Prescribing Morse scalar curvatures: subcritical blowing-up solutions

Andrea Malchiodi and Martin Mayer

Scuola Normale Superiore, Piazza dei Cavalieri 7, 50126 Pisa, ITALY
andrea.malchiodi@sns.it, martin.mayer@sns.it

December 22, 2018

Abstract

Prescribing conformally the scalar curvature of a Riemannian manifold as a given function consists in solving an elliptic PDE involving the critical Sobolev exponent. One way of attacking this problem consist in using subcritical approximations for the equation, gaining compactness properties. Together with the results in [30], we completely describe the blow-up phenomenon in case of uniformly bounded energy and zero weak limit in positive Yamabe class. In particular, for dimension greater or equal to five, Morse functions and with non-zero Laplacian at each critical point, we show that subsets of critical points with negative Laplacian are in one-to-one correspondence with such subcritical blowing-up solutions.

Key Words: Conformal geometry, sub-critical approximation, blow-up analysis.

Contents

1 Introduction

Consider a compact manifold (M^n, g_0) with $n \geq 3$ and a *conformal metric* $g = u^{\frac{4}{n-2}} g_0$, $u > 0$: with this notation the scalar curvature transforms in the following way (see [4])

$$R_{g_u} u^{\frac{n+2}{n-2}} = L_{g_0} u := -c_n \Delta_{g_0} u + R_{g_0} u \qquad c_n = \frac{4(n-1)}{(n-2)},$$

with Δ_{g_0} the Laplace-Beltrami operator of g_0. L_{g_0} is called the *conformal Laplacian* and transforms according to the law $L_g(u\,\phi) = u^{\frac{n+2}{n-2}} L_{g_0}(\phi)$.

In the 70's, Kazdan and Warner considered in [28] the problem of prescribing the scalar curvature of manifolds via conformal deformation of the metric, see also [26], [27]. By the above transformation law, if one wishes to prescribe R_g as a given function $K(x)$ then would need to solve

$$L_{g_0} u = K(x) u^{\frac{n+2}{n-2}} \qquad \text{on} \qquad (M, g_0). \tag{1.1}$$

There are rather easy obstructions to the solvability of (1.1): for example, if the sign of K is constant, it has to coincide with that of the first eigenvalue of L_{g_0}. Depending on the latter sign, which is conformally invariant, a conformal class of metrics is said to be of *negative, zero or positive Yamabe class.* We will discuss for simplicity the case of function K with constant sign, despite in the literature there are many interesting papers dealing with changing-sign functions.

In [28], Kazdan and Warner proved some existence results for zero or negative Yamabe classes using the sub- and super-solution method. For positive Yamabe class instead, they found a now well-known obstruction to existence on the sphere, namely that if u solves (1.1), then one must have

$$\int_{S^n} \langle \nabla K, \nabla f \rangle_{g_{S^n}} u^{\frac{2n}{n-2}} \, d\mu_{g_{S^n}} = 0, \tag{1.2}$$

and hence, for conformal curvatures K, the function $\langle \nabla K, \nabla f \rangle_{g_{S^n}}$ must change sign.

Later on, some existence results were found under conditions that would imply *topological richness* of the sub-levels of K, contrary to the above example. In two dimensions, where (1.1) is replaced by an equation in exponential form, J. Moser showed that the problem is solvable on the standard sphere if K is antipodally symmetric. In higher dimensions, existence results under the action of symmetry groups were proven in [20] and [21], [22].

A general difficulty in studying (1.1) is the lack of compactness due to the presence of the critical exponent. A typical phenomenon encountered here is that of *bubbling. Bubbles* are solutions of (1.1) on S^n with $K \equiv 1$: these arise as profiles of general diverging solutions and were classified in [11], see also [3], [36]. From the variational point of view, bubbles generate diverging Palais-Smale sequences for the Euler-Lagrange energy of (1.1), given by $J = J_K$:

$$J(u) = \frac{\int_M \left(c_n |\nabla u|^2_{g_0} + R_{g_0} u^2 \right) d\mu_{g_0}}{\left(\int_M K u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}}}.$$

From a formal expansion of J on a finite sum of bubbles, see e.g. the introduction in [30], one sees a role of the dimension in the strength of the mutual interaction among bubbles, which is weaker as n increases: a consequence of this fact is that in three dimensions only one bubble can form. Exploiting this fact, after some work on S^2 by A. Chang and P. Yang in [16], [17], A. Bahri and J.M. Coron proved an existence result in [6] on S^3 assuming that K is a Morse function satisfying the following two properties

$$\{\nabla K = 0\} \cap \{\Delta K = 0\} = \emptyset; \tag{1.3}$$

$$\sum_{\{x \in M \,:\, \nabla K(x) = 0, \Delta K(x) < 0\}} (-1)^{m(x,K)} \neq -1, \tag{1.4}$$

where $m(x, K)$ stands for the Morse index of K at x, see also [12] and [35] for more general related results. The above existence statement was extended to arbitrary dimensions in [24] for functions satisfying a suitable flatness condition, and in [18], [1], [29] for functions K close to a positive constant in the C^2-sense.

In four dimensions, see [7] and [25], it was shown that even if multiple bubbles can form, they cannot be too close to each-other; such phenomenon is usually refereed to as *isolated simple blow-up.* Results of different kind were also proven in [19] for $n = 2$ and in [9] [8], [10]: see also Chapter 6 in [4].

Two main approaches have been used to understand the blow-up phenomenon: sub-critical approximations or the construction of pseudo-gradient flows. In this paper we focus on the former, while the other one will be the subject of [32], where a one-to-one correspondence of blowing-up solutions with bounded energy (and zero weak limit) and *critical points at infinity* is shown. Consider the problem

$$-c_n \Delta_{g_0} u + R_{g_0} u = K(x) \, u^{\frac{n+2}{n-2} - \tau}, \qquad 0 < \tau \ll 1, \tag{1.5}$$

which, up to a proper dilation, is the Euler-Lagrange equation for the functional

$$J_\tau(u) = \frac{\int_M \left(c_n|\nabla u|^2_{g_0} + R_{g_0}u^2\right) d\mu_{g_0}}{\left(\int_M K u^{p+1} d\mu_{g_0}\right)^{\frac{2}{p+1}}}, \quad u \in \mathcal{A}. \tag{1.6}$$

Being now the exponent lower than critical, solutions can be easily found, even though one could lose uniform estimates as τ tends to zero. In [12], [35], [24], the single-bubbling behaviour for diverging solutions of (1.5) was proved. Then, by degree- or Morse-theoretical arguments it was shown that under (1.4) there must be families of solutions that stay uniformly bounded, therefore converging to solutions of (1.1). For this argument to work, one crucial step was to completely characterize blowing-up solutions of (1.5), showing that in three dimensions single blow-ups occur at any critical point of K with negative laplacian and that they are unique. On four-dimensional spheres, a similar property was proved in [25] for multiple blow-ups (see also [7]), assuming a suitable condition related to the multi-bubble interactions.

For Morse functions, if $n \geq 5$ the situation is more involved, and blow-ups might be possibly of infinite energy, see e.g. [13], [14], [15], [37]. In [30] it was however proved that if a sequence of blowing-up solutions has uniformly-bounded $W^{1,2}$-energy and zero weak limit, then blow-ups are still isolated simple. Although the result is similar to the case of dimensions three and four, the phenomenon is somehow opposite since it is *driven* by the function K rather than from the mutual bubble interactions. Both assumptions, zero weak limit and bounded energy, are indeed natural: if the former fails then problem (1.1) would have a solution; the second one instead is usually found when using min-max or Morse-theoretical arguments, as it will be done in [31]. However, differently from $n = 3, 4$, in [30] no restriction is proven on the number or location of blow-up points, provided they occur at critical points of K with negative Laplacian.

The goal of this paper is to show that the characterization of the above blow-ups in [30] is sharp, namely that they can occur at arbitrary subsets of $\{\nabla K = 0\} \cap \{\Delta K < 0\}$. Furthermore, we prove uniqueness of such solutions, their non-degeneracy and determine their Morse index. Our main result is the following one, that follows from Proposition 3.1, Corollary 4.1 and Theorem 1 in [30].

Theorem 1. *Let (M, g) be a compact manifold of dimension $n \geq 5$ with positive Yamabe class, and let $K : M \to \mathbb{R}$ be a positive Morse function satisfying (1.3). Let $x_1, \ldots, x_q$ be distinct critical points of K with negative Laplacian. Then, as $\tau \to 0$, there exists a unique solution $u_{\tau,x_1,\ldots,x_q}$ developing a simple bubble at each point x_i and converging weakly to zero in $W^{1,2}(M, g)$ as $\tau \to 0$. Moreover, up to scaling by constants, $u_{\tau,x_1,\ldots,x_q}$ is non-degenerate for J_τ and $m(J_\tau, u_{\tau,x_1,\ldots,x_q}) = (q-1) + \sum_{i=1}^q (n - m(K, x_i))$. Furthermore, all blow-ups with uniformly bounded energy and zero weak limit are of the above type.*

As it will be shown in [31], for $n \geq 5$ there cannot be a direct counterpart of (1.4), which is an index-counting condition. However, existence results of different type will be derived there.

Remark 1.1. *(i) A more precise expression for $u_{\tau,x_1,\ldots,x_q}$ is given by the following formula*

$$\Big\| u_m - \sum_{j=1}^q \alpha_{j,m}\delta_{\lambda_{j,m},a_{j,m}} \Big\|_{W^{1,2}(M,g_0)} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty,$$

$$\alpha_{j,m} = \frac{\Theta}{K(x_j)^{\frac{n-2}{4}}} + o(1), \quad a_{j,m} \longrightarrow x_j \quad \text{and} \quad \lambda_{j,m} \simeq \lambda_{\tau_m} = \tau_m^{-\frac{1}{2}}.$$

Here the multiplicative constant Θ depends on the blowing-up solutions but it is independent of j. For this and more precise formulas we refer to Section 3 and Theorem 2 in the Appendix. If $n = 4$, the same conclusions hold replacing $\Delta K(a_j) < 0$ for all j with (iv) of Theorem 2 in [30].

(ii) Even though upon scaling the above solutions $u_{\tau,x_1,\ldots,x_q}$ are non-degenerate, they Hessian of J_τ there has $\sum_{i=1}^q (n - m(K, x_i))$ eigenvalues approaching zero as $\tau \to 0$, see Section 4.

(iii) Theorem 1 gives a one-to-one correspondence of zero weak limit subcritical blow-up solutions to subsets of critical points of K with negative Laplacian, while in [32] this correspondence will be shown with zero weak limit, i.e. pure critical points at infinity, according to the terminology in [5], see also [33]

The proof of Theorem 1 relies on the estimates in [30] and a finite-dimensional reduction, see e.g. [2], with a careful asymptotic analysis. In dimension four, this approach was used in Section 2 of [25]: here we show that in higher dimensions blow-up might occur at arbitrary critical points of K with negative Laplacian, which affects the global structure of the solutions of problem (1.1). Via careful expansions, we also determine the Hessian of the Euler-Lagrange functional and the Morse index of these solutions, which we prove to be non-degenerate.

The solutions we consider here lie in a set $V(q,\varepsilon)$ in the functional space $W^{1,2}(M,g_0)$ which contains a manifold of approximate solutions for (1.5), $\sum_{i=1}^q \alpha^i \varphi_{a_i,\lambda_i}$, which is transversally non-degenerate (see Section 2 for the notation used here). This allows to solve (1.5) orthogonally to this manifold via a proper transversal correction to the approximate solutions, see Definition 3.1 and Lemma 3.1, and reduce to the study of the tangent component. By Theorem 2 from [30] we can reduce ourselves to a smaller set $\bar{V}(q,\varepsilon)$, see (3.1), where more precise estimates hold for the gradient of J_τ. These allow us to use an orthogonal correction $\bar{v}$ small in size, solve also for the tangent component and to estimate the second differential of J_τ at $\sum_{i=1}^q \alpha^i \varphi_{a_i,\lambda_i} + \bar{v}$, see Section 4. Finally, this allows in turn to compute the Morse index of the solutions $u_{\tau,x_1,\dots,x_q}$ and to prove their uniqueness. In this step we show that, even though the correction $\bar{v}$ is of the same order of the small eigenvalues of the Hessian of J_τ, some cancellation occurs in the estimate of the Morse index.

The plan of the paper is the following: in Section 2 we collect some preliminary material concerning approximate solutions and the finite-dimensional reduction of the problem, which is then worked-out in detail in Section 3. In Section 4 we study the Hessian of the Euler-Lagrange functional J_τ in $\bar{V}(q,\varepsilon)$, finding a proper base with respect to which the Hessian nearly diagonalizes. Finally, we collect in an Appendix some useful and technical estimates from [30] and a table of constants.

Acknowledgments. A.M. has been supported by the project *Geometric Variational Problems* and *Finanziamento a supporto della ricerca di base* from Scuola Normale Superiore and by MIUR Bando PRIN 2015 2015KB9WPT$_{001}$. He is also member of GNAMPA as part of INdAM.

2 Preliminaries

In this section we collect some background and preliminary material, concerning the variational properties of the problem and some estimates on highly-concentrated approximate solutions of bubble type.

We consider a smooth, closed riemannian manifold $M=(M^n,g_0)$ with volume measure μ_{g_0} and scalar curvature R_{g_0}. Letting $\mathcal{A}=\{u\in W^{1,2}(M,g_0)\mid u\geq 0, u\not\equiv 0,\}$ the *Yamabe invariant* is defined as

$$Y(M,g_0)=\inf_{\mathcal{A}}\frac{\int\left(c_n|\nabla u|_{g_0}^2+R_{g_0}u^2\right)d\mu_{g_0}}{\left(\int u^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{\frac{n-2}{n}}};\qquad c_n=4\frac{n-1}{n-2},$$

and it turns out to depend only on the conformal class of g_0. We will assume from now on that the invariant is positive, namely that (M,g_0) is of *positive Yamabe class*. As a consequence, the *conformal Laplacian* $L_{g_0}=-c_n\Delta_{g_0}+R_{g_0}$ is a positive and self-adjoint operator. Without loss of generality we assume $R_{g_0}>0$ and denote by $G_{g_0}: M\times M\setminus\Delta\longrightarrow\mathbb{R}_+$ the Green's function of L_{g_0}. Considering a conformal metric $g=g_u=u^{\frac{4}{n-2}}g_0$, there holds

$$d\mu_{g_u}=u^{\frac{2n}{n-2}}d\mu_{g_0}\qquad\text{and}\qquad R=R_{g_u}=u^{-\frac{n+2}{n-2}}(-c_n\Delta_{g_0}u+R_{g_0}u)=u^{-\frac{n+2}{n-2}}L_{g_0}u.$$

Note that

$$c\|u\|_{W^{1,2}(M,g_0)}\leq\int u\,L_{g_0}u\,d\mu_{g_0}=\int\left(c_n|\nabla u|_{g_0}^2+R_{g_0}u^2\right)d\mu_{g_0}\leq C\|u\|_{W^{1,2}(M,g_0)}.$$

In particular we may define

$$\|u\|^2=\|u\|_{L_{g_0}}^2:=\int u\,L_{g_0}u\,d\mu_{g_0}$$

and use $\|\cdot\|$ as an equivalent norm on $W^{1,2}(M,g_0)$. Setting $R=R_u$ for $g=g_u=u^{\frac{4}{n-2}}g_0$, we have

$$r=r_u=\int R d\mu_{g_u}=\int uL_{g_0}ud\mu_{g_0}, \tag{2.1}$$

and hence

$$J_\tau(u)=\frac{r}{k_\tau^{\frac{2}{p+1}}} \quad \text{with} \quad k_\tau=\int K\, u^{p+1}d\mu_{g_0}. \tag{2.2}$$

The first- and second-order derivatives of the functional J_τ are given by

$$\partial J_\tau(u)v=\frac{2}{k_\tau^{\frac{2}{p+1}}}\Big[\int L_{g_0}uvd\mu_{g_0}-\frac{r}{k_\tau}\int Ku^pvd\mu_{g_0}\Big]; \tag{2.3}$$

$$\begin{aligned}\partial^2 J_\tau(u)vw=&\frac{2}{k_\tau^{\frac{2}{p+1}}}\Big[\int L_{g_0}vwd\mu_{g_0}-p\frac{r}{k_\tau}\int Ku^{p-1}vwd\mu_{g_0}\Big]\\&-\frac{4}{k_\tau^{\frac{2}{p+1}+1}}\Big[\int L_{g_0}uvd\mu_{g_0}\int Ku^pwd\mu_{g_0}+\int L_{g_0}uwd\mu_{g_0}\int Ku^pvd\mu_{g_0}\Big]\\&+\frac{2(p+3)r}{k_\tau^{\frac{2}{p+1}+2}}\int Ku^pvd\mu_{g_0}\int Ku^pwd\mu_{g_0}.\end{aligned} \tag{2.4}$$

In particular, J_τ is of class $C^{2,\alpha}_{loc}(\mathcal{A})$ and, for $\varepsilon>0$, uniformly Hölder continuous on each set of the form $U_\varepsilon=\{u\in\mathcal{A}\mid \varepsilon<\|u\|,\ J_\tau(u)\le\varepsilon^{-1}\}$.

