WEAK FORMULATION OF ELASTODYNAMICS IN DOMAINS WITH GROWING CRACKS

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Abstract. In this paper we formulate and study the system of elastodynamics on domains with arbitrary growing cracks. This includes homogeneous Neumann conditions on the crack sets and mixed general Dirichlet-Neumann conditions on the boundary. The only assumptions on the crack sets are to be $(n-1)$-rectifiable with finite surface measure, and increasing in the sense of set inclusions. In particular they might be dense, hence the weak formulation must fall outside the usual context of Sobolev spaces and Korn's inequality. We prove existence of a solution both for the damped and undamped systems, while in the damped case we are also able to prove uniqueness and an energy balance.

Keywords: second order linear hyperbolic system, dynamic fracture mechanics, cracking domains, boundary conditions, bounded deformation

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1. Introduction

The theory of dynamic fracture mechanics contains basically three principles that can be resumed as follows

- elastodynamics off the cracks;
- energy-dissipation balance which includes also the surface energy dissipated by the crack;
- a principle dictating when a crack must grow.

For the first two conditions we refer to [15], while the third one is discussed in [16] in some more details and a maximal dissipation condition is proposed.

In this paper we focus on the first issue. Precisely we fix a time interval $[0, T]$ and we consider $\Omega \subset \mathbb{R}^n$ a regular domain as reference configuration, a *fixed* family of growing-in-time crack sets $\Gamma(t)$ contained in Ω, and $u(t, x)$ the displacement which might be essentially discontinuous for $x \in \Gamma(t)$. Then, given initial conditions, and mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$, we want to find a solution to the system of (possibly damped) *elastodynamics*

$$\ddot{u}(t) - \operatorname{div}[\mathbb{C}\,\mathcal{E}u(t)] - \gamma \operatorname{div}[\mathbb{B}\,\mathcal{E}\dot{u}(t)] = f(t), \text{ in } \Omega \setminus \Gamma(t) \tag{1}$$

where $\mathbb{C}$ and $\mathbb{B}$ are the elasticity tensor and the viscosity tensor, respectively, $\mathcal{E}u$ denotes the symmetric part of the gradient of u, div denotes the divergence operator acting on the rows of matrices, $f(t)$ is a vector field representing the volume force, and at each time t the system (1) is complemented with homogeneous Neumann condition on the crack $\Gamma(t)$. This last condition reflects the fact that no external forces are acting on the crack lips. The parameter γ can take value only in $\{0,1\}$, and in particular for $\gamma = 1$ the system is called damped, while for $\gamma = 0$ the system is called undamped.

In the corresponding quasi-static models, all the known existence results for the coupled problem $(u(t), \Gamma(t))$ without a-priori assumptions on $\Gamma(t)$, are obtained by minimizing a weak form of the Griffith's energy on function spaces with no regularity on the jump sets except the $(n-1)$-rectifiability (see [11] [14] [10] [7]). The existence of a solution with $\Gamma(t)$ closed is obtained only in particular cases through a regularity argument (see [5] [4]). Therefore, also in the dynamic case we expect that in dealing with any general existence results, no a-priori regularity assumptions on the crack sets $\Gamma(t)$ should be assumed. For this reason we assume only that the cracks $\Gamma(t)$ are $(n-1)$-rectifiable with finite $(n-1)$-dimensional Hausdorff measure.

In this paper we prove that, both in the undamped and damped case, a solution actually exists.

The first issue is to give a weak formulation to the system written in (1). The presence of the cracks forces at each time to solve the system on the set $\Omega \setminus \Gamma(t)$. Therefore we need to introduce suitable function spaces V_t, containing for each time t the solution $u(t)$ as well as the test functions. The scalar case, i.e. when (1) reduces to the wave equation, has been treated by Dal Maso, Larsen in [8]. Since the structure of the equation implies no bound on the amplitude of the jump of u, but only on the L^2-norm of the gradient

$$\int_{\Omega\setminus\Gamma(t)} |\nabla u(t)|^2 \, dx, \tag{2}$$

they defined the problem in the context of $GSBV(\Omega)$ (for a definition we refer to [2, Definition 4.26]). Precisely in [8] it has been shown the existence of a weak solution $u(t)$ living at each time t in the space $GSBV_2^2(\Omega;\Gamma(t))$, composed of all functions $u \in GSBV(\Omega) \cap L^2(\Omega)$ whose jump sets are contained in $\Gamma(t)$ and such that (2) is finite.

In our case the structure of the equation leads to an estimate of

$$\int_{\Omega\setminus\Gamma(t)} |\mathcal{E}u(t)|^2 \, dx.$$

Hence V_t needs to include all the displacements in $L^2(\Omega,\mathbb{R}^n)$ whose jump sets are contained in $\Gamma(t)$ and with square integrable symmetric gradient away form the cracks. Since we assume no regularity on the cracks, in this general context a Korn's type inequality is not true. This means that we cannot control the L^2-norm of the gradient of $u(t)$ with the L^2-norm of its symmetric part. As a consequence we are forced to formulate our problem in the context of BD functions, and precisely to define $V_t = GSBD_2^2(\Omega;\Gamma(t))$ (see Definition 2.3) and $V_t^* = GSBD_2^2(\Omega;\Gamma(t))^*$ its dual. Note that if $\Gamma(t)$ are closed sets in Ω, then $GSBD_2^2(\Omega;\Gamma(t))$ reduces to the space of square integrable vector fields, whose symmetric gradients in the sense of distribution on $\Omega \setminus \Gamma(t)$ are square integrable.

The weak formulation of the system will be

$$\langle \ddot{u}(t), \phi \rangle_t^* + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot{u}(t), \mathcal{E}\phi \rangle_{H_n} = \langle f(t), \phi \rangle_H \;\; \forall \phi \in V_t \tag{3}$$

for a.e. $t \in [0,T]$, where $\langle \cdot,\cdot \rangle_t^*$ denotes the duality pairing between V_t and V_t^*, $\langle \cdot,\cdot \rangle_H$ $\langle \cdot,\cdot \rangle_{H_n}$ denote the scalar product in $L^2(\Omega,\mathbb{R}^n)$ and in $L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$, respectively.

We want to emphasize that one of the most serious mathematical issues arises because these spaces are varying (increasingly) in time, so that test functions at some time t are not necessarily admissible test functions for times $s < t$. Moreover since $u(t)$ lives on each time t in different spaces V_t, we need to give a meaning to the second derivative in time $\ddot{u}(t)$ as an element of V_t^*.

