$

This relies on the well known fact that all the eigenvalues of an elliptic operator with Dirichlet boundary conditions are decreasing functions of the domain. Therefore, summarizing, from theorem 2.4 we can deduce the following result.

Corollary 2.5 *Let $\Phi : \mathbb{R}^N \to [0, +\infty]$ be a function which is lower semicontinuous in the sense of (6), and monotone decreasing in the sense of (7). Then the optimization problem*

$$\min\{\Phi(\lambda(\Omega)) : |\Omega| = m, \Omega \in \mathcal{A}\}$$

admits at least a solution in $\mathcal{A}$.

3 Geometrical constraints

If the functional F is not decreasing with respect to the set inclusion, one has to impose some geometrical constraints on the admissible domains, in order to obtain the γ-convergence of a minimizing sequence to a domain, in other words to obtain the γ-compactness of minimizing sequences. This will be done by endowing the admissible class $\mathcal{A}$ with a suitable topology τ and by imposing to elements of $\mathcal{A}$ suitable geometrical constraints. The role of these constraints is double : on one side they make the topological space $(\mathcal{A}, \tau)$ compact, and on the other side they provide the γ-convergence of any τ-convergent sequence.

Some good tools to deal with this problem are the so called Mosco conditions, which are equivalent to the γ-convergence for a sequence of open sets. Suppose that $(\Omega_n)_n$ is a sequence of open sets contained in a bounded design region D. It is said that the sequence of spaces $H_0^1(\Omega_n)$ converges in the sense of Mosco to the space $H_0^1(\Omega)$ if the following conditions are satisfied:

1. For all $\phi \in H_0^1(\Omega)$ there exists a sequence $\phi_n \in H_0^1(\Omega_n)$ such that ϕ_n converges strongly in $H_0^1(D)$ to ϕ.
2. For any sequence $\phi_{n_k} \in H_0^1(\Omega_{n_k})$ weakly convergent in $H_0^1(D)$ to a function ϕ we have $\phi \in H_0^1(\Omega)$.

If we choose as τ the Hausdorff complementary topology on $\mathcal{A}$, given by the metric

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^N} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|,$$

then it is easy to see that the first Mosco condition is fulfilled for every sequence $\Omega_n \xrightarrow{\tau} \Omega$. Moreover, as it is well known, the space $(\mathcal{A}, \tau)$ is compact, even if it does not turn out to be γ-compact.

In order to satisfy the second Mosco condition, and not to lose compactness, one has to impose some geometrical constraints. We recall now some classical results having these properties. The following classes are γ-compact, listed in an order going from the strongest constraints to the weakest ones.

- The class of convex sets contained in D.
- The class of Lipschitz subsets of D with the Lipschitz constant a priori bounded (see [22], [15]).
- The class of domains satisfying a uniform exterior cone property, i.e. such that for every point x_0 on the boundary of any $\Omega \in \mathcal{A}$ there is a cone, with uniform height and opening, and with vertex in x_0, lying in the complement of Ω.
- The class of domains satisfying a uniform flat cone condition (see Bucur, Zolésio [12]), i.e. as above, but with the weaker requirement that the cone may be flat, that is of dimension $N-1$.
- The class of domains satisfying a uniform capacitary density condition, i.e. such that there exist $c, r > 0$ such that for any $\Omega \in \mathcal{A}$, and for any $x \in \partial\Omega$, we have

  $$\forall t \in (0, r) \qquad \frac{\operatorname{cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\operatorname{cap}(B_{x,t}, B_{x,2t})} \geq c,$$

  where $B_{x,s}$ denotes the ball of radius s centered at x.
- The class of domains satisfying a uniform Wiener condition, i.e. domains satisfying for any $\Omega \in \mathcal{A}$ and for any point $x \in \partial\Omega$

  $$\int_r^R \frac{\operatorname{cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\operatorname{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r, R, x) \text{ for every } 0 < r < R < 1$$

  where $g : (0,1) \times (0,1) \times B \to I\!R_+$ is fixed, such that for every $R \in (0,1)$ and every $x \in B$, $\lim_{r \to 0} g(r, R, x) = +\infty$ locally uniformly on x.

Another interesting γ-compact class which is only of topological type, was given by Sverak [23] and consists in the following.

- For $N = 2$, the class of all open subsets Ω of D for which the number of connected components of $\overline{D} \setminus \Omega$ is uniformly bounded.

In fact, it can be shown (see Bucur [5]) that this result is a consequence of the capacitary density condition, and in N dimensions it was generalized by Bucur and Trebeschi [11] for the p-Laplacian, with $p \in (N-1, N]$.

