ON THE GLOBAL BIFURCATION DIAGRAM OF THE GEL'FAND PROBLEM

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Abstract. For domains of first kind [7, 13] we describe the qualitative behavior of the global bifurcation diagram of the unbounded branch of solutions of the Gel'fand problem crossing the origin. At least to our knowledge this is the first result about the exact monotonicity of the branch of non-minimal solutions which is not just concerned with radial solutions [28] and/or with symmetric domains [23]. Toward our goal we parametrize the branch not by the $L^\infty(\Omega)$-norm of the solutions but by the energy of the associated mean field problem. The proof relies on a carefully modified spectral analysis of mean field type equations.

Keywords: Global bifurcation, Gelfand problem, Mean field equation.

1. Introduction

We are concerned with the global bifurcation diagram of solutions of,

$$\begin{cases} -\Delta v = \mu e^{v} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \qquad (1)_\mu$$

where $\Omega \subset \mathbb{R}^2$ is any smooth, open and bounded domain and $\mu \in \mathbb{R}$. Problem $(1)_\mu$, also known as the Gel'fand problem [19], arises in many applications, such as for example the thermal ignition of gases [8], the dynamics of self-interacting particles [2] and of chemiotaxis aggregation [37], the statistical mechanics of point vortices [12] and of self-gravitating objects with cylindrical symmetries [24], [34]. A basic question seems unanswered so far concerning the qualitative behavior of the unbounded continuum [35] of solutions of $(1)_\mu$,

$$\Gamma_\infty(\Omega) = \Big\{ (\mu, v_\mu) \in \mathbb{R} \times C^{2,\alpha}_0(\,\overline{\Omega}\,) : v_\mu \text{ solves } (1)_\mu \text{ for some } \mu \in \mathbb{R} \Big\}, \quad (\mu, v_\mu) = (0,0) \in \Gamma_\infty(\Omega),$$

emanating from the origin $(\mu, v_\mu) = (0,0)$. Under which conditions on Ω, $\Gamma_\infty(\Omega)$ takes the same form (see Fig. 1) as that corresponding to a disk $\Omega = B_R$? Here $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ and in this case solutions are radial [20] and can be evaluated explicitly, see for example [36].
For a general domain, classical results [25] show that v_μ is a monotonic increasing function of μ as far as the first eigenvalue of the associated linearized problem is strictly positive. This is the so called branch of minimal solutions which is well understood and naturally described in the $(\mu, \|v_\mu\|_{L^\infty(\Omega)})$-plane for $\mu < \mu_\star(\Omega) \in (0, +\infty)$. In fact, this nice behavior breaks down at some positive value $\mu_\star(\Omega)$, which is the least upper bound of those $\mu_\star$ such that $(1)_\mu$ has solutions for any $\mu < \mu_\star$, see [26]. In particular, the first eigenvalue of the linearized problem at $\mu_\star(\Omega)$ is zero and $\mu_\star(\Omega)$ is known to be a bending point, see [37].
The situation for non-minimal solutions on $\Gamma_\infty(\Omega)$, i.e. after the first bending point, is more involved. Besides classical facts (see [1], [36] and [31] for a complete discussion and references),

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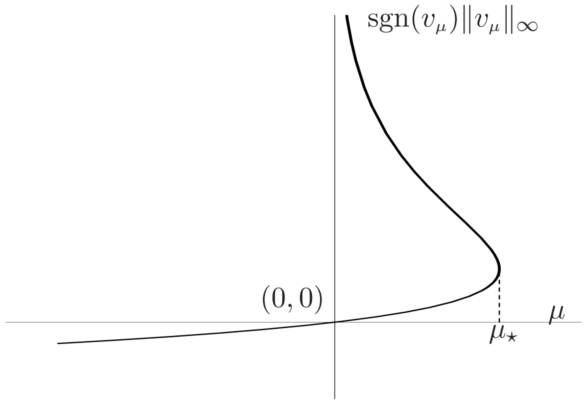

FIGURE 1. The graph of $\Gamma_\infty(B_1)$

at least to our knowledge there are only two rather general result concerning this problem. The first one is in [36], where it is shown that, for a certain class of simply connected domains (see Remark 1.3 below), $\Gamma_\infty(\Omega)$ is a smooth curve with only one bending point which makes 1-point blow up [33] as $\mu \to 0^+$. The second one is an unpublished but straightforward corollary of some results in [7], which implies that the result in [36] holds even for a large class of domains with holes, see Remark 1.3 below. Therefore it is natural in this situation to guess that $\Gamma_\infty(\Omega)$ takes the same form shown in Fig. 1. However, a subtle point arises since after the bending, even in this situation where we know that the bifurcation curve cannot bend back to the right, the first eigenvalue of the linearized equation for $(1)_\mu$ is negative (while the second eigenvalue is positive [36, 7]) and then the monotonicity of $\|v_\mu\|_\infty = \|v_\mu\|_{L^\infty(\Omega)}$, depending in a tricky way on certain changing sign quantities, cannot be taken for granted. Of course, one expects that, as is the case for radial solutions, $\|v_\mu\|_\infty$ is still a monotone function of μ, and well known pointwise estimates ([14, 30]) for blow up solutions suggest that this is the case for $\mu \searrow 0^+$ small enough. At least to our knowledge there are no proofs of this fact. Actually, under some symmetry assumptions on Ω, by the result in [23], for any $m \in (0,+\infty)$ there exists one and only one solution of $(1)_\mu$ such that $\|v_\mu\|_\infty = m$ and $\Gamma_\infty(\Omega)$ is a smooth curve which contains all solutions of $(1)_\mu$. Therefore, for these symmetric domains, the results in [36] and [23] together show that indeed $\|v_\mu\|_\infty$ is monotone along $\Gamma_\infty(\Omega)$, which answer to our question in this case. Finally, it seems that there is no gain in replacing $\|v_\mu\|_\infty$ with other seemingly natural quantities, as for example $\int\limits_\Omega |\nabla v_\mu|^2$ or either $\int\limits_\Omega e^{v_\mu}$, as one is always left with the problem of possibly sign changing terms.

We attack this problem here by a new method based on some ideas recently introduced in [4], that is, to parametrize the curve $\Gamma_\infty(\Omega)$ not by $\|v_\mu\|_\infty$ but with the energy $\mathbb{E}(\mu)$ of the associated mean field equation, naturally arising in the Onsager description of two-dimensional turbulence [12]. Our proof works for domains of "first kind" (see Definition 1.2 below), initially introduced in statistical mechanics [12] and then sharpened and fully characterized in [13] and in [7]. For any pair $(\mu, v_\mu) \in (\mathbb{R}, C^{2,\alpha}_0(\overline{\Omega}))$ solving $(1)_\mu$ we define,

$$\mathbb{E}(\mu) = \begin{cases} \dfrac{1}{2\mu \int\limits_\Omega e^{v_\mu}} \displaystyle\int\limits_\Omega \dfrac{e^{v_\mu}}{\int\limits_\Omega e^{v_\mu}} v_\mu, & \mu \neq 0, \\ \\ \frac{1}{2|\Omega|^2} \int\limits_\Omega G(x,y)dxdy, & \mu = 0, \end{cases}$$

where $G(x,y)$ is the Green function for $-\Delta$ with Dirichlet boundary conditions. For later use let us set,

$$E_0 = E_0(\Omega) := \mathbb{E}(0) = \frac{1}{2|\Omega|^2} \int\limits_{\Omega} G(x,y)dxdy.$$

We say that $f : I \to X$, where $I \subseteq \mathbb{R}$ is an open set and X is a Banach space, is real analytic [11] if for each $t_0 \in I$ it admits a power series expansion in t, which is totally convergent in the X-norm in a suitable neighborhood of t_0. Our main result is the following:

