LOCALITY OF THE PERIMETER IN CARNOT GROUPS AND CHAIN RULE

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Abstract. In the class of Carnot groups we study fine properties of sets of finite perimeter. Improving a recent result by Ambrosio-Kleiner-Le Donne, we show that the perimeter measure is local, i.e., that given any pair of sets of finite perimeter their perimeter measures coincide on the intersection of their essential boundaries. This solves a question left open in [4]. As a consequence we prove a general chain rule for BV functions in this setting.

Introduction

In this paper we deal with sets of finite perimeter in a Carnot group $\mathbb{G}$. In this context one can define a good notion of sets of finite perimeter and a left-invariant Carnot-Carathéodory distance d_c and a surface measure $|D\chi_E|$ by fixing a metric in the horizontal layer V_1 of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. More precisely, given an orthonormal basis $X_1, \ldots, X_m$ of V_1, one defines the vector valued distribution

$$(X_1\chi_E, \ldots, X_m\chi_E)$$

(here we think to X_i as derivations) and says that E has finite perimeter if the distribution is representable by a vector-valued measure. Then, $|D\chi_E|$ is the total variation of this measure, and plays the role of surface measure in this context. Recall that in this context the Lie algebra stratification provides an integer Q that plays the role of metric dimension (the so-called homogeneous dimension) of the group: indeed, the volume measure of the group is a constant multiple of $\mathcal{S}^Q$, where $\mathcal{S}^Q$ is the Q-dimensional Hausdorff measure induced by d_c.

The structure of $|D\chi_E|$ has been deeply analyzed in [1], [10], [11]. In [1] it has been proved, actually in a much more general context, that $|D\chi_E|$ can be represented as

$$|D\chi_E|(B) = \int_{B\cap\partial^* E} \theta_E \, d\mathcal{S}^{Q-1} \qquad \forall\ B \subseteq \mathbb{G} \text{ Borel},$$

(here $\partial^* E$ is the measure-theoretic boundary of E, see Definition 6) but no constructive formula for θ_E is provided: its existence is ensured only by the Radon-Nikodym theorem. First in the Heisenberg groups [10], and then in groups of step 2 [11], Franchi-Serapioni-Serra Cassano made a much more precise analysis of $|D\chi_E|$, that leads to a precise identification of θ_E in terms of $\partial^* E$; however this analysis depends on the fact that, as in the classical De Giorgi's theorem [6], tangent sets to E at x are, for $|D\chi_E|$-a.e. x, halfspaces. Presently this information is available *only* on step 2 groups, but some progress on this problem has recently been made in [2] by A.-Kleiner-Le Donne: they proved in all Carnot groups $\mathbb{G}$ that, for $|D\chi_E|$-a.e. x, an halfspace belongs to the collection of tangents sets to E at x. It still remains to prove, however, that halfspaces are the *unique* tangent sets.

In this paper we slightly improve the techniques used in [2] to show a locality property of the perimeter measure left open in [4]. Precisely, we show that

$$|D\chi_E|(B) = |D\chi_F|(B) \qquad \text{for all } B \subset \partial^* E \cap \partial^* F \text{ Borel} \tag{0.1}$$

for any pair of sets of locally finite perimeter E and F. The strategy is to consider pairs of sets (E_1, F_1) as tangents to (E, F); a more precise analysis of the techniques in [2] allows to show the existence of a pair of halfspaces among the tangents. This fact provides equality of normals, up to the sign, and eventually (0.1).

Using the locality property, it is not difficult to define in a consistent way a surface measure ρ_S associated to sets S contained in a countable union of essential boundaries of sets of finite perimeter: among all measures ρ vanishing on $\mathcal{S}^{Q-1}$-negligible sets, ρ_S is uniquely characterized by the property $\rho_S(B) = |D\chi_E|(B)$ whenever E has locally finite perimeter and $B \subset \partial^* E$.

As an application we improve the (weak) chain rule formula in $BV(\mathbb{G})$ proved in [4], proving a result completely analogous to Euclidean case. The chain rule formula involves ρ_{S_u}, where S_u is the approximate discontinuity set of u.

1. Carnot Groups

1.1. Definitions and basic properties.

Definition 1. Throughout this paper $\mathbb{G}$ denotes a Carnot group of step $s \geq 1$, whose unit element shall be denoted by e. Precisely, $\mathbb{G}$ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step s stratification, i.e.,

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s \tag{1.2}$$

with $[V_j, V_1] = V_{j+1}$, $1 \leq j \leq s$, $V_{s+1} = 0$. We use the notation $n = \sum_i \dim V_i$ for the topological dimension of $\mathbb{G}$, and we denote by

$$Q := \sum_{i=1}^{s} i \dim V_i \tag{1.3}$$

the so called homogeneous dimension of $\mathbb{G}$.

Definition 2. Consider a family of inhomogeneous dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\delta_\lambda(\sum_{i=1}^{s} v_i) := \sum_{i=1}^{s} \lambda^i v_i \qquad \lambda \geq 0 \tag{1.4}$$

where $X = \sum_{i=1}^{s} v_i$ with $v_i \in V_i$, $1 \leq i \leq s$. The dilations δ_λ belong to $GL(\mathfrak{g})$ and are uniquely determined by the homogeneity conditions

$$\delta_\lambda X = \lambda^k X \qquad \forall X \in V_k. \tag{1.5}$$

On a Carnot group $\mathbb{G}$, the Carnot-Carathéodory distance is a left invariant distance defined as follows: Denote by m the dimension of V_1, fix an inner product in V_1 and an orthonormal basis $X_1, \dots, X_m$ of V_1. Then, the CC distance d is defined as

$$d^2(x,y) := \inf \left\{ \int_0^1 \sum_{i=1}^m |a_i(t)|^2 dt : \gamma(0) = x,\ \gamma(1) = y \right\}, \tag{1.6}$$

where the infimum is made among all Lipschitz curves $\gamma : [0,1] \to \mathbb{G}$ (where in $\mathbb{G}$ a left-invariant Riemannian distance is fixed) with the property $\gamma'(t) = \sum_{i=1}^m a_i(t)(X_i)_{\gamma(t)}$ for a.e. $t \in [0,1]$.

Remark 1. A Carnot group $\mathbb{G}$ is clearly nilpotent, hence the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism. So any element $g \in \mathbb{G}$ can be identified with $\exp(X)$ for some $X \in \mathfrak{g}$, and uniquely written in the form

$$\exp(\sum_{i=1}^s v_i), \quad v_i \in V_i,\ 1 \le i \le s. \tag{1.7}$$

With this identification we can define a family of intrinsic dilations $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, $\lambda \ge 0$, by

$$\delta_\lambda(\exp(\sum_{i=1}^s v_i)) := \exp(\sum_{i=1}^s \lambda^i v_i), \tag{1.8}$$

or we can write it more briefly $\exp \circ \delta_\lambda = \delta_\lambda \circ \exp$. The Carnot-Carathéodory distance is well-behaved under these dilations, namely

$$d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x,y) \qquad \forall x, y \in \mathbb{G}. \tag{1.9}$$

In exponential coordinates $p = \exp(p_1 X_1 + \cdots + p_n X_n)$, we might identify p with the n-uple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and consequently identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Baker-Campbell-Hausdorff formula. However, we shall avoid the use of this identification whenever possible.

Carnot groups are nilpotent and so unimodular, thus the right and the left Haar measures coincide, up to constants. We shall denote by $\mathcal{H}^k$ the Hausdorff k-dimensional measure associated to the Carnot-Carathéodory distance on $\mathbb{G}$. The Hausdorff measure $\mathcal{H}^Q$ is a Radon measure in $\mathbb{G}$ and, by the left invariance of the CC distance, is an Haar measure on $\mathbb{G}$: it will be also denoted by $\mathrm{vol}_{\mathbb{G}}$. Moreover, in exponential coordinates, this measure coincides with a constant multiple of the Lebesgue measure on $\mathbb{R}^n$. From these considerations it follows that

$$\mathrm{vol}_{\mathbb{G}}(\delta_\lambda(A)) = \lambda^Q \mathrm{vol}_{\mathbb{G}}(A),$$

for all Borel sets $A \subseteq \mathbb{G}$. In particular

$$\mathrm{vol}_{\mathbb{G}}(B_\rho(g)) = c\rho^Q \tag{1.10}$$

for some constant $c > 0$ independent of g.

