# Prescribing Morse scalar curvatures: blow-up analysis

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**Abstract**

We study finite-energy blow-ups for prescribed Morse scalar curvatures in both the subcritical and the critical regime. After general considerations on Palais-Smale sequences we determine precise blow up rates for subcritical solutions: in particular the possibility of tower bubbles is excluded in all dimensions. In subsequent papers we aim to establish the sharpness of this result, proving a converse existence statement, together with a one to one correspondence of blowing-up subcritical solutions and *critical points at infinity.* This analysis will be then applied to deduce new existence results for the geometric problem.

*Key Words:* Conformal geometry, sub-critical approximation, blow-up analysis.

# Contents



# 1 Introduction

The problem of prescribing the scalar curvature of a manifold conformally has a long history, starting from [33], see also [31], [32]. In case of the round sphere, this is known as *Nirenberg's problem.*

Given a closed manifold $(M, g_0)$ of dimension $n \geq 3$ and a conformal metric $g = u^{\frac{4}{n-2}} g_0$ for a positive function $u > 0$ on $M$, the conformal change of the scalar curvature is given by

$$R_{g_u} u^{\frac{n+2}{n-2}} = L_{g_0} u,$$

where by definition

$$L_{g_0} u = -c_n \Delta_{g_0} u + R_{g_0} u, \quad c_n = \frac{4(n-1)}{n-2}$$

is the *conformal Laplacian*, while $\Delta_{g_0}$ is the Laplace-Beltrami operator with respect to $g_0$. Thus, in order to prescribe a function $K$ on $M$ as the scalar curvature with respect to $g$, one needs to solve

$$L_{g_0} u = K u^{\frac{n+2}{n-2}}, \quad u > 0 \tag{1.1}$$

pointwise on $M$, see [3]. The exponent on the right-hand side is critical with respect to Sobolev's embedding, which makes the problem particularly challenging. In contrast to the *Yamabe problem*, which amounts to finding a constant scalar curvature metric, for $K$ varying on $M$ there are obstructions to the existence for (1.1). For example Kazdan and Warner proved in [33] that on the round sphere $(S^n, g_{S^n})$ every solution $u$ of (1.1) must satisfy

$$\int_{S^n} \langle \nabla K, \nabla f \rangle_{g_{S^n}} u^{\frac{2n}{n-2}} \, d\mu_{g_{S^n}} = 0$$

for any restriction $f$ to $S^n$ of an affine function on $\mathbb{R}^{n+1}$. In particular, since $u$ is positive, a necessary condition for the existence of solutions is that the function $\langle \nabla K, \nabla f \rangle_{g_{S^n}}$ changes sign.

One of the first answers to Nirenberg's problem was given by J. Moser in [41] for two dimensions, where the counterpart of (1.1) has an exponential form. He proved that for $K$ being an even function on $S^2$ a solution always exists. A related result was given by J. Escobar and R. Schoen in [23], showing existence of solutions when $K$ is invariant under some group $G$ acting without fixed points, under suitable *flatness assumptions* of order $n-2$. In the same paper some results were also found for non-spherical manifolds using positivity of the mass. Other sufficient conditions for the existence in case of $G$-invariant functions were given by E. Hebey and M. Vaugon in [25], [26], allowing the possibility of fixed points.

Other existence results were obtained by A. Chang and P. Yang, see [18], [19], for the case $n = 2$ without requiring any symmetry of $K$. One condition, for which they obtained existence, is the following. First they assumed, that $K$ is a positive Morse function satisfying

$$\{\nabla K = 0\} \cap \{\Delta K = 0\} = \emptyset, \tag{1.2}$$

where here and in the following $\nabla = \nabla_{g_0}$ and $\Delta = \Delta_{g_0}$, cf. (2.5) and below. Secondly, they supposed that $K$ possesses $p$ local maxima and $q$ saddle points with negative Laplacian and $p \neq q + 1$. The latter condition was used to prove the result via a Leray-Schauder degree-theoretical argument. In the same papers other results were given, requiring conditions only at some prescribed levels of $K$. Typically $K$ must possess two maxima $x_0$ and $x_1$, $K(x_1) \leq K(x_0)$, which are connected by some path $x(t)$ for which

$$x \text{ saddle point for } K \ \wedge \ \inf_t K(x(t)) \leq K(x) < K(x_0) \ \Rightarrow \ \Delta K(x) > 0.$$

Statements of this last kind have been obtained in [21] for $n = 2$ and in [9] for $n \geq 3$. Another existence result was given by A. Bahri and J.M. Coron in [6] for $n = 3$ and a Morse function $K$ satisfying (1.2) and

$$\sum_{x \in \{\nabla K = 0\} \cap \{\Delta K < 0\}} (-1)^{m(x,K)} \neq -1. \tag{1.3}$$

Here $m(x, K)$ denotes the Morse index of $K$ at $x$, cf. also [12]. The result of Bahri and Coron, which relies on a topological argument, has been extended in several directions.

An extension of condition (1.3), based on Morse's inequalities, was given by Schoen and Zhang in [45] for the case $n = 3$. For a Morse function $K$ satisfying (1.2) and setting

$$c_q = \sharp\{x \in M \ : \ \nabla K(x) = 0, \ \Delta K(x) < 0 \ \text{ and } \ m(K, x) = 3 - q\}$$

they required that either $c_0 - c_1 + c_2 \neq 1$ or $c_0 - c_1 > 1$. Note that the first condition is equivalent to (1.3) and the second one for $n = 2$ corresponds to the condition $p + 1 > q$ in [18].

Other results of perturbative type and relying on finite-dimensional reductions were given by A. Chang and P. Yang in [20] and by A. Ambrosetti, J. Garcia-Azorero and I. Peral in [1], see also [35]. The authors considered the case in which $K$ is close to a constant and satisfies an analogue of (1.3), i.e.

$$\sum_{x \in \{\nabla K = 0\} \cap \{\Delta K < 0\}} (-1)^{m(x,K)} \neq (-1)^n.$$

In [28] Y.Y. Li proved existence of solutions for every dimension, if the function $K$ near each critical point has a Morse-type structure, but with a flatness of order $\beta \in (n-2, n)$. His proof relied on a homotopy argument: considering $K_t = tK + (1-t)$, $t \in [0,1]$ the author used the degree-counting formula of [20] for $t$ small, and then a refined blow-up analysis of equation (1.1), when $t$ tends to 1. A different degree formula under more general flatness conditions was introduced in [16]. Other results obtained by different approaches can also be found in [8], [10], [22].

A useful tool for the above results is a subcritical approximation of (1.1), namely

$$-c_n \Delta_{g_0} u + R_{g_0} u = K\, u^{\frac{n+2}{n-2}-\tau}, \quad 0 < \tau \ll 1. \tag{1.4}$$

The advantage of (1.4), compared to (1.1), is that the lower exponent makes the problem compact, so it is easier to construct solutions. However, the interesting point is passing to the limit of solutions for $\tau \longrightarrow 0$ and in general one expects some of them to diverge with zero weak limit. The approach in [12], [45], [28] was to understand in detail the behaviour of blowing-up solutions and then to use degree- or Morse-theoretical arguments to show that some solutions stay bounded.

Consider now a Morse function $K$ on the sphere satisfying (1.2). In dimension $n = 3$ or under a flatness condition in higher dimensions, it turns out that blowing-up solutions to (1.4) develop a single bubble at critical points of $K$ with negative Laplacian. *Bubbles* correspond to solutions of (1.1) on $S^n$ with $K \equiv 1$ and were classified in [11], see also [2], [47], and after proper dilation represent the profiles of diverging solutions, cf. Section 2 for precise formulas.

The single-bubble phenomenon can be qualitatively explained exploiting the variational features of the problem, which admits the Euler-Lagrange energy $J = J_K$ given by

$$J(u) = \frac{\int_M \left(c_n |\nabla u|^2_{g_0} + R_{g_0} u^2\right) d\mu_{g_0}}{\left(\int K u^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}}},$$

see also (2.1) regarding (1.4). Denote by $\delta_{a,\lambda}$ a bubble centered at $a \in S^n$ with dilation parameter $\lambda$. Then for distinct and fixed points $a_1, a_2$ and $\lambda$ large one has the expansions

$$\int_{S^n} K(\delta_{a_1,\lambda} + \delta_{a_2,\lambda})^{\frac{2n}{n-2}} d\mu_{g_{S^n}} \simeq K(a_1) + K(a_2) + \frac{c_1}{\lambda^{n-2}}, \quad \int_{S^n} K \delta_{a_i,\lambda}^{\frac{2n}{n-2}} d\mu_{g_{S^n}} \simeq c_2 K(a_i) - \frac{c_3}{\lambda^2} \Delta K(a_i) \tag{1.5}$$

with constants $c_i > 0$, where $c_1$ depends on $a_1$ and $a_2$. We refer to Section 5 for more accurate results. Terms similar to the above ones appear in the expression of $J_\tau$. By the latter formulas and for $\lambda \longrightarrow \infty$ and $n = 3$ the *interaction* of the bubbles with $K$ is *dominated* by the mutual interactions among bubbles. This causes multiple bubbles to suppress each other allowing only one blow-up point at a time, which has to be close to at critical points of $K$ with negative Laplacian due to a Pohozaev identity.

This analysis was carried over in [29] also on $S^4$. In this case the above interactions are of the same order and multiple blow-ups occur. It was also shown there that multiple bubbles cannot accumulate at a single point. Using a terminology from [43], [44] such blow-ups are called *isolated simple.* In four dimensions a different constraint on multiple blow-up points replaces $\Delta K < 0$, depending on the least eigenvalue of a matrix constructed out of $K$ and the location of the blow-up points, cf. (0.8) in [29]. On general four-dimensional manifolds there is an extra term due to the *mass* of the manifold leading to similar phenomena, but with modified formulas, see [7].

The goal of this paper is to investigate the blow-up behaviour in an opposite regime, when the dimension $n \geq 5$ and the function $K$ is Morse. In this case the second term in (1.5) dominates the first one, so it is drastically different from situation of low-dimensions or with flat curvatures. However we can still show that blow-ups are isolated simple, which is important in understanding the Morse-theoretical structure of the energy functional. Here is our main result.

**Theorem 1.** *Let $(M^n, g_0)$, $n \geq 5$ be a closed manifold of positive Yamabe invariant and $K : M \longrightarrow \mathbb{R}$ a smooth positive Morse function satisfying* (1.2). *Then positive sequences of solutions to* (1.4) *for $\tau_m \searrow 0$ with uniformly bounded $W^{1,2}$-energy and zero weak limit have only isolated simple blow-ups at critical points of $K$ with negative Laplacian.*

The above theorem follows from Proposition 3.1, where a general characterization of blowing-up Palais-Smale sequences for (1.4) as $\tau \longrightarrow 0$ is given, and from Theorem 2, where a lower bound on the norm of the gradient of the Euler-Lagrange functional $J_\tau$ for (1.4) is proved, see (2.1).

**Remark 1.1.** *Solutions of* (1.4) *can be found as suitably normalized critical points of the scaling-invariant energy $J_\tau$ in* (2.1). *For a sequence of critical points $(u_m)$ of $J_{\tau_m}$, with $\tau_m$ as in Theorem 1, there exist up to subsequences $q \in \mathbb{N}$ and distinct points $x_1, \ldots, x_q \in M$ with $\nabla K(x_j) = 0$ and $\Delta K(x_j) < 0$ such that*

$$\left\| u_m - \sum_{j=1}^{q} \alpha_{j,m} \delta_{\lambda_{j,m}, a_{j,m}} \right\|_{W^{1,2}(M,g_0)} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty$$

*for some*

$$\alpha_{j,m} = \frac{\Theta}{K(x_j)^{\frac{n-2}{4}}} + o(1), \quad a_{j,m} \longrightarrow x_j \quad \text{and} \quad \lambda_{j,m} \simeq \lambda_{\tau_m} = \tau_m^{-\frac{1}{2}},$$

*where the multiplicative constant $\Theta$ reflects the scaling invariance of $J_{\tau_m}$, see* (2.1), *and can be fixed for instance by prescribing the conformal volume, cf. Remark 6.2. In Theorem 2 we will show much more precise estimates, that will be crucial for [36]. For example, if $n \geq 6$, we find*

$$\lambda_{j,m} = c_1 \sqrt{\frac{\Delta K(x_j)}{K(x_j)\tau}}, \quad a_{j,m} = c_2 (\nabla^2 K(x_j))^{-1} \frac{\nabla \Delta K(x_j)}{\lambda_{j,m}^3}, \quad \alpha_j = \Theta \cdot \sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_{j,m})}}$$

*up to errors of order $o(\lambda_{\tau_m}^{-3})$, where $c_1, c_2$ are dimensional constants and we identify by a slight abuse of notation $a_{j,m}$ with its image in conformal normal coordinates at $x_j$, cf. [27]. Hence all the finite dimensional variables, i.e. $\alpha_{j,m}, a_{j,m}$ and $\lambda_{j,m}$ are determined to a precision of order $o(\lambda_{\tau_m}^{-3})$.*

**Remark 1.2.** *We next compare Theorem 1 to some existing literature and add further comments.*

*(a) On $S^3$ and $S^4$ the isolated-simpleness of solutions was proved in [12], [28], [29], [45] for arbitrary sequences of solutions by a refined blow-up analysis. The uniform $W^{1,2}$-bound is then derived a-posteriori. In dimension $n \geq 5$ the latter bound may not hold true in general - we refer the reader to [13], [14], [15], where in some cases it is shown that blowing-up solutions for the purely critical equation* (1.1) *must have diverging energy and blow-ups of diverging energies and towering bubbles are also constructed, cf. also [34], [42], [48]. However, in the forthcoming paper [37] we will construct solutions to* (1.4) *via min-max or Morse theory with the purpose of finding a non-zero weak limit. These will indeed satisfy the required energy bound. This will allow us to obtain existence results under less stringent conditions compared to some others in the literature, as in [9] and [17].*

*(b) On manifolds not conformally equivalent to $S^n$ a-priori estimates were proved in [30] for $n = 3$ in both critical and subcritical cases. Our analysis carries over for $n = 4$ as well, where the matrix in Definition 6.1, introduced in [7], [29] and also involving the* mass, *gives constraints on the location of multiple blow-up points. The main new aspect of our result is the isolated simple blow-up behaviour in dimension $n \geq 5$, so we chose to state Theorem 1 in a simple form only for this case. We refer to Theorem 2 for a more precise version of the result: here we derive indeed estimates on solutions with high precision as $\tau \longrightarrow 0$, as well as estimates that are uniform in this parameter.*

*(c) In [36] we will show a converse statement. Given any distinct points $p_1, \ldots, p_k$ in $\{\nabla K = 0\} \cap \{\Delta K < 0\}$ and $\tau_i \searrow 0$ there exist solutions $(u_i)_i$ to* (1.4) *blowing-up at $p_1, \ldots, p_k$ exactly as described above. Thence the characterization of Theorem 1 is optimal. We refer to [28], [29] for the counterparts on three- and four-spheres. Proposition A2 in [5] regards the construction of a* pseudo gradient flow *for problem* (1.1) *ruling out multiple bubble formation at the same point for any $n$, although we believe the proof there is not complete. We refer to [39] for details and for the proof of a one-to-one correspondence of blowing-up sequences and* critical points at infinity, *cf. [4]. See also [40] for some delicate relations between $L^2$- and pseudo gradient flows.*

*(d) We expect the same conclusion of Theorem 1 should hold true replacing the energy bound with a Morse index bound. It would also be interesting to understand the case of non-zero weak limits.*

We discuss next some heuristics about the proof of Theorem 1. First we show a quantization result for Palais-Smale sequences of solutions to (1.4) as $\tau \longrightarrow 0$. We are inspired in this step from a result by M. Struwe in [46], where the same was proved for $\tau = 0$: in our case we need extra work in the limiting process, due to a different dilation covariance of subcritical equations.

We then prove that we are in a perturbative regime and every solution to (1.4) for $\tau$ sufficiently small can be written as a finite sum of highly peaked bubbles and an error term small in $W^{1,2}$-norm, which we prove to have a minor effect in the expansions. Performing a careful analysis of the interactions of the bubbles among themselves and with $K$, it is not difficult to see that for $n \geq 5$ blow-ups should occur at critical points of $K$ with negative Laplacian only, cf. also Theorem 1.1 in [14], and we are left with excluding multiple bubbles towering at the same limit point, which is the crucial result in our paper.

We give an idea of this fact in some particular cases, that are easy to describe. Let $J_\tau$ be the Euler-Lagrange energy of (1.4), see (2.1). For a critical point $a$ of $K$, the following expansion holds for $J_\tau$ on a *bubble* concentrated at $a$

$$J_\tau(\delta_{a,\lambda}) \simeq \frac{1}{K^{\frac{n-2}{n}}(a)}(\lambda^\tau - \frac{\Delta K(a)}{K(a)\lambda^2}), \tag{1.6}$$

cf. Proposition 5.1. By elementary considerations one checks that for $\Delta K(a) < 0$ the function in the right-hand side has a non-degenerate minimum point at $\lambda = \lambda_\tau \simeq \tau^{-\frac{1}{2}}$, see also Proposition 2.1 in [45]. Since bubbles have an *attractive interaction* , cf. the first equation in (1.5), even in terms of dilations centering more bubbles at the point $a$ would make all dilation parameters *collapse* at $\lambda = \lambda_\tau$, see Figure 1. For the same reason, still by (1.6), one would get collapse with respect to the center points of multiple bubbles distributed along the unstable directions from a critical point of $K$, since points with lager values of $K$ have smaller energy, due to (1.6), see Figure 2. We consider then the case of bubbles centered at two

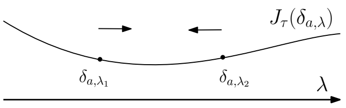


Figure 1: two bubbles with same center, different $\lambda$'s

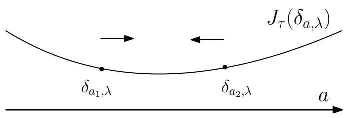


Figure 2: two bubbles along unstable direction of $K$, same $\lambda$

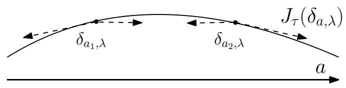


Figure 3: two bubbles along stable direction of $K$, same $\lambda$

points $a_1, a_2$ symmetrically located at distance $d$ from a critical point $\bar{p}$ such that $\Delta K(\bar{p}) < 0$, and along a stable direction of $K$, with the same $\lambda$'s. Here in principle the attractive force among bubbles could compensate the *repulsive* interaction from the critical point $\bar{p}$ of $K$, see Figure 3. For this configuration one gets an energy expansion of the form

$$J_\tau(\delta_{a_1,\lambda} + \delta_{a_2,\lambda}) \simeq \frac{c_0}{K^{\frac{n-2}{n}}(a_1)}(\lambda^\tau - \frac{\Delta K(a_1)}{K(a_1)\lambda^2}) - c_1\frac{1}{d^{n-2}\lambda^{n-2}} \simeq (c_2 - c_3 d^2)\left(\lambda^\tau + c_4\lambda^{-2}\right) - c_1\frac{1}{d^{n-2}\lambda^{n-2}}$$

with $c_i > 0$. From the analysis in Proposition 3.1 it turns out that $\lambda^\tau \simeq 1$, so imposing criticality in both $\lambda$ and $d$ one finds the relations

$$\frac{1}{\lambda^2} \simeq \tau + \frac{1}{(\lambda d)^{n-2}} \quad \text{and} \quad d \simeq \frac{1}{\lambda^{n-2}d^{n-1}}.$$

These asymptotics imply that $\lambda^{-2} \simeq \tau + \lambda^{-\frac{2(n-2)}{n}}$, which is impossible for $\lambda$ large. The general case is rather involved to study and will be treated by a top-down cascade of estimates in Section 6.

The plan of the paper is the following. In Section 2 we introduce the variational setting of the problem and list some preliminary results. We then study some approximate solutions of (1.1), highly concentrated at arbitrary points of $M$. From these one can carry out a reduction procedure of the problem, which is done later in the paper. In Section 3 we prove a general quantization result for Palais-Smale sequences of (1.4) with uniformly bounded $W^{1,2}$-energy. In Section 4 we reduce the problem to a finite-dimensional one, while in Section 5 we derive some precise asymptotic expansions of the Euler-Lagrange energy. Section 6 is then devoted to proving suitable bounds on the gradient to exclude tower bubbles and prove our main result. We finally collect in the appendix the proofs of some useful technical estimates as well as a list of relevant constants appearing.

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# 2 Variational setting and preliminaries

In this section we collect some background and preliminary material, concerning the variational properties of the problem and some estimates on highly-concentrated approximate solutions of bubble type.

We consider a smooth, closed Riemannian manifold $M = (M^n, g_0)$ with volume measure $\mu_{g_0}$ and scalar curvature $R_{g_0}$. Letting $\mathcal{A} = \{u \in W^{1,2}(M, g_0) \mid u \geq 0, u \not\equiv 0\}$ the *Yamabe invariant* is defined as

$$Y(M, g_0) = \inf_{\mathcal{A}} \frac{\int \left(c_n|\nabla u|^2_{g_0} + R_{g_0}u^2\right) d\mu_{g_0}}{\left(\int u^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}}}, \quad c_n = 4\frac{n-1}{n-2}.$$

We will assume from now on that the invariant is positive. As a consequence the *conformal Laplacian*

$$L_{g_0} = -c_n\Delta_{g_0} + R_{g_0}$$

is a positive and self-adjoint operator. Without loss of generality we assume $R_{g_0} > 0$ and denote by

$$G_{g_0} : M \times M \setminus \Delta \longrightarrow \mathbb{R}_+$$

the Green's function of $L_{g_0}$. Considering a conformal metric $g = g_u = u^{\frac{4}{n-2}} g_0$ there holds

$$d\mu_{g_u} = u^{\frac{2n}{n-2}} d\mu_{g_0} \qquad \text{and} \qquad R = R_{g_u} = u^{-\frac{n+2}{n-2}}(-c_n\Delta_{g_0}u + R_{g_0}u) = u^{-\frac{n+2}{n-2}} L_{g_0}u.$$

Note that

$$c\|u\|^2_{W^{1,2}(M,g_0)} \leq \int u\, L_{g_0}u\, d\mu_{g_0} = \int \left(c_n|\nabla u|^2_{g_0} + R_{g_0}u^2\right) d\mu_{g_0} \leq C\|u\|^2_{W^{1,2}(M,g_0)}.$$

In particular we may define and use $\|u\|^2 = \|u\|^2_{L_{g_0}} = \int u\, L_{g_0}u\, d\mu_{g_0}$ as an equivalent norm on $W^{1,2}$. For

$$p = \frac{n+2}{n-2} - \tau \;\text{ and }\; 0 \leq \tau \longrightarrow 0$$

we want to study the scaling-invariant functionals

$$J_\tau(u) = \frac{\int_M \left(c_n|\nabla u|^2_{g_0} + R_{g_0}u^2\right) d\mu_{g_0}}{\left(\int K u^{p+1} d\mu_{g_0}\right)^{\frac{2}{p+1}}}, \quad u \in \mathcal{A}. \tag{2.1}$$

Since the conformal scalar curvature $R = R_u$ for $g = g_u = u^{\frac{4}{n-2}} g_0$ satisfies

$$r = r_u = \int R d\mu_{g_u} = \int u L_{g_0} u d\mu_{g_0}, \tag{2.2}$$

we have

$$J_\tau(u) = \frac{r}{k_\tau^{\frac{2}{p+1}}} \qquad \text{with} \qquad k_\tau = \int K\, u^{p+1} d\mu_{g_0}. \tag{2.3}$$

The first- and second-order derivatives of the functional are given by

$$\partial J_\tau(u)v = \frac{2}{k_\tau^{\frac{2}{p+1}}}\left[\int L_{g_0}uv d\mu_{g_0} - \frac{r}{k_\tau}\int K u^p v d\mu_{g_0}\right];$$

$$\begin{aligned}\partial^2 J_\tau(u)vw =&\frac{2}{k_\tau^{\frac{2}{p+1}}}\Big[\int L_{g_0}vwd\mu_{g_0}-p\frac{r}{k_\tau}\int Ku^{p-1}vwd\mu_{g_0}\Big]\\&-\frac{4}{k_\tau^{\frac{2}{p+1}+1}}\Big[\int L_{g_0}uvd\mu_{g_0}\int Ku^pwd\mu_{g_0}+\int L_{g_0}uwd\mu_{g_0}\int Ku^pvd\mu_{g_0}\Big]\\&+\frac{2(p+3)r}{k_\tau^{\frac{2}{p+1}+2}}\int Ku^pvd\mu_{g_0}\int Ku^pwd\mu_{g_0}.\end{aligned}$$

In particular $J_\tau$ is of class $C^{2,\alpha}_{loc}(\mathcal{A})$ and uniformly Hölder continuous on each set of the form

$$U_\epsilon=\{u\in\mathcal{A}\mid \epsilon<\|u\|,\ J_\tau(u)\le\epsilon^{-1}\}.$$

Indeed $u\in U_\epsilon$ implies

$$\epsilon^2\le r\le\epsilon^{-2}\ \text{ and }\ c\,\epsilon^3\le k_\tau^{\frac{1}{p+1}}=J_\tau(u)^{-1}r_u\le C\epsilon^{-3}.$$

Thus uniform Hölder continuity on $U_\epsilon$ follows from the standard pointwise estimates

$$\begin{cases}||a|^p-|b|^p|\le C_p|a-b|^p & \text{in case } 0<p<1\\ ||a|^p-|b|^p|\le C_p\max\{|a|^{p-1},|b|^{p-1}\}|a-b| & \text{in case } p\ge 1\end{cases}\tag{2.4}$$

We consider next some approximate solutions to (1.1), highly concentrated at arbitrary points of $M$. As we will see, for suitable values of $\lambda$ these are also approximate solutions of (1.4). Let us recall the construction of *conformal normal coordinates* from [27]. Given $a\in M$, one chooses a special conformal metric

$$g_a=u_a^{\frac{4}{n-2}}g_0\quad\text{ with }\quad u_a=1+O(d^2_{g_0}(a,\cdot)),\tag{2.5}$$

whose volume element in $g_a$-geodesic normal coordinates coincides with the Euclidean one, see also [24]. In particular

$$(exp_a^{g_0})^{-}\circ\exp_a^{g_a}(x)=x+O(|x|^3)$$

for the exponential maps centered at $a$, which e.g. implies

$$\nabla_{g_0}K(a)=\nabla_{g_a}K(a),\ \nabla^2_{g_0}K(a)=\nabla^2_{g_a}K(a),$$

and in case $\nabla K(a)=0$ also

$$\nabla^3_{g_0}K(a)=\nabla^3_{g_a}K(a).$$

Moreover by smoothness of the exponential map $\exp_{g_a}=\exp_a^{g_a}$ with respect to $a$ there holds

$$\nabla_a\exp_{g_a}(x)=id+O(|x|^2)\tag{2.6}$$

in a $g_a$-normal chart, as seen from the corresponding geodesic equation. We then denote by $r_a$ the geodesic distance from $a$ with respect to the metric $g_a$ just introduced. With this choice the expression of the Green's function $G_{g_a}$ with pole at $a\in M$, denoted by $G_a=G_{g_a}(a,\cdot)$, for the conformal Laplacian $L_{g_a}$ simplifies considerably. From Section 6 in [27] one may expand

$$G_a=\frac{1}{4n(n-1)\omega_n}(r_a^{2-n}+H_a),\ r_a=d_{g_a}(a,\cdot),\ H_a=H_{r,a}+H_{s,a}\ \text{ for }\ g_a=u_a^{\frac{4}{n-2}}g_0,\tag{2.7}$$

where $\omega_n=|S^{n-1}|$. Here $H_{r,a}\in C^{2,\alpha}_{loc}$, while the *singular* error term satisfies

$$H_{s,a}=O\begin{pmatrix}0 & \text{for } n=3\\ r_a^2\ln r_a & \text{for } n=4\\ r_a & \text{for } n=5\\ \ln r_a & \text{for } n=6\\ r_a^{6-n} & \text{for } n\ge 7\end{pmatrix}.$$

Precisely the leading term in $H_{s,a}$ for $n = 6$ is $-\frac{|\mathbb{W}(a)|^2}{288c_n}\ln r$, where $\mathbb{W}$ denotes the Weyl tensor. Let

$$\varphi_{a,\lambda} = u_a \left(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}, \quad G_a = G_{g_a}(a,\cdot), \quad \gamma_n = (4n(n-1)\omega_n)^{\frac{2}{2-n}} \quad \text{for} \quad \lambda > 0. \tag{2.8}$$

We notice that the constant $\gamma_n$ is chosen so that

$$\gamma_n G_a^{\frac{2}{2-n}}(x) = d^2_{g_a}(a,x) + o(d^2_{g_a}(a,x)) \quad \text{as} \quad x \longrightarrow a.$$

Evaluating the conformal Laplacian on such functions shows that they are approximate solutions.

