A TWO-END FAMILY OF SOLUTIONS FOR THE INHOMOGENEOUS ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

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Abstract. In this work we construct a family of entire bounded solutions for the singularly perturbed inhomogeneous Allen-Cahn equation $\varepsilon^2 \mathrm{div}(a(x)\nabla u) - a(x)F'(u) = 0$ in $\mathbb{R}^2$, where $\varepsilon \to 0$. The nodal set of these solutions is close to a "nondegenerate" curve which is asymptotically two non-parallel straight lines. Here F' is a double-well potential and a is a smooth positive function. We also provide examples of curves and functions a where our result applies. This work is in connection with the results found in [9], [7] and [12], handling the compact case.

1. Introduction

In this paper, we consider the singularly perturbed Inhomogeneous Allen-Cahn equation

$$\varepsilon^2 \,\mathrm{div}(a(x)\nabla u(x)) - a(x)F'(u) = 0, \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

To fix ideas we can think of equation (1.1) as the

$$\varepsilon^2 \Delta_g u(x) - F'(u) = 0, \quad \text{in } M \tag{1.2}$$

where (M, g) is the whole plane endowed with a planar metric induced by the potential a. In this setting Δ_g denotes the Laplace Beltrami operator on M and the function $F : \mathbb{R} \to \mathbb{R}$ is a smooth double-well potential, i.e a function satisfying

$$F(s) > 0, \quad \forall s \neq \pm 1, \tag{1.3}$$

$$F(\pm 1) = 0, \tag{1.4}$$

$$\sigma_\pm^2 := F''(\pm 1) > 0. \tag{1.5}$$

The typical example for F corresponds to the balanced and bi-stable twin-pit

$$F(u) = \frac{1}{4}(1-u^2)^2, \quad -F'(u) = u - u^3$$

and $\sigma_\pm = \sqrt{2}$.

Equation (1.2) arises in the *Gradient Theory of Phase Transitions* [1]. In this physical model, there are two different states of a material represented by the values $u = \pm 1$. It is of interest to study nontrivial configurations in which these two states try to coexist. Hence, the function u represents a smooth realization of the phase, which except for a narrow region, is expected to take values close to ± 1, namely the global minima of F.

Let us consider the energy functional

$$J_\varepsilon(u) = \int_M \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\right] dV_g. \tag{1.6}$$

We are interested in critical points of (1.6), which correspond to solutions of (1.2). For any set $\Lambda \subset M$ the function

$$u^*_\Lambda := \chi_\Lambda - \chi_{M\setminus\Lambda}, \quad \text{in} \quad M \tag{1.7}$$

minimizes the second term in (1.6), however, it is not a smooth solution. By considering an $\varepsilon-$regularization of u^*_Λ, say $u_{\Lambda,\varepsilon}$, it can be checked that

$$J_\varepsilon(u_{\Lambda,\varepsilon}) \approx \int_{\partial\Lambda} 1 dS_g \tag{1.8}$$

for $\varepsilon > 0$ small, where dS_g denotes the area element in $\partial\Lambda$. Relation (1.8) entails that, transitions varying from -1 to $+1$ must be selected, for instance, in such way that $\partial\Lambda$ minimizes the perimeter functional

$$\int_{\partial\Lambda} 1 dS_g \tag{1.9}$$

In the case that $\partial\Lambda$ is also a smooth submanifold of M, we say that $\partial\Lambda$ is a *minimal submanifold* of M if $\partial\Lambda$ is critical for the area functional (1.9). In particular, it is easy to see that minimizing submanifolds are critical for (1.9), and therefore they are minimal submanifolds of M.

The intuition behind the previous remarks, was first observed by Modica in [11], based upon the fact that when $\partial\Lambda$ is a smooth submanifold of M, then transitions varying from -1 to $+1$ take place along the normal direction of $\partial\Lambda$ in M, having a $1-$dimensional profile in this direction. This profile corresponds to a function w, which is the heteroclinic solution to the ODE

$$\begin{cases} w''(t) - F'(w(t)) = 0, \quad \text{in} \quad \mathbb{R} \\ w(\pm\infty) = \pm 1 \end{cases}$$

connecting the two states. The existence of w is ensured by conditions (1.3)-(1.4)-(1.5) of F. As for the twin pit nonlinearity, we have that

$$w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right), \quad t \in \mathbb{R}.$$

This intuition gave a great impulse to the Calculus of Variations, and the theory of the $\Gamma-$Convergence in the 70th's. Modica stated that if $M = \Omega \subset \mathbb{R}^N$ is a smooth Euclidean domain and $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of local minimizers of (1.6) with uniformly bounded energy, then up to subsequence, u_ε converges in a $L^1_{loc}(\Omega)$-sense to some limit u^*_Λ of the form (1.7).

There are a number of important works regarding existence and asymptotic behavior of solutions to (1.2), under a variety of different settings. In [12] Pacard and Ritore studied equation (1.2) in the case that (M, g) is a compact Riemannian Manifold, they construct a family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to (1.2) having transition from -1 to $+1$ on a region $\varepsilon-$close to a compact minimal submanifold N, with positive Ricci-curvature

$$k_N := |A_N|^2 + Ric(\nu_N, \nu_N) > 0.$$

Under the same conditions del Pino, Kowalczyk, Wei and Yang constructed in [5] a sequence of solutions with multiple clustered layers collapsing onto N. A gap condition is needed, related with the interaction between interfaces.

There are related other results under a similar setting for M and N, regarding the equation

$$\varepsilon^2\Delta_g u - V(z)F'(u) = 0, \quad \text{in} \quad M$$

done by B.Lai and Z.Du in [9] where a family of solutions with a single transition is constructed. Additionally L.Wang and Z.Du dealt in [8] with the same problem, considering multiple transitions this time. In both works the minimal character and nondegeneracy properties of N, are with respect to the weighted area functional $\int_M V^{1/2}$. In the same line, it is worth to mention a recent work due to Z.Du and C.Gui [7] where they build a smooth solution to the Neumann problem

$$\varepsilon^2\Delta u - V(z)F'(u) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

having a single transition near a smooth closed curve $\Gamma \subset \Omega$, nondegenerate geodesic relative to the arclength $\int_\Gamma V^{1/2}$. Here, Ω is a smooth bounded domain in $\mathbb{R}^2$, and V is an uniformly positive smooth potential.

As for the noncompact case, recently in [4] the authors considered equation (1.2) when $M = \mathbb{R}^3$. Here the authors build for any small $\varepsilon > 0$ a family of solutions with transitions close to a non-degenerate complete embedded minimal surface with finite total curvature.

1.1. **The Main Result.** We consider equation (1.2) in a slightly general form, but we restrict ourselves to dimension $N = 2$. More precisely, let us consider the equation

$$\varepsilon^2 \operatorname{div}(a(x)\nabla u(x)) - a(x)F'(u) = 0, \quad \text{in } \mathbb{R}^2 \tag{1.10}$$

where the function $a(x)$ is a smooth positive function.

As far as our knowledge goes, little is known about entire solutions to (1.10) having a single transition close to a noncompact curve, in the case that $a(x)$ is not identically constant.

In this work, we will consider a smooth noncompact curve $\Gamma = \gamma(\mathbb{R})$, where $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$ is parameterized by arc-length. We denote by $\nu : \Gamma \to \mathbb{R}^2$ the choice of the normal vector to Γ, so that the curve is positively oriented.

In order to state the main result, let us first list our set of assumptions on the function $a(x)$ and the curve Γ.

As for the function $a : \mathbb{R}^2 \to \mathbb{R}$, we assume that

$$a \in C^5(\mathbb{R}^2), \quad 0 < m < a(x) \leq M \tag{1.11}$$

for some positive constants m, M. We also assume that the curvature $k(s)$ of Γ in arch-length variable satisfies

$$|k(s)| + |k'(s)| + |k''(s)| \leq \frac{C}{(1+|s|)^{1+\alpha}} \tag{1.12}$$

for some $\alpha > 0$. In particular, condition (1.12) implies that

$$\dot{\gamma}_{\pm} := \lim_{s \to \pm\infty} \dot{\gamma}(s)$$

are well defined directions in $\mathbb{R}^2$. We must assume further some non-parallelism condition, namely

$$-1 \leq \dot{\gamma}_+ \cdot \dot{\gamma}_- < 1. \tag{1.13}$$

Points $x \in \mathbb{R}^2$ that are $\delta-$close to this curve, with δ small, can be represented using Fermi coordinates as follows

$$x = \gamma(s) + z\nu(s) =: X(s,z), \quad |z| < \delta, \quad s \in \mathbb{R}.$$

Condition (1.13) implies that the mapping $x = X(s,z)$ provides local coordinates in a region of the form

$$\mathcal{N}_\delta := \{x = X(s,z) : |z| < \delta + c_0|s|\}$$

for some fixed constant c_0 and the mapping $x \mapsto (s,z)$ defines a local diffeomorphism.

Finally, abusing notation, by setting $a(s,z) = a(X(s,z))$, we assume that

$$|\nabla_{s,z} a(s,z)| \leq \frac{C}{(1+|s|)^{1+\alpha}}, \quad |D^2_{s,z}\, a(s,z)| \leq \frac{C}{(1+|s|)^{2+\alpha}} \tag{1.14}$$

where $\alpha > 0$ is as in (1.12).

Any smooth curve $\delta-$close to Γ in a C^m-topology can be parameterized by

$$\gamma_h(s) = \gamma(s) + h(s)\nu(s)$$

where h is a small C^m-function. The weighted length of Γ_h, where $\gamma_h : \mathbb{R} \to \Gamma_h$ is given by

$$\begin{aligned} l_\Gamma(h) &:= \int_{\Gamma_h} a(x)d\vec{r} = \int_{-\infty}^{+\infty} a\left(\gamma_h(s)\right)|\dot{\gamma}_h(s)|ds \\ &= \int_{-\infty}^{+\infty} a(s,h(s))|\dot{\gamma} + h\dot{\nu} + h'\nu|ds. \end{aligned}$$

Since $|\dot{\gamma}| = 1$ and $\dot{\nu}(s) = k(s)\dot{\gamma}(s)$, where k is the signed curvature of Γ, we find that

$$l_h(h) = \int_{-\infty}^{+\infty} a(s, h(s))[(1 + kh)^2 + |h'|^2]^{1/2} ds.$$

We say that Γ is a stationary (or a critical) curve respect to the function $a(x)$, if and only if,

$$l'_\Gamma[h] = \int_{\Gamma_h} (\partial_z a(s,0) - a(s,0)k(s))h(s)ds = 0, \quad \forall h \in C_c^\infty(\mathbb{R})$$

which is equivalent to say that the curve Γ satisfies

$$\partial_z a(s,0) = k(s)a(s,0), \quad s \in \mathbb{R}. \tag{1.15}$$

Regarding the stability properties of the stationary curve Γ and the second variation of the length functional l_Γ,

$$l''_\Gamma(h,h) = \int_{-\infty}^{+\infty} \big(a(s,0)|h'(s)|^2 + [\partial_{zz} a(s,0) - 2k^2(s)]h^2(s) \big) ds$$

we have the Jacobi operator of Γ, corresponding to the linear differential operator

$$\mathcal{J}_{a,\Gamma}(h) = h''(s) + \frac{\partial_s a(s,0)}{a(s,0)} h'(s) - \left[\partial_{zz} a(s,0) - 2k^2(s)\right] h(s). \tag{1.16}$$

We say that the stationary curve Γ is also nondegenerate respect to the potential $a(x)$, if the bounded kernel of $\mathcal{J}_{a,\Gamma}$ is the trivial one. The nondegeneracy condition basically implies that $\mathcal{J}_{a,\Gamma}$ has an appropriate right inverse, so that the curve is isolated in a properly chosen topology.

Next, we proceed to state the main result.

Theorem 1. *Assume that $a(x)$ is a smooth potential and let Γ be a smooth stationary and nondegenerate curve respect to the length functional $\int_\Gamma a(x)d\vec{r}$. Assume also that conditions* (1.11)-(1.14) *are satisfied. Then for any $\varepsilon > 0$ small enough, there exists a smooth bounded solution u_ε to the inhomogeneous Allen-Cahn equation* (1.10)*, such that*

$$u_\varepsilon(x) = w\left(\frac{z - h(s)}{\varepsilon}\right) + O(\varepsilon^2), \quad for \quad x = X(s,z), \quad |z| < \delta$$

where the function h satisfies

$$\|h\|_{C^1(\mathbb{R})} \leq C\varepsilon.$$

This solution converges to ± 1, away from Γ, namely

$$u_\varepsilon^2(x) \to \pm 1, \quad dist(\Gamma, x) \to \infty.$$

Remark 1.1. *Throughout the proof of this theorem, we obtain an explicit description for u_ε and its derivatives. We apply infinite dimensional reduction method in the spirit of the pioneering work due to Floer and Weinstein* [10].

The paper is organized as follows. Section 2 deals with the geometrical setting need to set up the proof of theorem 1. In section 3 we present the invertibility theory for the Jacobi operator of the curve Γ while in section 4 we give some examples of the function $a(x)$ and the curve Γ, for which our result applies. Section 5 is devoted to find a good approximation of a solution to (1.10). Next,we sketch the proof of theorem 1 in section 6, while leaving complete details for subsequent sections.

2. Geometrical background

In this section we write the differential operator

$$\varepsilon^2 \Delta_{\bar{x}} u + \varepsilon^2 \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_{\bar{x}} u \tag{2.1}$$

involved in equation (1.10), in some appropriate coordinate system.

First, observe that the obvious scaling $\bar{x} = \varepsilon x$ and setting $v(x) := u(\varepsilon x)$, transforms (2.1) into

$$\Delta_x v + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_x v \tag{2.2}$$

Let us consider a large dilation of the curve Γ, namely $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$, for $\varepsilon > 0$ small. Next, we introduce local translated Fermi coordinates near Γ_ε

$$\begin{aligned} X_{\varepsilon,h}(s,t) &= X_\varepsilon(s, t + h(\varepsilon s)) \\ &= \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s))\, \nu(\varepsilon s) \end{aligned}$$

where $h \in C^2(\mathbb{R})$ is a bounded smooth function. From assumption (1.13), we see that the mapping $X_{\varepsilon,h}(s,t)$ gives local coordinates in the region

$$\mathcal{N}_{\varepsilon,h} = \left\{ x = X_{\varepsilon,h}(s,t) \in \mathbb{R}^2 : |t + h(\varepsilon s)| < \frac{\delta}{\varepsilon} + c_0|s| \right\}$$

which is a dilated tubular neighborhood around Γ_ε translated in h.

Now, we give an expression for the euclidean laplacian in terms of the coordinates $X_{\varepsilon,h}$. A detailed proof of this fact can be found in the Appendix.

Lemma 2.1. *On the open neighborhood $\mathcal{N}_{\varepsilon,h}$ of Γ_ε, the euclidean laplacian has the following expression when is computed in the coordinate $x = X_{\varepsilon,h}(s,t)$, which reads as*

$$\begin{aligned} \Delta_{X_{\varepsilon,h}} &= \partial_{tt} + \partial_{ss} - 2\varepsilon h'(\varepsilon s)\partial_{st} - \varepsilon^2 h''(\varepsilon s)\partial_t + \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt} \\ &\quad - \varepsilon[k(\varepsilon s) + \varepsilon(t + h(\varepsilon s))k^2(\varepsilon s)]\partial_t + D_{\varepsilon,h}(s,t) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} D_{\varepsilon,h}(s,t) := {} & \varepsilon(t+h)A_0(\varepsilon s, \varepsilon(t+h))[\partial_{ss} - 2\varepsilon h'(\varepsilon s)\partial_{ts} - \varepsilon^2 h''(\varepsilon s)\partial_t + \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt}] \\ & + \varepsilon^2 (t+h) B_0(\varepsilon s, \varepsilon(t+h))[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] \\ & + \varepsilon^3 (t+h)^2 C_0(\varepsilon s, \varepsilon(t+h))\partial_t \end{aligned}$$

for which

$$A_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = 2k(\varepsilon s) + \varepsilon O(|[t + h(\varepsilon s)]k^2(\varepsilon s)|) \tag{2.4}$$

$$B_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = \dot{k}(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))\dot{k}(\varepsilon s) \cdot k^2(\varepsilon s)|) \tag{2.5}$$

$$C_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = k^3(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))k^4(\varepsilon s)|) \tag{2.6}$$

are smooth functions and these relations can be differentiated.

Next, we derive an expression for the second term in equation (2.2), in terms of the Fermi coordinates. We collect the computations in the following lemma, whose proof can be found also in the Appendix.

Lemma 2.2. *In the open neighborhood $\mathcal{N}_{\varepsilon,h}$ of Γ_ε, we have the validity of the following expression*

$$\begin{aligned} \varepsilon \frac{\nabla_X a}{a} \cdot \nabla_{X_{\varepsilon,h}} = {} & \varepsilon \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) + \varepsilon(t + h(\varepsilon s)) \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) \right|^2 \right) \right] \partial_t \\ & + \varepsilon \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, 0)[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] + E_{\varepsilon,h}(s,t) \end{aligned}$$

where

$$E_{\varepsilon,h}(s,t) := \varepsilon^2(t+h(\varepsilon s))D_0(\varepsilon s,\varepsilon(t+h))[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] + \varepsilon^3(t+h(\varepsilon s))^2 F_0(\varepsilon s,\varepsilon(t+h))\partial_t \tag{2.7}$$

and for which the next functions are smooth

$$D_0(\varepsilon s,\varepsilon(t+h)) = \partial_{\boldsymbol{t}}\left[\frac{\partial_{\boldsymbol{s}} a}{a}\right](\varepsilon s,0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right]\right) + A_0(\varepsilon s,\varepsilon(t+h))\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,\varepsilon(t+h))$$

$$F_0(\varepsilon s,\varepsilon(t+h)) = \frac{1}{2}\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right](\varepsilon s,0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{ttt}}\left[\frac{\partial_t a}{a}\right]\right)$$

and where $A_0(\varepsilon s,\varepsilon(t+h))$ *given in* (2.4)*. Further, these relations can be differentiated.*

Using lemmas 2.1 and 2.2, the fact that the curve Γ satisfies condition (1.15), we can write expression (2.2) in coordinates $x = X_{\varepsilon,h}(s,t)$ as

$$\Delta_{X_{\varepsilon,h}} + \varepsilon\frac{\nabla_X a(\varepsilon x)}{a(\varepsilon x)}\cdot\nabla_{X_{\varepsilon,h}} =$$

$$\partial_{tt} + \partial_{ss} + \varepsilon\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)\partial_s$$

$$-\varepsilon^2\left\{h''(\varepsilon s) + \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)\,h'(\varepsilon s) + \left[2\,k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0)\right]h(\varepsilon s)\right\}\partial_t$$

$$-\varepsilon^2\left[k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0) + \left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right]t\,\partial_t - 2\varepsilon h'(\varepsilon s)\partial_{st} + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}$$

$$+\varepsilon(t+h(\varepsilon s))A_0(\varepsilon s,\varepsilon(t+h))[\partial_{ss} - 2\varepsilon h'(\varepsilon s)\partial_{ts} - \varepsilon^2 h''(\varepsilon s)\partial_t + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}$$

$$+\varepsilon^2\,(t+h(\varepsilon s))\tilde{B}_0(\varepsilon s,\varepsilon(t+h))[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] + \varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s,\varepsilon(t+h))\partial_t \tag{2.8}$$

where

$$\tilde{B}_0(\varepsilon s,\varepsilon(t+h)) := B_0(\varepsilon s,\varepsilon(t+h)) + D_0(\varepsilon s,\varepsilon(t+h)) \tag{2.9}$$

$$\tilde{C}_0(\varepsilon s,\varepsilon(t+h)) := C_0(\varepsilon s,\varepsilon(t+h)) + F_0(\varepsilon s,\varepsilon(t+h)). \tag{2.10}$$

We postpone detailed computations to the Appendix, in order to keep the presentation as clear as possible.