1.2. X**-derivative.** As in Definition 2.6 of [2], we introduce the X-derivative in a Carnot group $\mathbb{G}$. Given a vector field $X \in \Gamma(T\mathbb{G})$ we define the divergence $\mathrm{div}X$ in the sense of distributions as follows:

$$\int_{\mathbb{G}} Xu \, d\,\mathrm{vol}_{\mathbb{G}} = -\int_{\mathbb{G}} u \,\mathrm{div}X \, d\,\mathrm{vol}_{\mathbb{G}} \qquad \forall u \in C_c^\infty(\mathbb{G}). \tag{1.11}$$

Definition 3. Let $u \in L^1_{\rm loc}(\mathbb{G})$ and let $X \in \Gamma(T\mathbb{G})$ be divergence-free. We denote by Xu the distribution

$$\langle Xu, v\rangle := -\int_{\mathbb{G}} uXv \, d\,\mathrm{vol}_{\mathbb{G}}, \qquad v \in C^\infty_c(\mathbb{G}).$$

If $f \in L^1_{\rm loc}(\mathbb{G})$, we write $Xu = f$ if $\langle Xu, v\rangle = \int_{\mathbb{G}} vf \, d\,\mathrm{vol}_{\mathbb{G}}$ for all $v \in C^\infty_c(\mathbb{G})$. Analogously, if μ is a Radon measure on $\mathbb{G}$, we write $Xu = \mu$ if $\langle Xu, v\rangle = \int_{\mathbb{G}} v \, d\mu$ for all $v \in C^\infty_c(\mathbb{G})$.

Given $X \in \Gamma(T\mathbb{G})$ we denote by $\varphi_X : \mathbb{G} \times \mathbb{R} \to \mathbb{G}$ the flow of X, assuming that X is sufficiently smooth to ensure its global existence and uniqueness.

Theorem 1.1 (2.12 of [2])**.** *Let $u \in L^1_{\rm loc}(\mathbb{G})$ be satisfying $Xu = 0$ in the sense of distributions. Then, for all $t \in \mathbb{R}$, $u = u \circ \Phi_X(\cdot, t)$ $vol_{\mathbb{G}}$-a.e. in $\mathbb{G}$.*

One can prove that all $X \in \mathfrak{g}$ are divergence free, using the invariance of the right Haar measure with respect to the flow of X (see Remark 2.13 of [2] for details).

Now we recall the definition of the adjoint map. For $k \in \mathbb{G}$ the conjugation map C_k is the composition of L_k with $R_{k^{-1}}$. The *adjoint* representation Ad of $\mathbb{G}$ maps $\mathbb{G}$ in $\mathrm{Aut}(\mathfrak{g})$ as follows

$$Ad_k(X) := (C_k)_* X,$$

if we consider the elements of $\mathfrak{g}$ as left-invariant vector fields on $\mathbb{G}$. Equivalently

$$Ad_k(X) f(x) = X(f \circ C_k)(C_k^{-1}(x)),$$

if X is seen as a derivation on $C^\infty(\mathbb{G})$.

We recall the following result, proved in Proposition 2.17 of [2].

Proposition 1.2. *Assume that $\mathbb{G}$ is a connected, simply connected nilpotent Lie group. Let $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$ satisfying $\dim(\mathfrak{g}') + 2 \le \dim(\mathfrak{g})$, and assume that $W := \mathfrak{g}' \oplus \{\mathbb{R}X\}$ generates the whole Lie algebra $\mathfrak{g}$ for some $X \notin \mathfrak{g}'$. Then, there exists $k \in \exp(\mathfrak{g}')$ such that $Ad_k(X) \notin W$.*

1.3. **Measure theoretic tools.** In this section we recall some facts and definitions of measure theory, the main references for this section are [8] and [5].

Definition 4. For any set $E \subset \mathbb{G}$, denote by χ_E the characteristic function of E. In the class of Borel set of $\mathbb{G}$ we considere the local convergence in measure:

$$E_h \to E \iff \mathrm{vol}_{\mathbb{G}}(K \cap [(E_h \setminus E) \cup (E \setminus E_h)]) \to 0 \qquad \text{for all } K \subseteq \mathbb{G} \text{ compact.}$$

Notice that the local convergence in measure of E_h to E is equivalent to the convergence $\chi_{E_h} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{G})$.

Let $\mathcal{M}(\mathbb{G})$ be the class of signed Radon measures in $\mathbb{G}$; any Borel proper map (i.e., such that the inverse image of bounded set is bounded) $f : \mathbb{G} \to \mathbb{G}$ induces a push-forward operator $f_\sharp : \mathcal{M}(\mathbb{G}) \to \mathcal{M}(\mathbb{G})$ defined by

$$f_\sharp \mu(B) := \mu\big(f^{-1}(B)\big) \qquad \text{for all } B \subset \mathbb{G} \text{ bounded Borel.}$$

In the space $\mathcal{M}(\mathbb{G})$ we consider the weak* convergence induced by the duality with $C_c(\mathbb{G})$, more precisely $\mu_k \rightharpoonup \mu$ if

$$\lim_{k\to\infty} \int_{\mathbb{G}} u d\mu_k = \int_{\mathbb{G}} u d\mu \qquad \forall u \ \in C_c(\mathbb{G}).$$

We will use also vector-valued Radon measures, representable as a vector $(\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathcal{M}(\mathbb{G})$. Recall that the total variation $|\mu|$ of a $\mathbb{R}^n$-valued measure μ is defined by

$$|\mu|(B) := \sup\left\{ \sum_{i=1}^{\infty} |\mu(B_i)| : \quad \{B_i\} \text{ Borel partition of } B \right\}.$$

For a nonnegative Radon measure μ we have the following useful implications: for all $t > 0$ and $B \subseteq \mathbb{G}$ Borel it holds

$$\limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \geq t \quad \forall x \in B \qquad \Longrightarrow \qquad \mu(B) \geq t\mathcal{S}^k(B) \tag{1.12}$$

$$\limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \leq t \quad \forall x \in B \qquad \Longrightarrow \qquad \mu(B) \leq t\mathcal{S}^k(B). \tag{1.13}$$

Here ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$ and $\mathcal{S}^k$ is the spherical Hausdorff k-dimensional measure.

Definition 5. A nonnegative Radon measure μ in $\mathbb{G}$ is said to be asymptotically doubling if:

$$\limsup_{r\downarrow 0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{G}. \tag{1.14}$$

For asymptotically doubling measures a Lebesgue differentiation theorem holds, for a proof see 2.8.17 in [8].

Theorem 1.3. *Assume that μ is asymptotically doubling and $\nu \in \mathcal{M}(\mathbb{G})$ is absolutely continuous with respect to μ. Then the limit*

$$f(x) := \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \tag{1.15}$$

exists and it is finite for μ-a.e. point $x \in supp\mu$. Moreover $f \in L^1_{\text{loc}}(\mu)$ and $\nu = f\mu$, namely $\nu(B) = \int_B f d\mu$ for all bounded Borel sets $B \subseteq \mathbb{G}$.

Remark 2. As a consequence of the previous theorem, if μ is an asymptotically doubling Radon measure and $L \subseteq \mathbb{G}$ is a Borel set we have

$$\lim_{r\downarrow 0} \frac{\mu(L \cap B_r(x))}{\mu(B_r(x))} = \chi_L(x) \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{G}. \tag{1.16}$$

Given $E \subseteq \mathbb{G}$, we recall the definition of the essential boundary $\partial^* E$ of E.