Theorem 1.1. *Let Ω be a domain of first kind (see Definition 1.2). For any $E \in (0, +\infty)$, the equation*

$$\mathbb{E}(\mu) = E \qquad (\mu, v_\mu) \in \Gamma_\infty(\Omega) \qquad \textbf{(E)}$$

admits a unique solution $\mu = \mu_\infty(E)$.
In particular, $\mu_\infty : (0, +\infty) \to (-\infty, +\infty)$ and $v_\mu|_{\mu=\mu_\infty(E)} : (0, +\infty) \to C^{2,\alpha}_0(\overline{\Omega})$ are real analytic functions of E and $(\mu, v_\mu)|_{\mu=\mu_\infty(E)}$ is a parametrization of $\Gamma_\infty(\Omega)$. Finally $\mu_\infty(E)$ has the following properties:
(i) $\mu_\infty(E) \to -\infty$ as $E \to 0^+$, $\mu_\infty(E_0) = 0$, $\mu_\infty(E) \to 0^+$ as $E \to +\infty$;
(ii) $\frac{d\mu_\infty(E)}{dE} > 0$ for $E < E_*$, $\frac{d\mu_\infty(E_*)}{dE} = 0$, $\frac{d\mu_\infty(E)}{dE} < 0$ for $E > E_*$, where $E_* = E_*(\Omega) > E_0(\Omega)$ is uniquely defined by $E_*(\Omega) = \mathbb{E}(\mu_\star(\Omega))$, that is $\mu_\infty(E_*) = \mu_\star(\Omega)$.

Therefore, on domains of first kind, we have found a global parametrization of $\Gamma_\infty(\Omega)$,

$$\Gamma_\infty(\Omega) = \left\{ (\mu, v_\mu) \in [0, \mu_\star(\Omega)] \times C^{2,\alpha}_0(\overline{\Omega}) \,:\, \mu = \mu_\infty(E),\, E \in (0, +\infty) \right\},$$

which takes the form depicted in Fig. 2, as claimed. At least to our knowledge, this is the first global result (i.e. including non-minimal solutions) about the monotonicity of the bifurcation diagram for an elliptic equation with superlinear growth in dimension $n = 2$, which is not just concerned with radially symmetric solutions [28], [29], and/or with domains sharing some kind of symmetries [23]. The situation in higher dimension is far more subtle, see for example [22] and more recently [17, 18].

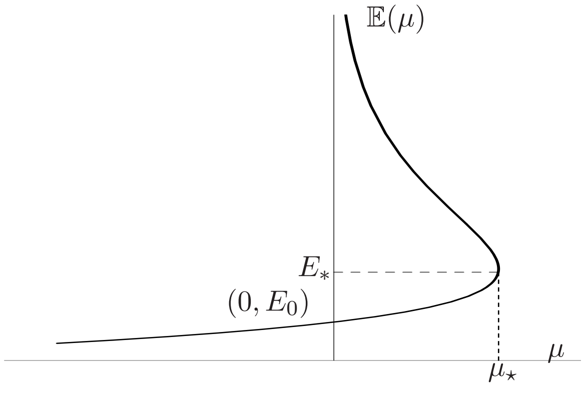

FIGURE 2. The graph of $\Gamma_\infty(\Omega)$ on domains of first kind

Although its definition in the context of the Gel'fand problem looks rather unnatural, it turns out that indeed $\mathbb{E}(\mu)$ is just the energy in the Onsager mean field model, when expressed as a function of μ. For $\lambda \in (-\infty, 8\pi)$ we consider the mean field equation [12],

$$\begin{cases} -\Delta \psi_\lambda = \dfrac{e^{\lambda \psi_\lambda}}{\int\limits_{\Omega} e^{\lambda \psi_\lambda}} & \Omega \\ \psi_\lambda = 0 & \partial\Omega \end{cases} \qquad \textbf{(P}_\lambda\textbf{)}$$

To simplify the notations, for fixed $\lambda \in (-\infty, 8\pi)$, here and in the rest of this paper we set,

$$\rho_\lambda = \frac{e^{\lambda\psi_\lambda}}{\int\limits_\Omega e^{\lambda\psi_\lambda}}, \qquad < f >_\lambda = \int\limits_\Omega \rho_\lambda f, \qquad f_0 = f - < f >_\lambda .$$

The energy associated with the density ρ_λ is by definition (see [12]),

$$\mathcal{E}(\rho_\lambda) = \frac{1}{2}\int\limits_\Omega\int\limits_\Omega \rho_\lambda(y)G(x,y)\rho_\lambda(x)dydx.$$

Clearly, if ψ_λ solves $(\mathbf{P}_\lambda)$, then $v_\mu = \lambda\psi_\lambda$ is a solution of $(1)_\mu$, for some $\mu \in \mathbb{R}$ which satisfies,

$$\mu = \mu_\lambda = \frac{\lambda}{\int\limits_\Omega e^{\lambda\psi_\lambda}},$$

and then from $(\mathbf{P}_\lambda)$ we see that,

$$E_\lambda := \mathbb{E}(\mu_\lambda) = \mathcal{E}(\rho_\lambda) = \frac{1}{2} < \psi_\lambda >_\lambda = \frac{1}{2}\int\limits_\Omega |\nabla\psi_\lambda|^2.$$

Therefore, in particular the energy $\mathbb{E}(\mu_\lambda)$ is just the Dirichlet energy of ψ_λ when expressed in terms of μ.

The uniqueness of solutions of $(\mathbf{P}_\lambda)$ is easy to prove for $\lambda < 0$, see Proposition 2.1 below. On the other side it is well known that $(\mathbf{P}_\lambda)$ admits a unique solution for any $\lambda \in (0, 8\pi)$ (see [36] for simply connected domains and [7] for general domains). The existence/non existence problem for $\lambda = 8\pi$ is a subtle issue, since $(\mathbf{P}_\lambda)$ is critical with respect to the Moser-Trudinger inequality [32]. For a complete discussion of this problem see [13] for simply connected domains and [7] for general domains. This is why we need the following definition,

Definition 1.2. *A domain Ω is of first kind if $(\mathbf{P}_\lambda)$ has no solution for $\lambda = 8\pi$ and is of second kind otherwise.*

Remark 1.3. *It has been proved in [13] and [7] that Ω is of first kind if and only if the unique solutions of $(\mathbf{P}_\lambda)$ for $\lambda < 8\pi$ blow up [9, 33], as $\lambda \to 8\pi^-$. As a consequence, it turns out that the domains considered in Theorem 2 in [36] are exactly the simply connected domains of first kind. However the estimates about the first and second eigenvalue in [36] has been extended in [7] to the case of any connected domain. As an immediate consequence, the result in [36] hold for any domain of first kind. It is well known that any disk $\Omega = B_R$ is of first kind. Actually any regular polygon is of first kind, see [13]. It has been proved in [5] that there exists a universal constant $I > 4\pi$ such that any convex domain whose isoperimetric ratio is larger than I is of second kind. If $\Omega_{a,b}$ is a rectangle of sides $a \leq b$, then there exists $\xi \in (0,1)$ such that $\Omega_{a,b}$ is of first kind if and only if $\frac{a}{b} \leq \xi$, see [13]. If $\Omega_r = B_1 \setminus \overline{B_r(x_0)}$ with $x_0 \in B_1$, $x_0 \neq 0$, then there exists $r_0 < \min\{|x_0|, 1-|x_0|\}$ such that Ω_r is of first kind for any $r < r_0$, see [7]. We refer to [13] and [7] for other equivalent characterizations of domains of first kind and a complete discussion concerning this point. Among other things it is proved there that the set of domains of first kind is closed in the C^1-topology. Therefore, for example domains of first kind need not be symmetric.*

Remark 1.4. *It is well known [15],[27] that if Ω is not simply connected, then there exist countably many distinct families of blow up solutions of $(1)_\mu$ as $\mu \to 0^+$. These families of blow up solutions are unique [6] and nondegenerate [21] under suitable nondegeneracy assumptions. Therefore, it is not true in general that $\Gamma_\infty(\Omega)$ contains all solution of $(1)_\mu$. However, as remarked above, this is the case for domains with certain symmetries, see [23].*

Let us sketch the argument of the proof. We describe $\Gamma_\infty(\Omega)$ by using the unique solutions of $(\mathbf{P}_\lambda)$ for $\lambda \in (-\infty, 8\pi)$. Any solution ψ_λ of $(\mathbf{P}_\lambda)$ yield a solution of $(1)_\mu$ with $\mu_\lambda = \lambda(\int\limits_\Omega e^{\lambda\psi_\lambda})^{-1}$.