While in [8] only homogeneous Neumann boundary condition was considered, in the present paper we consider also non-homogeneous mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$. This introduces another difficulty when the crack sets approach the boundary, and as a consequence possible problems may occur with the boundary conditions. Indeed, when we have non-homogeneous Neumann boundary condition on a part of $\partial\Omega$, we might think that when the elastic material between this part of the boundary and the crack sets is infinitesimally small, then the elastic reaction to the traction forces will be infinitesimal too. From a mathematical point of view, the difficulty is due to the lack of continuity of the trace operator acting on functions having jump sets close to the boundary. In order to solve this problem, we make use of the results obtained in [18], which allow us to restrict our attention to a suitable space of traction forces F.

We also show an energy balance and uniqueness for the damped problem. The energy balance we are able to prove in the damped case, is a conservation of kinetic plus elastic plus dissipated energy due to the damping. For the undamped problem the energy balance, where only the kinetic plus the elastic energy are considered, is clearly false. This can be seen using the results of [3]. In the undamped case, the uniqueness is still an open problem.

This paper is organized as follows: in Section 2 we define the function spaces. Precisely, we define V_t, we show some functional properties about these spaces, and then we introduce the space of admissible traction forces F appearing in the Neumann part of the boundary. Then we give a precise definition of $\ddot{u}(t)$ for a.e. $t \in [0,T]$ as an element of V_t^*, and we show that under some regularity assumptions on the test functions, $\ddot{u}$ satisfies an integration by parts formula in time.

In Section 3 we first give the definition of weak solution. Then we show an existence result for the damped equation complemented with boundary conditions, by a discrete in time approximation technique and passing to the limit when the time step goes to zero (see Theorem 3.2). More precisely to define the discrete approximate solution u_k in the time interval $(t_k^i, t_k^{i+1}]$, suppose that we have already defined u_k for $t \le t_k^i$, and let u_k^{i+1} be the minimizer in $V_{t_{i+1}} + w(t_k^i)$[1] of

$$u \mapsto \left\| \frac{u - u_k^i}{\tau_k} - \frac{u_k^i - u_k^{i-1}}{\tau_k} \right\|^2 + \langle \mathbb{C}\,\mathcal{E}u, \mathcal{E}u \rangle + \frac{1}{\tau_k}\langle \mathbb{B}\,(\mathcal{E}u - \mathcal{E}u_k^i), \mathcal{E}u - \mathcal{E}u_k^i \rangle - 2\langle f_k^i, u \rangle,$$

where $u_k^i = u_k(t_k^i)$, $\|\cdot\|$ is the norm in L^2 and $\langle\cdot,\cdot\rangle$ is the L^2-scalar product, f_k^i is a suitable discrete approximation of f and τ_k is the time step. We define u_k on $(t_k^i, t_k^{i+1}]$ as the linear interpolation between u_k^i and u_k^{i+1}.

Then we show that the limit u of the u_k, satisfies the energy balance (38). Precisely for each k, u_k satisfies a discrete energy balance which converges to the desired energy balance for u as $k \to \infty$ (see Proposition 3.4 and Proposition 3.8). As a consequence we deduce existence and uniqueness for the damped problem (see Theorem 3.7).

[1] $V_{t_{i+1}} + w(t_k^i)$ is the space of functions that jump on $\Gamma(t_{i+1})$ with Dirichlet boundary condition $w(t_k^i)$ on $\partial_D\Omega \subset \partial\Omega$.

Finally, in Section 4 we show the existence of a weak solution to the undamped equation complemented with boundary conditions, always by a discrete in time approximation technique and passing to the limit when the time step goes to zero (Theorem 4.2).

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2. Notation and preliminary results

We denote the space of $n \times n$ matrices with real entries as $\mathbb{M}^{n\times n}$ endowed with the euclidean scalar product

$$\xi \cdot \eta := \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_{ij}\eta_{ij} \right),$$

and we denote as $|\cdot|$ the associated norm; the subspace of symmetric $n\times n$ matrices is denoted by $\mathbb{M}^{n\times n}_{sym}$. $\mathcal{L}(\mathbb{M}^{n\times n})$ is the space of continuous linear maps of $\mathbb{M}^{n\times n}$ into itself.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote the space $L^2(\Omega,\mathbb{R}^n)$ as H, with scalar product $\langle\cdot,\cdot\rangle_H$ and with associated norm $\|\cdot\|_H$. Analogously we denote the space $L^2(\Omega,\mathbb{M}^{n\times n})$ as H_n, with scalar product $\langle\cdot,\cdot\rangle_{H_n}$ and with associated norm $\|\cdot\|_{H_n}$.

Definition 2.1. We say that $\mathbb{C}\colon\Omega \to \mathcal{L}(\mathbb{M}^{n\times n})$ is a bounded symmetric and positive definite tensor field, if it is $\mathcal{L}^n$-measurable and

- $\|\mathbb{C}\|_{L^\infty} < \infty$,
- $\mathbb{C}(x)\xi \in \mathbb{M}^{n\times n}_{sym}, \quad \forall \xi \in \mathbb{M}^{n\times n},$ for a.e. $x\in\Omega$
- $\mathbb{C}(x)\xi\cdot\eta = \xi\cdot\mathbb{C}(x)\eta, \quad \forall\xi,\eta\in\mathbb{M}^{n\times n},$ for a.e. $x\in\Omega$ (symmetry),
- $\mathbb{C}(x)\xi\cdot\xi \geq \gamma_0|\xi|^2, \quad \forall\xi\in\mathbb{M}^{n\times n}_{sym},$ for a.e. $x\in\Omega$ ($\gamma_0>0$) (positiveness),

(which are the usual assumptions in linear elasticity). The strictly positive number γ_0 is called ellipticity constant of $\mathbb{C}$. Under the previous assumptions on $\mathbb{C}$, given any Lebesgue-measurable functions $\xi\colon\Omega\to\mathbb{M}^{n\times n}$, we write

$$\|\xi\|_{H_n^{\mathbb{C}}} := \int_\Omega \mathbb{C}(x)\xi(x)\cdot\xi(x)\, dx.$$

Remark 2.2. Thanks to the symmetry and positiveness properties of $\mathbb{C}$, it follows that the function $\|\cdot\|_{H_n^{\mathbb{C}}}$ defined on the real vector space of all measurable functions $\xi\colon\Omega\to\mathbb{M}^{n\times n}_{sym}$ is a norm. Moreover, by using also the L^∞-bound, the norm $\|\cdot\|_{H_n^{\mathbb{C}}}$ is equivalent to the norm $\|\cdot\|_{H_n}$.

We recall that $GSBD^2_2(\Omega)$ is the space of vector fields $u\in GSBD(\Omega)$ (see [6] for the definition of $GSBD(\Omega)$) such that $u\in L^2(\Omega,\mathbb{R}^n)$, and their symmetric approximate gradients $\mathcal{E}u$ belong to $L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$.