3. The Jacobi Operator $\mathcal{J}_{a,\Gamma}$

This section is meant to provide a complete proof of proposition 6.4. Recall that the Jacobi operator of the curve Γ associated to the potential a, corresponds to the linear operator

$$\mathcal{J}_{a,\Gamma}[h](\boldsymbol{s}) = h''(\boldsymbol{s}) + \frac{\partial_{\boldsymbol{s}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)}h'(\boldsymbol{s}) - Q(\boldsymbol{s})h(\boldsymbol{s}) \tag{3.1}$$

where we recall that

$$Q(\boldsymbol{s}) := \left[\frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s})\right] \tag{3.2}$$

Recall also that we are assuming the curve Γ to be nondegenerate, which means that the only bounded solution to

$$\mathcal{J}_{a,\Gamma}[h](\boldsymbol{s}) = 0, \quad \forall \boldsymbol{s}\in\mathbb{R}$$

is the trivial one.

3.1. **The kernel of $\mathcal{J}_{a,\Gamma}$.** In order to find accurate information on the kernel of (3.1), we consider the auxiliary equation

$$\frac{d}{d\boldsymbol{s}}\left(p(\boldsymbol{s})\frac{d}{d\boldsymbol{s}}h\right) - q(\boldsymbol{s})h = 0, \quad \text{in } \mathbb{R} \tag{3.3}$$

where we assume that $p, q : \mathbb{R} \to \mathbb{R}$ satisfy the following

$$p \in C^1[0,\infty) \cap L^\infty[0,\infty), \;\; q \in C^1[0,\infty) \tag{3.4}$$

$$p(\boldsymbol{s}) \geq p_0 > 0, \quad \forall \boldsymbol{s} \geq 0 \tag{3.5}$$

$$\lim_{\boldsymbol{s}\to\pm\infty} p(\boldsymbol{s}) =: p(\pm\infty) > 0 \tag{3.6}$$

$$|p(\boldsymbol{s})| + (1+|\boldsymbol{s}|)^{2+\alpha}|p'(\boldsymbol{s})| \leq C, \quad \forall \boldsymbol{s} \geq 0 \tag{3.7}$$

$$|q(\boldsymbol{s})| + |q'(\boldsymbol{s})| \leq \frac{C}{1+|\boldsymbol{s}|^{2+\alpha}}, \quad \forall \boldsymbol{s} \geq 0 \tag{3.8}$$

for some constants $\alpha > -1$, $\beta_0 > 0$ and $C > 0$.

The first result concerns the decay for the derivative of a solution to the auxiliary equation, provided that p and q decay sufficiently fast.

Lemma 3.1. *Suppose $\alpha > -1$, and consider a one-sided bounded solution $h \in L^\infty[0,\infty)$ of* (3.3)*, for which functions p and q fulfill* (3.4) *to* (3.8)*. Then there is a constant $C = C(p,q,\alpha,h) > 0$ such that*

$$|h'(\boldsymbol{s})| \leq \frac{C}{|\boldsymbol{s}|^{1+\alpha}}, \quad \forall \boldsymbol{s} > 0$$

where $C(p,q,\alpha,h) = \|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.

Proof. Observe first that thanks to assumptions (3.4)-(3.6), it holds

$$p(\boldsymbol{s}) = p(+\infty) - \int_{\boldsymbol{s}}^{+\infty} p'(\xi)d\xi \tag{3.9}$$

Now, since h solves the equation, then for $\boldsymbol{s}_1 > \boldsymbol{s}_2 > 0$ we have

$$\begin{aligned}|p(\boldsymbol{s}_1)h'(\boldsymbol{s}_1) - p(\boldsymbol{s}_2)h'(\boldsymbol{s}_2)| &\leq \int_{\boldsymbol{s}_2}^{\boldsymbol{s}_1} |q(\boldsymbol{s})h(\boldsymbol{s})| \\ &\leq \|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)}\left|\int_{\boldsymbol{s}_2}^{\boldsymbol{s}_1}\frac{1}{1+|\xi|^{2+\alpha}}d\xi\right| \\ &\leq C(q,h)\left(\frac{1}{|\boldsymbol{s}_2|^{1+\alpha}} - \frac{1}{|\boldsymbol{s}_1|^{1+\alpha}}\right)\end{aligned}$$

where $C(q,h) := C\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)} < \infty$ is fixed.

In particular using that $1+\alpha > 0$, it follows that

$$\lim_{\boldsymbol{s}_1\to+\infty}|p(\boldsymbol{s}_1)h'(\boldsymbol{s}_1)| \leq |p(\boldsymbol{s}_2)h'(\boldsymbol{s}_2)| + C(q,h)\frac{1}{|\boldsymbol{s}_2|^{1+\alpha}} < +\infty$$

which implies that $p(+\infty)h'(\infty) \in \mathbb{R}$. From this, we can rewrite equation (3.3) in its integral form

$$p(\boldsymbol{s})h'(\boldsymbol{s}) = p(+\infty)h'(+\infty) - \int_{\boldsymbol{s}}^{+\infty} q(\xi)h(\xi)d\xi. \tag{3.10}$$

but using (3.9), we find that

$$p(+\infty)h'(\boldsymbol{s}) - h'(\boldsymbol{s})\int_{\boldsymbol{s}}^{+\infty} p'(\xi)d\xi = p(+\infty)h'(+\infty) - \int_{\boldsymbol{s}}^{+\infty} q(\xi)h(\xi)d\xi$$

and so

$$p(+\infty)h'(s) = p(+\infty)h'(+\infty) + h'(s)\int_s^{+\infty} p'(\xi)d\xi - \int_s^{+\infty} q(\xi)h(\xi)d\xi.$$

Integrating again between 0 and $\boldsymbol{s}$, we obtain an expression for the solution h of (3.3)

$$\begin{aligned} p(+\infty)h(s) &= p(+\infty)h(0) + p(\infty)h'(+\infty)s \\ &\quad + \underbrace{\int_0^s h'(\xi)\int_\xi^{+\infty} p'(\tau)d\tau d\xi}_{I} - \underbrace{\int_0^s \int_\xi^{+\infty} q(\tau)h(\tau)d\tau d\xi}_{II} \end{aligned} \tag{3.11}$$

Let us estimate these integrals. We first estimate integral I

$$\begin{aligned} |I| &\le \int_0^s |h'(\xi)|\int_\xi^{+\infty} |p'(\tau)|d\tau \\ &\le C\|h'\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})p'\|_{L^\infty[0,\infty)}\int_0^{\boldsymbol{s}}\int_\xi^{+\infty}\frac{1}{1+|\tau|^{2+\alpha}}d\tau d\xi \\ &\le C_{h',p',\alpha}\int_0^{\boldsymbol{s}}\frac{1}{1+|\xi|^{1+\alpha}}d\xi = O(1+|\boldsymbol{s}|^{-\alpha}) \end{aligned}$$

where $C_{h',p',\alpha} := C\|h'\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})p'\|_{L^\infty[0,\infty)}$.

In the same way, we estimate II

$$\begin{aligned} |II| &\le \int_0^{\boldsymbol{s}}\int_\xi^{+\infty} |q(\tau)|\ |h(\tau)|d\tau d\xi \\ &\le C\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})q\|_{L^\infty[0,\infty)}\int_0^{\boldsymbol{s}}\int_\xi^{+\infty}\frac{d\tau d\xi}{1+|\tau|^{2+\alpha}} \\ &\le C_{h,q,\alpha}(1+|\boldsymbol{s}|)^{-\alpha} \end{aligned}$$

with $C_{h,q,\alpha} := C\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.

Since h is bounded, we deduce from (3.11) that

$$O(1) = p(+\infty)h(0) + p(+\infty)h'(+\infty)\boldsymbol{s} + O(1+|\boldsymbol{s}|^{-\alpha}). \tag{3.12}$$

Dividing (3.12) by $\boldsymbol{s} > 0$ and taking $\boldsymbol{s} \to +\infty$, we get that

$$0 = p(+\infty)h'(+\infty)$$

provided that $\alpha > -1$. From (3.6), it follows that $h'(+\infty) = 0$.

In particular, the latter fact together with formula (3.10), imply that

$$p(\boldsymbol{s})h'(\boldsymbol{s}) = \int_{\boldsymbol{s}}^{\infty} q(\xi)h(\xi)d\xi$$

and consequently

$$|h'(\boldsymbol{s})| \le C\|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})q\|_{L^\infty[0,\infty)}\frac{1}{1+|\boldsymbol{s}|^{1+\alpha}}$$

which completes the proof of the estimate. □

The core of this section is reflected in the next result.

Lemma 3.2. *Let $\alpha > 0$, and suppose function q satisfies (3.4)-(3.8). Then the equation*

$$u''(\boldsymbol{s}) - q(\boldsymbol{s})u(\boldsymbol{s}) = 0, \quad in \quad \mathbb{R} \tag{3.13}$$

has two linearly independent smooth solutions $u, \tilde{u}$, so that as $s \to +\infty$

$$u(\boldsymbol{s}) = \boldsymbol{s} + O(1) + O(|\boldsymbol{s}|^{-1-\alpha}), \qquad \tilde{u}(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$

$$u'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), \qquad \tilde{u}'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha})$$

Proof. To begin with, we look for a solution $u(\boldsymbol{s}) = \boldsymbol{s}v(\boldsymbol{s})$, so that, multiplying equation (3.13) by $\boldsymbol{s}$, we find that v satisfies

$$\frac{d}{d\boldsymbol{s}}\left(\boldsymbol{s}^2 v'(\boldsymbol{s})\right) - q(\boldsymbol{s})\boldsymbol{s}^2 v(\boldsymbol{s}) = 0 \tag{3.14}$$

Now, consider the functions

$$x(\boldsymbol{s}) := \boldsymbol{s}^2 v'(\boldsymbol{s}), \quad y(\boldsymbol{s}) := v(\boldsymbol{s})$$

so that, equation (3.14) becomes the linear system of differential equations

$$\begin{cases} x'(\boldsymbol{s}) = q(\boldsymbol{s})\boldsymbol{s}^2 y(\boldsymbol{s}) \\ y'(\boldsymbol{s}) = \dfrac{1}{\boldsymbol{s}^2} x(\boldsymbol{s}) \end{cases}, \quad \forall \boldsymbol{s} \in [\boldsymbol{s}_0, +\infty)$$

Integrating this system between $\boldsymbol{s}_0$ and $\boldsymbol{s}$ we obtain the identities

$$y(\boldsymbol{s}) = y(\boldsymbol{s}_0) + \int_{\boldsymbol{s}_0}^{s} \frac{1}{\xi^2} x(\xi) d\xi$$

$$x(\boldsymbol{s}) = x(\boldsymbol{s}_0) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} q(\xi)\xi^2 y(\xi) d\xi \tag{3.15}$$

In particular, we deduce an explicit formula for $y(\boldsymbol{s})$, given by

$$y(\boldsymbol{s}) = y(\boldsymbol{s}_0) + x(\boldsymbol{s}_0)\left(\frac{1}{\boldsymbol{s}_0} - \frac{1}{\boldsymbol{s}}\right) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\tau^2\left(\frac{1}{\tau} - \frac{1}{\boldsymbol{s}}\right) d\tau \tag{3.16}$$

In this way, we can estimate $y(\boldsymbol{s})$ for $\boldsymbol{s} \geq \boldsymbol{s}_0$ as

$$|y(\boldsymbol{s})| \leq |y(\boldsymbol{s}_0)| + |x(\boldsymbol{s}_0)|\left(\frac{1}{\boldsymbol{s}_0} - \frac{1}{\boldsymbol{s}}\right) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} |y(\tau)|\ |q(\tau)|\tau\left(1 - \frac{\tau}{\boldsymbol{s}}\right) d\tau.$$

From Gronwall's inequality we find the estimate

$$|y(\boldsymbol{s})| \leq \left(|y(\boldsymbol{s}_0)| + \frac{2|x(\boldsymbol{s}_0)|}{\boldsymbol{s}_0}\right) \exp\left(\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} |q(\tau)|\tau\left(1 - \frac{\tau}{\boldsymbol{s}}\right) d\tau\right)$$

Notice that, for any $\boldsymbol{s} \geq \tau > \boldsymbol{s}_0 :\ \left|\tau\left(1 - \frac{\tau}{\boldsymbol{s}}\right)\right| \leq 2\tau = O(\tau)$. This fact combined with the decay of $q(\boldsymbol{s})$, leads to

$$|y(\boldsymbol{s})| \leq C_{q,\alpha}(|y(\boldsymbol{s}_0)| + \frac{2}{\boldsymbol{s}_0}|x(\boldsymbol{s}_0)|) \tag{3.17}$$

where $C_{q,\alpha} := C\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[\boldsymbol{s}_0,+\infty)} \int_{\boldsymbol{s}_0}^{\infty} |\tau|^{-1-\alpha} d\tau$.

From (3.16) it follows that for any $\boldsymbol{s}_1 > \boldsymbol{s}_2 \geq \boldsymbol{s}_0 > 0$:

$$|y(\boldsymbol{s}_1) - y(\boldsymbol{s}_2)| \leq |x(\boldsymbol{s}_0)|\left(\frac{1}{\boldsymbol{s}_2} - \frac{1}{\boldsymbol{s}_1}\right) + C\int_{\boldsymbol{s}_2}^{\boldsymbol{s}_1} |q(\tau)|\tau d\tau$$

implying that $y(+\infty) \in \mathbb{R}$. Moreover, same formula (3.16) yields

$$y(+\infty) = y(\boldsymbol{s}_0) + \frac{x(\boldsymbol{s}_0)}{\boldsymbol{s}_0} + \int_{\boldsymbol{s}_0}^{+\infty} y(\tau)q(\tau)\tau d\tau$$

which allows us to write

$$y(\boldsymbol{s}) - y(+\infty) = -\frac{x(\boldsymbol{s}_0)}{\boldsymbol{s}_0} - \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau - \int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau$$

In particular, by choosing the constants to be $y(+\infty) = 1,\ x(\boldsymbol{s}_0) = 0$, we finally deduce

$$y(\boldsymbol{s}) = 1 - \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau - \int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau \tag{3.18}$$

Additionally, the derivative $y'(\boldsymbol{s}) = v'(\boldsymbol{s})$ can be obtained from $x(\boldsymbol{s})$ using relation (3.15), as

$$v'(\boldsymbol{s}) = \frac{x(\boldsymbol{s})}{\boldsymbol{s}^2} = \frac{0}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}^2}\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} q(\xi)\xi^2 y(\xi)d\xi. \tag{3.19}$$

Now that $y(\boldsymbol{s})$ is bounded in $[\boldsymbol{s}_0, +\infty)$, similar arguments as shown in (3.17) imply the same estimates for the integrals in (3.18)-(3.19), since

$$\left|\int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau\right| = O(|\boldsymbol{s}|^{\alpha}), \quad \left|\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau\right| = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$

From these estimates, we conclude that

$$v(\boldsymbol{s}) = y(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$

$$v'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-2} + |\boldsymbol{s}|^{-1-\alpha})$$

So the asymptotic behavior of the first solution follows, as $\alpha > 0$ and by definition of u:

$$\begin{array}{l} u(\boldsymbol{s}) = \boldsymbol{s}\left(1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})\right) = \boldsymbol{s} + O(1 + |\boldsymbol{s}|^{1-\alpha}) \\ u'(\boldsymbol{s}) = v(\boldsymbol{s}) + \boldsymbol{s}v'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \end{array}, \quad \boldsymbol{s} \geq \boldsymbol{s}_0$$

which finishes the analysis of the profile of the first solution to equation (3.13).

To conclude, we find $\tilde{u}$ using reduction of order formula, to find that

$$\tilde{u}(\boldsymbol{s}) = \left(\int_{\boldsymbol{s}}^{\infty} u^{-2}(\xi)d\xi\right) \cdot u(\boldsymbol{s})$$

and directly from this, one gets

$$\begin{array}{l} \tilde{u}(\boldsymbol{s}) = C(\boldsymbol{s})u(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ \tilde{u}'(\boldsymbol{s}) = C'(\boldsymbol{s})u(\boldsymbol{s}) + C(\boldsymbol{s})u'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-2} + |\boldsymbol{s}|^{-1-\alpha}) \end{array}, \quad \text{for } \boldsymbol{s} >> \boldsymbol{s}_0$$

which concludes the proof of lemma 3.2. □

Now proceed to state the main result of this section, which characterize the profile of the kernel of the Jacobi operator.