This might seem not a good point of view since, $(\mathbf{P}_\lambda)$ being a constrained problem, the associated linearized equation is more difficult to analyze. In general the first eigenvalue $\widehat{\sigma}_{1,\lambda}$ is not simple and the first eigenfunctions may change sign, see [3] or the Appendix below for an example of this sort. Therefore, even if we know that $\widehat{\sigma}_{1,\lambda} > 0$ (see [13] and [7]), it is not clear how to use this information to establish the monotonicity of μ_λ. However, as recently observed in [4], it is possible to modify the standard spectral theory relative to the linearization of $(\mathbf{P}_\lambda)$, see (2.2) below, and build a complete set of eigenfunctions which span the space of functions of vanishing mean. All the main steps of the proof rely on this modified spectral setting. The modified first eigenvalue $\sigma_{1,\lambda}$ is strictly positive for $\lambda \in (-\infty, 8\pi)$ and the set of solutions of $(\mathbf{P}_\lambda)$ can be shown to be locally an analytic curve (λ, ψ_λ) with no bifurcation points. A crucial fact which follows from $\sigma_{1,\lambda} > 0$ is that the energy E_λ is a (real analytic) strictly increasing function of λ. Therefore, to understand the monotonicity of μ as a function of E, it is enough to evaluate the sign of $\frac{d\mu_\lambda}{d\lambda}$. The difficult part is to show that there exists $\lambda_* \in (0, 8\pi)$ such that $\frac{d\mu_\lambda}{d\lambda} > 0 \iff \lambda < \lambda_*$. Indeed, a major problem arises in the proof of $\frac{d\mu_\lambda}{d\lambda} < 0$ along the non-minimal branch of solutions, that is for $\lambda > \lambda_*$. We solve this problem by two non-trivial facts about the quantity which controls the sign of $\frac{d\mu_\lambda}{d\lambda}$, which is $g(\lambda)$ in Lemma 4.1 below. First of all, still by exploiting $\sigma_{1,\lambda} > 0$, we obtain a version of the maximum principle based on the sign of $g(\lambda)$. This is not at all obvious since $\sigma_{1,\lambda} > 0$ does not imply that the maximum principle holds for the linearized problem relative to $(\mathbf{P}_\lambda)$. The second fact is a remarkable formula for $g(\lambda)$: it turns out that it satisfies a first order non-homogeneous O.D.E. (see (5.6) below). In particular we conclude that $\frac{d\mu_\lambda}{d\lambda}$ changes sign only once in $(0, 8\pi)$.

The description of $\Gamma_\infty(\Omega)$ on domains of second kind is more difficult. Indeed, solutions on a certain part of $\Gamma_\infty(\Omega)$ correspond to solutions of $(\mathbf{P}_\lambda)$ with $\lambda > 8\pi$, a region where solutions are not unique and $\sigma_{1,\lambda}$ is not anymore positive. Therefore it is not easy to understand the monotonicity of E_λ, see [4] for further details concerning this point.

This paper is organized as follows. In section 2 we first introduce the modified spectral analysis and collect some important preliminary results concerning the linearized mean field equation. In section 3 as a first step toward the proof of the main result we deduce the monotonicity of the energy (3.1). Then, in section 4 we prove the main Theorem 1.1 postponing the proof of the key Lemma 4.1 to section 5. Finally, further discussion on the modified spectral analysis with an explicit example is given in the Appendix.

2. Spectral decomposition of linearized mean field type equations

For any ψ_λ solving $(\mathbf{P}_\lambda)$, we introduce the linearized operator,

$$L_\lambda \phi := -\Delta \phi - \lambda \rho_\lambda \phi_0, \quad \phi \in H^1_0(\Omega) \tag{2.1}$$

where we recall that

$$\phi_0 = \phi - < \phi >_\lambda .$$

We say that $\sigma = \sigma(\lambda, \psi_\lambda) \in \mathbb{R}$ is an eigenvalue of the linearized operator (2.1) if the equation,

$$-\Delta \phi - \lambda \rho_\lambda \phi_0 = \sigma \rho_\lambda \phi_0, \tag{2.2}$$

admits a non-trivial weak solution $\phi \in H^1_0(\Omega)$. This definition of the eigenvalues requires some comments. Let ψ_λ be a fixed solution of $(\mathbf{P}_\lambda)$ and let us define,

$$Y_0 := \left\{ \varphi \in \{L^2(\Omega), < \cdot, \cdot >_\lambda\} \; : \; \int\limits_\Omega \rho_\lambda \varphi = 0 \right\},$$

where $< \cdot, \cdot >_\lambda$ denotes the scalar product $< f, g >_\lambda := < fg >_\lambda$. Let us also define

$$T(\phi) := G[\rho_\lambda \phi], \quad \phi \in L^2(\Omega), \text{ where } G[f] = \int_\Omega G(x,y) f(y) dy.$$

By the results in [9, 30] and standard elliptic regularity theory we see that ρ_λ is a smooth function for $\lambda < 8\pi$, so these definitions are well posed. Clearly Y_0 is an Hilbert space, and, since $T(Y_0) \subset W^{2,2}(\Omega)$, then it is not difficult to see that the linear operator,

$$T_0 : Y_0 \to Y_0, \; T_0(\varphi) = G[\rho_\lambda \varphi] - < G[\rho_\lambda \varphi] >_\lambda, \tag{2.3}$$

is self-adjoint and compact. As a consequence, standard results concerning the spectral decomposition of self-adjoint, compact operators on Hilbert spaces show that Y_0 is the Hilbertian direct sum of the eigenfunctions of T_0, which can be represented as

$$\varphi_k = \phi_{k,0} := \phi_k - < \phi_k >_\lambda, \; k \in \mathbb{N} = \{1, 2, \cdots\},$$

$$Y_0 = \overline{\text{Span}\,\{\phi_{k,0}, \; k \in \mathbb{N}\}},$$

for some $\phi_k \in H^1_0(\Omega)$, $k \in \mathbb{N} = \{1, 2, \cdots\}$. In fact φ_k is an eigenfunction whose eigenvalue is $\mu_k = \frac{1}{\lambda + \sigma_k} \in \mathbb{R} \setminus \{0\}$, that is,

$$\varphi_k = (\lambda + \sigma_k) \left(G[\rho_\lambda \varphi_k] - < G[\rho_\lambda \varphi_k] >_\lambda \right),$$

if and only if the function ϕ_k,

$$\phi_k := (\lambda + \sigma_k) G[\rho_\lambda \varphi_k],$$

is in $H^1_0(\Omega)$ and weakly solves,

$$-\Delta \phi_k = (\lambda + \sigma_k) \rho_\lambda \phi_{k,0} \quad \text{in} \quad \Omega. \tag{2.4}$$

At this point, standard arguments in the calculus of variations show that,

$$\sigma_1 = \sigma_1(\lambda, \psi_\lambda) = \inf_{\phi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 - \lambda \int_\Omega \rho_\lambda \phi_0^2}{\int_\Omega \rho_\lambda \phi_0^2}. \tag{2.5}$$

The ratio in the right hand side of (2.5) is well defined in $H^1_0(\Omega)$ by the Jensen inequality, which implies that

$$\int_\Omega \rho_\lambda \phi_0^2 = < \phi^2 >_\lambda - < \phi >_\lambda^2 \geq 0,$$

where the equality holds if and only if $\phi \equiv 0$ a.e. in Ω. Higher eigenvalues are defined inductively via the variational problems,

$$\sigma_k = \sigma_k(\lambda, \psi_\lambda) = \inf_{\phi \in H^1_0(\Omega) \setminus \{0\}, < \phi_0, \phi_{k,0} >_\lambda = 0, \; m \in \{1, \ldots, k-1\}} \frac{\int_\Omega |\nabla \phi|^2 - \lambda \int_\Omega \rho_\lambda \phi_0^2}{\int_\Omega \rho_\lambda \phi_0^2}. \tag{2.6}$$