Definition 2.3. Let $\Gamma\subset\Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set (see [13, Definition 3.2.14]) with $\mathcal{H}^{n-1}(\Gamma)<\infty$. We define

$$GSBD^2_2(\Omega;\Gamma) := \{u\in GSBD^2_2(\Omega) \mid J_u\subset\Gamma\}.$$

Proposition 2.4. *Let Ω be an open set of $\mathbb{R}^n$, and let $\Gamma\subset\Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma)<\infty$. Then the space $GSBD^2_2(\Omega;\Gamma)$ endowed with the scalar product*

$$\langle u,v\rangle_2 = \langle u,v\rangle_H + \langle\mathcal{E}u,\mathcal{E}v\rangle_{H_n}, \tag{4}$$

is a separable Hilbert space. Moreover, we denote by $\|\cdot\|$ the associated norm to the scalar product $\langle\cdot,\cdot\rangle_2$.

Proof. Thanks to [6, Remark 4.6] we know that $GSBD(\Omega)$ is a real vector space, and as a consequence $GSBD^p_p(\Omega)$ is a real vector space too. The fact that $GSBD^p_p(\Omega;\Gamma)$ is also a real vector space follows once we prove that given $u,v\in GSBD(\Omega)$ then $J_{u+v}\subset J_u\cup J_v$ $\mathcal{H}^{n-1}$-a.e.. To see this, fix Ξ an orthonormal basis of $\mathbb{R}^n$, say $\{\xi_1,\dots,\xi_n\}$, and consider the directions in $\mathbb{S}^{n-1}$ defined by

$$C(\Xi,\delta) := \{x\in\mathbb{S}^{n-1} \mid |x\cdot\xi_i| > (1/\sqrt{n}-\delta)|x|, \quad \text{for every } \xi_i\in\Xi\},$$

where δ is any real number in $(0,1/\sqrt{n})$. We claim that

$$A := \{x\in J_{u+v} \mid \nu_{u+v}(x)\in C(\Xi,\delta)\}\subset J_u\cup J_v, \quad \mathcal{H}^{n-1}\text{-a.e..} \tag{5}$$

Notice that for every $\xi\in\mathbb{S}^{n-1}$ and for every $y\in\xi^\perp$

$$[\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y\cap(J_u)^\xi_y=\varnothing, \quad \text{and} \quad [\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y\cap(J_v)^\xi_y=\varnothing,$$

and thanks to [6, Theorem 8.1] we deduce that for every $\xi\in\mathbb{S}^{n-1}$

$$[\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y\cap J_{\hat{u}^\xi_y}=\varnothing, \ \text{and} \ [\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y\cap J_{\hat{v}^\xi_y}=\varnothing, \ \mathcal{H}^{n-1}\text{-a.e. } y\in\xi^\perp. \tag{6}$$

Since the one dimensional slices of u and v are SBV_{loc}-functions, by (6) we deduce that for $\mathcal{H}^{n-1}$-a.e. $y\in\xi^\perp$ the sets $[\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y$ is contained in the set of Lebesgue points of $\hat{u}^\xi_y+\hat{v}^\xi_y$ which in turn is contained in the set of Lebesgue points of $((u+v)\cdot\xi)^\xi_y$. By using again [6, Theorem 8.1], this means that for every $\xi\in\mathbb{S}^{n-1}$ we have

$$[\mathbb{R}^n\smallsetminus(J_u\cup J_v)]^\xi_y\cap(J^\xi_{u+v})^\xi_y=\varnothing, \quad \mathcal{H}^{n-1}\text{-a.e. } y\in\xi^\perp. \tag{7}$$

Now suppose that (5) does not hold. This means that there exists a set $A'\subset A$ with $\mathcal{H}^{n-1}(A')>0$ but $\mathcal{H}^{n-1}((J_u\cup J_v)\cap A')=0$. Since Ξ is a basis of $\mathbb{R}^n$, then there must exists $\xi_i\in\Xi$ such that

$$\mathcal{H}^{n-1}(A'\cap\{((u+v)^+-(u+v)^-)\cdot\xi_i\neq 0\})>0.$$

By using also that for $\mathcal{H}^{n-1}$-a.e. $x\in A'\cap\{((u+v)^+-(u+v)^-)\cdot\xi_i\neq 0\}$ we have $\nu_{u+v}(x)\cdot\xi_i>0$ (simply by definition of A), and by using Coarea Formula applied to the projection map

$$\pi^\xi\colon A'\cap\{((u+v)^+-(u+v)^-)\cdot\xi_i\neq 0\}\to\xi^\perp$$

we deduce that if we set $A'_i := A'\cap\{((u+v)^+-(u+v)^-)\cdot\xi_i\neq 0\}$, then $\mathcal{H}^{n-1}(\pi^\xi(A'_i))>0$ and

$$\mathcal{H}^0([A'\cap\{((u+v)^+-(u+v)^-)\cdot\xi_i\neq 0\}]^\xi_y)>0, \quad \mathcal{H}^{n-1}\text{-a.e. } y\in\pi^\xi(A'_i).$$

But since $A'_i\subset J^{\xi_i}_{u+v}$, by (7) this means also that

$$\mathcal{H}^0((J_u\cup J_v)^\xi_y)>0, \quad \mathcal{H}^{n-1}\text{-a.e. } y\in\pi^\xi(A'_i),$$

which is a contradiction since we assumed that $\mathcal{H}^{n-1}((J_u\cup J_v)\cap A')=0$ and proves our claim. Finally, thanks to the arbitrariness of Ξ, relation (5) proves exactly that $J_{u+v}\subset J_u\cup J_v$ $\mathcal{H}^{n-1}$-a.e..

To prove the completeness we argue in the following way. Suppose that $(u_k)\subset GSBD^p_p(\Omega;\Gamma)$ is a Cauchy sequence. In particular $(u_k)_k$ converges strongly in $L^p(\Omega,\mathbb{R}^n)$ to some u . Since

$$\sup_{k\in\mathbb{N}}\big(\|u_k\|_{L^p}+\|\mathcal{E}u_k\|_{L^p}+\mathcal{H}^{n-1}(J_{u_k})\big)<\infty,$$

by Theorem [6, Theorem 11.3], eventually passing through a subsequence, we know that there exists $v \in GSBD(\Omega)$ such that $u_k \to v$ pointwise a.e. on Ω and $\mathcal{E}u_k \rightharpoonup \mathcal{E}v$ weakly in $L^1(\Omega)$. This implies $u = v$ and thanks to the lower semicontinuity of the L^p norm with respect to the weak convergence also

$$\|\mathcal{E}u\|_{L^p} \le \liminf_{k\to\infty} \|\mathcal{E}u_k\|_{L^p},$$

hence $u \in GSBD^p_p(\Omega)$.