**Lemma 2.1.** *There holds $L_{g_0}\varphi_{a,\lambda} = O(\varphi_{a,\lambda}^{\frac{n+2}{n-2}})$. More precisely on a geodesic ball $B_\alpha(a)$ for $\alpha > 0$ small*

$$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} - 2nc_n r_a^{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + \frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}} + o(r_a^{n-2})\varphi_{a,\lambda}^{\frac{n+2}{n-2}},$$

*where $r_a = d_{g_a}(a,\cdot)$. Since $R_{g_a} = O(r_a^2)$ in conformal normal coordinates, cf. [27], we obtain*

*(i)* $$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)[1 - \tfrac{c_n}{2}r_a^{n-2}(H_a(a) + n\nabla H_a(a)x)]\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O\begin{pmatrix} \lambda^{-\frac{3}{2}}\varphi_{a,\lambda}^{\frac{n-1}{n-2}} & \textit{for } n=3 \\ \frac{\ln r}{\lambda^2}\varphi_{a,\lambda}^{\frac{n-1}{n-2}} & \textit{for } n=4 \\ \lambda^{-2}\varphi_{a,\lambda} & \textit{for } n=5 \end{pmatrix};$$

*(ii)* $$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} = 4n(n-1)[1 + \tfrac{c_n}{2}W(a)\ln r]\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O(\lambda^{-2}\varphi_{a,\lambda}) \quad \textit{for } n=6;$$

*(iii)* $$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O(\lambda^{-2}\varphi_{a,\lambda}) \quad \textit{for } n \geq 7.$$

*The expansions stated above persist upon taking $\lambda\partial\lambda$ and $\frac{\nabla_a}{\lambda}$ derivatives.*

*Proof.* A straightforward calculation shows that

$$\Delta_{g_a}\Big(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}}\Big)^{\frac{n-2}{2}} = \frac{n}{2-n}\gamma_n\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n+2}{n-2}}|\nabla G_a|^2_{g_a}G_a^{2\frac{n-1}{2-n}} + \gamma_n\lambda\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n}{n-2}}G_a^{\frac{n}{2-n}}\Delta_{g_a}G_a,$$

which is due to $|\nabla G_a|^2_{g_a}G_a^{2\frac{n-1}{2-n}} = (n-2)^2|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a}$ and $c_n\Delta_{g_a}G_a = -\delta_a + R_{g_a}G_a$ with $\delta_a$ denoting the Dirac measure at $a$. This is equivalent to

$$\Delta_{g_a}\Big(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}}\Big)^{\frac{n-2}{2}} = n(2-n)\gamma_n\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n+2}{n-2}}|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} + \frac{R_{g_a}\gamma_n}{c_n}\lambda\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n}{n-2}}G_a^{\frac{2}{2-n}}.$$

Since $L_{g_a} = -c_n\Delta_{g_a} + R_{g_a}$ with $c_n = 4\frac{n-1}{n-2}$, we obtain

$$L_{g_a}\frac{\varphi_{a,\lambda}}{u_a} = 4n(n-1)\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n+2}{n-2}}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} + \frac{R_{g_a}}{\lambda}\Big(\frac{\varphi_{a,\lambda}}{u_a}\Big)^{\frac{n}{n-2}}.$$

By conformal covariance we also get

$$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} + \frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}},$$

in particular $L_{g_0}\varphi_{a,\lambda} = O(\varphi_{a,\lambda}^{\frac{n+2}{n-2}})$. Expanding $G_a$ as $G_a = \frac{1}{4n(n-1)\omega_n}(r_a^{2-n} + H_a)$, $r_a = d_{g_a}(a,\cdot)$ we find

$$\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} = |\nabla(r_a(1+r_a^{n-2}H_a)^{\frac{1}{2-n}})|^2_{g_a} = 1 - \frac{2}{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)r_a^{n-2} + o(r_a^{n-2}),$$

and conclude that

$$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} - 2nc_n((n-1)H_a + r_a\partial_{r_a}H_a)r_a^{n-2}\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + o(r_a^{n-2}\varphi_{a,\lambda}^{\frac{n+2}{n-2}}) + \frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}}.$$

Clearly these calculations transcend to the $\lambda$ and $a$ derivatives. Then the claim follows from the above expansion of the Green's function. $\square$

After introducing some notation we state a useful lemma, which will be proved in the first appendix.

**Notation.** Given an exponent $p \geq 1$ we will denote by $L^p_{g_0}$ the set of functions of class $L^p$ with respect to the measure $d\mu_{g_0}$. Recall also that for $u \in W^{1,2}(M, g_0)$ we set $r_u = \int u L_{g_0} u d\mu_{g_0}$, while for a point $a \in M$ we denote by $r_a$ the geodesic distance from $a$ with respect to the metric $g_a$ introduced above. For a set of points $\{a_i\}_i$ of $M$ we will denote by $K_i, \nabla K_i$ and $\Delta K_i$ for instance

$$K(a_i),\ \nabla K(a_i) = \nabla_{g_0} K(a_i) \quad \text{and} \quad \Delta K(a_i) = \Delta_{g_0} K(a_i).$$

For $k, l = 1, 2, 3$ and $\lambda_i > 0$, $a_i \in M$, $i = 1, \dots, q$ let

(i) $\varphi_i = \varphi_{a_i,\lambda_i}$ and $(d_{1,i}, d_{2,i}, d_{3,i}) = (1, -\lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})$;

(ii) $\phi_{1,i} = \varphi_i,\ \phi_{2,i} = -\lambda_i \partial_{\lambda_i}\varphi_i,\ \phi_{3,i} = \frac{1}{\lambda_i}\nabla_{a_i}\varphi_i$, so $\phi_{k,i} = d_{k,i}\varphi_i$.

Note that with the above definitions the $\phi_{k,i}$'s are uniformly bounded in $W^{1,2}(M, g_0)$.

**Lemma 2.2.** *Let $\theta = \frac{n-2}{2}\tau$ and $k, l = 1, 2, 3$ and $i, j = 1, \dots, q$. Then for*

$$\varepsilon_{i,j} = \Big(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j)\Big)^{\frac{2-n}{2}} \tag{2.9}$$

*there holds uniformly as $0 \leq \tau \longrightarrow 0$*

*(i)* $|\phi_{k,i}|, |\lambda_i\partial_{\lambda_i}\phi_{k,i}|, |\frac{1}{\lambda_i}\nabla_{a_i}\phi_{k,i}| \leq C\varphi_i$;

*(ii)* $\lambda_i^\theta \int \varphi_i^{\frac{4}{n-2}-\tau}\phi_{k,i}\phi_{k,i}d\mu_{g_0} = c_k \cdot id + O(\tau + \frac{1}{\lambda_i^{n-2+\theta}} + \frac{1}{\lambda_i^{2+\theta}}),\ c_k > 0$;

*(iii)* *for $i \neq j$ up to some error of order* $O(\tau^2 + \sum_{i\neq j}(\frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}} + \varepsilon_{i,j}^{\frac{n+2}{n}}))$

$$\lambda_i^\theta \int \varphi_i^{\frac{n+2}{n-2}-\tau}\phi_{k,j}d\mu_{g_0} = b_k d_{k,i}\varepsilon_{i,j} = \int \varphi_i^{1-\tau} d_{k,j}\varphi_j^{\frac{n+2}{n-2}}d\mu_{g_0};$$

*(iv)* $\lambda_i^\theta \int \varphi_i^{\frac{4}{n-2}-\tau}\phi_{k,i}\phi_{l,i}d\mu_{g_0} = O(\frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i^2})$ *for $k \neq l$ and for $k = 2, 3$*

$$\lambda_i^\theta \int \varphi_i^{\frac{n+2}{n-2}-\tau}\phi_{k,i}d\mu_{g_0} = O\left(\tau + \begin{pmatrix} \lambda_i^{2-n} & \text{for } n \leq 5 \\ \frac{\ln\lambda_i}{\lambda_i^4} & \text{for } n = 6 \\ \lambda_i^4 & \text{for } n \geq 7 \end{pmatrix}\right);$$

*(v)* $\lambda_i^\theta \int \varphi_i^{\alpha-\tau}\varphi_j^\beta d\mu_{g_0} = O(\varepsilon_{i,j}^\beta)$ *for $i \neq j$, $\alpha + \beta = \frac{2n}{n-2}$, $\alpha - \tau > \frac{n}{n-2} > \beta \geq 1$;*

*(vi)* $\int \varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}}d\mu_{g_0} = O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}),\ i \neq j$;

*(vii)* $(1, \lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})\varepsilon_{i,j} = O(\varepsilon_{i,j}),\ i \neq j$.

*with constants* $b_k = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}}$ *for $k = 1, 2, 3$ and*

$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n},\quad c_2 = \frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{(r^2-1)^2dx}{(1+r^2)^{n+2}},\quad c_3 = \frac{(n-2)^2}{n}\int_{\mathbb{R}^n}\frac{r^2dx}{(1+r^2)^{n+2}}.$$

## 3 Blow-up analysis

In this section we prove a result related to a well-known one in [46]. We obtain indeed similar conclusions, but allowing the exponent in the equation to vary along a sequence of approximate solutions.

**Proposition 3.1.** *Let $(u_m)_m \subset W^{1,2}(M,g_0)$ be a sequence with $u_m \geq 0$ and $k_{\tau_m} = 1$ satisfying*

$$J_{\tau_m}(u_m) = r_{u_m} \longrightarrow r_\infty \quad \text{and} \quad \partial J_{\tau_m}(u_m) \longrightarrow 0 \ \text{in} \ W^{-1,2}(M,g_0).$$

*Then up to a subsequence there exist $u_\infty : M \longrightarrow [0,\infty)$ smooth, $q \in \mathbb{N}_0$ and for $i = 1,\dots,q$ sequences*

$$M \supset (a_{i,m}) \longrightarrow a_{i_\infty} \quad \text{and} \quad \mathbb{R}_+ \supset \lambda_{i,m} \longrightarrow \infty \quad \text{as} \quad m \longrightarrow \infty$$

*such that $u_m = u_\infty + \sum_{i=1}^q \alpha_i \varphi_{a_{i,m},\lambda_{i,m}} + v_m$ with*

$$\partial J_0(u_\infty) = 0, \quad \|v_m\| \longrightarrow 0, \quad \lambda_{i,m}^{\tau_m} \longrightarrow 1 \quad \text{and} \quad \frac{r_\infty K(a_{i_\infty})\alpha_i^{\frac{4}{n-2}}}{4n(n-1)} = 1$$

*and $(\varepsilon_{i,j})_m \longrightarrow 0$ as $m \longrightarrow \infty$ for each pair $1 \leq i < j \leq q$ .*

*Proof.* Setting $J = J_{\tau_m}$, by our assumptions we have

$$J(u_m) = \int u_m L_{g_0} u_m d\mu_{g_0} \longrightarrow r_\infty \ \text{ and } \ \partial J(u_m) = L_{g_0} u_m - r_\infty K u_m^{p_m} = o(1) \text{ in } W^{-1,2}(M,g_0).$$

In particular $(u_m) \subset W^{1,2}(M,g_0)$ is bounded, hence $u_m \rightharpoonup u_\infty$ weakly in $W^{1,2}(M,g_0)$ and strongly in $L^q(M,g_0)$, $q < \frac{2n}{n-2}$. Notice that $u_\infty \geq 0$ is a critical point of $J_0$ and therefore it is a smooth function. We may then write $u_m = u_\infty + u_{1,m}$ with $u_{1,m} \rightharpoonup 0$ weakly, and strongly in $L^q(M,g_0)$. Thus

$$r_\infty \longleftarrow J(u_m) = \int u_\infty L_{g_0} u_\infty d\mu_{g_0} + \int u_{1,m} L_{g_0} u_{1,m} d\mu_{g_0} + o(1),$$

whence $\int u_{1,m} L_{g_0} u_{1,m} d\mu_{g_0} \longrightarrow r_{1,\infty} \geq 0$ and secondly, due to (2.4), that

$$E(u_{1,m}) := L_{g_0} u_{1,m} - r_\infty K u_{1,m}^{p_m} = o(1) \quad \text{in } W^{-1,2}(M,g_0). \tag{3.1}$$

We may assume $r_{1,\infty} > 0$, since otherwise we are done. We now claim the concentration behavior

$$\forall\, 0 < \varepsilon \ll 1 \ \exists\, \lambda_m \longrightarrow \infty \ : \ \sup_{x \in M} \int_{B_{\frac{1}{\lambda_m}}(x)} |\nabla u_{1,m}|_{g_0}^2 d\mu_{g_0} \geq \varepsilon. \tag{3.2}$$

Indeed we have for a fixed cut-off function

$$\begin{aligned} o(1) =& \langle E(u_{1,m}), u_{1,m}\eta^2 \rangle = \int \left[ (\eta u_{1,m}) L_{g_0}(\eta u_{1,m}) - r_\infty K |\eta u_{1,m}|^2 u_{1,m}^{p_m-1} \right] d\mu_{g_0} + o(1) \\ \geq& \|\nabla(\eta u_{1,m})\|^2 - r_\infty K_{\min} \|\eta u_{1,m}\|^2_{L^{p_m+1}_{\mu_{g_0}}} \|u_{1,m}\|^{p_m-1}_{L^{p_m+1}_{\mu_{g_0}}(supp(\eta))} + o(1). \end{aligned}$$

Using Hölder's inequality and Sobolev's embedding we obtain

$$o(1) \geq \|\nabla(\eta u_{1,m})\|^2 (1 - C\|u_{1,m}\|^{p_m-1}_{L^{p_m+1}_{\mu_{g_0}}(supp(\eta))}) + o(1).$$

Thus, if $u_{1,m}$ does not concentrate in $L^{p_m+1}(M,g_0)$ similarly to (3.2), then by a covering argument

$$\int |\nabla u_{1,m}|_{g_0}^2 d\mu_{g_0} \longrightarrow 0$$

contradicting $r_{1,\infty} > 0$. By (3.1) concentration in $L^{p_m+1}(M,g_0)$ is equivalent to concentration in $L^2$-norm for the gradient, which had to be shown. Fixing $\varepsilon > 0$ small, we measure the rate of concentration via

$$\Lambda_{1,m} = \sup \left\{ \lambda > 0 \ \mid \ \max_{x \in M} \int_{B_{\frac{1}{\lambda}}(x)} |\nabla u_{1,m}|_{g_0}^2 d\mu_{g_0} = \varepsilon \right\} \longrightarrow \infty,$$

and choose for any $\lambda_{1,m} \nearrow \infty$ with $1 \leq \lim_{m\to\infty} \frac{\Lambda_{1,m}}{\lambda_{1,m}} = \delta < \infty$ up to a subsequence

$$(a_{1,m}) \subset M \ : \ \int_{B_{\frac{1}{\lambda_{1,m}}}(a_{1,m})} |\nabla u_{1,m}|^2_{g_0} d\mu_{g_0} = \sup_{x\in M} \int_{B_{\frac{1}{\lambda_{1,m}}}(x)} |\nabla u_{1,m}|^2_{g_0} d\mu_{g_0} \geq c$$

for some positive $c = c(\varepsilon, \delta)$ to be specified later. On a suitably small ball $B_\rho(a_{1,m})$ we then rescale

$$w_{1,m} = \lambda_{1,m}^{\frac{2-n}{2}} u_{1,m}\Big( \exp_{g_{a_{1,m}}} \frac{\cdot}{\lambda_{1,m}}\Big).$$

The function $w_{1,m}$ is well defined on $B_{\rho\lambda_{1,m}}(0)$ and satisfies, with $\theta_m = \frac{n-2}{2}\tau_m$,

$$-c_n\Delta w_{1,m} - \frac{r_\infty K(a_{i,m})}{\lambda_{1,m}^{\theta_m}} w_{1,m}^{p_m} = o(1) \quad \text{in } W^{-1,2}_{loc}(\mathbb{R}^n), \quad \Delta = \Delta_{\mathbb{R}^n}.$$

Since $\int |\nabla u_{1,m}|^2 d\mu_{g_0}$ is bounded, so it is $\int_{B_{\rho\lambda_{1,m}}(0)} |\nabla w_{1,m}|^2 dx$ for any $\rho > 0$. Hence

$$w_{1,m} \rightharpoonup w_{1,\infty} \text{ weakly in } W^{-1,2}_{loc}(\mathbb{R}^n) \ \text{ with } \ -\Delta w_{1,\infty} = \sigma_1 r_\infty \kappa_1 w_{1,\infty}^{\frac{n+2}{n-2}},$$

where

$$\kappa_1 = \lim_{m\to\infty} K(a_{1,m}) \ \text{ and } \ \sigma_1 = \lim_{m\to\infty} \lambda_{1,m}^{-\theta_m} \in [0,1].$$

Given a compactly supported cut-off $\eta$, we calculate

$$\begin{aligned}
0 \longleftarrow & \int_{\mathbb{R}^n} (w_{1,m} - w_{1,\infty})\eta^2\big(\Delta w_{1,m} + \frac{r_\infty K}{\lambda_{1,m}^{\theta_m}} w_{1,m}^{p_m}\big)dx \\
= & \int_{\mathbb{R}^n} (w_{1,m} - w_{1,\infty})\eta^2\big(\Delta(w_{1,m} - w_{1,\infty}) + \sigma_1 r_\infty K(w_{1,m}^{p_m} - w_{1,\infty}^{\frac{n+2}{n-2}})\big)dx + o(1) \\
\leq & - \int_{\mathbb{R}^n} |\nabla((w_{1,m} - w_{1,\infty})\eta)|^2 dx + \sigma_1 r_\infty \int_{\mathbb{R}^n} K\eta^2 |w_{1,m} - w_{1,\infty}|^{p_m+1} dx + o(1) \\
= & - \int_{\mathbb{R}^n} |\nabla((w_{1,m} - w_{1,\infty})\eta)|^2 dx + \sigma_1 r_\infty \int_{\mathbb{R}^n} K\eta^2 |w_{1,m} - w_{1,\infty}|^{p_m+1} dx + o(1).
\end{aligned} \tag{3.3}$$

The main step here is the inequality in the above formula. Passing from $\frac{n+2}{n-2}$ to $p_m = \frac{n+2}{n-2} - \tau_m$ in the exponent is easy, as $w_{1,\infty}$ is fixed. Since $w_{1,m} \to w_{1,\infty}$ in $L^p(supp(\eta))$, $p < \frac{2n}{n-2}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} K\eta^2 (w_{1,m} - w_{1,\infty})(w_{1,m}^{p_m} - w_{1,\infty}^{p_m})dx = & \int_{\mathbb{R}^n} K\eta^2 (w_{1,m}^{p_m+1} - w_{1,\infty}^{p_{m+1}})dx \\
= \int_{\mathbb{R}^n} K\eta^2 \Big[ - \int_0^1 \partial_s |w_{1,m} - s w_{1,\infty}|^{p_m+1} ds - & w_{1,\infty}^{p_m+1} + |w_{1,m} - w_{1,\infty}|^{p_{m+1}} \Big] dx.
\end{aligned}$$

Therefore the main inequality follows from observing that

$$\begin{aligned}
& \Big| \int_{\mathbb{R}^n} K\eta^2 \Big[ - \int_0^1 \partial_s |w_{1,m} - s w_{1,\infty}|^{p_m+1} ds - w_{1,\infty}^{p_m+1}\Big] dx \Big| \\
& \quad \leq \int_0^1 ds \int_{\mathbb{R}^n} K\eta^2 [(p_{m+1})(w_{1,m} - s w_{1,\infty})|w_{1,m} - s w_{1,\infty}|^{p_m-1} w_{1,\infty} - w_{1,\infty}^{p_{m+1}}] dx \\
& \quad \longrightarrow \int_0^1 ds \int_{\mathbb{R}^n} K\eta^2 [(p_{m+1})(1-s)^{p_m} w_{1,\infty}^{p_m+1} - w_{1,\infty}^{p_{m+1}}] dx = 0.
\end{aligned}$$

Hence (3.3) is justified and we obtain as before

$$\int_{\mathbb{R}^n} |\nabla((w_{1,m} - w_{1,\infty})\eta)|^2 (1 - C\|w_{1,m} - w_{1,\infty}\|^{p_m - 1}_{L^{p_m+1}(supp(\eta))})dx \leq o(1).$$

Thus $w_{1,m} \longrightarrow w_{1,\infty}$ locally strongly, unless $w_{1,m}$ concentrates in $L^{p_m+1}$, but by our choice of $\Lambda_{1,m}$

$$\varepsilon = \sup_{x \in M} \int_{B_{\frac{1}{\Lambda_{1,m}}}(x)} |\nabla u_{1,m}|^2_{g_0} d\mu_{g_0} \geq \sup_{x \in B_{c\lambda_{1,m}}(0) \subset \mathbb{R}^n} \int_{B_{\frac{\lambda_{1,m}}{\Lambda_{1,m}}}(x)} |\nabla w_{1,m}|^2 dx$$

and $1 \geq \frac{\lambda_{1,m}}{\Lambda_{1,m}} \not\longrightarrow 0$, so the $L^2$-gradient norm does not concentrate beyond $\varepsilon$ and, since

$$-c_n \Delta_{\mathbb{R}^n} w_{1,m} - \frac{r_\infty K(a_{1,m})}{\lambda_{1,m}^{\theta_m}} w_{1,m}^{p_m} = o(1) \text{ locally strongly in } W^{-1,2}_{loc}(\mathbb{R}^n),$$

neither the $L^{p_m+1}$-norm does. Thus $w_{1,m} \longrightarrow w_{1,\infty}$ locally strongly. In particular

$$\int_{B_1(0)} |\nabla w_{1,\infty}|^2 dx \longleftarrow \int_{B_{\frac{1}{\lambda_{1,m}}}(a_{1,m})} |\nabla u_{1,m}|^2_{g_0} d\mu_{g_0} \geq c = c(\varepsilon, \delta).$$

But $\sigma_1 = 0$ implies $w_{1,\infty} = 0$ by harmonicity, so $\sigma_1 \in (0,1]$, cf. (3.3), and we easily show $w_{1,\infty} > 0$ and

$$w_{1,\infty} = \alpha_1 \left( \frac{\tilde{\lambda}_1}{1 + \tilde{\lambda}_1^2 r_a^2} \right)^{\frac{n-2}{2}} \quad \text{with } \alpha_1 > 0, \; r_a = |x - a|, \; a \in \mathbb{R}^n \text{ and } \tilde{\lambda}_1 > 0.$$

Note that $-\Delta_{\mathbb{R}^n} w_{1,\infty} = \sigma_1 r_\infty \kappa_1 w_{1,\infty}^{\frac{n+2}{n-2}}$ implies $\sigma_1 r_\infty \kappa_1 \alpha_1^{\frac{4}{n-2}} = 4n(n-1)$. Moreover by construction

$$\int_{B_1(0)} |\nabla w_{1,m}|^2 dx \geq \sup_{x \in B_{c\lambda_{1,m}}(0)} \int_{B_1(x)} |\nabla w_{1,m}|^2 dx,$$

which transfers to $w_{1,\infty}$ by locally strong convergence. This implies $a = 0$ and

$$\frac{\tilde{\lambda}_1^n}{1 + \tilde{\lambda}_1^n} \sim \int_{B_1(0)} \left| \nabla \left( \frac{\tilde{\lambda}_1}{1 + \tilde{\lambda}_1^2 r^2} \right)^{\frac{n-2}{2}} \right|^2 dx = \varepsilon \alpha_1^{-2} = \varepsilon (\sigma_1 r_\infty \kappa_1)^{\frac{n-2}{2}}.$$

By $\lim_{m \to \infty} \lambda_{1,m}^{-\theta_m} = \sigma_1 \in (0,1]$ and $0 < \varepsilon \ll 1$ we get $\tilde{\lambda}_1 \sim \lim_{m \to \infty} \lambda_{1,m}^{\frac{2-n}{2n}\theta_m}$. Dilating back we may then write

$$u_m = u_\infty + \alpha_1 \varphi_{1,m} + u_{2,m}, \qquad \varphi_{1,m} = \varphi_{a_{1,m}, \bar{\lambda}_{1,m}}, \qquad \bar{\lambda}_{1,m} = \tilde{\lambda}_1 \lambda_{1,m}.$$

Moreover we know that $u_{2,m} \rightharpoonup 0$ weakly in $W^{1,2}(M, g_0)$ and

$$w_{2,m} = (\bar{\lambda}_{1,m})^{\frac{2-n}{2}} u_{2,m} \left( \exp_{g_{a_{1,m}}} \frac{\cdot}{\bar{\lambda}_{1,m}} \right) \longrightarrow 0 \text{ locally strongly in } W^{1,2}(\mathbb{R}^n).$$

Since the initial sequence $(u_m)$ was non-negative, it follows that $u_\infty \geq 0$ and the negative part of $u_{2,m}$ tends to zero as $m \longrightarrow \infty$ in $W^{1,2}$-norm. Using a dilation argument, the latter property and the above formula, it is easy to show that, if $\alpha, \beta \geq 1$ with $\alpha + \beta = \frac{2n}{n-2}$, then

$$\int \varphi_{1,m}^\alpha |u_{2,m}|^\beta d\mu_{g_0} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{3.4}$$

and that also $\int u_{2,m} L_{g_0} \varphi_{1,m} d\mu_{g_0} = o(1)$. Thence as before for $u_{1,m}$

$$r_\infty \longleftarrow J_{\tau_m}(u_m) = \int u_\infty L_{g_0} u_\infty d\mu_{g_0} + \alpha_1^2 \int \varphi_{1,m} L_{g_0} \varphi_{1,m} d\mu_{g_0} + \int u_{2,m} L_{g_0} u_{2,m} d\mu_{g_0}$$

and therefore $\int u_{2,m}L_{g_0}u_{2,m}d\mu_{g_0} \longrightarrow r_{2,\infty} \geq 0$. Likewise

$$E(u_{2,m}) = L_{g_0}u_{2,m} - r_\infty K u_{2,m}^{p_m} = o(1) \quad \text{ in } W_{loc}^{-1,2}(\mathbb{R}^n)$$

since by expansion of the non-linear term of $\partial J_{\tau_m}(u_m)$ we find

$$\begin{aligned} o(1) =& L_{g_0}(u_\infty + \alpha_1\varphi_{1,m} + u_{2,m}) - r_\infty K(u_\infty + \alpha_1\varphi_{1,m} + u_{2,m})^{p_m} \\ =& L_{g_0}u_\infty - r_\infty K u_\infty^{p_m} + \alpha_1 L_{g_0}\varphi_{1,m} - r_\infty K \alpha_1^{p_m}\varphi_{1,m}^{p_m} \\ &+ L_{g_0}u_{2,m} - r_\infty K u_{2,m}^{p_m} + o(1) = L_{g_0}u_{2,m} - r_\infty K u_{2,m}^{p_m} + o(1) \quad \text{ in } W^{-1,2}(M, g_0). \end{aligned}$$

The second equality follows from applying the latter formulas to any test function in $W^{1,2}(M, g_0)$ and then applying Sobolev's and Hölder's inequalities together with (3.4). We may therefore iterate the afore going and find for a finite sum $u_m = \sum_i \alpha_i\varphi_{i,m} + v_m$, with energy

$$r_\infty \longleftarrow J(u_m) \geq \int u_\infty L_{g_0}u_\infty d\mu_{g_0} + \sum_i \alpha_i^2 \int \varphi_{i,m}L_{g_0}\varphi_{i,m}d\mu_{g_0}.$$

But all $\alpha_i$ are uniformly lower bounded due to

$$\sigma_i r_\infty \kappa_i \alpha_i^{\frac{4}{n-2}} = 1, \sigma_i = \lim_{m\to\infty} \lambda_{i,m}^{-\theta_m} \in (0,1] \text{ and } \kappa_i = \lim_{m\to\infty} K(a_{i,m}),$$

thence the iteration has to stop after finitely-many steps. In particular $v_m$ does not concentrate locally and consequently vanishes strongly as $m \longrightarrow \infty$. Now take any fixed index $j$ and recall that

$$w_{j,m} = \bar{\lambda}_{j,m}^{\frac{2-n}{2}} u_{j,m}\left(\exp_{g_{a_{j,m}}} \frac{\cdot}{\bar{\lambda}_{j,m}}\right)$$

and that by construction $\frac{\bar{\lambda}_{k,m}}{\bar{\lambda}_{l,m}} \not\longrightarrow 0$ for $k < l$. We had seen

$$w_{j,m} \longrightarrow w_{j,\infty} \text{ weakly and locally strongly, where } \quad -c_n\Delta w_{j,\infty} - \sigma_j r_\infty \kappa_j w_{j,\infty}^{\frac{n+2}{n-2}} = 0.$$

On the other hand

$$w_{j,m} = \alpha_j\left(\frac{1}{1+r^2}\right)^{\frac{n-2}{2}} + \sum_{i>j} u_{a_{i,m}}(a_j)\alpha_i\left(\frac{\frac{\bar{\lambda}_{i,m}}{\bar{\lambda}_{j,m}}}{1 + \bar{\lambda}_{i,m}^2\gamma_n G_{a_{i,m}}^{\frac{2}{2-n}}\left(\exp_{g_{a_{j,m}}} \frac{\cdot}{\bar{\lambda}_{j,m}}\right)}\right)^{\frac{n-2}{2}}$$

up to some error of order $o(1)$ locally in $W^{1,2}$, and the latter sum has to vanish, which is equivalent to

$$\frac{\bar{\lambda}_{j,m}}{\bar{\lambda}_{i,m}} \longrightarrow \infty \text{ or } \bar{\lambda}_{i,m}\bar{\lambda}_{j,m}G_{a_{i,m}}(a_{j,m}) \longrightarrow \infty.$$

Recalling (2.9), this shows that $(\varepsilon_{i,j})_m \longrightarrow 0$ for all $i \neq j$. We are left with proving $\bar{\lambda}_{i,m}^{\tau_m} \longrightarrow 1$. Ordering

$$\bar{\lambda}_{1,m} \geq \ldots \geq \bar{\lambda}_{q,m}$$

up to a subsequence, let

$$1 \leq \bar{q} = \sharp\left\{l = 1, \ldots, q \mid \lim_{m\to\infty} \frac{\bar{\lambda}_{1,m}}{\bar{\lambda}_{l,m}} < \infty\right\}.$$

Then $\frac{\bar{\lambda}_{k,m}}{\bar{\lambda}_{l,m}} \longrightarrow \infty$ for $k \leq \bar{q} < l$ and $c \leq \lim_{m\to\infty} \frac{\bar{\lambda}_{k,m}}{\bar{\lambda}_{l,m}} \leq C$ for $k, l \leq \bar{q}$. Select a half-ball $B_\delta^+(a_{k,m})$ with

$$1 \leq k \leq \bar{q} \text{ and } 0 < \delta \ll 1 \text{ such that } B_\delta^+(a_{k,m}) \cap \{a_{l,m} \mid 1 \leq l \leq \bar{q}, l \neq k\} = \emptyset$$

up to a subsequence, where for some affine function $\nu_{k,m}$ with unit gradient we have set

$$B_\delta^+(a_{k,m}) = B_\delta(a_{k,m}) \cap \{\nu_{k,m} > 0\}$$

in a local coordinate system. Then rescaling $u_m$ on $B_\delta^{a_{k,m}} \cap \{\nu_{k,m} > \frac{1}{\bar\lambda_{k,m}}\}$ we find

$$w_{k,m} = \bar\lambda_{k,m}^{\frac{2-n}{2}} u_m\Big(\exp_{g_{a_i}} \frac{\cdot}{\bar\lambda_{k,m}}\Big) = \alpha_l \Big(\frac{1}{1+r^2}\Big)^{\frac{n-2}{2}} + o(1) \;\text{ on }\; B_{c\bar\lambda_{k,m}}(0) \cap \{x_1 > 1\}.$$

On the other hand side, $w_{k,m}$ solves

$$-c_n \Delta w_{k,m} - \frac{r_\infty \kappa_k}{\bar\lambda_{k,m}^{\theta_m}} w_{k,m}^{p_m} = o(1),\; \kappa_k = \lim_{m\to\infty} K(a_{k,m}) \;\text{ on }\; B_{c\bar\lambda_{k,m}}(0).$$

Recalling that $p_m = \frac{n+2}{n-2} - \tau_m$ and $\theta_m = \frac{n-2}{2}\tau_m$, this implies, that up to rotating coordinates

$$(1+r^2)^{\theta_m} \;\text{ is nearly constant on }\; B^+_{c\bar\lambda_{k,m}}(0) \cap \{x_1 > 1\}.$$

Thus $\bar\lambda_{k,m}^{\theta_m} \longrightarrow 1$. The claim follows, since $\lim_{m\to\infty} \frac{\bar\lambda_{k,m}}{\bar\lambda_{l,m}} \geq c$ for all $l = 1, \ldots, q$. $\square$

# 4 Reduction and v-part estimates

In this section we will consider a sequence $u_m$ as in Proposition 3.1, with zero weak limit. We will recall some well-known facts about finite-dimensional reductions and derive preliminary error estimates and on suitable components of the gradient of $J_\tau$.