Obviously a base $\{\phi_{i,0}\}_{i \in \mathbb{N}}$ of Y_0 can be constructed to satisfy,

$$< \phi_{i,0}, \phi_{j,0} >_\lambda = 0, \; i \neq j. \tag{2.7}$$

The eigenvalues form a countable nondecreasing sequence $\sigma_1(\lambda, \psi_\lambda) \leq \sigma_2(\lambda, \psi_\lambda) \leq .... \leq \sigma_k(\lambda, \psi_\lambda) \leq ...$, where in particular,

$$\lambda + \sigma_k \geq \lambda + \sigma_1 > 0, \; \forall \, k \in \mathbb{N}, \tag{2.8}$$

the last inequality being an immediate consequence of (2.5).
Obviously, by the Fredholm alternative,

$$\text{if } 0 \notin \{\sigma_j\}_{j \in \mathbb{N}}, \text{ then } I - \lambda T_0 \text{ is an isomorphism of } Y_0 \text{ onto itself.} \tag{2.9}$$

Finally, any $f \in L^2(\Omega)$ admits the Fourier series expansion,

$$f = \alpha_0 + \sum_{j=1}^{+\infty} \alpha_j \phi_{j,0}, \qquad \int\limits_{\Omega} \rho_\lambda \phi_{j,0}^2 = 1, \tag{2.10}$$

where

$$\alpha_j = \alpha_j(f) = < \phi_{j,0}, f >_\lambda .$$

If we let $\widehat{\sigma}_1$ be the standard first eigenvalue defined by,

$$\widehat{\sigma}_1 = \inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int\limits_{\Omega} |\nabla \phi|^2 - \lambda \int\limits_{\Omega} \rho_\lambda (\phi^2 - < \phi >_\lambda^2)}{\int\limits_{\Omega} \rho_\lambda \phi^2}, \tag{2.11}$$

then we see that, as far as $\widehat{\sigma}_1(\lambda, \psi_\lambda) \geq 0$, we have,

$$\sigma_1 \geq \widehat{\sigma}_1, \tag{2.12}$$

and then, in view of the results in [36], as later improved in [13] and [7], we have the following:

Theorem A.([7], [13], [36]) *Let Ω be a smooth and bounded domain. For any $\lambda \in [0, 8\pi)$, the first eigenvalue of* (2.1) *is strictly positive, that is, $\sigma_1(\lambda, \psi_\lambda) > 0$, for each $\lambda \in [0, 8\pi)$.*

The curve,

$$(-\infty, 8\pi) \supseteq I \ni \lambda \mapsto (\lambda, \psi_\lambda) \in (-\infty, 8\pi) \times C_0^{2,\alpha}(\overline{\Omega}),$$

from an open interval $I \subseteq (-\infty, 8\pi)$ is said to be analytic if for each $\lambda_0 \in I$, ψ_λ admits a power series expansion as a function of λ, which is totally convergent in the $C_0^{2,\alpha}(\overline{\Omega})$-norm in a suitable neighborhood of λ_0. We have the following,

Proposition 2.1. *Let Ω be a smooth and bounded domain.*
(i) For any $\lambda \in (-\infty, 8\pi)$ it holds $\sigma_1 = \sigma_1(\lambda, \psi_\lambda) > 0$.
(ii) If ψ_λ solves $(\mathbf{P}_\lambda)$ *for some $\lambda = \lambda_0 \in (-\infty, 8\pi)$, then there exists an open neighborhood of $(\lambda_0, \psi_{\lambda_0})$ in the product topology, $I \times B = \mathcal{U} \subset \mathbb{R} \times C_0^{2,\alpha}(\overline{\Omega})$, such that the set of solutions of* $(\mathbf{P}_\lambda)$ *in $\mathcal{U}$ is an analytic curve $I \ni \lambda \mapsto (\lambda, \psi_\lambda) \in \mathcal{U}$, for suitable neighborhoods I of λ_0 in $\mathbb{R}$ and B of ψ_{λ_0} in $C_0^{2,\alpha}(\overline{\Omega})$.*
(iii) For any $\lambda \in (-\infty, 8\pi)$ there exists a unique solution of $(\mathbf{P}_\lambda)$.

Proof. If ψ_λ solves $(\mathbf{P}_\lambda)$ for some $\lambda \in (-\infty, 8\pi)$, then Theorem A, (2.12) and (2.8) immediately imply that $\sigma_1 = \sigma_1(\lambda, \psi_\lambda) > 0$, which proves (i). As a consequence of (i), (2.9) and the analytic implicit function theorem [10, 11], it can be shown by standard arguments that (ii) holds, see Lemma 2.4 in [4]. Therefore also (ii) is proved.
Putting $u_\lambda = \lambda \psi_\lambda$, then solutions of $(\mathbf{P}_\lambda)$ are critical points of

$$J_\lambda(u) = \frac{1}{2} \int\limits_{\Omega} |\nabla u|^2 - \lambda \log \left(\int\limits_{\Omega} e^u \right), \; u \in H_0^1(\Omega).$$

The existence of at least one minimizer for J_λ for any $\lambda \in (-\infty, 8\pi)$ is a well known consequence of the Moser-Trudinger inequality [32]. The uniqueness of solutions of $(\mathbf{P}_\lambda)$ for $\lambda = 0$ is trivial, while for $\lambda \in (0, 8\pi)$ it has been proved in [7], [13], [36]. For $\lambda < 0$ the second variation of J_λ is strictly positive definite. Indeed, by (2.11), the first eigenvalue of the associated quadratic form, which is $\widehat{\sigma}_{1,\lambda}$, is strictly positive. Therefore J_λ admits at most one critical point, which concludes the proof of (iii). $\square$

3. Monotonicity of the energy

The proof of Theorem 1.1 relies on Proposition 2.1 and on the following result about the global structure of the bifurcation diagram of $(\mathbf{P}_\lambda)$ for $\lambda \in (-\infty, 8\pi)$ and the asymptotic behavior of the energy. This is an extension of some ideas first introduced in [4].

Proposition 3.1. *Let Ω be a domain of first kind. For each $\lambda \in (-\infty, 8\pi)$ there exists a unique solution ψ_λ of $(\mathbf{P}_\lambda)$ and*

$$\mathcal{G}_{8\pi} = \{(\lambda, \psi_\lambda) \,:\, \lambda \in (-\infty, 8\pi),\, \psi_\lambda \text{ solves } (\mathbf{P}_\lambda)\},$$

is an analytic curve in $(-\infty, 8\pi) \times C^{2,\alpha}_0(\overline{\Omega})$. In particular

$$E_\lambda = \mathbb{E}(\mu_\lambda) = \frac{1}{2}\int\limits_\Omega \rho_\lambda \psi_\lambda = \frac{1}{2} < \psi_\lambda >_\lambda = \int\limits_\Omega |\nabla \psi_\lambda|^2, \tag{3.1}$$

is real analytic in $(-\infty, 8\pi)$ and satisfies

$$E_\lambda \to 0^+ \text{ as } \lambda \to -\infty,\ E_\lambda \to +\infty, \text{ as } \lambda \to 8\pi^-,\ E_\lambda|_{\lambda=0} = E_0, \tag{3.2}$$

and

$$\frac{dE_\lambda}{d\lambda} > 0, \forall\, \lambda \in (-\infty, 8\pi). \tag{3.3}$$