It remains to prove that $J_u \subseteq \Gamma$. Using [6, Theorem, 11.3], for every open set $U \subset \Omega$ we have

$$\mathcal{H}^{n-1}(J_u \cap U) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \cap U) \ .$$

Since the measure $\mathcal{H}^{n-1} \llcorner \Gamma$ is inner regular, for every $\epsilon > 0$ we can find a compact set $K \subset \Gamma$, such that $\mathcal{H}^{n-1}(\Gamma \setminus K) \le \epsilon$, and so

$$\mathcal{H}^{n-1}(J_u \setminus K) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \setminus K) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(\Gamma \setminus K) \le \epsilon.$$

For the arbitrariness of ϵ, we conclude that $\mathcal{H}^{n-1}(J_u \setminus \Gamma) = 0$.

To prove the separability consider the embedding $j: GSBD^2_2(\Omega;\Gamma) \to L^2(\mathbb{R}^n \times \mathbb{R}^{n^2})$ defined by $j(u) := (u, \mathcal{E}u)$. By the well known fact that subspaces of a separable metric space are separable, since j is an embedding, we deduce that also $GSBD^2_2(\Omega;\Gamma)$ is separable. □

The dual $GSBD^2_2(\Omega;\Gamma)^*$ will not be identified with the underlying Hilbert space, but instead will be endowed with a pairing consistent with the L^2 inner product, as is usually done for the duals of Sobolev spaces. Since

$$GSBD^2_2(\Omega;\Gamma) \subset L^2(\Omega, \mathbb{R}^n)$$

is a dense embedding, we have

$$L^2(\Omega, \mathbb{R}^n) = L^2(\Omega, \mathbb{R}^n)^* \subset GSBD^2_2(\Omega;\Gamma)^*,$$

and $L^2(\Omega, \mathbb{R}^n)$ is densely embedded in $GSBD^2_2(\Omega;\Gamma)^*$.

In the case Ω has also finite perimeter, the trace operator $Tr(\cdot)$ can be extended to the space $GSBD(\Omega;\Gamma)$, using the notion of *approximate limit* on the point of the reduced boundary $\mathcal{F}\Omega$ (see [18, Definition 3.9]). Moreover the following theorem holds true.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set, with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then there exists a measurable functions $\Theta: \mathcal{F}\Omega \to \mathbb{R}^+$ such that*

(a) $\mathcal{H}^{n-1}(\{\Theta = 0\}) = 0$ and $\Theta \in L^\infty(\mathcal{F}\Omega, \mathcal{H}^{n-1})$ (in particular $\|\Theta\|_\infty \le 1$);
(b) For every $u \in GSBD^2_2(\Omega;\Gamma)$ we have

$$\int_{\mathcal{F}\Omega} |Tr(u)|^2 \Theta \ d\mathcal{H}^{n-1} \le C(n,2)(\|u\|_H + \|\mathcal{E}u\|_{H_n})^2, \tag{8}$$

where $C(n,2)$ is a constant depending only on n and 2.

Proof. Let Θ^+ be the weight function given by [18, Theorem 3.2], if we define

$$\Theta(x) := \Theta^+(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}\Omega,$$

then (a) and (b) are a direct consequences of [18, Theorem 3.2] with $p = 2$. □

Now let $\partial_N\Omega$ be a Borel subset of $\mathcal{F}\Omega$. In order to impose a Neumann boundary condition in equation (1), we are led to study the continuity property of the following linear form

$$u \mapsto \int_{\partial_N\Omega} F\cdot Tr(u)\; d\mathcal{H}^{n-1} \;\; u\in GSBD^2_2(\Omega;\Gamma), \tag{9}$$

where $F\colon\partial_N\Omega\to\mathbb{R}^n$ is some measurable vector field. In view of inequality (8) we introduce the following space.

Definition 2.6 (Admissible Neumann term)**.** Let Ω, Γ, and Θ be as in the previous theorem, and let $\partial_N\Omega$ be a Borel subset of $\mathcal{F}\Omega$. We define $N_\Theta := L^2(\partial_N\Omega,\Theta\mathcal{H}^{n-1})$, and we denote by N_Θ^* its dual. We identify N_Θ^* with the space of measurable vector fields $F\colon\partial_N\Omega\to\mathbb{R}^n$ such that

$$\int_{\partial_N\Omega}\frac{|F|^2}{\Theta}\; d\mathcal{H}^{n-1}<\infty,$$

and we consider the corresponding duality pairing between N_Θ^* and N_Θ given by

$$\langle F,g\rangle_\Theta := \int_{\partial_N\Omega} F\cdot g\; d\mathcal{H}^{n-1}\;\; (g\in N_\Theta).$$

The induced norm is denoted by $\|\cdot\|^*_{N_\Theta}$.

Putting together the definition of N_Θ and Theorem 2.5 we have the following result.

Proposition 2.7. *Let Ω, Γ, and Θ be as in Theorem 2.5. Let $\partial_N\Omega$ be a Borel subset of $\mathcal{F}\Omega$. If $F\in N_\Theta^*$ then the linear form defined in (9) belongs to $GSBD^2_2(\Omega;\Gamma)^*$.*

Proof. It is enough to use inequality (8) to have

$$\int_{\partial_N\Omega} F\cdot Tr(u)\; d\mathcal{H}^{n-1}\le \|F\|^*_{N_\Theta}\|Tr(u)\|_{N_\Theta}\le C(n,2)\|F\|^*_{N_\Theta}\big(\|u\|_H+\|\mathcal{E}u\|_{H_n}\big).$$

□

Our choice of Neumann forces, in some sense, is natural. In fact looking at the construction of Θ made in [18], roughly speaking, it turns out that Θ measures the "closeness" of Γ to the boundary. From a physical point of view, this might be interpreted as the fact that, when the elastic material between the Neumann boundary and the crack is infinitesimally small, then its elastic reaction can only balance traction forces which decrease their intensity (proportionally to Θ).

The following proposition is useful to prescribe a Dirichlet boundary condition on a certain portion of the boundary $\partial_D\Omega\subseteq\mathcal{F}\Omega$.

Proposition 2.8. *Let Ω, Γ be as in Proposition 2.7, let $\partial_D\Omega$ be a Borel subset of $\mathcal{F}\Omega$, and let $g:\partial_D\Omega\to\mathbb{R}$ be a measurable function. Then the set $\{u\in GSBD^2_2(\Omega;\Gamma)\mid Tr(u)=g$ on $\partial_D\Omega\}$ is an affine closed subspace of $GSBD^2_2(\Omega;\Gamma)$*

Proof. The only non trivial fact is to show that it is closed with respect to the norm induced by the scalar product (4). But this is a direct consequence of [18, Theorem 4.1]. □

Definition 2.9. Let $\partial_D\Omega\subseteq\mathcal{F}\Omega$ be a Borel set. We define

$$GSBD^2_{2,D}(\Omega;\Gamma) := \{u\in GSBD^2_2(\Omega;\Gamma)\mid Tr(u)=0,\ \text{on }\partial\Omega_D\}. \tag{10}$$

Thanks to our previous proposition, $GSBD^2_{2,D}(\Omega;\Gamma)$ is actually an Hilbert space with scalar product inherited as a subspace of $GSBD^2_2(\Omega;\Gamma)$.