Proposition 3.1. *Let $\Gamma \subset \mathbb{R}^2$ be a stationary non-degenerate curve as in respect to a. Assume also that conditions* (1.11)-(1.14) *are satisfied, for $\alpha > 0$ and additionally the potential stabilizes on the curve at infinity, namely*

$$a(\pm\infty, 0) := \lim_{\boldsymbol{s}\to\pm\infty} a(\boldsymbol{s}, 0) > 0 \in \mathbb{R}.$$

Then, there are two linearly independent elements in the kernel of h_1, h_2 of (3.1) *satisfying that*

$$\begin{array}{l} h_i(\boldsymbol{s}) = |\boldsymbol{s}| + O(1) + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ h_i'(\boldsymbol{s}) = O(1) + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha}) \end{array}, \quad as \quad (-1)^i\boldsymbol{s} \to +\infty \tag{3.20}$$

and they are bounded functions as $(-1)^{i+1}\boldsymbol{s} \to \infty$. Furthermore, in the region where the latter happens, it holds

$$|h_i(\boldsymbol{s})| + (1 + |\boldsymbol{s}|^{1+\alpha})|h_i'(\boldsymbol{s})| \leq C, \quad as \quad (-1)^{i+1}\boldsymbol{s} \to +\infty \tag{3.21}$$

Proof. We look for solutions $h(s) = a(s,0)^{-1/2} \cdot u(s)$ to (3.1), which means that u solves the auxiliary equation

$$u''(\boldsymbol{s}) - \tilde{q}(\boldsymbol{s})u(\boldsymbol{s}) = 0, \quad \text{in } \mathbb{R}$$

where

$$\tilde{q}(\boldsymbol{s}) := \frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s}) + \frac{1}{2}\frac{\partial_{\boldsymbol{ss}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - \frac{1}{4}\left|\frac{\partial_{\boldsymbol{s}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)}\right|^2$$

Now, thanks to the assumptions we have made on $a(\boldsymbol{s},\boldsymbol{t})$ and Γ, it follows that

$$(1+|\boldsymbol{s}|)^{2+\alpha}|\tilde{q}(\boldsymbol{s})| \leq C.$$

Therefore, applying lemma 3.2 on the region $[0,\infty)$, we deduce the existence of two solutions linearly independent of equation (3.1) in $\mathbb{R}$, denoted by $u(\boldsymbol{s})$ and $\tilde{u}(\boldsymbol{s})$, which satisfies the right-sided asymptotic behavior as $s \to +\infty$

$$u(\boldsymbol{s}) = s + O(1) + O(|\boldsymbol{s}|^{1-\alpha}), \qquad \tilde{u}(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$
$$u'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), \qquad \tilde{u}'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha})$$

Applying Lemma 3.2 again, but this time on the region $(-\infty, 0]$, we obtain two other solutions $v(s)$ and $\tilde{v}(s)$ linearly independent of equation (3.1) in $\mathbb{R}$, that now satisfy the left-sided asymptotic behavior as $s \to -\infty$

$$v(\boldsymbol{s}) = |\boldsymbol{s}| + O(1) + O(|\boldsymbol{s}|^{1-\alpha}), \qquad \tilde{v}(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$
$$v'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), \qquad \tilde{v}'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha})$$

We remark that the non-degeneracy of curve Γ, implies that $\tilde{u}(\boldsymbol{s})$ cannot be bounded on $(-\infty, 0]$. Recall also that $\{u, \tilde{u}\}$ and $\{v, \tilde{v}\}$ represent two different basis of the vector space of solutions to the equation (3.1). So that, for some constats α_i, for $i = 1, \ldots, 4$, we have that

$$\forall \boldsymbol{s} \in \mathbb{R}: \quad u(\boldsymbol{s}) = \alpha_1 v(\boldsymbol{s}) + \alpha_2 \tilde{v}(\boldsymbol{s}), \quad \tilde{u}(\boldsymbol{s}) = \alpha_3 v(\boldsymbol{s}) + \alpha_4 \tilde{v}(\boldsymbol{s})$$

From the previous discussion about $\tilde{u}$, we observe that not only that $\tilde{u}$ grows at most at a linear rate on $(-\infty, 0]$, but also that the non-degeneracy property implies $\alpha_3 \neq 0$. Hence, the function $h_1(\boldsymbol{s}) := \alpha_3^{-1} a(\boldsymbol{s},0)^{-1/2}\tilde{u}(\boldsymbol{s})$ belongs to the kernel of $\mathcal{J}_{a,\Gamma}$, satisfying (3.20)-(3.21) for $i = 1$.

Likewise, the same argument can be applied to $\tilde{v}(\boldsymbol{s})$ to find the function $h_2(\boldsymbol{s}) := a(\boldsymbol{s},0)^{-1/2}u(\boldsymbol{s})$ behaving as predicted and clearly, being linear independent with h_1. This completes the proof of the proposition. □

Once we have described the kernel of (3.1), it is straightforward to check the following proposition, whose proof is left to the readers.

Proposition 3.2. *Under the same set of assumptions as in proposition 3.1 and given $\alpha > 0$, $\lambda \in (0,1)$ and a function f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < \infty$, then the equation*

$$\mathcal{J}_a[h](\boldsymbol{s}) = f(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{R}$$

has a unique bounded solution, given by the variation of parameters formula

$$h(\boldsymbol{s}) = -h_1(\boldsymbol{s})\int_{-\infty}^{\boldsymbol{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi - h_2(\boldsymbol{s})\int_{\boldsymbol{s}}^{+\infty} a(\xi,0)h_1(\xi)f(\xi)d\xi$$

In addition, there is some positive constant $C = C(a,\Gamma,\alpha)$ such that

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

where

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1+|s|)^{1+\alpha}h'\|_{L^\infty(\mathbb{R})} + \sup_{s\in\mathbb{R}}(1+|\boldsymbol{s}|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(\boldsymbol{s}-1,\boldsymbol{s}+1)}$$

4. Examples:

It is of interest to mention that Theorem 1 relies on a very important fact, whose nature is essentially geometrical. This concerns the existence of a curve $\Gamma \subset \mathbb{R}^2$, given a fixed suitable potential $a(x)$ of the Allen-Cahn equation (1.10), that be a stationary curve with respect to the weighted length functional $l_{a,\Gamma}$ in the sense of (1.15) and also be a non-degenerate curve as in definition (1.16). To get a better understanding of the geometrical settings of this problem, it would be useful to present some examples that portray the nature of the curves and of the potentials we are thinking of and how they interact in order for this configuration to be admissible.

The following section is devoted to provide concrete examples of such curves associated to some nontrivial potential $a(x)$, in a way that they meet all the hypothesis of Theorem 1.

4.1. Characterization of Non-degeneracy.

Now we state a result that provides precise conditions on a and Γ, where in such case the non-degeneracy property of the curve holds.

Corollary 4.1. *Non-degeneracy in the minimizing case*
Let $\alpha > -1/2$, Γ be a stationary curve with respect to $l_{a,\Gamma}$ as in (1.15)*, and let the potential $a(\bar{s}, t)$ along with Γ be such that $Q(\bar{s}) := \partial_{tt} a(\bar{s}, 0)/a(\bar{s}, 0) - 2k^2(\bar{s})$ satisfies the following conditions*

$$Q(\bar{s}) \geq 0, \quad \text{and} \quad Q(\bar{s}) \not\equiv 0 \tag{4.1}$$

and the asymptotic polinomial decay

$$|Q(\bar{s})| \leq \frac{C}{(1 + |\bar{s}|)^{2+\alpha}}, \quad \text{for} \quad |\bar{s}| >> \bar{s}_0 \tag{4.2}$$

then the curve Γ is non-degenerate, in the sense of (1.16)*.*

Proof. Let h be a bounded element in the kernel of the Jacobi operator, so that $h \in L^\infty(\mathbb{R})$ solves $\mathcal{J}_a[h](\bar{s}) = 0$ in $\mathbb{R}$. Lemma 3.1 assures the existence of a constant $C > 0$ such that

$$|h'(\bar{s})| \leq \frac{C}{1 + |\bar{s}|^{1+\alpha}}, \quad \forall \bar{s} \in \mathbb{R} \tag{4.3}$$

which implies $h' \in L^2(\mathbb{R})$, as $\alpha > -1/2$. Furthermore, condition (4.2) plus $\alpha > -1/2$ imply the integrability of $Q(s)$ in $\mathbb{R}$ since

$$\int_{\mathbb{R}} |Q(\bar{s})| d\bar{s} = \int_{|\bar{s}| > \bar{s}_0} Q(\bar{s}) d\bar{s} + \int_{|\bar{s}| \leq \bar{s}_0} Q(\bar{s}) d\bar{s} \leq C \int_{|\bar{s}| > \bar{s}_0} \frac{d\bar{s}}{1 + |\bar{s}|^{2+\alpha}} + \int_{|\bar{s}| \leq \bar{s}_0} Q(\bar{s}) d\bar{s}$$

In particular, the foregoing guarantees that the following expression is well defined

$$\begin{aligned} \|h\|_Q := \langle \mathcal{J}_a[h], h \rangle_{L^2(a(s,0))} &= \int_{\mathbb{R}} [h''(\bar{s}) + \frac{\partial_{\boldsymbol{s}} a(\bar{s}, 0)}{a(\bar{s}, 0)} h'(\bar{s}) - Q(\bar{s}) h(\bar{s})] h(\bar{s}) a(\bar{s}, 0) d\bar{s} \\ &= a(\bar{s}, 0) h'(\bar{s}) h(\bar{s}) \Big|_{\bar{s}=-\infty}^{\bar{s}=+\infty} - \int_{\mathbb{R}} a(\bar{s}, 0)[|h'(\bar{s})|^2 + Q(\bar{s}) h^2(\bar{s})] d\bar{s} \end{aligned}$$

where the boundary value terms at infinity vanishes, as h is bounded and from the decay (4.3). Finally as h solves $\mathcal{J}_a[h] = 0$, the latter together with hypothesis (4.1) imply

$$0 = \int_{\mathbb{R}} a(\bar{s}, 0)[|h'(\bar{s})|^2 + Q(\bar{s}) h^2(\bar{s})] d\bar{s} \geq \int_{\mathbb{R}} a(\bar{s}, 0) |h'(\bar{s})|^2 d\bar{s}$$

As $a(\bar{s}, 0) > 0$, we deduce that $h' = 0$ a.e. in $\mathbb{R}$. Moreover, the smoothness of h guarantees that $h' \equiv 0$ in $\mathbb{R}$. Using again the last inequality we get that

$$0 = \int_{\mathbb{R}} Q(\bar{s}) h^2(\bar{s}) d\bar{s}$$

However, condition (4.1) on Q assures that $Q(\bar{s}) > 0$ on a neighborhood of some point $\bar{s}_0 \in \mathbb{R}$. Therefore, last equality gives that $h(\bar{s}) = 0$ in this neighborhood, but as $h \equiv C$ is a constant function in the entire space, we conclude $h = 0$, which concludes the proof of Corollary 4.1. ∎

4.2. **Geodesics for $l_{a,\Gamma}$ in Euclidean coordinates.** In what follows, we will consider a curve Γ that can be represented as the graph of some function. Let us consider a smooth function $f : \mathbb{R} \to \mathbb{R}$, $f = f(\boldsymbol{x})$, and a parametrized curve $\Gamma := \{\gamma(\boldsymbol{x}) / \ \boldsymbol{x} \in \mathbb{R}\} \subset \mathbb{R}^2$ such that

$$\gamma(\boldsymbol{x}) = (\boldsymbol{x}, f(\boldsymbol{x})), \quad \dot{\gamma}(\boldsymbol{x}) = (1, f'(\boldsymbol{x})) \tag{4.4}$$

In addition, let the normal field ν of Γ be oriented negatively, meaning that vector $\dot{\gamma}(\boldsymbol{x}) \wedge \nu(\boldsymbol{x})$ points in the opposite direction than e_3, the generator of the $z-$axis in $\mathbb{R}^3$. This forces

$$\nu(\boldsymbol{x}) = \frac{1}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} (f'(\boldsymbol{x}), -1)$$

Let us also consider a potential defined in Euclidean coordinates $a = a(\boldsymbol{x}, \boldsymbol{y})$. Recall from the criticality condition (1.15), that in order for Γ to be a stationary curve with respect to the weighted arc-length $l_{a,\Gamma}$, is necessary that the potential a and the curvature k satisfy the equation

$$\partial_{\boldsymbol{t}} a(\boldsymbol{s}, 0) = k(\boldsymbol{s}) \cdot a(\boldsymbol{s}, 0) \ , \quad \text{a.e. } \boldsymbol{s} \in \mathbb{R} \tag{4.5}$$

Denoting $X(\boldsymbol{x}, \boldsymbol{t}) := \gamma(\boldsymbol{x}) + \boldsymbol{t}\nu(\boldsymbol{x})$, we can now set the potential written in this coordinates as

$$\tilde{a}(\boldsymbol{x}, \boldsymbol{t}) := a \circ X(\boldsymbol{x}, \boldsymbol{t}) = a \left(\boldsymbol{x} + \frac{\boldsymbol{t} f'(\boldsymbol{x})}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}, \ f(\boldsymbol{x}) - \frac{\boldsymbol{t}}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} \right) \tag{4.6}$$

Accordingly, relation (4.6) implies that the criticality condition (4.5) amounts to the following equation in Euclidean coordinates

$$\frac{\partial_{\boldsymbol{x}} a(\boldsymbol{x}, f(\boldsymbol{x})) f'(\boldsymbol{x})}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} - \frac{\partial_{\boldsymbol{y}} a(\boldsymbol{x}, f(\boldsymbol{x}))}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} = \frac{f''(\boldsymbol{x})}{(1 + |f'(\boldsymbol{x})|^2)^{3/2}} \cdot a(\boldsymbol{x}, f(\boldsymbol{x})) \tag{4.7}$$

where it has been used the classical formula for the curvature of Γ as given in (4.4),

$$k(\boldsymbol{x}) = f''(\boldsymbol{x})(1 + |f'(\boldsymbol{x})|^2)^{-3/2}$$

Example 1: The x-axis. We consider $\Gamma \subset \mathbb{R}^2$ to be the x-axis. Nonetheless, the stationarity of this line must be with respect to some nontrivial potential $a(\boldsymbol{x}, \boldsymbol{y}) \not\equiv 1$ that does not represent the classic Euclidean metric in $\mathbb{R}^2$, case in which all straight lines are trivially known as stationary curves.

With this purpose, let us set the function $f(\boldsymbol{x}) \equiv 0$ in (4.4), implying that $\Gamma = \overrightarrow{0X}$. In particular, we have that $\nu(\boldsymbol{x}) \equiv e_2$, thus the Fermi coordinates are reduced simply to the Euclidean coordinates, namely $X(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{x} e_1 + \boldsymbol{t} e_2 = (\boldsymbol{x}, \boldsymbol{t})$.

In this simplified context, it turns out that the criticality condition (4.7) is reduced to

$$\partial_{\boldsymbol{y}} a(\boldsymbol{x}, 0) = 0, \quad \forall \boldsymbol{x} \in \mathbb{R}. \tag{4.8}$$

Therefore, we only need to find a nontrivial potential $\tilde{a}(\boldsymbol{x}, \boldsymbol{t}) = a(\boldsymbol{x}, \boldsymbol{y})$ in such way the x-axis becomes a stationary curve, and also a nondegenerate curve as in the sense of 1.16.

Claim 4.1. *Given any $\alpha > 0$, the following potential*

$$a(\boldsymbol{x}, \boldsymbol{y}) := 1 + \frac{1}{(1 + |\boldsymbol{x}|)^{2+\alpha}} \cdot \left(\frac{\boldsymbol{y}^2}{\cosh(\boldsymbol{y})} \right) \tag{4.9}$$

satisfies all the requirements previously indicated, in relation with the curve $\Gamma = \overrightarrow{0X}$. See Figure 1.

Proof. Let us note that $a(\boldsymbol{x}, \boldsymbol{y})$ is smooth, globally bounded, and bounded below far away from zero. Further, it is direct that $\overrightarrow{0X}$ is a stationary curve relative to $l_{a,\Gamma}$ since solves equation (4.8)

$$\partial_{\boldsymbol{y}} a(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{(1 + |\boldsymbol{x}|)^{2+\alpha}} \left(\frac{2\boldsymbol{y} - \boldsymbol{y}^2 \sinh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})} \right) \quad \Rightarrow \quad \partial_{\boldsymbol{y}} a(\boldsymbol{x}, 0) = 0, \quad \forall \boldsymbol{x} \in \mathbb{R}$$

Now to see that $\overrightarrow{0X}$ is a nondegenerate curve, just note that the potential achieves it minimum exactly on the region defined by the $\boldsymbol{x}$-axis, and moreover, around this curve the potential is strictly convex in the $\boldsymbol{y}$-direction. The latter translates in the fact that $\partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)>0$, since

$$\partial_{\boldsymbol{yy}}a(\boldsymbol{x},\boldsymbol{y})=\frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}}\left(\frac{2-2\boldsymbol{y}\sinh(\boldsymbol{y})-\boldsymbol{y}^2\cosh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})}-\frac{2(2\boldsymbol{y}-\boldsymbol{y}^2\sinh(\boldsymbol{y}))\sinh(\boldsymbol{y})}{\cosh^3(\boldsymbol{y})}\right)$$

and

$$\partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)=\frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}}>0,\quad \forall\boldsymbol{x}\in\mathbb{R}$$

Taking this into account, note that $a(\boldsymbol{x},\boldsymbol{y})$ and $k(\boldsymbol{x})\equiv 0$ are such that term

$$Q(\boldsymbol{x}):=\frac{\partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)}{a(\boldsymbol{x},0)}-2k^2(\boldsymbol{x})$$

fulfills the following conditions

$$Q(\boldsymbol{x})>0,\qquad\text{and}\qquad |Q(\boldsymbol{x})|\le\frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}},\quad\forall\boldsymbol{x}\in\mathbb{R}$$

Hence we deduce from Corollary 4.1 that $\Gamma=\overrightarrow{0X}$ is a nondegenerate curve with respect to the potential $a(\boldsymbol{x},\boldsymbol{y})$ given in (4.9), finishing the proof of the claim. □

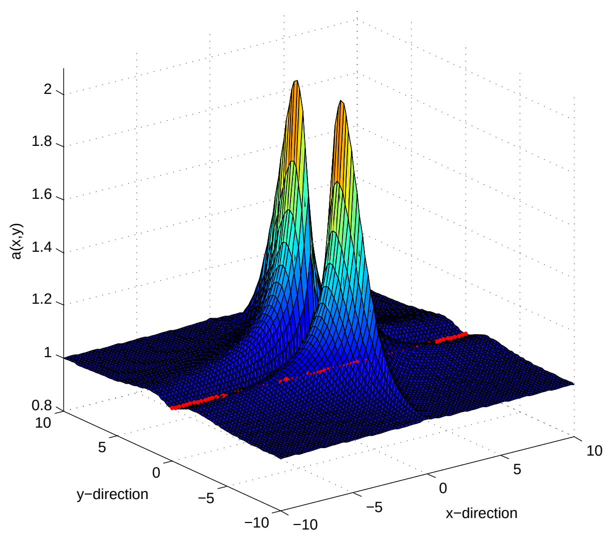

FIGURE 1. Potential $a(\boldsymbol{x},\boldsymbol{y})$ (4.9) with geodesic $\Gamma=0\vec{X}$, for $\alpha=0.01$.