Proof. The claim about the uniqueness and regularity of ψ_λ is just Proposition 2.1$(ii) - (iii)$. It is easy to see from $(\mathbf{P}_\lambda)$ that (3.1) holds. Since ψ_λ is real analytic in $(-\infty, 8\pi)$, then E_λ, being the composition of real analytic functions, is also real analytic as a function λ in $(-\infty, 8\pi)$, see for example Theorem 4.5.7 in [11].
Next, let us prove the inequality in (3.3). Since ψ_λ is real analytic, then in particular it is differentiable as a function of λ, and then by $(\mathbf{P}_\lambda)$ we conclude that $\eta_\lambda = \frac{d\psi_\lambda}{d\lambda} \in C^{2,\alpha}_0(\overline{\Omega})$ is a classical solution of,

$$-\Delta \eta_\lambda = \rho_\lambda \psi_{\lambda,0} + \lambda \rho_\lambda \eta_{\lambda,0}, \qquad \text{in} \qquad \Omega \tag{3.4}$$

where

$$\psi_{\lambda,0} = \psi_\lambda - < \psi_\lambda >_\lambda \text{ and } \eta_{\lambda,0} = \eta_\lambda - < \eta_\lambda >_\lambda .$$

By using (3.1), we also conclude that,

$$\frac{dE_\lambda}{d\lambda} = \int\limits_\Omega (\nabla \eta_\lambda, \nabla \psi_\lambda) = -\int\limits_\Omega \eta_\lambda (\Delta \psi_\lambda) = < \eta_\lambda >_\lambda . \tag{3.5}$$

Next, by using $(\mathbf{P}_\lambda)$ and (3.4), we have,

$$< \eta_\lambda >_\lambda = \int\limits_\Omega \rho_\lambda \eta_\lambda = \int\limits_\Omega -(\Delta \psi_\lambda)\eta_\lambda = \int\limits_\Omega -\psi_\lambda(\Delta \eta_\lambda) = < \psi^2_{\lambda,0} >_\lambda + \lambda < \psi_{\lambda,0}\eta_{\lambda,0} >_\lambda . \tag{3.6}$$

Let,

$$\psi_{\lambda,0} = \sum_{j=1}^{+\infty} \alpha_j \phi_{j,0}, \quad \eta_{\lambda,0} = \sum_{j=1}^{+\infty} \beta_j \phi_{j,0},$$

be the Fourier expansions (2.10) of $\psi_{\lambda,0}$ and $\eta_{\lambda,0}$. Multiplying (3.4) by $\phi_{j,0}$, integrating by parts and using (2.2), we conclude that,

$$\sigma_j \int\limits_\Omega \rho_\lambda \phi_{j,0}\eta_{\lambda,0} = \int\limits_\Omega \rho_\lambda \phi_{j,0}\psi_{\lambda,0}, \text{ that is } \sigma_j \beta_j = \alpha_j, \tag{3.7}$$

where $\sigma_j = \sigma_j(\lambda, \psi_\lambda)$. As a consequence, in view of Proposition 2.1(i), for any $\lambda \in (-\infty, 8\pi)$ we conclude that,

$$< \psi_{\lambda,0}, \eta_{\lambda,0} >_\lambda = \sum_{j=1}^{+\infty} \alpha_j \beta_j = \sum_{j=1}^{+\infty} \sigma_j (\beta_j)^2 \geq \sigma_1 \sum_{j=1}^{+\infty} (\beta_j)^2 \geq \sigma_1 < \eta^2_{\lambda,0} >_\lambda, \tag{3.8}$$

and

$$< \psi_{\lambda,0}, \eta_{\lambda,0} >_\lambda = \sum_{j=1}^{+\infty} \alpha_j \beta_j = \sum_{j=1}^{+\infty} \frac{\alpha_j^2}{\sigma_j} \leq \frac{1}{\sigma_1} \sum_{j=1}^{+\infty} \alpha_j^2 = \frac{1}{\sigma_1} < \psi_{\lambda,0}^2 >_\lambda, \tag{3.9}$$

By using (3.5), (3.6), (3.8), (3.9) and Proposition 2.1(i), we finally obtain,

$$\frac{dE_\lambda}{d\lambda} \geq < \psi_{\lambda,0}^2 >_\lambda + \lambda\sigma_1 < \eta_{\lambda,0}^2 >_\lambda > 0, \ \forall \lambda \in [0, 8\pi), \tag{3.10}$$

and

$$\frac{dE_\lambda}{d\lambda} \geq < \psi_{\lambda,0}^2 >_\lambda + \frac{\lambda}{\sigma_1} < \psi_{\lambda,0}^2 >_\lambda = \frac{\sigma_1 + \lambda}{\sigma_1} < \psi_{\lambda,0}^2 >_\lambda > 0, \ \forall \lambda \in (-\infty, 0),$$

where in the last inequality we used (2.8). The last two inequalities prove the inequality in (3.3). Next let us prove that $E_\lambda \to +\infty$ as $\lambda \to 8\pi^-$. Indeed, if by contradiction there exist $\lambda_n \to 8\pi^-$ and $\psi_n = \psi_{\lambda_n}$, such that $E_{\lambda_n} \leq C$, $\forall\, n \in \mathbb{N}$, then by (3.1) we would conclude that $\int\limits_\Omega |\nabla \psi_n|^2 \leq C$ for any $n \in \mathbb{N}$. Therefore, since ψ_λ is positive for $\lambda > 0$, by the Jensen and Moser-Trudinger inequalities [32], we would also conclude that ρ_λ with $\lambda = \lambda_n$ is uniformly bounded in $L^p(\Omega)$ for any $p > 1$. At this point, by standard elliptic estimates, we could obtain a uniform bound for ψ_n as $\lambda_n \to 8\pi^-$ and then in particular, passing to the limit along a subsequence in $(\mathbf{P}_\lambda)$, we could conclude that $(\mathbf{P}_\lambda)$ admits a solution for $\lambda = 8\pi$. This is impossible since Ω is of first kind and so by Definition 1.2 there is no solution of $(\mathbf{P}_\lambda)$ for $\lambda = 8\pi$. This contradiction proves that $E_\lambda \to +\infty$ as $\lambda \to 8\pi^-$. Obviously, by the uniqueness of solutions, we also have $E_\lambda|_{\lambda=0} = E_0$. We are just left with the proof of $E_\lambda \to 0^+$ as $\lambda \to -\infty$. By contradiction, in view of the monotonicity of E_λ, we can assume that $E_\lambda \to E_\infty > 0$ as $\lambda \to -\infty$. By using once more the monotonicity of E_λ and $E_\lambda \to +\infty$ as $\lambda \to 8\pi^-$, we conclude that there is no solution of $(\mathbf{P}_\lambda)$ with $\lambda \in (-\infty, 8\pi)$ whose energy is less than E_∞. This is impossible since a well known result in [12] states that for any $E > 0$ there is a solution of $(\mathbf{P}_\lambda)$ with $\lambda < -8\pi$ whose energy is E. □

4. The proof of Theorem 1.1

The proof of Theorem 1.1.
The crux of the proof of Theorem 1.1 is to control the monotonicity of μ as a function of E. The first part of this task has been accomplished in Proposition 3.1, which is based on the study of solutions of $(\mathbf{P}_\lambda)$. To conclude the proof we will analyze the monotonicity of μ as a function of λ. Actually, it is easier to do this in terms of $u_\lambda = \lambda\psi_\lambda$ which solves

$$\begin{cases} -\Delta u_\lambda = \lambda \dfrac{e^{u_\lambda}}{\int\limits_\Omega e^{u_\lambda}} & \Omega \\ u_\lambda = 0 & \partial\Omega \end{cases} \qquad (\mathbf{Q}_\lambda)$$

if and only if ψ_λ solves $(\mathbf{P}_\lambda)$. Clearly, by Proposition 3.1, u_λ is a real analytic function of λ, which in particular shows that this correspondence holds also for $\lambda = 0$, in the sense that $u_\lambda \equiv 0$ if and only if $\lambda = 0$ and ψ_λ is the unique solution of $(\mathbf{P}_\lambda)$ with $\lambda = 0$.