Now fix $T > 0$, and fix $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Consider for $t \in [0,T]$ an increasing family of cracks $t \mapsto \Gamma(t)$

$$\Gamma(s) \subseteq \Gamma(t) \subseteq \Gamma \ \text{ if } s \le t,$$

For simplicity of notation, we denote $GSBD^2_2(\Omega;\Gamma)$ by V and $GSBD^2_{2,D}(\Omega;\Gamma(t))$ by V_t. The norm in V is denoted by $\|\cdot\|$, the norm in V_t with $\|\cdot\|_t$. Note that for $s < t$ we have $V_s \subset V_t \subset V$, and as we have already mentioned, since $V \subset H$ (remember $H = L^2(\Omega,\mathbb{R}^n)$) is densely embedded in H, we have the embedding $H \subset V^*$ and the density of H in V^*. Similarly H is a dense subspace of V_t^* for every $t \in [0,T]$. We denote the pairing between V^* and V by $\langle\cdot,\cdot\rangle$, and the associated dual norm $\|\cdot\|^*$, we denote the pairing between V_t^* and V_t by $\langle\cdot,\cdot\rangle_t$, and the associated dual norm $\|\cdot\|_t^*$. We note that these pairings are the unique continuous bilinear maps on $V^* \times V$ and $V_t^* \times V_t$ such that $\langle f, v\rangle = \langle f, v\rangle_H$ and $\langle f, v_t\rangle_t = \langle f, v_t\rangle_H$ whenever $f \in H, v \in V, v_t \in V_t$.
If $s < t$ then V_s is not dense in V_t and so V_t^* is not embedded in V_s^*. Anyway we can introduce the projection operators from V_t^* to V_s^* in the following way.

Definition 2.10. Let $s < t$ and let $i : V_s \to V_t$ denote the embedding $V_s \subset V_t$. Let f be an element of V_t^*. Then we define the projection map P_{st} of V_t^* onto V_s^* as

$$\langle P_{st} f, v_s\rangle_s := \langle f, i(v_s)\rangle_t \text{ for any } v_s \in V_s. \tag{11}$$

Note that the projection maps defined above are continuous and in particular $\|P_{st} f\|_s^* \le \|f\|_t^*$. When there is no misunderstanding, we omit the notation $P_{st} f$, since the action of $f \in V_t^*$ on elements of $V_s \subset V_t$ is clear from the context.

Lemma 2.11. *Let $u \in W^{1,\infty}(0,T;H)$. Assume that there exists a positive function $g \in L^2(0,T)$, such that for every $s,t \in [0,T]$ with $s < t$, we have*

$$u \in W^{2,2}(t,T;V_s^*), \ \text{ and } \ \|\ddot u(r)\|_s^* \le g(r) \text{ for a.e. } r \in (t,T). \tag{12}$$

Then there exists a set $E \subset [0,T]$ of full measure, such that for every $t \in E$ there exists $w(t) \in V_t^*$ with the following properties

$$\|w(t)\|_t^* \le \tilde g(t), \tag{13}$$

where $\tilde g(t) = \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} g(r)\, dr$, and

$$\lim_{h\to 0^+} \frac{\dot u(t+h) - \dot u(t)}{h} = w(t), \ \text{ weakly in } V_t^*, \tag{14}$$

and

$$\lim_{h\to 0} \frac{\dot u(t+h) - \dot u(t)}{h} = w(t), \ \text{ strongly in } V_s^* \text{ for every } s < t. \tag{15}$$

In particular for every $s \in [0,T]$ the functions $t \mapsto u(t)$ and $t \mapsto P_{st} w(t)$, considered as a functions from (s,T) to V_s^*, belong respectively to $W^{2,2}(s,T;V_s^*)$ and $L^2(s,T;V_s^*)$, and satisfy $\ddot u(t) = P_{st} w(t)$ in V_s^* for a.e. $t \in (s,T)$.

Remark 2.12. Under the previous hypothesis on u, for every $t \in [0,T]$ $\dot u(t)$ is a well defined element of H. More precisely the functions $\dot u : [0,T] \to H$ is weakly continuous, i.e. for every $t_k \to t \in [0,T]$ we have

$$\dot u(t_k) \rightharpoonup \dot u(t) \text{ weakly in } H, \text{ as } k \to \infty. \tag{16}$$

Indeed, thanks to our hypothesis $\dot{u}(t)$ is a well defined element of V_0^* for every $t \in [0,T]$ and $\dot{u}(t_k) \rightharpoonup \dot{u}(t)$ weakly in V_0^* whenever $t_k \to t$ as $k \to \infty$. Moreover, since $\dot{u} \in L^\infty(0,T;H)$, by using the lower semicontinuity of the norm $\|\cdot\|_H$ with respect to the weak convergence in H, and the fact that $H \subset V_0^*$ is an embedding, we deduce that actually $\dot{u}(t) \in H$ for every $t \in [0,T]$. This shows that $\dot{u}(t)$ is a well defined element of H for every $t \in [0,T]$. Finally, by arguing as before it is easy to see that $\dot{u}(t_k) \rightharpoonup \dot{u}(t)$ weakly in H as $k \to \infty$.

Remark 2.13. In the proof of Lemma 2.11, we are able to show that the convergence in (14) holds when we consider the incremental quotients *only for positive* h.

In the proof of Lemma 2.11 we shall use the following result on increasing sequences of subspaces of separable Hilbert spaces proved in [6, Lemma 2.3].

Lemma 2.14. *Let* $\{X_t \mid t \in [0,T]\}$ *be an increasing family of closed linear subspaces of a separable Hilbert space* X*. Then, there exists a countable set* $S \subset [0,T]$ *such that for all* $t \in [0,T] \setminus S$*, we have*

$$X_t = \overline{\bigcup_{s<t} X_s}.$$

Proof. (Lemma 2.11) Let $D \subset [0,T]$ be a countably dense set. Choose $s \in D$, then thanks to (12) for a.e. $t > s$ there exists $\ddot{u}(t)$ as an element of V_s^* and $\|\ddot{u}(t)\|_s^* \le \tilde{g}(t)$. By the fact that D is countable we have a set $E' \subset (0,T)$ of full measure, such that if $t \in E'$ $\ddot{u}(t)$ exists as an element of V_s^* and $\|\ddot{u}(t)\|_s^* \le \tilde{g}(t)$ for every $s \in (0,t) \cap D$. Moreover, by density, for any $s_1 < t$ there exists $s_2 \in D$ with $s_1 < s_2 < t$ and thanks to the continuity of the projection map $P_{s_1 s_2}$, we have the relation between $\ddot{u}(t)$ computed in $V_{s_2}^*$ and in $V_{s_1}^*$, given by

$$\lim_{h\to 0} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = \ddot{u}(t) \text{ in } V_{s_2}^* \Rightarrow \lim_{h\to 0} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = P_{s_1 s_2} \ddot{u}(t) \text{ in } V_{s_1}^*, \tag{17}$$

This means that for every $t \in E'$ and for every $s < t$

$$\ddot{u}(t) \text{ exists in } V_s^*, \ \|\ddot{u}(t)\|_s^* \le \tilde{g}(t), \tag{18}$$

and precisely two derivatives computed on different V_s^* are related by (17). Hence for every $t \in E'$, $\ddot{u}(t)$ is well defined as an element of V_s^* for every $s < t$.