4 The first two eigenvalues

In this section we consider the very special case of minimum problems involving only the first two eigenvalues, that is

$$\min\{\Phi(\lambda_1(\Omega), \lambda_2(\Omega)) : |\Omega| \leq m, \Omega \in \mathcal{A}\}. \tag{8}$$

As in the previous situations we take the a priori bounded case

$$\mathcal{A} = \{\Omega \ \text{ quasi-open, } \ \Omega \subseteq D\}$$

being D a given bounded open subset of $I\!R^N$. We assume for simplicity that D is a large ball or at least that D contains a sufficiently large ball.

The structure of the set E of all attainable points of $I\!R^2$ whose first and second coordinates are respectively the first and second eigenvalues of $-\Delta$ on $H^1_0(\Omega)$, with $|\Omega| \leq m$ and $\Omega \in \mathcal{A}$, has been recently studied by Bucur, Buttazzo and Figueiredo [7]. Using a result by Ashbaugh and Benguria [1] it is possible to show that

- the set E is bounded by the lines

  $$x = \lambda_1(B), \quad y = \lambda_1(A), \quad x = y, \quad y = x\frac{\lambda_2(B)}{\lambda_1(A)}$$

  being B the ball of measure m and A the union of two disjoint balls of measure $\frac{m}{2}$ each.

In the paper [7] we prove that

- the set E is closed in $I\!R^2$.

As a consequence we obtain the following existence result.

Theorem 4.1 *Let $\Phi : \mathbb{R}^2 \to [0, +\infty]$ be any lower semicontinuous function, and define*

$$\Phi(+\infty, +\infty) = \liminf_{x,y\to\infty} \Phi(x, y). \tag{9}$$

Then the optimization problem (8) admits at least a solution.

Proof Take a minimizing sequence (Ω_n); if $\lambda_2(\Omega_n) \to +\infty$, due to the structure of the set E we also have $\lambda_1(\Omega_n) \to +\infty$. Therefore, if for the empty set we define $\lambda_1(\emptyset) = \lambda_2(\emptyset) = +\infty$, thanks to the position (9) we may conclude that the empty set is a minimizer for problem (8).

Assume on the contrary that $\lambda_2(\Omega_n)$ is bounded. The existence of a minimizer for problem (8) then follows straightforward, since the set E is closed and the function Φ is lower semicontinuous. $\square$

Remark 4.2 An interesting question is to see if the subset E of $\mathbb{R}^2$ is convex. When the dimension $N = 2$ there is a numerical evidence of this fact (see Keller and Wolf [25]) but up to now we do not know any proof or counterexample.

Several other open problems can be raised, in relation to the quite surprising result of theorem 4.1. Some of them are listed below:

1. Is the set E still closed if instead of λ_1 and λ_2 we consider λ_j and λ_k, or more generally $(\lambda_j)_{j\in K}$ where K is a given set of integer numbers? In particular, if $K = \mathbb{N}$, is the set E of all sequences $\lambda(\Omega)$ with $\Omega \in \mathcal{A}$ and $|\Omega| \le m$ closed in $\mathbb{R}^{\mathbb{N}}$?

2. Is the set E still closed if instead of the Laplace operator $-\Delta$ we consider an elliptic operator of the form $-\mathrm{div}(a(x)\nabla u)$ with the matrix $a(x)$ bounded and uniformly elliptic?

3. Do points on ∂E correspond to smooth domains Ω?

Some more numerical computations about problems involving the first two eigenvalues can be found in Finzi Vita [17].

5 Further remarks and open problems

According to theorem 2.4 of section 2, the existence of an optimal domain minimizing the functional $\Omega \mapsto F(\Omega)$ is obtained for quasi-open sets $\Omega \subseteq D$ with $|\Omega| = m$, where D is a bounded design region. It is natural to think that the minimizing set depends on D.

Two natural questions can be formulated.

Problem 1 *If by $\mathcal{A}$ we denote the family of all bounded quasi-open sets of $\mathbb{R}^N$ does problem (2) still have a solution?*

Problem 2 *If a minimizer exists, is this set open and smooth?*

Only for few functionals F these problems were solved. This is the case of λ_1, λ_2, but in general, for functionals satisfying the assumptions of theorem 2.4, the answer is negative.

Consider for example the divergence operator

$$Au = -\mathrm{div}(a(x)\nabla u)$$

where $a : \mathbb{R}^N \to \mathbb{R}$ is given by $a(x) = \frac{2+|x|}{1+|x|}$. Denoting by $F(\Omega)$ the first eigenvalue of A on Ω, one can easily remark that F satisfies the assumptions of theorem 2.4 and that

$$\inf\{F(\Omega) : |\Omega| = m, \Omega \in \mathcal{A}\} = \lambda_1(B)$$

where $\lambda_1(B)$ is the first eigenvalue of the Laplacian on the ball of measure m. A simple computation proves that this infimum is not attained, and hence problem 1 does not have in general a solution if we drop the assumption that admissible domains have to be contained in a bounded design region D.