Definition 6. Set $m_E(x,\rho) = \text{vol}_{\mathbb{G}}(E \cap B_\rho(x))$, we say that $x \in \partial^* E$ if

$$\limsup_{\rho\downarrow 0} \frac{m_E(x,\rho)}{\text{vol}_{\mathbb{G}}(B_\rho(x))} > 0 \qquad \text{and} \qquad \limsup_{\rho\downarrow 0} \frac{m_{E^c}(x,\rho)}{\text{vol}_{\mathbb{G}}(B_\rho(x))} > 0.$$

1.4. **Sets of finite perimeter.**

Definition 7. Let $f \in L^1_{\rm loc}(\mathbb{G})$. We shall denote by $\mathrm{Reg}(f)$ the vector subspace of $\mathfrak{g}$ made by vectors X such that Xf is representable by a Radon measure. We shall denote by $\mathrm{Inv}(f)$ the subspace of $\mathrm{Reg}(f)$ corresponding to the vector fields X such that $Xf = 0$, and by $\mathrm{Inv}_0(f)$ the subset made by homogeneous directions, i.e.,

$$\mathrm{Inv}_0(f) := \mathrm{Inv}(f) \cap \bigcup_{i=1}^{s} V_i.$$

Proposition 1.4 (4.7 of [2])**.** *Let* $f \in L^1_{\rm loc}(\mathbb{G})$*. Then* $\mathrm{Reg}(f)$*,* $\mathrm{Inv}(f)$*,* $\mathrm{Inv}_0(f)$ *are invariant under left translations, and* $\mathrm{Inv}_0(f)$ *is invariant under dilations. Moreover:*

(1) $\mathrm{Inv}(f)$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[\mathrm{Inv}_0(f), \mathrm{Inv}_0(f)] \subseteq \mathrm{Inv}_0(f)$*,*
(2) *If* $X \in \mathrm{Inv}(f)$ *and* $k = \exp(X)$*, then* Ad_k *maps* $\mathrm{Reg}(f)$ *into* $\mathrm{Reg}(f)$ *and* $\mathrm{Inv}(f)$ *into* $\mathrm{Inv}(f)$*, more precisely*

$$Ad_k(Y)f = (R_{k^{-1}})_\sharp Y f \qquad \forall Y \in \mathrm{Reg}(f).$$

We will consider regular, and invariant directions of characteristic functions, therefore we set

$$\mathrm{Reg}(E) := \mathrm{Reg}(\chi_E), \quad \mathrm{Inv}(E) := \mathrm{Inv}(\chi_E), \quad \mathrm{Inv}_0(E) := \mathrm{Inv}_0(\chi_E).$$

We may now define halfspaces as the subsets of $\mathbb{G}$ having invariance along a codimension 1 space of directions, and monotonicity along one direction. We call vertical halfspace an halfspace which is invariant along all non-horizontal directions:

Definition 8 (4.2 of [2])**.** We say that a Borel set $H \subseteq \mathbb{G}$ is a vertical halfspace if $\mathrm{Inv}_0(H) \supseteq \cup_2^s V_i$, $V_1 \cap \mathrm{Inv}(H)$ is a codimension one subspace of V_1 and there is $\bar{X} \in V_1 \setminus \mathrm{Inv}_0(H)$ such that $\bar{X}\chi_H \geq 0$.

Identifying the Lie group $\mathbb{G}$ with $\mathbb{R}^n$, vertical halfspaces are images by the exponential map of halfspaces in $\mathbb{R}^n$, as stated in the following proposition, for a proof see Proposition 4.4 of [2] (recall that m denotes the dimension of V_1 and that $X_1, \ldots, X_m$ is a given orthonormal basis of V_1).

Proposition 1.5. $H \subseteq \mathbb{G}$ *is a vertical halfspace if there exist* $c \in \mathbb{R}$ *and a unit vector* $\nu \in \mathbf{S}^{m-1}$ *such that* $H = H_{c,\nu}$*, where*

$$H_{c,\nu} := \exp\left(\left\{\sum_{i=1}^{m} a_i X_i + \sum_{i=2}^{s} v_i : \ v_i \in V_i, \ a \in \mathbb{R}^n, \ \sum_{i=1}^{m} a_i \nu_i \leq c\right\}\right). \tag{1.17}$$

Now, we can define the class of sets of locally finite perimeter:

Definition 9 (Essential boundary)**.** A Borel set $E \subseteq \mathbb{G}$ has locally finite perimeter if $X\chi_E$ is a Radon measure for any $X \in V_1$. Given a set E of locally finite perimeter, we denote by $D\chi_E$ the vector-valued measure

$$D\chi_E = (X_1\chi_E, \ldots, X_m\chi_E).$$

Let us introduce the reduced boundary $\mathcal{F}E$ of a finite perimeter set E: in the rest of the paper we will use mainly this notion of boundary, only when necessary we will refer to $\partial^* E$ (see also Remark 4).

Definition 10 (De Giorgi's reduced boundary)**.** Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. We denote by $\mathcal{F}E$ the set of points $x \in \operatorname{supp}|D\chi_E|$ where:

(1) the limit $\nu_E(x) = (\nu_{E,1}(x), \ldots, \nu_{E,m}(x)) := \lim_{r\downarrow 0} \dfrac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$ exists,
(2) $|\nu_E(x)| = 1$.

We know that Carnot groups are Ahlfors Q-regular spaces, by (1.10). Moreover, see for instance Proposition 11.17 in [12], they support a 1-Poincaré inequality, thus applying the theory in [1] to the particular case of Carnot groups we obtain that the perimeter measure is asymptotically doubling and it is concentrated on $\partial^* E$:

Theorem 1.6 (4.16 of [2])**.** *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then $|D\chi_E|$ is concentrated on $\partial^* E$. In addition $|D\chi_E|$ is asymptotically doubling, and more precisely the following property holds: for $|D\chi_E|$-a.e. $x \in \mathbb{G}$ there exists $\overline{r}(x) > 0$ satisfying*

$$\ell_{\mathbb{G}} r^{Q-1} \leq |D\chi_E|(B_r(x)) \leq L_{\mathbb{G}} r^{Q-1} \qquad \forall r \in (0, \overline{r}(x)), \tag{1.18}$$

with $\ell_{\mathbb{G}} > 0$ and $L_{\mathbb{G}}$ depending on $\mathbb{G}$ only.

Remark 3. Therem 1.6 also implies, by the density estimates (1.12), (1.13), an upper and lower bound for $|D\chi_E|$ with respect to the spherical Hausdorff measure $\mathcal{S}^{Q-1}$, i.e.,

$$\frac{\ell_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1}(B \cap \partial^* E) \leq |D\chi_E|(B) \leq \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1}(B \cap \partial^* E), \tag{1.19}$$

for all Borel sets $B \subseteq \mathbb{G}$. A similar result holds also for $\mathcal{H}^{Q-1}$, since $\mathcal{H}^{Q-1} \leq \mathcal{S}^{Q-1} \leq 2^{Q-1}\mathcal{H}^{Q-1}$.

Remark 4. The density lower bounds at points $x \in \mathcal{F}E$ in [11] imply that $\mathcal{F}E$ is contained in $\partial^* E$; on the other hand, since we know from Theorem 1.3 and the asymptotic doubling estimates that $|D\chi_E|$ is concentrated on $\mathcal{F}E$, we can choose $B = \partial^* E \setminus \mathcal{F}E$ in (1.19) to obtain that $\mathcal{H}^{Q-1}(\partial^* E \setminus \mathcal{F}E) = 0$.

1.5. **Tangent sets.** Now we define, as in Definition 5.1 of [2], the tangent set at x of the set $E \subset \mathbb{G}$:

Definition 11 (Tangent set)**.** Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \mathcal{F}E$. Denote by $\operatorname{Tan}(E, x)$ all limit points, in the topology of local convergence in measure, of the translated and rescaled family of sets $\left\{\delta_{1/r}(x^{-1}E)\right\}_{r>0}$ as $r \downarrow 0$.
If $F \in \operatorname{Tan}(E, x)$ we say that F is tangent to E at x. We also set

$$\operatorname{Tan}(E) := \bigcup_{x \in \mathcal{F}E} \operatorname{Tan}(E, x).$$

We also define the iterated tangent sets $\operatorname{Tan}^k(E, x)$ as

$$\operatorname{Tan}^{k+1}(E, x) := \bigcup \left\{\operatorname{Tan}(E^k) : \ E^k \in \operatorname{Tan}^k(E, x)\right\}.$$

The following proposition provides a first list of properties fulfilled by all tangent sets.

Proposition 1.7 ([11]). *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E$ the following properties hold:*

(1) $0 < \liminf_{r\downarrow 0} |D\chi_E|(B_r(x))/r^{Q-1} \le \limsup_{r\downarrow 0} |D\chi_E|(B_r(x))/r^{Q-1} < \infty$;
(2) *if $(r_i) \downarrow 0$ there exists a subsequence $(r_{i(k)})$ such that $\delta_{1/r_{i(k)}}(x^{-1}E)$ locally converge in measure, so that in particular $\mathrm{Tan}(E,x) \neq \emptyset$;*
(3) *if $E_i \to F$ locally in measure, with $E_i = \delta_{1/r_i}(x^{-1}E)$ and (r_i) infinitesimal, then $|D\chi_{E_i}|$ and $D\chi_{E_i}$ weakly* converge respectively to $|D\chi_F|$ and $D\chi_F$;*
(4) *for all $E_1 \in \mathrm{Tan}(E,x)$ we have that $e \in \mathrm{supp}\,|D\chi_{E_1}|$ and*

$$\nu_{E_1}(y) = \nu_E(x) \qquad \text{for } |D\chi_{E_1}|\text{-a.e. } y \in \mathbb{G}.$$

In particular $V_1 \cap \mathrm{Inv}_0(E_1)$ coincides with the codimension 1 subspace of V_1

$$\left\{ \sum_{i=1}^m a_i X_i : \ \sum_{i=1}^m a_i \nu_{E,i}(x) = 0 \right\}$$

and, setting $X := \sum_{i=1}^m \nu_{E,i}(x) X_i \in \mathfrak{g}$, $X\chi_{E_1}$ is a nonnegative Radon measure.