For $\varepsilon > 0$, $q \in \mathbb{N}$, $u \in W^{1,2}(M, g_0)$ and $(\alpha^i, \lambda_i, a_i) \in (\mathbb{R}^q_+, \mathbb{R}^q_+, M^q)$ we define

(i) $\quad A_u(q,\varepsilon) = \{(\alpha^i, \lambda_i, a_i) \mid \underset{i\neq j}{\forall}\; \lambda_i^{-1}, \lambda_j^{-1}, \varepsilon_{i,j}, \Big|1 - \frac{r\alpha_i^{\frac{4}{n-2}} K(a_i)}{4n(n-1)k_\tau}\Big|, \|u - \alpha^i \varphi_{a_i,\lambda_i}\| < \varepsilon,\; \lambda_i^\tau < 1+\varepsilon\};$

(ii) $\quad V(q,\varepsilon) = \{u \in W^{1,2}(M, g_0) \mid A_u(q,\varepsilon) \neq \emptyset\},$

cf. (2.2), (2.3) and (2.8). For both conditions $\lambda_i > \varepsilon^{-1}, \lambda_i^\tau < 1+\varepsilon$ to hold, we will always assume that $\tau \ll \varepsilon$ and this is consistent with the statement of Proposition 3.1. Under the above conditions on the parameters $\alpha_i, a_i$ and $\lambda_i$ the functions $\sum_{i=1}^q \alpha^i \varphi_{a_i,\lambda_i}$ form a smooth manifold in $W^{1,2}(M, g_0)$, which implies the following well known result, cf. [4].

**Proposition 4.1.** *For every $\varepsilon_0 > 0$ there exists $\varepsilon_1 > 0$ such that for $u \in V(q,\varepsilon)$ with $\varepsilon < \varepsilon_1$*

$$\inf_{(\tilde\alpha_i, \tilde a_i, \tilde\lambda_i) \in A_u(q, 2\varepsilon_0)} \int (u - \tilde\alpha^i \varphi_{\tilde a_i, \tilde\lambda_i}) L_{g_0} (u - \tilde\alpha^i \varphi_{\tilde a_i, \tilde\lambda_i}) d\mu_{g_0}$$

*admits an unique minimizer $(\alpha_i, a_i, \lambda_i) \in A_u(q, \varepsilon_0)$ depending smoothly on $u$ and we set*

$$\varphi_i = \varphi_{a_i,\lambda_i}, \quad v = u - \alpha^i \varphi_i, \quad K_i = K(a_i). \tag{4.1}$$

The term $v = u - \alpha^i\varphi_i$ is orthogonal to all $\varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i$, with respect to the product

$$\langle \cdot, \cdot \rangle_{L_{g_0}} = \langle L_{g_0} \cdot, \cdot \rangle_{L^2_{g_0}}.$$

For $u \in V(q,\varepsilon)$ let

$$H_u(q,\varepsilon) = \langle \varphi_i, \lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{L_{g_0}}}. \tag{4.2}$$

We next have an estimate on the projection of the gradient of $J_\tau$ onto $H_u$.

**Lemma 4.1.** *For $u \in V(q,\varepsilon)$ with $k_\tau = 1$, cf. (2.3), and $\nu \in H_u(q,\varepsilon)$ there holds*

$$\partial J_\tau(\alpha^i\varphi_i)\nu = O\Big(\Big[\sum_r \frac{\tau}{\lambda_r^\theta} + \sum_r \frac{|\nabla K_r|}{\lambda_r^{1+\theta}} + \sum_r \frac{1}{\lambda_r^{2+\theta}} + \sum_r \frac{1}{\lambda_r^{n-2+\theta}} + \sum_{r\neq s} \frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\Big]\|\nu\|\Big).$$

*Proof.* Due to the fact that $k_\tau = 1$ and $\nu \in H_u(q,\varepsilon)$ we have

$$-\frac{1}{2}\partial J_\tau(\alpha^i\varphi_i)\nu = r_{\alpha^i\varphi_i}\int K(\alpha^i\varphi_i)^p \nu d\mu_{g_0},$$

and therefore

$$\partial J_\tau(\alpha^i\varphi_i)\nu \simeq \int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0}.$$

Decomposing iteratively $M$ as $\{\alpha_j\varphi_j > \sum_{i>j}\alpha_i\varphi_i\} \cup \{\alpha_j\varphi_j \leq \sum_{i>j}\alpha_i\varphi_i\}$, we find

$$\int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0} = \sum_i \int K(\alpha_i\varphi_i)^p\nu d\mu_{g_0} + O(\sum_{r\neq s}\int_{\{\alpha_s\varphi_s \leq \alpha_r\varphi_r\}} (\alpha^r\varphi_r)^{p-1}\alpha^s\varphi_s|\nu|d\mu_{g_0}).$$

Using Hölder's inequality with exponents $1 = \frac{1}{p} + \frac{1}{q} = \frac{n+2}{2n} + \frac{n-2}{2n}$ and Lemma 2.2 $(v)$ applied to the latter error term, where the inequality $\varphi_s \lesssim \varphi_r$ can be used to apply it with $\beta \geq 1$, we get

$$\int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0} = \sum_i \int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0} + O\Big(\sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\|\nu\|\Big),$$

and by a simple expansion we also obtain

$$\int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0} = \sum_i K_i\alpha_i^p\int \varphi_i^p\nu d\mu_{g_0} + O\Big(\Big[\sum_r \frac{|\nabla K_r|}{\lambda_r^{1+\theta}} + \sum_r \frac{1}{\lambda_r^{2+\theta}} + \sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\Big]\|\nu\|\Big). \tag{4.3}$$

Note that

$$\begin{aligned}
\|\lambda_i^{-\theta}\varphi_i^{\frac{n+2}{n-2}} - \varphi_i^p\|^{\frac{2n}{n+2}}_{L_{g_0}^{\frac{2n}{n+2}}} &= \int \varphi_i^{\frac{2n}{n-2}-\frac{2n}{n+2}\tau}|1-\lambda_i^{-\theta}\varphi_i^\tau|^{\frac{2n}{n+2}}d\mu_{g_0} \\
&\lesssim \int_{B_c(0)}\Big(\frac{\lambda_i}{1+\lambda_i^2r^2}\Big)^{n-\frac{2n}{n+2}\theta}\Big|1-\Big(\frac{1}{1+\lambda_i^2O(r^2)}\Big)^\theta\Big|^{\frac{2n}{n+2}}dx + O\Big(\frac{1}{\lambda_i^{n-\frac{2n}{n+2}\theta}}\Big) \\
&= \lambda_i^{-\frac{2n}{n+2}\theta}\int_{B_{c\lambda_i}(0)}\Big(\frac{1}{1+r^2}\Big)^{n-\frac{2n}{n+2}\theta}\Big|1-\Big(\frac{1}{1+O(r^2)}\Big)^\theta\Big|^{\frac{2n}{n+2}}dx + O\Big(\frac{1}{\lambda_i^{n-\frac{2n}{n+2}\theta}}\Big),
\end{aligned}$$

whence

$$\|\lambda_i^{-\theta}\varphi_i^{\frac{n+2}{n-2}} - \varphi_i^p\|_{L_{g_0}^{\frac{2n}{n+2}}} = O\Big(\frac{\theta}{\lambda_i^\theta} + \frac{1}{\lambda_i^{n-\frac{2n}{n+2}\theta}}\Big). \tag{4.4}$$

Thus up to some $O([\sum_r \frac{\tau}{\lambda_r^\theta} + \sum_r \frac{|\nabla K_r|}{\lambda_r^{1+\theta}} + \sum_r \frac{1}{\lambda_r^{2+\theta}} + \sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}]\|\nu\|)$ we arrive at

$$\int K(\alpha^i\varphi_i)^p\nu d\mu_{g_0} = \sum_i K_i\lambda_i^{-\theta}\alpha_i^{\frac{n+2}{n-2}}\int \varphi_i^{\frac{n+2}{n-2}}\nu d\mu_{g_0}.$$

Finally from Lemma 2.1 and the fact that $\nu \in H_u(q,\varepsilon)$ (hence $\int \nu L_{g_0}\varphi_i d\mu_{g_0} = 0$) we obtain

$$\Big|\int \varphi_i^{\frac{n+2}{n-2}}\nu d\mu_{g_0}\Big| \leq \|v\|\Big\|\frac{L_{g_0}\varphi_i}{4n(n-1)} - \varphi_i^{\frac{n+2}{n-2}}\Big\|_{L_{g_0}^{\frac{2n}{n+2}}} = O\begin{pmatrix} \lambda_i^{-1} & \text{for } n=3 \\ \lambda_i^{-2} & \text{for } n=4 \\ \lambda_i^{-3} & \text{for } n=5 \\ \ln^{\frac{2}{3}}\lambda_i\lambda_i^{-\frac{10}{3}} & \text{for } n=6 \\ \lambda_i^{-4} & \text{for } n\geq 7\end{pmatrix}\|v\|, \tag{4.5}$$

so the claim follows. $\square$

**Lemma 4.2.** *For $u \in V(q,\varepsilon)$ with $k_\tau = 1$ and $v$ is as in* (4.1) *there holds*

$$\|v\| = O\Big(\sum_r \frac{\tau}{\lambda_r^\theta} + \sum_r \frac{|\nabla K_r|}{\lambda_r^{1+\theta}} + \sum_r \frac{1}{\lambda_r^{2+\theta}} + \sum_r \frac{1}{\lambda_r^{n-2+\theta}} + \sum_{r\neq s} \frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta} + |\partial J_\tau(u)|\Big).$$

*Proof.* Since the Hessian of $J_\tau$ is uniformly Hölder continuous on bounded sets of $W^{1,2}$, we have

$$\partial J_\tau(u)v = \partial J_\tau(\alpha^i\varphi_i)v + \partial^2 J_\tau(\alpha^i\varphi_i)v^2 + o(\|v\|^2) = \partial J_\tau(\alpha^i\varphi_i)v + \partial^2 J_\tau(u)v^2 + o(\|v\|^2);$$

$$\begin{aligned}\partial^2 J_\tau(u)v^2 =& 2\Big[\int vL_{g_0}vd\mu_{g_0} - pr_u Ku^{p-1}v^2 d\mu_{g_0}\Big] - 8\int uL_{g_0}vd\mu_{g_0}\int Ku^p v d\mu_{g_0}\\ &+ 2(p+3)r\int Ku^p v d\mu_{g_0}\int Ku^p v d\mu_{g_0}.\end{aligned} \tag{4.6}$$

Since $v \in H_u(q,\varepsilon)$, by similar expansions we then find (also replacing $p$ with $\frac{n+2}{n-2}$ with an error $o(1)$)

$$\begin{aligned}&\partial^2 J_\tau(u)v^2 = 2\Big[\int vL_{g_0}vd\mu_{g_0} - pr_u\int Ku^{p-1}v^2 d\mu_{g_0}\Big]\\ =&2\Big[\int vL_{g_0}vd\mu_{g_0} - \frac{n+2}{n-2}\big(\int(\alpha^i\varphi_i)L_{g_0}(\alpha^j\varphi_j)d\mu_{g_0}\big)\int K(\alpha^i\varphi_i)^{p-1}v^2 d\mu_{g_0}\Big]\\ =&2\Big[\int vL_{g_0}vd\mu_{g_0} - \frac{n+2}{n-2}\sum_{i,j}\frac{K_i\alpha_i^{\frac{4}{n-2}}\alpha_j^2\int\varphi_j L_{g_0}\varphi_j d\mu_{g_0}}{\lambda_i^\theta}\int\varphi_i^{\frac{4}{n-2}}v^2 d\mu_{g_0}\Big]\end{aligned}$$

up to some $o(\|v\|^2)$. Furthermore by definition of $V(q,\varepsilon)$ there holds $\lambda_i^\theta = 1 + o(1)$ and

$$K_i\alpha_i^{\frac{4}{n-2}} = \frac{1}{r_{\alpha^i\varphi_i}} + o(1) = \frac{1}{\int\sum_j \alpha_j^2\varphi_j L_{g_0}\varphi_j d\mu_{g_0}} + o(1).$$

Thus

$$\partial^2 J_\tau(u)v^2 = 2\Big[\int vL_{g_0}vd\mu_{g_0} - \frac{n+2}{n-2}\int\varphi_i^{\frac{4}{n-2}}v^2 d\mu_{g_0}\Big] + o(\|v\|^2).$$

This quadratic form is positive definite for $\varepsilon$ sufficiently small on the subspace $v$ belongs to, cf. [4], so

$$\|v\|^2(1+o(1)) \le C\partial^2 J_\tau(u)v^2 \le C[\partial J_\tau(\alpha^i\varphi_i)v + |\partial J_\tau(u)|^2].$$

Therefore the claim follows from Lemma 4.1. □

We now establish cancellations testing the gradient of $J_\tau$ orthogonally to $H_u(q,\varepsilon)$.

**Lemma 4.3.** *For $u \in V(q,\varepsilon)$ with $k_\tau = 1$ the quantity $\partial J_\tau(u)\phi_{k,i}$ expands as*

$$\partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} + O\Big(\sum_r \frac{\tau^2}{\lambda_r^{2\theta}} + \sum_r \frac{|\nabla K_r|^2}{\lambda_r^{2+2\theta}} + \sum_r \frac{1}{\lambda_r^{4+2\theta}} + \sum_r \frac{1}{\lambda_r^{2(n-2)+2\theta}} + \sum_{r\neq s} \frac{\varepsilon_{r,s}^{\frac{n+2}{n}}}{\lambda_r^{2\theta}} + |\partial J_\tau(u)|^2\Big).$$

*Proof.* By the mean value theorem and (4.6) we have, with some $\sigma \in [0,1]$

$$\begin{aligned}&\partial J_\tau(u)\phi_{k,i} - \partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} = \partial^2 J_\tau(\alpha^j\varphi_j + \sigma v)\phi_{k,i}v\\ =&2(1+O(\|v\|))\Big[\int vL_{g_0}\phi_{k,i}d\mu_{g_0} - pr_{\alpha^i\varphi_i}(1+O(\|v\|))\int K(\alpha^j\varphi_j+\sigma v)^{p-1}v\phi_{k,i}d\mu_{g_0}\Big]\\ &-4(1+O(\|v\|))\Big[\int(\alpha^j\varphi_j+\sigma v)L_{g_0}vd\mu_{g_0}\int K(\alpha^j\varphi_j+\sigma v)^p\phi_{k,i}d\mu_{g_0}\\ &\qquad+\int(\alpha^j\varphi_j+\sigma v)L_{g_0}\phi_{k,i}d\mu_{g_0}\int K(\alpha^j\varphi_j+\sigma v)^p v d\mu_{g_0}\Big]\\ &+2(p+3)r_{\alpha^i\varphi_i}(1+O(\|v\|))\int K(\alpha^j\varphi_j+\sigma v)^p v d\mu_{g_0}\int K(\alpha^j\varphi_j+\sigma v)^p\phi_{k,i}d\mu_{g_0}.\end{aligned}$$

Therefore, since $v \in H_u(q,\varepsilon)$, up to some $O(\|v\|^2)$ we also get

$$\begin{aligned}
&\partial J_\tau(u)\phi_{k,i} - \partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} = -2pr_{\alpha^i\varphi_i}\int K(\alpha^j\varphi_j+\sigma v)^{p-1}v\phi_{k,i}d\mu_{g_0}\\
&-4\int(\alpha^j\varphi_j)L_{g_0}\phi_{k,i}d\mu_{g_0}\int K(\alpha^j\varphi_j+\sigma v)^p v d\mu_{g_0}\\
&+2(p+3)r_{\alpha^i\varphi_i}\int K(\alpha^j\varphi_j+\sigma v)^p v d\mu_{g_0}\int K(\alpha^j\varphi_j+\sigma v)^p\phi_{k,i}d\mu_{g_0}.
\end{aligned}$$

Decomposing now $M$ as $\{\alpha^j\varphi_j \le 2\|v\|\}\cup\{\alpha^j\varphi_j\ge 2\|v\|\}$, and using $|\phi_{k,i}|\le C\alpha_i\varphi_i\le C\alpha^j\varphi_j$, we find

$$\begin{aligned}
&\partial J_\tau(u)\phi_{k,i} - \partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} = -2pr_{\alpha^i\varphi_i}\int K(\alpha^j\varphi_j)^{p-1}v\phi_{k,i}d\mu_{g_0}\\
&-4\int(\alpha^j\varphi_j)L_{g_0}\phi_{k,i}d\mu_{g_0}\int K(\alpha^j\varphi_j)^p v d\mu_{g_0}\\
&+2(p+3)r_{\alpha^i\varphi_i}\int K(\alpha^j\varphi_j)^p v d\mu_{g_0}\int K(\alpha^j\varphi_j)^p\phi_{k,i}d\mu_{g_0}+O(\|v\|^2).
\end{aligned}$$

Now, arguing as for (4.3) and using Lemma 2.2 $(iv)$, we have

$$\int K(\alpha^j\varphi_j)^p v d\mu_{g_0} = \sum_j K_j\alpha_j^p\int\varphi_j^p v d\mu_{g_0} + O\bigg(\bigg[\sum_r\frac{|\nabla K_r|}{\lambda_r^{1+\theta}}+\sum_r\frac{1}{\lambda_r^{2+\theta}}+\sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\bigg]\|v\|\bigg);$$

$$\int K(\alpha^j\varphi_j)^{p-1}\phi_{k,i}v d\mu_{g_0} = K_i\alpha_i^{p-1}\int\varphi_i^{p-1}\phi_{k,i}v d\mu_{g_0} + O\bigg(\bigg[\sum_r\frac{|\nabla K_r|}{\lambda_r^{1+\theta}}+\sum_r\frac{1}{\lambda_r^{2+\theta}}+\sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{2n}}}{\lambda_r^\theta}\bigg]\|v\|\bigg),$$

whence

$$\begin{aligned}
\partial J_\tau(u)\phi_{k,i} - \partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} =& -2pr_{\alpha^i\varphi_i}K_i\alpha_i^{p-1}\int\varphi_i^{p-1}\phi_{k,i}v d\mu_{g_0}\\
&-4\alpha_i\int\varphi_i L_{g_0}\phi_{k,i}d\mu_{g_0}\sum_j K_j\alpha_j^p\int\varphi_j^p v d\mu_{g_0}\\
&+2(p+3)r_{\alpha^i\varphi_i}K_i\alpha_i^p\int\varphi_i^p\phi_{k,i}d\mu_{g_0}\sum_j K_j\alpha_j^p\int\varphi_j^p v d\mu_{g_0}
\end{aligned}$$

up to some $O\Big(\sum_r\frac{|\nabla K_r|^2}{\lambda_r^{2+2\theta}}+\sum_r\frac{1}{\lambda_r^{4+2\theta}}+\sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{n}}}{\lambda_r^{2\theta}}+\|v\|^2\Big)$. Using (4.4) and (4.5) we arrive at

$$\begin{aligned}
&\partial J_\tau(u)\phi_{k,i} - \partial J_\tau(\alpha^j\varphi_j)\phi_{k,i} = -2pr_{\alpha^i\varphi_i}K_i\alpha_i^{p-1}\int\varphi_i^{p-1}\phi_{k,i}v d\mu_{g_0}\\
&\quad+O\bigg(\sum_r\frac{\tau^2}{\lambda_r^{2\theta}}+\sum_r\frac{|\nabla K_r|^2}{\lambda_r^{2+2\theta}}+\sum_r\frac{1}{\lambda_r^{4+2\theta}}+\sum_r\frac{1}{\lambda_r^{2(n-2)+2\theta}}+\sum_{r\neq s}\frac{\varepsilon_{r,s}^{\frac{n+2}{n}}}{\lambda_r^{2\theta}}+\|v\|^2\bigg).
\end{aligned}$$

Yet also the first summand on the right hand side is of the same order as the second one, arguing as for (4.4) and (4.5). Combining this with Lemma 4.2, we obtain the conclusion. □

# 5 The functional and its derivatives

For $u \in V(q,\varepsilon)$ and $\varepsilon>0$ sufficiently small let

$$\alpha^2=\sum_i\alpha_i^2,\quad \alpha_{K,\tau}^s=\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^s,\quad \theta=\frac{n-2}{2}\tau. \tag{5.1}$$

Recalling the notation from the previous section we may expand the Euler-Lagrange energy as follows.

**Proposition 5.1.** *For $u = \alpha^i\varphi_i + v \in V(q,\varepsilon)$ and $\varepsilon > 0$, both $J_\tau(u)$ and $J_\tau(\alpha^i\varphi_i)$ can be written as*

$$\frac{\hat{c}_0\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\left(1 - \hat{c}_1\tau - \hat{c}_2\sum_i \frac{\Delta K_i}{K_i\lambda_i^2}\frac{\alpha_i^2}{\alpha^2} - \hat{b}_1\sum_{i\neq j}\frac{\alpha_i\alpha_j}{\alpha^2}\varepsilon_{i,j} - \hat{d}_1\sum_i\frac{\alpha_i^2}{\alpha^2}\begin{pmatrix} \frac{H_i}{\lambda_i} & \text{for } n=3 \\ \frac{H_i + O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2} & \text{for } n=4 \\ \frac{H_i}{\lambda_i^3} & \text{for } n=5 \\ \frac{W_i\ln\lambda_i}{\lambda_i^4} & \text{for } n=6 \\ 0 & \text{for } n\geq 7\end{pmatrix}\right)$$

*with positive constants $\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{b}_1, \hat{d}_1$ up to errors of the form*

$$O(\tau^2 + \sum_r \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2).$$

*Proof.* The above expansion for $J_\tau(\alpha^i\varphi_i)$ implies the one for $J_\tau(u)$ via Lemmata 4.1 and 4.2 expanding

$$J_\tau(u) = J_\tau(\alpha^i\varphi_i) + \partial J_\tau(\alpha^i\varphi_i)v + O(\|v\|^2).$$

We next start analyzing $J_\tau(\alpha^i\varphi_i)$ from the denominator. Decomposing iteratively $M$ as

$$M = \{\alpha_j\varphi_j > \sum_{i>j}\alpha_i\varphi_i\} + \{\alpha_j\varphi_j \leq \sum_{i>j}\alpha_i\varphi_i\}$$

we may expand

$$\begin{aligned}\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0} =& \sum_i \alpha_i^{p+1}\int K\varphi_i^{p+1}d\mu_{g_0} + (p+1)\sum_{i\neq j}\alpha_i^p\alpha_j\int K\varphi_i^p\varphi_j d\mu_{g_0} \\ &+ O\Big(\sum_{r\neq s}\int_{\{\alpha_r\varphi_r\geq\alpha_s\varphi_s\}}(\alpha_r\varphi_r)^p\alpha_s\varphi_s d\mu_{g_0}\Big).\end{aligned}$$

Recalling $\lambda_r^\theta \sim 1$ and the boundedness of $\alpha_r$ by the definition of $V(q,\varepsilon)$, using Lemma 2.2 and reasoning as for the proof of Lemma 4.1, the latter term is of order $O(\sum_{r\neq s}\varepsilon_{i,j}^{\frac{n+2}{n}})$, and also

$$\begin{aligned}\int K\varphi_i^p\varphi_j d\mu_{g_0} =& K_i\int\varphi_i^p\varphi_j d\mu_{g_0} + O\Big(\int_{B_c(a_i)}(|\nabla K_i|r_{a_i} + r_{a_i}^2)\varphi_i^p\varphi_j d\mu_{g_0}\Big) + O\Big(\frac{1}{\lambda_i^{\frac{n+2}{n-2}-\theta}\lambda_j^{\frac{n-2}{2}}}\Big) \\ =& K_i\int\varphi_i^p\varphi_j d\mu_{g_0} + O\Big(\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\Big).\end{aligned}$$

Indeed we for example have

$$\int_{B_c(a_i)} r_{a_i}\varphi_i^p\varphi_j d\mu_{g_0} = \int_{B_c(a_i)} r_{a_i}\varphi_i^{\frac{n+2}{n-2}-\frac{n+2}{2n}}\varphi_i^{\frac{n+2}{2n}}\varphi_j d\mu_{g_0} \leq C\varepsilon_{i,j}^{\frac{n+2}{2n}}\|r\varphi_{0,\lambda_i}^{\frac{n+2}{n-2}-\frac{n+2}{2n}}\|_{L_{\mu_{g_0}}^{(\frac{2n}{(n+2)})^2}}$$

with the latter norm that can be controlled by

$$\int_{\mathbb{R}^n} r^{(\frac{2n}{n+2})^2}(\frac{\lambda_i}{1+\lambda_i^2r^2})^n dx \leq C\lambda_i^{-(\frac{2n}{n+2})^2}\Big(1 + \int_1^\infty r^{-1+n+(\frac{2n}{n+2})^2-2n}dr\Big) = O\Big(\Big(\frac{1}{\lambda_i}\Big)^{(\frac{2n}{n+2})^2}\Big).$$

Thus Lemma 2.2, where $b_1$ is defined, yields

$$\int K\varphi_i^p\varphi_j d\mu_{g_0} = b_1\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j} + O\Big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}}\Big), \tag{5.2}$$

and we arrive at

$$\begin{aligned}\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0} =&\sum_i \alpha_i^{p+1}\int K\varphi_i^{p+1}d\mu_{g_0}+(p+1)\sum_{i\neq j}\alpha_i^p\alpha_j b_1\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j}\\ =&\sum_i \alpha_i^{p+1}\int K\varphi_i^{p+1}d\mu_{g_0}+\bar b_1\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_j\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j},\quad \bar b_1=\frac{2n}{n-2}b_1\end{aligned}\tag{5.3}$$

up to an error $O(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}})$. Finally, recalling our notation in Section 2 and denoting by $x^i$ a generic polynomial of degree $i$ in the $x$-variables, we expand

$$\begin{aligned}\int K\varphi_i^{p+1}d\mu_{g_0}=&\int_{B_c(a_i)}K\varphi_i^{p+1}d\mu_{g_0}+O\big(\frac{1}{\lambda_i^{n-\theta}}\big)\\ =&K_i\int_{B_c(a_i)}\varphi_i^{p+1}d\mu_{g_0}+\nabla K_i\int_{B_c(a_i)}x\varphi_i^{p+1}d\mu_{g_0}\\ &+\frac{\nabla^2}{2}K_i\int_{B_c(a_i)}x^2\varphi_i^{p+1}d\mu_{g_0}+\frac{\nabla^3}{6}K_i\int_{B_c(a_i)}x^3\varphi_i^{p+1}d\mu_{g_0}+O\Big(\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}}\Big)\end{aligned}\tag{5.4}$$

with an extra error of order $O\left(\frac{\ln\lambda}{\lambda^4}\right)$ if $n=4$. For the first term on the right-hand side up to some $O(\tau^2+\frac{1}{\lambda_i^4})$ we may pass integrating with respect to conformal normal coordinates. Indeed

$$\int\limits_{B_c(a_i)}\varphi_i^{p+1}d\mu_{g_0}=\int\limits_{B_c(a_i)}u_{a_i}^{-\tau}(\frac{\varphi_i}{u_{a_i}})^{\frac{2n}{n-2}-\theta}d\mu_{g_{a_i}}=\int\limits_{B_c(a_i)}(\frac{\varphi_i}{u_{a_i}})^{\frac{2n}{n-2}-\theta}d\mu_{g_{a_i}}+O(\tau\int_{B_c(a_i)}r_{a_i}^2(\frac{\varphi_i}{u_{a_i}})^{\frac{2n}{n-2}-\theta}d\mu_{g_{a_i}})$$

and the latter term is of order $O(\frac{\tau}{\lambda_i^{2+\theta}})$. From (2.8) we find

$$\begin{aligned}\int\varphi_i^{p+1}d\mu_{g_0}=&\int_{B_c(a_i)}(\frac{\lambda_i}{1+\lambda_i^2r_{a_i}^2(1+r_{a_i}^{n-2}H_{a_i})^{\frac{2}{2-n}}})^{n-\theta}d\mu_{g_{a_i}}\\ =&\int_{B_c(0)}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{n-\theta}(1+\frac{2(n-\theta)}{n-2}\frac{\lambda_i^2r^nH_{a_i}}{1+\lambda_i^2r^2})dx,\end{aligned}$$

up to some $O(\tau^2+\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^n})$. Clearly

$$\begin{aligned}\int_{B_c(0)}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{n-\theta}dx=&\lambda_i^{-\theta}\int_{B_{c\lambda_i}}\frac{dx}{(1+r^2)^{n-\theta}}=\lambda_i^{-\theta}\int\limits_{\mathbb{R}^n}\frac{dx}{(1+r^2)^{n-\theta}}+O(\lambda_i^{-n})\\ =&\frac{1}{\lambda^\theta}\int\limits_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n}+\frac{\theta}{\lambda^\theta}\int\limits_{\mathbb{R}^n}\frac{\ln(1+r^2)dx}{(1+r^2)^n}+O\Big(\tau^2+\frac{1}{\lambda_i^4}+O\Big(\frac{1}{\lambda_i^{2(n-2)}}\Big)\Big)\\ =&\frac{\bar c_0}{\lambda_i^\theta}+\frac{\bar c_1\tau}{\lambda_i^\theta}+O\Big(\tau^2+\frac{1}{\lambda_i^4}+O\Big(\frac{1}{\lambda_i^{2(n-2)}}\Big)\Big)\end{aligned}$$

letting

$$\bar c_0=\int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n}\quad\text{and}\quad \bar c_1=\frac{n-2}{2}\int_{\mathbb{R}^n}\frac{\ln(1+r^2)}{(1+r^2)^n}dx.\tag{5.5}$$