Example 2: Asymptotic straight line. This time we consider the function $f(\boldsymbol{x}):=\sqrt{1+\omega^2x^2}$, so that Γ converges asymptotically to straight lines as $|\boldsymbol{x}|\to\infty$. We have to exhibit some nontrivial potential $a(\boldsymbol{x},\boldsymbol{y})$ for which Γ is a nondegenerate geodesic relative to the arclength $\int_\Gamma a(\vec{x})$.

We will assume a weaker dependence of the potential in Euclidean variables, namely $a=a(\boldsymbol{y})$, so that criticality condition (4.7) amounts to

$$\frac{a'(f(\boldsymbol{x}))}{a(f(\boldsymbol{x}))}=\frac{-f''(\boldsymbol{x})}{1+|f'(\boldsymbol{x})|^2}=g(f(x))\tag{4.10}$$

with $g(\boldsymbol{y}):=-\omega^2[(1+\omega^2)\boldsymbol{y}^3-\omega^2\boldsymbol{y}]^{-1}$.
We can solve directly this ordinary differential equation (4.10), for a in $\boldsymbol{y}-$variable.

$$\log(a(\boldsymbol{y}))=\int g(\boldsymbol{y})d\boldsymbol{y}+M\quad\Leftrightarrow\quad a(\boldsymbol{y})=M\exp\left(\int\frac{-\omega^2d\boldsymbol{y}}{(1+\omega^2)\boldsymbol{y}^3-\omega^2\boldsymbol{y}}\right)$$

This integral can be computed using partial fraction decomposition, that leads to

$$a(\boldsymbol{y}) = \frac{M\sqrt{1+\omega^2}\boldsymbol{y}}{\sqrt{(1+\omega^2)\boldsymbol{y}^2-\omega^2}}$$

For this construction, we will need to consider a slight modification of function a as follows. We say that the potential $\hat{a} : \mathbb{R}^2 \to \mathbb{R}$ is an *admissible left-extension* of function $a(x, y)$, provided that

- $\hat{a}$ be smooth bounded function, of at least $C^2(\mathbb{R}^2)$ class.
- $\hat{a}(x, y) = a(x, y)$ for points with $y \geq \omega^2/(1+\omega^2)$.
- $\hat{a}$ is uniformly positive, bounded below away from zero.

We state the following

Claim 4.2. *Given $|\omega| \leq 1/\sqrt{2}$, any admissible left-extension of the potential given below*

$$a(\boldsymbol{x}, \boldsymbol{y}) := \frac{\sqrt{1+\omega^2}\boldsymbol{y}}{\sqrt{(1+\omega^2)\boldsymbol{y}^2-\omega^2}} \tag{4.11}$$

induces a metric in $\mathbb{R}^2$ for which $\Gamma = \left\{(\boldsymbol{x}, \sqrt{1+\omega^2\boldsymbol{x}^2})\right\}_{\boldsymbol{x}\in\mathbb{R}}$ is a nondegenerate geodesic. See Figure 2.

Proof. Regardless the value of the parameter $\omega \neq 0$, it can be readily checked that within the region $y \geq 2\omega/\sqrt{1+\omega^2}$, function (4.11) is smooth, bounded, and uniformly positive. Moreover, this potential satisfies the asymptotic stability on the curve Γ, since $f(\boldsymbol{x}) \to +\infty$ as $|\boldsymbol{x}| \to +\infty$ and additionally $\lim_{\boldsymbol{y}\to+\infty} a(\boldsymbol{x}, \boldsymbol{y}) = 1$, $\forall \boldsymbol{x} \in \mathbb{R}$. The previous construction of $a(\boldsymbol{x}, \boldsymbol{y})$ was intended to build a potential satisfying the criticality condition (4.10) for the curve generated by $f(\boldsymbol{x}) = \sqrt{1+\omega^2 x^2}$. Thus Γ is a geodesic for the arclength $\int_\Gamma a(\vec{x})$. All these features of a ensure that any *admissible left-extension* will provide a potential with the desired properties to induce a smooth metric in $\mathbb{R}^2$, fulfilling hypothesis (1.15) of Theorem 1. Moreover, a tedious but simple calculation shows that

$$a'(\boldsymbol{y}) = \frac{-\omega^2\sqrt{1+\omega^2}}{(\boldsymbol{y}^2+\omega^2(\boldsymbol{y}^2-1))^{3/2}}, \quad a''(\boldsymbol{y}) = \frac{3\omega^2(1+\omega^2)^{3/2}\boldsymbol{y}}{[(1+\omega^2)\boldsymbol{y}^2-\omega^2]^{5/2}}$$

Therefore, taking into account the decay of the derivatives of $f(\boldsymbol{x})$, it follows that this potential satisfies condition (1.14) of Theorem 1, for $\alpha = 2 > 0$. It only remains to prove the nondegeneracy property of the curve Γ, and to do this, we will make use of Corollary 4.1. It can be checked the positiveness of the term $Q(\boldsymbol{x})$, in fact

$$2k^2(\boldsymbol{x}) = \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}, \quad \partial_{\boldsymbol{tt}}\tilde{a}(\boldsymbol{x}, 0) = a''(f(\boldsymbol{x}))\frac{1+\omega^2\boldsymbol{x}^2}{1+(\omega^2+\omega^4)\boldsymbol{x}^2}$$

so by the definition $Q(\boldsymbol{x}) = \partial_{tt}\tilde{a}(\boldsymbol{x}, 0)/\tilde{a}(\boldsymbol{x}, 0) - 2k^2(\boldsymbol{x})$ we obtain

$$\begin{aligned}
Q(\boldsymbol{x}) &\geq C_a\left(\frac{3\omega^2(1+\omega^2)^{3/2}(1+\omega^2\boldsymbol{x}^2)^{1/2}}{(1+\omega^2)^{5/2}(1+\omega^2\boldsymbol{x}^2)^{5/2}} - \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}\right) \\
&\geq C_a\left(\frac{3\omega^2}{(1+\omega^2)(1+\omega^2\boldsymbol{x}^2)^2} - \frac{2\omega^2}{(1+\omega^2\boldsymbol{x}^2)^3}\right) \\
&= \frac{C_a\omega^2}{(1+\omega^2\boldsymbol{x}^2)^2}\left(\frac{3}{1+\omega^2} - \frac{2}{1+\omega^2\boldsymbol{x}^2}\right).
\end{aligned}$$

Hence, choosing $\omega \in \mathbb{R}\setminus\{0\}$ with $|\omega| \leq 1/\sqrt{2}$, we get that $Q(\boldsymbol{x}) > 0$ in the entire domain $\mathbb{R}$. Finally, the term $Q(\boldsymbol{x})$ decays polynomially at a rate $O((1+|\boldsymbol{x}|)^{-4})$ as a consequence of the decay of the potential and the squared curvature, which finishes the proof of Claim 4.2. ☐

Remark 4.1. *We emphasize the fact that the criticality condition for Γ and the nondegeneracy property are tested only within the semi-space $\boldsymbol{y} \geq 1$, which involve only the part* (4.11) *of the admissible left-extension, since $\hat{a}(\boldsymbol{x}, \boldsymbol{y}) = a(\boldsymbol{y})$ in this region and the curve complies $|f(\boldsymbol{x})| \geq 1$.*

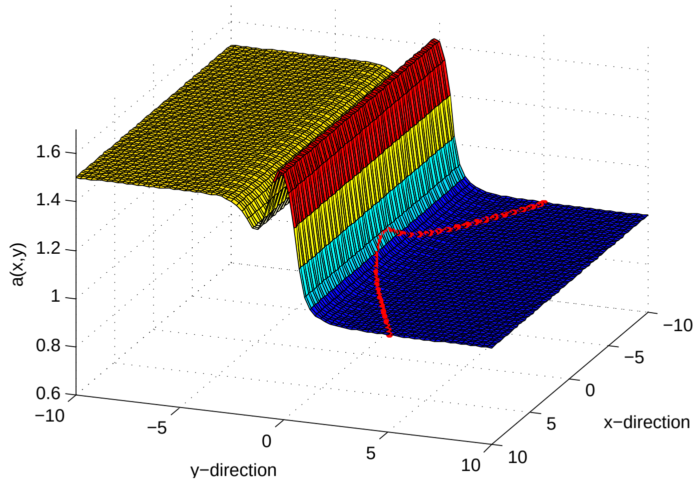

FIGURE 2. Potential $\hat{a}(\boldsymbol{x}, \boldsymbol{y})$ (4.11) with Γ_ω as nondegenerate geodesic, for $\omega = 1/2$.

5. Approximation of the solution and preliminary discussion

Here and after we assume

$$-F'(s) := s(1 - s^2), \quad s \in \mathbb{R}$$

but we remark that the developments we make follows, for a general double well potential F satisfying (1.3)-(1.4)-(1.5), with no significant changes.

5.1. **The approximation local Approximation.** To begin with, let us consider the parameter function $h \in C^2(\mathbb{R})$, for which we assume further $h = h(\boldsymbol{s})$ satisfies the apriori estimate

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1 + |\,\boldsymbol{s}\,|)^{1+\alpha} h'\|_{L^\infty(\mathbb{R})} + \sup_{\boldsymbol{s} \in \mathbb{R}} (1 + |\,\boldsymbol{s}\,|)^{2+\alpha} \|h''\|_{C^{0,\lambda}(\boldsymbol{s}-1,\boldsymbol{s}+1)} \leq \mathcal{K}\varepsilon \quad (5.1)$$

for a certain constant $\mathcal{K} > 0$ that will be chosen later, but independent of $\varepsilon > 0$.

Let us consider $w(t)$, the solution to the ODE

$$w''(t) - F'(w(t)) = 0, \quad w'(t) > 0, \quad w(\pm\infty) = \pm 1.$$

As mentioned in the previous section, by an obvious rescaling, equation (1.10) becomes

$$S(v) := \Delta_x v + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x v + v(1 - v^2) = 0, \quad \text{in } \mathbb{R}^2$$

Let us choose in the region $\mathcal{N}_{\varepsilon,h}$, as first approximate for a solution to (1.10), the function

$$v_0(x) := w(z - h(\varepsilon s)) = w(t), \quad x = X_{\varepsilon,h}(s,t) \in \mathcal{N}_{\varepsilon,h}$$

where $z = t + h(\varepsilon s)$ designates the normal coordinate to $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$.

Using expression (2.8) we compute the error $S(v_0)$ in the region $\mathcal{N}_{\varepsilon,h}$ to find that

$$S(v_0) \;=\; -\varepsilon^2 \mathcal{J}_a[h](\varepsilon s) w'(t) - \varepsilon^2 \left[2k^2(\varepsilon s) \;-\; \frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) \right] t w'(t) \;+\; \varepsilon^2 |h'(\varepsilon s)|^2 w''(t)$$

$$+\, \varepsilon(t + h(\varepsilon s)) A_0(\varepsilon s, \varepsilon(t+h))[-\varepsilon^2 h''(\varepsilon s) w'(t) + \varepsilon^2 |h'(\varepsilon s)|^2 w''(t)]$$

$$+\, \varepsilon^2 (t + h(\varepsilon s)) \tilde{B}_0(\varepsilon s, \varepsilon(t+h)) \left(-\varepsilon h'(\varepsilon s) w'(t)\right)$$

$$+\,\varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s,\varepsilon(t+h))w'(t) \tag{5.2}$$

with $A_0, \tilde{B}_0, \tilde{C}_0$ are given in (2.4)-(2.9) -(2.10), respectively. We emphasize that we have broken formula (5.2) into powers of ε, keeping in mind that $h = O(\varepsilon)$.

For every fixed $s \in \mathbb{R}$, let us consider the L^2-projection, given by

$$\Pi(s) := \int_{-\infty}^{+\infty} S(v_0)(s,t)w'(t)dt$$

where for simplicity we are assuming that coordinates are defined for all t, since the difference with the integration taken in all the actual domain for t, produces only exponentially small terms.

From (5.2), we observe that

$$\Pi(\varepsilon s) = -\varepsilon^2 \mathcal{J}_a[h](\varepsilon s)\int_{\mathbb{R}} w'(t)^2 dt$$
$$-\,\varepsilon^3\int_{\mathbb{R}}(t+h)A_0(\varepsilon s,\varepsilon(t+h))\left[h''(\varepsilon s)\,|w'(t)|^2 - |h'(\varepsilon s)|^2\,w''(t)w'(t)\right]dt$$
$$-\varepsilon^3\int_{\mathbb{R}}\left[(t+h)\tilde{B}_0(\varepsilon s,\varepsilon(t+h))h'(\varepsilon s)\;-\;(t+h)^2\tilde{C}_0(\varepsilon s,\varepsilon(t+h))\right]|w'(t)|^2 dt \tag{5.3}$$

where we used that $\int_{-\infty}^{+\infty} tw'(t)^2dt = 0, \int_{-\infty}^{+\infty} w''(t)w'(t)dt = 0$, to get rid of the terms of order ε^2.

Making this projection equal to zero is equivalent to the nonlinear differential equation for h

$$\mathcal{J}_{a,\Gamma}[h] = h''(\boldsymbol{s}) + \frac{\partial_s a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)}h'(\boldsymbol{s}) - Q(\boldsymbol{s})h(\boldsymbol{s}) = G_0[h]\ ,\ \forall \boldsymbol{s}\in\mathbb{R} \tag{5.4}$$

where we have set

$$Q(\boldsymbol{s}) = \left[\frac{\partial_{tt}a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s})\right]$$

and G_0 consists in the remaining terms of (5.3).

G_0 is easily checked to be a Lipschitz in h, with small Lipschitz constant. Here is where the *nondegeneracy condition* on the curve Γ makes its appearance, as we need to invert the operator $\mathcal{J}_{a,\Gamma}$, in such way that equation (5.4) can be set as a fixed problem for a contraction mapping of a ball of the form (5.1).

It will be necessary to pay attention to the size of $S(v_0)$ up to $O(\varepsilon^2)$, because the solvability of the nonlinear Jacobi equation (5.4) depends strongly on the fact that the error created by our choice of the approximation, is sufficiently small in $\varepsilon > 0$.

We improve our choice of the approximation throughout the following argument. Let us consider the ODE

$$\psi_0''(t) - F''(w(t))\psi_0(t) = tw'(t)$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the variation of parameters formula

$$\psi_0(t) = w'(t)\int_0^t w'(\tau)^{-2}\int_{-\infty}^{\tau} sw'(s)^2 ds d\tau.$$

Since $\int_{\mathbb{R}} sw'(s)^2ds = 0$, the function $\psi_0(s)$ satisfies that

$$\|e^{\sigma|t|}\partial^j\psi_0\|_{L^\infty(\mathbb{R})} \le C_{j,\sigma}, \quad j = 1,2,\dots, \quad 0 \le \sigma < \sqrt{2}.$$

Due to the finer topology we are considering for h, we must also improve the term

$$-\varepsilon^3\left[k^3(\varepsilon s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]t^2w'(t)$$

present in $\varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s,\varepsilon(t+h))w'(t)$ in expression (5.2). Hence, let us consider $g(t) := t^2w'(t)$ and note that we can write

$$g = C_g w' + g_\perp$$

where $g_\perp$ denotes the orthogonal projection of g onto w' in $L^2(\mathbb{R})$, given by

$$g_\perp(t) := t^2 w'(t) - \frac{\int_{\mathbb{R}} \tau^2 w'(\tau)^2 d\tau}{\int_{\mathbb{R}} w'(\tau)^2 d\tau} w'(t).$$

Thus by setting

$$\psi_1(t) = w'(t) \int_0^t w'(\tau)^{-2} \int_{-\infty}^{\tau} g_\perp(s) w'(s) ds d\tau$$

this formula provides a bounded smooth solution to $\psi_1''(t)F''(w(t))\psi_1(t) = g_\perp(t)$ with

$$\|e^{\sigma|t|}\partial^j \psi_1\|_{L^\infty(\mathbb{R})} \leq C_{j,\sigma}, \quad j = 1, 2, \ldots, \quad 0 \leq \sigma < \sqrt{2}.$$

Hence, we choose as new approximation, the function

$$v_1(s,t) := v_0(s,t) + \varphi_1(s,t) = w(t) + \varphi_1(s,t) \tag{5.5}$$

where

$$\begin{aligned} \varphi_1(s,t) \quad := \quad & \varepsilon^2 \left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}} a(\varepsilon s, 0)}{a(\varepsilon s, 0)} \right] \psi_0(t) \\ & -\varepsilon^3 \left[k^3(\varepsilon s) + \frac{1}{2} \partial_{tt} \left(\frac{\partial_t a}{a} \right)(\varepsilon s, 0) \right] \psi_1(t) \end{aligned}$$

and which can be easily seen to behave like $\varphi_1(s,t) = O(\varepsilon^2(1+|\varepsilon s|)^{-2-\alpha} e^{-\sigma|t|})$, for sigma $0 < \sigma < \sqrt{2}$. This is due to the assumptions (1.12)-(1.14) we have made on the curve Γ and the potential a, and to the previous observation on $\psi_0(t)$, $\psi_1(t)$.