The next Lemma, proved in Lemma 5.8 of [2], shows how to build an invariant direction starting from a regular one.

Lemma 1.8. *Let $E \subseteq \mathbb{G}$ be of locally finite perimeter, let $Z = \sum_{i=1}^l v_i \ \in \mathrm{Reg}(E)$, where $v_i \in V_i$. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E$, $v_l \in \mathrm{Inv}_0(F)$ for all $F \in \mathrm{Tan}(E,x)$.*

Now we state a Lemma which gives us an iterating process to increase the dimension, in higher tangents, of the set of invariant measures, the proof is based on Proposition 1.2 and Lemma 1.8 and it makes more precise the arguments implicit in Lemma 5.9 of [2].

Lemma 1.9. *Let $E \subseteq \mathbb{G}$ be of locally finite perimeter such that*

$$\dim(\mathrm{span}(\mathrm{Inv}_0(E))) \le n-2$$

and assume that $\mathrm{Inv}(E)$ has codimension 1 in V_1. Then for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E$ we have

$$\mathrm{span}(\mathrm{Inv}_0(E_1)) \supsetneq \mathrm{span}(\mathrm{Inv}_0(E)) \qquad \text{for all } E_1 \in \mathrm{Tan}(E,x).$$

Proof. As in Lemma 5.9 of [2], first we prove the existence of

$$Z \in \mathrm{Reg}(E) \setminus \big(\mathrm{span}(\mathrm{Inv}_0(E)) + V_1\big).$$

Indeed, applying Proposition 1.2 with $\mathfrak{g}' = \mathrm{span}(\mathrm{Inv}_0(E))$ and $X = \sum_1^m \nu_{E,i}(x) X_i$ we obtain $X' \in \mathfrak{g}'$ such that

$$Z := Ad_{\exp(X')}(X) \notin \mathrm{span}(\mathrm{Inv}_0(E)) \oplus \{\mathbb{R}X\} = \mathrm{span}(\mathrm{Inv}_0(E)) + V_1,$$

where the equality follows by the codimension 1 property. By Proposition 1.4(2) we have that $Z \in \mathrm{Reg}(E)$.

Since Z has no horizontal component we can write

$$Z = v_{i_1} + \cdots + v_{i_l}, \quad v_{i_j} \in V_{i_j}, i_j \ge 2, \quad i_j \le i_{j+1}.$$

Now choose the largest k such that $v_{i_k} \notin \mathrm{Inv}_0(E)$ and consider $Z' := v_{i_1} + \cdots + v_{i_k}$. Notice that Z' still belongs to $\mathrm{Reg}(E)$, because $v_{i_{k+1}} + \cdots + v_{i_l} \in \mathrm{span}(\mathrm{Inv}_0(E))$. Choose a point $x \in \mathcal{F}E$ where Lemma 1.8 holds (with $Z = Z'$) to obtain that $v_{i_k} \in \mathrm{Inv}_0(E_1)$ for all $E_1 \in \mathrm{Tan}(E,x)$. On the other hand, since

$$\mathrm{span}(\mathrm{Inv}_0(E)) \cap V_j \subseteq \mathrm{Inv}(E) \cap V_j \subseteq \mathrm{Inv}_0(E) \qquad \forall j = 1,\dots,s$$

we have that $v_{i_k} \notin \mathrm{span}(\mathrm{Inv}_0(E))$. □

2. Locality of the perimeter measure

In this section E, $F \subseteq \mathbb{G}$ are sets of finite perimeter. Our strategy to prove the locality property (0.1) is to show first that at $\mathcal{H}^{Q-1}$-a.e. point $x \in \mathcal{F}E \cap \mathcal{F}F$ the unit inner normals coincide up to sign, i.e., $\nu_E(x) = \pm\nu_F(x)$, then the locality property will follow from a further blow-up argument. The equality of normals will be proved first in the case when one set is contained in another, by iterating the tangent operator (which preserves the inclusion) to both sets, until halfspaces are reached. At the level of halfspaces, set-theoretic inclusion obviously implies equality of the normals.

The locality property requires instead a more sophisticated argument: it will be achieved by showing the existence of family of scales on which *both* sets E and F are close to hyperplanes. In order to get this result, we obviously need to generalize Definition 11 of tangent space, considering couples of tangent sets at common scales.

Definition 12. Let $E, F \subseteq \mathbb{G}$ be sets of locally finite perimeter and $x \in \mathcal{F}E \cap \mathcal{F}F$. We denote by $\mathrm{Tan}(E,F,x)$ all limit points, in the topology of local convergence in measure, of the translated and rescaled family of set pairs $(\delta_{1/r}(x^{-1}E), \delta_{1/r}(x^{-1}F))$ as $r \downarrow 0$.
If $(E_1,F_1) \in \mathrm{Tan}(E,F,x)$ we say that (E_1,F_1) is tangent to (E,F) at x. We also set

$$\mathrm{Tan}(E,F) := \bigcup_{x \in \mathcal{F}E \cap \mathcal{F}F} \mathrm{Tan}(E,F,x).$$

Remark 5. By the definition of $\mathrm{Tan}(E,F,x)$ it follows directly that $(E_1,F_1) \in \mathrm{Tan}(E,F,x)$ implies $E_1 \in \mathrm{Tan}(E,x)$ and $F_1 \in \mathrm{Tan}(F,x)$; conversely, if $E_1 \in \mathrm{Tan}(E,x)$, $F_1 \in \mathrm{Tan}(F,x)$, and if

$$\delta_{1/r_i}(x^{-1}E) \to E_1, \qquad \delta_{1/r_i}(x^{-1}F) \to F_1$$

for a *common* sequence $r_i \to 0$, then it follows that $(E_1,F_1) \in \mathrm{Tan}(E,F,x)$. Thus, whenever a property holds for every element of $\mathrm{Tan}(E,x)$ and of $\mathrm{Tan}(F,x)$ we can extend it to $\mathrm{Tan}(E,F,x)$.

Remark 6. As a consequence of Remark 5 and Proposition 1.7, we know that for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$, $\mathrm{Tan}(E,F,x) \neq \emptyset$ and for every element $(E_1,E_2) \in \mathrm{Tan}(E,F,x)$, $\mathrm{Inv}(E_1)$ and $\mathrm{Inv}(F_1)$ have codimension 1 in V_1. Moreover, if we define $X := \sum_{i=1}^m \nu_{E,i}(x)X_i$ and $Y := \sum_{i=1}^m \nu_{F,i}(x)X_i$, we have $X\chi_{E_1} \geq 0$ and $Y\chi_{F_1} \geq 0$.

Lemma 2.1. *Let $E, F \subseteq \mathbb{G}$ as above, let $X \in \mathrm{Reg}(E)$ and $Y \in \mathrm{Reg}(F)$ and write $X = \sum_{i=1}^l v_i$ and $Y = \sum_{i=0}^h w_i$, where $v_i, w_i \in V_i$. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$, $v_l \in \mathrm{Inv}_0(E_1)$ and $w_h \in \mathrm{Inv}_0(F_1)$ for all $(E_1,F_1) \in \mathrm{Tan}(E,F,x)$.*

Proof. The lemma is a direct consequence of Remark 5 and Lemma 1.8. □

As in Definition 5.1 of [2] we can define the iterated tangent spaces.

Definition 13. Let $x \in \mathcal{F}E \cap \mathcal{F}F$. We define $\mathrm{Tan}^1(E, F, x) := \mathrm{Tan}(E, F, x)$ and

$$\mathrm{Tan}^{k+1}(E, F, x) := \bigcup \left\{ \mathrm{Tan}(E_1^k, F_1^k) : \ (E_1^k, F_1^k) \in \mathrm{Tan}^k(E, F, x) \right\}.$$

In order to prove that regular directions become invariant in a sufficiently high tangent space, we need to look at the reduced boundary of a couple in $\mathrm{Tan}(E, F, x)$.