Moreover

$$\int\limits_{B_c(0)}\frac{\lambda_i^{n+4-\theta}r^{2n}H_{a_i}^2}{(1+\lambda_i^2r^2)^{n+2-\theta}}dx\leq\int\limits_{B_c(0)}\frac{\lambda_i^{n-\theta}r^{2(n-2)}H_{a_i}^2}{(1+\lambda_i^2r^2)^{n-\theta}}dx\leq C\int\limits_{B_c(0)}\frac{\lambda_i^{n-\theta}r^{2(n-2)}}{(1+\lambda_i^2r^2)^{n-\theta}}\begin{pmatrix}1 & \text{for } n=3\\ 1 & \text{for } n=4\\ 1 & \text{for } n=5\\ \ln^2 r & \text{for } n=6\\ r^{2(6-n)} & \text{for } n\geq 7\end{pmatrix}dx$$

up to some $O(\frac{1}{\lambda_i^{2(n-2)}}+\frac{1}{\lambda_i^4})$ and with an extra error of order $O\left(\frac{\ln\lambda}{\lambda^4}\right)$ if $n=4$, and

$$\int\limits_{B_c(0)}\left(\frac{\lambda_i}{1+\lambda_i^2r^2}\right)^{n-\theta}\frac{\lambda_i^2r^nH_{a_i}}{1+\lambda_i^2r^2}dx=\int\limits_{B_c(0)}\left(\frac{\lambda_i}{1+\lambda_i^2r^2}\right)^{n-\theta}\frac{\lambda_i^2r^2}{1+\lambda_i^2r^2}r^{n-2}\begin{pmatrix}H_i+\nabla H_ix+O(r^2)\\ H_i+\nabla H_ix+O(r^2\ln r)\\ H_i+O(r)\\ -W_i\ln r+O(r\ln r)\\ O(r^{6-n})\end{pmatrix}dx,$$

whence up to some $O(\tau^2+\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}})$

$$\int_{B_c(0)}\left(\frac{\lambda_i}{1+\lambda_i^2r^2}\right)^{n-\theta}\frac{\lambda_i^2r^nH_{a_i}}{1+\lambda_i^2r^2}dx=\bar{d}_1\begin{pmatrix}\frac{H_i}{\lambda_i^{1+\theta}}\\ \frac{H_i}{\lambda_i^{2+\theta}}+O(\frac{\ln\lambda_i}{\lambda_i^{4+\theta}})\\ \frac{H_i}{\lambda_i^{3+\theta}}\\ \frac{W_i\ln\lambda_i}{\lambda_i^{4+\theta}}\\ 0\end{pmatrix},\ \bar{d}_1=\int\limits_{\mathbb{R}^n}\frac{r^ndx}{(1+r^2)^{n+1}}. \tag{5.6}$$

Likewise by radial symmetry and, since we may assume $d\mu_{g_{a_i}}\equiv 1$, cf. [24], we find

(1) $\int_{B_c(a_i)}x^3\varphi_i^{p+1}d\mu_{g_{a_i}}=O\left(\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}}\right)$;

(2) $\frac{\nabla^2}{2}K_i\int_{B_c(a_i)}x^2\varphi_i^{p+1}d\mu_{g_0}=\frac{\Delta K_i}{2n\lambda_i^{2+\theta}}\int\limits_{\mathbb{R}^n}\frac{r^2dx}{(1+r^2)^n}+O\left(\tau^2+\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}}\right)$;

(3) $\int_{B_c(a_i)}x\varphi_i^{p+1}d\mu_{g_0}=O\left(\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}}\right)$

with an extra error of order $O\left(\frac{\ln\lambda}{\lambda^4}\right)$ if $n=4$. Collecting all terms we arrive at

$$\int K\varphi_i^{p+1}d\mu_{g_0}=\frac{\bar{c}_0K_i}{\lambda_i^\theta}+\bar{c}_1\frac{K_i\tau}{\lambda_i^\theta}+\bar{c}_2\frac{\Delta K_i}{\lambda_i^{2+\theta}}+\bar{d}_1K_i\begin{pmatrix}\frac{H_i}{\lambda_i^{1+\theta}}\\ \frac{H_i}{\lambda_i^{2+\theta}}+O(\frac{\ln\lambda_i}{\lambda_i^{4+\theta}})\\ \frac{H_i}{\lambda_i^{3+\theta}}\\ \frac{W_i\ln\lambda_i}{\lambda_i^{4+\theta}}\\ 0\end{pmatrix},\ \bar{c}_2=\frac{1}{2n}\int\limits_{\mathbb{R}^n}\frac{r^2dx}{(1+r^2)^n} \tag{5.7}$$

up to an error $O(\tau^2+\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}})$, and thus obtain

$$\begin{aligned}\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}=&\sum_i\left(\bar{c}_0\frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1}+\bar{c}_1\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\tau+\bar{c}_2\frac{\Delta K_i}{\lambda_i^{2+\theta}}\alpha_i^{\frac{2n}{n-2}}\right)\\ &+\bar{d}_1\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i+O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2}\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}+\bar{b}_1\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_j\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j}\end{aligned} \tag{5.8}$$

up to some $O(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}})$. Consequently up to the same error

$$J_\tau(\alpha^i\varphi_i) = \frac{\alpha^i\alpha^j\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0}}{(\int K(\sum_i\alpha_i\varphi_i)^{p+1})^{\frac{2}{p+1}}} = \frac{\alpha^i\alpha^j\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0}}{(\bar{c}_0\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1})^{\frac{2}{p+1}}}\Bigg(1-\bar{c}_1\sum_i\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\tau$$

$$-\bar{c}_2\sum_i\frac{\Delta K_i}{\lambda_i^{2+\theta}}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} - \bar{d}_1\sum_i\frac{K_i}{\lambda_i^\theta}\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i+O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2}\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} - \bar{b}_1\sum_{i\neq j}\frac{\alpha_i^{\frac{n+2}{n-2}}\alpha_j}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j}\Bigg). \tag{5.9}$$

Next for $i\neq j$ using Lemma 2.1 we get

$$\int\frac{\varphi_i L_{g_0}\varphi_j}{4n(n-1)}d\mu_{g_0} = \int\varphi_i^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0} + O\Big(\frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}} + \varepsilon_{i,j}^{\frac{n+2}{n}}\Big).$$

For example to check the error term, we may estimate

$$\int_{B_c(a_i)} r_{a_i}^{n-2}\varphi_i^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0} \leq \|r_{a_i}^{n-2}\varphi_i^{\frac{n+2}{n-2}-\frac{n+2}{2n}}\|_{L_{B_c(a_i)}^{(\frac{2n}{n+2})^2}}\|\varphi_i^{\frac{n+2}{2n}}\varphi_j\|_{L^{\frac{4n^2}{(3n+2)(n-2)}}},$$

which is of order $O(\frac{\varepsilon_{i,j}^{\frac{n+2}{2n}}}{\lambda_i^{n-2}})$ thanks to Lemma 2.2, and likewise for e.g. $n\geq 7$

$$\int\varphi_i\varphi_j d\mu_{g_0} \lesssim \|\varphi_i\varphi_j\|_{L_{g_0}^{\frac{n}{n-2}}} = O(\varepsilon_{i,j}\ln^{\frac{n-2}{n}}\varepsilon_{i,j}),$$

whence $\lambda_i^{-2}\int\varphi_i\varphi_j d\mu_{g_0} = o(\frac{\varepsilon_{i,j}^{\frac{n+2}{2n}}}{\lambda_i^2})$. Thus Lemma 2.2 shows that

$$\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0} = \tilde{b}_1\varepsilon_{i,j} + O(\sum_{r\neq s}\frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n}{n}}), \quad \tilde{b}_1 = 4n(n-1)b_1. \tag{5.10}$$

Finally from (2.8) and Lemma 2.1 we find

$$\int\frac{\varphi_i L_{g_0}\varphi_i}{4n(n-1)}d\mu_{g_0} = \int\varphi_i^{\frac{2n}{n-2}}d\mu_{g_0} - \frac{c_n}{2}\int_{B_c(0)}\frac{\lambda_i^2 r^{n-2}}{(1+\lambda_i^2r^2)^n}\begin{pmatrix}H_i+n\nabla H_i x\\ H_i+n\nabla H_i x\\ H_i+n\nabla H_i x\\ -W_i\ln r\\ 0\end{pmatrix}$$

up to some error terms of order $O(\lambda_i^{-3},\lambda_i^{-4}\ln\lambda_i,\lambda_i^{-4},\lambda_i^{-4},\lambda_i^{-4})$, whence

$$\int\frac{\varphi_i L_{g_0}\varphi_i}{4n(n-1)}d\mu_{g_0} = \int\varphi_i^{\frac{2n}{n-2}}d\mu_{g_0} - \tilde{d}_1\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i}{\lambda_i^2}+O(\frac{\ln\lambda_i}{\lambda_i^4})\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}, \ \tilde{d}_1 = \frac{c_n}{2}\int_{\mathbb{R}^n}\frac{r^{n-2}dx}{(1+r^2)^n},$$

up to $O(\frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}})$. Recalling (5.7), we obtain

$$\int\frac{\varphi_i L_{g_0}\varphi_i}{4n(n-1)}d\mu_{g_0} = \bar{c}_0 + 4n(n-1)(\bar{d}_1-\tilde{d}_1)\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i}{\lambda_i^2}+O(\frac{\ln\lambda_i}{\lambda_i^4})\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix} \tag{5.11}$$

up to some $O(\tau^2+\frac{1}{\lambda_i^4}+\frac{1}{\lambda_i^{2(n-2)}})$. As $\bar{d}_1=\tilde{d}_1$, cf. (5.6), we simply get

$$\alpha^i\alpha^j\int\varphi_i L_{g_0}\varphi_j d\mu_{g_0} = 4n(n-1)\bar{c}_0\sum_i\alpha_i^2+\tilde{b}_1\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{i,j} \tag{5.12}$$

up to an error of order $O\Big(\tau^2+\sum_r\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}}\Big)$. Plugging this into (5.9), we obtain

$$\begin{aligned}
J_\tau(\alpha^i\varphi_i) = & \frac{4n(n-1)\bar{c}_0^{\frac{p-1}{p+1}}\sum_i\alpha_i^2}{(\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{p+1})^{\frac{2}{p+1}}}\Bigg(1-\bar{c}_1\sum_i\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\tau-\bar{c}_2\sum_i\frac{\Delta K_i}{\lambda_i^{2+\theta}}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} \\
& -\bar{d}_1\sum_i\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i+O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2}\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}-\sum_{i\neq j}(\bar{b}_1\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{n+2}{n-2}}\alpha_j}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}-\frac{\tilde{b}_1}{\bar{c}_0}\frac{\alpha_i\alpha_j}{\alpha^2})\varepsilon_{i,j}\Bigg)
\end{aligned}$$

up to some $O(\tau^2+\sum_r\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}})$. Recalling

$$\bar{b}_1=\frac{2n}{n-2}b_1,\quad \tilde{b}_1=4n(n-1)b_1,\quad \alpha^2=\sum_i\alpha_i^2,\quad \alpha_{K,\tau}^{\frac{2n}{n-2}}=\frac{n}{n-2}\bar{c}_0\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}},$$

and setting

$$\hat{c}_0=4n(n-1)\bar{c}_0^{\frac{p-1}{p+1}},\ \hat{c}_1=\frac{\bar{c}_1}{\bar{c}_0},\ \hat{c}_2=\frac{\bar{c}_2}{\bar{c}_0},\ \hat{d}_1=\frac{\bar{d}_1}{\bar{c}_0},\ \hat{b}_1=2\frac{b_1}{\bar{c}_0} \tag{5.13}$$

we may rewrite this as

$$\begin{aligned}
J_\tau(u)=J_\tau(\alpha^i\varphi_i)= & \frac{\hat{c}_0\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\Bigg(1-\hat{c}_1\sum_i\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\tau-\hat{c}_2\sum_i\frac{\Delta K_i}{\lambda_i^{2+\theta}}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}} \\
& -\hat{d}_1\sum_i\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i+O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2}\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}-\hat{b}_1\sum_{i\neq j}(\frac{K_i}{\lambda_i^\theta}\frac{\alpha_i^{\frac{n+2}{n-2}}\alpha_j}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}-\frac{\alpha_i\alpha_j}{2\alpha^2})\varepsilon_{i,j}\Bigg).
\end{aligned}$$

Then the claim follows from Lemma 5.1. □

We next state three lemmas with some expansions for the derivatives of the functionals with respect to the parameters involved (recall our notation from Section 2). The proofs are given in appendix B.

**Lemma 5.1.** *For $u\in V(q,\varepsilon)$ and $\varepsilon>0$ sufficiently small the three quantities $\partial J_\tau(u)\phi_{1,j}$, $\partial J_\tau(\alpha^i\varphi_i)\phi_{1,j}$, $\partial_{\alpha_j}J_\tau(\alpha^i\varphi_i)$ can be written as*

$$\begin{aligned}
& \frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\Bigg(\grave{c}_0\big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\big)-\grave{c}_2\big(\frac{\Delta K_j}{K_j\lambda_j^2}-\sum_k\frac{\Delta K_k}{K_k\lambda_k^2}\frac{\alpha_k^2}{\alpha^2}\big) \\
& +\grave{b}_1\Big(\sum_{k\neq l}\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l}-\sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\varepsilon_{i,j}\Big)-\grave{d}_1\begin{pmatrix}\frac{H_j}{\lambda_j}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k} & \text{for } n=3\\ \frac{H_j}{\lambda_j^2}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k^2}+O(\sum_r\frac{\ln\lambda_r}{\lambda_r^4}) & \text{for } n=4\\ \frac{H_j}{\lambda_j^3}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k^3} & \text{for } n=5\\ \frac{W_j\ln\lambda_j}{\lambda_i^4}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{W_k\ln\lambda_k}{\lambda_k^4} & \text{for } n=6\\ 0 & \text{for } n\geq 7\end{pmatrix}\Bigg)
\end{aligned}$$

*with positive constants* $\grave{c}_0, \grave{c}_2, \grave{b}_1, \grave{d}_1$ *up to an error of order*

$$O\big(\tau^2 + \sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2\big). \tag{5.14}$$

*In particular for all* $j$

$$\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1} = 1 + O\big(\tau + \sum_{r \neq s} \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \varepsilon_{r,s} + |\partial J_\tau(u)|\big).$$

**Lemma 5.2.** *For* $u \in V(q, \varepsilon)$ *and* $\varepsilon > 0$ *sufficiently small the three quantities* $\partial J_\tau(u)\phi_{2,j}$*,* $\partial J_\tau(\alpha^i \varphi_i)\phi_{2,j}$ *and* $\frac{\lambda_j}{\alpha_j} \partial_{\lambda_j} J_\tau(\alpha^i \varphi_i)$ *can be written as*

$$\frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \Big( \tilde{c}_1 \tau + \tilde{c}_2 \frac{\Delta K_j}{K_j \lambda_j^2} - \tilde{b}_2 \sum_{j \neq i} \frac{\alpha_i}{\alpha_j} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} + \tilde{d}_1 \begin{pmatrix} \frac{H_j}{\lambda_j} & \text{for } n = 3 \\ \frac{H_j}{\lambda_j^2} + O(\frac{\ln \lambda_j}{\lambda_j^4}) & \text{for } n = 4 \\ \frac{H_j}{\lambda_j^3} & \text{for } n = 5 \\ \frac{W_j \ln \lambda_j}{\lambda_j^4} & \text{for } n = 6 \\ 0 & \text{for } n \geq 7 \end{pmatrix} \Big),$$

*with positive constants* $\tilde{c}_1, \tilde{c}_2, \tilde{d}_1, \tilde{b}_2$ *up to some error of the form*

$$O\big(\tau^2 + \sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2\big). \tag{5.15}$$

**Lemma 5.3.** *For* $u \in V(q, \varepsilon)$ *and* $\varepsilon > 0$ *sufficiently small the three quantities* $\partial J_\tau(u)\phi_{3,j}$*,* $\partial J_\tau(\alpha^i \varphi_i)\phi_{3,j}$ *and* $\frac{\nabla_{a_j}}{\alpha_j \lambda_j} J_\tau(\alpha^i \varphi_i)$ *can be written as*

$$-\frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \left( \check{c}_3 \frac{\nabla K_j}{K_j \lambda_j} + \check{c}_4 \frac{\nabla \Delta K_j}{K_j \lambda_j^3} + \check{b}_3 \sum_{j \neq i} \frac{\alpha_i}{\alpha_j} \frac{\nabla_{a_j}}{\lambda_j} \varepsilon_{i,j} \right),$$

*with positive constants* $\check{c}_3, \check{c}_4, \check{b}_3$ *up to some error of the form*

$$O\big(\tau^2 + \sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2\big). \tag{5.16}$$

# 6 Gradient bounds

Theorem 2 will give suitable lower norm-bounds on the gradient of $J_\tau$, yielding Theorem 1 as a corollary. We recall that on $S^3$ and $S^4$ the result was proved in [12], [28], [29], [45] in more generality.

**Definition 6.1.** *Let* $H$ *be as in (2.7). We call a positive Morse function* $K$ *on* $M$ non-degenerate

*(i)* of degree $q \in \mathbb{N}$ *in case* $n = 4$*, if* $\{\nabla K = 0\} \cap \{\tilde{c}_2 \frac{\Delta K}{K} + \tilde{c}_3 H = 0\} = \emptyset$ *and if for every* $1 \leq k \leq q$ *and every subset* $\{x_1, \ldots, x_k\} \subseteq \{\nabla K = 0\} \cap \{\tilde{c}_2 \frac{\Delta K}{K} + \tilde{c}_3 H < 0\}$ *the matrices*

$$\mathcal{M}_{x_1,\ldots,x_k} = - \begin{pmatrix} \tilde{c}_2 \frac{\Delta K(x_1)}{K(x_1)^2} + \tilde{c}_3 \frac{H(x_1)}{K(x_1)} & \tilde{c}_4 \frac{G_0(x_1,x_2)}{\gamma_n (K(x_1)K(x_2))^{\frac{1}{2}}} & \cdots & \tilde{c}_4 \frac{G_0(x_1,x_k)}{\gamma_n (K(x_1)K(x_k))^{\frac{1}{2}}} \\ \tilde{c}_4 \frac{G_0(x_2,x_1)}{\gamma_n (K(x_2)K(x_1))^{\frac{1}{2}}} & \ddots & & \vdots \\ \vdots & & & \vdots \\ \vdots & & \ddots & \tilde{c}_4 \frac{G_0(x_{k-1},x_k)}{\gamma_n (K(x_{k-1})K(x_k))^{\frac{1}{2}}} \\ \tilde{c}_4 \frac{G_0(x_k,x_1)}{\gamma_n (K(x_k)K(x_1))^{\frac{1}{2}}} & \cdots & & \tilde{c}_2 \frac{\Delta K(x_k)}{K(x_k)^2} + \tilde{c}_3 \frac{H(x_k)}{K(x_k)} \end{pmatrix}$$

*have non-vanishing least eigenvalues, where* $\tilde{c}_2 = \sqrt{3\omega_4}$, $\tilde{c}_3 = 24\sqrt{3\omega_4} = \tilde{c}_4$. *We say that* $K$ *is* non-degenerate, *if it is non-degenerate of all degrees.*

*(ii) in case* $n \geq 5$, *if* $\{\nabla K = 0\} \cap \{\Delta K = 0\} = \emptyset$, *i.e.* (1.2) *holds.*

**Remark 6.1.** *Non-degeneracy in case* $n = 4$ *implies the existence of a least eigenvalue*

$$\mathcal{M}_{x_1,\ldots,x_k}\mathbf{x}_{x_1,\ldots,x_k} = \lambda_{x_1,\ldots,x_k}\mathbf{x}_{x_1,\ldots,x_k} \quad \textit{with} \quad \lambda_{x_1,\ldots,x_k} \neq 0$$

*and such that* $\lambda_{x_1,\ldots,x_k}$ *is simple and has a positive eigenvector, i.e.*

$$\mathbf{x}_{x_1,\ldots,x_k} = (x^1_{x_1,\ldots,x_k}, \ldots, x^k_{x_1,\ldots,x_k}) \ \textit{ with } \ \mathbf{x}^l_{x_1,\ldots,x_k} > 0 \ \textit{ for all } 1 \leq l \leq k.$$

**Theorem 2.** *Let* $\mathcal{M}_{x_1,\ldots,x_k}$ *be as in Definition 6.1, and suppose that*

$$\begin{Bmatrix} K \ \textit{ is non-degenerate of degree } q & \textit{for } \ n = 4 \\ K \ \textit{ is non-degenerate} & \textit{for } \ n \geq 5 \end{Bmatrix}.$$

*Then for* $\varepsilon > 0$ *sufficiently small there exists* $c > 0$ *such that for any* $u \in V(q, \varepsilon)$ *with* $k_\tau = 1$ *there holds*

$$|\partial J_\tau(u)| \geq c\big(\tau + \sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \big|1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_r}{\lambda_r^\theta} \alpha_r^{p-1}\big| + \varepsilon_{r,s}\big),$$

*cf.* (5.1), *unless there is a violation of at least one of the four conditions*

*(i)* $\quad \tau > 0$;

*(ii)* $\quad$ *there exists* $x_i \neq x_j \in \begin{Bmatrix} \{\nabla K = 0\} \cap \{\tilde{c}_2 \frac{\Delta K}{K} + \tilde{c}_3 H < 0\} & \textit{for } \ n = 4, \\ \{\nabla K = 0\} \cap \{\Delta K < 0\} & \textit{for } \ n \geq 5 \end{Bmatrix}$ *and* $d(a_i, x_i) = O(\frac{1}{\lambda_i})$;

*(iii)*
$$\begin{Bmatrix} \alpha_j = \Theta \cdot \left( \frac{\lambda_j^\theta}{K_j} \left( 1 + \frac{1}{8} \big( \frac{\Delta K_j}{K_j \lambda_j^2} - 60 \frac{H_j}{\lambda_j^2} - \frac{\sum_k (\frac{\Delta K_k}{K_k^2 \lambda_k^2} - 60 \frac{H_k}{K_k \lambda_k^2})}{\sum_k \frac{1}{K_k}} \big) \right) \right)^{\frac{1}{p-1}} + o(\frac{1}{\lambda_j^2}) & \textit{for } \ n = 4, \\ \alpha_j = \Theta \cdot (\frac{\lambda_j^\theta}{K_j})^{\frac{1}{p-1}} + o(\frac{1}{\lambda_j^2}) & \textit{for } \ n \geq 5 \end{Bmatrix};$$

*(iv)*
$$\begin{Bmatrix} \mathcal{M}_{x_1,\ldots,x_q} > 0 \ \textit{ and } \ \lambda_j = \frac{\sigma_j + o(1)}{\sqrt{\tau}} & \textit{for } \ n = 4, \\ \tilde{c}_1 \tau = -\tilde{c}_2 \frac{\Delta K_k}{K_k \lambda_k^2} + o(\frac{1}{\lambda_k^2}) & \textit{for } \ n \geq 5 \end{Bmatrix}$$

*for all* $j \neq i, j = 1, \ldots, q$, *where* $\sigma = (\sigma_1, \ldots, \sigma_q)$ *in case* $n = 4$ *is the unique solution of*

$$\tilde{c}_1 \begin{pmatrix} \frac{\sigma_1}{K(x_1)} \\ \vdots \\ \frac{\sigma_q}{K(x_q)} \end{pmatrix} = \mathcal{M}_{x_1,\ldots,x_q} \begin{pmatrix} \frac{1}{\sigma_1} \\ \vdots \\ \frac{1}{\sigma_q} \end{pmatrix} \quad \textit{with } \ \sigma_j > 0,$$

*while* $\Theta$ *is given in Remark 6.2. In the latter case there holds* $\lambda_1 \simeq \ldots \lambda_q \simeq \lambda = \frac{1}{\sqrt{\tau}}$ *and setting*

$$a_j = \exp_{g_{x_j}}(\bar{a}_j)$$

*we still have up to an error* $o(\frac{1}{\lambda^3})$ *the lower bound*

$$\begin{aligned} |\partial J(u)| \gtrsim & \sum_j (|\tau + \frac{1}{2} \frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} + 12[\frac{H(x_j)}{\lambda_j^2} + \sum_{j \neq i} \sqrt{\frac{K(x_j)}{K(x_i)}} \frac{G_{g_0}(x_i, x_j)}{\gamma_n \lambda_i \lambda_j}]|) \\ & + \sum_j (|\frac{\bar{a}_j}{\lambda_j} + \frac{1}{3} (\nabla^2 K(x_j))^{-1} \frac{\nabla \Delta K(x_j)}{\lambda_j^3} + 8 \sum_{j \neq i} \sqrt{\frac{K^3(x_j)}{K(x_i)}} (\nabla^2 K(x_j))^{-1} \frac{\nabla_{x_j} G_{g_0}(x_i, x_j)}{\gamma_n \lambda_i \lambda_j^2}|) \\ & + \sum_j |\alpha_j - \Theta \cdot \sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)} (1 + \frac{1}{8} \left( \frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} - 60 \frac{H(x_j)}{\lambda_j^2} - \frac{\sum_k (\frac{\Delta K(x_k)}{K(x_k)^2 \lambda_k^2} - 60 \frac{H(x_k)}{K(x_k)\lambda_k^2})}{\sum_k \frac{1}{K(x_k)}} \right)} )| \end{aligned}$$

*in case $n = 4$ and*

$$|\partial J(u)| \gtrsim \sum_j |\tau + \frac{2}{9}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} + \frac{512}{9\pi}[\frac{H(x_j)}{\lambda_j^3} + \sum_{j\neq i}\sqrt{\frac{K(x_j)}{K(x_i)}}\frac{G_{g_0}(x_i,x_j)}{\gamma_n(\lambda_i\lambda_j)^{\frac{3}{2}}}]|$$
$$+\sum_j |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}|$$
$$+\sum_j |\alpha_j - \Theta\cdot \sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}(1-\frac{1}{90}\left(\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2} + \frac{2816}{\pi}\frac{H(x_j)}{\lambda_j^3} - \frac{\sum_k(\frac{\Delta K(x_k)}{K(x_k)^2\lambda_k^2} + \frac{2816}{\pi}\frac{H(x_k)}{K(x_k)\lambda_k^3})}{\sum_k \frac{1}{K(x_k)}}\right)})|$$

*in case $n = 5$ and*

$$|\partial J_\tau(u)| \gtrsim \sum_j (|\tau + \frac{\tilde{c}_2}{\tilde{c}_1}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}| + |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}| + |\alpha_j - \Theta\cdot\sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}}|)$$

*in case $n \geq 6$. The constants appearing above are defined by $\bar{c}_0 = \int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n}$,*

$$\tilde{c}_1 = \frac{n(n-1)(n-2)^2}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln\frac{1}{1+r^2}dx, \quad \tilde{c}_2 = -\frac{(n-1)(n-2)}{\bar{c}_0^{\frac{n-2}{n}}}\int_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+1}}dx$$

*and*

$$\check{c}_3 = \int_{\mathbb{R}^n}\frac{4(n-1)(n-2)}{(1+r^2)^n}dx, \quad \check{c}_4 = \int_{\mathbb{R}^n}\frac{2(n-1)r^2}{(1+r^2)^n}dx.$$

The differences in the above expressions for $n = 5$ and $n \geq 6$ is caused by a different decay of bubble functions causing stronger mutual interactions in lower dimension.