Now, to analyze the error term $S(v_1)$, notice that

$$S(v_0 + \varphi_1) = S(v_0) \; + \; \Delta_x \varphi_1 + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \varphi_1 - F''(v_0)\varphi_1 + N_0(\varphi_1)$$

where

$$N_0(\varphi_1) = -F'(v_0 + \varphi_1) + F'(v_0) + F''(v_0)\varphi_1 \tag{5.6}$$

From the definition of φ_1, we find that

$$\begin{aligned} S(v_1) \quad = \quad & S(v_0) \; + \; \varepsilon^2 \left[2k^2(\varepsilon s) - \frac{\partial_{tt} a(\varepsilon s, 0)}{a(\varepsilon s, 0)} \right] t w'(t) \\ & - \varepsilon^3 \left[k^3(\varepsilon s) + \frac{1}{2} \partial_{tt} \left(\frac{\partial_t a}{a} \right)(\varepsilon s, 0) \right] g_\perp(t) \\ & + \left[\Delta_x + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x - \partial_{tt} \right] \varphi_1 \; + \; N_0(\varphi_1). \end{aligned} \tag{5.7}$$

Analyzing the new error created by φ_1, we readily check using the expansions for the differential operators (2.3)-(2.7) and the definition (5.6), that

$$\begin{gathered} \left[\Delta_x + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x - \partial_{tt} \right] \varphi_1 \; + \; N_0(\varphi_1) \; = \\ -\varepsilon^4 \, Q''(\varepsilon s)\psi_0 + \varepsilon^4 \left[\mathcal{J}_a[h](\varepsilon s) - t Q(\varepsilon s) \right] Q(\varepsilon s) \psi_0' \\ -\varepsilon^4 \left(\frac{\partial_{\boldsymbol{s}} a(\varepsilon s, 0)}{a(\varepsilon s, 0)} Q'(\varepsilon s)\psi_0 + 2h_1'(-Q'(\varepsilon s)\psi_0') + |h_1'|^2 Q(\varepsilon s)\psi_0'' \right) \\ + O(\varepsilon^4 (1 + |\varepsilon s|)^{-4-\alpha} e^{-\sigma|t|}) \end{gathered} \tag{5.8}$$

where we recall the convention

$$Q(\boldsymbol{s}) = \frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s}, 0)}{a(\boldsymbol{s}, 0)} - 2k^2(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{R}$$

and have used that the error terms in the differential operator evaluated in φ_1, associated to $A_0(\varepsilon s, \varepsilon(t+h))$, $\tilde{B}_0(\varepsilon s, \varepsilon(t+h))$, $\tilde{C}_0(\varepsilon s, \varepsilon(t+h))$ behave like $O(\varepsilon^5(1+|\varepsilon s|)^{-4-2\alpha}e^{-\sqrt{2}|t|})$, given that h has a bounded size is εs by (5.1), and since $\varphi_1(s,t)$ has smooth dependence in εs with size $O(\varepsilon^2(1+|\varepsilon s|)^{-2-\alpha}e^{-\sigma|t|})$.

Therefore, the error (5.8) is can be written as

$$\left[\Delta_x + \varepsilon\frac{\nabla_{\bar{x}} a}{a}\nabla_x - \partial_{tt}\right]\varphi_1 + N_0(\varphi_1) = \varepsilon^4 Q(\varepsilon s)\psi_0'(t)h''(\varepsilon s) + R_0(\varepsilon s, t, h)$$

where the function $R_0 = R_0(\varepsilon s, t, h(\varepsilon s), h'(\varepsilon s))$ has Lipschitz dependence in variables h, h' on the ball

$$\|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})} \le \mathcal{K}\varepsilon.$$

Moreover, under our set of assumptions and the observation made on ψ_0, it turns out that for any $\lambda \in (0,1)$:

$$\|R_0(\varepsilon s, t, h)\|_{C^{0,\lambda}(B_1(s,t))} \le C\varepsilon^4(1+|\varepsilon s|)^{-4-\alpha}e^{-\sigma|t|}.$$

With this remarks, we can write the error created by v_1 in (5.7), as

$$S(v_1) = -\varepsilon^2\mathcal{J}_a[h](\varepsilon s)w'(t) + \varepsilon^3\left[k^3(\varepsilon s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s, 0)\right]\frac{\int_{\mathbb{R}}\tau^2 w'(\tau)^2 d\tau}{\int_{\mathbb{R}} w'(\tau)^2 d\tau}w'(t)$$

$$+\varepsilon^4 Q(\varepsilon s)\psi_0'(t)h''(\varepsilon s) - \varepsilon^3(t+h)A_0(\varepsilon s, \varepsilon(t+h))h''(\varepsilon s)w'(t) \; + \; R_1(\varepsilon s, t, h(\varepsilon s), h'(\varepsilon s)) \tag{5.9}$$

where

$$R_1 = \varepsilon^2|h'|^2 w''(t) + R_0(\varepsilon s, t) + \varepsilon^3(t+h)A_0(\varepsilon s, \varepsilon(t+h))|h'|^2 w''(t)$$

$$- \varepsilon^3(t+h)\tilde{B}_0(\varepsilon s, \varepsilon(t+h))h'w'(t) + \varepsilon^4(t+h)O\left(\partial_{\boldsymbol{ttt}}\left(\frac{\partial_{\boldsymbol{t}} a}{a}\right) + k^4\right)t^2 w'(t). \tag{5.10}$$

Furthermore, $R_1 = R_1(\varepsilon s, t, h(\varepsilon s), h'(\varepsilon s))$ satisfies that

$$|\partial_\imath R_1(\varepsilon s, t, \imath, \jmath)| + |\partial_\jmath R_1(\varepsilon s, t, \imath, \jmath)| + |R_1(\varepsilon s, t, \imath, \jmath)| \le C\varepsilon^4(1+|\varepsilon s|)^{-2-2\alpha}e^{-\sqrt{2}|t|}$$

with the constant C above depending on the number $\mathcal{K}$ of condition (5.1), but independent of $\varepsilon > 0$.

We can summarize this discussion by saying that

$$S(v_1) \; + \; \varepsilon^2\mathcal{J}_a[h](\varepsilon s)w'(t) - \varepsilon^3\left[k^3(\varepsilon s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s, 0)\right]\frac{\int_{\mathbb{R}}\tau^2 w'(\tau)^2 d\tau}{\int_{\mathbb{R}} w'(\tau)^2 d\tau}w'(t) \; = \; O(\varepsilon^4(1+|\varepsilon s|)^{-2-\alpha}e^{-\sigma|t|})$$

for $x = X_{\varepsilon,h}(s,t) \in \mathcal{N}_{\varepsilon,h}$.

5.2. **The global approximation.** The approximation $v_1(x)$ in (5.5) will be sufficient for our purposes. However, it is defined only in the region

$$\mathcal{N}_{\varepsilon,h} = \left\{x = X_{\varepsilon,h}(s,t) \in \mathbb{R}^2 / \; |t + h(\varepsilon s)| < \frac{\delta}{\varepsilon} + c_0|s| =: \rho_\varepsilon(s)\right\} \tag{5.11}$$

Since we are assuming that Γ is a connected and simple and that it also possesses two ends departing from each other, it follows that $\mathbb{R}^2 \setminus \Gamma_\varepsilon$ consists of precisely two components S_+ and S_-. Let us use the convention that ν_ε points towards S_+. The previous comments allow us to define in $\mathbb{R}^2 \setminus \Gamma_\varepsilon$ the function

$$\mathbb{H}(x) := \begin{cases} +1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases}$$

Let us consider $\eta(s)$ a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_3(x) := \begin{cases} \eta(|t + h(\varepsilon s)| - \rho_\varepsilon(s) + 3) & \text{if } x \in \mathcal{N}_{\varepsilon,h} \\ 0 & \text{if } x \notin \mathcal{N}_{\varepsilon,h} \end{cases}$$

where ρ_ε is defined in (5.11).

Next, we consider as global approximation the function $\mathtt{w}(x)$ defined as

$$\mathtt{w} := \zeta_3 \cdot v_1 + (1 - \zeta_3) \cdot \mathbb{H} \tag{5.12}$$

where $u_1(x)$ is given by (5.5).

Using that $\mathbb{H}(\varepsilon^{-1}\,\bar{x})$ is an exact solution to (1.10) in $\mathbb{R}^2 \setminus \Gamma$, the error of global approximation can be computed as

$$S(\mathtt{w}) \;=\; \Delta_x \mathtt{w} + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} - F'(\mathtt{w}) \;=\; \zeta_3 S(v_1) + E \tag{5.13}$$

where $S(v_1)$ is computed in (5.9) and the term E is given by

$$E = \Delta_x \zeta_3 (v_1 - \mathbb{H}) + 2\nabla_x \zeta_3 \nabla_x (v_1 - \mathbb{H}) + (v_1 - \mathbb{H}) \frac{\nabla_{\bar{x}} a}{a} \nabla_x \zeta_3 \\ - F'\big(\zeta_3 v_1 + (1 - \zeta_3)\mathbb{H}\big) + \zeta_3 F'(v_1) \tag{5.14}$$

It is worth to mention that the from the form of the neighborhood $\mathcal{N}_{\varepsilon,h}$ in (5.11), and from the choice of v_1, one can readily check that for every $x = X_{\varepsilon,h} \in \mathcal{N}_{\varepsilon,h}$

$$|v_1(x) - \mathbb{H}(x)| \le e^{-\sqrt{2}|t + h(\varepsilon s)|}, \quad \rho_\varepsilon - 2 < |t + h(\varepsilon s)| < \rho_\varepsilon - 1$$

and therefore

$$|E| \le C\; e^{-\sqrt{2}|t + h(\varepsilon s)|} \le C e^{-\sqrt{2}\delta/\varepsilon} \cdot e^{-c|s|} e^{-\sigma|t|}$$

for some $0 < \sigma < \sqrt{2}$ and $c > 0$ small.

6. The proof of Theorem 1

In this section we sketch the proof of Theorem 1 leaving the detailed proofs of every proposition mentioned here for subsequent sections.

We look for a solution u of the inhomogeneous Allen-Cahn equation (2.2) in the form

$$u = \mathtt{w} + \varphi$$

where $\mathtt{w}$ is the global approximation defined in (5.12) and φ is small in some suitable sense. We find that φ must solve the following nonlinear equation

$$\Delta_x \varphi + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \varphi - F''(\mathtt{w})\varphi \;=\; -S(\mathtt{w}) \;-\; N_1(\varphi) \tag{6.1}$$

where

$$S(\mathtt{w}) := \Delta_x \mathtt{w} + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} - F'(\mathtt{w}) \tag{6.2}$$

$$N_1(\varphi) := -F'(\mathtt{w} + \varphi) + F'(\mathtt{w}) + F''(\mathtt{w})\varphi \tag{6.3}$$

We introduce several norms that will allow us to set up an appropriate functional scheme to solve (6.1). Let us consider $\eta(s)$, a cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta = 0$ for $s > 2$, we define

$$\zeta_n(x) := \begin{cases} \eta\left(|t + h(\varepsilon s)| - \rho_\varepsilon(s) + n\right) & \text{if } x \in \mathcal{N}_{\varepsilon,h} \\ 0 & \text{if } x \notin \mathcal{N}_{\varepsilon,h} \end{cases} \tag{6.4}$$

where ρ_ε and $\mathcal{N}_{\varepsilon,h}$ are set in (5.11).

Let us consider $\lambda \in (0,1)$, $b_1, b_2 > 0$ fixed and satisfying that $b_1^2 + b_2^2 < (\sqrt{2} - \tau)/2$ for $\tau > 0$. Define the weight function $K(x)$, for $x = (x_1, x_2) \in \mathbb{R}^2$, as follows

$$K(x) := \zeta_2(x) \left[e^{\sigma|t|/2} (1 + |\varepsilon s|)^\mu \right] + (1 - \zeta_2(x)) e^{b_1|x_1| + b_2|x_2|}. \tag{6.5}$$

For a function $g(x)$ defined in $\mathbb{R}^2$, we set the norms

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{L^\infty(B_1(x))} \tag{6.6}$$

$$\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{C^{0,\lambda}(B_1(x))} \tag{6.7}$$

On the other hand, consider $\varepsilon > 0$, $\mu \geq 0$ and $0 < \sigma < \sqrt{2}$. For functions $g(s,t)$ and $\phi(s,t)$ defined in whole $\mathbb{R}\times\mathbb{R}$, we set

$$\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} (1+|\varepsilon s|)^\mu e^{\sigma|t|}\|g\|_{C^{0,\lambda}(B_1(s,t))} \tag{6.8}$$

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} \tag{6.9}$$

Finally, given $\alpha > 0$ and $\lambda \in (0,1)$, consider,for a function f defined in $\mathbb{R}$, the norm

$$\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} := \sup_{\boldsymbol{s}\in\mathbb{R}} (1+|\boldsymbol{s}|)^{2+\alpha}\|f\|_{C^{0,\lambda}(\boldsymbol{s}-1,\boldsymbol{s}+1)}. \tag{6.10}$$

Recall also that the parameter function $h(\bar{s})$ satisfies for some $\lambda \in (0,1)$

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\varepsilon$$

where

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1+|\boldsymbol{s}|)^{1+\alpha}h'\|_{L^\infty(\mathbb{R})} + \|h''\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}. \tag{6.11}$$

4.1 The gluing procedure. In order to solve (6.1), let us look for a solution φ of problem, having the form

$$\varphi(x) = \zeta_3(x)\phi(s,t) + \psi(x), \quad \text{for } x \in \mathbb{R}^2$$

where ϕ is defined in $\Gamma_\varepsilon \times \mathbb{R}$ and ψ is defined in entire $\mathbb{R}^2$. Using that $\zeta_3 \cdot \zeta_4 = \zeta_4$, we get that (6.1) reads as

$$\begin{aligned} S(\mathtt{w}+\varphi) = \zeta_3 &\left[\Delta_x\phi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi + f'(u_1)\phi\right] \\ &+ \zeta_4\,[\,[f'(u_1) - f'(H(t))]\psi + N_1(\psi+\phi) + S(u_1)] \\ &+\Delta_x\psi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi + [(1-\zeta_4)f'(u_1) + \zeta_4 f'(H(t))]\psi + (1-\zeta_3)S(\mathtt{w}) \\ &+(1-\zeta_4)N_1(\psi+\zeta_3\phi) + 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3 \end{aligned}$$

where $H(t)$ is some increasing smooth function satisfying

$$H(t) = \begin{cases} +1 & \text{if } t > 1 \\ -1 & \text{if } t < -1. \end{cases} \tag{6.12}$$

In this way, we will have constructed a solution $\varphi = \zeta_3\phi + \psi$ to problem (6.1) if we require that the pair (ϕ,ψ) satisfies the coupled system below

$$\begin{aligned} &\Delta_x\phi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi + f'(u_1)\phi \\ &\qquad + \zeta_4[f'(u_1) - f'(H(t))]\psi + \zeta_4 N_1(\psi+\phi) + S(v_1) = 0 \ \text{ for } |t| < \frac{\delta}{\varepsilon} \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\Delta_x\psi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi + [(1-\zeta_4)f'(u_1) + \zeta_4 f'(H(t))]\psi + (1-\zeta_3)S(\mathtt{w}) \\ &\qquad + (1-\zeta_4)N_1(\psi+\zeta_3\phi) \ + \ 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3 = 0, \quad \text{in} \quad \mathbb{R}^2 \end{aligned} \tag{6.14}$$

Next, we will extend equation (6.13) to entire $\mathbb{R}\times\mathbb{R}$. To do so, let us set

$$\mathtt{B}(\phi) = \zeta_0\tilde{\mathtt{B}}_0(\phi) := \zeta_0[\Delta_x - \partial_{tt} - \partial_{ss}]\phi \tag{6.15}$$

where Δ_x is expressed in local coordinates, using formula (2.3), and $\mathtt{B}(\phi)$ is understood to be zero for $|t+h(\varepsilon s)| > \rho_\varepsilon(s,t)-2$. Thus equation (6.13) is extended as

$$\partial_{tt}\phi+\partial_{ss}\phi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi+\mathtt{B}(\phi)+f'(w(t))\phi=-\tilde{S}(u_1)$$
$$-\left\{[f'(u_1)-f'(w)]\phi+\zeta_4[f'(u_1)-f'(H(t))]\psi+\zeta_4 N_1(\psi+\phi)\right\}, \quad \text{in } \mathbb{R}\times\mathbb{R} \tag{6.16}$$

where we have denoted

$$\tilde{S}(u_1)(s,t)=-\varepsilon^2\mathcal{J}_a[h](\varepsilon s)w'(t)-\varepsilon^3\left[k^3(\varepsilon s)+\frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]\frac{\int_{\mathbb{R}}\tau^2 w'(\tau)^2 d\tau}{\int_{\mathbb{R}} w'(\tau)^2 d\tau}w'(t)$$
$$+\varepsilon^4 Q(\varepsilon s)\psi_0'(t)\cdot h''(\varepsilon s)+\zeta_0\left\{\varepsilon^3(t+h)A_0(\varepsilon s,\varepsilon(t+h))h''(\varepsilon s)\cdot w''(t)+R_1\right\}. \tag{6.17}$$

Recall from (5.10) that

$$R_1=R_1(\varepsilon s,t,h(\varepsilon s),h'(\varepsilon s))$$

satisfies

$$|\partial_\imath R_1(\varepsilon s,t,\imath,\jmath)|+|\partial_\jmath R_1(\varepsilon s,t,\imath,\jmath)|+|R_1(\varepsilon s,t,\imath,\jmath)|\le C\varepsilon^4(1+|\varepsilon s|)^{-2-2\alpha}e^{-\sqrt{2}|t|} \tag{6.18}$$

In order to solve the resulting system (6.14)-(6.16), we focus first on solving equation (6.14) in ψ for a fixed and small ϕ. We make use of the important observation that the term $[(1-\zeta_4)f'(u_1)+\zeta_4 f'(H)]$, is uniformly negative and so the operator in (6.14) is qualitatively similar to $\Delta_x+\varepsilon\nabla_{\bar{x}}a/a\cdot\nabla_x-2$. A direct application of the contraction mapping principle lead us to the existence of a solution $\psi=\Psi(\phi)$, according to the next proposition whose detailed proof is carried out in Section 5.

Proposition 6.1. *Let $\lambda\in(0,1)$, $\sigma\in(0,\sqrt{2})$, $\mu\in(0,2+\alpha)$. There is $\varepsilon_0>0$, such that for any small $\varepsilon\in(0,\varepsilon_0)$ the following holds. Given ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\le 1$, there is a unique solution $\psi=\Psi(\phi)$ to equation* (6.14) *with*

$$\|\psi\|_X:=\|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)}+\|D\psi\|_{L^\infty_K(\mathbb{R}^2)}+\|\psi\|_{L^\infty_K(\mathbb{R}^2)}\le Ce^{-\sigma\delta/2\varepsilon}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1)-\Psi(\phi_2)\|_X\le Ce^{-\sigma\delta/2\varepsilon}\|\phi_1-\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{6.19}$$

where the norms $L^\infty_K, C^{0,\lambda}_K, C^{2,\lambda}_{\mu,\sigma}$ are defined in (6.6)-(6.7)-(6.9).