To understand the blow-up phenomenon, it is convenient to consider some highly concentrated approximate solutions to (1.1). Let us first recall the construction of *conformal normal coordinates* from [23]: given $a\in M$, these are defined as geodesic normal coordinates for a suitable conformal metric $g_a\in[g_0]$. Let r_a be the geodesic distance from a with respect to the metric g_a: with this choice, the expression of the Green's function G_{g_a} for the conformal Laplacian L_{g_a} with pole at $a\in M$, denoted by $G_a=G_{g_a}(a,\cdot)$, simplifies considerably. In Section 6 of [23] one can find the expansion

$$G_a=\frac{1}{4n(n-1)\omega_n}(r_a^{2-n}+H_a),\ r_a=d_{g_a}(a,\cdot),\ H_a=H_{r,a}+H_{s,a}\ \text{ for }\ g_a=u_a^{\frac{4}{n-2}}g_0. \tag{2.5}$$

Here $H_{r,a}\in C^{2,\alpha}_{loc}$, while the *singular* error term is of the type:

$$H_{s,a}=O\begin{pmatrix} r_a & \text{for } n=5\\ \ln r_a & \text{for } n=6\\ r_a^{6-n} & \text{for } n\ge 7\end{pmatrix}.$$

The leading term in $H_{s,a}$ for $n=6$ is $-\frac{|\mathbb{W}(a)|^2}{288c_n}\ln r$, with $\mathbb{W}$ the Weyl tensor. For $\lambda>0$ large define

$$\varphi_{a,\lambda}=u_a\left(\frac{\lambda}{1+\lambda^2\gamma_nG_a^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}},\quad G_a=G_{g_a}(a,\cdot),\quad \gamma_n=(4n(n-1)\omega_n)^{\frac{2}{n-2}}. \tag{2.6}$$

We notice that the constant γ_n is chosen so that

$$\gamma_nG_a^{\frac{2}{2-n}}(x)=d^2_{g_a}(a,x)+o(d^2_{g_a}(a,x))\ \text{ as }\ x\longrightarrow a.$$

Such functions are approximate solutions of (1.1), see Lemma 5.1, and for suitable values of λ depending on τ these are also approximate solutions of (1.5), see Lemma 5.7 for a multi-bubble version.

Notation. For $p\ge1$, $L^p_{g_0}$ will stand for the family of functions of class L^p with respect to the measure $d\mu_{g_0}$. Recall also that for $u\in W^{1,2}(M,g_0)$ we have set $r_u=\int uL_{g_0}ud\mu_{g_0}$, while for $a\in M$ we denote by r_a the geodesic distance from a with respect to the conformal metric g_a introduced before. For a finite set of points $\{a_i\}_i$ of M we will denote by $K_i,\nabla K_i,W_i$, the quantities $K(a_i),\nabla K(a_i),|\mathbb{W}(a_i)|^2$, etc..

For $k,l=1,2,3$ and $\lambda_i>0$, $a_i\in M$, $i=1,\dots,q$, let

(i) $\varphi_i = \varphi_{a_i,\lambda_i}$ and $(d_{1,i}, d_{2,i}, d_{3,i}) = (1, -\lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})$;

(ii) $\phi_{1,i} = \varphi_i,\ \phi_{2,i} = -\lambda_i \partial_{\lambda_i}\varphi_i,\ \phi_{3,i} = \frac{1}{\lambda_i}\nabla_{a_i}\varphi_i$, so $\phi_{k,i} = d_{k,i}\varphi_i$.

With these definitions, the $\phi_{k,i}$'s are uniformly bounded in $W^{1,2}(M, g_0)$ for every value of the λ_i's.

We next recall a standard finite-dimensional reduction for functions that are close in $W^{1,2}$ to a finite sum of bubbles. It is useful to define the following quantity

$$\varepsilon_{i,j} := \left(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j) \right)^{\frac{2-n}{2}}. \tag{2.7}$$

Given $\varepsilon > 0,\ q \in \mathbb{N},\ u \in W^{1,2}(M, g_0)$ and $(\alpha^i, \lambda_i, a_i) \in (\mathbb{R}^q_+, \mathbb{R}^q_+, M^q)$, we set

(i) $A_u(q, \varepsilon) = \{(\alpha^i, \lambda_i, a_i) \mid \underset{i \neq j}{\forall}\ \lambda_i^{-1}, \lambda_j^{-1}, \varepsilon_{i,j}, \left|1 - \frac{r\alpha_i^{\frac{4}{n-2}} K(a_i)}{4n(n-1)k_\tau}\right|, \|u - \alpha^i \varphi_{a_i,\lambda_i}\| < \varepsilon,\ \lambda_i^\tau < 1 + \varepsilon\}$;

(ii) $V(q, \varepsilon) = \{u \in W^{1,2}(M, g_0) \mid A_u(q, \varepsilon) \neq \emptyset\}$,

see (2.1), (2.2) and (2.6). For $A_u(q, \varepsilon)$ to be non-empty, we will always assume that $\tau \ll \varepsilon$. Under the above conditions on the parameters α_i, a_i and λ_i, the functions $\sum_{i=1}^q \alpha^i \varphi_{a_i,\lambda_i}$ constitute a smooth manifold in $W^{1,2}(M, g_0)$, which implies the following well known result (see e.g. [5]).

Proposition 2.1. *Given $\varepsilon_0 > 0$ there exists $\varepsilon_1 > 0$ such that for $u \in V(q, \varepsilon)$ with $\varepsilon < \varepsilon_1$, the problem*

$$\inf_{(\tilde{\alpha}_i, \tilde{a}_i, \tilde{\lambda}_i) \in A_u(q, 2\varepsilon_0)} \int (u - \tilde{\alpha}^i \varphi_{\tilde{a}_i, \tilde{\lambda}_i}) L_{g_0} (u - \tilde{\alpha}^i \varphi_{\tilde{a}_i, \tilde{\lambda}_i}) d\mu_{g_0}$$

admits an unique minimizer $(\alpha_i, a_i, \lambda_i) \in A_u(q, \varepsilon_0)$ and we set

$$\varphi_i = \varphi_{a_i,\lambda_i}, \qquad v = u - \alpha^i \varphi_i, \qquad K_i = K(a_i). \tag{2.8}$$

Moreover, $(\alpha_i, a_i, \lambda_i)$ depends smoothly on u.

The term $v = u - \alpha^i \varphi_i$ is orthogonal to all $\varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i$, with respect to the product

$$\langle \cdot, \cdot \rangle_{L_{g_0}} = \langle L_{g_0} \cdot, \cdot \rangle_{L^2_{g_0}}.$$

Finally, for $u \in V(q, \varepsilon)$ let

$$H_u = H_u(q, \varepsilon) = \langle \varphi_i, \lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{L_{g_0}}}. \tag{2.9}$$

3 Existence of subcritical solutions

Theorem 2, from [30], describes in detail the behaviour as $\tau \to 0$ of blowing-up solutions to (1.5) with uniformly bounded energy and zero weak limit in $V(q, \varepsilon)$, providing positive lower bounds on $\|\partial J_\tau\|$ in a suitable subset of the functional space. In view of this, we can restrict our attention to *centers* $a_1, \ldots, a_q$ close to distinct critical points $x_1, \ldots, x_q$ of K with negative Laplacian: more precisely, for $n \geq 6$ we can assume the following conditions (for $n = 5$ they are slightly modified: see the above-mentioned statement)

(i) $|\alpha_j - \Theta \sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}}| < \frac{\epsilon}{\lambda^3}$;

(ii) $|\frac{\bar{a}_j}{\lambda_j} + c_1 (\nabla^2 K(x_j))^{-} \frac{\nabla \Delta K(x_j)}{\lambda_j^3}| \leq \frac{\epsilon}{\lambda^3}$;

(iii) $|\lambda_j^2 + c_2 \frac{\Delta K(x_j)}{K(x_j)\tau}| \leq \frac{\epsilon}{\lambda^2}$, for $\lambda^2 = \frac{1}{\tau}$ and some $x_j \in \{\nabla K = 0\} \cap \{\Delta K < 0\}$ with $x_i \neq x_j$, $i \neq j$. Here, $\Theta > 0$ (uniformly bounded and bounded away from zero) depends on the function in $V(q, \varepsilon)$, determined in Remark 6.2 of [30].

We next define the following (refined) neigbourhood of potential subcritical blowing-up solutions as

$$\bar{V}(q, \varepsilon) = \{u \in V(q, \varepsilon) \mid \quad \text{(i), (ii) and (iii) above hold true.}\} \tag{3.1}$$

From Lemmata 5.4, 5.5 and 5.6 it follows that (recalling (2.2)) there exists $\tilde{\epsilon} > 0$, tending to zero as $\varepsilon \to 0$, such that

$$|\partial J_\tau(u)| \gtrsim \frac{\tilde{\epsilon}}{\lambda^3} \quad \text{for} \quad u \in V(q, \varepsilon) \setminus \bar{V}(q, \varepsilon) \quad \text{with} \quad k_\tau = 1,$$

so this justifies to look for solutions in $\bar{V}(q, \varepsilon)$ only.

For $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$ with $c < \alpha_i < C$, we have the expansion

$$J_\tau(\alpha^i \varphi_i + v) = J_\tau(\alpha^i \varphi_i) + \partial J_\tau(\alpha^i \varphi_i) v + \frac{1}{2} \partial^2 J_\tau(\alpha^i \varphi_i) v^2 + O(\|v\|^3). \tag{3.2}$$

Recall the uniform positivity of $\partial^2 J_\tau(\alpha^i \varphi_i)$ on $H_u(q, \varepsilon)$ (see (2.9) and [5]), which justifies the following

Definition 3.1. *For $\alpha^i \varphi_i \in V(q, \varepsilon)$ we define $\bar{v}$ as the unique solution of the minimization problem*

$$J_\tau(\alpha^i \varphi_i + \bar{v}) = \min_{v \in H_{\alpha^i \varphi_i}, \|v\| < \varepsilon} J_\tau(\alpha^i \varphi_i + v). \tag{3.3}$$

Lemma 3.1. *Let $\bar{v}$ be as in the above definition. Then one has the following properties*

(i) for $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$ there holds $\|\bar{v}\| \lesssim \frac{1}{\lambda^2} \simeq \tau$;

(ii) if $u \in V(q, \varepsilon)$ is such that $\partial J_\tau(u) = 0$, then $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$ and $u = \alpha^i \varphi_i + \bar{v}$.

Moreover, for $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$ one has that

$$\partial J_\tau(\alpha^i \varphi_i + \bar{v}) = O(\frac{\tilde{\epsilon}}{\lambda^3}), \qquad \text{where } \tilde{\epsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{3.4}$$

Proof. Let us denote by $\Pi_{H_{\alpha^i \varphi_i}}$ the projection onto $H_{\alpha^i \varphi_i}$: we need to solve $\Pi_{H_{\alpha^i \varphi_i}} \partial J_\tau(\alpha^i \varphi_i + \bar{v}) = 0$. Since $\partial^2 J_\tau$ is invertible on this subspace, we can write $\Pi_{H_{\alpha^i \varphi_i}} \partial J_\tau(\alpha^i \varphi_i + \bar{v}) = 0$ as

$$\bar{v} = -(H_{\alpha^i \varphi_i} \partial^2 J_\tau(\alpha^i \varphi_i))^{-1} \left[\partial J_\tau(\alpha^i \varphi_i) + \left(\partial J_\tau(\alpha^i \varphi_i + \bar{v}) - \partial J_\tau(\alpha^i \varphi_i) - \partial^2 J_\tau(\alpha^i \varphi_i) \bar{v} \right) \right].$$

We know from Lemma 5.7 that for $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$ one has $\|\partial J_\tau(\alpha^i \varphi_i)\| \lesssim \frac{1}{\lambda^2}$. Since by Hölder's continuity the quantity within round brackets in the last formula is of order $o(\|\bar{v}\|)$, we can use a contraction argument in a ball of size $\frac{1}{\lambda^2}$ to get the existence of a solution to $\Pi_{H_{\alpha^i \varphi_i}} \partial J_\tau(\alpha^i \varphi_i + \bar{v}) = 0$, with the estimate (i). By the definition of $\bar{v}$ and the above contraction argument we have that

$$\partial^2 J_\tau(\alpha^i \varphi_i) \bar{v} = -\partial J_\tau(\alpha^i \varphi_i) + o(\frac{1}{\lambda^2}) \quad \text{on} \quad \langle \phi_{k,i} \rangle^{\perp_{L_{g_0}}}. \tag{3.5}$$

Testing thus $\partial J_\tau(\alpha^i \varphi_i)$ on $\langle \phi_{k,i} \rangle$, we find from Lemmata 5.4, 5.5 and 5.6, again for $\alpha^i \varphi_i \in \bar{V}(q, \varepsilon)$

$$|\partial J_\tau(\alpha^i \varphi_i) \phi_{k,i}| \leq \frac{\tilde{\epsilon}}{\lambda^3}.$$

It is easy to see from (2.4) and Lemma 5.1 that $\partial^2 J_\tau \phi_{k,i} = o(\frac{1}{\lambda})$, and since $\|\bar{v}\| \lesssim \frac{1}{\lambda^2}$ we have that

$$\partial^2 J_\tau(\alpha^i \varphi_i) \bar{v} \phi_{k,i} = o(\frac{1}{\lambda^3}), \tag{3.6}$$

More in general, one finds also that

$$\partial^2 J(\alpha^i\varphi_i+\theta\bar{v})\bar{v}\phi_{k,j}=o(\frac{1}{\lambda^3})$$

for any $\theta\in(0,1)$. To see this, since $\bar{v}\in\langle\phi_{k,i}\rangle^{\perp_{L_{g_0}}}$, recalling (2.4) it is sufficient to show that

$$\int K(\alpha^i\varphi_i+\theta\bar{v})^{p-1}\bar{v}\varphi_j d\mu_{g_0}-\int K(\alpha^i\varphi_i)^{p-1}\bar{v}\varphi_j d\mu_{g_0}=O(\frac{1}{\lambda^3}).$$

This, in turn, can be verified by dividing the domain of integration into $\{|\bar{v}|\leq\alpha^i\varphi_i\}$ and its complementary set, using Hölder's inequality and the fact that $\|\bar{v}\|\lesssim\frac{1}{\lambda^2}$. Consequently

$$\partial J_\tau(\alpha^i\varphi_i+\bar{v})=\partial J_\tau(\alpha^i\varphi_i+\bar{v})\lfloor_{\langle\phi_{k,i}\rangle}=\partial J_\tau(\alpha^i\varphi_i)\lfloor_{\langle\phi_{k,i}\rangle}+o(\frac{1}{\lambda^3})=O(\frac{\tilde{\epsilon}}{\lambda^3}),$$

where $\tilde{\epsilon}$ tends to zero as ε does. Finally, if a solution $\partial J_\tau(u)=0$ exists on $V(q,\varepsilon)$, then we may write

$$u=\alpha^i\varphi_i+\bar{v}+\tilde{v}\ \text{ with }\ \tilde{v}\perp_{L_{g_0}}\langle\phi_{k,i}\rangle.$$