The previous bound in (18) implies in particular

$$u \in W^{2,2}(s,T;V_s^*) \text{ for every } s \in (0,T). \tag{19}$$

Now define

$$E := E' \cup \{t \in [0,T] \mid \overline{\bigcup_{s<t} V_s} = V_t \text{ and } \tilde{g}(t) < \infty\}. \tag{20}$$

By Lemma 2.14, Lebesgue's Differentiation Theorem, and the definition of E', it holds that E has still full measure.

Now let $t \in E$. In order to prove (13), we notice that for every $\phi \in V_t$ there exists an increasing sequence $(s_n)_{n\in\mathbb{N}}$ converging to t, and a sequence of functions $(\phi_{s_n})_{n\in\mathbb{N}}$ with $\phi_{s_n} \in V_{s_n}$ strongly converging to ϕ in V_t. Hence we can define $w(t)\colon V_t \to \mathbb{R}$ as

$$\langle w(t), \phi \rangle_t := \lim_{n\to\infty} \langle \ddot{u}(t), \phi_{s_n} \rangle_{s_n} \text{ for every } \phi \in V_t. \tag{21}$$

We have to show that the previous limit exists and does not depend on the approximating sequence $(\phi_{s_n})_{n\in\mathbb{N}}$. It is enough to notice that if $n > m$ then

$$\begin{aligned} \langle \ddot u(t), \phi_{s_n}\rangle_{s_n} - \langle \ddot u(t), \phi_{s_m}\rangle_{s_m} &= \langle \ddot u(t), \phi_{s_n} - \phi_{s_m}\rangle_{s_n} \\ &\le \|\ddot u(t)\|^*_{s_n} \|\phi_{s_n} - \phi_{s_m}\|_{s_n} \\ &\le \tilde g(t)\|\phi_{s_n} - \phi_{s_m}\|_t. \end{aligned} \tag{22}$$

This defines a continuous linear functional on V_t and moreover $\|w(t)\|^*_t \le \tilde g(t)$. This is exactly (13).

To prove (14) let $t \in E$. We fix $\epsilon > 0$ and $\phi \in V_t$, then we can find $s < t$ and $\phi_s \in V_s$ such that $\|\phi_s - \phi\|_t \le \epsilon$. Hence

$$\begin{aligned} \lim_{h\to 0^+} \left\langle \frac{\dot u(t+h) - \dot u(t)}{h} - w(t), \phi \right\rangle_t &= \lim_{h\to 0^+} \left\langle \frac{\dot u(t+h) - \dot u(t)}{h} - w(t), \phi_s + (\phi - \phi_s) \right\rangle_t \\ &\le \limsup_{h\to 0^+} \frac{1}{h} \int_t^{t+h} \langle \ddot u(r) - w(t), \phi - \phi_s\rangle_t \, dr \\ &\le \limsup_{h\to 0+} \left(\frac{1}{h}\int_t^{t+h} g(r)\, dr + \tilde g(t) \right) \|\phi - \phi_s\|_t \\ &\le 2\tilde g(t)\epsilon, \end{aligned} \tag{23}$$

where we used the fact that $u \in W^{2,2}(t,T;V_t^*)$ and the Fundamental Theorem of Calculus. The arbitrariness of ϵ gives assertion (14) and concludes the proof. □

Definition 2.15. Under the assumption of Lemma 2.11, the element $w(t)$ of V_t^* defined in (14) for a.e. $t \in [0,T]$ is denoted by $\ddot u(t)$.

Lemma 2.16. *Let u be as in Lemma 2.11 and consider $\varphi \in L^2(0,T;V) \cap W^{1,2}(0,T;H)$ such that $\varphi(t) \in V_t$ for every $t \in [0,T]$. Then the map $t \mapsto \langle \dot u(t), \varphi(t)\rangle_H$ is absolutely continuous on $[0,T]$ and more precisely*

$$\langle \dot u(t_2), \varphi(t_2)\rangle_H - \langle \dot u(t_1), \varphi(t_1)\rangle_H = \int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi(\tau)\rangle_\tau \, d\tau, \tag{24}$$

for every $0 \le t_1 < t_2 \le T$.

Proof. By Remark 2.12 we know that $t \mapsto \dot u(t)$ is weakly continuous in H. Therefore, since $t \mapsto \varphi(t)$ is strongly continuous in H, we deduce that $t \mapsto \langle \dot u(t), \varphi(t)\rangle_H$ is a continuous real valued map.

First of all we prove our assertion for $\varphi(\cdot)_h := \varphi(\cdot - h)$ instead of $\varphi(\cdot)$. Fix any $t \in [h, T-h]$. Since $\varphi(\cdot - h) \in V_t$ on the time interval $[t, t+h]$ and $u \in W^{2,2}(t, t+h; V_t^*)$, we easily deduce that

$$\langle \dot u(t_2), \varphi_h(t_2)\rangle_H - \langle \dot u(t_1), \varphi_h(t_1)\rangle_H = \int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi_h(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi_h(\tau)\rangle_\tau \, d\tau, \tag{25}$$

for every $t_1, t_2 \in (t, t+h)$ $(t_1 < t_2)$. Since t was arbitrary and $\langle \dot u(\cdot), \varphi_h(\cdot)\rangle_H$ is continuous, we can actually obtain (25) for every $t_1, t_2 \in [h, T-h]$ $(t_1 < t_2)$.

Finally thanks to the fact $\varphi \in W^{1,2}(0,T;H)$ the left hand side of (25) converges to $\langle \dot u(t_2), \varphi(t_2)\rangle_H - \langle \dot u(t_1), \varphi(t_1)\rangle_H$ as $h \to 0^+$, while using also $\varphi \in L^2(0,T;V)$ (in particular the continuity of the translations in L^2), the right hand side of (25) converges to $\int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi(\tau)\rangle_\tau \, d\tau$ and we are done. □

3. The damped system of elastodynamics

From now on we consider the following standing assumptions:

(a) $\Omega \subset \mathbb{R}^n$ is an open set of finite perimeter;

(b) $(\Gamma(t))_{t\in[0,T]}$ is an increasing family of crack sets:

$$\Gamma(s) \subseteq \Gamma(t) \subseteq \Gamma \quad \text{for } s < t, \tag{26}$$

where $\Gamma \subseteq \Omega$ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$;

(c) $\partial_D\Omega$, $\partial_N\Omega$ are two disjoint Borel subsets of $\mathcal{F}\Omega$, respectively the Dirichlet and the Neumann part of the reduced boundary, such that $\partial_D\Omega \cup \partial_N\Omega = \mathcal{F}\Omega$.