**Remark 6.2.** *Under non-degeneracy conditions, Theorem 2 has the following immediate implications.*

1. *In case $\tau = 0$ there are no solutions of $\partial J(u) = \partial J_0(u) = 0$ in $V(q,\varepsilon)$, cf. Theorem 1.4 in [14].*
2. *In case $\tau > 0$ every solution $\partial J_\tau(u) = 0$ in $V(q,\varepsilon)$ satisfies*
$$\lambda_1 \simeq \ldots \simeq \lambda_q \simeq \frac{1}{\sqrt{\tau}}$$
*and has isolated simple blow-ups occurring close to*
$$\begin{cases} \{\nabla K = 0\} \cap \{\tilde{c}_2\frac{\Delta K}{K} + \tilde{c}_3 H < 0\} & \text{for } n = 4 \\ \{\nabla K = 0\} \cap \{\Delta K < 0\} & \text{for } n \geq 5. \end{cases}$$
3. *The $\alpha_j, \lambda_j$ and $a_j$'s are determined to a precision $o(\tau^{\frac{3}{2}}) + O(|\partial J_\tau(u)|)$. Indeed, for e.g. $n = 6$*
$$|\tau + \frac{\tilde{c}_2}{\tilde{c}_1}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}|$$
*determines $\lambda_j$ up to the latter error from $\tau$ and $x_j$, whence $a_j$ is determined as well by*
$$|\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}|$$
*from $\lambda_j$ and $x_j$, and finally up to the multiplicative constant $\Theta$ also $\alpha_j$ is determined by*
$$|\alpha_j - \Theta\cdot\sqrt[p-1]{\frac{\lambda_j^\theta}{K(a_j)}}|$$

*from $\lambda_j, a_j$ and $\tau$, recalling $\theta = \frac{n-2}{2}\tau$ and $p = \frac{n+2}{n-2} - \tau$. As for the multiplicative constant we have*

$$1 = k_\tau = \int K(\alpha^i \varphi_i + v)^{p+1} d\mu_{g_0} = \int K(\alpha^i \varphi_i)^{p+1} = \sum_i \frac{K(a_i)}{\lambda_i^\theta} \alpha_i^{p+1} \left( \bar{c}_0 + \bar{c}_1 \tau + \bar{c}_2 \frac{\Delta K(x_i)}{K(x_i)\lambda_i^2}) \right)$$

*up to some $o(\tau^{\frac{3}{2}})$, cf.* (4.5), *Lemma 4.2, Lemma 2.2 and* (5.8), *whence*

$$1 = \Theta^{p-1} \sum_i \alpha_i^2 \left( \bar{c}_0 + \bar{c}_1 \tau + \bar{c}_2 \frac{\Delta K(x_i)}{K(x_i)\lambda_i^2}) \right) = \Theta^{p+1} \sum_i \left( \frac{\lambda_j^\theta}{K(a_j)} \right)^{\frac{2}{p-1}} \left( \bar{c}_0 + \bar{c}_1 \tau + \bar{c}_2 \frac{\Delta K(x_i)}{K(x_i)\lambda_i^2}) \right)$$

*up to the same error and so the multiplicative constant $\Theta$ is determined as well.*

*Proof of Theorem 2.* First we note that $k_\tau = 1$ implies, that all the $\alpha_i$ do not tend to infinity and least one of them does not approach zero. Hence by definition of $V(q,\varepsilon)$ all the $\alpha_i$ are uniformly bounded away from zero and infinity. Secondly, if for some index $j = 1, \ldots, q$ we have

$$\left|1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1}\right| \gg \tau + \sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s},$$

then the claim follows from Lemma 5.1, whence we may henceforth assume that for all $j = 1, \ldots, q$

$$\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1} = 1 + O\big(\tau + \sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s}\big). \tag{6.1}$$

Thus we have to show

$$|\partial J_\tau(u)| \gtrsim \tau + \sum_{j=1}^q \frac{|\nabla K_j|}{\lambda_j} + \frac{1}{\lambda_j^2} + \sum_{r \neq s} \varepsilon_{r,s} \tag{6.2}$$

and arguing by contradiction we may assume that

$$|\partial J_\tau(u)| \lesssim \tau + \sum_{j=1}^q \frac{|\nabla K_j|}{\lambda_j} + \frac{1}{\lambda_j^2} + \sum_{r \neq s} \varepsilon_{r,s}.$$

Then by Lemmata 5.2 and 5.3 we have

$$(a) \quad \partial J_\tau(u) \phi_{3,j} = \frac{-\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} (\check{c}_3 \frac{\nabla K_j}{K_j \lambda_j} + \check{b}_3 \sum_{j \neq i} \frac{\alpha_i}{\alpha_j} \frac{\nabla_{a_j}}{\lambda_j} \varepsilon_{i,j});$$

$$(\lambda) \quad \partial J(u) \phi_{2,j} = \frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \left( \tilde{c}_1 \tau + \tilde{c}_2 \frac{\Delta K_j}{K_j \lambda_j^2} - \tilde{b}_2 \sum_{j \neq i} \frac{\alpha_i}{\alpha_j} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \right)$$

up to some errors of the form $O(\frac{1}{\lambda_j^3}) + O(\tau^2 + \sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}})$, where we have to add for $(\lambda)$ $\tilde{d}_1 \frac{H_i}{\lambda_i^2}$ to $\tilde{c}_2 \frac{\Delta K_i}{K_i \lambda_i{}^2}$ in case $n = 4$. Ordering indices so that $\lambda_1 \geq \ldots \geq \lambda_q \Longleftrightarrow \frac{1}{\lambda_1} \leq \ldots \leq \frac{1}{\lambda_q}$ and recalling (2.9), we have

$$-\lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = \frac{n-2}{2} \frac{\frac{\lambda_j}{\lambda_i} - \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j)}{(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j))^{\frac{n}{2}}}$$

and therefore

$$\lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = \frac{2-n}{2} \varepsilon_{i,j} + O(\frac{1}{\lambda_j^4} + \varepsilon_{i,j}^{\frac{n+2}{n}}) \quad \text{in case } \; j < i \; \text{ or } \; d_{g_0}(a_i, a_j) \neq o(1). \tag{6.3}$$

From $(a)$ and $(\lambda)$ above we find uniformly bounded vector fields $A_1, \Lambda_1$ on $V(q,\varepsilon)$ such that

$$(\mathbf{A_1}) \quad \partial J_\tau(u)A_1 \gtrsim \frac{|\nabla K_1|}{\lambda_1} + O\big(\frac{1}{\lambda_1^3} + \sum_{1\neq i}\varepsilon_{1,i}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big);$$

$$(\mathbf{\Lambda_1}) \quad \partial J_\tau(u)\Lambda_1 \simeq \tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_1}{K_1\lambda_1^2} + \tilde{c}_4\sum_{1\neq j}\frac{\alpha_i}{\alpha_1}\varepsilon_{1,i} + O\big(\frac{1}{\lambda_1^3}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big)$$

with $\tilde{c}_4 = \frac{n-2}{2}\tilde{b}_2$, and combining $X_1 = \Lambda_1 + \epsilon A_1$ with some $\epsilon > 0$ small and fixed such that we keep a positive coefficient in front of $\varepsilon_{1,i}$, we get

$$(\mathbf{C_1}) \quad B_1 = \partial J_\tau(u)X_1 \gtrsim \big(\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_1}{K_1\lambda_1^2}\big) + \epsilon\big(\frac{|\nabla K_1|}{\lambda_1} + \sum_{1\neq i}\varepsilon_{1,i}\big) + O\big(\frac{1}{\lambda_1^3}\big) + O\big(\tau^2 + \sum_{\substack{r\neq s\\ r,s>1}}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Likewise from $(a)$ and $(\lambda)$ we find uniformly bounded vector fields $A_2, \Lambda_2$ defined on $V(q,\varepsilon)$ such that

$$(\mathbf{A_2}) \quad \partial J_\tau(u)A_2 \gtrsim \frac{|\nabla K_2|}{\lambda_2} + O\big(\frac{1}{\lambda_2^3} + \sum_{2\neq i}\varepsilon_{1,i}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big);$$

$$(\mathbf{\Lambda_2}) \quad \partial J_\tau(u)\Lambda_2 \simeq \tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_2}{K_2\lambda_2^2} + \tilde{c}_4\sum_{2<i}\frac{\alpha_i}{\alpha_2}\varepsilon_{1,i} + O\big(\frac{1}{\lambda_2^3} + \varepsilon_{1,2}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big)$$

and combining them as $X_2 = \Lambda_2 + \epsilon A_2$ with $\epsilon > 0$ small we obtain

$$B_2 = \partial J_\tau(u)X_2 \gtrsim \big(\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_2}{K_2\lambda_2^2}\big) + \epsilon\big(\frac{|\nabla K_2|}{\lambda_2} + \sum_{2<i}\varepsilon_{2,i}\big) + O\big(\frac{1}{\lambda_2^3} + \varepsilon_{1,2}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Therefore combining $B_1$ and $B_2$ so that the coefficient of $\varepsilon_{i,j}$ is positive

$$(\mathbf{C_2}) \quad B_1 + \epsilon B_2 \gtrsim \sum_{j=1}^{2}\big[\epsilon^j\big(\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2}\big) + \epsilon^{j+1}\big(\frac{|\nabla K_j|}{\lambda_j} + \sum_{j\neq i}\varepsilon_{j,i}\big)\big] + O\big(\frac{1}{\lambda_2^3}\big) + O\big(\tau^2 + \sum_{\substack{r\neq s\\ r,s>2}}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Iteratively, for all $k = 1, \ldots, q$ we can find uniformly bounded vector fields $A_k, \Lambda_k$ such that

$$(\mathbf{A_k}) \quad \partial J_\tau(u)A_k \gtrsim \frac{|\nabla K_k|}{\lambda_k} + O\big(\frac{1}{\lambda_k^3} + \sum_{k\neq i}\varepsilon_{k,i}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big);$$

$$(\mathbf{\Lambda_k}) \quad \partial J_\tau(u)\Lambda_k \simeq \tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_k}{K_k\lambda_k^2} + \tilde{c}_4\sum_{k<i}\frac{\alpha_i}{\alpha_k}\varepsilon_{k,i} + O\big(\frac{1}{\lambda_k^3} + \sum_{k>i}\varepsilon_{k,i}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big);$$

$$(\mathbf{C_k}) \quad \sum_{j=1}^{k}\epsilon^j B_j \gtrsim \sum_{j=1}^{k}\big[\epsilon^j\big(\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2}\big) + \epsilon^{j+1}\big(\frac{|\nabla K_j|}{\lambda_j} + \sum_{j\neq i}\varepsilon_{j,i}\big)\big] + O\big(\frac{1}{\lambda_k^3}\big) + O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}^{\frac{n+2}{n}}\big),$$

where we have to add $\tilde{c}_3\frac{H_j}{\lambda_j^2}$ to $\tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2}$ in case $n = 4$, where

$$\tilde{c}_3 = \tilde{d}_1 \tag{6.4}$$

In particular

$$(\mathbf{C_q})\quad \sum_{j=1}^{k}\epsilon^j B_j \gtrsim \sum_{j=1}^{k}\big[\epsilon^j\big(\tilde c_1\tau+\tilde c_2\frac{\Delta K_j}{K_j\lambda_j^2}\big)+\epsilon^{j+1}\big(\frac{|\nabla K_j|}{\lambda_j}+\sum_{j\neq i}\varepsilon_{j,i}\big)\big]+O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Then, if either

$$\frac{1}{\lambda_q^2}\ll\tau+\sum_{j=1}^{q}\frac{|\nabla K_j|}{\lambda_j}+\sum_{r\neq s}\varepsilon_{r,s}\qquad\text{or}\qquad\frac{1}{\lambda_q^2}\gg\tau+\sum_{j=1}^{q}\frac{|\nabla K_j|}{\lambda_j}+\sum_{r\neq s}\varepsilon_{r,s},$$

we obviously have (6.2) from $(\mathbf{C_q})$. Thus we may assume

$$\frac{1}{\lambda_q^2}\simeq\tau+\sum_{j=1}^{q}\frac{|\nabla K_j|}{\lambda_j}+\sum_{r\neq s}\varepsilon_{r,s},\tag{6.5}$$

whence we may simplify the above formulas to

$$(\mathbf{A_k})\quad \partial J_\tau(u)A_k\gtrsim\frac{|\nabla K_k|}{\lambda_k}+O\big(\sum_{k\neq i}\varepsilon_{k,i}\big)+o\big(\frac{1}{\lambda_q^2}\big);$$

$$(\mathbf{\Lambda_k})\quad \partial J_\tau(u)\Lambda_k\simeq\tilde c_1\tau+\tilde c_2\frac{\Delta K_k}{K_k\lambda_k^2}+\tilde c_4\sum_{k<i}\frac{\alpha_i}{\alpha_k}\varepsilon_{k,i}+O\big(\sum_{k>i}\varepsilon_{k,i}\big)+o\big(\frac{1}{\lambda_q^2}\big);$$

$$(\mathbf{C_k})\quad \sum_{j=1}^{k}\epsilon^j B_j\gtrsim\sum_{j=1}^{k}\big[\epsilon^j\big(\tilde c_1\tau+\tilde c_2\frac{\Delta K_j}{K_j\lambda_j^2}\big)+\epsilon^{j+1}\big(\frac{|\nabla K_j|}{\lambda_j}+\sum_{j\neq i}\varepsilon_{j,i}\big)\big]+o\big(\frac{1}{\lambda_q^2}\big),$$

adding $\tilde c_3\frac{H_j}{\lambda_j^2}$ to $\tilde c_2\frac{\Delta K_j}{K_j\lambda_j^2}$ for $n=4$. We first consider the pair $(q-1,q)$. Suppose

$$\frac{1}{\lambda_{q-1}^2}=o\big(\frac{1}{\lambda_q^2}\big).$$

To prove (6.2) we then may assume from $(\mathbf{C_{q-1}})$ and (6.5) that also $\tau+\sum_{r\neq s}\varepsilon_{r,s}=o(\frac{1}{\lambda_q^2})$, since

$$\sum_{j=1}^{q-1}\sum_{j\neq i}\varepsilon_{i,j}=\sum_{q-1\geq r\neq s}\varepsilon_{r,s}=\sum_{r\neq s}\varepsilon_{r,s}.$$

As the coefficient of $\lambda_q^{-2}$ in $(\mathbf{\Lambda_q})$ is non zero by non-degeneracy, (6.2) follows. So we may assume

$$\frac{1}{\lambda_{q-1}}\simeq\frac{1}{\lambda_q},$$

and therefore, still by (6.5),

$$|\nabla K_{q-1}|\lesssim\frac{1}{\lambda_{q-1}},\quad |\nabla K_q|\lesssim\frac{1}{\lambda_q}.$$

So, if $a_{q-1}$ is close to $a_q$, these points are close to the same critical point of $K$, which, as $K$ is Morse, implies $d(a_{q-1},a_q))\lesssim\frac{1}{\lambda_q}\simeq\frac{1}{\lambda_{q-1}}$. This however contradicts the fact that by Proposition 3.1

$$\varepsilon_{q-1,q}\simeq\frac{1}{(\lambda_{q-1}\lambda_q d^2(a_{q-1},a_q))^{\frac{n-2}{2}}}\longrightarrow 0.$$

Therefore for the pair $(q-1,q)$ we may assume

$$|\nabla K_{q-1}|, |\nabla K_q| \lesssim \frac{1}{\lambda_{q-1}} \simeq \frac{1}{\lambda_q}, \quad \text{ and } \quad d(a_{q-1},a_q) > c.$$

In particular in case $n \geq 5$ we have $\varepsilon_{q-1,q} \simeq \frac{1}{\lambda_q^{n-2}} = o(\frac{1}{\lambda_q^2})$, whereas in case $n=4$

$$\varepsilon_{q-1,q} = \frac{G_{g_0}(a_{q-1},a_q)}{\gamma_n \lambda_{q-1}\lambda_q} + O(\frac{1}{\lambda_q^4}).$$

We turn to consider the triple $(q-2,q-1,q)$. Suppose that $\frac{1}{\lambda_{q-2}^2} = o(\frac{1}{\lambda_{q-1}^2})$. To get (6.2) we then may assume from $(\mathbf{C_{q-2}})$ and (6.5) that

$$\tau + \sum_{q-2 \geq r \neq s} \varepsilon_{r,s} = o(\frac{1}{\lambda_q^2})$$

as well. But then clearly in case $n \geq 5$ we obtain (6.2) from $(\mathbf{\Lambda_{q-1}})$ or $(\mathbf{\Lambda_q})$, since $\varepsilon_{q-1,q} = o(\lambda_q^{-2})$ is already known. In case $n=4$ we have to argue more subtly. From $(\lambda)$ we find

$$\partial J(u)\phi_{2,q-1} = \frac{\alpha_{q-1}}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \left( \tilde{c}_2 \frac{\Delta K_{q-1}}{K_{q-1}\lambda_{q-1}^2} + \tilde{c}_3 \frac{H_{q-1}}{\lambda_{q-1}^2} + \tilde{c}_4 \frac{\alpha_q}{\alpha_{q-1}} \frac{G_{g_0}(a_{q-1},a_q)}{\gamma_n \lambda_{q-1}\lambda_q} \right)$$

and

$$\partial J(u)\phi_{2,q} = \frac{\alpha_q}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \left( \tilde{c}_2 \frac{\Delta K_q}{K_q\lambda_q^2} + \tilde{c}_3 \frac{H_q}{\lambda_q^2} + \tilde{c}_4 \frac{\alpha_{q-1}}{\alpha_q} \frac{G_{g_0}(a_{q-1},a_q)}{\gamma_n \lambda_{q-1}\lambda_q} \right)$$

up to an error of order $o(\frac{1}{\lambda_q^2})$, cf. (6.3). Obviously (6.2) then follows if either

$$\tilde{c}_2 \frac{\Delta K_q}{K_q\lambda_q^2} + \tilde{c}_3 \frac{H_q}{\lambda_q^2} > 0 \quad \text{ or } \quad \tilde{c}_2 \frac{\Delta K_{q-1}}{K_{q-1}\lambda_{q-1}^2} + \tilde{c}_3 \frac{H_{q-1}}{\lambda_{q-1}^2} > 0.$$

We may thus assume both summands to be negative. Recalling (6.1), we then obtain

$$\partial J_\tau(u) \begin{pmatrix} \beta_{q-1}\phi_{2,q-1} \\ \beta_q\phi_{2,q} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_{q-1}} & 0 \\ 0 & \frac{1}{\lambda_q} \end{pmatrix} \begin{pmatrix} \tilde{c}_2 \frac{\Delta K_{q-1}}{K_{q-1}^2} + \tilde{c}_3 \frac{H_{q-1}}{K_{q-1}} & \tilde{c}_4 \frac{G_0(a_{q-1},a_q)}{\gamma_n (K_{q-1}K_q)^{\frac{1}{2}}} \\ \tilde{c}_4 \frac{G_0(a_{q-1},a_q)}{\gamma_n (K_{q-1}K_q)^{\frac{1}{2}}} & \tilde{c}_2 \frac{\Delta K_q}{K_q^2} + \tilde{c}_3 \frac{H_q}{K_q} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_{q-1}} \\ \frac{1}{\lambda_q} \end{pmatrix}$$

up to an error $o(\frac{1}{\lambda_q^2})$ letting

$$K_j\alpha_j\beta_j = (\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \quad \text{for} \quad j = q-1,q,$$

and thus $|\partial J_\tau(u)| \gtrsim \lambda_q^{-2}$, since otherwise $a_{q-1},a_q$ close to $x_{q-1},x_q \in \{\nabla K = 0\} \cap \{\tilde{c}_2 \frac{\Delta K}{K} + \tilde{c}_3 H < 0\}$ and

$$\mathcal{M}_{q-1,q} = \begin{pmatrix} \tilde{c}_2 \frac{\Delta K(x_{q-1})}{K(x_{q-1})^2} + \tilde{c}_3 \frac{H(x_{q-1})}{K(x_{q-1})} & \tilde{c}_4 \frac{G_0(x_{q-1},x_q)}{\gamma_n (K(x_{q-1})K(x_q))^{\frac{1}{2}}} \\ \tilde{c}_4 \frac{G_0(x_{q-1},x_q)}{\gamma_n (K(x_{q-1})K(x_q))^{\frac{1}{2}}} & \tilde{c}_2 \frac{\Delta K(x_q)}{K(x_q)^2} + \tilde{c}_3 \frac{H(x_q)}{K(x_q)} \end{pmatrix}$$

would have after a blow-up for $\tau \longrightarrow 0$ a vanishing eigenvalue with strictly positive eigenvector, which by Remark 6.1 is impossible. So (6.2) again follows. We may thus assume

$$\frac{1}{\lambda_{q-2}} \simeq \frac{1}{\lambda_{q-1}} \simeq \frac{1}{\lambda_q}$$

and therefore by (6.5)

$$|\nabla K_{q-2}| \lesssim \frac{1}{\lambda_{q-2}}, \quad |\nabla K_{q-1}| \lesssim \frac{1}{\lambda_{q-1}}, \quad |\nabla K_q| \lesssim \frac{1}{\lambda_q}.$$

So, if $a_{q-2}$ is close to either $a_{q-1}$ or $a_q$, these points are close to the same critical point of $K$, whence

$$\varepsilon_{q-2,q-1} \simeq 1 \ \text{ or } \ \varepsilon_{q-2,1} \simeq 1$$

as before, contradicting Proposition 3.1. Thus for $(q-2, q-1, q)$ we may assume

$$|\nabla K_{q-2},|, |\nabla K_{q-1}|, |\nabla K_q| \lesssim \frac{1}{\lambda_{q-2}} \simeq \frac{1}{\lambda_{q-1}} \simeq \frac{1}{\lambda_q}$$

and

$$d(a_{q-2}, a_{q-1}), d(a_{q-2}, a_q), d(a_{q-1}, a_q) > c$$

analogously to the previous case of the pair $(q-1, 1)$. In particular in case $n \geq 5$

$$\varepsilon_{q-2,q-1}, \varepsilon_{q-2,q}, \varepsilon_{q-1,q} \simeq \frac{1}{\lambda_q^{n-2}} = o(\frac{1}{\lambda_q^2}),$$

whereas in case $n = 4$ up to an error $O(\frac{1}{\lambda_q^4})$

$$\varepsilon_{q-2,q-1} = \frac{G_{g_0}(a_{q-2}, a_{q-1})}{\gamma_n \lambda_{q-2}\lambda_{q-1}}, \varepsilon_{q-2,q} = \frac{G_{g_0}(a_{q-2}, a_q)}{\gamma_n \lambda_{q-2}\lambda_q}, \varepsilon_{q-1,q} = \frac{G_{g_0}(a_{q-1}, a_q)}{\gamma_n \lambda_{q-1}\lambda_q}.$$

Iteratively, we then may assume for all $k \neq l = 1, \ldots, q$

$$|\nabla K_k| \lesssim \frac{1}{\lambda_k} \simeq \frac{1}{\lambda_l} \quad \text{and} \quad d(a_k, a_l) > c.$$

In particular $\varepsilon_{k,l} = o(\frac{1}{\lambda_q^2})$ for $n \geq 5$ and $\varepsilon_{k,l} = \frac{G_{g_0}(a_k,a_l)}{\lambda_k\lambda_l}$ for $n = 4$. But then

$$(\mathbf{\Lambda_k}) \quad \partial J_\tau(u)\Lambda_k \simeq (\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_k}{K_k\lambda_k^2}) + o(\frac{1}{\lambda_q^2})$$

in case $n \geq 5$ and thus

$$|\partial J_\tau(u)| \gtrsim \left| \tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_k}{K_k\lambda_k^2} \right|$$

up to some $o(\frac{1}{\lambda_q^2})$. Therefore (6.2) holds unless $\tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_k}{K_k\lambda_k^2} = o(\frac{1}{\lambda_q^2})$, while now for $n = 4$

$$\partial J(u)\phi_{2,j} = \frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \left( \tilde{c}_1\tau + \tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2} + \tilde{c}_3\frac{H_j}{\lambda_j^2} + \tilde{c}_4\sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\frac{G_{g_0}(a_i, a_j)}{\gamma_n\lambda_i\lambda_j} \right)$$

up to some $o(\frac{1}{\lambda_q^2})$, cf. (6.3), for all $j = 1, \ldots, q$. Obviously (6.2) then follows, if for some $j = 1, \ldots, q$

$$\tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2} + \tilde{c}_3\frac{H_j}{\lambda_j^2} > 0,$$

whence we may assume all these summands to be negative, proving (ii). From $(\lambda)$ and (6.1) we then have

$$\partial J(u)(\beta_j\phi_{2,j}) = \tilde{c}_1\frac{\tau}{K_j} + \tilde{c}_2\frac{\Delta K_j}{K_j^2\lambda_j^2} + \tilde{c}_3\frac{H_j}{K_j\lambda_j^2} + \tilde{c}_4\sum_{j\neq i}\frac{G_{g_0}(a_i, a_j)}{\gamma_n\sqrt{K_iK_j}\lambda_i\lambda_j}$$

up to some $o(\frac{1}{\lambda_q^2})$ letting as before $\beta_j = \frac{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}{K_j\alpha_j}$. Therefore

$$|\partial J(u)| \gtrsim \left| \begin{pmatrix} \frac{\tilde{c}_1\tau}{K_1} \\ \vdots \\ \frac{\tilde{c}_1\tau}{K_q} \end{pmatrix} - diag(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_q})\mathcal{M}_{a_1,\ldots,a_q} \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_q} \end{pmatrix} \right|$$

up to the same error. This implies that (6.2) holds true, unless we can solve

$$\begin{pmatrix} \frac{\tilde{c}_1\tau\lambda_1}{K_1} \\ \vdots \\ \frac{\tilde{c}_1\tau\lambda_q}{K_q} \end{pmatrix} = \mathcal{M}_{a_1,\ldots,a_q} \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_q} \end{pmatrix} + o(\frac{1}{\lambda_q}) \tag{6.6}$$

and we may already assume, by (ii), that $a_j$ is close to

$$x_j \in \{\nabla K = 0\} \cap \{\tilde{c}_2 \frac{\Delta K}{K} + \tilde{c}_3 H < 0\}.$$

In particular (6.2) follows in case $\tau = 0$ by the non-degeneracy condition on $K$, proving (i). In case $\tau > 0$, writing $\sigma_j = \sqrt{\tau}\lambda_j$, we find passing to the limit $\tau \longrightarrow 0$, that there has to exist a solution to

$$\tilde{c}_1 \begin{pmatrix} \frac{\sigma_1}{K(x_1)} \\ \vdots \\ \frac{\sigma_q}{K(x_q)} \end{pmatrix} = \mathcal{M}_{x_1,\ldots,x_q} \begin{pmatrix} \frac{1}{\sigma_1} \\ \vdots \\ \frac{1}{\sigma_q} \end{pmatrix}. \tag{6.7}$$

In particular, testing the above relation with $\mathbf{x} = \mathbf{x}_{x_1,\ldots,x_q}$, cf. Remark 6.1, we find

$$0 \leq \tilde{c}_1 \sum_j \frac{x_j\sigma_j}{K_j} = \lambda \sum_j \frac{x_j}{\sigma_j},$$

where $\lambda = \lambda_{x_1,\ldots,x_q}$ is the least eigenvalue of $\mathcal{M}_{x_1,\ldots,x_q}$. Thus necessarily $\mathcal{M}_{x_1,\ldots,x_q} > 0$. Since

$$F(\sigma) = \mathcal{M}_{x_1,\ldots,x_q} \begin{pmatrix} \frac{1}{\sigma_1} \\ \vdots \\ \frac{1}{\sigma_q} \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} \\ \vdots \\ \frac{1}{\sigma_q} \end{pmatrix} + 2\tilde{c}_1 \sum \frac{\sigma_i}{K_i}$$

is a sum of convex functions, there exists a unique critical point of $F$ satisfying (6.7). Hence we have comparability $\lambda_1 \simeq \ldots \simeq \lambda_1 \simeq 1/\sqrt{\tau} \simeq \lambda$ like in case $n \geq 5$. Thus (iv) follows upon checking constants for $n = 4$, i.e. $\bar{c}_0 = \int_{\mathbb{R}^n} (\frac{1}{1+r^2})^n = \frac{\omega_4}{12}$ and

$$\begin{aligned}
&\mathbf{1.}\quad \tilde{c}_1 = \frac{n(n-1)(n-2)^2}{\bar{c}_0^{\frac{n-2}{n}}} \int_{\mathbb{R}^n} \frac{1-r^2}{(1+r^2)^{n+1}} \ln \frac{1}{1+r^2} dx = 2\sqrt{3\omega_4};\\
&\mathbf{2.}\quad \tilde{c}_2 = -\frac{(n-1)(n-2)}{\bar{c}_0^{\frac{n-2}{n}}} \int_{\mathbb{R}^n} \frac{r^2(1-r^2)}{(1+r^2)^{n+1}} dx = \sqrt{3\omega_4};\\
&\mathbf{3.}\quad \tilde{c}_3 = \tilde{d}_1 = -\frac{4n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}} \int_{\mathbb{R}^n} \frac{r^n(n+2-nr^2)}{(1+r^2)^{n+2}} = 24\sqrt{3\omega_4};\\
&\mathbf{4.}\quad \tilde{c}_4 = \frac{n-2}{2}\tilde{b}_2 = \frac{2n(n-1)(n-2)}{\bar{c}_0^{\frac{n-2}{n}}} \int_{\mathbb{R}^n} \frac{1}{(1+r^2)^{\frac{n+2}{2}}} = 24\sqrt{3\omega_4},
\end{aligned} \tag{6.8}$$

cf. (7.14) from the corresponding Lemma 5.2. We turn next to (iii). In case $n \geq 5$ we may now assume

$$\tilde{c}_1\tau + \tilde{c}_2 \frac{\Delta K_k}{K_k\lambda_k^2} = o(\frac{1}{\lambda^2}) \text{ and } \varepsilon_{k,l} = o(\frac{1}{\lambda^2}) \text{ for } \lambda_k \simeq \lambda_l \simeq \lambda,$$

which by Lemma 5.1 implies

$$|\partial J_\tau(u)| \gtrsim \left| 1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1} \right| + o(\frac{1}{\lambda^2}).$$

Note that $\alpha_j^{p-1} = \Theta^{p-1} \cdot \frac{\lambda_j^\theta}{K_j}$ is modulo scaling the unique and non-degenerate maximum of

$$\alpha = (\alpha_1, \ldots, \alpha_q) \longrightarrow \frac{\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}} = \frac{\sum \alpha_i^2}{(\sum \frac{K_i}{\lambda_i^\theta} \alpha_i^{p+1})^{\frac{2}{p+1}}}.$$

Now (6.2) follows, unless $\alpha_j^{p-1} = \Theta^{p-1} \cdot \frac{\lambda_j^\theta}{K_j} + o(\frac{1}{\lambda^2})$ and there holds

$$|\partial J_\tau(u)| \gtrsim \left| \alpha_j - \Theta \cdot \sqrt[p-1]{\frac{\lambda_j^\theta}{K_j}} \right| + o(\frac{1}{\lambda^2}).$$

In case $n=4$ we may rewrite Lemma 5.1 up to some $o(\frac{1}{\lambda^2})$ with constant given below as

$$\begin{aligned} \partial J_\tau(u)\phi_{1,j} = \frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \Bigg( & \grave{c}_0 (1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1}) \\ & - K_j (\grave{c}_2 \frac{\Delta K_j}{K_j^2 \lambda_j^2} + \grave{d}_1 \frac{H_j}{K_j \lambda_j^2} + \grave{b}_1 \sum_{j \neq i} \frac{G_{g_0}(a_i, a_j)}{\gamma_n \sqrt{K_i K_j} \lambda_i \lambda_j}) \\ & + \frac{\alpha_K^{\frac{2n}{n-2}}}{(\alpha^2)^2} (\grave{c}_2 \sum_k \frac{\Delta K_k}{K_k^2 \lambda_k^2} + \grave{d}_1 \sum_k \frac{H_k}{K_k \lambda_k^2} + \grave{b}_1 \sum_{k \neq l} \frac{G_{g_0}(a_k, a_l)}{\gamma_n \sqrt{K_k K_l} \lambda_k \lambda_l} \Bigg) \end{aligned} \tag{6.9}$$

using (6.1) and $\lambda_j^\theta \simeq (\frac{1}{\sqrt{\tau}})^{\frac{n-2}{2}\tau} = 1 + O(\frac{\ln \lambda}{\lambda^2})$. Moreover, up to an error $o(1)$ there holds

$$\frac{(\alpha^2)^2}{\alpha_K^{\frac{2n}{n-2}}} = \frac{\alpha^2 \sum_i \alpha_i^2}{\alpha_K^{\frac{2n}{n-2}}} = \frac{\alpha^2 \sum_i \frac{\alpha_K^{\frac{2n}{n-2}}}{\alpha^2} \frac{1}{K_i}}{\alpha_K^{\frac{2n}{n-2}}} = \sum_i \frac{1}{K_i},$$

and due to (6.6)

$$\tilde{c}_2 \sum_k \frac{\Delta K_k}{K_k^2 \lambda_k^2} + \tilde{c}_3 \sum_k \frac{H_k}{K_k \lambda_k^2} + \tilde{c}_4 \sum_{k \neq l} \frac{G_{g_0}(a_k, a_l)}{\gamma_n \sqrt{K_k K_l} \lambda_k \lambda_l} = \mathcal{M}_{a_1, \ldots, a_q} \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_q} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_q} \end{pmatrix} = \tilde{c}_1 \sum_i \frac{\tau}{K_i}$$

and

$$\tilde{c}_2 \frac{\Delta K_j}{K_j^2 \lambda_j^2} + \tilde{c}_3 \frac{H_j}{K_j \lambda_j^2} + \tilde{c}_4 \sum_{j \neq i} \frac{G_{g_0}(a_i, a_j)}{\gamma_n \sqrt{K_i K_j} \lambda_i \lambda_j} = \mathcal{M}_{a_1, \ldots, a_q} \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_q} \end{pmatrix} \frac{e_j}{\lambda_j} = \tilde{c}_1 \frac{\tau}{K_j}$$

up to some $o(\frac{1}{\lambda^2})$. We may therefore cancel out the interaction terms in (6.9) and obtain

$$\begin{aligned} \partial J_\tau(u)\phi_{1,j} = \frac{\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \Bigg( & \grave{c}_0 (1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1}) - K_j ((\grave{c}_2 - \frac{\grave{b}_1}{\tilde{c}_4} \tilde{c}_2) \frac{\Delta K_j}{K_j^2 \lambda_j^2} + (\grave{d}_1 - \frac{\grave{b}_1}{\tilde{c}_4} \tilde{c}_3) \frac{H_j}{K_j \lambda_j^2}) \\ & + \frac{1}{\sum_k \frac{1}{K_k}} ((\grave{c}_2 - \frac{\grave{b}_1}{\tilde{c}_4} \tilde{c}_2) \sum_k \frac{\Delta K_k}{K_k^2 \lambda_k^2} + (\grave{d}_1 - \frac{\grave{b}_1}{\tilde{c}_4} \tilde{c}_3) \sum_k \frac{H_k}{K_k \lambda_k^2} \Bigg). \end{aligned} \tag{6.10}$$