But then

$$0=\partial J_\tau(\alpha^i\varphi_i+\bar{v}+\tilde{v})\tilde{v}=\partial J_\tau(\alpha^i\varphi_i+\bar{v})\tilde{v}+\partial^2 J_\tau(\alpha^i\varphi_i+\bar{v})\tilde{v}\tilde{v}+o(|\tilde{v}|^2),$$

whence necessarily $\tilde{v}=0$ by uniform positivity of $\partial^2 J_\tau(\alpha^i\varphi_i)$ on $\langle\phi_{k,i}\rangle^{\perp_{L_{g_0}}}$. Thus

$$\partial J_\tau(u)=0\ \text{ with }\ u\in\bar{V}(q,\varepsilon)\quad\Longrightarrow\quad u=\alpha^i\varphi_i+\bar{v}$$

where $\bar{v}=\bar{v}_{\alpha,a,\lambda}$ is the unique solution to (3.3), for which $\alpha^i\varphi_i+\bar{v}\in\bar{V}(q,\varepsilon)$. ■

Remark 3.1. *For $\alpha^i\varphi_i\in\bar{V}(q,\varepsilon)$ and $\nu\in W^{1,2}(M,g_0)$ with $\|\nu\|=1$ it can be shown that*

$$\begin{aligned}\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^i\varphi_i}}{8n(n-1)}\partial J_\tau(\alpha^i\varphi_i)\nu=&-\alpha^i\tau\int_{B_\varepsilon(a_i)}\Big(\varphi_i^{\frac{n+2}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar{c}_1}{c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\Big)\nu d\mu_{g_0}\\&+\alpha^i\tau\int_{B_\varepsilon(a_i)}\Big(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2r^2}{2n}\varphi_i^{\frac{n+2}{n-2}}-\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\Big)\nu d\mu_{g_0}\\&-\alpha^i\int_{B_\varepsilon(a_i)}(\frac{\nabla^2_{k,l}K_i}{2K_i}x^kx^l-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}\nu d\mu_{g_0}+o(\frac{1}{\lambda^2}),\end{aligned}$$

referring to the table at the end of the paper for the definition of the constants. As a consequence of these formulas, one can prove that $\bar{v}$ is indeed of order $\frac{1}{\lambda^2}$ and not smaller, as well as determine the leading order in its expansion. Anyway, due to some cancellation properties, this will not substantially affect the eigenvalues of the Hessian of J_τ at $\alpha^i\varphi_i+\bar{v}$, estimated in the next section.

Let us now set $(d_{1,i},d_{2,i},d_{3,i})=(1,-\lambda_i\partial_{\lambda_i},\frac{1}{\lambda_i}\nabla_{a_i})$, for $i=1,\ldots,q$.

Lemma 3.2. *For $u=\alpha^i\varphi_i+\bar{v}\in\bar{V}(q,\varepsilon)$ there holds*

$$\|\bar{v}\|,\|d_{l,j}\bar{v}\|=O(\frac{1}{\lambda^2}).$$

PROOF. The bound on $\|\bar{v}\|$ follows from Lemma 3.1. Differentiating $\langle\phi_{k,i},\bar{v}\rangle_{L_{g_0}}=0$ we obtain

$$\langle\phi_{k,i},d_{l,j}\bar{v}\rangle_{L_{g_0}}=-\langle d_{l,j}\phi_{k,i},\bar{v}\rangle_{L_{g_0}}=O(\|\bar{v}\|),$$

whence denoting by $\Pi_{\langle\phi_{k,i}\rangle}$ the orthogonal projection onto $\Pi_{\langle\phi_{k,i}\rangle}$ we have $\|\Pi_{\langle\phi_{k,i}\rangle}\bar{v}\| \simeq \frac{1}{\lambda^2}$ due to $\|\bar{v}\| \lesssim \frac{1}{\lambda^2}$. Moreover, since $\partial J_\tau(\alpha^i\varphi_i + \bar{v})v = 0$ for every smoothly-varying vector field $v \in \langle\phi_{k,i}\rangle^{\perp_{L_{g_0}}}$ of unit norm we have

$$0 = d_{l,j}(\partial J_\tau(\alpha^i\varphi_i + \bar{v})v) = \partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})d_{l,j}(\alpha^i\varphi_i + \bar{v})v + \partial J_\tau(\alpha^i\varphi_i + \bar{v})d_{l,j}v$$

and we can estimate the last summand above as

$$\partial J_\tau(\alpha^i\varphi_i + \bar{v})d_{l,j}v = \partial J_\tau(\alpha^i\varphi_i + \bar{v})\Pi_{\langle\phi_{k,i}\rangle}(d_{l,j}v) = O(|\partial J_\tau(\alpha^i\varphi_i + \bar{v})|\|v\|),$$

since $\langle\phi_{k,i}, d_{l,j}v\rangle = \langle d_{l,j}\phi_{k,i}, v\rangle = O(\|v\|)$. Thence, $\partial J_\tau(\alpha^i\varphi_i + \bar{v}) = O(\frac{1}{\lambda^2})$ implies

$$\partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})v d_{l,j}\bar{v} = -\partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})v d_{l,j}(\alpha^i\varphi_i) + O(\frac{1}{\lambda^2}).$$

Then the claim would follow from $\|\Pi_{\langle\phi_{k,i}\rangle}(d_{l,j}\bar{v})\| \simeq \frac{1}{\lambda^2}$, which we had seen before, and the uniform positivity of $\partial^2 J_\tau(\alpha^i\varphi_i)$ on $\langle\phi_{k,i}\rangle^{\perp_{L_{g_0}}}$, provided we show

$$\partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})\phi_{l,j}v = O(\frac{1}{\lambda^2}), \tag{3.7}$$

cf. (4.1) and (4.7) for weaker statements. Let us prove (3.7) for $l = 1$. We next claim that

$$\partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})\varphi_j v = \partial^2 J_\tau(\alpha^i\varphi_i)\varphi_j v + O(\frac{1}{\lambda^2}).$$

From (2.4), since $v \in \langle\phi_{k,i}\rangle^{\perp_{L_{g_0}}}$, it is sufficient to show that we must show (see the proof of Lemma 3.1)

$$\int K(\alpha^i\varphi_i + \bar{v})^{p-1}v\varphi_j d\mu_{g_0} - \int K(\alpha^i\varphi_i)^{p-1}v\varphi_j d\mu_{g_0} = O(\frac{1}{\lambda^2}).$$

Again, this can be seen considering the set $\{|\bar{v}| \leq \alpha^i\varphi_i\}$ and its complementary, using Hölder's inequality and $\|\bar{v}\| \lesssim \frac{1}{\lambda^2}$. Thus, from the above claim and (2.4) we find, due to the orthogonalities $\langle\phi_{k,i}, v\rangle_{L_{g_0}} = 0$,

$$\begin{aligned}\partial^2 J_\tau(\alpha^i\varphi_i)\varphi_i v = & -\frac{2p}{(k_\tau)_{\alpha^i\varphi_i}^{\frac{2}{p+1}}}\frac{r_{\alpha^i\varphi_i}}{(k_\tau)_{\alpha^i\varphi_i}}\int K(\alpha^i\varphi_i)^{p-1}\varphi_j v d\mu_{g_0} \\ & -\frac{4}{(k_\tau)_{\alpha^i\varphi_i}^{\frac{2}{p+1}+1}}\int L_{g_0}(\alpha^i\varphi_i)\varphi_j d\mu_{g_0}\int K(\alpha^i\varphi_i)^p v d\mu_{g_0} \\ & +\frac{2(p+3)r_{\alpha^i\varphi_i}}{(k_\tau)_{\alpha^i\varphi_i}^{\frac{2}{p+1}+2}}\int K(\alpha^i\varphi_i)^p\varphi_j d\mu_{g_0}\int K(\alpha^i\varphi_i)^p v d\mu_{g_0}.\end{aligned}$$

By definition of $\bar{V}(q, \varepsilon)$ we have $\tau \simeq \frac{1}{\lambda^2}$ and recalling (5.2) and (5.5) we may simplify this to

$$\begin{aligned}\partial^2 J_\tau(\alpha^i\varphi_i)\varphi_j v \simeq & -4n(n-1)\frac{n+2}{n-2}\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\int K(\alpha^i\varphi_i)^{\frac{4}{n-2}}\varphi_j v d\mu_{g_0} \\ & -\frac{2}{\bar{c}_0\alpha_{K,\tau}^{\frac{2n}{n-2}}}\int L_{g_0}(\alpha^i\varphi_i)\varphi_j d\mu_{g_0}\int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}v d\mu_{g_0} \\ & +4n(n-1)\frac{(\frac{n+2}{n-2}+3)\alpha^2}{\bar{c}_0(\alpha_{K,\tau}^{\frac{2n}{n-2}})^2}\int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0}\int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}v d\mu_{g_0}\end{aligned}$$

up to error $O(\frac{1}{\lambda^2})$. Moreover, from (5.3) and (5.4) we have

$$\int L_{g_0}(\alpha^i\varphi_i)\varphi_j d\mu_{g_0} = 4n(n-1)\bar{c}_0\alpha_j + O(\frac{1}{\lambda^2})$$

and since $d(a_i, a_j) \simeq 1$, we find by expanding and using Lemma 5.2

$$\int K(\alpha^i\varphi_i)^{\frac{4}{n-2}}\varphi_j v d\mu_{g_0} = \alpha_j^{\frac{4}{n-2}}\int K\varphi_j^{\frac{n+2}{n-2}} v d\mu_{g_0}; \qquad \int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}} v d\mu_{g_0} = \sum_i \alpha_i^{\frac{n+2}{n-2}}\int K\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0};$$

$$\int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0} = \alpha_j^{\frac{n+2}{n-2}}\int K\varphi_j^{\frac{2n}{n-2}} d\mu_{g_0}; \qquad \int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}} v d\mu_{g_0} = \sum_i \alpha_i^{\frac{n+2}{n-2}}\int K\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0},$$

up to errors of order $O(\frac{1}{\lambda^2})$. Therefore, since $\frac{|\nabla K_i|}{\lambda_i} = O(\frac{1}{\lambda^2})$ due to (3.1), we obtain

$$\begin{aligned}\partial^2 J_\tau(\alpha^i\varphi_i)\varphi_j v \simeq & -4n(n-1)\frac{n+2}{n-2}\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}K_i\alpha_j^{\frac{4}{n-2}}\int\varphi_j^{\frac{n+2}{n-2}} v d\mu_{g_0} \\ & - \frac{8n(n-1)\alpha_j}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\sum_i K_i\alpha_i^{\frac{n+2}{n-2}}\int\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0} \\ & + 4n(n-1)\frac{(\frac{n+2}{n-2}+3)\alpha^2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^2}\alpha_j^{\frac{n+2}{n-2}}K_j\sum_i K_i\alpha_i^{\frac{n+2}{n-2}}\int\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0}\end{aligned}$$

up to an error $O(\frac{1}{\lambda^2})$. Therefore using again (3.1) we have

$$\partial^2 J_\tau(\alpha^i\varphi_i)\varphi_j v \simeq -\frac{n+2}{n-2}\int\varphi_j^{\frac{n+2}{n-2}} v d\mu_{g_0} - 2\sum_i\frac{\alpha_i\alpha_j}{\alpha^2}\int\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0} + (\frac{n+2}{n-2}+3)\sum_i\frac{\alpha_i\alpha_j}{\alpha^2}\int\varphi_i^{\frac{n+2}{n-2}} v d\mu_{g_0}$$

up to the same error. Thus, $\partial^2 J_\tau(\alpha^i\varphi_i)\varphi_j v = O(\frac{1}{\lambda^2})$ using (5.6), obtaining (3.7) for $l = 1$.

For $l = 2, 3$ one can reason analogously. ■

Theorem 1 follows from the next proposition, based on the analysis of Section 4, and Corollary 4.1.

Proposition 3.1. *Let $n \geq 5$ and let $K : M \to \mathbb{R}$ be a positive Morse function satisfying* (1.3). *Then, for every subset $\{x_1, \ldots, x_q\}$ of $\{\nabla K = 0\} \cap \{\Delta K < 0\}$, as $\tau \to 0$ there exists a unique*

$$u = \alpha^i\varphi_{a_i,\lambda_i} + \bar{v} \in V(q,\varepsilon) \ \text{ with } \ \|u\|^2_{L_{g_0}} = 1, \ d(a_i, x_i) = o(1) \ \text{ and } \ \partial J_\tau(u) = 0.$$

Proof. Due to (3.4), we have

$$|\partial J| \leq \frac{\tilde{\epsilon}}{\lambda^3} \ \text{ on } \ \bar{V}(q,\varepsilon) \ \text{ and } \ |\partial J| \geq \frac{\hat{\epsilon}}{\lambda^3} \ \text{ on } \ \partial\bar{V}(q,\varepsilon)$$

as long as $c < \alpha_j < C$. Thus, by (*ii*) in Lemma 3.1, it is sufficient to look for critical points in the set

$$\tilde{\mathcal{C}} := \{\tilde{u}(\alpha,\lambda,a) := \alpha^i\varphi_i + \bar{v}(\alpha,\lambda,a) \in \bar{V}(q,\varepsilon) \mid \|\tilde{u}\|^2_{L_{g_0}} = 1\},$$

which is a smooth $(3(n+2)-1)$-dimensional manifold in $W^{1,2}(M, g_0)$.

Vice-versa, we claim that a critical point of $J_\tau\lfloor_{\tilde{\mathcal{C}}}$ is indeed a critical point of J_τ. In fact, by Lagrange multiplier's rule, the gradient of J_τ at a constrained critical point $\tilde{u}_0$ must be orthogonal to $\tilde{\mathcal{C}}$. Since J_τ is dilation-invariant, its gradient on $\mathcal{C}$ must be tangent to the unit sphere in the $\|\cdot\|_{L_{g_0}}$ norm. On the other hand, by construction of $\bar{v}$, the gradient of J_τ at $\tilde{u}_0$ is tangent to $\mathcal{C} := \{\alpha^i\varphi_i \in \bar{V}(q,\varepsilon) \mid \|u\|^2_{L_{g_0}} = 1\}$ at the point u_0 such that $\tilde{u}_0 = u_0 + \bar{v}_0$ (with obvious notation). By the estimate on the derivatives of $\bar{v}$ in Lemma 3.2, $T_{\tilde{u}_0}\tilde{\mathcal{C}}$ is nearly parallel to $T_{u_0}\mathcal{C}$, which implies that $\partial J_\tau(\tilde{u}_0) = 0$, as desired.

It remains to prove existence and uniqueness of critical points of $J_\tau\lfloor_{\tilde{\mathcal{C}}}$. For the existence part, one can use the expansions in Lemmas 5.4, 5.5 and 5.6, together with the definition of $\bar{V}(q,\varepsilon)$ to show that ∂J_τ is non-vanishing on the boundary of $\tilde{\mathcal{C}}$. For example (see (iii) in the definition of $\bar{V}(q,\varepsilon)$), suppose

$$\lambda_j^2 = -c_2\frac{\Delta K(x_j)}{K(x_j)\tau} + \frac{\varepsilon}{\lambda^2}; \qquad \frac{1}{\lambda^2} = \tau.$$

From Lemma 5.5 one deduces that there exists $\tilde{\epsilon} > 0$, tending to zero as $\varepsilon \to 0$, such that

$$\lambda_j \partial_{\lambda_j} J_\tau(\alpha^i \varphi_i) > \frac{\tilde{\epsilon}}{\lambda^3}.$$

From Lemmas 3.1 and 3.2 one has also that

$$\lambda_j \partial_{\lambda_j} J_\tau(u(\alpha, \lambda, a)) > \frac{1}{2} \frac{\tilde{\epsilon}}{\lambda^3},$$

with a similar reversed inequality, with opposite sign, if $\lambda_j^2 = -c_2 \frac{\Delta K(x_j)}{K(x_j)\tau} - \frac{\varepsilon}{\lambda^2}$. Analogous estimates can be derived for the $\alpha-$ and $a-$derivatives, yielding that the degree of ∂J_τ on $\tilde{\mathcal{C}}$ is well-defined and non-zero. This shows the existence of a critical point for $J_\tau \lfloor_{\tilde{\mathcal{C}}}$, which is (freely) critical for J_τ by the above discussion. Since by construction the negative part of the above solutions is small in $W^{1,2}$ norm, it is possible to show from Sobolev's inequality that it has to vanish identically, so full positivity follows then from the maximum principle.