(d) $\mathbb{C}$ and $\mathbb{B}$ are bounded symmetric and positive definite tensor fields with ellipticity constant γ_0 and γ_1, respectively (see Definition 2.1).

In this section we deal with the damped system of elastodynamics:

$$\ddot{u}(t) - \operatorname{div}[\mathbb{C}\,\mathcal{E}u(t)] - \operatorname{div}[\mathbb{B}\,\mathcal{E}\dot{u}(t)] = f(t). \tag{27}$$

Now we shall give the precise definition of weak solution:

Definition 3.1. Assume (a), (b), (c) and (d). With the notation introduced in Section 2, let $f \in L^2(0,T;V^*)$, let $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;H^1(\Omega)^n)$ and let $F \in L^2(0,T;N_\Theta)$ where Θ is the function relative to the crack set Γ given by Theorem 2.5. We say that u is a weak solution to (27) on the time dependent domain $t \mapsto \Omega \setminus \Gamma(t)$ with Dirichlet boundary condition $w(t)$ on $\partial_D\Omega$, Neumann boundary condition $F(t)$ on $\partial_N\Omega$, and homogeneous Neumann boundary condition on $\Gamma(t)$, if

$$u \in W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;V). \tag{28}$$

$$\text{For every } t \in [0,T]\ u(t) - w(t) \in V_t. \tag{29}$$

$$\text{For every } s \in [0,T)\ u \in W^{2,2}(s,T;V_s^*), \text{ and} \tag{30}$$

$$\|P_{st}\ddot{u}(t)\|_s^* \le g(t) \text{ for a.e. } t \in (s,T), \text{ for some } g \in L^2(0,T). \tag{31}$$

$$\lim_{h\to 0^+} \int_h^T \frac{\|\dot{u}(t) - \dot{u}(t-h)\|_H^2}{h}\, dt = 0. \tag{32}$$

For a.e. $t \in [0,T]$

$$\begin{aligned} &\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi\rangle_{H_n} - \langle F(t), Tr(\phi)\rangle_\Theta = \\ &= \langle f(t), \phi\rangle_t, \text{ for every } \phi \in V_t \end{aligned} \tag{33}$$

where $\ddot{u}(t)$ is the one given by Definition 2.15.

Given $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, since $t \mapsto u(t)$ is strongly continuous in V the initial value for u is well defined as element of $V_0 + w(0)$. Moreover we are able to prescribe the initial conditions for $\dot{u}(0)$ asking

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \|\dot{u}(t) - u^1\|_H^2\, dt = 0. \tag{34}$$

Since $\mathcal{E}u(t)$ and $\mathcal{E}\dot{u}(t)$ are in general only elements of H_n, it does not make sense to talk about their traces. For this reason the Neumann boundary conditions, $F(t)$ on $\partial_N\Omega$ and homogeneous on both sides of $\Gamma(t)$, have to be intended in a weak sense by means of integration by parts in equation (33). Moreover, condition (32) is technical and is related to the presence of the damping term. In fact, it plays a crucial role in view of the energy balance (38) (see Proposition 3.4).

The work of the external forces on the solution u over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\mathcal{W}_{load}(u; t_1, t_2) := \int_{t_1}^{t_2} \langle f(t), \dot{u}(t) \rangle_t \, dt \tag{35}$$

which is well defined by (28) and the fact that $f \in L^2(0, T; V^*)$. One would expect that the work on the solution u due to the varying Dirichlet boundary conditions w and Neumann boundary conditions F over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\begin{aligned} \mathcal{W}_{bdry}(u; t_1, t_2) := &\int_{t_1}^{t_2} \left(\int_{\partial_D \Omega} (\mathbb{C}\mathcal{E}u(t) + \mathbb{B}\mathcal{E}u(t))\nu \cdot \dot{w}(t) \, d\mathcal{H}^{n-1} \right) dt \\ &+ \int_{t_1}^{t_2} \langle F(t), \dot{u}(t) \rangle_\Theta dt. \end{aligned} \tag{36}$$

Unfortunately, under the assumptions (28)-(31) the trace of the normal derivative $\partial_\nu u(t)$ cannot be defined, not even in a weaker sense, because $\mathcal{E}u(t)$ in general belongs only to H_n. We decide to solve this problem following [9, Proposition 3.1], by using the weak formulation of the work due to the Dirichlet boundary conditions:

$$\begin{aligned} \mathcal{W}^D_{bdry}(u; t_1, t_2) := \langle \dot{u}(t_2), \dot{w}(t_2) \rangle_H - \langle \dot{u}(t_1), \dot{w}(t_1) \rangle_H - \int_{t_1}^{t_2} \langle \ddot{w}(t), \dot{u}(t) \rangle_H \, dt \\ - \int_{t_1}^{t_2} \langle F(t), \dot{w}(t) \rangle_\Theta \, dt - \int_{t_1}^{t_2} \langle f(t), \dot{w}(t) \rangle_t \, dt + \int_{t_1}^{t_2} \langle \mathbb{C}\, \mathcal{E}u(t) + \mathbb{B}\, \mathcal{E}\dot{u}(t), \mathcal{E}\dot{w}(t) \rangle_{H_n} \, dt. \end{aligned} \tag{37}$$

With this notation, the energy balance that we are able to prove for the solution u to (27) has the following form:

$$\begin{aligned} &\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|^2_{H_n^{\mathbb{C}}} + \frac{1}{2}\int_0^t \frac{1}{2}\|\mathcal{E}\dot{u}(\tau)\|^2_{H_n^{\mathbb{B}}} \, d\tau = \\ &\quad = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u; 0, t) + \mathcal{W}_{bdry}(u; 0, t). \end{aligned} \tag{38}$$

Now we prove the main result.

Theorem 3.2 (Partial existence)**.** *Assume (a), (b), (c) and (d). Let f, w and F be as in Definition 3.1. Then, given two initial conditions $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, there exists a function u satisfying (28)-(31) and (33) of Definition 4.1, with initial conditions $u(0) = u^0$ and (34). Moreover u satisfies the energy inequality*

$$\begin{aligned} &\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|^2_{H_n^{\mathbb{C}}} + \frac{1}{2}\int_0^t \frac{1}{2}\|\mathcal{E}\dot{u}(\tau)\|^2_{H_n^{\mathbb{B}}} \, d\tau \le \\ &\quad \le \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u; 0, t) + \mathcal{W}_{bdry}(u; 0, t) \end{aligned} \tag{39}$$

for a.e. $t \in [0, T]$.