Since, by Proposition 2.1(iii), for $\lambda \in (-\infty, 8\pi)$ $(\mathbf{Q}_\lambda)$ admits a unique solution, then we can define the map $\lambda \mapsto \mu_\lambda$ as follows,

$$\mu_\lambda = \frac{\lambda}{\int\limits_\Omega e^{u_\lambda}}. \tag{4.1}$$

Clearly μ_λ is well defined as a function of λ for $\lambda \in (-\infty, 8\pi)$ and obviously, for fixed λ, the pair (μ, v_μ), where $\mu = \mu_\lambda$ and $v_\mu = u_\lambda$ solves $(1)_\mu$. Although not need it here, it is understood that $(\mu, v_\mu)|_{\mu_\lambda}$, for $\lambda > 0$ small enough, is just a parametrization of a portion of the minimal branch [25] of solutions of $(1)_\mu$ for $\mu > 0$ small enough.

Since u_λ is real analytic in $(-\infty, 8\pi)$, then, in view of (4.1), we also have that μ_λ is a real analytic function of λ and it holds,

$$\frac{d\mu_\lambda}{d\lambda} = \left(\int\limits_\Omega e^{u_\lambda}\right)^{-2} \left(\int\limits_\Omega e^{u_\lambda} - \lambda \int\limits_\Omega e^{u_\lambda} z_\lambda\right), \lambda \in (-\infty, 8\pi), \tag{4.2}$$

where

$$z_\lambda = \frac{du_\lambda}{d\lambda} = u_\lambda^{'}.$$

Therefore we readily find that,

$$\left(\int\limits_\Omega e^{u_\lambda}\right) \frac{d\mu_\lambda}{d\lambda} = 1 - \lambda < z_\lambda >_\lambda, \quad \lambda \in (-\infty, 8\pi).$$

The crux of the proof of Theorem 1.1 is the following,

Lemma 4.1. *Let* $g(\lambda) := 1 - \lambda < z_\lambda >_\lambda$, $\lambda \in (-\infty, 8\pi)$. *There exists* $\lambda_* \in [4\pi, 8\pi)$ *such that* $g(\lambda) > 0$ *for* $\lambda \in (-\infty, \lambda_*)$, $g(\lambda_*) = 0$, $g(\lambda) < 0$ *for* $\lambda \in (\lambda_*, 8\pi)$. *In particular* $g(\lambda) \to +\infty$ *as* $\lambda \to -\infty$, $g(0) = 1$ *and* $g(\lambda) \to -\infty$ *as* $\lambda \to 8\pi^-$.

We first conclude the proof of Theorem 1.1 and then get back in the next section to that of Lemma 4.1.

Let E_λ be as defined in Proposition 3.1 and $E_* = E_\lambda|_{\lambda=\lambda_*}$. By (3.3) in Proposition 3.1, we see that E_λ is a strictly increasing and analytic function of λ in $(-\infty, 8\pi)$ and in particular that $E = E_*$ if and only if $\lambda = \lambda_*$. Therefore, it is well defined the inverse of E_λ, $\lambda_\infty = \lambda_\infty(E)$, and we define

$$\mu_\infty(E) = \mu_\lambda|_{\lambda=\lambda_\infty(E)}.$$

In particular, since μ_λ is a real analytic function of λ, and since $E_\lambda^{'}$ is strictly positive, then λ_∞ is a real analytic function of E and so does μ_∞.
As a consequence of Lemma 4.1 we conclude that $\frac{d\mu_\lambda}{d\lambda} > 0$ iff $\lambda < \lambda_*$, where λ_* is the unique value of $\lambda \in [4\pi, 8\pi)$ such that $1 - \lambda < z_\lambda >_\lambda = 0$. Next, by (3.2) in Proposition 3.1, we have $E_\lambda \to 0^+$ as $\lambda \to -\infty$. Since solutions of $(1)_\mu$ are uniformly bounded in $[\widehat{\mu}, 0]$ for any $\widehat{\mu} < 0$, since $(1)_\mu$ admits a unique solution for any $\mu < 0$, and by using the definition of $\mathbb{E}(\mu)$, then it is not difficult to see that necessarily $\mu_\infty \to -\infty$ as $E \to 0^+$. Also, since $\mu_\lambda \to 0$ as $\lambda \to 0$, and

$$\mathbb{E}(\mu_\lambda) = E_\lambda \to E_0, \text{ as } \lambda \to 0,$$

then $\mu_\infty(E_0) = 0$. By (3.2) in Proposition 3.1 we also have $\mathbb{E}(\mu_\lambda) = E_\lambda \to +\infty$ as $\lambda \to 8\pi^-$ and by Lemma 4.1 $\frac{d\mu_\lambda}{d\lambda} < 0$ iff $\lambda > \lambda_*$. Therefore μ_∞ decreases monotonically in $(E_*, +\infty)$ and then we also conclude that $\mu_\infty(E) \to 0^+$ as $E \to +\infty$. Indeed, this is an immediate consequence of (4.1) and of the following well known estimate for blow up solutions of mean field type equations, $\int\limits_\Omega e^{u_\lambda} \to +\infty$ as $\lambda \to 8\pi^-$, see for example [30].
As a consequence, to conclude the proof, it just remains to show that $E_* = E_*(\Omega) = \mathbb{E}(\mu_\star(\Omega))$, that is $\mu_\infty(E_*) = \mu_\star(\Omega)$. Clearly, by the definition of $\mu_\star(\Omega)$, we have $\mu_\infty(E_*) \leq \mu_\star(\Omega)$. Therefore we are left to prove the following:
Claim: $\mu_\infty(E_*) \geq \mu_\star(\Omega)$**.**
Let us assume by contradiction that $\mu_\infty(E_*) < \mu_\star(\Omega)$. As E increases above E_*, then, because of (3.3), we see that λ is strictly increasing. Therefore $\lambda > \lambda_*$ for any $E > E_*$ and in particular $\mu_\lambda < \mu_\infty(E_*) < \mu_\star(\Omega)$ for any $\lambda \in (0, 8\pi)$. As a consequence there exists some open interval $I \subset (0, 8\pi)$ such that $\{(\mu_\lambda, v_{\mu_\lambda})\}_{\lambda \in I} \cap \Gamma_\infty(\Omega) = \emptyset$.

Therefore the solutions $(\mu_\lambda, u_{\mu_\lambda})$ form another smooth branch with no bifurcation points, say $\mathcal{M}_{8\pi}$, which however obviously emanates from $(\mu_\lambda, u_{\mu_\lambda})|_{\lambda=0} = (0,0)$.
However $\Gamma_\infty(\Omega)$ is a smooth branch with no bifurcation points as well which emanates from $(\mu, u_\mu) = (0,0)$, and this is a contradiction since then $(0,0)$ should be a bifurcation point where the two branches $\mathcal{M}_{8\pi}$ and $\Gamma_\infty(\Omega)$ meet. □

5. The proof of Lemma 4.1

To simplify the notations, in this section we set,

$$z_\lambda = \frac{du_\lambda}{d\lambda} = u_\lambda^{'}, \quad w_\lambda = \frac{dz_\lambda}{d\lambda} = z_\lambda^{'}, \quad < f >= \int\limits_{\Omega} \rho_\lambda f, \quad z_{\lambda,0} = z_\lambda - < z_\lambda > .$$

The derivative of $-g(\lambda) = \lambda < z_\lambda > -1$ takes the form,

$$-g^{'}(\lambda) = (\lambda < z_\lambda >)^{'} =< z_\lambda > + \lambda < z_\lambda >^{'} \tag{5.1}$$

or else, since $< z_\lambda >^{'} =< z_{\lambda,0}^2 > + < w_\lambda >$,

$$-g^{'}(\lambda) =< z_\lambda > + \lambda < z_{\lambda,0}^2 > + \lambda < w_\lambda > . \tag{5.2}$$

Clearly we have

$$-\Delta z_\lambda = \rho_\lambda + \lambda \rho_\lambda z_{\lambda,0}. \tag{5.3}$$

Since $(\mathbf{P}_\lambda)$ is a constrained problem, $\sigma_{1,\lambda} > 0$ does not imply that the maximum principle holds for the linearized problem (2.1) relative to $(\mathbf{P}_\lambda)$. Therefore, we can not claim that $\sigma_{1,\lambda} > 0$ implies $z_\lambda \geq 0$. However, as a consequence of Proposition 2.1(i) we are able to prove the following version of the maximum principle based on the sign of $g(\lambda)$:

Proposition 5.1.
(i) $< z_\lambda >> 0$, $\forall\, \lambda \in (-\infty, 8\pi)$.
(ii) *If* $\lambda \in (-\infty, 4\pi)$, *then* $g(\lambda) = 1 - \lambda < z_\lambda >> 0$.
(iii) *If* $\lambda \in (-\infty, 8\pi)$ *and* $1 - \lambda < z_\lambda > \geq 0$, *then* $z_\lambda \geq 0$.