Uniqueness follows from Lemma 3.2 and Proposition 4.1, implying the strict convexity or concavity of $J_\tau \lfloor_{\tilde{\mathcal{C}}}$ with respect to all parameters α's, λ's and the coordinates of the points a_i, provided they are chosen so that $\nabla^2 K(x_i)$ is diagonal. ■

4 The second variation

Let $\bar{V}(q, \varepsilon)$ be the open set defined in (3.1): the aim of this section is to find there a nearly diagonal form of the second differential of J_τ. Let us recall our notation from Section 2, and in particular that of the orthogonal space H_u in (2.9).

Proposition 4.1. *For $\alpha^i \varphi_i + \bar{v} \in \bar{V}(q, \varepsilon)$, consider the decomposition*

$$W^{1,2}(M, g_0) = H_{\alpha^i \varphi_i} \oplus \langle \varphi_i \rangle_{1 \le i \le q} \oplus \langle \lambda_i \partial_{\lambda_i} \varphi_i \rangle_{1 \le i \le q} \oplus \langle \frac{\nabla_{a_i}}{\lambda_i} \varphi_i \rangle_{1 \le i \le q} =: \mathcal{V} \oplus X_\alpha \oplus X_\lambda \oplus X_a.$$

Then there exists a basis $\mathbb{B}$ of $W^{1,2}(M, g_0)$, with elements in the subspaces of the above decomposition, such that the coefficients of the the second differential of J_τ with respect to $\mathbb{B}$ have the form

$$[\partial^2 J_\tau(\alpha^k \varphi_k + \bar{v})]_{\mathbb{B}} = \frac{1}{\lambda^2} \begin{pmatrix} \mathcal{V}_+ & 0 & 0 & 0 \\ 0 & \mathbb{A}_{q-1,0} & 0 & 0 \\ 0 & 0 & \mathbb{A}_+ & \\ 0 & 0 & 0 & -\frac{\nabla^2 K}{K} \end{pmatrix} + o(\frac{1}{\lambda^2}), \qquad \textit{where:}$$

(i) $\mathcal{V}_+$ represents the coefficients of a symmetric, positive-definite operator on $\mathcal{V}$ with eigenvalues uniformly bounded away from zero;

(ii) $\mathbb{A}_{q-1,0}$ has $q-1$ negative eigenvalues uniformly bounded away from zero and one-dimensional kernel;

(iii) $\mathbb{A}_+$ is positive-definite, with eigenvalues uniformly bounded away from zero;

(iv) $-\frac{\nabla^2 K}{K}$ stands for the diagonal matrix $-(\frac{\nabla^2 K_i}{K_i})_{i=1,\dots,q}$.

Remark 4.1. *The basis elements in $\mathbb{B}$ corresponding to the first two blocks have norms of order $\frac{1}{\lambda^2}$, while the ones corresponding to the last two blocks have norm of order 1. We made this choice to guarantee the* off-diagonal *terms in the above matrix to be of order $o(\frac{1}{\lambda^2})$.*

Proof of Proposition 4.1. We wish to analyse (2.4) for $u = \alpha^i \varphi_i + \bar{v} \in \bar{V}(q, \varepsilon)$. Recall that

$$W^{1,2}(M, g_0) = \langle \phi_{k,i} \rangle_{k,i} \oplus H_{\alpha^i \varphi_i},$$

see Section 2. We then choose a basis $\{\nu_0, \nu_1, \nu_2, \ldots\}$ for $H_{\alpha^i\varphi_i}$ which is orthonormal with respect to $\langle\cdot,\cdot\rangle^{\perp_{L_{g_0}}}$ and for $\lambda \simeq \lambda_1 \simeq \ldots \simeq \lambda_q \simeq \frac{1}{\sqrt{\tau}}$ define

$$\mathbb{B} = \{\tilde{\phi}_{k,i}, \tilde{\nu}_j\} := \{\frac{\varphi_i}{\lambda}, \lambda_i\partial_{\lambda_i}\varphi_i, \frac{\nabla_{a_i}}{\lambda_i}\varphi_i, \frac{\nu_j}{\lambda}\}; \qquad k = 1,2,3, \quad i = 1,\ldots,q.$$

It is not hard to see that, with this choice, the coefficients $[\partial^2 J_\tau(\alpha^k\varphi_k + \bar{v})]_{\mathbb{B}}$ are all of order $O(\frac{1}{\lambda^2})$, and our goal is to make their estimates more precise, considering different matrix blocks.

First block. The fact that $\partial^2 J_\tau(\alpha^i\varphi_i)$ is (uniformly) positive-definite on $H_{\alpha^i\varphi_i}$ is well-known, see e.g. [5]. The positivity of $\partial^2 J_\tau(\alpha^i\varphi_i + \varepsilon_v)$ on the same subspace follows from the Hölder continuity of the second differential and the fact that $\|\bar{v}\| = O(\frac{1}{\lambda^2})$.

First two blocks. Testing the second differential with $\tilde{\nu}_i$ and $\tilde{\phi}_{1,j} = \frac{\varphi_j}{\lambda}$ we get

$$\partial^2 J_\tau(\alpha^i\varphi_i + \bar{v})\tilde{\nu}_i\tilde{\phi}_{1,j} = o(\frac{1}{\lambda^2}) \tag{4.1}$$

using the orthogonality $\langle\tilde{\nu}_i, \tilde{\phi}_{1,j}\rangle_{L_{g_0}} = 0$, Lemma 5.1 and the fact that $\|\bar{v}\| \lesssim \frac{1}{\lambda^2}$. Moreover, from (2.4) and the fact that $\tilde{\phi}_{1,i}$ is of order $\frac{1}{\lambda}$, we find

$$\partial^2 J_\tau(\alpha^k\varphi_k + \bar{v})\tilde{\phi}_{1,i}\tilde{\phi}_{1,j} = \frac{16n(n-1)\bar{c}_0^{\frac{2}{n}}}{(n-2)(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\lambda^2}(-\delta_{k,l} + \frac{\alpha_k\alpha_l}{\alpha^2}) = \mathbb{A}_{i,j}; \qquad c_0 = \int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n}, \tag{4.2}$$

up to an error of order $o(\frac{1}{\lambda^2})$. Let us compare the above expression to

$$f(\alpha) = \frac{\alpha^2}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}; \qquad \alpha := \sum_{i=1}^q \alpha_i^2, \quad \alpha_K^{\frac{2n}{n-2}} := \sum_{i=1}^q K_i\alpha_i^{\frac{2n}{n-2}},$$

with first- and second-order derivatives given by

$$\frac{1}{2}\partial_{\alpha_i}f(\alpha) = \frac{\alpha_i}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} - \frac{\alpha^2K_i\alpha_i^{\frac{n+2}{n-2}}}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}} = \frac{\alpha_i}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}(1 - \frac{\alpha^2}{\alpha_K^{\frac{2n}{n-2}}}K_i\alpha_i^{\frac{4}{n-2}});$$

$$\begin{aligned}\frac{1}{2}\partial_{\alpha_i}\partial_{\alpha_j}f(\alpha) = {}& \delta_{i,j}\frac{1}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}(1 - \frac{n+2}{n-2}\frac{\alpha^2}{\alpha_K^{\frac{2n}{n-2}}}K_i\alpha_i^{\frac{4}{n-2}}) + 2\frac{\alpha_i\alpha_j}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}\frac{\alpha^2}{\alpha_K^{\frac{2n}{n-2}}}K_i\alpha_i^{\frac{4}{n-2}}K_j\alpha_j^{\frac{4}{n-2}} \\ & - 2\frac{\alpha_i\alpha_j}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}(K_i\alpha_i^{\frac{4}{n-2}} + K_j\alpha_j^{\frac{4}{n-2}}) + \frac{2n}{n-2}\frac{\alpha^2}{(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}+2}}K_j\alpha_j^{\frac{n+2}{n-2}}K_i\alpha_i^{\frac{n+2}{n-2}}.\end{aligned}$$

The function f is scaling invariant and restricted to $\{\alpha_K^{\frac{2n}{n-2}} = 1\}$ attains its maximum at $(\alpha_i)_i$ satisfying

$$\frac{\alpha^2}{\alpha_K^{\frac{2n}{n-2}}}K_i\alpha_i^{\frac{4}{n-2}} = 1 \qquad \text{for all } \ i = 1,\ldots,q,$$

where we have

$$\frac{1}{2}\partial_{\alpha_i}\partial_{\alpha_j}f(\alpha) = \frac{4}{(n-2)(\alpha_K^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}(-\delta_{i,j} + \frac{\alpha_i\alpha_j}{\alpha^2}). \tag{4.3}$$

Comparing (4.2) and (4.3) we conclude, with obvious notation

$$[\partial^2 J_\tau(\alpha^k\varphi_k + \bar{v})]_{\mathbb{B}} = \begin{pmatrix} \frac{1}{\lambda^2}\mathbb{V}_+ & 0 & \partial^2 J_\tau\tilde{\nu}\tilde{\phi}_2 & \partial^2 J_\tau\tilde{\nu}\tilde{\phi}_3 \\ 0 & \frac{1}{\lambda^2}\mathbb{A}_{q-1,0} & \partial^2 J_\tau\tilde{\phi}_1\tilde{\phi}_2 & \partial^2 J_\tau\tilde{\phi}_1\tilde{\phi}_3 \\ \partial^2 J_\tau\tilde{\phi}_2\tilde{\nu} & \partial^2 J_\tau\tilde{\phi}_2\tilde{\phi}_1 & \partial^2 J_\tau\tilde{\phi}_2\tilde{\phi}_2 & \partial^2 J_\tau\tilde{\phi}_2\tilde{\phi}_3 \\ \partial^2 J_\tau\tilde{\phi}_3\tilde{\nu} & \partial^2 J_\tau\tilde{\phi}_3\tilde{\phi}_1 & \partial^2 J_\tau\tilde{\phi}_3,\tilde{\phi}_2 & \partial^2 J_\tau\tilde{\phi}_3\tilde{\phi}_3 \end{pmatrix} + o(\frac{1}{\lambda^2}).$$

Terms off 2x2 blocks. Let us consider next the interaction of $\tilde{\nu}_i$ with $\tilde{\phi}_{k,j} = \phi_{k,j}$ for $k = 2, 3$. Since

$$\bar{v} = O(\frac{1}{\lambda^2}), \quad \tilde{\nu}_i = O(\frac{1}{\lambda}), \quad \langle \varphi_k, \phi_{k,j} \rangle_{L_{g_0}} = O(\frac{1}{\lambda^2}) \quad \text{and} \quad \langle \nu_i, \phi_{k,j} \rangle_{L_{g_0}} = 0$$

we simply find for (2.4)

$$\partial^2 J_\tau(\alpha^l \varphi_l + \bar{v}) \tilde{\nu}_i \tilde{\phi}_{j,k} = \partial^2 J_\tau(\alpha^l \varphi_l) \tilde{\nu}_i \tilde{\phi}_{j,k} = -\frac{2pr_{\alpha^i \varphi_i}}{k_\tau^{\frac{2}{p+1}+1}} \int K(\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0}, \tag{4.4}$$

up to an error of order $o(\frac{1}{\lambda^2})$. Indeed, by (2.4), the crucial estimates needed to verify (4.4) are

$$\int K(\alpha^l \varphi_l)^p \tilde{\nu}_i d\mu_{g_0} = o(\frac{1}{\lambda^2}) = \int K(\alpha^l \varphi_l)^p \tilde{\phi}_{k,j} d\mu_{g_0}. \tag{4.5}$$

These however follow easily by expansion and interaction estimates using

$$\langle \varphi_l, \phi_{k,j} \rangle_{L_{g_0}} = O(\frac{1}{\lambda^2}), \qquad \langle \nu_i, \phi_{k,j} \rangle_{L_{g_0}} = 0, \qquad L_{g_0} \varphi_l = 4n(n-1) \varphi_l^{\frac{n+2}{n-2}} + o(1) \text{ in } W^{-1,2}(M, g_0)$$

and Lemma 5.3. For the remaining integral in (4.4), we then have

$$\begin{aligned}
\int K(\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} = & K_j \int (\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} + o(\frac{1}{\lambda^2}) \\
= & K_j \int_{\{\varphi_j > \sum_{j \neq l} \alpha^l \varphi_l\}} (\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} + O(\frac{1}{\lambda} \sum_{j \neq l} \| \varphi_l^{p-1} \varphi_j \|_{L^{\frac{p+1}{p}}}) + o(\frac{1}{\lambda^2}) \\
= & K_j \alpha_j^{p-1} \int_{\{\varphi_j > \sum_{j \neq l} \alpha^l \varphi_l\}} \varphi_j^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} + O(\frac{1}{\lambda} \sum_{j \neq l} \| \varphi_l^{p-1} \varphi_j + \varphi_l \varphi_j^{p-1} \|_{L^{\frac{p+1}{p}}}) + o(\frac{1}{\lambda^2})
\end{aligned} \tag{4.6}$$

and therefore, using Lemma 5.2 (with $p = \frac{n+2}{n-2} - \tau$)

$$\int K(\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} = K_j \alpha_j^{p-1} \int \varphi_j^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} + o(\frac{1}{\lambda^2}).$$

Then, since $\|\tilde{\nu}_i\| = O(\frac{1}{\lambda}), \tau = O(\frac{1}{\lambda^2})$ and $\varepsilon_{r,s} = O(\frac{1}{\lambda^{n-2}})$, we find

$$\int K(\alpha^l \varphi_l)^{p-1} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} = K_j \alpha_j^{\frac{4}{n-2}} \int \varphi_j^{\frac{4}{n-2}} \tilde{\nu}_i \tilde{\phi}_{j,k} d\mu_{g_0} + o(\frac{1}{\lambda^2}) = o(\frac{1}{\lambda^2}),$$

where the last inequality follows from Lemma 5.1 and $\langle \phi_{k,j}, \tilde{\nu}_i \rangle_{L_{g_0}} = 0$. Thus

$$\partial^2 J_\tau(\alpha^l \varphi_l + \bar{v}) \tilde{\nu}_i \tilde{\phi}_{k,j} = o(\frac{1}{\lambda^2}) \text{ for } k = 2, 3. \tag{4.7}$$

By exactly the same arguments with $\tilde{\phi}_{1,i} = O(\frac{1}{\lambda})$ as for (4.5) there holds

$$\partial^2 J_\tau(\alpha^l \varphi_l + \bar{v}) \tilde{\phi}_{1,i} \tilde{\phi}_{k,j} = \partial^2 J_\tau(\alpha^l \varphi_l + \bar{v}) \frac{\phi_{1,i}}{\lambda} \phi_{k,j} = \frac{1}{\lambda} \partial^2 J_\tau(\alpha^l \varphi_l) \varphi_i \phi_{k,j} = o(\frac{1}{\lambda^2})$$

for $k = 2, 3$. Thus we arrive at

$$[\partial^2 J_\tau(\alpha^l \varphi_l + \bar{v})]_{\mathbb{B}} = \begin{pmatrix} \frac{1}{\lambda^2} \mathbb{V}_+ & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda^2} \mathbb{A}_{q-1,0} & 0 & 0 \\ 0 & 0 & \partial^2 J_\tau \tilde{\phi}_2 \tilde{\phi}_2 & \partial^2 J_\tau \tilde{\phi}_2 \tilde{\phi}_3 \\ 0 & 0 & \partial^2 J_\tau \tilde{\phi}_3 \tilde{\phi}_2 & \partial^2 J_\tau \tilde{\phi}_3, \tilde{\phi}_3 \end{pmatrix} + o(\frac{1}{\lambda^2}).$$