For a counterexample to problem 2 it is sufficient to consider a bounded quasi-open quasi-connected set A which is not open, and with measure equal to m. Setting $F(\Omega) = \lambda_1(\Omega \cap A)$ where $\lambda_1(\Omega \cap A)$ is the first eigenvalue of the Laplacian $-\Delta$ on $\Omega \cap A$, we have that F satisfies the assumptions of theorem 2.4 and has the unique (up to zero capacity sets) minimizer $\Omega = A$. Hence a minimizer exists, but it is not open.

However, for minimization problems of functions of eigenvalues, of the form (1) with $D = \mathbb{R}^N$ we do not know an answer to problems 1 and 2, even if Φ is monotone in the sense of (7).

For some particular functionals F, a slightly different version of problem 1 is solved in [9]. It is quite natural to extend the family of admissible domains to quasi-open sets of $I\!R^N$ of measure equal to m, but which are not necessarily bounded. Following [24], the compact injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ holds, and hence the spectrum of the Laplacian consists of eigenvalues. For example, in [9] Bucur and Henrot prove the existence of a minimizer for λ_3 in this class. The key tool is a concentration-compactness theorem in relation with the γ-convergence (see [6]). In fact, one can prove that if a minimizer in $\mathcal{A}$ exists and is bounded for λ_h with $h = 1, 2, ..., k$, then a minimizer exists for λ_{k+1}. We refer to [6], [9] for further details.

The concentration-compactness principle for the γ-convergence, can solve the minimization problem (8) if the family of admissible domains is extended to all quasi-open sets of $I\!R^N$ with measure less than or equal to m (see [6]).

Several other questions about the minimization of some particular functionals of eigenvalues are discussed in a recent paper by Ashbaugh [2].

A different question, more related to the regularity of minimizers is the following one. When minimizing λ_1 in the family of quasi-open sets of fixed measure contained in a bounded design region D, then theorem 2.4 proves the existence of a minimizer. It is obvious that if the design region D is large "enough", to contain a ball B of measure m, then the minimizer is B. If D does not contain such a ball, we can prove (by using a quite involved argument which relies on results by Brock [4] and Hayouni [18]) that every minimizer must touch the boundary. When the minimizer Ω is smooth and simply connected, a more direct procedure is to argue by contradiction and to perform the shape derivative which gives

$$\left\{ \begin{array}{rcl} -\Delta u & = & \lambda_1(\Omega) u \text{ on } \Omega \\ u & = & 0 \text{ on } \partial\Omega \\ \frac{\partial u}{\partial n} & = & c \text{ on } \partial\Omega. \end{array} \right. \tag{10}$$

This is the Serrin problem and it is known that Ω must be a ball. Hence one can formulate the following problems.

Problem 3 *Prove (or disprove) the existence of a smooth minimizer Ω in the case D smooth, or more generally that $D \cap \partial\Omega$ is smooth for every (smooth or not) D.*

It will be also interesting to prove (or disprove) that in the cases Ω touches the boundary ∂D, $\partial\Omega$ arrives tangentially on ∂D.

Problem 4 *If a smooth minimizer exists, prove (or disprove) that all connected components of $D \cap \partial\Omega$ have constant curvature.*

Problem 5 *If problem 4 has a positive answer, prove (or disprove) that the constant is the same for each connected component.*

In fact, one can perform a local shape derivative on each connected component of $D \cap \partial\Omega$ and obtain $\frac{\partial u}{\partial n}$ = constant, and the question then becomes a "local" version of the Serrin problem.

Other questions which are related to different versions of the Serrin problem are the following.

Problem 6 *Is the minimizer simply connected if we assume D simply connected?*

Problem 7 *Is the minimizer convex if we assume D convex?*

Note that the regularity problem 3 can be raised for every minimization problem of functions of eigenvalues, like (1).

An interesting question is to find a "continuous" deformation of an arbitrary open set into a ball of same measure, such that the first eigenvalue is decreasing and the measure is kept constant. The continuity here might be of different types: continuity of the second eigenvalue, continuity of all the spectrum, γ-continuity, etc. It is not difficult to see that the results on the set E obtained in [7] give the existence of a deformation $t \mapsto \Omega_t$, $0 \le t \le 1$, satisfying the following conditions:

1. $\Omega_0 = \Omega$, $\Omega_1 = B$;
2. $|\Omega_t| = m$;
3. $t \mapsto \lambda_1(\Omega_t)$ is continuous and not increasing;
4. $t \mapsto \lambda_2(\Omega_t)$ is continuous.

A more difficult question, is to find a transformation satisfying properties 1), 2),3) above and which is γ-continuous. Of course, such a transformation satisfies condition 4). A partial answer was given by Bucur and Henrot in [10], where only the γ-continuity on the left was proved. Nevertheless, this result seems to be true, and the proof will appear in a subsequent article. The main tool is the fine study of the continuous Steiner symmetrization introduced in Brock [3].

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