Proposition 2.2. *For $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ the following property holds: for all $(E_1, F_1) \in \mathrm{Tan}(E, F, x)$ the reduced boundaries $\mathcal{F}E_1$ and $\mathcal{F}F_1$ are $\mathcal{H}^{Q-1}$ equivalent (i.e., their symmetric difference is $\mathcal{H}^{Q-1}$-negligible).*

Proof. Let us prove the inclusion $\mathcal{F}E_1 \subseteq \mathcal{F}F_1$ up to $\mathcal{H}^{Q-1}$-negligible sets (the proof of the opposite one being similar). Set $E_{x,r} = \delta_{1/r}(x^{-1}E)$, $F_{x,r} = \delta_{1/r}(x^{-1}F)$. By (1.19) and Remark 4 we have $|D\chi_{E_{x,r}}| \leq \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1} \llcorner \mathcal{F}E_{x,r}$. We decompose the right hand side of the previous inequality in two terms, one concentrated on $\mathcal{F}F_{x,r}$ and the other one concentrated on $\mathcal{F}E_{x,r} \setminus \mathcal{F}F_{x,r}$:

$$|D\chi_{E_{x,r}}| \leq \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1} \llcorner \mathcal{F}F_{x,r} + \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1} \llcorner (\mathcal{F}E_{x,r} \setminus \mathcal{F}F_{x,r}).$$

Using again (1.19) we have

$$|D\chi_{E_{x,r}}| \leq \frac{L_{\mathbb{G}}}{\ell_{\mathbb{G}}} |D\chi_{F_{x,r}}| + \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathcal{S}^{Q-1} \llcorner (\mathcal{F}E_{x,r} \setminus \mathcal{F}F_{x,r}). \tag{2.20}$$

If we prove that the second term in the right hand part of (2.20) is infinitesimal (i.e., its mass in any ball $B_R(e)$ is infinitesimal), choosing a sequence $(r_i) \downarrow 0$ such that $(E_{x,r_i}, F_{x,r_i}) \to (E_1, F_1)$ we get $|D\chi_{E_1}| \leq \frac{L_{\mathbb{G}}}{\ell_{\mathbb{G}}} |D\chi_{F_1}|$, hence (1.19) again gives $\mathcal{F}E_1 \subset \mathcal{F}F_1$ up to $\mathcal{H}^{Q-1}$-negligible sets.

Consider the measure $\mathcal{S}^{Q-1} \llcorner \mathcal{F}E$ which is asymptotically doubling as a consequence of (1.18) and (1.19). Choose $L = \mathbb{G} \setminus \mathcal{F}F$ in Remark 2, so that

$$\lim_{r \downarrow 0} \frac{\mathcal{S}^{Q-1}(B_r(x) \cap \mathcal{F}E \setminus \mathcal{F}F)}{\mathcal{S}^{Q-1}(B_r(x) \cap \mathcal{F}E)} = 0 \tag{2.21}$$

for $\mathcal{S}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$. By the scaling property of $\mathcal{S}^{Q-1}$ and the reduced boundary we have

$$\frac{\mathcal{S}^{Q-1}(B_R(e) \cap \mathcal{F}E_{x,r} \setminus \mathcal{F}F_{x,r})}{R^{Q-1}} = \frac{\mathcal{S}^{Q-1}(B_{Rr}(x) \cap \mathcal{F}E \setminus \mathcal{F}F)}{\mathcal{S}^{Q-1}(B_{Rr}(x) \cap \mathcal{F}E)} \frac{\mathcal{S}^{Q-1}(B_{Rr}(x) \cap \mathcal{F}E)}{(Rr)^{Q-1}}.$$

Choosing a point x where (2.21) holds, and taking into account that $\mathcal{S}^{Q-1}(B_r(x) \cap \mathcal{F}E)/r^{Q-1}$ is bounded above as $r \downarrow 0$ we obtain

$$\mathcal{S}^{Q-1}(B_R(e) \cap \mathcal{F}E_{x,r} \setminus \mathcal{F}F_{x,r}) \to 0 \qquad \forall R > 0.$$

□

The following Lemma is the fundamental step in order to prove the locality property: we can use it to start an iterating process that leads to the existence of a pair of vertical halfspaces in the iterated tangent sets $\mathrm{Tan}^k(E, F, x)$.

Lemma 2.3. *Let $E, F \subseteq \mathbb{G}$ be sets of locally finite perimeter such that*

$$i := \min\{\dim(\mathrm{span}(\mathrm{Inv}_0(E))), \dim(\mathrm{span}(\mathrm{Inv}_0(F)))\} \leq n-2$$

and assume that $\mathrm{Inv}(E)$, $\mathrm{Inv}(F)$ *have codimension* 1 *in* V_1. *Then, for* $\mathcal{H}^{Q-1}$*-a.e* $x \in \mathcal{F}E \cap \mathcal{F}F$, *for all* $(E_1, F_1) \in \mathrm{Tan}(E, F, x)$ *we have*

$$\min\{\dim(\mathrm{span}(\mathrm{Inv}_0(E_1))), \dim(\mathrm{span}(\mathrm{Inv}_0(F_1)))\} > i.$$

Proof. Assume first that *both* $\mathrm{span}(\mathrm{Inv}_0(E))$ and $\mathrm{span}(\mathrm{Inv}_0(F))$ have dimension less than $n-2$. Choose $\bar{x} \in \mathcal{F}E \cap \mathcal{F}F$ such that Lemma 1.9 holds for E and F, this means

$$\mathrm{span}(\mathrm{Inv}_0(E_1)) \supsetneq \mathrm{span}(\mathrm{Inv}_0(E)) \quad \forall\, E_1 \in \mathrm{Tan}(E, x). \tag{2.22}$$

$$\mathrm{span}(\mathrm{Inv}_0(F_1)) \supsetneq \mathrm{span}(\mathrm{Inv}_0(F)) \quad \forall\, F_1 \in \mathrm{Tan}(F, x). \tag{2.23}$$

Combining (2.22), (2.23) we get that for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ we have $\mathrm{span}(\mathrm{Inv}_0(E_1)) \supsetneq \mathrm{span}(\mathrm{Inv}_0(E))$ and $\mathrm{span}(\mathrm{Inv}_0(F_1)) \supsetneq \mathrm{span}(\mathrm{Inv}_0(F))$ for all $(E_1, F_1) \in \mathrm{Tan}(E, F, x)$.

If one of the dimensions, say the one of $\mathrm{span}(\mathrm{Inv}_0(E))$, exceeds $n-2$, then it must be $n-1$ and E is an halfspace. Since vertical halfspaces H are self-similar (i.e., $x^{-1}H = H$ for all $x \in \partial H$ and $\delta_\lambda H = H$ for all $\lambda > 0$) iterating once more the blow-up procedure improves the number of invariant directions of the second set of the pair. □

Remark 7. The condition on the codimension of $\mathrm{Inv}(E)$ and $\mathrm{Inv}(F)$ in V_1 will be automatically satisfied when dealing with tangent spaces, by Proposition 1.7.