Last 2x2 block. We are left with the estimate of

$$\partial^2 J_\tau(\alpha^k\varphi_k+\bar v)\tilde\phi_{k,i}\tilde\phi_{l,j}=\partial^2 J_\tau(\alpha^k\varphi_k+\bar v)\phi_{k,i}\phi_{l,j}$$

for $k,l=2,3$. Using the fact that

$$\int\phi_{k,i}L_{g_0}(\alpha^m\varphi_m+\bar v)d\mu_{g_0}=o(\frac{1}{\lambda})=\int\phi_{k,i}K(\alpha^m\varphi_m+\bar v)^p d\mu_{g_0}\ \text{ for }\ k=2,3,$$

which follows from $\|\bar v\|=O(\frac{1}{\lambda^2})$, Lemma 5.1 and Lemma 5.2, we find for (2.4)

$$\begin{aligned}\partial^2 J_\tau&(\alpha^m\varphi_m+\bar v)\phi_{k,i}\phi_{l,j}\\ &=\frac{2}{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}\int\left[\phi_{k,i}L_{g_0}\phi_{l,j}-p\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}K(\alpha^m\varphi_m+\bar v)^{p-1}\phi_{k,i}\phi_{l,j}\right]d\mu_{g_0}\\ &=:\frac{2}{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}I=:\frac{2}{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}(I_1-I_2)=\frac{2}{(c_0\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}(I_1-I_2)+o\left(\frac{1}{\lambda^2}\right).\end{aligned}\tag{4.8}$$

In the latter formula, recalling (2.2) and the definition of $\bar V(q,\varepsilon)$, we have used the fact that

$$(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}=(c_0\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}+o(1)$$

and that both I_1,I_2 are of order $\frac{1}{\lambda^2}$. Let us first compute I_2, for which we clearly have

$$I_2=p\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}\int K(\alpha^m\varphi_m)^{p-1}\phi_{k,i}\phi_{l,j}d\mu_{g_0}+p(p-1)\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}\int K(\alpha^m\varphi_m)^{p-2}\phi_{k,i}\phi_{l,j}\bar v d\mu_{g_0}$$

up to an error $o(\frac{1}{\lambda^2})$, as $\|\bar v\|=O(\frac{1}{\lambda^2})$, and therefore still up to an error $o(\frac{1}{\lambda^2})$

$$\begin{aligned}I_2=&p\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}\int K(\alpha^m\varphi_m)^{p-1}\phi_{k,i}\phi_{l,j}d\mu_{g_0}\\ &+4n(n-1)\frac{n+2}{n-2}\frac{4}{n-2}\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\int K(\alpha^m\varphi_m)^{\frac{6-n}{n-2}}\phi_{k,i}\phi_{l,j}\bar v d\mu_{g_0}.\end{aligned}$$

As due to $d(a_i,a_j)\simeq 1$ for $i\neq j$, the interactions terms $\varepsilon_{i,j}$ in (2.7) are of order $\frac{1}{\lambda^{n-2}}=o(\frac{1}{\lambda^2})$, we find

$$\begin{aligned}I_2=&p\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}\delta_{i,j}\alpha_i^{p-1}\int K\varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\ &+4n(n-1)\frac{n+2}{n-2}\frac{4}{n-2}\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\delta_{i,j}\alpha_i^{\frac{6-n}{n-2}}\int K\varphi_i^{\frac{6-n}{n-2}}\phi_{k,i}\phi_{l,i}\bar v d\mu_{g_0}\end{aligned}$$

up to an error $o(\frac{1}{\lambda^2})$. Using (3.1), up to the same error we may simplify this to

$$\begin{aligned}I_2=&p\frac{r_{\alpha^m\varphi_m+\bar v}}{(k_\tau)_{\alpha^m\varphi_m+\bar v}}\delta_{i,j}K_i\alpha_i^{p-1}\int\varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0}+4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\int_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\ &+4n(n-1)\frac{n+2}{n-2}\frac{4}{n-2}\delta_{i,j}\alpha_i^{-1}\int\varphi_i^{\frac{6-n}{n-2}}\phi_{k,i}\phi_{l,i}\bar v d\mu_{g_0}\end{aligned}$$

for some $\varepsilon>0$ small and fixed. Moreover, by orthogonality and (5.12)

$$\frac{r_{\alpha^i\varphi_i+\bar v}}{(k_\tau)_{\alpha^i\varphi_i+\bar v}}=\frac{r_{\alpha^i\varphi_i}}{(k_\tau)_{\alpha^i\varphi_i}}=4n(n-1)\frac{\alpha^2}{\alpha_{K,\theta}^{p+1}}(1-(\frac{\bar c_1}{\bar c_0}-\frac{\tilde c_1}{\tilde c_2}\frac{\bar c_2}{\bar c_0})\tau)+o(\frac{1}{\lambda^2}),$$

whence by (3.1) and the fact that $p = \frac{n+2}{n-2} - \tau$ we arrive at

$$\begin{aligned} I_2 =& 4n(n-1)\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\int \varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0} \\ &+ 4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2 K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}d\mu_{g_0} \\ &+ 4n(n-1)\frac{n+2}{n-2}\frac{4}{n-2}\delta_{i,j}\alpha_i^{-1}\int \varphi_i^{\frac{6-n}{n-2}}\phi_{k,i}\phi_{l,i}\bar{v}d\mu_{g_0}. \end{aligned}$$

Let us compute the last integral above, which is of order $O(\frac{1}{\lambda^2})$, as it is $\|\bar{v}\|$. There holds

$$\begin{aligned} \frac{4}{n-2}\int \varphi_i^{\frac{6-n}{n-2}}\phi_{k,i}\phi_{l,i}\bar{v}d\mu_{g_0} =& \int d_{k,i}\varphi_i^{\frac{4}{n-2}}\phi_{l,i}\bar{v}d\mu_{g_0} \\ =& d_{k,i}\int \varphi_i^{\frac{4}{n-2}}\phi_{l,i}\bar{v}d\mu_{g_0} - \int \varphi_i^{\frac{4}{n-2}}d_{k,i}\phi_{l,i}\bar{v}d\mu_{g_0} - \int \varphi_i^{\frac{4}{n-2}}\phi_{l,i}d_{k,i}\bar{v}d\mu_{g_0}. \end{aligned}$$

Due to orthogonality, the first integral above is of order $o(\frac{1}{\lambda^2})$ and denoting by

$$\widehat{w} = \Pi_{\langle \phi_{k,i}\rangle^{\perp_{L_{g_0}}}} w \ \text{ for } \ w \in W^{1,2}(M, g_0) \tag{4.9}$$

the orthogonal projection onto $\langle \phi_{k,i}\rangle^{\perp_{L_{g_0}}}$ we have, up to an error $o(\frac{1}{\lambda^2})$

$$\int \varphi_i^{\frac{4}{n-2}}d_{k,i}\phi_{l,i}\bar{v}d\mu_{g_0} = \int \varphi_i^{\frac{4}{n-2}}\widehat{d_{k,i}\phi_{l,i}}\bar{v}d\mu_{g_0}$$

due to the orthogonalities $\langle \bar{v}, \phi_{k,i}\rangle_{L_{g_0}} = 0$ and the fact that $\|\bar{v}\| = O(\frac{1}{\lambda^2})$. Hence, using the same notation as in (4.9), we arrive at

$$\begin{aligned} I_2 =& 4n(n-1)\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\int \varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0} \\ &+ 4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\int_{B_c(a_i)}\frac{\nabla^2 K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}d\mu_{g_0} \\ &- 4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\alpha_i^{-1}(\int \varphi_i^{\frac{4}{n-2}}\widehat{d_{k,i}\phi_{l,i}}\bar{v}d\mu_{g_0} + \int \varphi_i^{\frac{4}{n-2}}\phi_{l,i}d_{k,i}\bar{v}d\mu_{g_0}). \end{aligned}$$

Due to the fact that $\|\bar{v}\| = O(\frac{1}{\lambda^2})$ we have, still up to a $o(\frac{1}{\lambda^2})$

$$\partial^2 J_\tau(\alpha^m\varphi_m)\bar{v} = \frac{8n(n-1)}{(\bar{c}_0\alpha_{K,\tau}^{p+1})^{\frac{n-2}{n}}}\left(\frac{L_{g_0}}{4n(n-1)}\bar{v} - \frac{n+2}{n-2}\sum_m \varphi_m^{\frac{4}{n-2}}\bar{v}\right),$$

and we recall from (3.5) that $\partial^2 J_\tau(\alpha^m\varphi_m)\bar{v} = -\partial J_\tau(\alpha^m\varphi_m) + o(\frac{1}{\lambda^2})$ on $\langle \phi_{l,j}\rangle^{\perp_{L_{g_0}}}$. From this we deduce, again by smallness of interactions terms $\varepsilon_{i,j}$

$$\frac{n+2}{n-2}\int \varphi_i^{\frac{4}{n-2}}\widehat{d_{k,i}\phi_{l,i}}\bar{v}d\mu_{g_0} = \frac{(\bar{c}_0\alpha_{K,\tau}^{p+1})^{\frac{n-2}{n}}}{8n(n-1)}\partial J_\tau(\alpha^m\varphi_m)\widehat{d_{k,i}\phi_{l,i}} + \frac{\langle \bar{v}, \widehat{d_{k,i}\phi_{l,i}}\rangle_{L_{g_0}}}{4n(n-1)}$$

and, by orthogonality and Lemma 5.1, there holds up to an error $o(\frac{1}{\lambda^2})$

$$\begin{aligned} \langle \bar{v}, \widehat{d_{k,i}\phi_{l,i}}\rangle_{L_{g_0}} =& -\langle d_{k,i}\bar{v}, \phi_{l,i}\rangle_{L_{g_0}} = -4n(n-1)\int \bar{d}_{k,i}vd_{l,i}\varphi^{\frac{n+2}{n-2}}d\mu_{g_0} \\ =& -4n(n-1)\frac{n+2}{n-2}\int \bar{\varphi}_i^{\frac{4}{n-2}}d_{k,i}v\phi_{l,i}d\mu_{g_0}. \end{aligned}$$

We therefore conclude that, up to an error $o(\frac{1}{\lambda^2})$,

$$
\begin{aligned}
I_2 =&4n(n-1)\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\int\varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\
&+4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\
&-4n(n-1)\delta_{i,j}\alpha_i^{-1}\frac{(\bar{c}_0\alpha_{K,\tau}^{p+1})^{\frac{n-2}{n}}}{8n(n-1)}\partial J_\tau(\alpha^m\varphi_m)\widehat{d_{k,i}\phi_{l,i}},
\end{aligned}
$$

at which point $\bar{v}$ has been eliminated from the main terms in the expansion. By Lemma 3.1 we then have

$$
\partial J_\tau(\alpha^m\varphi_m)\lfloor_{\langle\phi_{k,i}\rangle}=o(\frac{1}{\lambda^2}),
$$

so we may pass from $\widehat{d_{k,i}\phi_{l,i}}$ to $d_{k,i}\phi_{l,i}$ in the above formulas and, as $\partial J_\tau(\alpha^m\varphi_m)=O(\frac{1}{\lambda^2})$, we obtain

$$
\begin{aligned}
&\frac{(\bar{c}_0\alpha_{K,\tau}^{p+1})^{\frac{n-2}{n}}}{8n(n-1)}\partial J_\tau(\alpha^m\varphi_m)d_{k,i}\phi_{l,i}\\
&\quad=-\alpha^m\tau\int\limits_{B_\varepsilon(a_m)}\left(\varphi_m^{\frac{n+2}{n-2}}\ln(1+\lambda_m^2r^2)^{\frac{n-2}{2}}-\frac{\bar{c}_1}{c_1}\varphi_m^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_m^{\frac{4}{n-2}}\lambda_m\partial_{\lambda_m}\varphi_m\right)d_{k,i}\phi_{l,i}d\mu_{g_0}\\
&\quad\quad+\alpha^m\tau\int\limits_{B_\varepsilon(a_m)}\left(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_m^2r^2}{2n}\varphi_m^{\frac{n+2}{n-2}}-\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_1}\varphi_m^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_m^{\frac{4}{n-2}}\lambda_m\partial_{\lambda_m}\varphi_m\right)d_{k,i}\phi_{l,i}d\mu_{g_0}\\
&\quad\quad-\alpha^m\int\limits_{B_\varepsilon(a_m)}(\frac{\nabla^2K_m}{2K_m}x^2-\frac{\Delta K_m}{2nK_m}r^2)\varphi_m^{\frac{n+2}{n-2}}d_{k,i}\phi_{l,i}d\mu_{g_0}.
\end{aligned}
$$

Still by the fact that $\varepsilon_{i,j}=o(\frac{1}{\lambda^2})$ we therefore arrive at

$$
\begin{aligned}
I_2 =&4n(n-1)\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\int\varphi_i^{p-1}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\
&+4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}d\mu_{g_0}\\
&-4n(n-1)\delta_{i,j}\Bigg(-\tau\int\limits_{B_\varepsilon(a_i)}\left(\varphi_i^{\frac{n+2}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar{c}_1}{c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\right)d_{k,i}\phi_{l,i}d\mu_{g_0}\\
&\quad+\tau\int\limits_{B_\varepsilon(a_i)}\left(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2r^2}{2n}\varphi_i^{\frac{n+2}{n-2}}-\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\right)d_{k,i}\phi_{l,i}d\mu_{g_0}\\
&\quad-\int\limits_{B_\varepsilon(a_i)}(\frac{\nabla^2K_i}{2K_i}x^2-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{l,i}d\mu_{g_0}\Bigg),
\end{aligned}
$$

up to some $o(\frac{1}{\lambda^2})$. By oddness, we may simplify this to

$$
\begin{aligned}
I_2 =& 4n(n-1)\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\delta_{k,l}\int\varphi_i^{p-1}\phi_{k,i}\phi_{k,i}d\mu_{g_0} \\
&+4n(n-1)\frac{n+2}{n-2}\delta_{i,j}\delta_{k,l}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0} \\
&-4n(n-1)\delta_{i,j}\delta_{k,l}\Big(-\tau\int\limits_{B_\varepsilon(a_i)}\Big(\varphi_i^{\frac{n+2}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar{c}_1}{c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\Big)d_{k,i}\phi_{k,i}d\mu_{g_0} \\
&\qquad+\tau\int\limits_{B_\varepsilon(a_i)}\Big(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2r^2}{2n}\varphi_i^{\frac{n+2}{n-2}}-\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_1}\varphi_i^{\frac{n+2}{n-2}}+\frac{2}{n-2}\frac{\tilde{c}_1}{c_2}\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\Big)d_{k,i}\phi_{k,i}d\mu_{g_0} \\
&\qquad-\int\limits_{B_\varepsilon(a_i)}(\frac{\nabla^2K_i}{2K_i}x^2-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}\Big)
\end{aligned}
$$

By Lemma 5.1 it follows that, up to some $o(\frac{1}{\lambda^2})$, for $k=2,3$

$$
\begin{aligned}
&4n(n-1)\frac{n+2}{n-2}\int\varphi_i^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i d_{k,i}\phi_{k,i}d\mu_{g_0}=\int L_{g_0}(\lambda_i\partial_{\lambda_i}\varphi_i)d_{k,i}\phi_{k,i}d\mu_{g_0} \\
=&\langle\lambda_i\partial_{\lambda_i}\varphi_i,(d_{k,i})^2\varphi_i\rangle_{L_{g_0}}=d_{k,i}\langle\lambda_i\partial_{\lambda_i}\varphi_i,d_{k,i}\varphi_i\rangle_{L_{g_0}}-\langle\lambda_i\partial_{\lambda_i}d_{k,i}\varphi_i,d_{k,i}\varphi_i\rangle_{L_{g_0}} \\
=&d_{k,i}\langle\phi_{2,i},\phi_{k,i}\rangle_{L_{g_0}}-\frac{1}{2}\lambda_i\partial_{\lambda_i}\|\phi_{k,i}^2\|_{L_{g_0}}=o(1),
\end{aligned}
$$

as $\langle\phi_{2,i},\phi_{k,i}\rangle_{L_{g_0}}$ and $\|\phi_{k,i}^2\|_{L_{g_0}}$ are almost constant in a_i and λ_i. So we simplify to

$$
\begin{aligned}
\frac{I_2}{4n(n-1)}=&\frac{n+2}{n-2}[(1-(\frac{n-2}{n+2}+\frac{\bar{c}_1}{\bar{c}_0}-\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau)]\lambda_i^{\theta}\delta_{i,j}\delta_{k,l}\int\varphi_i^{p-1}\phi_{k,i}\phi_{k,i}d\mu_{g_0} \\
+&\frac{n+2}{n-2}\delta_{i,j}\delta_{k,l}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}-\delta_{i,j}\delta_{k,l}\Big(-\tau\int\limits_{B_\varepsilon(a_i)}\Big(\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar{c}_1}{c_1}\Big)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0} \\
&+\tau\int\limits_{B_\varepsilon(a_i)}\Big(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2r^2}{2n}-\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_1}\Big)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}-\int\limits_{B_\varepsilon(a_i)}(\frac{\nabla^2K_i}{2K_i}x^2-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}\Big).
\end{aligned}
$$