4.2 Solving the Nonlinear Projected Problem. Using proposition 6.1, we solve (6.16) replacing ψ with the nonlocal operator $\psi=\Psi(\phi)$. Setting

$$\mathtt{N}(\phi):=\mathtt{B}(\phi)+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi+[f'(u_1)-f'(w)]\phi+\zeta_4[f'(u_1)-f'(H)]\Psi(\phi)+\zeta_4N_1(\Psi(\phi)+\phi)$$

our problem is reduced to find a solution ϕ to the following nonlinear, nonlocal problem

$$\partial_{tt}\phi+\partial_{ss}\phi+f'(w)\phi=-\tilde{S}(u_1)-\mathtt{N}(\phi) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{6.20}$$

Before solving (6.20), we consider the problem of finding a (ϕ,c) a solution to the following nonlinear projected problem

$$(NPP)\begin{cases}\partial_{tt}\phi+\partial_{ss}\phi+f'(w)\phi=-\tilde{S}(u_1)-\mathtt{N}(\phi)+c(s)w'(t) \quad \text{in } \mathbb{R}\times\mathbb{R}\\ \displaystyle\int_{\mathbb{R}}\phi(s,t)w'(t)dt=0, \quad \forall s\in\mathbb{R}.\end{cases} \tag{6.21}$$

Solving problem (6.21) amounts to eliminate the part of the right hand side in (6.20), that do not contribute to the projections onto $w'(t)$, namely $\int_{\mathbb{R}}[\tilde{S}(u_1)+N(\phi)]w'(t)dt$.

Since, we have that

$$\left\|\tilde{S}(u_1)+\varepsilon^2\mathcal{J}_a[h](\varepsilon s)\cdot w'(t)-\varepsilon^3\left[k^3(\varepsilon s)+\frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]\hat{c}w'(t)\right\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\leq C\varepsilon^4$$

where

$$\hat{c}=\|w'\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}t^2w'(t)^2dt$$

and due to the fact that $\mathtt{N}(\phi)$ defines a contraction within a ball centered at zero with radius $O(\varepsilon^4)$ in norm C^1, we conclude the existence of a unique small solution of problem (6.21) whose size is $O(\varepsilon^4)$ in this norm. This solution ϕ turns out to define an operator in h, namely $\phi=\Phi(h)$, which exhibits a Lipschitz character in norms $\|\cdot\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$. We collect the discussion in the following proposition.

Proposition 6.2. *Given $\lambda\in(0,1)$, $\mu\in(0,2+\alpha]$ and $\sigma\in(0,\sqrt{2})$, there exists a constant $K>0$ such that the nonlinear projected problem* (6.21) *has a unique solution $\phi=\Phi(h)$ with*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\leq K\varepsilon^4$$

Besides Φ has small a Lipschitz dependence on h satisfying condition (5.1)*, in the sense*

$$\|\Phi(h_1)-\Phi(h_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\leq C\varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\tag{6.22}$$

for any $h_1,h_2\in C^{2,\lambda}_{loc}(\mathbb{R})$ with $\|h_i\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\leq\mathcal{K}\varepsilon$.

The proof of this proposition is left to section 5, where a complete study of the linear theory needed to solve is discussed.

4.3 Adjusting the nodal set. In order to conclude the proof of Theorem 1, we have to adjust the parameter function h so that the nonlocal term

$$c(s)\int_{\mathbb{R}}|w'(t)|^2dt=\int_{\mathbb{R}}\tilde{S}(u_1)(\varepsilon s,t)w'(t)dt+\int_{\mathbb{R}}\mathtt{N}(\Phi(h))(s,t)w'(t)dt\tag{6.23}$$

becomes identically zero, and consequently we obtain a genuine solution to equation (1.10).

Setting $c_*:=\int_{\mathbb{R}}|w'(t)|^2dt$, using expression (6.17), and carrying out the same computation we did in (5.3), we obtain that

$$\int_{\mathbb{R}}\tilde{S}(u_1)(\varepsilon s,t)w'(t)dt=-c_*\varepsilon^2\mathcal{J}_a[h](\varepsilon s)+c_*\varepsilon^2G_1(h)(\varepsilon s)$$

where

$$\begin{aligned}c_*G_1(h)(\varepsilon s):=&-\varepsilon\left[k^3(\varepsilon s)+\frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]\frac{\int_{\mathbb{R}}\tau^2w'(\tau)^2d\tau}{\int_{\mathbb{R}}w'(\tau)^2d\tau}w'(t)\\&+\varepsilon h''(\varepsilon s)\int_{\mathbb{R}}\zeta_0(t+h)A_0(\varepsilon s,\varepsilon(t+h))w''(t)w'(t)dt\\&+\varepsilon^2Q(\varepsilon s)h''(\varepsilon s)\int_{\mathbb{R}}\psi_0'(t)w'(t)dt+\varepsilon^{-2}\int_{\mathbb{R}}\zeta_0\;R_1(\varepsilon s,t,h,h')w'(t)dt\end{aligned}\tag{6.24}$$

and we recall that R_1 is of size $O(\varepsilon^4)$ in the sense of (6.18). Thus setting

$$c_*G_2(h)(\varepsilon s):=\varepsilon^{-2}\int_{\mathbb{R}}\mathtt{N}(\Phi(h))(s,t)w'(t)dt,\quad \mathtt{G}(h)(\varepsilon s):=G_1(h)(\varepsilon s)+G_2(h)(\varepsilon s)\tag{6.25}$$

it turns out that equation (6.23) is equivalent to

$$c(s)\cdot c_*=-c_*\varepsilon^2\mathcal{J}_{a,\Gamma}[h](\varepsilon s)+c_*\varepsilon^2G_1(h)(\varepsilon s)+c_*\varepsilon^2G_2(h)(\varepsilon s)$$

Therefore the condition $c(s)=0$ is equivalent to the following nonlinear problem on h

$$\mathcal{J}_{a,\Gamma}[h](\varepsilon s)=h''(\varepsilon s)+\frac{\partial_{\boldsymbol{s}}a(\varepsilon s,0)}{a(\varepsilon s,0)}h'(\varepsilon s)-Q(\varepsilon s)h(\varepsilon s)=\mathtt{G}[h](\varepsilon s),\quad\text{in }\mathbb{R}\tag{6.26}$$

Consequently, we will have proved Theorem 1, if we find a function h, solving equation (6.26).

Hence, we need to devise a corresponding solvability theory for the linear problem

$$\mathcal{J}_a[h](\boldsymbol{s}) = f(\boldsymbol{s}), \quad \text{in } \mathbb{R}. \tag{6.27}$$

The next result addresses this matter.

Proposition 6.3. *Given $\alpha > 0$, $\lambda \in (0,1)$, and a function f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < \infty$, assume that Γ is a smooth curve satisfying* (1.15)*. If, Γ is nondegenerate respect to the potential a and conditions* (1.11)-(1.14) *hold, then there exists a unique bounded solution h of problem* (6.27)*, and there exists a positive constant $C = C(a,\Gamma,\alpha)$ such that*

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le C\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

with the norms defined in (6.10)-(6.11)*.*

In section 6, we study in detail the proof of this proposition. For the time being, let us note that, $\mathtt{G}$ is a small operator of size $O(\varepsilon)$ uniformly on functions h satisfying (5.1). Hence Proposition 6.3 plus the contraction mapping principle yield the next result, which ensures the solvability of the nonlinear Jacobi equation. Its detailed proof can be found in section 7.

Proposition 6.4. *Given $\alpha > 0$ and $\lambda \in (0,1)$, there exist a positive constant $\mathcal{K} > 0$ such that for any $\varepsilon > 0$ small enough the following holds. There is a unique solution h of* (6.26) *on the region* (6.11)*, namely $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le \mathcal{K}\varepsilon$.*

and this concludes the proof of Theorem 1.

The rest of the paper is devoted to give fairly detailed proofs of every result stated in this section.

7. Gluing reduction and solution to the projected problem

This section is devoted to give fairly detailed proves of propositions 6.1 and 6.2. In what follows, we refer to the notation and to the objects introduced in sections 3 and 4.

7.1. **The proof of proposition 6.1.**

In this part we prove proposition 6.1. To do so, let us first consider the linear problem

$$\Delta_x\psi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi - W_\varepsilon(x)\psi = g(x), \quad \text{in} \quad \mathbb{R}^2 \tag{7.1}$$

where

$$-W_\varepsilon(x) := [(1-\zeta_4)f'(u_1) + \zeta_4 f'(H(t))].$$

Observe that the dependence in ε is implicit on the cut-off function ζ_4, defined in (6.4).

Let us observe that for any $\varepsilon > 0$ small enough, the term W_ε satisfies the global estimate $0 < \beta_1 < W_\varepsilon(x) < \beta_2$ for a certain positive constants β_1, β_2. In fact, we can chose $\beta_1 := \sqrt{2} - \tau$ for any arbitrary small $\tau > 0$. To address the study of this equation, recall the definition of the weighted norms:

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{L^\infty(B_1(x))}, \quad \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{C^{0,\lambda}(B_1(x))}$$

with K is given by (6.5).

Lemma 7.1. *For any $\lambda \in (0,1)$, there are numbers $C > 0$, and $\varepsilon_0 > 0$ small enough, such that for $0 < \varepsilon < \varepsilon_0$ and any given continuous function $g = g(x)$ with $\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} < +\infty$, the equation* (7.1) *has a unique solution $\psi = \Psi(\phi)$ satisfying the a priori estimate:*

$$\|\psi\|_X := \|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le C\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)}$$

The proof of this lemma follows the same lines of lemma 7.1 in [4] with no significant changes. We leave details to the reader, but we do comment on the estimate

$$\|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le C\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \tag{7.2}$$

for $\varepsilon > 0$ small enough and any bounded solution ψ of (7.1). It follows directly from a sub-supersolution scheme, using that $b_1^2 + b_2^2 < (\sqrt{2} - \tau)/2$ and the fact that the function

$$\psi_0(x) := e^{R_0}\|\psi\|_\infty \cdot \left\{\zeta_3(x)[e^{-\sigma|t|/2}(1+|\varepsilon s|)^{-\mu}] + (1-\zeta_3(x))e^{-b_1|x_1|-b_2|x_2|}\right\}$$

ca be readily checked to be a positive supersolution of (7.1), provided that $R_0 > 0$ sufficiently large.

Hence, we can use the maximum principle within the annulus $B_{R_1}(\vec{0})\setminus B_{R_0}(\vec{0})$ with a barrier function of the form $\psi_0 + \theta e^{\sqrt{\beta_1/2}(|x_1|+|x_2|)}$ for $\theta > 0$ small, to find that

$$K(x)|\psi(x)| \le M\|\psi\|_{L^\infty(\mathbb{R}^2)} \le \tilde{M}\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)}, \quad x \in \mathbb{R}^2.$$

from which estimate (7.2) follows.

Now we have all the ingredients need for the proof of proposition 6.1. Let us set $\psi := \Upsilon(g)$ the solution of equation (7.1) predicted by lemma 7.1. We can write problem (6.14) as a fixed point problem in the space X of functions $\psi \in C^{2,\lambda}_{loc}(\mathbb{R}^2)$ with $\|\psi\|_X < \infty$, as

$$\psi = \Upsilon(g_1 + G(\psi)), \quad \psi \in X \tag{7.3}$$

where

$$g_1 := (1-\zeta_3)S(\mathtt{w}) + 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3,$$

$$G(\psi) := (1-\zeta_4)N_1(\psi + \zeta_3\phi).$$

Consider $\mu \in (0, 2+\alpha)$, $\sigma \in (0, \sqrt{2})$ and $\alpha > 0$ fixed and a function h satisfying (6.11). Consider also a function $\phi = \phi(s,t)$, satisfying $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le 1$.

Note that the derivatives of ζ_3 are nontrivial only within the region $\rho_\varepsilon - 2 < |t + h(\varepsilon s)| < \rho_\varepsilon - 1$, with ρ_ε defined in (5.11). Taking into account the weight $K(x)$ (6.5), we find that

$$\begin{aligned} K(x)\left|2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3\right| &\le C_a K(x)e^{-\sigma|t|}(1+|\varepsilon s|)^{-\mu}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\ &\le C_a\ e^{-\sigma\delta/2\varepsilon}e^{\sigma/2(-c_0|s|+2+|h|)}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \end{aligned}$$

provided that

$$c_0 < \frac{b_2\delta}{a_2}, \qquad \frac{c_0\,\theta}{1-\theta} \le b_2$$

conditions that holds, since we can take $c_0 > 0$ small enough,independent of $\varepsilon > 0$ and θ small depending maybe on c_0. At the end, there are some constants $\tilde{c}_0$ and $\tilde{\delta} > 0$, depending on Γ and $a(x,y)$, such that the right hand side satisfies for $x \in \mathbb{R}^2$

$$\left\|2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3\right\|_{C^{0,\lambda}B(x,1)} \le C_{a,\Gamma}\ e^{-\sigma\tilde{\delta}/\varepsilon}e^{-\tilde{c}_0|x|}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

where these constants are explicitly $\tilde{\delta} := \delta - c_0a_2/b_2$, $\tilde{c}_0 := \sigma\theta c_0/b_2$, and where we emphasize that $C_{a,\Gamma}$ does not depend on ε.

Expressions (5.13)-(5.14) for $S(\mathtt{w})$ imply that $\|S(\mathtt{w})\|_{C^{0,\lambda}_{\mu,\sqrt{2}}(\mathbb{R}^2)} \le C\varepsilon^3$. In particular, the exponential decay exhibited by w', w'', ψ_0, ψ_1 in $t-$variable imply

$$|(1-\zeta_3)S(\mathtt{w})| = |(1-\zeta_3)\zeta_3 S(u_1) + (1-\zeta_3)E| \le C_a\ e^{-\sigma|t|}(1+|\varepsilon s|)^{-2-\alpha}$$

Now since this error term is vanishing everywhere but on the region $\rho_\varepsilon - 2 < |t + h(\varepsilon s)| < \rho_\varepsilon - 1$, we can use the definition (6.5) of the weight function $K(x)$ to prove that

$$\begin{aligned} K(x)|(1-\zeta_3)S(\mathtt{w})(x)| &\le e^{\sigma|t|/2}(1+|\varepsilon s|)^{\mu-2-\alpha}\ C_a e^{-\sigma|t|/2} e^{-(\sqrt{2}-\sigma/2)|t|} \\ &\le C_a e^{-(\sqrt{2}-\sigma/2)(\delta/\varepsilon + c_0|s| - |h| - 2)} \le C e^{-\sigma\tilde{\delta}/\varepsilon} \end{aligned}$$

where we have used the expression (5.11) for ρ_ε, and we set $\tilde{\delta} := (\sqrt{2}/\sigma - 1/2)\delta >> \delta/2$. Further, the regularity in the $s-$variable of the functions involved in g_1, imply that

$$\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C e^{-\sigma\delta/2\varepsilon}$$

On the other hand, consider the set for $A > 0$ large

$$\Lambda = \{\psi \in X :\ \|\psi\|_X \le A \cdot e^{-\sigma\delta/2\varepsilon}\} \tag{7.4}$$

The definitions of N_1 in (6.3) and G in (7.1), lead us to the following computations

$$\begin{aligned} &(1-\zeta_4)|N_1(\Psi(\phi_1)+\zeta_3\phi_1) - N_1(\Psi(\phi_2)+\zeta_3\phi_2)| \le \\ &C_{\mathtt{w}}(1-\zeta_4)\sup_{t\in(0,1)} |t\Psi(\psi_1) + (1-t)\Psi(\psi_2) + \zeta_3(t\phi_1 + (1-t)\phi_2)| \cdot |\Psi(\psi_1) - \Psi(\psi_2)| \end{aligned}$$

together with

$$\begin{aligned} |G(\psi_1) - G(\psi_2)| &\le (1-\zeta_4)\sup_{\xi\in(0,1)} |DN_1(\xi\psi_1 + (1-\xi)\psi_2 + \zeta_3\phi)[\psi_1 - \psi_2]| \\ &\le C\|f''(\mathtt{w})\|_\infty (1-\zeta_4)\sup_{\xi\in(0,1)} |\xi\psi_1 + (1-\xi)\psi_2 + \zeta_3\phi| \cdot |\psi_1 - \psi_2| \end{aligned}$$

The latter, plus the regularity in the $s-$variable leads the Lipschitz character of G:

$$\|G(\psi_1) - G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C\, A\, e^{-\sigma\delta/\varepsilon}\|\psi_1 - \psi_2\|_{C^{0,\lambda}_K(\mathbb{R}^2)}$$

while

$$\|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C_{\mathtt{w}}\|(1-\zeta_4)\zeta_3^2\phi^2\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C e^{-\sigma\delta/\varepsilon}$$

In order to use the fixed point theorem, we need to estimate the size of the nonlinear operator

$$\begin{aligned} \|\Upsilon(g_1 + G(\psi))\|_X &\le \|\Upsilon(g_1 + G(\psi) - G(0))\|_X + \|\Upsilon(G(0))\|_X \\ &\le C(\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|G(\psi) - G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\le C(C_a\ e^{-\sigma\delta/2\varepsilon} + e^{-\sigma\delta/\varepsilon}\|\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\le C e^{-\sigma\delta/2\varepsilon}(1 + \|\psi\|_X) \end{aligned}$$

additionally, we also have

$$\begin{aligned} \|\Upsilon(g_1 + G(\psi_1)) - \Upsilon(g_1 + G(\psi_2))\|_X &\le C\|G(\psi_1) - G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \\ &\le C e^{-\sigma\delta/\varepsilon}\|\psi_1 - \psi_2\|_X \end{aligned}$$

where in both inequalities we used that Υ is a linear and bounded operator.

This means that the right hand side of equation (7.3) defines a contraction mapping on Λ into itself, provided that the number A in definition (7.4) is taken large enough and $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}} \le 1$. Hence applying Banach fixed point theorem follows the existence of a unique solution $\psi = \Psi(\phi) \in \Lambda$.

In addition, it is direct to check that

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le C_a\ e^{-\sigma\delta/2\varepsilon}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R})} + C\ e^{-\sigma\delta/\varepsilon}\|\Psi(\phi_1) - \Psi(\phi_2)\|_X$$

from where the Lipschitz dependence (6.19) of $\Psi(\phi)$ follows and this concludes the proof of Lemma 6.1

7.2. **The proof of proposition 6.2.** The purpose of the whole section is to give a proof of proposition 6.2, which deals with the solvability of the nonlinear projected problem (6.21) for ϕ.

At the core of the proof of proposition 6.2, is the fact that the heteroclinic solution $w(t)$ of the ODE

$$w''(t) + w(t)(1 - w^2(t)) = 0, \quad w'(t) > 0, \quad w(\pm\infty) = \pm 1$$

is $L^\infty-$nondegenerate in the sense of the following lemma.