Theorem 2.4. *Let $E, F \subseteq \mathbb{G}$ be sets of locally finite perimeter. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$*

$$(H_{0,\nu_E(x)}, H_{0,\nu_F(x)}) \in \mathrm{Tan}^k(E, F, x) \quad \textit{with} \quad k := 1 + (n-m).$$

Proof. By Proposition 1.7, sets in $\mathrm{Tan}(E, F, x)$ are invariant in at least $m-1$ directions and have at least a non-invariant direction, provided by the inner normals $\nu_E(x)$, $\nu_F(x)$. Define the integers $w(T) = \dim(\mathrm{span}(\mathrm{Inv}_0(T)))$, $T \subseteq \mathbb{G}$, and i_k as follows:

$$i_k = \max\{\min\{w(E_1), w(F_1)\},\ (E_1, F_1) \in \mathrm{Tan}^k(E, F, x)\}, \qquad k \geq 1.$$

Then $i_1 \geq m-1$, and by Lemma 2.3 if follows that $i_{k+1} > i_k$ as long as $i_k \leq n-2$. Indeed, Proposition 2.2 ensures that the reduced boundaries of sets $(E_{k+1}, F_{k+1}) \in \mathrm{Tan}(E_k, F_k, y_k)$ (for $k \geq 0$, $E_0 = E$, $F_0 = F$) are $\mathcal{H}^{Q-1}$-equivalent, so we can repeatedly apply Lemma 2.3. On the other hand, if $i_k = n-1$, then there exists $(E_1, F_1) \in \mathrm{Tan}^k(E, F, x)$ with $w(E_1) = w(F_1) = n-1$, hence both E_1 and F_1 are halfspaces. □

Corollary 2.5. *Let $F \subseteq E \subseteq \mathbb{G}$ be sets of locally finite perimeter. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ we have $\nu_E(x) = \nu_F(x)$.*

Proof. Consider the set $N \subset \mathcal{F}E \cap \mathcal{F}F$ where the property stated in Theorem 2.4 holds and fix an $x \in N$. Then $(H_{0,\nu_E(x)}, H_{0,\nu_F(x)}) \in \mathrm{Tan}^k(E, F, x)$ for some k. Since $L \subseteq M$ implies $L_1 \subseteq M_1$ whenever $(L_1, M_1) \in \mathrm{Tan}(L, M, y)$, we know that $H_{0,\nu_F(x)} \subseteq H_{0,\nu_E(x)}$. In exponential coordinates this means

$$\left\{\sum_{i=1}^n y_i X_i :\ \sum_{i=1}^m y_i \nu_{F,i}(x) \leq 0\right\} \subseteq \left\{\sum_{i=1}^n y_i X_i :\ \sum_{i=1}^m y_i \nu_{E,i}(x) \leq 0\right\},$$

where $(X_1, \dots, X_n)$ is a basis of $\mathfrak{g}$ extending the basis $(X_1, \dots, X_m)$ of V_1. This implies $\nu_E(x) = \nu_F(x)$. □

In order to prove the locality property we need to extend the previous corollary to all possible sets $E, F \subseteq \mathbb{G}$ of locally finite perimeter, not necessarily one contained in the other.

Corollary 2.6. *Let $F, E \subseteq \mathbb{G}$ be sets of locally finite perimeter. Then for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ we have $\nu_E(x) = \pm\nu_F(x)$.*

Proof. Consider the set $E \cap F$, clearly $E \cap F \subseteq F$ and the same holds for E. Then Corollary 2.5 yields

$$\nu_{E\cap F} = \nu_E \quad \mathcal{H}^{Q-1}\text{-a.e. on } \mathcal{F}(E\cap F)\cap\mathcal{F}E, \qquad \nu_{E\cap F} = \nu_F \quad \mathcal{H}^{Q-1}\text{-a.e. on } \mathcal{F}(E\cap F)\cap\mathcal{F}F.$$

Noticing that $\nu_F = -\nu_{\mathbb{G}\setminus F}$, Corollary 2.5 also yields

$$\nu_{E\setminus F} = \nu_E \quad \mathcal{H}^{Q-1}\text{-a.e. on } \mathcal{F}(E\setminus F)\cap\mathcal{F}E, \qquad \nu_{E\setminus F} = -\nu_F \quad \mathcal{H}^{Q-1}\text{-a.e. on } \mathcal{F}(E\setminus F)\cap\mathcal{F}F.$$

Since analogous relations hold for $F \setminus E$, it follows that $\nu_E = \pm\nu_F$ $\mathcal{H}^{Q-1}$-a.e. on $(\mathcal{F}(E\cap F) \cup \mathcal{F}(E\setminus F) \cup \mathcal{F}(F\setminus E)) \cap (\mathcal{F}E \cap \mathcal{F}F)$. It remains to show that

$$\mathcal{F}(E\cap F) \cup \mathcal{F}(E\setminus F) \cup \mathcal{F}(F\setminus E) \supseteq \mathcal{F}E \cap \mathcal{F}F$$

up to $\mathcal{H}^{Q-1}$-negligible sets. In order to show this fact we shall work with the essential boundaries, equivalent in $\mathcal{H}^{Q-1}$-measure to the reduced boundaries by Remark 4. Writing $E = (E\cap F) \cup (E \setminus F)$ since the union is disjoint we have the obvious inclusion

$$\partial^* E \subseteq \partial^*(E\cap F) \cup \partial^*(E\setminus F),$$

and the same relation holds also for F. Therefore taking the intersection we get

$$\partial^* E \cap \partial^* F \subseteq \partial^*(E\cap F) \cap (\partial^*(E\setminus F) \cup \partial^*(F\setminus E)),$$

and the proof is complete. □

By the locality property of the outer normal, a blow-up argument and a measure differentiation we can prove the locality property of the perimeter measure.

We say that a measure $\mu \in \mathcal{M}^m(\mathbb{G})$ is *asymptotically q-regular* if

$$0 < \liminf_{r\downarrow 0} \frac{|\mu|(B_r(x))}{r^q} \le \limsup_{r\downarrow 0} \frac{|\mu|(B_r(x))}{r^q} < +\infty \qquad \text{for } |\mu|\text{-a.e. } x \in \mathbb{G}. \tag{2.24}$$

Notice that asymptotically q-regular measures are asymptotically doubling, and that the perimeter measure $|D\chi_E|$ is asymptotically $(Q-1)$-regular, thanks to Theorem 1.6.

In the sequel we shall denote the scaling map $y \mapsto \delta_{1/r}(x^{-1}y)$ by $I_{x,r}$.

Definition 14 (Tangents to a measure)**.** Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. We denote by $\mathrm{Tan}(\mu, x)$ the family of all measures $\nu \in \mathcal{M}^m(\mathbb{G})$ that are weak* limit point as $r \downarrow 0$ of the family of measures $r^{-q}(I_{x,r})_\sharp\mu$.

The following theorem shows the principle that iterated tangents are tangents, see [14] or [13, Theorem 14.16]; see Theorem 6.4 of [2] for the proof of the statement given below, involving vector-valued measures in Carnot groups.

Theorem 2.7 ([2]). *Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. Then, for $|\mu|$-a.e. x, the following property holds:*

$$\mathrm{Tan}(\nu, y) \subseteq \mathrm{Tan}(\mu, x) \qquad \forall \nu \in \mathrm{Tan}(\mu, x),\ \forall y \in \mathrm{supp}\,|\nu|.$$

Theorem 2.8. *Let $F, E \subseteq \mathbb{G}$ be sets with locally finite perimeter. Then for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ we have*

$$\bigcup_{k=1}^{\infty} \mathrm{Tan}^k(E, F, x) \subseteq \mathrm{Tan}(E, F, x).$$

Proof. Consider the vector-valued measure $\mu = (D\chi_E, D\chi_F) \in \mathcal{M}^{2m}(\mathbb{G})$. Then, for $x \in \mathcal{F}E \cap \mathcal{F}F$ we have the equivalence

$$(E_1, F_1) \in \mathrm{Tan}(F, E, x) \Longleftrightarrow (D\chi_{E_1}, D\chi_{F_1}) \in \mathrm{Tan}(\mu, x),\ D\chi_{E_1} \neq 0,\ D\chi_{F_1} \neq 0. \tag{2.25}$$

Indeed, assume without loss of generality that $x = e$ and that $(D\chi_{E_1}, D\chi_{F_1})$ is the weak* limit of $(r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_E, r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_F)$, with $r_i \downarrow 0$ as $i \to \infty$ and $D\chi_{E_1} \neq 0$, $D\chi_{F_1} \neq 0$. Set $E_i = \delta_{1/r_i}E$, $F_i = \delta_{1/r_i}F$, by the compactness properties of finite perimeter sets we can assume that $(E_i, F_i) \to (E', F')$ locally in measure. Then $(r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_E, r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_F) = (D\chi_{E_i}, D\chi_{F_i})$ weakly* converge to $(D\chi_{E'}, D\chi_{F'})$ so that $(D\chi_{E'}, D\chi_{F'}) = (D\chi_{E_1}, D\chi_{F_1})$. Since $\chi_{E_1} - \chi_{E'}$ has zero horizontal distributional derivative, it is equivalent to a constant (here we use the validity of the Sobolev-Poincarè inequality in $BV(\mathbb{G})$). This holds only when $E_1 = E'$ or $E_1 = \mathbb{G} \setminus E'$ and the second possibility is ruled out because it implies $0 = D\chi_{E_1} - D\chi_{E'} = 2D\chi_{E_1}$. Clearly the same is true also for F_1 and we have proved that $(E_1, F_1) \in \mathrm{Tan}(E, F, e)$. The converse implication follows easily by a scaling argument.