Next, for the first summand above we find that, up to an error $o(\frac{1}{\lambda^2})$

$$
\begin{aligned}
\lambda_i^{\theta}\int\varphi_i^{p-1}\phi_{k,i}\phi_{k,i}d\mu_{g_0}=&\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}+\int\limits_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}(\lambda_i^{\theta}\varphi_i^{-\tau}-1)\phi_{k,i}\phi_{k,i}d\mu_{g_0} \\
=&\frac{n-2}{n+2}\int d_{k,i}\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}d\mu_{g_0}+\int\limits_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}((1+\lambda_i^2r^2)^{\theta}-1)\phi_{k,i}\phi_{k,i}d\mu_{g_0} \\
=&\frac{1}{4n(n-1)}\frac{n-2}{n+2}\langle\phi_{k,i},\phi_{k,i}\rangle_{L_{g_0}}+\theta\int\limits_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)\phi_{k,i}\phi_{k,i}d\mu_{g_0}
\end{aligned}
$$

using Lemma 5.1 and properly expanding. Recalling (4.8), we thus conclude

$$
\begin{aligned}
&\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar v)\phi_{k,i}\phi_{l,j}=\int\frac{L_{g_0}}{4n(n-1)}\phi_{k,i}\phi_{l,j}d\mu_{g_0}-\frac{I_2}{4n(n-1)}\\
=&\delta_{i,j}\delta_{k,l}\Bigg((1+\frac{n+2}{n-2}(\frac{\bar c_1}{\bar c_0}-\frac{\tilde c_1}{\tilde c_2}\frac{\bar c_2}{\bar c_0}))\tau\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}-\frac{n+2}{n-2}\tau\int_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}\\
&-\tau\int_{B_\varepsilon(a_i)}\left(\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar c_1}{c_1}\right)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}+\tau\int_{B_\varepsilon(a_i)}\left(\frac{\tilde c_1}{\tilde c_2}\frac{\lambda_i^2r^2}{2n}-\frac{\tilde c_1\bar c_2}{\tilde c_2c_1}\right)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}\\
&-\int_{B_\varepsilon(a_i)}(\frac{\nabla^2K_i}{2K_i}x^2-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}d_{k,i}\phi_{k,i}d\mu_{g_0}-\frac{n+2}{n-2}\int_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}\Bigg)
\end{aligned}
\tag{4.10}
$$

and in particular for $i=1,\ldots,q$, and $j=1,\ldots,n$ we have, up to an error $o(\frac{1}{\lambda^2})$

$$
[\partial^2J_\tau(\alpha^k\varphi_k+\bar v)]_{\mathbb B}=\begin{pmatrix}\frac{1}{\lambda^2}\mathcal V_+ & 0 & 0 & 0\\ 0 & \frac{1}{\lambda^2}\mathbb A_{q-1,0} & 0 & 0\\ 0 & 0 & \partial^2J_\tau\lambda_i\partial_{\lambda_i}\varphi_i\lambda_i\partial_{\lambda_i}\varphi_i & 0\\ 0 & 0 & 0 & \partial^2J_\tau\frac{(\nabla_{a_i})_j}{\lambda_i}\varphi_i\frac{(\nabla_{a_i})_j}{\lambda_i}\varphi_i\end{pmatrix}.
$$

Last diagonal terms. Concerning λ-derivatives, we first notice that mixed derivatives in different λ_i's are of order λ^{2-n}, which is a $o(\lambda^{-2})$ since $n\geq5$. Therefore it is sufficient to compute second derivatives with respect to the same λ_i. This corresponds to

$$
\begin{aligned}
&\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}{8n(n-1)}\partial^2J_\tau(\alpha^m\varphi_m+\bar v)(\lambda_i\partial_{\lambda_i}\varphi_i)^2=(1+\frac{n+2}{n-2}(\frac{\bar c_1}{\bar c_0}-\frac{\tilde c_1}{\tilde c_2}\frac{\bar c_2}{\bar c_0}))\tau\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}d\mu_{g_0}\\
&-\frac{n+2}{n-2}\tau\int_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2d\mu_{g_0}\\
&-\tau\int_{B_\varepsilon(a_i)}\left(\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar c_1}{c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}+\tau\int_{B_\varepsilon(a_i)}\left(\frac{\tilde c_1}{\tilde c_2}\frac{\lambda_i^2r^2}{2n}-\frac{\tilde c_1\bar c_2}{\tilde c_2c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}\\
&-\int_{B_\varepsilon(a_i)}(\frac{\nabla^2K_i}{2K_i}x^2-\frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}-\frac{n+2}{n-2}\int_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2d\mu_{g_0}.
\end{aligned}
$$

The second-last summand vanishes and $\int\varphi_i^{p-1}\phi_{k,i}\phi_{k,i}d\mu_{g_0}=c_k+o(1)$, cf. Lemma 5.2, whence

$$
\begin{aligned}
&\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar v}}{8n(n-1)}\partial^2J_\tau(\alpha^m\varphi_m+\bar v)(\lambda_i\partial_{\lambda_i}\varphi_i)^2=c_2(1+\frac{n+2}{n-2}(\frac{\bar c_1}{\bar c_0}-\frac{\tilde c_1}{\tilde c_2}\frac{\bar c_2}{\bar c_0}))\tau\\
&-\frac{n+2}{n-2}\tau\int_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2d\mu_{g_0}-\tau\int_{B_\varepsilon(a_i)}\left(\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}-\frac{\bar c_1}{c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}\\
&+\tau\int_{B_\varepsilon(a_i)}\left(\frac{\tilde c_1}{\tilde c_2}\frac{\lambda_i^2r^2}{2n}-\frac{\tilde c_1\bar c_2}{\tilde c_2c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}-\frac{n+2}{n-2}\frac{\Delta K_i}{2nK_i}\int_{B_\varepsilon(a_i)}r^2\varphi_i^{\frac{4}{n-2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2d\mu_{g_0}.
\end{aligned}
$$

Moreover,

$$
\int\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_id\mu_{g_0}=\lambda_i\partial_{\lambda_i}\int\varphi_i^{\frac{n+2}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_id\mu_{g_0}-\frac{n+2}{n-2}\int\varphi_i^{\frac{4}{n-2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2d\mu_{g_0}=-\frac{n+2}{n-2}c_2+o(1),
$$

and

$$\frac{n+2}{n-2}\int\limits_{B_\varepsilon(a_i)} r^2\varphi_i^{\frac{4}{n-2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2 d\mu_{g_0} = \int\limits_{B_\varepsilon(a_i)} r^2\lambda_i\partial_{\lambda_i}\varphi_i\lambda_i\partial_{\lambda_i}\varphi_i^{\frac{n+2}{n-2}}d\mu_{g_0}$$

$$=\lambda_i\partial_{\lambda_i}\int\limits_{B_\varepsilon(a_i)} r^2\lambda_i\partial_{\lambda_i}\varphi_i\varphi_i^{\frac{n+2}{n-2}}d\mu_{g_0} - \int\limits_{B_\varepsilon(a_i)} r^2\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_i d\mu_{g_0}.$$

Thus, recalling (3.1), in particular $\tilde{c}_1\tau+\tilde{c}_2\frac{\Delta K_i}{K_i\lambda_i^2}=o(\frac{1}{\lambda^2})$, we arrive at

$$\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\lambda_i\partial_{\lambda_i}\varphi_i)^2 = c_2\tau-\frac{n+2}{n-2}\tau\int\limits_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2 d\mu_{g_0}$$

$$-\tau\int\limits_{B_\varepsilon(a_i)}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_i d\mu_{g_0}+\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\tau}{2n}\lambda_i^3\partial_{\lambda_i}\int\limits_{B_\varepsilon(a_i)} r^2\lambda_i\partial_{\lambda_i}\varphi_i\varphi_i^{\frac{n+2}{n-2}}d\mu_{g_0},$$

and for the last integral above we find passing to integration over $\mathbb{R}^n$

$$\lambda_i\partial_{\lambda_i}\int\limits_{B_\varepsilon(a_i)} r^2\lambda_i\partial_{\lambda_i}\varphi_i\varphi_i^{\frac{n+2}{n-2}}d\mu_{g_0}=\lambda_i\partial_{\lambda_i}\int_{\mathbb{R}^n} r^2\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx$$

$$=\frac{n-2}{2n}(\lambda_i\partial_{\lambda_i})^2\int r^2\delta_{0,\lambda_i}^{\frac{2n}{n-2}}dx=\frac{n-2}{2n}(\lambda_i\partial_{\lambda_i})^2(\lambda_i^{-2}\int_{\mathbb{R}^n}\frac{r^2}{(1+r^2)^n}dx)=\frac{n-2}{8n}\frac{\bar{c}_2}{\lambda_i^2}$$

up to some error of order $o(1)$. Consequently,

$$\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\lambda_i\partial_{\lambda_i}\varphi_i)^2 = c_2(1+\frac{n-2}{16n^2}\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_2})\tau$$
$$-\frac{n+2}{n-2}\tau\int_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2 d\mu_{g_0}-\tau\int\limits_{B_\varepsilon(a_i)}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}\varphi_i^{\frac{n+2}{n-2}}(\lambda_i\partial_{\lambda_i})^2\varphi_i d\mu_{g_0}.$$

Finally, we calculate passing to integration over $\mathbb{R}^n$ and up to a $o(1)$

$$\frac{n+2}{n-2}\int\limits_{B_\varepsilon(a_i)}\varphi_i^{\frac{4}{n-2}}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}|\lambda_i\partial_{\lambda_i}\varphi_i|^2 d\mu_{g_0}$$
$$=\int_{\mathbb{R}^n}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx=\lambda_i\partial_{\lambda_i}\int_{\mathbb{R}^n}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx$$
$$-(n-2)\int_{\mathbb{R}^n}\frac{\lambda_i^2r^2}{1+\lambda_i^2r^2}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx-\int_{\mathbb{R}^n}\ln(1+\lambda_i^2r^2)^{\frac{n-2}{2}}(\lambda_i\partial_{\lambda_i})^2\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx,$$

where the first summand above vanishes by rescaling, and we are reduced to

$$\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\lambda_i\partial_{\lambda_i}\varphi_i)^2 = c_2(1+\frac{n-2}{16n^2}\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_2})\tau+(n-2)\tau\int_{\mathbb{R}^n}\frac{\lambda_i^2r^2}{1+\lambda_i^2r^2}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx,$$

where, up to some $o(1)$,

$$\int\limits_{\mathbb{R}^n}\frac{\lambda_i^2r^2}{1+\lambda_i^2r^2}\lambda_i\partial_{\lambda_i}\delta_{0,\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}dx=\frac{n-2}{2}\int\limits_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+2}}dx=-\frac{n-2}{2}\hat{c}_3,\ \hat{c}_3=-\int\limits_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+2}}dx. \tag{4.11}$$

By an explicit computation (all the above constants can be explicitly evaluated), we conclude that up to an error $o(\frac{1}{\lambda^2})$

$$\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\lambda_i\partial_{\lambda_i}\varphi_i)^2=\left(c_2(1+\frac{n-2}{16n^2}\frac{\tilde{c}_1\bar{c}_2}{\tilde{c}_2c_2})-\frac{(n-2)^2}{2}\hat{c}_3\right)\tau=\frac{(n-2)^2\Gamma^2(\frac{n}{2})}{128n\Gamma(n+1)}\tau.$$

Thence we arrive at (with $i = 1, \dots, q$ and $j = 1, \dots, n$)

$$[\partial^2 J_\tau(\alpha^k \varphi_k + \bar{v})]_{\mathbb{B}} = \begin{pmatrix} \frac{1}{\lambda^2}\mathcal{V}_+ & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda^2}\mathbb{A}_{q-1,0} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda^2}\mathbb{A}_+ & \\ 0 & 0 & 0 & \partial^2 J_\tau \frac{(\nabla_{a_i})_j}{\lambda_i}\varphi_i \frac{(\nabla_{a_i})_j}{\lambda_i}\varphi_i \end{pmatrix}$$

up to $o(\frac{1}{\lambda^2})$, where $\mathbb{A}_+ > 0$ is as in the statement. We are left with the computation of the terms

$$\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m \varphi_m + \bar{v})\frac{\nabla_{a_i}}{\lambda_i}\varphi_i \frac{\nabla_{a_i}}{\lambda_i}\varphi_i,$$

for instance we consider

$$\begin{aligned}
&\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m \varphi_m + \bar{v})(\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i)^2 \\
=&(1 + \frac{n+2}{n-2}(\frac{\bar{c}_1}{\bar{c}_0} - \frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0}))\tau \int \varphi_i^{\frac{4}{n-2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0} - \frac{n+2}{n-2}\tau \int_{B_\varepsilon(a_i)} \varphi_i^{\frac{4}{n-2}} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0} \\
&- \tau \int_{B_\varepsilon(a_i)} \left(\ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} - \frac{\bar{c}_1}{c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} + \tau \int_{B_\varepsilon(a_i)} \left(\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2 r^2}{2n} - \frac{\tilde{c}_1 \bar{c}_2}{\tilde{c}_2 c_1}\right)\varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} \\
&- \int_{B_\varepsilon(a_i)} (\frac{\nabla^2 K_i}{2K_i}x^2 - \frac{\Delta K_i}{2nK_i}r^2)\varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} - \frac{n+2}{n-2}\int_{B_\varepsilon(a_i)} \frac{\nabla^2 K_i}{2K_i}x^2 \varphi_i^{\frac{4}{n-2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0}.
\end{aligned}$$

At this point some simplifications occur. From the relation

$$\tilde{c}_1 \tau + \tilde{c}_2 \frac{\Delta K_i}{K_i \lambda_i^2} = o(\frac{1}{\lambda^2})$$

we obtain cancellation of the terms involving ΔK_i and $\frac{\tilde{c}_1}{\tilde{c}_2}\frac{\lambda_i^2 r^2}{2n}$. Using as well the relations

$$\int \varphi_i^{\frac{4}{n-2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0} = \frac{c_3}{n} + o(1); \qquad \int \varphi_i^{\frac{n+2}{n-2}} (\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} = -\frac{n+2}{n-2}\frac{c_3}{n} + o(1)$$

together with $(\frac{\bar{c}_1}{\bar{c}_0} - \frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\frac{c_3}{n} = (\frac{\bar{c}_1}{c_1} - \frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{c_1})\frac{c_2}{n}$, due to the fact that $\bar{c}_0 = c_1$ and $c_2 = c_3$, to obtain

$$\begin{aligned}
&\frac{(k_\tau)^{\frac{2}{p+1}}_{\alpha^m\varphi_m+\bar{v}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m \varphi_m + \bar{v})(\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i)^2 \\
=&\frac{c_3}{n}\tau - \frac{n+2}{n-2}\tau \int_{B_\varepsilon(a_i)} \varphi_i^{\frac{4}{n-2}} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0} - \tau \int_{B_\varepsilon(a_i)} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} \varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} \\
&- \int_{B_\varepsilon(a_i)} \frac{\nabla^2 K_i}{2K_i}x^2 \varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0} - \frac{n+2}{n-2}\int_{B_\varepsilon(a_i)} \frac{\nabla^2 K_i}{2K_i}x^2 \varphi_i^{\frac{4}{n-2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0}.
\end{aligned}$$