Checking constants for $n=4$, i.e. with $\bar{c}_0 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n} = \frac{\omega_4}{12}$

1. $\grave{c}_0 = 8n(n-1)(\int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n})^{\frac{2}{n}} = 16\sqrt{3\omega_4}$, $\quad \grave{c}_2 = \frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}} \frac{1}{2n} \int_{\mathbb{R}^n} \frac{r^2}{(1+r^2)^n} = 4\sqrt{3\omega_4}$;
2. $\grave{d}_1 = \frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}} \int_{\mathbb{R}^n} \frac{r^n}{(1+r^2)^{n+1}} = 24\sqrt{3\omega_4}$, $\quad \grave{b}_1 = \frac{8n(n-1)(n+2)}{\bar{c}_0^{\frac{n-2}{n}} (n-2)} \int_{\mathbb{R}^n} \frac{1}{(1+r^2)^{\frac{n+2}{2}}} = 144\sqrt{3\omega_4}$,

cf. (7.9) from the corresponding Lemma 5.1, we then find

$$|\partial J_\tau(u)| \gtrsim \left| 1 - \frac{\alpha^2 K_j \alpha_j^{p-1}}{\alpha_{K,\tau}^{p+1} \lambda_j^\theta} + \frac{1}{8} \left( \frac{\Delta K_j}{K_j \lambda_j^2} - 60 \frac{H_j}{\lambda_j^2} - \frac{\sum_k (\frac{\Delta K_k}{K_k^2 \lambda_k^2} - 60 \frac{H_k}{K_k \lambda_k^2})}{\sum_k \frac{1}{K_k}} \right) \right| + o(\frac{1}{\lambda^2}).$$

Note that setting

$$E_j = \frac{1}{8}\left(\frac{\Delta K_j}{K_j\lambda_j^2} - 60\frac{H_j}{\lambda_j^2} - \frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2} - 60\frac{H_k}{K_k\lambda_k^2})}{\sum_k \frac{1}{K_k}}\right),$$

there holds $E_j = O(\frac{1}{\lambda^2})$, $\sum_j \frac{E_j}{K_j} = 0$, and $\alpha_j^{p-1} = \Theta^{p-1}\frac{\lambda_j^\theta}{K_j}(1+E_j)$ is modulo scaling the unique and non-degenerate maximum of

$$\alpha = (\alpha_1, \ldots, \alpha_q) \longrightarrow \frac{\alpha^2}{(\alpha^{p+1}_{\frac{K}{1+E},\tau})^{\frac{2}{p+1}}} = \frac{\sum \alpha_i}{(\sum \frac{K_i}{\lambda_i^\theta(1+E_i)}\alpha_i^{p+1})^{\frac{2}{p+1}}},$$

and satisfies

$$\begin{aligned}\frac{\alpha^2}{\alpha^{p+1}_{K,\tau}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1} =& \Theta^{p-1}\cdot\frac{\alpha^2}{\alpha^{p+1}_{K,\tau}}(1+E_j) = \frac{\sum[\frac{\lambda_i^\theta}{K_i}(1+E_i)]^{\frac{2}{p-1}}}{\sum\frac{K_i}{\lambda_i^\theta}[\frac{\lambda_i^\theta}{K_i}(1+E_i)]^{\frac{p+1}{p-1}}}(1+E_j) \\ =& \frac{\sum(\frac{\lambda_i^\theta}{K_i})^{\frac{2}{p-1}} + \frac{2}{p-1}\sum(\frac{\lambda_i^\theta}{K_i})^{\frac{2}{p-1}}E_i}{\sum(\frac{\lambda_i^\theta}{K_i})^{\frac{2}{p-1}} + \frac{p+1}{p-1}\sum(\frac{\lambda_i^\theta}{K_i})^{\frac{2}{p-1}}E_i}(1+E_j) = 1 + E_j + o(\frac{1}{\lambda^3})\end{aligned}$$

due to $(\frac{\lambda_i^\theta}{K_i})^{\frac{2}{p-1}} = \frac{1}{K_i} + O(\frac{\ln\lambda}{\lambda^2})$. Thus (6.2) follows unless, up to some $o(\frac{1}{\lambda^2})$,

$$|\partial J_\tau(u)| \gtrsim \left|\alpha_j - \Theta\sqrt[p-1]{\frac{\lambda_j^\theta}{K_j}\left(1 + \frac{1}{8}(\frac{\Delta K_j}{K_j\lambda_j^2} - 60\frac{H_j}{\lambda_j^2} - \frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2} - 60\frac{H_k}{K_k\lambda_k^2})}{\sum_k\frac{1}{K_k}})\right)}\right|. \tag{6.11}$$

We have therefore proved (i)-(iv), which will be used for showing the second statement of the proposition. In this case the error terms in Lemmata 5.1, 5.2 and 5.3 are of type $o(\lambda^{-3}) + O(|\partial J_\tau(u)|^2)$. This follows immediately in case $n \geq 5$, while the terms $\varepsilon_{r,s}^{\frac{n+2}{n}} \simeq \lambda^{-3}$ in case $n = 4$, for which however the underlying estimates can be improved to derive a quadratic error in $\varepsilon_{r,s}$, cf. [38]. Let us first treat the lower bounds arising from Lemma 5.3. In case $n \geq 5$ we find from the latter lemma

$$|\partial J_\tau(u)| \gtrsim |\check{c}_3\frac{\nabla K_j}{K_j\lambda_j} + \check{c}_4\frac{\nabla\Delta K_j}{K_j\lambda_j^3}| \gtrsim |\check{c}_3\frac{\nabla K(a_j)}{\lambda_j} + \check{c}_4\frac{\nabla\Delta K(x_j)}{\lambda_j^3}|$$

up to some $o(\lambda^{-3})$ and therefore, writing $a_j = \exp_{g_{x_j}}(\bar{a}_j)$, that

$$|\partial J_\tau(u)| \gtrsim |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}| + o(\frac{1}{\lambda^3}).$$

Similarly in case $n = 4$ we find up to some $o(\lambda^{-3})$

$$|\partial J_\tau(u)| \gtrsim |\check{c}_3\frac{\nabla K_j}{K_j\lambda_j} + \check{c}_4\frac{\nabla\Delta K_j}{K_j\lambda_j^3} + \check{b}_3\sum_{j\neq i}\frac{\alpha_i}{\alpha_j}\frac{\nabla_{a_j}G_{g_0}(a_i,a_j)}{\gamma_n\lambda_i\lambda_j^2}|.$$

From (iii) we have $\alpha_i = \Theta(\frac{\lambda_i^\theta}{K_i})^{\frac{1}{p-1}} + O(\frac{1}{\lambda^2})$, which by $\theta = \frac{n-2}{2}\tau$ and $\lambda_i \simeq \tau^{-\frac{1}{2}}$ due to (iv) becomes $\alpha_i = \frac{\Theta}{\sqrt{K_i}} + O(\frac{\ln\lambda}{\lambda^2})$. Thus, still up to some $o(\lambda^{-3})$

$$\begin{aligned}|\partial J_\tau(u)| \gtrsim& |\frac{\nabla K(a_j)}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}\frac{\nabla\Delta K(x_j)}{\lambda_j^3} + \frac{\check{b}_3}{\check{c}_3}\sum_{j\neq i}\sqrt{\frac{K^3(x_j)}{K(x_i)}}\frac{\nabla_{x_j}G_{g_0}(x_i,x_j)}{\gamma_n\lambda_i\lambda_j^2}| \\ \gtrsim& |\frac{\bar{a}_j}{\lambda_j} + \frac{\check{c}_4}{\check{c}_3}(\nabla^2 K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3} + \frac{\check{b}_3}{\check{c}_3}\sum_{j\neq i}\sqrt{\frac{K^3(x_j)}{K(x_i)}}(\nabla^2 K(x_j))^{-1}\frac{\nabla_{x_j}G_{g_0}(x_i,x_j)}{\gamma_n\lambda_i\lambda_j^2}|,\end{aligned}$$

and checking constants from Lemma 5.3, cf. (7.20), we have

$$\check{c}_3=\int_{\mathbb{R}^n}\frac{4(n-1)(n-2)dx}{(1+r^2)^n}=3\omega_4,\ \check{c}_4=\int_{\mathbb{R}^n}\frac{2(n-1)r^2dx}{(1+r^2)^n}=\omega_4,\ \check{b}_3=\int_{\mathbb{R}^n}\frac{8n(n-1)dx}{(1+r^2)^{\frac{n+2}{2}}}=24\omega_4.$$

We conclude that, up to some $o(\frac{1}{\lambda^3})$

$$|\partial J_\tau(u)|\gtrsim\begin{pmatrix}|\frac{\bar{a}_j}{\lambda_j}+\frac{1}{3}(\nabla^2K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}+8\sum_{j\neq i}\sqrt{\frac{K^3(x_j)}{K(x_i)}}(\nabla^2K(x_j))^{-1}\frac{\nabla_{x_j}G_{g_0}(x_i,x_j)}{\gamma_n\lambda_i\lambda_j^2}| & \text{for } n=4\\ |\frac{\bar{a}_j}{\lambda_j}+\frac{\check{c}_4}{\check{c}_3}(\nabla^2K(x_j))^{-1}\frac{\nabla\Delta K(x_j)}{\lambda_j^3}| & \text{for } n\geq 5\end{pmatrix} \tag{6.12}$$

By this, i.e. $\bar{a}_j=O(\frac{1}{\lambda^2})$, and $\alpha_i=\frac{\Theta}{\sqrt{K_i}}+O(\frac{\ln\lambda}{\lambda^2})$ we then infer from Lemma 5.2 that up to some $o(\frac{1}{\lambda^3})$

$$|\partial J_\tau(u)|\gtrsim|\tilde{c}_1\tau+\tilde{c}_2\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}+\frac{n-2}{2}\tilde{b}_2\sum_{j\neq i}\sqrt{\frac{K(x_j)}{K(x_i)}}\frac{G_{g_0}(x_i,x_j)}{\gamma_n(\lambda_i\lambda_j)^{\frac{n-2}{2}}}+\tilde{d}_1\frac{H(x_j)}{\lambda_j^{n-2}}|$$

with constants, cf. above, given for $n=4,5$ respectively by

1. $\frac{\tilde{c}_2}{\tilde{c}_1}=-\frac{\int_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+1}}dx}{n(n-2)\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln(\frac{1}{1+r^2})dx}=\frac{1}{2},\frac{2}{9};$
2. $\frac{\tilde{c}_3}{\tilde{c}_1}=\frac{\tilde{d}_1}{\tilde{c}_1}=\frac{4}{(n-2)^2}\frac{\int_{\mathbb{R}^n}\frac{r^n(n+2)-nr^2}{(1+r^2)^{n+2}}dx}{\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln(\frac{1}{1+r^2})dx}=12,\frac{512}{9\pi};$
3. $\frac{n-2}{2}\frac{\tilde{b}_2}{\tilde{c}_1}=\frac{\tilde{c}_4}{\tilde{c}_1}=\frac{2}{n-2}\frac{\int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^{\frac{n+2}{2}}}}{\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln(\frac{1}{1+r^2})dx}=12,\frac{512}{9\pi},$

we conclude

$$|\partial J_\tau(u)|\gtrsim\begin{pmatrix}|\tau+\frac{1}{2}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}+12[\frac{H(x_j)}{\lambda_j^2}+\sum_{j\neq i}\sqrt{\frac{K(x_j)}{K(x_i)}}\frac{G_{g_0}(x_i,x_j)}{\gamma_n\lambda_i\lambda_j}]| & \text{for } n=4\\ |\tau+\frac{2}{9}\frac{\Delta K(x_j)}{K(x_j)\lambda_j^2}+\frac{512}{9\pi}[\frac{H(x_j)}{\lambda_j^3}+\sum_{j\neq i}\sqrt{\frac{K(x_j)}{K(x_i)}}\frac{G_{g_0}(x_i,x_j)}{\gamma_n(\lambda_i\lambda_j)^{\frac{3}{2}}}]| & \text{for } n=5\\ |\tau+\frac{\tilde{c}_2}{\tilde{c}_1}\frac{\Delta K_j}{K_j\lambda_j^2}| & \text{for } n\geq 6\end{pmatrix}. \tag{6.13}$$

By similar reasoning, using $\bar{a}_j=O(\frac{1}{\lambda^2})$ and $\alpha_i=\frac{\Theta}{\sqrt{K_i}}+O(\frac{\ln\lambda}{\lambda^2})$ we finally have, up to some $o(\frac{1}{\lambda^3})$

$$|\partial J_\tau(u)|\gtrsim\begin{pmatrix}|1-\frac{\alpha^2K_j\alpha_j^{p-1}}{\alpha_{K,\tau}^{p+1}\lambda_j^\theta}+\frac{1}{8}(\frac{\Delta K_j}{K_j\lambda_j^2}-60\frac{H_j}{\lambda_j^2}-\frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2}-60\frac{H_k}{K_k\lambda_k^2})}{\sum_k\frac{1}{K_k}})| & \text{for } n=4\\ |1-\frac{\alpha^2K_j\alpha_j^{p-1}}{\alpha_{K,\tau}^{p+1}\lambda_j^\theta}-\frac{1}{90}(\frac{\Delta K_j}{K_j\lambda_j^2}+\frac{2816}{\pi}\frac{H_j}{\lambda_j^3}-\frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2}+\frac{2816}{\pi}\frac{H_k}{K_k\lambda_k^3})}{\sum_k\frac{1}{K_k}})| & \text{for } n=5\\ |1-\frac{\alpha^2K_j\alpha_j^{p-1}}{\alpha_{K,\tau}^{p+1}\lambda_j^\theta}| & \text{for } n\geq 6\end{pmatrix}.$$

This follows in case $n\geq 6$ immediately from Lemma 5.1 and for $n=4$ by repeating the arguments leading to (6.9) and (6.10), while the case $n=5$ follows by arguing as in case $n=4$ using (6.13) to cancel out the interaction terms when passing from (6.9) to (6.10). Then arguing as for the passage from (6.10) to (6.11) we finally obtain that up to some $o(\frac{1}{\lambda^3})$

$$|\partial J_\tau(u)|\gtrsim\begin{pmatrix}|\alpha_j-\Theta\sqrt[p-1]{\frac{\lambda_j^\theta}{K_j}(1+\frac{1}{8}(\frac{\Delta K_j}{K_j\lambda_j^2}-60\frac{H_j}{\lambda_j^2}-\frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2}-60\frac{H_k}{K_k\lambda_k^2})}{\sum_k\frac{1}{K_k}}))}| & \text{for } n=4\\ |\alpha_j-\Theta\sqrt[p-1]{\frac{\lambda_j^\theta}{K_j}(1-\frac{1}{90}(\frac{\Delta K_j}{K_j\lambda_j^2}+\frac{2816}{\pi}\frac{H_j}{\lambda_j^3}-\frac{\sum_k(\frac{\Delta K_k}{K_k^2\lambda_k^2}+\frac{2816}{\pi}\frac{H_k}{K_k\lambda_k^3})}{\sum_k\frac{1}{K_k}}))}| & \text{for } n=5\\ |\alpha_j-\Theta\sqrt[p-1]{\frac{\lambda_j^\theta}{K_j}}| & \text{for } n\geq 6\end{pmatrix}. \tag{6.14}$$

Thus the second statement of the theorem follows from combining (6.12), (6.13) and (6.14). $\square$

In [36] the next result will be needed.

**Lemma 6.1.** *For every $u \in V(q,\varepsilon)$ there holds*

$$|\partial J_\tau(u)| \lesssim \tau + \sum_{r\neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + |1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_r}{\lambda_r^\theta} \alpha_r^{p-1}| + \varepsilon_{r,s}^{\frac{n+2}{2n}} + \|v\|.$$

*Proof.* Recalling (4.2) we can find $|\beta_{k.i}|, |\beta| = O(1)$ and $\nu \in H_u(p,\varepsilon), \|\nu\| = 1$ such that

$$|\partial J_\tau(u)| \lesssim |\beta^{k,i}||\partial J_\tau(u)\phi_{k,i}| + |\beta||\partial J_\tau(u)\nu| \lesssim \sum_{k,i} |\partial J_\tau(u)\phi_{k,i}| + |\partial J_\tau(u)\nu|.$$

From Lemmata 5.1, 5.2 and 5.3 we then find

$$\sum_{k,i} |\partial J_\tau(u)\phi_{k,i}| \lesssim \tau + \sum_{j=1}^q \frac{|\nabla K_j|}{\lambda_j} + \frac{1}{\lambda_j^2} + \frac{1}{\lambda_j^{n-2}} + |1 - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}} \frac{K_j}{\lambda_j^\theta} \alpha_j^{p-1}| + \sum_{r\neq s} \varepsilon_{r,s} + |\partial J_\tau(u)|^2,$$

whereas from Lemma 4.1 we have

$$\partial J_\tau(u)\nu = \partial J_\tau(\alpha^i \varphi_i)\nu + O(\|v\|) = O(\tau + \sum_r \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s} \varepsilon_{r,s}^{\frac{n+2}{2n}} + \|v\|).$$

From this the claim follows. □

# 7 Appendix

## 7.1 Interactions

*Proof of Lemma 2.2.* (i) follows using straightforwardly the expression of $\phi_{k,i}$.

(ii) ($\alpha$) **Case $k = 1$.** We have $\phi_{k,i} = \varphi_i$ for $k = 1$, and thus for $c > 0$ small

$$\int \varphi_i^{\frac{2n}{n-2}-\tau} d\mu_{g_0} = \int_{B_c(a_i)} u_{a_i}^{-\tau} \left( \frac{\lambda_i}{1 + \lambda_i^2 \gamma_n G_{a_i}^{\frac{2}{2-n}}} \right)^{n-\theta} d\mu_{g_{a_i}} + O\left(\frac{1}{\lambda_i^{n-\theta}}\right).$$

On $B_c(a_i)$ one has $u_{a_i}^{-\tau} = 1 + O(\tau|x-a_i|^2)$, and by (2.8)

$$\gamma_n G_{a_i}^{\frac{2}{2-n}} = r^2 + O\begin{pmatrix} r^3 & \text{for } n = 3 \\ r^4 & \text{for } n = 4 \\ r^5 & \text{for } n = 5 \\ r^6 \ln r & \text{for } n = 6 \\ r^6 & \text{for } n \geq 7 \end{pmatrix},$$

whence passing to normal coordinates at $a_i$

$$\int \varphi_i^{\frac{2n}{n-2}-\tau} d\mu_{g_0} = \int_{B_{c\lambda_i}(0)} \frac{\lambda_i^{-\theta} dx}{(1+r^2)^{n-\theta}} + O\begin{pmatrix} \frac{1}{\lambda^{1+\theta}} & \text{for } n = 3 \\ \frac{1}{\lambda^{2+\theta}} & \text{for } n = 4 \\ \frac{1}{\lambda^{3+\theta}} & \text{for } n = 5 \\ \frac{\ln \lambda}{\lambda^{4+\theta}} & \text{for } n = 6 \\ \frac{1}{\lambda^{4+\theta}} & \text{for } n \geq 7 \end{pmatrix}$$

up to some error $O(\frac{\tau}{\lambda_i^{2+\theta}})$, whence the claim follows with $c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n}$.

($\beta$) **Case $k = 2$.** The proof works analogously to the one of case $k = 1$ above.

$(\gamma)$ **Case $k=3$.** We have $\phi_{k,i} = \frac{2-n}{2} u_{a_i} \frac{\lambda_i \gamma_n \nabla_{a_i} G_{a_i}^{\frac{2}{2-n}}}{1+\lambda_i^2 \gamma_n G_{a_i}^{\frac{2}{2-n}}} \varphi_i + \frac{\nabla_{a_i} u_{a_i}}{\lambda_i} \varphi_i$, whence

$$\gamma_n(\nabla_{a_i} G_{a_i}^{\frac{2}{2-n}})(x) = -2x + O(r^2, r^3, r^4, r^5 \ln r, r^5) \text{ for } n = 3, \ldots, 6 \text{ and } n \geq 7.$$

Moreover $u_{a_i} = 1 + O(r_{a_i}^2)$, implies $\nabla_{a_i} u_{a_i} = O(r_{a_i})$. Thus

$$\int \varphi_i^{\frac{4}{n-2}-\tau} |\phi_{k,i}|^2 d\mu_{g_0} = \frac{(n-2)^2}{n} \int_{\mathbb{R}^n} \frac{\lambda_i^{-\theta} r^2 dx}{(1+r^2)^{n+2-\theta}} + O\big(\frac{1}{\lambda_i^{2+\theta}}\big) + O\begin{pmatrix} \lambda^{-1-\theta} & \text{for } n=3 \\ \lambda^{-2-\theta} & \text{for } n=4 \\ \lambda^{-3-\theta} & \text{for } n=5 \\ \frac{\ln \lambda}{\lambda^{4+\theta}} & \text{for } n=6 \\ \lambda^{-4-\theta} & \text{for } n \geq 7 \end{pmatrix}.$$

From this the claim follows.

(iii) We just prove the case $k=2$ and start showing that

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0} = -\lambda_i^\theta \lambda_j \int \varphi_i^{1-\tau} \partial_{\lambda_j} \varphi_j^{\frac{n+2}{n-2}} d\mu_{g_0} \tag{7.1}$$

up to some $O\big(\tau^2 + \sum_{i \neq j} \big(\frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}} + \varepsilon_{i,j}^{\frac{n+2}{n}}\big)\big)$, so we may evaluate either of these integrals. Clearly

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0} = -\lambda_i^\theta \lambda_j \int_{B_c(a_i)} \varphi_i^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0}$$

up to an error $O(\frac{1}{\lambda_i^{\frac{n+2}{2}}} \frac{1}{\lambda_j^{\frac{n-2}{2}}})$, whence using Lemma 2.1 we find

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0} = -\lambda_i^\theta \lambda_j \int_{B_c(a_i)} \varphi_i^{-\tau} \partial_{\lambda_j} \varphi_j \frac{L_{g_0} \varphi_i}{4n(n-1)} d\mu_{g_0}$$

up to $O(\lambda_i^{-2} + \lambda_i^{-(n-2)}) \varepsilon_{i,j}^{\frac{n+2}{2n}})$. Indeed we clearly have $\lambda_i^{-\frac{n+2}{2}} \lambda_j^{-\frac{n-2}{2}} = O\big(\lambda_i^{-1} \varepsilon_{i,j}^{\frac{n+2}{2n}}\big)$, and the difference from $L_{g_0} \varphi_i$ to $4n(n-1) \varphi_i^{\frac{n+2}{n-2}}$ can be estimated by Lemma 2.1 via quantities of the type

$$\int_{B_c(a_i)} r_{a_i}^\alpha \varphi_i^\beta \varphi_j d\mu_{g_0} = \int_{B_c(a_i)} r_{a_i}^\alpha \varphi_i^{\beta - \frac{n+2}{2n}} \varphi_i^{\frac{n+2}{2n}} \varphi_j d\mu_{g_0} = O\big(\varepsilon_{i,j}^{\frac{n+2}{2n}} \| r^\alpha \varphi_{0,\lambda}^{\beta - \frac{n+2}{2n}} \|_{L^{(\frac{2n}{n+2})^2}}\big),$$

thanks to case (v). Passing back to integrating on the whole manifold $M$ we find , estimating also mixed products of gradients of $\varphi_i$ and $\varphi_j$,

$$\begin{aligned} -\lambda_i^\theta \lambda_j \int \varphi_i{}^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0} = & -(1+O(\tau)) \lambda_i^\theta \lambda_j \int_{B_c(a_i)} \varphi_i^{1-\tau} \partial_{\lambda_j} \frac{L_{g_0} \varphi_j}{4n(n-1)} d\mu_{g_0} \\ & + O\big(\lambda_i^\theta \int \varphi_i \Delta_{g_0} \varphi_i^{-\tau} \varphi_j d\mu_{g_0}\big) + O\big(\big(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}}\big) \varepsilon_{i,j}^{\frac{n+2}{2n}}\big). \end{aligned}$$

By direct calculation $\Delta_{g_0} \varphi_i^{-\tau} = O(\tau \varphi_i^{\frac{4}{n-2}-\tau})$, whence

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau} \partial_{\lambda_j} \varphi_j d\mu_{g_0} = -\lambda_i^\theta \lambda_j \int \varphi_i^{1-\tau} \partial_{\lambda_j} \frac{L_{g_0} \varphi_j}{4n(n-1)} d\mu_{g_0} + O\big(\big(\tau + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}}\big) \varepsilon_{i,j}^{\frac{n+2}{2n}}\big).$$

Now applying Lemma 2.1 as before, but in differentiated form, (7.1) follows. Let

$$R_{i,j} = O\big(\tau^2 + \sum_{i \neq j} \big(\frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}} + \varepsilon_{i,j}^{\frac{n+2}{n}}\big)\big)$$

denote a quantity such order. We now assume the non-exclusive alternative

$$\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j} \quad \vee \quad \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i\lambda_j d^2(a_i,a_j) \gg \frac{\lambda_j}{\lambda_i}. \tag{7.2}$$

For $c>0$ small and fixed we have by the expression in (2.8)

$$\begin{aligned}
& -\lambda_i^{\theta}\lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau}\partial_{\lambda_j}\varphi_j d\mu_{g_0} \\
& = \frac{n-2}{2}\lambda_i^{\theta}\int_{B_c(a_i)} (\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n+2}{2}-\theta}\frac{u_{a_j}}{u_{a_i}^{1+\tau}}(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}+1}d\mu_{g_{a_i}} + R_{i,j},
\end{aligned}$$

whence passing to $g_{a_i}$-normal coordinates and recalling (2.8) we find

$$\begin{aligned}
-\lambda_i^{\theta}\lambda_j \int \varphi_i^{\frac{n+2}{n-2}-\tau}\partial_{\lambda_j}\varphi_j d\mu_{g_0} = & \frac{n-2}{2}\int_{B_{c\lambda_i}(0)} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}-\theta}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})+1} \\
& \left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\right)^{\frac{n-2}{2}} d\mu_{g_{a_i}}
\end{aligned} \tag{7.3}$$

up to the error $R_{i,j}$. Indeed for e.g. $n\geq 7$ (2.8) tells us that on $B_c(0)$

$$(\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n+2}{2}-\theta} = (\frac{\lambda_i}{1+\lambda_i^2 r^2})^{\frac{n+2}{2}-\theta}(1+O(\frac{\lambda^2 r^4}{1+\lambda^2 r^2})) = (\frac{\lambda_i}{1+\lambda_i^2 r^2})^{\frac{n+2}{2}-\theta}(1+r^2)$$

in conformal normal coordinates, whence by Hölder's inequality and Lemma 2.2

$$\int_{B_c(a_i)} r_{a_i}^2\varphi_i^{\frac{n+2}{n-2}-\tau}\varphi_j d\mu_{g_{a_i}} \leq \|r^2\varphi_{0,\lambda}^{\frac{n+2}{n-2}-\frac{n+2}{2n}-\tau}\|_{L^{(\frac{2n}{n+2})^2}}\varepsilon_{i,j}^{\frac{n+2}{2n}} = O(\frac{\varepsilon_{i,j}^{\frac{n+2}{2n}}}{\lambda_i^{2+\theta}}).$$

Due to (7.2) we have that either

$$\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j) \quad \text{or} \quad \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j},$$

and for $\epsilon>0$ sufficiently small may expand on

$$\mathcal{A} = \left\{|\frac{x}{\lambda_i}| \leq \epsilon\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}\right\} \cup \left\{|\frac{x}{\lambda_i}| \leq \epsilon\frac{1}{\lambda_j}\right\} \subset B_{c\lambda_i}(0)$$

the integrand in (7.3) as

$$\begin{aligned}
& \frac{1}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i}))^{\frac{n-2}{2}}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})+1} \\
& \quad = (\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{2-n}{2}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)+1} \\
& \quad + \frac{2-n}{2}\frac{\gamma_n\nabla G_{a_j}^{\frac{2}{2-n}}(a_i)\lambda_j x}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)+1} \\
& \quad + \frac{2}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}\frac{\gamma_n\nabla G_{a_j}^{\frac{2}{2-n}}(a_i)\lambda_j x}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} + \frac{O(\frac{\lambda_j}{\lambda_i}|x|^2)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}.
\end{aligned}$$

Using radial symmetry we then get, with $\tilde{b}_2 = \frac{n-2}{2}\int\limits_{\mathbb{R}^n} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}} = \frac{n-2}{2}b_1$,

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0} = \frac{\tilde{b}_2 u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} \frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i) - 1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i) + 1}$$

up to errors of the form $R_{i,j}$ and $I_{\mathcal{A}^c}$, where

$$I_{\mathcal{A}^c} \simeq \int\limits_{\mathcal{A}^c} \frac{1}{(1+r^2)^{\frac{n+2}{2}-\theta}} \Big(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} d\mu_{g_{a_i}}.$$