Lemma 7.2. *Let ϕ be a bounded and smooth solution of the problem*

$$L(\phi) = 0, \quad in \;\; \mathbb{R}^2$$

Then necessarily $\phi(s,t) = Cw'(t)$, with $C \in \mathbb{R}$.

For a detailed proof of this lemma we refer the reader to [4] and references there in.
Next, let us consider the linear projected problem

$$\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = g(s,t) + c(s)w'(t) \;\; \text{in} \;\; \mathbb{R}\times\mathbb{R} \\ \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \forall s \in \mathbb{R} \end{cases} \tag{7.5}$$

Assuming that the corresponding operations can be carried out, for every fixed s, we can multiply the equation by $w'(t)$ and integrate by parts, to find that

$$c(s) = -\frac{\int_{\mathbb{R}} g(s,t)w'(t)dt}{\int_{\mathbb{R}} |w'(t)|^2 dt} \tag{7.6}$$

Hence, if ϕ solves problem (7.5), then ϕ eliminates the part of g which does not contribute to the projection onto $w'(t)$. This means, that ϕ solves the same equation, but with g replaced by $\tilde{g}$, where

$$\tilde{g}(s,t) = g(s,t) - \frac{\int_{\mathbb{R}} g(s,\tau)w'(\tau)d\tau}{\int_{\mathbb{R}} |w'(\tau)|^2 d\tau} w'(t)$$

Observe that the term $c(s)$ in problem (6.21) has a similar role, except that we cannot find it so explicitly, since this time the PDE in ϕ is nonlinear and nonlocal.

Now, we show that the linear problem (7.5) has a unique solution ϕ, which respects the size of g in norm (6.8), up to its second derivatives. We collect the discussion in the following proposition, whose proof is basically that contained in [6], [4].

Proposition 7.1. *Given $\mu \geq 0$ and $0 < \sigma < \sqrt{2}$, there is a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ the following holds. For any g with $\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$, the problem* (7.5) *with $c(s)$ defined in* (7.6)*, has a unique solution ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$. Furthermore, this solution satisfies the estimate*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Now, we are in a position to proof proposition 6.2. Recall from section 4, that proposition 6.2 refers to the solvability of the projected problem

$$\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(s)w'(t) \quad \text{in} \quad \mathbb{R}\times\mathbb{R} \\ \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \text{for all } s \in \mathbb{R}. \end{cases} \tag{7.7}$$

where we recall that $\mathtt{N}(\phi)$ is made out of the operators $H(t)$ and $\mathtt{B}$, given in (6.3)-(6.15), like

$$\mathtt{N}(\phi) := \underbrace{\mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi + \varepsilon\nabla_{\bar{x}}a/a \cdot \nabla_x\phi}_{\mathtt{N}_1(\phi)} + \underbrace{\zeta_4[f'(u_1) - f'(H(t))]\Psi(\phi)}_{\mathtt{N}_2(\phi)} + \underbrace{\zeta_4 N_1(\Psi(\phi) + \phi)}_{\mathtt{N}_3(\phi)} \tag{7.8}$$

Let us define $\phi := T(g)$ as the operator providing the solution predicted in proposition 7.1. Then (7.7) can be recast as the fixed point problem

$$\phi = T(-\tilde{S}(u_1) - \varepsilon^2 \mathcal{J}_{a,\Gamma}[h]\, w'(t) - \mathtt{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq K\varepsilon^4 \tag{7.9}$$

Claim 7.1. *Given $\alpha > 0$, $0 < \mu < 2 + \alpha$ and $0 < \sigma < \sqrt{2}$, there is some constant $C > 0$, possibly depending on the constant $\mathcal{K}$ of* (6.11) *but independent of ε, such that for $M > 0$ and ϕ_1, ϕ_2 satisfying*

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq M\varepsilon^4, \quad i = 1, 2.$$

then the nonlinearity $\mathtt{N}$ behaves locally Lipschitz, as

$$\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \; \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{7.10}$$

where the operator $\mathtt{N}$ is given in (7.8)*.*

To prove this claim, we analyze each of its components $\mathtt{N}_i$ from (7.8). Let us start with $\mathtt{N}_1$. Note that its first term corresponds to a second order linear operator with coefficients of order ε plus a decay of order at least $O((1 + |\varepsilon s|)^{-1-\alpha})$. In particular, recall from (6.15) that $\mathtt{B} = \zeta_0 \tilde{\mathtt{B}}_0$, where in coordinates $[\Delta_x - \partial_{tt} - \partial_{ss}]$ amounts

$$\begin{aligned}
\tilde{B}_0 = &- 2\varepsilon h \partial_{st} - \varepsilon[k(\varepsilon s) + \varepsilon(t+h)k^2(\varepsilon s)]\partial_t + \varepsilon(t+h)A_0(\varepsilon s, \varepsilon(t+h)) \\
&\cdot [\partial_{ss} - 2h'\partial_t + \varepsilon^2 |h'|^2 \partial_{tt}] + \varepsilon^2 (t+h) B_0(\varepsilon s, \varepsilon(t+h))[\partial_s - \varepsilon h' \partial_t] \\
&- \varepsilon^2 h'' \partial_t + \varepsilon^2 |h'|^2 \partial_{tt} \\
&+ \varepsilon^3 (t+h)^2 C_0(\varepsilon s, \varepsilon(t+h))\partial_t
\end{aligned}$$

Analyzing each term, leads to

$$\|\mathtt{B}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Thus, $\mathtt{N}_1$ satisfies

$$\|\mathtt{N}_1(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}. \tag{7.11}$$

On the other hand, consider functions ϕ_i, with

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq M\varepsilon^3, \quad i = 1, 2$$

Now, let us analyze $\mathtt{N}_2$, by noting that for any $(s,t) \in \mathbb{R}^2$ the definition (5.5) implies that

$$\begin{aligned}
&\|\mathtt{N}_3(\phi_1) - \mathtt{N}_3(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\qquad \leq C \sup_{(s,t)\in\mathbb{R}^2} e^{(\sigma/2 - \sqrt{2})|t|} \sup_{x\in\mathbb{R}^2} K(x) \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))} \\
&\qquad \leq C \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}_{K}(\mathbb{R}^2)} = Ce^{-\sigma\delta/2\varepsilon} \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned} \tag{7.12}$$

In order to analyze $\mathtt{N}_4$, note that the definition (6.3) of N_1 also implies

$$\begin{aligned}
|\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)| &\leq |\zeta_4 N_1(\Psi(\phi_1) + \phi_1) - \zeta_4 N_1(\Psi(\phi_2) + \phi_2)| \\
&\leq C\zeta_4 \sup_{\xi\in(0,1)} |\xi(\Psi(\phi_1) + \phi_1) + (1-\xi)(\Psi(\phi_2) + \phi_2)| \cdot (|\phi_1 - \phi_2| + |\Psi(\psi_1) - \Psi(\psi_2)|)
\end{aligned}$$

taking into account the region of $\mathbb{R}^2$ we are considering, it is possible to make appear de weight $K(x)$ in (6.5). Therefore thanks to the hypothesis on ϕ_i and Lemma 6.1, we obtain

$$\begin{aligned}
&\|\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\quad \le C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ e^{\sigma|t|/2} [\|\phi_1\|_{C^{0,\lambda}(B_1(s,t))} + \|\phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \\
&\qquad \cdot e^{\sigma|t|/2}(1+|\varepsilon s|)^{\mu} [\|\phi_1 - \phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \Big\} \\
&\le C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ \Big[\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)(\|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}) \Big] \\
&\qquad \cdot (e^{-\sigma|t|/2} \|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}) \Big\} \\
&\le C(\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\Psi(\phi_1)\|_X + \|\Psi(\phi_2)\|_X)[\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}} + \|\Psi(\phi_1) - \Psi(\phi_1)\|_X] \\
&\le 2C(\varepsilon^3 + e^{-\sigma\delta/2\varepsilon}) \Big\{ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + e^{-\sigma\delta/2\varepsilon} \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \Big\} \qquad (7.13)
\end{aligned}$$

To reach a conclusion, we note from (7.11)-(7.12) and (7.13) that choosing $\varepsilon > 0$ small enough we obtain the validity of inequality (7.10). The proof of Claim 7.1 is concluded. □

In order to conclude the proof of proposition6.2,we make the observation that the formula (6.17) and estimate (6.18) ensure that, for any $0 < \mu \le 2 + \alpha$, $\sigma \in (0, \sqrt{2})$ and $\lambda \in (0, 1)$ it holds

$$\left\| \tilde{S}(u_1) + \varepsilon^2 \mathcal{J}_a[h](\varepsilon s) \cdot w'(t) - \varepsilon^3 \left[k^3(\varepsilon s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s, 0)\right] \hat{c} w'(t) \right\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \quad \le \quad C\varepsilon^4 \qquad (7.14)$$

Let us assume now that $\phi_1, \phi_2 \in B_\varepsilon$, where

$$B_\varepsilon := \{\phi \in C^{2,\lambda}_{loc}(\mathbb{R}^2) \ / \ \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le K\varepsilon^4\}$$

for a constant K to be chosen. Note that using Claim 7.1, we are able to estimate the size of $\mathtt{N}(\phi)$ for any $\varepsilon > 0$ sufficiently small, as follows

$$\begin{aligned}
\|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &\le C\|\mathtt{N}(0)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\varepsilon\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&= C\|\zeta_4[f'(u_1) - f'(H)]\Psi(0) + \zeta_4 N_1(\Psi(0))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\varepsilon\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le C \sup_{t\in\mathbb{R}} e^{-\sigma\delta/2\varepsilon} \cdot \|\Psi(0)\|_X + \|\Psi(0)\|_X^2 + C\varepsilon \cdot K\varepsilon^4 \\
&\le \tilde{C}\varepsilon^5 \quad \forall\, \phi \in B_\varepsilon \qquad (7.15)
\end{aligned}$$

for some constant $\tilde{C}$, independent of K.

Then from the estimates (7.14)-(7.15) follows that the right hand side of the projected problem (7.7) defines an operator $\mathcal{T}$ applying the ball B_ε into itself, provided K is fixed sufficiently large and independent of $\varepsilon > 0$. Indeed using the definition of $\mathcal{T}$ from (7.9), and Proposition 7.1, we can easily find an estimate for the size of ϕ, through

$$\begin{aligned}
\|\mathcal{T}(\phi)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \|T(-\tilde{S}(u_1) - \varepsilon^2 \mathcal{J}_a[h]w' - \mathtt{N}(\phi))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le \|T\|(\|\tilde{S}(u_1) + \varepsilon^2 \mathcal{J}_a[h]w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}) \le C\varepsilon^4
\end{aligned}$$

Further, $\mathcal{T}$ is also a contraction mapping of B_ε in norm $C^{2,\lambda}_{\mu,\sigma}$ provided that $\mu \leq 2+\alpha$, since Claim (7.1) asserts that $\mathtt{N}$ has Lipschitz dependence in ϕ:

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = \| - T(\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$
$$\leq C\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \; \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

So by taking $\varepsilon > 0$ small, we can use the contraction mapping principle to deduce the existence of a unique fixed point ϕ to equation (7.9), and thus ϕ turns out to be the only solution of problem (7.7). This justify the existence of ϕ, as required.

On the other hand, the Lipschitz dependence (6.22) of Φ in h, follows from the fact that

$$\|\mathcal{T}(\Phi(h_1)) - \mathcal{T}(\Phi(h_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \; \leq \; C(\|\tilde{S}(u_1,h_1) - \tilde{S}(u_1,h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(\Phi_1) - \mathtt{N}(\Phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)})$$

A series of lengthy but straightforward computations, leads to (6.22) and so the proof is complete.

8. The proof of proposition 6.4

In this section, we will finish the proof of Theorem 1 by proving proposition 6.4. Recall that the reduced problem (6.26) reads as

$$\mathcal{J}_a[h](\varepsilon s) := h''(\varepsilon s) + \frac{\partial_{\boldsymbol{s}} a(\varepsilon s, 0)}{a(\varepsilon s, 0)} h'(\varepsilon s) - Q(\varepsilon s) h(\varepsilon s) \; = \; \mathtt{G}(h)(\varepsilon s) \quad \text{in } \mathbb{R} \tag{8.1}$$

where $Q(\boldsymbol{s})$ was defined in (3.2), and the operator $\mathtt{G} = G_1 + G_2$ was given in (6.24)-(6.25).

We will make use of the following technical lemma, whose proof is left to the reader.

Lemma 8.1. *Let $\Theta = \Theta(s,t)$ be a function defined in $\mathbb{R}\times\mathbb{R}$, such that, for any $\lambda \in (0,1)$, $\mu \in (1, 2+\alpha]$ and $\sigma \in (0, \sqrt{2})$*

$$\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} e^{\sigma|t|}(1+|\varepsilon s|)^\mu \|\Theta\|_{C^{0,\lambda}(B_1(s,t))} < +\infty$$

Then the function defined in $\mathbb{R}$ as

$$Z(\varepsilon s) := \int_{\mathbb{R}} \Theta(s,t) w'(t) dt$$

satisfies for some constant $C = C(w,\mu,\sigma) > 0$ the following estimate:

$$\|Z\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-1} \|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{8.2}$$

Let us apply Lemma 8.1 to the function $\Theta(s,t) := \mathtt{N}(\Phi(h))(s,t)$, to estimate the size of the operator G_2 in (6.25), where we recall that

$$G_2(h)(\varepsilon s) := c_*^{-1}\varepsilon^{-2} \int_{\mathbb{R}} \mathtt{N}(\Phi(h))(s,t) w'(t) dt$$

We can estimate the size of the projection of $\mathtt{N}$ using the previous estimate (8.2), and the bound (7.15) for the size of $\mathtt{N}$:

$$\|G_2(h)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-3} \|\mathtt{N}(\Phi(h))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^2 \tag{8.3}$$

Likewise, for $\phi_i = \Phi(h_i)$, $i = 1,2$ it holds similarly that

$$\|G_2(h_1) - G_2(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-3} \|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Nonetheless, using (7.10) and proposition 6.2, it follows that

$$\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^4 \|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}.$$

The previous estimates allow us to deduce

$$\|G_2(h_1) - G_2(h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

Furthermore, from (8.3) we also have that

$$\|G_2(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon^2 \tag{8.4}$$

for some $C > 0$ possibly depending on $\mathcal{K}$.

Next, we consider

$$\begin{aligned} c_* G_1(h_1) = &-\varepsilon\left[k^3(\boldsymbol{s}) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\boldsymbol{s},0)\right] c^* \hat{c} w'(t) \\ &+ \varepsilon h_1''(\boldsymbol{s}) \int_{\mathbb{R}} \zeta_0(t+h_1) A_0(\boldsymbol{s}, \varepsilon(t+h_1)) w''(t) w'(t) dt \\ &+ \varepsilon^2 Q(\boldsymbol{s}) h_1''(\boldsymbol{s}) \int_{\mathbb{R}} \psi_0'(t) w'(t) dt + \varepsilon^{-2} \int_{\mathbb{R}} \zeta_0 \; R_1(\boldsymbol{s}, t, h_1, h_1') w'(t) dt \end{aligned}$$

It is direct to check, from (2.4) and (5.10) the following estimate on the Lipschitz character for $G_1(h)$

$$\|G_1(h_1) - G_1(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon \|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}.$$

Now, a simple but crucial observation we make is that

$$c_* G_1(0) = \varepsilon^{-2} \int_{\mathbb{R}} \zeta_0 \; R_1(\varepsilon s, t, 0, 0) w'(t) dt$$

has the size

$$\|G_1(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon^{-2} \cdot \varepsilon^{-1} \|R_1\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R})} \le C_2 \varepsilon \tag{8.5}$$

for some universal constant $C_2 > 0$. Therefore, the entire operator $\mathtt{G}(h)$ inherits a Lipschitz character in h, from those of G_1, G_2:

$$\|\mathtt{G}(h_1) - \mathtt{G}(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon \|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}. \tag{8.6}$$

Moreover, estimates (8.4)-(8.5) imply that $\mathtt{G}$ is such

$$\|\mathtt{G}(0)\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \le 2C_2\varepsilon \tag{8.7}$$

Now let $h = T(f)$ be the linear operator defined in Proposition 6.3, and let $\mathtt{G}$ be the nonlinear operator given in (6.25). Consider the Jacobi nonlinear equation (8.1), but this time written as a fixed point problem: Find some h such that

$$h = T(\mathtt{G}(h)), \quad \|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le \mathcal{K}\varepsilon \tag{8.8}$$

Observe that

$$\begin{aligned} \|T(\mathtt{G}(h))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} &\le C\left(\|\mathtt{G}(h) - \mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} + \|\mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}\right) \\ &\le C\,\varepsilon\left(1 + \|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}\right) \end{aligned}$$

where we made use of (8.6)-(8.7). Observe also that

$$\|T(\mathtt{G}(h_1)) - T(\mathtt{G}(h_2))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le C\|\mathtt{G}(h_1) - \mathtt{G}(h_2)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \le C\varepsilon\|h_1 - h_2\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

Hence choosing $\mathcal{K} > 0$, large enough but independent of $\varepsilon > 0$, we find that if ε is small, the operator $T \circ \mathtt{G}$ is a contraction on the ball $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le \mathcal{K}\varepsilon$. As a consequence of the Banach's fixed point theorem, obtain the existence of a unique fixed point of the problem (8.8). This finishes the proof of Proposition 6.4 and consequently, the proof of our theorem.

9. Appendix

This section is mainly oriented in finding expressions for each one of the terms in the Allen-Cahn equation (1.10) written in Fermi coordinates, that are suitable for the geometrical study of this equation. Since they define a local change of variables in a neighborhood of Γ, our effort will focus on finding an equivalent form of (1.10) in these coordinates.

9.1. Laplacian in Fermi Coordinates: Proof of Lemma 2.1.

In order to characterize the Euclidean Laplacian $\Delta_{x,y}$ in dilated and translated Fermi coordinates, we will follow the scheme from the appendix in [4].

For any $\delta > 0$ small but fixed, and a curve $\Gamma \subset \mathbb{R}^2$ parameterized by $\gamma \in C^2(\mathbb{R}, \mathbb{R}^2)$, let us consider first the local Fermi coordinates induced by Γ

$$X : \mathbb{R} \times (-\delta, \delta) \to \mathcal{N}_\delta \ , \quad X(\boldsymbol{s}, \boldsymbol{t}) = \gamma(\boldsymbol{s}) + \boldsymbol{t}\nu(\boldsymbol{s})$$

where $\nu(\boldsymbol{s})$ denotes the normal vector to the curve Γ at the point $\gamma(\boldsymbol{s})$.