Moreover we have, passing to integration over $\mathbb{R}^n$, up to an error $o(1)$

$$\begin{aligned}
\frac{n+2}{n-2}\int_{B_\varepsilon(a_i)} \varphi_i^{\frac{4}{n-2}} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} |\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2 d\mu_{g_0} =& \int_{\mathbb{R}^n} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} \frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}} \frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i} dx \\
=&-(n-2)\int_{\mathbb{R}^n} \frac{\lambda_i x_1}{1+\lambda_i^2 r^2}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}} \frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i} dx - \int_{\mathbb{R}^n} \ln(1+\lambda_i^2 r^2)^{\frac{n-2}{2}} \delta_{0,\lambda_i}^{\frac{n+2}{n-2}} (\frac{(\nabla_{a_i})_1}{\lambda_i})\delta_{0,\lambda_i} dx
\end{aligned}$$

and find for the first summand

$$(n-2)\int_{\mathbb{R}^n}\frac{\lambda_i x_1}{1+\lambda_i^2 r^2}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}dx=-\int_{\mathbb{R}^n}\delta_{0,\lambda_i}^{\frac{4}{n-2}}|\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}|^2dx=-\frac{c_3}{n}.$$

We therefore are left with

$$\begin{aligned}&\frac{(k_\tau)_{\alpha^m\varphi_m+\bar{v}}^{\frac{2}{p+1}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i)^2\\ &\quad=-\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\varphi_i d\mu_{g_0}-\frac{n+2}{n-2}\int\limits_{B_\varepsilon(a_i)}\frac{\nabla^2K_i}{2K_i}x^2\varphi_i^{\frac{4}{n-2}}|\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2d\mu_{g_0}.\end{aligned}$$

Finally, passing to integration over $\mathbb{R}^n$, up to some $o(1)$ there holds

$$\begin{aligned}\frac{n+2}{n-2}\int\limits_{B_\varepsilon(a_i)}x_l^2\varphi_i^{\frac{4}{n-2}}|\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i|^2d\mu_{g_0}=&\int_{\mathbb{R}^n}x_l^2\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}dx\\ =&-2\delta_{1,l}\int_{\mathbb{R}^n}\frac{x_1}{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}dx-\int_{\mathbb{R}^n}x_l^2\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}(\frac{(\nabla_{a_i})_1}{\lambda_i})^2\delta_{0,\lambda_i}dx,\end{aligned}$$

and similarly for $j=2,\dots,n$. Diagonalizing the Hessian we have $\nabla^2K_ix^2=\sum_{l=1}^n\partial_l^2K_ix_l^2$ and

$$\int_{\mathbb{R}^n}\frac{x_1}{\lambda_i}\delta_{0,\lambda_i}^{\frac{n+2}{n-2}}\frac{(\nabla_{a_i})_1}{\lambda_i}\delta_{0,\lambda_i}dx=-(n-2)\int_{\mathbb{R}^n}\delta_{0,\lambda_i}^{\frac{2n}{n-2}}\frac{x_1^2}{1+\lambda_i^2r^2}dx=-\frac{n-2}{n\lambda_i^2}\int_{\mathbb{R}^n}\frac{r^2}{(1+r^2)^{n+1}}dx,$$

(and similarly for $j=2,\dots,n$), so we conclude that

$$\frac{(k_\tau)_{\alpha^m\varphi_m+\bar{v}}^{\frac{2}{p+1}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})(\frac{(\nabla_{a_i})_1}{\lambda_i}\varphi_i)^2=-c\frac{\partial_1^2K_i}{K_i\lambda_i^2}.$$

Similarly, one can show analogous formula for any couple of indices

$$\frac{(k_\tau)_{\alpha^m\varphi_m+\bar{v}}^{\frac{2}{p+1}}}{8n(n-1)}\partial^2 J_\tau(\alpha^m\varphi_m+\bar{v})\frac{(\nabla_{a_i})_k}{\lambda_i}\varphi_i\frac{(\nabla_{a_i})_l}{\lambda_i}\varphi_i=-c\frac{\partial_{k,l}^2K_i}{K_i\lambda_i^2}.$$

The proof is thereby complete. ■

From Proposition 4.1 we deduce that the kernel of $\partial^2 J_\tau$ is exactly one-dimensional. The presence of a kernel is unavoidable due to the scaling invariance of J_τ, but this degeneracy turns out to be minimal. We can therefore restrict ourselves to some homogeneous constraint.

Corollary 4.1. *Let $I_\tau=J_\tau\lfloor_{[\|\cdot\|_{L_{g_0}}=1]}$ or $I_\tau=J_\tau\lfloor_{[\|\cdot\|_{k_\tau}=1]}$, and let $\tilde{u}$ be normalization of a solution u of (1.5) in $\bar{V}(q,\varepsilon)$. Then*

$$m(I_\tau,\tilde{u})=q-1+\sum_{i=1}^q(n-m(K,a_i)).$$

5 Appendix: some technical estimates

In this appendix, recalling our notation, we collect some useful statements and formulas proved in [30].

Lemma 5.1. *There holds $L_{g_0}\varphi_{a,\lambda}=O(\varphi_{a,\lambda}^{\frac{n+2}{n-2}})$. More precisely on a geodesic ball $B_\alpha(a)$ for $\alpha>0$ small*

$$L_{g_0}\varphi_{a,\lambda}=4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}-2nc_nr_a^{n-2}((n-1)H_a+r_a\partial_{r_a}H_a)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}+\frac{R_{g_a}}{\lambda}u_a^{\frac{2}{n-2}}\varphi_{a,\lambda}^{\frac{n}{n-2}}+o(r_a^{n-2})\varphi_{a,\lambda}^{\frac{n+2}{n-2}},$$

where $r_a=d_{g_a}(a,\cdot)$. Since $R_{g_a}=O(r_a^2)$ in conformal normal coordinates, cf. [23], we obtain *(i)* $L_{g_0}\varphi_{a,\lambda} = 4n(n-1)[1-\frac{c_n}{2}r_a^{n-2}(H_a(a)+n\nabla H_a(a)x)]\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O(\lambda^{-2}\varphi_{a,\lambda})$ *for* $n=5$;

(ii) $L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} = 4n(n-1)[1+\frac{c_n}{2}W(a)\ln r]\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O(\lambda^{-2}\varphi_{a,\lambda})$ *for* $n=6$;

(iii) $L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} = O(\lambda^{-2}\varphi_{a,\lambda})$ *for* $n \geq 7$,

where $W(a) = |\mathbb{W}(a)|^2$. *The expansions stated above persist upon taking* $\lambda\partial\lambda$ *and* $\frac{\nabla_a}{\lambda}$ *derivatives.*

Lemma 5.2. *Let* $\theta = \frac{n-2}{2}\tau$ *and* $k,l = 1,2,3$ *and* $i,j = 1,\ldots,q$. *Then, for* $\varepsilon_{i,j}$ *as in (2.7), there holds uniformly as* $0 \leq \tau \longrightarrow 0$

(i) $|\phi_{k,i}|, |\lambda_i\partial_{\lambda_i}\phi_{k,i}|, |\frac{1}{\lambda_i}\nabla_{a_i}\phi_{k,i}| \leq C\varphi_i$;

(ii) $\lambda_i^\theta \int \varphi_i^{\frac{4}{n-2}-\tau}\phi_{k,i}\phi_{k,i}d\mu_{g_0} = c_k \cdot id + O(\tau + \frac{1}{\lambda_i^{2+\theta}})$, $c_k > 0$;

(iii) *for* $i \neq j$ *up to some error of order* $O(\tau^2 + \sum_{i\neq j}(\frac{1}{\lambda_i^4} + \varepsilon_{i,j}^{\frac{n+2}{n}}))$

$$\lambda_i^\theta \int \varphi_i^{\frac{n+2}{n-2}-\tau}\phi_{k,j}d\mu_{g_0} = b_k d_{k,i}\varepsilon_{i,j} = \int \varphi_i^{1-\tau} d_{k,j}\varphi_j^{\frac{n+2}{n-2}}d\mu_{g_0};$$

(iv) $\lambda_i^\theta \int \varphi_i^{\frac{4}{n-2}-\tau}\phi_{k,i}\phi_{l,i}d\mu_{g_0} = O(\frac{1}{\lambda_i^2})$ *for* $k \neq l$ *and for* $k = 2,3$

$$\lambda_i^\theta \int \varphi_i^{\frac{n+2}{n-2}-\tau}\phi_{k,i}d\mu_{g_0} = O\left(\tau + \begin{pmatrix} \lambda_i^{2-n} & \text{for } n=5 \\ \frac{\ln\lambda_i}{\lambda_i^4} & \text{for } n=6 \\ \lambda_i^4 & \text{for } n\geq 7 \end{pmatrix}\right);$$

(v) $\lambda_i^\theta \int \varphi_i^{\alpha-\tau}\varphi_j^\beta d\mu_{g_0} = O(\varepsilon_{i,j}^\beta)$ *for* $i \neq j$, $\alpha+\beta = \frac{2n}{n-2}$, $\alpha - \tau > \frac{n}{n-2} > \beta \geq 1$;

(vi) $\int \varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}}d\mu_{g_0} = O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j})$, $i \neq j$;

(vii) $(1, \lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})\varepsilon_{i,j} = O(\varepsilon_{i,j})$, $i \neq j$.

with constants $b_k = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}}$ *for* $k = 1,2,3$ *and*

$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}, \qquad c_2 = \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{|r^2-1|^2dx}{(1+r^2)^{n+2}}, \qquad c_3 = \frac{(n-2)^2}{n}\int_{\mathbb{R}^n} \frac{r^2dx}{(1+r^2)^{n+2}}.$$

Lemma 5.3. *For* $u \in V(q,\varepsilon)$ *with* $k_\tau = 1$, *cf.* (2.2),*and* $\nu \in H_u(q,\varepsilon)$ *there holds*

$$\partial J_\tau(\alpha^i\varphi_i)\nu = O\Big(\Big[\sum_r \frac{\tau}{\lambda_r^\theta} + \sum_r \frac{|\nabla K_r|}{\lambda_r^{1+\theta}} + \sum_r \frac{1}{\lambda_r^{2+\theta}} + \sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\Big]\|\nu\|\Big).$$

Lemma 5.4. *For* $u \in V(q,\varepsilon)$ *and* $\varepsilon > 0$ *sufficiently small the three quantities* $\partial J_\tau(u)\phi_{1,j}$, $\partial J_\tau(\alpha^i\varphi_i)\phi_{1,j}$, $\partial_{\alpha_j}J_\tau(\alpha^i\varphi_i)$ *can be written as*

$$\frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\Big(\grave{c}_0\big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\big) - \grave{c}_2\big(\frac{\Delta K_j}{K_j\lambda_j^2} - \sum_k \frac{\Delta K_k}{K_k\lambda_k^2}\frac{\alpha_k^2}{\alpha^2}\big)$$

$$+\grave{b}_1\Big(\sum_{k\neq l}\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l} - \sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\varepsilon_{i,j}\Big) - \grave{d}_1\begin{pmatrix} \frac{H_j}{\lambda_j^3} - \sum_k \frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k^3} & \text{for } n=5 \\ \frac{W_j\ln\lambda_j}{\lambda_i^4} - \sum_k \frac{\alpha_k^2}{\alpha^2}\frac{W_k\ln\lambda_k}{\lambda_k^4} & \text{for } n=6 \\ 0 & \text{for } n\geq 7 \end{pmatrix}\Big)$$

up to an error of order $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$, *with positive constants* $\grave{c}_0,\grave{c}_2,\grave{b}_1,\grave{d}_1$

$$\grave{b}_1=\frac{8n(n-1)(n+2)}{\bar{c}_0^{\frac{n-2}{n}}(n-2)}b_1,\quad \grave{c}_2=\frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\bar{c}_2,\quad \grave{d}_1=\frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\bar{d}_1,\quad \grave{c}_0=8n(n-1)\bar{c}_0^{\frac{2}{n}}. \tag{5.1}$$

In particular for all j

$$\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}=1+O\big(\tau+\sum_{r\neq s}\frac{1}{\lambda_r^2}+\varepsilon_{r,s}+|\partial J_\tau(u)|\big).$$

Lemma 5.5. *For* $u\in V(q,\varepsilon)$ *and* $\varepsilon>0$ *sufficiently small the three quantities* $\partial J_\tau(u)\phi_{2,j}$, $\partial J_\tau(\alpha^i\varphi_i)\phi_{2,j}$ *and* $\frac{\lambda_j}{\alpha_j}\partial_{\lambda_j}J_\tau(\alpha^i\varphi_i)$ *can be written as*

$$\frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\left(\tilde{c}_1\tau+\tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2}-\tilde{b}_2\sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}+\tilde{d}_1\begin{pmatrix}\frac{H_j}{\lambda_j^3} & \text{for } n=5\\ \frac{W_j\ln\lambda_j}{\lambda_j^4} & \text{for } n=6\\ 0 & \text{for } n\geq 7\end{pmatrix}\right),$$

with positive constants $\tilde{c}_1,\tilde{c}_2,\tilde{d}_1,\tilde{b}_2$ *up to some error* $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$.

Lemma 5.6. *For* $u\in V(q,\varepsilon)$ *and* $\varepsilon>0$ *sufficiently small the three quantities* $\partial J_\tau(u)\phi_{3,j}$, $\partial J_\tau(\alpha^i\varphi_i)\phi_{3,j}$ *and* $\frac{\nabla_{a_j}}{\alpha_j\lambda_j}J_\tau(\alpha^i\varphi_i)$ *can be written as*

$$-\frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\left(\check{c}_3\frac{\nabla K_j}{K_j\lambda_j}+\check{c}_4\frac{\nabla\Delta K_j}{K_j\lambda_j^3}+\check{b}_3\sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\frac{\nabla_{a_j}}{\lambda_j}\varepsilon_{i,j}\right),$$

with positive constants $\check{c}_3,\check{c}_4,\check{b}_3$ *up to some error* $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$.