Let $x \in \mathcal{F}E \cap \mathcal{F}F$ be satisfying the property stated in Theorem 2.7 with $\mu = (D\chi_E, D\chi_F)$. Consider $(E_1, F_1) \in \mathrm{Tan}(E, F, x)$ and $(E_2, F_2) \in \mathrm{Tan}(E_1, F_1, y)$ for some $y \in \mathcal{F}E_1 \cap \mathcal{F}F_1$. By (2.25) we know that $(D\chi_{E_2}, D\chi_{F_2}) \in \mathrm{Tan}((D\chi_{E_1}, D\chi_{F_1}), y) \setminus \{(0,0)\}$ and $(D\chi_{E_1}, D\chi_{F_1}) \in \mathrm{Tan}(\mu, x) \setminus \{(0,0)\}$ hence $(D\chi_{E_2}, D\chi_{F_2}) \in \mathrm{Tan}(\mu, x)$. By applying (2.25) once more we get $(E_2, F_2) \in \mathrm{Tan}(E, F, x)$, and this ends the proof. □

Theorem 2.9 (Locality property). *Let $E, F \subseteq \mathbb{G}$ be sets of locally finite perimeter. Then*

$$|D\chi_E|(B) = |D\chi_F|(B) \qquad \text{for all } B \subset \mathcal{F}E \cap \mathcal{F}F \text{ Borel.} \tag{2.26}$$

Proof. If we prove that the density $K := |D\chi_E|/|D\chi_F|$ is constant and $\mathcal{H}^{Q-1}$-a.e. equal to 1 in $\mathcal{F}E \cap \mathcal{F}F$ the theorem follows, however we need to clearly define this density. Indeed, $|D\chi_E|$ and $|D\chi_F|$ have different supports, but we can overcome this difficulty.
We first claim that

$$\lim_{r \downarrow 0} \frac{|D\chi_E|((\mathcal{F}E \setminus \mathcal{F}F) \cap B_r(x))}{r^{Q-1}} = 0$$

for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$. Indeed, if $A \subset \mathcal{F}E \cap \mathcal{F}F$ is the Borel set where the property fails, we can find $A' \subseteq A$ and $\epsilon > 0$ such that $\mathcal{S}^{Q-1}(A') > 0$ and the lim sup of the ratio above is larger than ϵ at all $x \in A'$. Then (1.12) gives

$$|D\chi_E|((\mathcal{F}E \setminus \mathcal{F}F) \cap A') \geq \epsilon\omega_{Q-1}\mathcal{S}^{Q-1}(A') > 0.$$

This is impossible because $(\mathcal{F}E \setminus \mathcal{F}F) \cap A' = \emptyset$. Then write

$$|D\chi_E| = |D\chi_E| \llcorner (\mathcal{F}E \cap \mathcal{F}F) + |D\chi_E| \llcorner (\mathcal{F}E \setminus \mathcal{F}F) = \mu_E + \nu_E$$
$$|D\chi_F| = |D\chi_F| \llcorner (\mathcal{F}E \cap \mathcal{F}F) + |D\chi_F| \llcorner (\mathcal{F}E \setminus \mathcal{F}F) = \mu_F + \nu_F.$$

Notice that μ_E is absolutely continuous with respect to μ_F and both measures are asymptotically doubling, by Theorem 1.6 and by the fact that $\nu_E(B_r(x)) = o(r^{Q-1}) = o(\mu_E(B_r(x))))$ and $\nu_F(B_r(x)) = o(r^{Q-1}) = o(\mu_F(B_r(x)))$ at $\mathcal{H}^{Q-1}$-a.e. point. Then we can define $K(x) := \lim_{r\downarrow 0} \mu_E(B_r(x))/\mu_F(B_r(x))$ at points $x \in \mathcal{F}E \cap \mathcal{F}F$ where the limit exists (this happens $\mathcal{H}^{Q-1}$-a.e. on $\mathcal{F}E \cap \mathcal{F}F$ thanks to Theorem 1.3).

Fix a point $x \in \mathcal{F}E \cap \mathcal{F}F$ such that $\nu_E(x) = \nu_F(x)$ (the case $\nu_E(x) = -\nu_F(x)$ is analogous), $K(x)$ is defined and $\nu_E(B_r(x)) = o(\mu_E(B_r(x)))$, $\nu_F(B_r(x)) = o(\mu_F(B_r(x)))$. We obtain that also the quotient $|D\chi_E|(B_r(x))/|D\chi_F|(B_r(x))$ tends to $K(x)$ as $r \downarrow 0$, i.e.,

$$K(x) = \lim_{r\downarrow 0} \frac{\int_{B_r(x)} \langle \nu_E(y), D\chi_E(y)\rangle}{\int_{B_r(x)} \langle \nu_F(y), D\chi_F(y)\rangle}.$$

If we replace $\nu_E(y)$ with the constant $\nu_E(x)$ in the numerator, and $\nu_F(y)$ with $\nu_F(x)$ in the denominator, we still have

$$K(x) = \lim_{r\downarrow 0} \frac{\int_{B_r(x)} \langle \nu_E(x), d\,D\chi_E(y)\rangle}{\int_{B_r(x)} \langle \nu_F(x), d\,D\chi_F(y)\rangle}$$

because

$$\left| \int_{B_r(x)} \langle \nu_E(y) - \nu_E(x), d\,D\chi_E(y)\rangle \right| = \left| \int_{B_r(x)} \langle \nu_E(y) - \nu_E(x), \nu_E(y)\rangle \, d\,|D\chi_E| \right|$$
$$= |D\chi_E|(B_r(x)) - \left\langle \nu_E(x), \int_{B_r(x)} \nu_E(y) d\,|D\chi_E| \right\rangle = o(|D\chi_E|(B_r(x)))$$

and the same holds for F. Changing variables one has

$$K(x) = \lim_{r\downarrow 0} \frac{\int_{B_1(e)} \left\langle \nu_E(x), d\,D\chi_{E_{x,r}}(y)\right\rangle}{\int_{B_1(e)} \left\langle \nu_F(x), d\,D\chi_{F_{x,r}}(y)\right\rangle}$$

where $E_{x,r}$, $F_{x,r}$ are the left translated and rescaled sets of E and F respectively and e is the identity of the group. Since the limit exists, we can choose *any* sequence $(r_i) \downarrow 0$ to compute $K(x)$.
Setting $H := H_{0,\nu_E(x)} = H_{0,\nu_F(x)}$, we choose r_i such that $(E_{x,r_i}, F_{x,r_i}) \to (H, H)$. We know that

$$D\chi_{E_{x,r_i}} \rightharpoonup D\chi_H, \quad D\chi_{F_{x,r_i}} \rightharpoonup D\chi_H, \quad |D\chi_{E_{x,r_i}}| \rightharpoonup |D\chi_H|, \quad |D\chi_{F_{x,r_i}}| \rightharpoonup |D\chi_H|,$$

and the thesis follows if we prove that both $D\chi_{E_{x,r_i}}(B_1(e))$ and $D\chi_{F_{x,r_i}}(B_1(e))$ converge to $D\chi_H(B_1(e))$. Indeed, since $|D\chi_H|(\partial B_s(e)) = 0$ with at most countable many exceptions (by the finiteness of $|D\chi_H|$), by scaling invariance of H we have $|D\chi_H|(\partial B_1(e)) = 0$. Thus, applying Proposition 1.62(b) in [3] we get the desired convergence property. Therefore $K(x) = 1$ and the theorem is proved. □

2.1. **Consequences of locality.** As a first consequence we can define a surface measure ρ_S associated to Borel sets S contained in a countable union of essential boundaries of sets of locally finite perimeter.