Proof.
(i) Multiplying (5.3) by z_λ we conclude that,

$$\int\limits_{\Omega} |\nabla z_\lambda|^2 - \lambda \int\limits_{\Omega} \rho_\lambda z_{\lambda,0}^2 =< z_\lambda >, \tag{5.4}$$

that is, since by Proposition 2.1(i) the first eigenvalue σ_1 of L_λ is strictly positive for $\lambda \in (-\infty, 8\pi)$, then we have,

$$< z_\lambda > \geq \sigma_1 < z_{\lambda,0}^2 >> 0, \ \forall\, \lambda \in (-\infty, 8\pi).$$

(ii) Multiplying (5.3) by z_λ we also conclude that,

$$0 < \nu_1 < z_\lambda^2 > \leq \int\limits_{\Omega} |\nabla z_\lambda|^2 - \lambda < z^2 > =< z_\lambda > (1 - \lambda < z_\lambda >),$$

where ν_1 is the first eigenvalue of $-\Delta\phi - \lambda\rho_\lambda\phi$, which for $\lambda < 4\pi$ satisfies $\nu_1 > 0$, see [7] and [36]. Therefore, since $< z_\lambda >> 0$, then if $\lambda < 4\pi$ we conclude that $1 - \lambda < z_\lambda >> 0$.

(iii) We argue by contradiction and suppose that $z_- = z_\lambda \chi_{\Omega_-} \not\equiv 0$, where $\Omega_- = \{x \in \Omega \,:\, z_\lambda < 0\}$. Here χ_A is the characteristic function of the set A. Put $z_+ = z_\lambda \chi_{\Omega_+}$ and $\phi = \alpha z_+ + z_-$. Then ϕ satisfies

$$-\Delta\phi = \alpha\lambda\rho_\lambda z_+ + \alpha\rho_\lambda\chi_+ + \lambda\rho_\lambda z_- + \rho_\lambda\chi_- - \alpha\lambda\rho_\lambda < z_\lambda > \chi_+ - \lambda\rho_\lambda < z_\lambda > \chi_- =$$

$$\lambda\rho_\lambda\phi - \alpha\lambda\rho_\lambda < z_+ > \chi_+ - \alpha\lambda\rho_\lambda < z_- > \chi_+ - \lambda\rho_\lambda < z_- > \chi_- - \lambda\rho_\lambda < z_+ > \chi_- + \alpha\rho_\lambda\chi_+ + \rho_\lambda\chi_-$$

Multiplying this equation by ϕ and integrating by parts we find that,

$$\int\limits_{\Omega} |\nabla\phi|^2 = \lambda \int\limits_{\Omega} \rho_{\lambda}\phi^2 - \lambda < \alpha z_+ >^2 - \lambda < z_- >^2 - \lambda(\alpha^2+1) < z_- >< z_+ > + \alpha^2 < z_+ > + < z_- > .$$

Since

$$\left(\int\limits_{\Omega} \rho_{\lambda}\phi\right)^2 = < \alpha z_+ >^2 + < z_- >^2 + 2\alpha < z_- >< z_+ >,$$

we obtain that

$$\int\limits_{\Omega} |\nabla\phi|^2 - \lambda < \phi_0^2 >= P(\alpha) := -\lambda(\alpha-1)^2 < z_+ >< z_- > + \alpha^2 < z_+ > + < z_- > .$$

Then, by using once more that σ_1 is strictly positive, we conclude that the r.h.s of this inequality, which we call $P(\alpha)$, must be strictly positive if $\lambda < 8\pi$. Indeed, we see that,

$$P(\alpha) = \int\limits_{\Omega} |\nabla\phi|^2 - \lambda \int\limits_{\Omega} \rho_{\lambda} < \phi_0^2 > \geq \sigma_1 < \phi_0^2 >, \tag{5.5}$$

and that, since z_- does not vanish identically by assumption, $< \phi_0^2 >> 0$ for any $\alpha \in \mathbb{R}$. Therefore we have,

$$P(\alpha) = < z_+ > (1 - \lambda < z_- >)\alpha^2 + 2\lambda < z_+ >< z_- > \alpha + < z_- > (1 - \lambda < z_+ >) > 0, \forall\, \alpha \in \mathbb{R}.$$

The determinant of $P(\alpha)$ takes the form $-4 < z_+ >< z_- > (1 - \lambda < z_\lambda >)$ which is nonnegative whenever $(1 - \lambda < z_\lambda >) \geq 0$. Therefore, if $(1 - \lambda < z_\lambda >) \geq 0$, then $P(\alpha)$ must be nonpositive somewhere, which is impossible for $\lambda < 8\pi$. Therefore $z_- \equiv 0$ whenever $(1 - \lambda < z_\lambda >) \geq 0$ and $\lambda < 8\pi$, as claimed. □

It turns out that the analysis of the sign of $g(\lambda)$ is far from being trivial. Surprisingly enough, we will succeed in carrying out the latter analysis after proving that $g(\lambda)$ satisfies a first order O.D.E.

Lemma 5.2. *The function $g(\lambda) = 1 - \lambda < z_\lambda >$ satisfies*

$$g'(\lambda) = a(\lambda)g(\lambda) + b(\lambda), \quad \lambda \in (-\infty, 8\pi), \qquad g(0) = 1, \tag{5.6}$$

where

$$a(\lambda) = -(2\lambda < z_{\lambda,0}^2 > + \lambda < z_\lambda^2 > + < z_\lambda >), \quad b(\lambda) = -\lambda^2 < z_\lambda^3 > .$$

Proof. From (5.3) we find that,

$$-\Delta w_\lambda = 2\rho_{\lambda} z_{\lambda,0} + \lambda\rho_{\lambda}(z_{\lambda,0}^2)_0 + \lambda\rho_{\lambda} w_{\lambda,0}. \tag{5.7}$$

Multiplying this equation by z_λ, integrating by parts and using (5.3) we also find that,

$$< w_\lambda >= 2 < z_{\lambda,0}^2 > + \lambda < (z_{\lambda,0}^2)_0 z_{\lambda,0} > \equiv 2 < z_{\lambda,0}^2 > + \lambda < z_{\lambda,0}^3 > \tag{5.8}$$

By using (5.8) in (5.2), we reduce (5.2) to,

$$-g'(\lambda) = < z_\lambda > + 3\lambda < z_{\lambda,0}^2 > + \lambda^2 < z_{\lambda,0}^3 > . \tag{5.9}$$

Next, observing that

$$< z_{\lambda,0}^3 >= < z_\lambda^3 > - 3 < z_\lambda^2 >< z_\lambda > + 2 < z_\lambda >^3,$$

we see that (5.9) takes the form

$$-g'(\lambda) = \lambda^2 < z_\lambda^3 > - 3\lambda^2 < z_\lambda^2 >< z_\lambda > + 2\lambda^2 < z_\lambda >^3 + 3\lambda < z_\lambda^2 > - 3\lambda < z_\lambda >^2 + < z_\lambda >=$$
$$\lambda^2 < z_\lambda^3 > + 3\lambda < z_\lambda^2 > (1 - \lambda < z_\lambda >) + 2\lambda^2 < z_\lambda >^3 - 2\lambda < z_\lambda >^2 - \lambda < z_\lambda >^2 + < z_\lambda >=$$
$$\lambda^2 < z_\lambda^3 > + 3\lambda < z_\lambda^2 > (1 - \lambda < z_\lambda >) - 2\lambda < z_\lambda >^2 (1 - \lambda < z_\lambda >) + < z_\lambda > (1 - \lambda < z_\lambda >) =$$
$$\lambda^2 < z_\lambda^3 > + 2\lambda < z_{\lambda,0}^2 > (1 - \lambda < z_\lambda >) + \lambda < z_\lambda^2 > (1 - \lambda < z_\lambda >) + < z_\lambda > (1 - \lambda < z_\lambda >).$$