Lemma 5.7. *For every* $u\in V(q,\varepsilon)$ *there holds*

$$|\partial J_\tau(u)|\lesssim\tau+\sum_{r\neq s}\frac{|\nabla K_r|}{\lambda_r}+\frac{1}{\lambda_r^2}+|1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_r}{\lambda_r^\theta}\alpha_r^{p-1}|+\varepsilon_{r,s}^{\frac{n+2}{2n}}+\|v\|.$$

Theorem 2. *Suppose that* $n\geq 5$, $K:M\to\mathbb{R}$ *is positive, Morse and satisfies* (1.3). *Then for* $\varepsilon>0$ *sufficiently small there exists* $c>0$ *such that for any* $u\in V(q,\varepsilon)$ *with* $k_\tau=1$ *there holds*

$$|\partial J(u)|\geq c\big(\tau+\sum_{r\neq s}\frac{|\nabla K_r|}{\lambda_r}+\frac{1}{\lambda_r^2}+\big|1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_r}{\lambda_r^\theta}\alpha_r^{p-1}\big|+\varepsilon_{r,s}\big),$$

unless there is a violation of at least one of the four conditions

(i) $\tau>0$;

(ii) *there exists* $x_i\neq x_j\in\{\nabla K=0\}\cap\{\Delta K<0\}$ *and* $d(a_i,x_i)=O(\frac{1}{\lambda_i})$;

(iii) $\alpha_j=\Theta\cdot(\frac{\lambda_j^\theta}{K_j})^{\frac{1}{p-1}}+o(\frac{1}{\lambda_j^2})$;

(iv) $\tilde{c}_1\tau=-\tilde{c}_2\frac{\Delta K_k}{K_k\lambda_k^2}+o(\frac{1}{\lambda_k^2})$

where Θ *is a positive constant, uniformly bounded and bounded away from zero, that depends on* u *(see Remark 6.2 in [30]). In the latter case there holds* $\lambda_1\simeq\ldots\lambda_q\simeq\lambda=\frac{1}{\sqrt{\tau}}$ *and setting* $a_j=\exp_{g_{x_j}}(\bar{a}_j)$, *we still have up to an error* $o(\frac{1}{\lambda^3})$ *the lower bound*

$$
\begin{aligned}
|\partial J(u)| \gtrsim & \sum_j |\tau + \frac{2}{9}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} + \frac{512}{9\pi}[\frac{H(x_j)}{\lambda_j^3} + \sum_{j\neq i}\sqrt{\frac{K(x_j)}{K(x_i)}}\frac{G_{g_0}(x_i,x_j)}{\gamma_n(\lambda_i\lambda_j)^{\frac{3}{2}}}]| \\
& + \sum_j |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}| \\
& + \sum_j |\alpha_j - \Theta \cdot \sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}(1 - \frac{1}{90}\left(\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} + \frac{2816}{\pi}\frac{H(x_j)}{\lambda_j^3} - \frac{\sum_k(\frac{\Delta K(x_k)}{K(x_k)^2\lambda_k^2} + \frac{2816}{\pi}\frac{H(x_k)}{K(x_k)\lambda_k^3})}{\sum_k\frac{1}{K(x_k)}}\right)})|
\end{aligned}
$$

in case $n = 5$ *and*

$$
|\partial J(u)| \gtrsim \sum_j (|\tau + \frac{\tilde{c}_2}{\tilde{c}_1}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}| + |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}| + |\alpha_j - \Theta\cdot\sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}}|)
$$

in case $n \geq 6$*. The constants appearing above are defined by* $\bar{c}_0 = \int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n}$*,*

$$
\tilde{c}_1 = \frac{n(n-1)(n-2)^2}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln\frac{1}{1+r^2}dx, \quad \tilde{c}_2 = -\frac{(n-1)(n-2)}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+1}}dx;
$$

$$
\check{c}_3 = \int_{\mathbb{R}^n}\frac{4(n-1)(n-2)}{(1+r^2)^n}dx, \qquad \check{c}_4 = \int_{\mathbb{R}^n}\frac{2(n-1)r^2}{(1+r^2)^n}dx.;
$$

$$
\tilde{b}_2 = \frac{4n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^{\frac{n+2}{2}}}; \qquad \tilde{d}_1 = \frac{4n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n} r^n\frac{(n+2-nr^2)}{(1+r^2)^{n+2}}dx.
$$

From the proof of Proposition 5.1 and Sections 4,5 and 6 in [30] we will need the following estimates

(i) up to an error of order $O\left(\tau^2 + \sum_r \frac{1}{\lambda_r^4} + \sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}}\right)$ there holds ($\bar{b}_1 = \frac{2n}{n-2}b_1$)

$$
\begin{aligned}
\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0} = & \sum_i \left(\bar{c}_0\frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1} + \bar{c}_1\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\tau + \bar{c}_2\frac{\Delta K_i}{\lambda_i^{2+\theta}}\alpha_i^{\frac{2n}{n-2}}\right) \\
& + \bar{d}_1\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\begin{pmatrix}\frac{H_i}{\lambda_i^3} \\ \frac{W_i\ln\lambda_i}{\lambda_i^4} \\ 0\end{pmatrix} + \bar{b}_1\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_j\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j}; \qquad \bar{d}_1 = \int_{\mathbb{R}^n}\frac{r^n dx}{(1+r^2)^{n+1}};
\end{aligned} \tag{5.2}
$$

(ii) recalling (2.7), one has

$$
\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0} = \tilde{b}_1\varepsilon_{i,j} + O(\sum_{r\neq s}\frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}), \qquad \tilde{b}_1 = 4n(n-1)b_1; \tag{5.3}
$$

(iii) up to an error $O(\tau^2 + \frac{1}{\lambda_i^4})$, there holds

$$
\int\frac{\varphi_i L_{g_0}\varphi_i}{4n(n-1)}d\mu_{g_0} = \bar{c}_0; \tag{5.4}
$$

(iv) up to an error of order $O\left(\tau^2 + \sum_r\frac{1}{\lambda_r^4} + \sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}}\right)$. one has

$$
\alpha^i\alpha^j\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0} = 4n(n-1)\bar{c}_0\sum_i\alpha_i^2 + \tilde{b}_1\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{i,j}. \tag{5.5}
$$

(v) If φ_i is as in (2.6), then

$$\left|\int \varphi_i^{\frac{n+2}{n-2}} \nu d\mu_{g_0}\right| \leq \|v\| \left\| \frac{L_{g_0}\varphi_i}{4n(n-1)} - \varphi_i^{\frac{n+2}{n-2}} \right\|_{L_{g_0}^{\frac{2n}{n+2}}} = O\begin{pmatrix} \lambda_i^{-3} & \text{for } n=5 \\ \ln^{\frac{2}{3}} \lambda_i \lambda_i^{-\frac{10}{3}} & \text{for } n=6 \\ \lambda_i^{-4} & \text{for } n\geq 7 \end{pmatrix} \|v\|; \tag{5.6}$$

(vi) up to an error $O(\tau^2 + \frac{1}{\lambda_i^4})$ one has

$$\int K\varphi_i^{p+1} d\mu_{g_0} = \frac{\bar{c}_0 K_i}{\lambda_i^\theta} + \bar{c}_1 \frac{K_i \tau}{\lambda_i^\theta} + \bar{c}_2 \frac{\Delta K_i}{\lambda_i^{2+\theta}} + \bar{d}_1 K_i \begin{pmatrix} \frac{H_i}{\lambda_i^{3+\theta}} \\ \frac{W_i \ln \lambda_i}{\lambda_i^{4+\theta}} \\ 0 \end{pmatrix}, \ \bar{c}_2 = \frac{1}{2n} \int_{\mathbb{R}^n} \frac{r^2 dx}{(1+r^2)^n}; \tag{5.7}$$

(vii) up to an error or order $O(\tau^2 + \sum_{r\neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}})$ there holds

$$\begin{aligned} J_\tau(\alpha^i\varphi_i) = & \frac{\alpha^i\alpha^j \int \varphi_i L_{g_0}\varphi_j d\mu_{g_0}}{(\int K(\sum_i \alpha_i\varphi_i)^{p+1})^{\frac{2}{p+1}}} = \frac{\alpha^i\alpha^j \int \varphi_i L_{g_0}\varphi_j d\mu_{g_0}}{(\bar{c}_0 \sum_i \frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1})^{\frac{2}{p+1}}} \Bigg(1 - \bar{c}_1 \sum_i \frac{K_i}{\lambda_i^\theta} \frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} \tau \\ & - \bar{c}_2 \sum_i \frac{\Delta K_i}{\lambda_i^{2+\theta}} \frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} - \bar{d}_1 \sum_i \frac{K_i}{\lambda_i^\theta} \begin{pmatrix} \frac{H_i}{\lambda_i^3} \\ \frac{W_i \ln \lambda_i}{\lambda_i^4} \\ 0 \end{pmatrix} \frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} - \bar{b}_1 \sum_{i\neq j} \frac{\alpha_i^{\frac{n+2}{n-2}}\alpha_j}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} \frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j} \Bigg).; \end{aligned} \tag{5.8}$$

(viii) if $\varepsilon_{i,j}$ is as in (2.7), then

$$\lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = \frac{2-n}{2}\varepsilon_{i,j} + O(\frac{1}{\lambda_j^4} + \varepsilon_{i,j}^{\frac{n+2}{n}}) \quad \text{in case } j<i \text{ or } d_{g_0}(a_i,a_j) \neq o(1). \tag{5.9}$$

Finally, we derive one last technical estimate. Recalling (2.1), from (5.5) we have, up to an error $o(\frac{1}{\lambda^2})$,

$$r_{\alpha^i\varphi_i} = \alpha^i\alpha^j \int L_{g_0}\varphi_i\varphi_j d\mu_{g_0} = 4n(n-1)\bar{c}_0 \sum_i \alpha_i^2 = 4n(n-1)\bar{c}_0\alpha^2 \tag{5.10}$$

with $\bar{c}_0 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}$. From (5.2) instead, still up to an error $o(\frac{1}{\lambda^2})$, we get

$$\begin{aligned} \int K(\alpha^i\varphi_i)^{p+1} d\mu_{g_0} = & \sum_i \left(\bar{c}_0 \frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1} + \bar{c}_1 \frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\tau + \bar{c}_2 \frac{\Delta K_i}{\lambda_i^{2+\theta}}\alpha_i^{\frac{2n}{n-2}} \right) \\ = & \bar{c}_0 \alpha_{K,\theta}^{p+1} + \sum_i \frac{K_i \alpha_i^{\frac{2n}{n-2}}}{\lambda_i^\theta} \left(\bar{c}_1\tau + \bar{c}_2 \frac{\Delta K_i}{K_i\lambda_i^2} \right) \end{aligned}$$

with constants given by

$$\bar{c}_1 = \frac{2}{n-2} \int_{\mathbb{R}^n} \frac{\ln(1+r^2)}{(1+r^2)^n} dx, \qquad \text{and} \qquad \bar{c}_2 = \frac{1}{2n} \int_{\mathbb{R}^n} \frac{r^2}{(1+r^2)^n} dx. \tag{5.11}$$

Therefore

$$\frac{r_{\alpha^i\varphi_i}}{(k_\tau)_{\alpha^i\varphi_i}} = 4n(n-1)\frac{\alpha^2}{\alpha_{K,\theta}^{p+1}} - 4n(n-1)\frac{\alpha^2}{(\alpha_{K,\theta}^{p+1})^2} \sum_i \frac{K_i\alpha_i^{\frac{2n}{n-2}}}{\lambda_i^\theta} \left(\frac{\bar{c}_1}{\bar{c}_0}\tau + \frac{\bar{c}_2}{\bar{c}_0} \frac{\Delta K_i}{K_i\lambda_i^2} \right) + o(\frac{1}{\lambda^2})$$

and we conclude again from (3.1) that

$$\frac{r_{\alpha^i\varphi_i}}{(k_\tau)_{\alpha^i\varphi_i}} = 4n(n-1)\frac{\alpha^2}{\alpha_{K,\theta}^{p+1}} (1 - (\frac{\bar{c}_1}{\bar{c}_0} - \frac{\tilde{c}_1}{\tilde{c}_2}\frac{\bar{c}_2}{\bar{c}_0})\tau) + o(\frac{1}{\lambda^2}). \tag{5.12}$$

5.1 List of constants

For the reader's convenience, we display the equations where some dimensional constants appear.

		¯	ˆ	ˋ	˜	ˇ
c_0		(5.10)		(5.1)		
c_1	Lemma 5.2	(5.11)			Theorem 2	
c_2	Lemma 5.2	(5.11)		(5.1)	Theorem 2	
c_3	Lemma 5.2		(4.11)			Theorem 2
c_4						Theorem 2
d_1		(5.2)		(5.1)	Theorem 2	
b_1	Lemma 5.2	(5.2)		(5.1)	(5.3)	
b_2	Lemma 5.2				Theorem 2	
b_3	Lemma 5.2					

References

[1] Ambrosetti A., Garcia Azorero J., Peral A., Perturbation of $\Delta u + u^{\frac{(N+2)}{(N-2)}} = 0$, the Scalar Curvature Problem in $\mathbb{R}^N$ and related topics, Journal of Functional Analysis, 165 (1999), 117-149.

[2] Ambrosetti A., Malchiodi A., Perturbation Methods and Semilinear Elliptic Problems on R^n, Birkhäuser, 2006.

[3] Aubin T., Equations differentiélles non linéaires et Problème de Yamabe concernant la courbure scalaire, J. Math. Pures et Appl. 55 (1976), 269-296.

[4] Aubin T., *Some Nonlinear Problems in Differential Geometry*, Springer-Verlag, 1998.

[5] Bahri A., *Critical points at infinity in some variational problems*, Research Notes in Mathematics, 182, Longman-Pitman, London, 1989.

[6] Bahri A., Coron J.M., The Scalar-Curvature problem on the standard three-dimensional sphere, Journal of Functional Analysis, 95 (1991), 106-172.

[7] Ben Ayed M., Chen Y., Chtioui H., Hammami M., On the prescribed scalar curvature problem on 4-manifolds, Duke Mathematical Journal, 84 (1996), 633-677.

[8] Ben Ayed M., Chtioui H., Hammami M., The scalar-curvature problem on higher-dimensional spheres. Duke Math. J. 93 (1998), no. 2, 379-424.

[9] Bianchi G., The scalar curvature equation on $\mathbb{R}^n$ and on S^n, Adv. Diff. Eq. 1 (1996), 857-880.

[10] Bianchi G., Egnell H., A variational approach to the equation $\Delta\, u + K\, u^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$, Arch. Rat. Mech. Anal. 122 (1993), 159-182.

[11] Caffarelli L., Gidas B., Spruck J., Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (1989), no. 3, 271-297.

[12] Chang S. A., Gursky M. J., Yang P., The scalar curvature equation on 2- and 3- spheres, Calc. Var. 1 (1993), 205-229.

[13] Chen C.C., Lin C.S., Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. 50 (1997), no. 10, 971-1017.

[14] Chen C.C., Lin C.S., Estimate of the conformal scalar curvature equation via the method of moving planes. II. J. Diff. Geom. 49 (1998), no. 1, 115-178.

[15] Chen C.C., Lin C.S., Blowing up with infinite energy of conformal metrics on S^N. Comm. Partial Differential Equations 24 (1999), no. 5-6, 785-799.

[16] Chang S. A., Yang P., Prescribing Gaussian curvature on S^2, Acta Math. 159 (1987), 215-259.

[17] Chang S. A., Yang P., Conformal deformation of metrics on S^2, J. Diff. Geom. 27 (1988), 256-296.

[18] Chang S. A., Yang P., A perturbation result in prescribing scalar curvature on S^n, Duke Math. J. 64 (1991), 27-69.

[19] Chen W. X., Ding W., Scalar curvature on S^2, Trans. Amer. Math. Soc. 303 (1987), 365-382.

[20] Escobar J., Schoen R., Conformal metrics with prescribed scalar curvature, Inventiones Mathematicae, 86 (1986), 243-254.

[21] Hebey E., Changements de métriques conformes sur la sphère - Le problème de Nirenberg, Bull. Sci. Math. 114 (1990), 215-242.

[22] Hebey E., Vaugon M., Le probleme de Yamabe equivariant. [The equivariant Yamabe problem] Bull. Sci. Math. 117 (1993), no. 2, 241-286.

[23] Lee J., Parker T., The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37-91.

[24] Li Y.Y., Prescribing scalar curvature on S^n and related topics, Part I, J. Diff. Equat. 120 (1995), 319-410.

[25] Li Y.Y., Prescribing scalar curvature on S^n and related topics, Part II, Existence and compactness, Comm. Pure Appl. Math. 49 (1996), 437-477.

[26] Kazdan J.L., Warner F., Kazdan, Jerry L.; Warner, F. W. Curvature functions for compact 2-manifolds. Ann. of Math. (2) 99 (1974), 14-47.

[27] Kazdan J.L., Warner F., Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, Ann. of Math. 101 (1975), 317-331.

[28] Kazdan J.L., Warner F., Kazdan, Jerry L.; Warner, F. W. Scalar curvature and conformal deformation of Riemannian structure. J. Differential Geometry 10 (1975), 113-134.

[29] Malchiodi A., The Scalar Curvature problem on S^n: an approach via Morse Theory, Calc. Var., 2002.

[30] Malchiodi A., Mayer M., Prescribing Morse scalar curvature: blow-up analysis, preprint.

[31] Malchiodi A., Mayer M., Prescribing Morse scalar curvatures: pinching and Morse Theory, to appear.

[32] Mayer M., Prescribing Morse scalar curvatures: critical points at infinity, to appear.

[33] Mayer M., A scalar curvature flow in low dimensions. Calc. Var. Partial Differential Equations 56 (2017), no. 2, Art. 24, 41 pp.

[34] Moser J., On a nonlinear problem in differential geometry, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973, 273-280.

[35] Schoen R., Zhang D., Prescribed scalar curvature on the n-sphere, Calculus of Variations and Partial Differential Equations, 4 (1996), 1-25.

[36] Talenti G., Best constant in Sobolev Inequality, Ann. Mat. Pura Appl. 110 (1976) 353-372.

[37] Wei J., Yan S., Infinitely many solutions for the prescribed scalar curvature problem on S^N. J. Funct. Anal. 258 (2010), no. 9, 3048-3081.