Proposition 2.10. *Let S be a Borel set contained in a countable union of sets $\partial^* E_i$, with E_i of locally finite perimeter. Then there exists a unique σ-additive Borel measure ρ_S satisfying*

$$\rho_S(B) = |D\chi_E|(B) \qquad \text{for all } E \text{ with locally finite perimeter, } B \subseteq S \cap \partial^* E \text{ Borel.} \tag{2.27}$$

Proof. Write S a disjoint union of Borel sets S_i, each one contained in $\partial^* E_i$, with E_i of locally finite perimeter, and define

$$\rho_S(B) := \sum_i |D\chi_{E_i}|(S_i \cap B).$$

In order to check (2.27), fix a set with locally finite perimeter E and a Borel set $B \subseteq S \cap \partial^* E$ and assume, by σ-additivity of both sides, that $B \subseteq S_i$ for some i. Then the identity reduces to $|D\chi_{E_i}|(B) = |D\chi_E|(B)$, which follows by the locality property. The uniqueness of ρ_S is a direct consequence of (2.27) with $E = E_i$. □

3. A chain rule for BV functions on Carnot groups

In this section we present some fine properties of BV functions in Carnot groups, we refer to [4] for the general theory of BV functions on doubling metric measure spaces, for the theory in the Euclidean case we refer to the book [3].

Definition 15. Let $u : \mathbb{G} \to \mathbb{R}$ be a measurable function and let $x \in \mathbb{G}$; we define the upper and lower approximate limits of u at x respectively by

$$u^\vee(x) = \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{\rho \downarrow 0} \frac{\mathrm{vol}_{\mathbb{G}}(\{u > t\} \cap B_\rho(x))}{\mathrm{vol}_{\mathbb{G}}(B_\rho(x))} = 0 \right\}$$
$$u^\wedge(x) = \sup \left\{ t \in \overline{\mathbb{R}} : \lim_{\rho \downarrow 0} \frac{\mathrm{vol}_{\mathbb{G}}(\{u < t\} \cap B_\rho(x))}{\mathrm{vol}_{\mathbb{G}}(B_\rho(x))} = 0 \right\}.$$

If $u^\vee(x) = u^\wedge(x)$ we call their common value, denoted by $\tilde{u}(x)$, the approximate limit of u at x. We also set $S_u = \{x \mid u^\vee(x) > u^\wedge(x)\}$, the discontinuity set of u.

When $u = \chi_E$, then obviously $S_u = \partial^* E$. We have the following useful characterization of S_u:

Proposition 3.1 ([4])**.** *Let $u : \mathbb{G} \to \mathbb{R}$ be a measurable function then*

$$S_u = \bigcup_{t,s \in D, s \neq t} \partial^* \{u > t\} \cap \partial^* \{u > s\}, \tag{3.28}$$

where $D \subset \mathbb{R}$ is any dense set. In particular if $u \in BV(\mathbb{G})$ we can choose D such that for every $s \in D$ the set $\{u > s\}$ has finite perimeter. Furthermore, we have the implications:

$$t \in (u^\wedge(x), u^\vee(x)) \quad \Rightarrow \quad x \in \partial^* \{u > t\} \quad \Rightarrow \quad t \in [u^\wedge(x), u^\vee(x)]\,. \tag{3.29}$$

Remark 8. Assume that $u \in BV(\mathbb{G})$. Arguing as in Proposition 2.10, using the locality of the normal provided by Theorem 2.5 and the representation of S_u as a countable union of intersections of essential boundaries of sets of finite perimeter, one can define a Borel map $\nu_u : S_u \to \mathbf{S}^{m-1}$ such that

$$\nu_u = \nu_{\{u>t\}} \qquad \mathcal{H}^{Q-1}\text{-a.e. on } S_u \cap \partial^*\{u > t\} \tag{3.30}$$

for all $t \in \mathbb{R}$ such that $\{u > t\}$ has finite perimeter. It is also easy to check that (3.30) characterizes uniquely ν_u, up to $\mathcal{H}^{Q-1}$-negligible sets.

Proposition 3.2. *Let $u \in BV(\mathbb{G})$, then for all Borel set $B \subseteq S_u$ we have*

$$|Du|(B) = \int_B (u^\vee(x) - u^\wedge(x))\, d\rho_{S_u}(x), \qquad Du(B) = \int_B (u^\vee(x) - u^\wedge(x))\nu_u(x) d\rho_{S_u}(x) \tag{3.31}$$

where ρ_{S_u} and ν_u are defined respectively in (2.27) and (3.30)*.*

Proof. Set $E_t = \{u > t\}$. Notice first that $S_u = \bigcup_{s,t\in D,\ t\neq s} \partial^* E_s \cap \partial^* E_t$, by the characterization of S_u given in Proposition 3.1. Thus, for the set $S := S_u$ the measure ρ_S is defined. If we apply the coarea formula to $|Du|$, for every Borel set $B \subseteq S$ we have

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{E_t}|(B)\ dt. \tag{3.32}$$

Clearly, by the definition of ρ_S, we get

$$|D\chi_{E_t}|(B) = \rho_S(B \cap \partial^* E_t),$$

so that we can rewrite (3.32) as

$$|Du|(B) = \int_{-\infty}^{\infty} \rho_S(B \cap \partial^* E_t)\ dt.$$

Using Fubini's theorem and (3.29), we get

$$|Du|(B) = \int_B \int_{-\infty}^{\infty} \chi_{\{s: x\in\partial^* E_s\}}(t)\ dt d\rho_S(x) = \int_B \int_{u^\wedge(x)}^{u^\vee(x)} dt d\rho_S(x) = \int_B (u^\vee(x) - u^\wedge(x)) d\rho_S(x).$$

The proof of the second identity in (3.31) is analogous, and uses the identity $Du = \int_{-\infty}^{\infty} D\chi_{E_t}\, dt$ and (3.30). □

Now we prove the chain rule for BV functions on $\mathbb{G}$:

Proposition 3.3. *Let $u \in BV(\mathbb{G})$. Then for every $\psi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ the function $\psi \circ u$ belongs to $BV_{\mathrm{loc}}(\mathbb{G})$ and*

$$D(\psi \circ u) = \psi'(\tilde{u}(x)) Du \llcorner (\mathbb{G} \setminus S_u) + [\psi(u^\vee) - \psi(u^\wedge)]\nu_{S_u}\rho_{S_u}. \tag{3.33}$$

Proof. We can assume with no loss of generality $\psi(0) = 0$. Under this assumption $\psi \circ u \in L^1(\mathbb{G})$ and it is easy to prove that $\psi \circ u \in BV(\mathbb{G})$: indeed, by the Meyers-Serrin type theorem for anisotropic Sobolev spaces proved in [9], we can find a sequence $(u_h) \subseteq C^1(\mathbb{G})$ with $u_h \to u$ in $L^1(\mathbb{G})$ and $|Du_h|(\mathbb{G}) \to |Du|(\mathbb{G})$. Since for C^1 functions v the total variation $|Dv|$ is the L^1 norm of the horizontal gradient, the classical chain rule gives that

$|D(\psi \circ u_h)|(\mathbb{G})$ is uniformly bounded. As a consequence, the $L^1(\mathbb{G})$ limit of $\psi \circ u_h$, namely $\psi \circ u$, belongs to $BV(\mathbb{G})$.

Since any $\psi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ can be written as the difference of two strictly increasing functions with the same properties, by the linearity of (3.33) we can assume in the rest of the proof that ψ is strictly increasing. Under this assumption, $S_{\psi \circ u} = S_u$, $\nu_{\psi \circ u} = \nu_u$, $(\psi \circ u)^\vee = \psi(u^\vee)$, $(\psi \circ u)^\wedge = \psi(u^\wedge)$, hence the validity of (3.33) on Borel sets $B \subseteq S_u$ is a direct consequence of (3.31).

Let now $B \subset \mathbb{G} \setminus S_u$ be a Borel set. Appling the coarea formula to $\psi \circ u$ one has

$$D(\psi \circ u)(B) = \int_{-\infty}^{\infty} D\chi_{E_t}(B)\ dt,$$

where $E_t = \{\psi \circ u > t\}$. The change of variables $t = \psi(s)$ gives

$$\int_{-\infty}^{\infty} D\chi_{E_t}(B)\ dt = \int_{-\infty}^{\infty} \psi'(s) D\chi_{\{u>s\}}(B)\ ds.$$

Since B does not intersect S_u, for $x \in B$ we have that $x \in \partial^* \{u > s\}$ only if $s = \tilde{u}(x)$ (by the definitions of $u^\vee$ and $u^\wedge$), thus we can rewrite the integral as

$$\int_{-\infty}^{\infty} \psi'(s) D\chi_{\{u>s\}}(B)\ ds = \int_{-\infty}^{\infty} \int_B \psi'(\tilde{u}(x))\ dD\chi_{\{u>s\}}\ ds.$$

Finally, using the coarea formula once more we have

$$D(\psi \circ u)(B) = \int_B \psi'(\tilde{u})\ dDu.$$

□

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