In particular we see that,

$$-g'(\lambda) = \lambda^2 < z_\lambda^3 > +2\lambda < z_{\lambda,0}^2 > g(\lambda) + \lambda < z_\lambda^2 > g(\lambda) + < z_\lambda > g(\lambda), \quad g(0) = 1,$$

for any $\lambda \in (-\infty, 8\pi)$, as claimed. □

At this point we can conclude the proof of Lemma 4.1.
By (5.6) we find that

$$e^{-A(\lambda)} g(\lambda) = 1 + \int_0^\lambda e^{-A(t)} b(t) dt = 1 - \int_0^\lambda e^{-A(t)} t^2 < z_t^3 > dt, \tag{5.10}$$

where $A(\lambda) = \int_0^\lambda a(t)dt$. By Proposition 5.1$(ii)$ we have $g(\lambda) > 0$ for $\lambda < 4\pi$. Since $z_\lambda = u_\lambda' = (\lambda\psi_\lambda)'$, then $< z_\lambda > = < \psi_\lambda > + \lambda < \eta_\lambda > = 2E_\lambda + \lambda E_\lambda'$, where we used (3.1) and (3.5). Thus, as a consequence of (3.3), we also conclude that

$$< z_\lambda > \geq 2E_\lambda \to +\infty, \text{ as } \lambda \to 8\pi^-,$$

where we used (3.2). Therefore $g(\lambda) \to -\infty$ as $\lambda \to 8\pi^-$. Since $g(0) = 1$ and $g(\lambda)$ is continuous, then there exists at least one value of $\lambda = \lambda_0 \in [4\pi, 8\pi)$ such that $g(\lambda_0) = 0$. Let

$$\lambda_* = \sup\{\lambda > 0 \, : \, g(\tau) > 0, \forall \, 0 \leq \tau < \lambda\}.$$

Then $\lambda_* \geq 4\pi$, $g(\lambda_*) = 0$ and we claim that $g(\lambda) < 0$ for any $\lambda > \lambda_*$. Indeed, by Proposition 5.1 (iii) we have $z_\lambda \geq 0$ for $\lambda \leq \lambda_*$. Therefore, by continuity, $< z_\lambda^3 >> 0$ in a small enough right neighborhood of λ_* and so, by (5.10), $g(\lambda) < 0$ in a small enough right neighborhood of λ_* as well. We argue by contradiction and suppose that the claim is false. Therefore, putting

$$\widehat{\lambda} = \sup\{\lambda > \lambda_* \, : \, g(\tau) < 0, \forall \, \lambda_* < \tau < \lambda\},$$

we must have that $\widehat{\lambda} \in (\lambda_*, 8\pi)$. Since clearly $g(\widehat{\lambda}) = 0$ and $g(\lambda) < 0$ for $\lambda_* < \lambda < \widehat{\lambda}$, then $g'(\widehat{\lambda}) \geq 0$ and then by (5.6)

$$(\widehat{\lambda})^2 < z_\lambda^3 > \big|_{\lambda = \widehat{\lambda}} = -g'(\widehat{\lambda}) \leq 0.$$

However this is impossible since by Proposition 5.1 (iii) we have $z_\lambda\big|_{\lambda=\widehat{\lambda}} \geq 0$ which implies that $< z_\lambda^3 > \big|_{\lambda=\widehat{\lambda}} > 0$. This contradiction shows that in fact $g(\lambda) < 0$ for $\lambda > \lambda_*$ as claimed. □

6. Appendix

We present here a further discussion on the modified spectral analysis introduced in section 2. To this end we consider a simplified linear problem which however share the same structure as (2.1):

$$-\Delta\phi = \sigma \left(\phi - \fint_{B_1} \phi \right) \quad \text{in} \quad B_1 \tag{6.1}$$

with $\phi = 0$ on ∂B_1, and $\fint_{B_1} \phi = \frac{1}{|B_1|} \int_{B_1} \phi$. Passing to the new variable $\phi_0 = \phi - \fint_{B_1} \phi$ we are reduced to calculate the spectrum of

$$-\Delta\phi_0 - \sigma\phi_0 = 0 \quad \text{in} \quad B_1, \tag{6.2}$$

with boundary conditions

$$\text{(I)} \quad \phi_0 = \text{constant} \quad \text{on} \quad \partial B_1 \quad \text{and} \quad \text{(II)} \quad \fint_{B_1} \phi_0 = 0. \tag{6.3}$$

Passing to polar coordinates, we first consider solutions of (6.2) of the form

$$\psi_n(r,\theta) = (A\cos(n\theta) + B\sin(n\theta))\, J_n(\sqrt{\sigma}r)$$

where J_n is the Bessel function of order n. Now either $n = 0$ and then (6.3)-(I) is always satisfied or $n \geq 1$ and then (6.3)-(I) is satisfied if and only if $\sigma = \sigma_{n,m} = \mu_{n,m}^2$, where $\mu_{n,m}$ is the m-th zero of J_n. Next, if $n \geq 1$ then (6.3)-(II) is always satisfied, while if $n = 0$ then (6.3)-(II) is equivalent to $\int\limits_0^1 J_0(\sqrt{\sigma}r)rdr = 0$. Since $(J_1(r)r)^{'} = rJ_0(r)$ this is equivalent to $\sqrt{\sigma}J_1(\sqrt{\sigma}) = 0$, that is $\sigma = \mu_{1,m}^2$. Therefore the eigenvalues of (6.1) are

$$\sigma_{n,m} = \mu_{n,m}^2,$$

where $\sigma_{1,m}$ admits three eigenfunctions, $\{J_0(\mu_{1,m}r) - J_0(\mu_{1,m}), \cos(\theta)J_1(\mu_{1,m}r), \sin(\theta)J_1(\mu_{1,m}r)\}$ and $\sigma_{n,m}$ with $n \geq 2$ that admits two eigenfunctions $\{\cos(n\theta)J_n(\mu_{n,m}r), \sin(\theta)J_n(\mu_{n,m}r)\}$). Observe that these are the eigenfunctions of (6.1). In particular the first eigenvalue $\sigma_{1,1} = \mu_{1,1}^2 \simeq (3.83)^2$ admits three eigenfunctions, one of which is radial,

$$\phi_1 = \phi_1(r) = J_0(\mu_{1,1}r) - J_0(\mu_{1,1}),$$

and satisfies $\phi_1(1) = 0$ and $\phi_1^{'}(1) = \mu_{1,1}J_0^{'}(\mu_{1,1}) = 0$. This is not in contradiction with the Hopf Lemma or with general unique continuation principles, since $\fint\limits_{B_1}\phi_1 = -J_0(\mu_{1,1}) \neq 0$ and the identically zero function $\phi \equiv 0$ is not a solution of (6.1) with $\fint\limits_{B_1}\phi \neq 0$. Alternatively, in terms of ϕ_0, the eigenfunction $\phi_{1,0} = \phi_1 - \fint\limits_{B_1}\phi_1$ satisfies $\phi_{1,0}(1) = J_0(\mu_{1,1}) \neq 0$ and $\phi_{1,0}^{'}(1) = \mu_{1,1}J_0^{'}(\mu_{1,1}) = 0$ but still the function $\phi_0 \equiv J_0(\mu_{1,1}) \neq 0$ is not a solution of (6.2), (6.3).

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