It can be seen that X defines a local change of variables on the tubular open neighborhood

$$\mathcal{N}_\delta := \{(\bar{x}, \bar{y}) = \gamma(\boldsymbol{s}) + \boldsymbol{t}\nu(\boldsymbol{s}) \ / \ \boldsymbol{s} \in \mathbb{R}, \ |\boldsymbol{t}| < \delta + \varepsilon \cdot 2c_0|\boldsymbol{s}|\}$$

of Γ, where $c_0 > 0$ is a fixed number, and $|\boldsymbol{t}| = \mathrm{dist}((\bar{x}, \bar{y}), \Gamma)$ for every $(\bar{x}, \bar{y}) = X(\boldsymbol{s}, \boldsymbol{t})$.

Given that $X(\mathcal{N}_\delta) \subset \mathbb{R}^2$ is a 2-dimensional manifold, we can employ a formula from Differential Geometry that allow us to compute the Euclidean Laplacian in terms of Fermi coordinates, for points $(\bar{x}, \bar{y}) = X(\boldsymbol{s}, \boldsymbol{t}) \in \mathcal{N}_\delta$ as follows

$$\Delta_X = \frac{1}{\sqrt{\det(g(\boldsymbol{s}, \boldsymbol{t}))}} \, \partial_i \left(\sqrt{\det(g(\boldsymbol{s}, \boldsymbol{t}))} \cdot g^{ij}(\boldsymbol{s}, \boldsymbol{t}) \ \partial_j \right), \quad i, j = \boldsymbol{s}, \boldsymbol{t} \tag{9.1}$$

where $g_{ij}(\boldsymbol{s}, \boldsymbol{t}) = < \partial_{\boldsymbol{i}} X(\boldsymbol{s}, \boldsymbol{t}), \partial_{\boldsymbol{j}} X(\boldsymbol{s}, \boldsymbol{t}) >$ corresponds to the ijth entry of metric g of Γ, and we regard $g^{ij} = (g^{-1})_{i,j}$ as the respective entry for the inverse of the metric. Performing explicit calculations, and using the relations between the tangent and the normal to the curve Γ, it follows

$$\partial_{\boldsymbol{s}} X(\boldsymbol{s}, \boldsymbol{t}) = \dot{\gamma}(\boldsymbol{s}) + \boldsymbol{t}\dot{\nu}(\boldsymbol{s}), \quad \partial_{\boldsymbol{t}} X(\boldsymbol{s}, \boldsymbol{t}) = \nu(\boldsymbol{s}) \tag{9.2}$$

And so by (9.2), the metric g can be computed as

$$\begin{aligned} g_{ss}(\boldsymbol{s}, \boldsymbol{t}) &= |\dot{\gamma}(\boldsymbol{s})|^2 + 2\boldsymbol{t}\dot{\gamma}(\boldsymbol{s}) \cdot \dot{\nu}(\boldsymbol{s}) + \boldsymbol{t}^2|\dot{\nu}(\boldsymbol{s})|^2 = (1 - \boldsymbol{t}k(\boldsymbol{s}))^2 \\ g_{st}(\boldsymbol{s}, \boldsymbol{t}) &= g_{ts}(\boldsymbol{s}, \boldsymbol{t}) = 0, \quad g_{tt}(\boldsymbol{s}, \boldsymbol{t}) = 1 \end{aligned}$$

Hence, the components of the g^{-1} are

$$g^{ss}(\boldsymbol{s}, \boldsymbol{t}) = \frac{1}{(1 - \boldsymbol{t}k(\boldsymbol{s}))^2}, \ \ g^{st}(\boldsymbol{s}, \boldsymbol{t}) = g^{ts}(\boldsymbol{s}, \boldsymbol{t}) = 0, \ \ g^{tt}(\boldsymbol{s}, \boldsymbol{t}) = 1 \tag{9.3}$$

Replacing formula (9.1) and using values obtained in (9.3), we get

$$\Delta_x = \partial_{\boldsymbol{tt}} + g^{ss}\partial_{\boldsymbol{ss}} + \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g})\partial_{\boldsymbol{t}} + \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g} \cdot g^{ss})\partial_{\boldsymbol{s}} \tag{9.4}$$

where $\sqrt{\det g} = 1 - \boldsymbol{t}k(\boldsymbol{s})$, and with

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g}) = \frac{-k(\boldsymbol{s})}{1 - \boldsymbol{t}k(\boldsymbol{s})}, \quad \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g} \cdot g^{ss}) = \frac{\boldsymbol{t}\dot{k}(\boldsymbol{s})}{(1 - \boldsymbol{t}k(\boldsymbol{s}))^3}$$

Since k vanishes at infinity, we can make an approximation of this operator at main order,

$$\frac{1}{(1-\boldsymbol{t}k)^2} = \underbrace{\left(\sum_{m=0}^{\infty}(\boldsymbol{t}k)^m\right)^2}_{\boldsymbol{t}\text{ small}} = [1+\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]^2 = [1+2\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]$$

$$\frac{1}{(1-\boldsymbol{t}k)^3} = \underbrace{\left(\sum_{m=0}^{\infty}(\boldsymbol{t}k)^m\right)^3}_{\boldsymbol{t}\text{ small}} = [1+\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]^3 = [1+3\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)].$$

In this way, we deduce the expansion

$$g^{ss} = 1+\boldsymbol{t}A_0(\boldsymbol{s},\boldsymbol{t}) \tag{9.5}$$

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g}) = -k(\boldsymbol{s})-\boldsymbol{t}k^2(\boldsymbol{s})+t^2C_0(\boldsymbol{s},\boldsymbol{t}) \tag{9.6}$$

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g}\cdot g^{ss}) = \boldsymbol{t}B_0(\boldsymbol{s},\boldsymbol{t}) \tag{9.7}$$

where $A_0(\boldsymbol{s},\boldsymbol{t})$, $B_0(\boldsymbol{s},\boldsymbol{t})$ and $C_0(\boldsymbol{s},\boldsymbol{t})$ are are the smooth functions described in (2.4), (2.5) and (2.6).

So using relations (9.4) to (9.7), we get the Euclidean Laplacian in Fermi coordinates

$$\Delta_X = \partial_{\boldsymbol{tt}}+\partial_{\boldsymbol{ss}}-[k(\boldsymbol{s})+\boldsymbol{t}k^2(\boldsymbol{s})]\partial_{\boldsymbol{t}}+\boldsymbol{t}A_0(\boldsymbol{s},\boldsymbol{t})\partial_{\boldsymbol{ss}}+\boldsymbol{t}B_0(\boldsymbol{s},\boldsymbol{t})\partial_{\boldsymbol{s}}+\boldsymbol{t}^2C_0(\boldsymbol{s},\boldsymbol{t})\partial_{\boldsymbol{t}} \tag{9.8}$$

Next, we consider the dilated curve $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ by $\gamma_\varepsilon : s \mapsto \varepsilon^{-1}\gamma(\varepsilon s)$, and we define associated local **dilated Fermi coordinates** in $\mathbb{R}^2$ by

$$X_\varepsilon(s,t) := \frac{1}{\varepsilon}X(\varepsilon s,\varepsilon t) = \frac{1}{\varepsilon}\gamma(\varepsilon s)+t\nu(\varepsilon s)$$

on a dilated tubular neighborhood $\varepsilon^{-1}\mathcal{N}_\delta$ of the curve Γ_ε

$$\mathcal{N}_\varepsilon = \left\{(x,y) = X_\varepsilon(s,t)\in\mathbb{R}^2/\ s\in\mathbb{R},\ |t|<\frac{\delta}{\varepsilon}+2c_0|s|)\right\}$$

where $c_0>0$ is a fixed number, and in such way that X_ε defines a local change of variables. Consequently, scaling formula (9.8) yields the expression

$$\begin{aligned}\Delta_{X_\varepsilon} := {} & \partial_{tt}+\partial_{ss}-\varepsilon[k(\varepsilon s)+\varepsilon tk^2(\varepsilon s)]\partial_t+\ \varepsilon tA_0(\varepsilon s,\varepsilon t)\partial_{ss} \\ & +\ \varepsilon^2tB_0(\varepsilon s,\varepsilon t)\partial_s+\ \varepsilon^3t^2C_0(\varepsilon s,\varepsilon t)\partial_t\end{aligned}$$

Setting $z = t-h(\varepsilon s)$, it is possible to compute $\Delta_{x,y}$ in terms of dilated and translated Fermi coordinates (s,t), as

$$\begin{aligned}\Delta_{X_{\varepsilon,h}} = {} & \partial_{tt}\ +\ \partial_{ss}-2\varepsilon h'(\varepsilon s)\partial_{st}-\varepsilon^2h''(\varepsilon s)\partial_t-\varepsilon[k(\varepsilon s)+\varepsilon(t+h(\varepsilon s))k^2(\varepsilon s)]\partial_t \\ & +\varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}+D_{\varepsilon,h}(s,t)\end{aligned}$$

where

$$\begin{aligned}D_{\varepsilon,h}(s,t) := {} & \varepsilon(t+h(\varepsilon s))A_0(\varepsilon s,\varepsilon(t+h))[\partial_{ss}v^*-2\varepsilon h'(\varepsilon s)\partial_{ts}v^*-\varepsilon^2h''(\varepsilon s)\partial_tv^*+\varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v^*] \\ & +\varepsilon^2(t+h(\varepsilon s))B_0(\varepsilon s,\varepsilon(t+h))[\partial_sv^*(s,t)-\varepsilon h'(\varepsilon s)\partial_tv^*(s,t)] \\ & +\varepsilon^3(t+h(\varepsilon s))^2C_0(\varepsilon s,\varepsilon(t+h))\partial_tv^*(s,t)\end{aligned}$$

thus finishing the proof of Lemma 2.1

9.2. **Proof of Lemma 2.2.** Analogously to what performed in the last section, our main goal is finding a characterization for the product $\varepsilon \nabla_{\bar{x}} a/a \cdot \nabla_x u$ in Fermi coordinates. To achieve this we will follow a scheme in 3 steps, for which the analysis is simplified.

Hereinafter, once again we adopt the convention $a = a(\bar{x}, \bar{y})$ and $u = u(\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y})$ denotes the non-dilated Euclidean coordinates of the space.

We had that $X(\boldsymbol{s}, \boldsymbol{t}) = \gamma(\boldsymbol{s}) + \boldsymbol{t}\nu(\boldsymbol{s})$ provided a local change of variables, implying that

$$(\bar{x}, \bar{y}) = X(X^{-1}(\bar{x}, \bar{y})), \qquad \text{for } (\bar{x}, \bar{y}) \in \mathcal{N}$$
$$\nabla_{\bar{x},\bar{y}} = \nabla_{\boldsymbol{s},\boldsymbol{t}} \cdot \left[D_{\boldsymbol{s},\boldsymbol{t}} X(X^{-1}(\bar{x}, \bar{y})) \right]^{-1}$$

Therefore, we obtain that

$$\frac{\nabla_{\bar{x},\bar{y}} a}{a} \nabla_{\bar{x},\bar{y}} = \frac{1}{(1 - \boldsymbol{t}k(\boldsymbol{s}))^2} \left(\frac{\partial_{\boldsymbol{s}} a}{a} \cdot \partial_{\boldsymbol{s}} v_u \right) + \frac{\partial_{\boldsymbol{t}} a}{a} \cdot \partial_{\boldsymbol{t}} v_u$$
$$= \frac{\partial_{\boldsymbol{s}} a}{a} \cdot \partial_{\boldsymbol{s}} + \frac{\partial_{\boldsymbol{t}} a}{a} \cdot \partial_{\boldsymbol{t}} + \boldsymbol{t} A_0(\boldsymbol{s}, \boldsymbol{t}) \frac{\partial_{\boldsymbol{s}} a}{a} \cdot \partial_{\boldsymbol{s}}$$

where we have made use of the expansion (9.5) of entry $g^{\boldsymbol{ss}}$ of the metric.

Further, using the above expression in the Taylor expansion for $\nabla a/a$ around the curve Γ, we find

$$\frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s}, \boldsymbol{t}) = \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s}, 0) + \boldsymbol{t}\, \partial_{\boldsymbol{t}} \left(\frac{\partial_{\boldsymbol{s}} a}{a} \right)(\boldsymbol{s}, 0) + O\left(\boldsymbol{t}^2 \partial_{\boldsymbol{tt}} \left(\frac{\partial_t a}{a} \right) \right)$$
$$\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, \boldsymbol{t}) = \frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) + \boldsymbol{t} \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\boldsymbol{s}, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) \right|^2 \right) + \frac{\boldsymbol{t}^2}{2} \partial_{\boldsymbol{tt}} \left(\frac{\partial_t a}{a}(\boldsymbol{s}, 0) \right) + O\left(\boldsymbol{t}^3 \partial_{\boldsymbol{ttt}} \left(\frac{\partial_t a}{a} \right) \right)$$

Replacing these expansions in the above equation we obtain

$$\frac{\nabla_X a}{a} \nabla_X = \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s}, 0) \partial_{\boldsymbol{s}} + \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) + \boldsymbol{t} \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\boldsymbol{s}, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) \right|^2 \right) \right] \partial_{\boldsymbol{t}}$$
$$+ \boldsymbol{t} D_0(\boldsymbol{s}, \boldsymbol{t}) \partial_{\boldsymbol{s}} + \boldsymbol{t}^2 F_0(\boldsymbol{s}, \boldsymbol{t}) \partial_{\boldsymbol{t}} \tag{9.9}$$

for which

$$D_0(\boldsymbol{s}, \boldsymbol{t}) = \partial_{\boldsymbol{t}} \left(\frac{\partial_{\boldsymbol{s}} a}{a} \right)(\boldsymbol{s}, 0) + O\left(\boldsymbol{t} \partial_{\boldsymbol{tt}} \left(\frac{\partial_t a}{a} \right) \right) + A_0(\boldsymbol{s}, \boldsymbol{t}) \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s}, \boldsymbol{t})$$
$$F_0(\boldsymbol{s}, \boldsymbol{t}) = \frac{1}{2} \partial_{\boldsymbol{tt}} \left(\frac{\partial_t a}{a}(\boldsymbol{s}, 0) \right) + O\left(\boldsymbol{t} \partial_{\boldsymbol{ttt}} \left(\frac{\partial_t a}{a} \right) \right)$$

and $A_0(\boldsymbol{s}, \boldsymbol{t})$ given by (2.4).

Let us recall the convention $(\bar{x}, \bar{y}) := (\varepsilon x, \varepsilon y)$, $(\boldsymbol{s}, \boldsymbol{t}) := (\varepsilon s, \varepsilon t)$ and observe that

$$\nabla_{x,y} = \varepsilon \nabla_{\bar{x},\bar{y}}$$

Directly from a this scaling and formula (9.9) we find that

$$\varepsilon \frac{\nabla_X a}{a}(\bar{x}, \bar{y}) \nabla_{X_\varepsilon} = \varepsilon^2 \Bigg\{ \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s}, 0) \partial_{\boldsymbol{s}} + \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) + \boldsymbol{t} \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\boldsymbol{s}, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s}, 0) \right|^2 \right) \right] \partial_{\boldsymbol{t}}$$
$$+ \boldsymbol{t} D_0(\boldsymbol{s}, \boldsymbol{t}) \partial_{\boldsymbol{s}} + \boldsymbol{t}^2 F_0(\boldsymbol{s}, \boldsymbol{t}) \partial_{\boldsymbol{t}} \Bigg\}$$

which can be written in terms of dilated Fermi coordinates (s,t)

$$\varepsilon\frac{\nabla_X a}{a}(\bar{x},\bar{y})\nabla_{X_\varepsilon} = \varepsilon\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)\partial_s + \varepsilon\left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0) + \varepsilon t\left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0) - \left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right)\right]\partial_t + E_\varepsilon(s,t)$$

where the derivatives of function a are with respect to shrink Fermi variables $(\boldsymbol{s},\boldsymbol{t})$, and with

$$E_\varepsilon(s,t) := \varepsilon^2 t D_0(\varepsilon s,\varepsilon t)\cdot\partial_s + \varepsilon^3 t^2 F_0(\varepsilon s,\varepsilon t)\cdot\partial_t$$

for which

$$D_0(\varepsilon s,\varepsilon t) = \partial_{\boldsymbol{t}}\left(\frac{\partial_{\boldsymbol{s}} a}{a}\right)(\varepsilon s,0) + \varepsilon O\left(t\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right]\right) + A_0(\varepsilon s,\varepsilon t)\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,\varepsilon t)$$
$$F_0(\varepsilon s,\varepsilon t) = \frac{1}{2}\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}(\varepsilon s,0)\right] + \varepsilon O\left(t\partial_{\boldsymbol{ttt}}\left(\frac{\partial_t a}{a}\right)\right)$$

and the function $A_0(\varepsilon s,\varepsilon t)$ given by (2.4)

Next, we observe that for $z = t - h(\varepsilon s)$, this product in dilated and translated Fermi coordinates amounts to

$$\varepsilon\frac{\nabla_X a}{a}(\varepsilon x,\varepsilon y)\nabla_{X_{\varepsilon,h}} = \varepsilon\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] + \varepsilon\left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0) + \varepsilon(t+h(\varepsilon s))\left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0) - \left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right)\right]\partial_t + E_{\varepsilon,h}(\varepsilon s,t)$$

where $E_{\varepsilon,h}$ is a small operator for $\varepsilon > 0$ small enough, and of the form

$$E_{\varepsilon,h}(\varepsilon s,t) := \varepsilon^2(t+h(\varepsilon s))D_0(\varepsilon s,\varepsilon(t+h(\varepsilon s)))[\partial_s v_u^*(s,t) - \varepsilon h'(\varepsilon s)\partial_t v_u^*(s,t)] + \varepsilon^3(t+h(\varepsilon s))^2 F_0(\varepsilon s,\varepsilon(t+h(\varepsilon s)))\partial_t v_u^*(s,t)$$

such that the following functions are smooth, and these relations can be derived.

$$D_0(\varepsilon s,\varepsilon(t+h)) = \partial_{\boldsymbol{t}}\left[\frac{\partial_{\boldsymbol{s}} a}{a}\right](\varepsilon s,0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right]\right) + A_0(\varepsilon s,\varepsilon(t+h))\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,\varepsilon(t+h))$$
$$F_0(\varepsilon s,\varepsilon(t+h)) = \frac{1}{2}\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right](\varepsilon s,0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{ttt}}\left[\frac{\partial_t a}{a}\right]\right)$$

this completes the proof of Lemma 2.2. □

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