In case $\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j}$, we obviously have

$$I_{\mathcal{A}^c} \leq C\Big(\frac{\lambda_j}{\lambda_i}\Big)^{\frac{n+2}{2}-2\theta} = o(\varepsilon_{i,j}^{\frac{n+2}{n}}).$$

Otherwise we may assume $\mathcal{A}^c \neq \emptyset$, thus $d(a_i,a_j) \ll 1$, and write $\mathcal{A}^c \subseteq \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 = \Big\{\epsilon\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \leq |\frac{x}{\lambda_i}| \leq E\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}\Big\} \text{ and } \mathcal{B}_2 = \Big\{E\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \leq |\frac{x}{\lambda_i}| \leq c\Big\}$$

for a sufficiently large constant $E>0$. We then may estimate

$$\begin{aligned} I_{\mathcal{B}_1} =& \int\limits_{\mathcal{B}_1} \frac{1}{(1+r^2)^{\frac{n+2}{2}-\theta}} \Big(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} d\mu_{g_{a_i}} \\ \leq& \frac{C(\frac{\lambda_i}{\lambda_j})^{\frac{n+2}{2}}}{(1+\lambda_i^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n+2}{2}-\theta}} \int\limits_{\Big\{|\frac{x}{\lambda_j}| \leq E\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}\Big\}} \Big(\frac{1}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_j})}\Big)^{\frac{n-2}{2}} d\mu_{g_{a_i}}. \end{aligned}$$

Changing coordinates via $d_{i,j} = \exp_{g_{a_i}}^{-1}\exp_{g_{a_j}}$, we get

$$I_{\mathcal{B}_1} \leq \frac{C}{(\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n+2}{2}-\theta}} \int\limits_{\Big\{|\frac{x}{\lambda_j}| \leq \tilde{E}d(a_i,a_j)\Big\}} \Big(\frac{1}{1+r^2}\Big)^{\frac{n-2}{2}} dx,$$

and thus $I_{\mathcal{B}_1} = O(\varepsilon_{i,j}^{\frac{n}{n-2}-\tau}) = o(\varepsilon_{i,j}^{\frac{n+2}{n}})$ using, (7.2). Moreover

$$\begin{aligned} I_{\mathcal{B}_2} =& \int\limits_{\mathcal{B}_2} \frac{1}{(1+r^2)^{\frac{n+2}{2}-\theta}} \Big(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} d\mu_{g_{a_i}} \\ \leq& \frac{C}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} \int\limits_{\Big\{|x| \geq \sqrt{\lambda_i^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}\Big\}} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}}. \end{aligned}$$

This shows $I_{\mathcal{A}^c} \lesssim I_{\mathcal{B}_1} + I_{\mathcal{B}_2} = o(\varepsilon_{i,j}^{\frac{n+2}{n}})$, and we arrive at

$$-\lambda_i^\theta \lambda_j \int \varphi_i^{\frac{n+2}{n-2}}\varphi_j d\mu_{g_0} = \frac{\tilde{b}_2 u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} \frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i) - 1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i) + 1}$$

up to some error of the form $R_{i,j}$. Due to conformal covariance, there holds

$$G_{a_j}(a_j,a_i) = u_{a_j}^{-1}(a_i)u_{a_j}^{-1}(a_j)G_{g_0}(a_i,a_j)$$

and we therefore conclude

$$-\lambda_i^\theta\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j d\mu_{g_0}=b_2\frac{\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)-\frac{\lambda_i}{\lambda_j}}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n}{2}}}+R_{i,j}. \tag{7.4}$$

We turn to the case left by (7.2), i.e.

$$\varepsilon_{i,j}^{\frac{2}{2-n}}\sim\frac{\lambda_j}{\lambda_i} \tag{7.5}$$

and, recalling (7.1), estimate for $c>0$ small

$$\begin{aligned}&-\lambda_i^\theta\lambda_j\int\varphi_i^{1-\tau}\partial_{\lambda_j}\varphi_j^{\frac{n+2}{n-2}}d\mu_{g_0}\\&=\frac{n+2}{2}\lambda_i^\theta\int_{B_c(a_j)}\Big(\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}}\Big)^{\frac{n-2}{2}-\theta}\frac{u_{a_i}^{1-\tau}}{u_{a_j}}\Big(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}}\Big)^{\frac{n+2}{2}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}+1}d\mu_{g_{a_j}}\end{aligned}$$

up to some error $R_{i,j}$, whence up to the same error

$$-\lambda_i^\theta\lambda_j\int\varphi_i\partial_{\lambda_j}\varphi_j^{\frac{n+2}{n-2}}d\mu_{g_0}=\frac{n+2}{2}\int_{B_{c\lambda_j}(0)}\frac{r^2-1}{r^2+1}\Big(\frac{1}{1+r^2}\Big)^{\frac{n+2}{2}}\frac{u_{a_i}^{1-\tau}(a_j)(\frac{\lambda_i}{\lambda_j})^\theta d\mu_{g_{a_i}}}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_j}))^{\frac{n-2}{2}-\theta}}.$$

On $\mathcal{A}=\Big\{|\frac{x}{\lambda_j}|\leq\varepsilon\sqrt{\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j)}\Big\}\cup\Big\{|\frac{x}{\lambda_j}|\leq\epsilon\frac{1}{\lambda_i}\Big\}$ we may expand for $\epsilon>0$ sufficiently small

$$\begin{aligned}\Big(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_i})\Big)^{\frac{2-n}{2}+\theta}=&\Big(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j)\Big)^{\frac{2-n}{2}+\theta}\\&+\Big(\frac{2-n}{2}+\theta\Big)\frac{\gamma_n\nabla G_{a_i}^{\frac{2}{2-n}}(a_j)\lambda_i x+O\big(\frac{\lambda_i}{\lambda_j}|x|^2\big)}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j))^{\frac{n}{2}-\theta}}.\end{aligned}$$

With analogous estimates as in the previous case we derive

$$-\lambda_i^\theta\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j d\mu_{g_0}=\bar b_2\frac{u_{a_i}^{1-\tau}(a_j)(\frac{\lambda_i}{\lambda_j})^\theta}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j))^{\frac{n-2}{2}-\theta}}+R_{i,j}$$

with

$$\bar b_2=\frac{n+2}{2}\int_{\mathbb{R}^n}\frac{r^2-1}{r^2+1}\Big(\frac{1}{1+r^2}\Big)^{\frac{n+2}{2}}dx \tag{7.6}$$

and indeed $\bar b_2=\tilde b_2=\frac{n-2}{2n}\omega_n$ whence, using conformal covariance, as before (7.5) implies

$$-\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j d\mu_{g_0}=\frac{\bar b_2}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n-2}{2}}}+R_{i,j}. \tag{7.7}$$

Now the claim follows comparing (7.4) under (7.2) and (7.7) under (7.5).

(iv) The first claim, i.e. that for $k\neq l$

$$\int\varphi_i^{\frac{4}{n-2}-\tau}\phi_{k,i}\phi_{l,i}d\mu_{g_0}=O\Big(\frac{1}{\lambda_i^{n-2+\theta}}+\frac{1}{\lambda_i^{2+\theta}}\Big)$$

follows like in case (ii), just with vanishing leading terms. The second one is proved analogously to (ii), cf. case $(\alpha)$ in the proof.

(v) The case $\tau = 0$ is known, cf. e.g. [38], Lemma 3.4. By Lemma 2.2 we therefore have

$$\int \varphi_i^{\alpha-\tau}\varphi_j^{\beta} d\mu_{g_0} = \int (\varphi_i^{\alpha-\tau} - \frac{1}{\lambda_i^{\theta}}\varphi_i^{\alpha})\varphi_j^{\beta} d\mu_{g_0} + O(\lambda_i^{-\theta}\varepsilon_{i,j}^{\beta}).$$

To estimate the integral in the above right-hand side, we write

$$\begin{aligned}\int \varphi_i^{\alpha-\tau}\Big|1-\frac{1}{\lambda_i^{\theta}}\varphi_i^{\tau}\Big|\varphi_j^{\beta} d\mu_{g_0} \leq & \int_0^1 d\sigma \int_{B_c(a_i)} \varphi_i^{\alpha-\tau}\Big|\partial_\sigma\Big(\frac{1}{1+\lambda_i^2 r_{a_i}^2}\Big)^{\sigma\theta}\Big|\varphi_j^{\beta} d\mu_{g_0} \\ \leq & \theta\int_{B_c(a_i)} \varphi_i^{\alpha-\tau}\Big|\ln\frac{1}{1+\lambda_i^2 r_{a_i}^2}\Big|\varphi_j^{\beta} d\mu_{g_0} \\ \leq & \theta\|\varphi_i^{\alpha-\beta-\varepsilon-\tau}\ln\frac{1}{1+\lambda_i^2 r_{a_i}^2}\|_{L^{\frac{2n}{2n-(n-2)(2\beta+\varepsilon)}}_{B_c(a_i),\mu_{g_0}}} \|\varphi_i^{\beta+\varepsilon}\varphi_j^{\beta}\|_{L^{\frac{2n}{(n-2)(2\beta+\varepsilon)}}_{\mu_{g_0}}}.\end{aligned}$$

From the case $\tau = 0$ and $\alpha+\beta = \frac{2n}{n-2}$ we then get

$$\int \varphi_i^{\alpha-\tau}\Big|1-\frac{1}{\lambda_i^{\theta}}\varphi_i^{\tau}\Big|\varphi_j^{\beta}, \mu_{g_0} \leq C\theta\varepsilon_{i,j}^{\beta}\Big\|\Big(\frac{\lambda_i}{1+\lambda_i^2 r^2}\Big)^{(n-2)\alpha-n-\frac{n-2}{2}(\varepsilon+\tau)}\ln\frac{1}{1+\lambda_i^2 r^2}\Big\|_{L^{\frac{n}{(n-2)\alpha-n-\frac{n-2}{2}\varepsilon}}_{B_c(0)}}.$$

By direct evaluation the latter norm is of order $\lambda_i^{-\theta}$ and the claim follows.

(vi) also follows from the same above reference in [38], while (vii) is a straightforward computation. □

## 7.2 Derivatives

In this appendix we give the remaining proofs from Section 5.

*Proof of Lemma 5.1.* First note that the equalities up to the error in (5.14)

$$\partial J_\tau(u)\phi_{1,j} = \partial J_\tau(\alpha^j\varphi_j)\phi_{1,i} = \partial_{\alpha_j} J_\tau(\alpha^i\varphi_i)$$

follow from Lemma 4.3 and the chain rule of differentiation. So we evaluate

$$\partial J_\tau(\alpha^i\varphi_i)\varphi_j = \frac{2}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\Big(\int \alpha^i\varphi_i L_{g_0}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}K(\alpha^i\varphi_i)^p\varphi_j d\mu_{g_0}\Big)$$

and start expanding

$$\begin{aligned}\int K(\alpha^i\varphi_i)^p\varphi_j d\mu_{g_0} = & \int_{\{\alpha_j\varphi_j > \sum_{j\neq i}\alpha_i\varphi_i\}} K\alpha_j^p\varphi_j^{p+1} + p\sum_{j\neq i} K\alpha_j^{p-1}\alpha_i\varphi_j^p\varphi_i d\mu_{g_0} \\ & + \int_{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\}} K(\sum_{j\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0} + O(\sum_{r\neq s}\int_{\{\varphi_r\geq\varphi_s\}}\varphi_r^{p-1}\varphi_s^2 d\mu_{g_0})d\mu_{g_0}.\end{aligned}$$

The above error term is of order $O\big(\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}}\big)$ by Lemma 2.2, whence

$$\int K(\alpha^i\varphi_i)^p\varphi_j d\mu_{g_0} = \int K\alpha_j^p\varphi_j^{p+1} + p\sum_{j\neq i}K\alpha_j^{p-1}\alpha_i\varphi_j^p\varphi_i d\mu_{g_0} + \int_{\{\alpha_j\varphi_j\leq\sum_{i\neq j}\alpha_i\varphi_i\}} K(\sum_{i\neq j}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0},$$

up to an error of order $O(\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}})$. Similarly

$$\int_{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\}} K(\sum_{j\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0} = \int_{\substack{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\} \\ \cap\ \{j\neq 1\}\ \cap \\ \{\alpha_1\varphi_1>\sum_{j,1\neq i}\alpha_i\varphi_i\}}} K\alpha_1^p\varphi_1^p\varphi_j d\mu_{g_0} + \int_{\substack{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\} \\ \cap\ \{j\neq 1\}\ \cap \\ \{\alpha_1\varphi_1\leq\sum_{j,1\neq i}\alpha_i\varphi_i\}}} K(\sum_{j,1\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0}$$

up to an error $O(\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}})$, and thus

$$\int\limits_{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\}} K(\sum_{j\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0} = \chi_{\{1\neq j\}}\int K\alpha_1^p\varphi_1^p\varphi_j d\mu_{g_0} + \chi_{\{1\neq j\}}\int\limits_{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\}} K(\sum_{j,1\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0}.$$

Iteratively we obtain $\int\limits_{\{\alpha_j\varphi_j\leq\sum_{j\neq i}\alpha_i\varphi_i\}} K(\sum_{j\neq i}\alpha_i\varphi_i)^p\varphi_j d\mu_{g_0} = \sum_{j\neq i}\int K\alpha_i^p\varphi_i^p\varphi_j d\mu_{g_0}$ and conclude

$$\int K(\alpha^i\varphi_i)^p\varphi_j d\mu_{g_0} = \sum_i \alpha_i^p\int K\varphi_i^p\varphi_j d\mu_{g_0} + p\sum_{j\neq i}K\alpha_j^{p-1}\alpha_i\varphi_j^p\varphi_i d\mu_{g_0} \tag{7.8}$$

up to an error of order $O(\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}})$. From this, we obviously have

$$\begin{aligned}\partial J_\tau(\alpha^i\varphi_i)\varphi_j =& \frac{2}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\left(\int\alpha_j\varphi_j L_{g_0}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\alpha_j^p K\varphi_j^{p+1}d\mu_{g_0}\right)\\ &+ \frac{2}{(\sum_k\alpha_k^{p+1}\int K\varphi_k^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\\ &\sum_{j\neq i}\left(\int\alpha_i L_{g_0}\varphi_j d\mu_{g_0} - \frac{\sum_k\alpha_k^2\int\varphi_k L_{g_0}\varphi_k d\mu_{g_0}}{\sum_k\alpha_k^{p+1}\int K\varphi_k^{p+1}d\mu_{g_0}}\alpha_i^p K\varphi_i^p\varphi_j d\mu_{g_0}\right)\\ &- \frac{2p\sum_k\alpha_k^2\int\varphi_k L_{g_0}\varphi_k d\mu_{g_0}}{(\sum_k\alpha_k^{p+1}\int K\varphi_k^{p+1}d\mu_{g_0})^{\frac{2}{p+1}+1}}\sum_{j\neq i}\alpha_j^{p-1}\alpha_i\int K\varphi_j^p\varphi_i d\mu_{g_0}\end{aligned}$$

up to some $O(\sum_{r\neq s}\varepsilon_{r,s}^{\frac{n+2}{n}})$. Then (5.8) and (5.11) applied to the second and third summands above show

$$\begin{aligned}\partial J_\tau(\alpha^i\varphi_i)\varphi_j =& \frac{2}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\left(\int\alpha_j\varphi_j L_{g_0}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\alpha_j^p K\varphi_j^{p+1}d\mu_{g_0}\right)\\ +& \frac{8n(n-1)\bar{c}_0^{\frac{p-1}{p+1}}b_1}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\sum_{j\neq i}\alpha_i\big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_i}{\lambda_i^\theta}\alpha_i^{p-1}\big)\varepsilon_{i,j} - \frac{8pn(n-1)\bar{c}_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\sum_{j\neq i}\alpha_i\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\varepsilon_{i,j}\end{aligned}$$

up to an $O(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}})$. Then applying (5.11) as well as (5.10) and Lemma 2.2 to the first summand above we find

$$\begin{aligned}\partial J_\tau(\alpha^i\varphi_i)\varphi_j =& \frac{8n(n-1)}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\left(\bar{c}_0\alpha_j - \frac{\bar{c}_0\alpha^2+b_1\sum_{k\neq l}\alpha_k\alpha_l\varepsilon_{k,l}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\alpha_j^p\int K\varphi_j^{p+1}d\mu_{g_0}\right)\\ +& \frac{8n(n-1)\bar{c}_0^{\frac{p-1}{p+1}}b_1}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\sum_{j\neq i}\alpha_i\big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_i}{\lambda_i^\theta}\alpha_i^{p-1}\big)\varepsilon_{i,j} - \frac{8pn(n-1)\bar{c}_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\sum_{j\neq i}\alpha_i\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\varepsilon_{i,j}.\end{aligned}$$

Using (5.8) for the first term in the right-hand side, we then get

$$\begin{aligned}\partial J_\tau(\alpha^i\varphi_i)\varphi_j =& \frac{8n(n-1)\bar{c}_0\alpha_j}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}\left(1-\frac{\alpha^2\alpha_j^{p-1}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\int K\varphi_j^{p+1}d\mu_{g_0}\right)\\ &- \frac{8n(n-1)\bar{c}_0^{-\frac{2}{p+1}}b_1}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\frac{\sum_{k\neq l}\alpha_k\alpha_l\varepsilon_{k,l}}{\sum_k\frac{K_k}{\lambda_k^\theta}\alpha_k^{p+1}}\alpha_j^p\frac{K_j}{\lambda_j^\theta} - \frac{8pn(n-1)\bar{c}_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\sum_{j\neq i}\alpha_i\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\varepsilon_{i,j}\end{aligned}$$

up to an error of order

$$O\big(\tau^2+\sum_{r\neq s}\big|1-\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{K_r}{\lambda_r^\theta}\alpha_r^{\frac{4}{n-2}}\big|^2+\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Applying now (5.8) to the first coefficient above we find

$$\begin{aligned}
\partial J_\tau(\alpha^i\varphi_i)\varphi_j= & \frac{8n(n-1)\bar c_0^{\frac{p-1}{p+1}}}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\alpha_j\Big(1-\frac{\alpha^2}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\alpha_j^{p-1}\int K\varphi_j^{p+1}d\mu_{g_0}\Big)\\
& -\frac{16n(n-1)}{p+1}\frac{\bar c_0^{-\frac{2}{p+1}}}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\alpha_j\Big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\alpha_j^\theta}\alpha_j^{p-1}\Big)\\
& \Bigg(\bar c_1\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\tau+\bar c_2\sum_i\frac{\Delta K_i}{\lambda_i^{2+\theta}}\alpha_i^{\frac{2n}{n-2}}+\bar d_1\sum_i\frac{K_i}{\lambda_i^\theta}\alpha_i^{\frac{2n}{n-2}}\begin{pmatrix}\frac{H_i}{\lambda_i}\\ \frac{H_i+O(\frac{\ln\lambda_i}{\lambda_i^2})}{\lambda_i^2}\\ \frac{H_i}{\lambda_i^3}\\ \frac{W_i\ln\lambda_i}{\lambda_i^4}\\ 0\end{pmatrix}+\bar b_1\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_j\frac{K_i}{\lambda_i^\theta}\varepsilon_{i,j}\Bigg)\\
& -\frac{8n(n-1)\bar c_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big(\sum_{k\neq l}\alpha_j\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l}+p\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big),
\end{aligned}$$

and obviously the second summand is of order of the previous error term. Thus

$$\begin{aligned}
\partial J_\tau(\alpha^i\varphi_i)\varphi_j= & \frac{8n(n-1)\bar c_0^{\frac{p-1}{p+1}}}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\alpha_j\Big(1-\frac{\alpha^2}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\alpha_j^{p-1}\int K\varphi_j^{p+1}d\mu_{g_0}\Big)\\
& -\frac{8n(n-1)\bar c_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big(\sum_{k\neq l}\alpha_j\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l}+p\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big)
\end{aligned}$$

up to the same error, and applying finally (5.7) and (5.8) we arrive at

$$\begin{aligned}
\partial J_\tau(\alpha^i\varphi_i)\varphi_j= & \frac{8n(n-1)\bar c_0^{\frac{p-1}{p+1}}}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\alpha_j\Big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big)\\
& -\frac{8n(n-1)\bar c_0^{-\frac{2}{p+1}}\alpha^2\alpha_j^p}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\Bigg(\bar c_1\frac{K_j\tau}{\lambda_j^\theta}+\bar c_2\frac{\Delta K_j}{\lambda_j^{2+\theta}}+\bar d_1K_j\begin{pmatrix}\frac{H_j}{\lambda_j^{1+\theta}}\\ \frac{H_j}{\lambda_j^{2+\theta}}+O(\frac{\ln\lambda_i}{\lambda_i^{4+\theta}})\\ \frac{H_j}{\lambda_j^{3+\theta}}\\ \frac{W_j\ln\lambda_j}{\lambda_i^{4+\theta}}\\ 0\end{pmatrix}\Bigg)\\
& +\frac{8n(n-1)\bar c_0^{-\frac{2}{p+1}}\alpha^2\alpha_j^p}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+2}}\frac{K_j}{\lambda_j^\theta}\quad\Bigg(\bar c_1\sum_k\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}\tau+\bar c_2\sum_k\frac{\Delta K_k}{\lambda_k^{2+\theta}}\alpha_k^{\frac{2n}{n-2}}\\
& \qquad+\bar d_1\sum_k\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}\begin{pmatrix}\frac{H_k}{\lambda_k}\\ \frac{H_k+O(\frac{\ln\lambda_i}{\lambda_k^2})}{\lambda_k^2}\\ \frac{H_k}{\lambda_k^3}\\ \frac{W_k\ln\lambda_k}{\lambda_k^4}\\ 0\end{pmatrix}+\bar b_1\sum_{k\neq l}\alpha_k^{\frac{n+2}{n-2}}\alpha_l\frac{K_k}{\lambda_k^\theta}\varepsilon_{k,l}\Bigg)\\
& -\frac{8n(n-1)\bar c_0^{-\frac{2}{p+1}}b_1\alpha^2}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big(\sum_{k\neq l}\alpha_j\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l}+p\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big),
\end{aligned}$$

again up to the same error term. Recalling that $\bar{b}_1=\frac{2n}{n-2}b_1$, we can rewrite this as

$$\begin{aligned}
\partial J_\tau(\alpha^i\varphi_i)\varphi_j = & \frac{8n(n-1)\bar{c}_0^{\frac{p-1}{p+1}}}{(\alpha_{K,\tau}^{p+1})^{\frac{2}{p+1}}}\alpha_j\Big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big)-\frac{8n(n-1)\bar{c}_0^{-\frac{n-2}{n}}\alpha^2\alpha_j^{\frac{n+2}{n-2}}}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}\frac{K_j}{\lambda_j^\theta} \\
& \Bigg(\bar{c}_1\big(1-\sum_k\frac{K_k}{\lambda_k^\theta}\frac{\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\big)\tau+\bar{c}_2\big(\frac{\Delta K_j}{K_j\lambda_j^2}-\sum_k\frac{\Delta K_k}{K_k\lambda_k^2}\frac{\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\big) \\
& +\bar{d}_1\begin{pmatrix}
\frac{H_j}{\lambda_j}-\sum_k\frac{\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{H_k}{\lambda_k} \\
\frac{H_j}{\lambda_j^2}-\sum_k\frac{\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{H_k+O(\sum_r\frac{\ln\lambda_r}{\lambda_r^2})}{\lambda_k^2} \\
\frac{H_j}{\lambda_j^3}-\sum_k\frac{\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{H_k}{\lambda_k^3} \\
\frac{W_j\ln\lambda_j}{\lambda_i^4}-\sum_k\frac{\frac{K_k}{\lambda_k^\theta}\alpha_k^{\frac{2n}{n-2}}}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{W_k\ln\lambda_k}{\lambda_k^4} \\
0
\end{pmatrix}\Bigg) \\
& -\frac{8n(n-1)\bar{c}_0^{-\frac{n-2}{n}}b_1\alpha^2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{\frac{4}{n-2}}\Big(\sum_{k\neq l}\alpha_j\big(\frac{\alpha_k\alpha_l}{\alpha^2}-\frac{2n}{n-2}\frac{K_k}{\lambda_k^\theta}\frac{\alpha_k^{\frac{n+2}{n-2}}\alpha_l}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\big)\varepsilon_{k,l}+\frac{n+2}{n-2}\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big)
\end{aligned}$$

up to an error of the form

$$O\big(\tau^2+\sum_{r\neq s}\big|1-\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{K_r}{\lambda_r^\theta}\alpha_r^{\frac{4}{n-2}}\big|^2+\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big).$$

Note that by (5.1) the coefficient of $\bar{c}_1$ in the above term vanishes. This then tells us in a first step, that

$$\forall\, i\ :\ 1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}=O\left(\tau^2+\sum_r\frac{1}{\lambda_r^2}+\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s}+|\partial J_\tau(u)|\right)$$

and therefore

$$\forall\, i\ :\ 1-\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{\frac{4}{n-2}}=O\left(\tau+\sum_r\frac{1}{\lambda_r^2}+\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s}+|\partial J_\tau(u)|\right).$$

Using this we derive up to an error of the form $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$

$$\begin{aligned}
\partial J_\tau(\alpha^i\varphi_i)\varphi_j = & \frac{8n(n-1)\bar{c}_0^{\frac{2}{n}}}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\alpha_j\Big(1-\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\Big) \\
& -\frac{8n(n-1)\bar{c}_0^{-\frac{n-2}{n}}\alpha_j}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\left(\bar{c}_2\big(\frac{\Delta K_j}{K_j\lambda_j^2}-\sum_k\frac{\Delta K_k}{K_k\lambda_k^2}\frac{\alpha_k^2}{\alpha^2}\big)+\bar{d}_1\begin{pmatrix}
\frac{H_j}{\lambda_j}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k} \\
\frac{H_j}{\lambda_j^2}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k+O(\sum_r\frac{\ln\lambda_r}{\lambda_r^2})}{\lambda_k^2} \\
\frac{H_j}{\lambda_j^3}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{H_k}{\lambda_k^3} \\
\frac{W_j\ln\lambda_j}{\lambda_i^4}-\sum_k\frac{\alpha_k^2}{\alpha^2}\frac{W_k\ln\lambda_k}{\lambda_k^4} \\
0
\end{pmatrix}\right) \\
& -\frac{8n(n-1)\bar{c}_0^{-\frac{n-2}{n}}b_1}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\quad\Big(\sum_{k\neq l}\alpha_j\big(\frac{\alpha_k\alpha_l}{\alpha^2}-\frac{2n}{n-2}\frac{\alpha_k\alpha_l}{\alpha^2}\big)\varepsilon_{k,l}+\frac{n+2}{n-2}\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big).
\end{aligned}$$

Finally note that the last summand can be simplified to

$$\frac{n+2}{n-2}\frac{8n(n-1)\bar{c}_0^{-\frac{n-2}{n}}b_1}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\Big(\sum_{k\neq l}\alpha_j\frac{\alpha_k\alpha_l}{\alpha^2}\varepsilon_{k,l}-\sum_{j\neq i}\alpha_i\varepsilon_{i,j}\Big).$$

From this the lemma follows setting

$$\grave{b}_1=\frac{8n(n-1)(n+2)}{\bar{c}_0^{\frac{n-2}{n}}(n-2)}b_1,\quad \grave{c}_2=\frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\bar{c}_2,\quad \grave{d}_1=\frac{8n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}\bar{d}_1,\quad \grave{c}_0=8n(n-1)\bar{c}_0^{\frac{2}{n}},\tag{7.9}$$

cf. (5.6), (5.7) and Lemma 2.2. □

*Proof of Lemma 5.2.* From Lemma 4.3 and the chain rule of differentiation we obtain

$$\partial J_\tau(u)\phi_{2,j}=\partial J_\tau(\alpha^i\varphi_i)\phi_{2,j}=\lambda_j\partial_{\lambda_j}J_\tau(\alpha^i\varphi_i),$$

up to the error in (5.15), and evaluate $\partial J_\tau(\alpha^i\varphi_i)\phi_{2,j}=\frac{2\Lambda}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}$ with

$$\Lambda=\int\alpha^i\varphi_iL_{g_0}\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}-\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}K(\alpha^i\varphi_i)^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}.$$

Arguing as for (7.8), we find

$$\begin{aligned}\Lambda=&\alpha_j\int\varphi_jL_{g_0}\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}-\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}K\alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}\\&+\sum_{j\neq i}\alpha_i\int\varphi_iL_{g_0}\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}-\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}K\alpha_i^p\varphi_i^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}\\&-p\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\sum_{j\neq i}\int K\alpha_j^{p-1}\alpha_i\varphi_j^{p-2}\varphi_i\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}\end{aligned}$$

and arguing as for (5.2) (5.10), (5.11) we see that

$$\int K\varphi_i^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=b_2\frac{K_i}{\lambda_i^\theta}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}+O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big),$$

and

$$\int\varphi_iL_{g_0}\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=\tilde{b}_2\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}+O\big(\sum_{r\neq s}\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big),\ \tilde{b}_2=4n(n-1)b_2\tag{7.10}$$

as well as $\int\varphi_jL_{g_0}\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=O\big(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big)$. Using these, we arrive at

$$\begin{aligned}\Lambda=&-\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\int K\alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}\\&+4n(n-1)b_2\sum_{j\neq i}\alpha_i\partial_{\lambda_j}\varepsilon_{i,j}-\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}b_2\frac{K_i}{\lambda_i^\theta}\alpha_i^p\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}\\&-p\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\sum_{j\neq i}\alpha_j^{p-1}\alpha_i\int K\varphi_j^{p-1}\varphi_i\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}\\&+O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}\big)\end{aligned}$$

Moreover, still arguing as for (5.10) and using Lemma 2.2, we have up to the same error as above

$$\int K\varphi_j^{p-1}\varphi_i\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} = \frac{b_2}{p}\frac{K_j}{\lambda_j^\theta}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}.$$

Combining this with (5.8), (5.10) and (5.11) we get with the same precision

$$\begin{aligned}\Lambda =& -4n(n-1)\frac{\alpha^2+\sum_{k\neq l}\alpha_k\alpha_l\varepsilon_{k,l}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\int K\alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} + 4n(n-1)b_2\sum_{j\neq i}\alpha_i\partial_{\lambda_j}\varepsilon_{i,j} \\ & -\frac{4n(n-1)\alpha^2}{\alpha_{K,\tau}^{p+1}}b_2\frac{K_i}{\lambda_i^\theta}\alpha_i^p\lambda_j\partial_{\lambda_j}\varepsilon_{i,j} - \frac{4n(n-1)b_2\alpha^2}{\alpha_{K,\tau}^{p+1}}\sum_{j\neq i}\alpha_i\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}.\end{aligned}$$

Using Lemma 5.1 we find by cancellation

$$\Lambda = -4n(n-1)\frac{\alpha^2+\sum_{k\neq l}\alpha_k\alpha_l\varepsilon_{k,l}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\int K\alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} - 4n(n-1)b_2\sum_{j\neq i}\alpha_i\lambda_j\partial_{\lambda_j}\varepsilon_{i,j},$$

up to some $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$. Moreover from Lemma 2.2 we have

$$\begin{aligned}\int K\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} =& K_j\int \alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} + O\big(\frac{|\nabla K_j|}{\lambda_j^{1+\theta}}+O(\frac{1}{\lambda_j^2})\big) \\ =& O\big(\frac{\tau}{\lambda_j^\theta}+\frac{1}{\lambda_j^{n-2+\theta}}+\frac{|\nabla K_j|}{\lambda_j^{1+\theta}}+O(\frac{1}{\lambda_j^{2+\theta}})\big),\end{aligned}$$

whence recalling (5.8) we get

$$\Lambda = -4n(n-1)\frac{\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}\int K\alpha_j^p\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} - 4n(n-1)b_2\sum_{j\neq i}\alpha_i\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}$$

up to some $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$. Therefore

$$\begin{aligned}&\partial J_\tau(\alpha^i\varphi_i)\phi_{2,j} = \frac{2\Lambda}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}} \\ =& -\frac{4n(n-1)\bar{c}_0^{-\frac{n-2}{n}}\alpha^2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}\alpha_j^p\int K\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} - \frac{4n(n-1)\bar{c}_0^{-\frac{n-2}{n}}b_2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\sum_{j\neq i}\alpha_i\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}\end{aligned} \tag{7.11}$$

up to the same error. Thus we are left with analysing

$$\begin{aligned}\int K\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_j d\mu_{g_0} =& \int_{B_c(a_j)}K\varphi_j^p\partial_{\lambda_j}\varphi_j d\mu_{g_0} + O\big(\frac{1}{\lambda_j^{n-\theta}}\big) \\ =& K_j\int_{B_c(a_j)}\varphi_j^p\partial_{\lambda_j}\varphi_j d\mu_{g_0} + \nabla K_j\int_{B_c(a_j)}x\varphi_j^p\partial_{\lambda_j}\varphi_j d\mu_{g_0} \\ & + \frac{\nabla^2}{2}K_j\int_{B_c(a_j)}x^2\varphi_j^p\partial_{\lambda_j}\varphi_j d\mu_{g_0} + \frac{\nabla^3}{6}K_j\int_{B_c(a_j)}x^3\varphi_j^p\partial_{\lambda_j}\varphi_j d\mu_{g_0} + O\big(\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big).\end{aligned}$$

Expanding the bubble $\varphi_j$ and its derivative $\lambda_j\partial_{\lambda_j}\varphi_j$ in conformal normal coordinates, i.e.

$$
\begin{aligned}
(p+1)\varphi_j^p\partial_j\varphi_j = \lambda_j\partial_j\varphi_j^{\frac{2n}{n-2}-\tau} &= u_{a_j}^{\frac{2n}{n-2}-\tau}\lambda_j\partial_{\lambda_j}\big(\frac{\lambda_j}{1+\lambda_j^2r_{a_j}^2(1+r_{a_j}^{n-2}H_{a_j})^{\frac{2}{2-n}}}\big)^{n-\theta} \\
=&(n-\theta)\big(\frac{\lambda_j}{1+\lambda_j^2r_{a_j}^2(1+r_{a_j}^{n-2}H_{a_j})^{\frac{2}{2-n}}}\big)^{n-\theta}\frac{1-\lambda_j^2r_{a_j}^2(1+r_{a_j}^{n-2}H_{a_j})^{\frac{2}{2-n}}}{1+\lambda_j^2r_{a_j}^2(1+r_{a_j}^{n-2}H_{a_j})^{\frac{2}{2-n}}} \\
=&(n-\theta)\big(\frac{\lambda_j}{1+\lambda_j^2r_{a_j}^2}\big)^{n-\theta}\frac{1-\lambda_j^2r_{a_j}^2}{1+\lambda_j^2r_{a_j}^2}+\frac{2(n-\theta)^2}{n-2}\big(\frac{\lambda_j}{1+\lambda_j^2r_{a_j}^2}\big)^{n-\theta}\frac{\lambda_j^2r_{a_j}^nH_{a_j}}{1+\lambda_j^2r_{a_j}^2}\frac{\frac{n+2-\theta}{n-2\theta}\lambda_j^2r_{a_j}^2-1}{1+\lambda_j^2r_{a_j}^2} \\
&+O\big(\big(\frac{\lambda_j}{1+\lambda_j^2r_{a_j}^2}\big)^{n-\theta}\frac{\lambda_j^4r_{a_j}^{2n}H_{a_j}^2}{(1+\lambda_j^2r_{a_j}^2)^2}\big)
\end{aligned}
$$

and arguing as for (5.4) we find using radial symmetry

(1) $\int_{B_c(a_j)}x\varphi_j^p\partial_{\lambda_j}\varphi_jd\mu_{g_0}$ , $\int_{B_c(a_j)}x^3\varphi_j^p\partial_{\lambda_j}\varphi_jd\mu_{g_0}=O\big(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big)$;

(2) $\frac{\nabla^2}{2}K_j\int_{B_c(a_j)}x^2\varphi_j^p\partial_{\lambda_j}\varphi_jd\mu_{g_0}=\frac{n-2}{4n}\frac{\Delta K_j}{\lambda_j^{2+\theta}}\int_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+1}}dx+O(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}})$,

Finally we have

$$
\begin{aligned}
\int_{B_c(a_j)}\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=&\frac{n-2}{2}\int_{B_c(0)}\big(\frac{\lambda_j}{1+\lambda_j^2r^2}\big)^{n-\theta}\frac{1-\lambda_j^2r^2}{1+\lambda_j^2r^2}dx \\
&+\int_{B_c(0)}\big(\frac{\lambda_j}{1+\lambda_j^2r^2}\big)^{n-\theta}\frac{\lambda_j^2r^nH_{a_j}}{1+\lambda_j^2r^2}\frac{n+2-n\lambda_j^2r^2}{1+\lambda_j^2r^2}dx
\end{aligned}
$$

up to some $O(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}})$, and see that for the first summand above there holds

$$
\frac{n-2}{2}\int_{B_c(0)}\big(\frac{\lambda_j}{1+\lambda_j^2r^2}\big)^{n-\theta}\frac{1-\lambda_j^2r^2}{1+\lambda_j^2r^2}dx=-\frac{n-2}{2}\frac{\theta}{\lambda_j^\theta}\int_{\mathbb{R}^n}\big(\frac{1}{1+r^2}\big)^n\frac{1-r^2}{1+r^2}\ln\frac{1}{1+r^2}dx,
$$

up to the same error. Defining

$$
\tilde{c}_1=\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{1-r^2}{(1+r^2)^{n+1}}\ln\frac{1}{1+r^2}dx,\qquad \tilde{c}_2=-\frac{n-2}{4n}\int_{\mathbb{R}^n}\frac{r^2(1-r^2)}{(1+r^2)^{n+1}}dx, \tag{7.12}
$$

it can be shown, that

$$
\tilde{c}_1=\frac{(n-2)^2}{48n}\omega_n\frac{\Gamma(n/2)^2}{\Gamma(n)}>0\ \text{ and }\ \tilde{c}_2=\frac{n-2}{4n}\omega_n\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)+\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}+2\right)}{2\Gamma(n+1)}>0,
$$

so we arrive at

$$
\int K\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=-\tilde{c}_1\frac{K_j}{\lambda_j^\theta}\tau-\tilde{c}_2\frac{\Delta K_j}{\lambda_j^{2+\theta}}+K_j\int_{B_c(0)}\big(\frac{\lambda_j}{1+\lambda_j^2r^2}\big)^{n-\theta}\frac{\lambda_j^2r^nH_{a_j}}{1+\lambda_j^2r^2}\frac{n+2-n\lambda_j^2r^2}{1+\lambda_j^2r^2}dx
$$

up to some $O(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}})$ and arguing as for (5.6) we find

$$
\begin{aligned}
&\int_{B_c(0)}(\frac{\lambda_j}{1+\lambda_j^2r^2})^{n-\theta}\frac{n+2-n\lambda_j^2r^2}{1+\lambda_j^2r^2}\frac{\lambda_j^2r^nH_{a_j}}{1+\lambda_j^2r^2}dx \\
&=\frac{1}{\lambda_j^{n-2+\theta}}\int_{B_{c\lambda_j}(0)}\frac{n+2-nr^2}{(1+r^2)^{n+2-\theta}}r^n\begin{pmatrix} H_j+\nabla H_j\frac{x}{\lambda_j}+O(\frac{r^2}{\lambda_j^2}) \\ H_j+\nabla H_j\frac{x}{\lambda_j}+O(\frac{r^2}{\lambda_j^2}\ln\frac{r}{\lambda_j}) \\ H_j+O(\frac{r}{\lambda_j}) \\ -W_j\ln\frac{r}{\lambda_j}+O(\frac{r}{\lambda_j}\ln\frac{r}{\lambda_j}) \\ O(\frac{r^{6-n}}{\lambda_j^{6-n}})\end{pmatrix}dx \\
&=-\tilde{d}_1\frac{\vartheta_j}{\lambda_j^\theta}+O\big(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big),\ \vartheta_j=\begin{pmatrix}\frac{H_j}{\lambda_j^{1+\theta}} \\ \frac{H_j}{\lambda_j^{2+\theta}}+O(\frac{\ln\lambda_j}{\lambda_j^{4+\theta}}) \\ \frac{H_j}{\lambda_j^{3+\theta}} \\ \frac{W_j\ln\lambda_j}{\lambda_j^{4+\theta}} \\ 0\end{pmatrix},\ \tilde{d}_1=-\int_{\mathbb{R}^n}\frac{r^n(n+2-nr^2)}{(1+r^2)^{n+2}}dx.
\end{aligned}
\tag{7.13}
$$

We conclude that

$$
\int K\varphi_j^p\lambda_j\partial_{\lambda_j}\varphi_jd\mu_{g_0}=-\tilde{c}_1\frac{K_j}{\lambda_j^\theta}\tau-\tilde{c}_2\frac{\Delta K_j}{\lambda_j^{2+\theta}}-\tilde{d}_1K_j\tilde{d}_1\frac{\vartheta_j}{\lambda_j^\theta}+O(\tau^2+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}).
$$

Plugging this into (7.11), we then have

$$
\begin{aligned}
\partial J_\tau(u)\phi_{2,j}=&\frac{4n(n-1)\bar{c}_0^{-\frac{n-2}{n}}\alpha^2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^p\big(\tilde{c}_1\tau+\tilde{c}_2\frac{\Delta K_j}{K_j\lambda_j^2}+\tilde{d}_1\vartheta_j\big) \\
&-\frac{4n(n-1)\bar{c}_0^{-\frac{n-2}{n}}b_2}{(\alpha_{K,\tau}^{\frac{2n}{n-2}})^{\frac{n-2}{n}}}\sum_{j\neq i}\alpha_i\lambda_j\partial_{\lambda_j}\varepsilon_{i,j}+O(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2).
\end{aligned}
$$

Now the claim follows from Lemma 5.1 by replacing the constants as follows

$$
(\tilde{c}_1,\tilde{c}_2,\tilde{d}_1,\tilde{b}_2)\rightsquigarrow\frac{4n(n-1)}{\bar{c}_0^{\frac{n-2}{n}}}(\tilde{c}_1,\tilde{c}_2,\tilde{d}_1,b_2),
\tag{7.14}
$$

cf. (7.10), (7.12) and (7.13) as well as Lemma 2.2. ☐

*Proof of Lemma 5.3.* From Lemma 4.3 and the chain rule we obtain up to the error in (5.16)

$$
\partial J_\tau(u)\phi_{3,j}=\partial J_\tau(\alpha^i\varphi_i)\phi_{3,j}=\frac{\nabla_{a_j}}{\lambda_j}J_\tau(\alpha^i\varphi_i)
$$

and write

$$
\partial J_\tau(\alpha^i\varphi_i)\phi_{2,j}=\frac{2A}{(\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0})^{\frac{2}{p+1}}}
\tag{7.15}
$$

with

$$
A=\int\alpha^iL_{g_0}\varphi_i\frac{\nabla_{a_j}}{\lambda_j}\varphi_j-\frac{\int L_{g_0}(\alpha^i\varphi_i)(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}K(\alpha^i\varphi_i)^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_jd\mu_{g_0}.
$$

Arguing as for (7.8), we find

$$
\begin{aligned}
A =&\alpha_j \int \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}} K\alpha_j^p\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} \\
&+\sum_{j\neq i}\alpha_i\int \varphi_i L_{g_0}\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}} K\alpha_i^p\varphi_i^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} \\
&- p\frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}}\sum_{j\neq i}\int K\alpha_j^{p-1}\alpha_i\varphi_j^{p-2}\varphi_i\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0}
\end{aligned}
$$

and arguing as for (5.2) and (5.10), in particular using Lemma 2.2, we obtain

$$
\begin{aligned}
A =&\alpha_j \int \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}} K\alpha_j^p\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} \\
&- 4n(n-1)b_3\sum_{j\neq i}\left(\alpha_i - \frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_i}{\lambda_i^\theta}\alpha_i^p - p\frac{\alpha^2}{\alpha_{K,\tau}^{p+1}}\frac{K_j}{\lambda_j^\theta}\alpha_j^{p-1}\alpha_i\right)\frac{\nabla_{a_j}}{\lambda_j}\varepsilon_{i,j} \\
=&\alpha_j \int \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} - \frac{\int(\alpha^i\varphi_i)L_{g_0}(\alpha^k\varphi_k)d\mu_{g_0}}{\int K(\alpha^i\varphi_i)^{p+1}d\mu_{g_0}} K\alpha_j^p\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} - 4n(n-1)b_3\sum_{j\neq i}\alpha_i\frac{\nabla_{a_j}}{\lambda_j}\varepsilon_{i,j}
\end{aligned}
$$

up to some

$$
O\big(\tau^2 + \sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2\big),
$$

using Lemma 5.1 for the last step. Consider a cut-off function $\eta$ such that

$$
\eta \in C^\infty(M,[0,1]),\ \eta = 1 \text{ on } B_c(a) \text{ and } \eta = 0 \text{ on } B_{2c}^c(a),
$$

with $c > 0$ sufficiently small and some $a \in M$ sufficiently close to $a_j$. Then

$$
\int K\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} = \int K\eta\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_j d\mu_{g_0} + O\big(\frac{1}{\lambda_j^{n-\theta}}\big) = \frac{1}{p+1}\frac{\nabla_{a_j}}{\lambda_j}\int K\eta\varphi_j^{p+1}d\mu_{g_0} + O\big(\frac{1}{\lambda_j^{n-\theta}}\big)
$$

and passing to conformal normal coordinates around $a_j$ we have

$$
\begin{aligned}
&\frac{\nabla_{a_j}}{\lambda_j}\int K\eta\varphi_j^{p+1}d\mu_{g_0} = \frac{\nabla_{a_j}}{\lambda_j}\int (K\eta)\circ \exp_{g_{a_j}}\big(\frac{\lambda_j}{1+\lambda_j^2r^2(1+r^{n-2}H_{a_j})^{\frac{2}{2-n}}}\big)^{n-\theta}dx \\
=&\int\left(\frac{\nabla_{a_j}\exp_{g_{a_j}}}{\lambda_j}(\eta\ K)\circ\exp_{g_{a_j}}\right)\big(\frac{\lambda_j}{1+\lambda_j^2r^2(1+r^{n-2}H_{a_j})^{\frac{2}{2-n}}}\big)^{n-\theta}dx \\
&-(n-\theta)\int(K\eta)\circ\exp_{g_{a_j}}\big(\frac{\lambda_j}{1+\lambda_j^2r^2(1+r^{n-2}H_{a_j})^{\frac{2}{2-n}}}\big)^{n-\theta}\frac{\lambda_j^2r^2\frac{\nabla_{a_j}}{\lambda_j}(1+r^{n-2}H_{a_j})^{\frac{2}{2-n}}}{1+\lambda_j^2r^2(1+r^{n-2}H_{a_j})^{\frac{2}{2-n}}}dx + O\big(\frac{1}{\lambda_j^{n-\theta}}\big) \\
=&\Gamma - (n-\theta)\mathfrak{M} + O\big(\frac{1}{\lambda_j^{n-\theta}}\big),
\end{aligned}
$$

where

$$
\mathfrak{M} = \frac{2}{2-n}\int_{B_c(a_j)}K(\exp_{g_{a_j}})\big(\frac{\lambda_j}{1+\lambda_j^2r^2}\big)^{n-\theta}\frac{\lambda_j^2r^n\frac{\nabla_{a_j}}{\lambda_j}H_{a_j}(1+O(r^{n-2}H_{a_j}))}{1+\lambda_j^2r^2}dx
$$

up to some $O(\frac{1}{\lambda_j^{n-\theta}})$. From (2.6) anda(2.8) and using radial symmetry we obtain

$$
\Gamma = \check{c}_3\frac{\nabla K_j}{\lambda_j^{1+\theta}} + \check{c}_4\frac{\nabla\Delta K_j}{\lambda_j^{3+\theta}} \quad\text{with}\quad \check{c}_3 = \int_{\mathbb{R}^n}\frac{dx}{(1+r^2)^n} \quad\text{and}\quad \check{c}_4 = \frac{1}{2n}\int_{\mathbb{R}^n}\frac{r^2dx}{(1+r^2)^n}. \tag{7.16}
$$

up to some $O\big(\tau^2+\frac{|\nabla K_j|^2}{\lambda_j^2}+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big)$ . By (2.8) we have $\nabla H_{a_j}H_{a_j}=O(1)$ for $n=3,4,5$ and

$$\nabla H_{a_j}H_{a_j}=O\begin{pmatrix}\ln^2 r & \text{for } \ n=6\\ r^{12-2n} & \text{for } \ n\geq 7\end{pmatrix},$$

whence up to some $O\big(\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big)$

$$\begin{aligned}\mathfrak{M}=&\frac{2}{2-n}\int_{B_c(a_j)}K(\exp_{g_{a_j}})(\frac{\lambda_j}{1+\lambda_j^2r^2})^{n-\theta}\frac{\lambda_j^2r^n\frac{\nabla_{a_j}}{\lambda_j}H_{a_j}}{1+\lambda_j^2r^2}d\mu_{g_0}\\ =&\frac{2}{2-n}\int_{B_c(a_j)}(K_j+\nabla K_jx+O(r^2))(\frac{\lambda_j}{1+\lambda_j^2r^2})^{n-\theta}\frac{\lambda_j^2r^n}{1+\lambda_j^2r^2}\begin{pmatrix}\frac{\nabla_{a_j}H_j}{\lambda_j}+\frac{\nabla_{a_j}\nabla H_jx}{\lambda_j}+O(\frac{r^2}{\lambda_j})\\ \frac{\nabla_{a_j}H_j}{\lambda_j}+\frac{\nabla_{a_j}\nabla H_jx}{\lambda_j}+O(\frac{r^2\ln r}{\lambda_j})\\ \frac{\nabla_{a_j}H_j}{\lambda_j}+O(\frac{r}{\lambda_j})\\ -\frac{\nabla_{a_j}W_j}{\lambda_j}\ln r+O(\frac{1}{\lambda_j})\\ O(\frac{r^{6-n}}{\lambda_j}),\end{pmatrix}dx,\end{aligned}$$

and we obtain

$$\begin{aligned}\mathfrak{M}=&\frac{2K_j}{2-n}\int_{B_c(a_j)}(\frac{\lambda_j}{1+\lambda_j^2r^2})^{n-\theta}\frac{\lambda_j^2r^n}{1+\lambda_j^2r^2}\begin{pmatrix}\frac{\nabla_{a_j}H_j}{\lambda_j}+O(\frac{r^2}{\lambda_j})\\ \frac{\nabla_{a_j}H_j}{\lambda_j}+O(\frac{r^2\ln r}{\lambda_j})\\ \frac{\nabla_{a_j}H_j}{\lambda_j}+O(\frac{r}{\lambda_j})\\ -\frac{\nabla_{a_j}W_j}{\lambda_j}\ln r+O(\frac{1}{\lambda_j})\\ O(\frac{r^{6-n}}{\lambda_j})\end{pmatrix}dx+O(\frac{1}{\lambda_j^4})\\ =&\check{d}_1K_j\frac{\vartheta_j}{\lambda_j^\theta}+O\big(\tau^2+\frac{1}{\lambda_j^4}\big),\quad \check{d}_1=\frac{2}{2-n}\int_{\mathbb{R}^n}\frac{r^n}{(1+r^2)^{n+1}}dx,\quad \vartheta_j=\begin{pmatrix}0\\ \frac{\nabla_{a_j}H_j}{\lambda_j^3}\\ 0\\ 0\\ 0\end{pmatrix}\end{aligned}$$

up to some $O(\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}})$. Collecting terms we arrive at

$$\int K\varphi_j^p\frac{\nabla_{a_j}}{\lambda_j}\varphi_jd\mu_{g_0}=\frac{\Gamma-(n-\theta)\mathfrak{M}}{p+1}=\frac{n-2}{2n}\big(\check{c}_3\frac{\nabla K_j}{\lambda_j^{1+\theta}}+\check{c}_4\frac{\nabla\Delta K_j}{\lambda_j^{3+\theta}}+n\check{d}_1K_j\frac{\vartheta_j}{\lambda_j^\theta}\big)\tag{7.17}$$

up to some $O\big(\tau^2+\frac{|\nabla K_j|^2}{\lambda_j^2}+\frac{1}{\lambda_j^4}+\frac{1}{\lambda_j^{2(n-2)}}\big)$ and conclude

$$\begin{aligned}A=&\alpha_j\int\varphi_jL_{g_0}\frac{\nabla_{a_j}}{\lambda_j}\varphi_jd\mu_{g_0}-\frac{4n(n-1)(n+2)b_3}{n-2}\sum_{j\neq i}\alpha_i\frac{\nabla_{a_j}}{\lambda_j}\varepsilon_{i,j}\\ &-\frac{2(n-1)(n-2)\alpha^2}{\alpha_{K,\tau}^{\frac{2n}{n-2}}}K_j\alpha_j^p\big(\check{c}_3\frac{\nabla K_j}{K_j\lambda_j^{1+\theta}}+\check{c}_4\frac{\nabla\Delta K_j}{\lambda_j^{3+\theta}}+n\check{d}_1\frac{\vartheta_j}{\lambda_j^\theta}\big)\end{aligned}$$

up to some $O\big(\tau^2+\sum_{r\neq s}\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{1}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\varepsilon_{r,s}^{\frac{n+2}{n}}+|\partial J_\tau(u)|^2\big)$. Applying Lemma 5.1 we find

$$\begin{aligned}A=&\alpha_j\int\varphi_jL_{g_0}\frac{\nabla_{a_j}}{\lambda_j}\varphi_jd\mu_{g_0}-4n(n-1)b_3\sum_{j\neq i}\alpha_i\frac{\nabla_{a_j}}{\lambda_j}\varepsilon_{i,j}\\ &-2(n-1)(n-2)\alpha_j\big(\check{c}_3\frac{\nabla K_j}{K_j\lambda_j}+\check{c}_4\frac{\nabla\Delta K_j}{K_j\lambda_j^3}+n\check{d}_1\vartheta_j\big)\end{aligned}\tag{7.18}$$

up to the same error. We are left with estimating

$$\int \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} = \int_{B_c(a_j)} \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} + O(\frac{1}{\lambda_j^{n-\theta}}).$$

Then from Lemma 2.1 we see that in case $n = 4, 5$

$$\int \frac{\varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j}{4n(n-1)} d\mu_{g_0} = \int_{B_c(a_j)} \varphi_j^{\frac{n+2}{n-2}} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} - \frac{c_n}{2} \int_{B_c(a_j)} r_{a_j}^{n-2} (H_j + n\nabla H_j x) \varphi_j^{\frac{n+2}{n-2}} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0}$$

up to some $O(\frac{1}{\lambda_j^4} + \frac{1}{\lambda_j^{2(n-2)}})$, and thus due to (7.17)

$$\int \frac{\varphi_j}{4n(n-1)} L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} = \frac{n-2}{2} \check{d}_1 \vartheta_j - \frac{c_n}{2} \int_{B_c(a_j)} r_{a_j}^{n-2} (H_j + n\nabla H_j x_{a_j}) \varphi_j^{\frac{n+2}{n-2}} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0}$$

up to some $O\big(\tau^2 + \frac{|\nabla K_j|^2}{\lambda_j^2} + \frac{1}{\lambda_j^4} + \frac{1}{\lambda_j^{2(n-2)}}\big)$. Finally we observe that

$$\frac{\nabla_{a_j}}{\lambda_j} \varphi_j = \frac{2-n}{2} u_{a_j} \big(\frac{\lambda_j}{1 + \lambda_j r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}}\big)^{\frac{n-2}{2}} \frac{\lambda_j \nabla_{a_j} (r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}})}{1 + \lambda_j^2 r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}} + O(\frac{r_{a_j}}{\lambda_j} \varphi_j),$$

and using the smoothness of conformal normal coordinates with respect to $a_j$ we find

$$\begin{aligned} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j = & \frac{2-n}{2} u_{a_j} \big(\frac{\lambda_j}{1 + \lambda_j r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}}\big)^{\frac{n-2}{2}} \frac{2\lambda_j x_{a_j}}{1 + \lambda_j^2 r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}} \\ & + O\big(\big(\frac{\lambda_j r_{a_j}^4}{1 + \lambda_j^2 r_{a_j}^2} + \frac{\lambda_j r_{a_j}^{n-1}}{1 + \lambda_j^2 r_{a_j}^2} + \frac{r_{a_j}}{\lambda_j}\big) \varphi_j\big). \end{aligned}$$

This gives

$$\int \frac{\varphi_j}{4n(n-1)} L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} = \frac{n-2}{2} \check{d}_1 \vartheta_j + \frac{(n-2)c_n}{2} \int\limits_{B_c(a_j)} \frac{\lambda_j x_{a_j} r_{a_j}^{n-2} (H_j + n\nabla H_j x) \varphi_j^{\frac{2n}{n-2}}}{1 + \lambda_j^2 r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}} d\mu_{g_0}$$

up to some $O\big(\tau^2 + \frac{|\nabla K_j|^2}{\lambda_j^2} + \frac{1}{\lambda_j^4} + \frac{1}{\lambda_j^{2(n-2)}}\big)$. Passing to conformal normal coordinates around $a_j$, we find

$$\int_{B_c(a_j)} \frac{\lambda_j x_{a_j} r_{a_j}^{n-2} (H_j + n\nabla H_j x) \varphi_j^{\frac{2n}{n-2}}}{1 + \lambda_j^2 r_{a_j}^2 (1 + r_{a_j}^{n-2} H_{a_j})^{\frac{2}{2-n}}} d\mu_{g_0} = \int_{B_c(0)} \frac{\lambda_j^{n+1} x r^{n-2} H_j}{(1 + \lambda_j^2 r^2)^{n+1}} dx = 0$$

up to some $O(\frac{1}{\lambda_j^4} + \frac{1}{\lambda_j^{2(n-2)}}))$. We therefore conclude

$$\int \varphi_j L_{g_0} \frac{\nabla_{a_j}}{\lambda_j} \varphi_j d\mu_{g_0} = 2n(n-1)(n-2) \check{d}_1 \vartheta_j$$

up to some $O\big(\tau^2 + \frac{|\nabla K_j|^2}{\lambda_j^2} + \frac{1}{\lambda_j^4} + \frac{1}{\lambda_j^{2(n-2)}}\big)$. Plugging into (7.18) we arrive at

$$A = -2(n-1)(n-2)\alpha_j \left( \check{c}_3 \frac{\nabla K_j}{K_j \lambda_j} + \check{c}_4 \frac{\nabla \Delta K_j}{K_j \lambda_j^3} \right) - 4n(n-1) b_3 \sum_{j \neq i} \alpha_i \frac{\nabla_{a_j}}{\lambda_j} \varepsilon_{i,j}, \tag{7.19}$$

up to some

$$O\big(\tau^2 + \sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^{\frac{n+2}{n}} + |\partial J_\tau(u)|^2\big).$$

Recalling (7.15) the claim follows by setting or replacing

$$(\check{c}_3, \check{c}_4, \check{b}_3) \rightsquigarrow 4(n-1)(n-2)(\check{c}_3, \check{c}_4, \frac{2n}{n-2} b_3), \tag{7.20}$$

cf. 7.16 and Lemma 2.2. □

## 7.3 List of constants

We give here a list of constants, referring to where they can be found.

| | | $\bar{}$ | $\hat{}$ | $\grave{}$ | $\tilde{}$ | $\check{}$ |
|---|---|---|---|---|---|---|
| $c_0$ | | (5.5) | (5.13) | (7.9) | | |
| $c_1$ | Lemma 2.2 | (5.5) | (5.13) | | (7.14) | |
| $c_2$ | Lemma 2.2 | (5.7) | (5.13) | (7.9) | (7.14) | |
| $c_3$ | Lemma 2.2 | | | | (6.4) | (7.20) |
| $c_4$ | | | | | (6.8) | (7.20) |
| $d_1$ | | (5.6) | (5.13) | (7.9) | (7.14) | |
| $b_1$ | Lemma 2.2 | (5.3) | (5.13) | (7.9) | (5.10) | |
| $b_2$ | Lemma 2.2 | (7.6) | | | (7.14) | |
| $b_3$ | Lemma 2.2 | | | | | (7.20) |

For instance, $c_2$ is found in Lemma 2.2, $\bar{c}_2$ in equation (5.7) and $\hat{d}_1$ in equation (5.13). For the empty fields the corresponding combination of accent and symbol is non-existent. As a caveat please note that we have within some proofs redefined constants for the sake for normalization, hence we point to the final definition, from which upwards mentioned constants can be easily recovered. Finally we recall that $c_n$ is the normalizing constants in the definition of the conformal laplacian

$$L_g = -c_n \Delta_g + R_g, \qquad c_n = \frac{4(n-1)}{n-2}.$$

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