Remark 3.3. Since $F \in L^2(0, T; N_\Theta)$, by Proposition 2.7 we have that $\langle F(t), Tr(\phi) \rangle_\Theta$ is actually a duality pairing between V_t^* and V_t. Therefore we can absorb the Neumann term into the forcing term defining

$$\langle \tilde{f}(t), \phi \rangle_t := \langle f(t), \phi \rangle_t + \langle F(t), Tr(\phi) \rangle_\Theta,$$

and we can reduce ourselves to prove Theorem 3.2 when (33) has the simplest form

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi\rangle_{H_n} = \langle f(t), \phi\rangle_t.$$

Proof. For $k \in \mathbb{N}$, we set $\tau_k := T/k$ and $t_k^j := j\tau_k$. For $j = 1, 2, \ldots, k$ we define $f_k^j \in V_T^*$ by

$$f_k^j := \frac{1}{\tau_k}\int_{t_k^j}^{t_k^{j+1}} f(\tau)\, d\tau, \tag{40}$$

and

$$w_k^j := w(t_k^j), \tag{41}$$

(we use $w \in W^{1,2}(0,T; H^1(\Omega)^n)$ implies that for every $t \in [0,T]$ $w(t)$ is well defined in $H^1(\Omega)^n$).

Inductively we define u_k^j for $j = -1, 0, \ldots, k$ by the following:

$$u_k^0 := u^0,\ \ u_k^{-1} := u^0 - \tau_k u^1; \tag{42}$$

then, for $j = 0, 1, \ldots, k-1$, the function u_k^{j+1} is the minimizer in $V_{t_k^{j+1}} + w_k^j$ of

$$u \mapsto \left\| \frac{u - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k}\right\|_H^2 + \|\mathcal{E}u\|^2_{H_n^{\mathbb{C}}} + \frac{1}{\tau_k}\|(\mathcal{E}u - \mathcal{E}u_k^j)\|^2_{H_n^{\mathbb{B}}} - 2\langle f_k^j, u\rangle_{t_k^{j+1}}. \tag{43}$$

Thanks to the ellipticity hypothesis on $\mathbb{C}$ and $\mathbb{B}$, at each step the above functional is coercive in $V_{t_k^{j+1}} + w_k^j$ because it is greater than

$$c_k[\|u\|_H^2 + (\gamma_0 + \gamma_1)\|\mathcal{E}u\|^2_{H_n}] - 2\|f_k^j\|^*_{t_k^{j+1}}(\|u\|_H + \|\mathcal{E}u\|_{H_n}) - a_k^{j+1} \tag{44}$$

where $c_k := \min\{1, 1/\tau_k^2\}$, a_k^{j+1} is a constant depending only on k j, and $\gamma_0, \gamma_1 > 0$ are the ellipticity constants of $\mathbb{C}$ and $\mathbb{B}$, respectively. By using also that the first three terms in (43) are lower-semicontinuous (here we use the symmetry and positiveness of $\mathbb{C}$ and $\mathbb{B}$) while the term $\langle f_k^j, u\rangle_{t_k^{j+1}}$ is even continuous with respect to the weak convergence in the closed affine subspace $V_{t_k^{j+1}} + w_k^{j+1}$, we deduce that the functional in (43) admits a minimizer $u_k^{j+1} \in V_{t_k^{j+1}} + w_k^{j+1}$. The Euler equation for the minimizer u_k^{j+1} is

$$\left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k}, \frac{\phi}{\tau_k}\right\rangle_H + \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}\phi\rangle_{H_n} + \frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}\phi\rangle_{H_n} = \langle f_k^j, \phi\rangle_{t_k^{j+1}} \tag{45}$$

for every $\phi \in V_{t_k^{j+1}}$. Then using $u_k^{j+1} - u_k^j - (w_k^{j+1} - w_k^j) \in V_{t_k^{j+1}}$ as ϕ we can write

(46)

$$\left\|\frac{u_k^{j+1} - u_k^j}{\tau_k}\right\|_H^2 - \left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k}, \frac{u_k^j - u_k^{j-1}}{\tau_k}\right\rangle_H - \left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1} - w_k^j}{\tau_k}\right\rangle_H +$$

$$+ \|\mathcal{E}u_k^{j+1}\|^2_{H_n^{\mathbb{C}}} - \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}u_k^j\rangle_{H_n} - \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \frac{1}{\tau_k}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|^2_{H_n} +$$

$$- \frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} = \langle f_k^j, u_k^{j+1} - u_k^j\rangle_{t_k^{j+1}} - \langle f_k^j, w_k^{j+1} - w_k^j\rangle_{t_k^{j+1}}.$$

Now using the identity $\|a\|^2 - \langle a, b\rangle = \frac{1}{2}\|a\|^2 + \frac{1}{2}\|a-b\|^2 - \frac{1}{2}\|b\|^2$, multiplying by 2, and rearranging, we get

(47)
$$\begin{aligned}
&\left\|\frac{u_k^{j+1}-u_k^j}{\tau_k}\right\|_H^2 + \left\|\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 - 2\left\langle \frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1}-w_k^j}{\tau_k}\right\rangle_H + \\
&\quad + \|\mathcal{E}u_k^{j+1}\|^2_{H_n^{\mathbb{C}}} + \|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|^2_{H_n^{\mathbb{C}}} + 2\langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \\
&\quad + 2\frac{1}{\tau_k}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|^2_{H_n^{\mathbb{B}}} - 2\frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} = \\
&\quad = \left\|\frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 + \|\mathcal{E}u_k^j\|_{H_n^{\mathbb{C}}} + 2\langle f_k^j,\ u_k^{j+1}-u_k^j\rangle_{t_k^{j+1}} - 2\langle f_k^j, w_k^{j+1}-w_k^j\rangle_{t_k^{j+1}}.
\end{aligned}$$

Summing from $j = 0$ to $i \in \{1, \dots, k\}$ and using (42), we get

(48)
$$\begin{aligned}
&\left\|\frac{u_k^{i+1}-u_k^i}{\tau_k}\right\|_H^2 + \|\mathcal{E}u_k^{i+1}\|^2_{H_n^{\mathbb{C}}} + \sum_{j=0}^{i}\left\|\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 + \sum_{j=0}^{i}\|\mathcal{E}u_n^{j+1} - \mathcal{E}u_k^j\|^2_{H_n^{\mathbb{C}}} + \\
&+ 2\frac{1}{\tau_k}\sum_{j=0}^{i}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|^2_{H_n^{\mathbb{B}}} = \|u^1\|_H^2 + \|\mathcal{E}u^0\|^2_{H_n^{\mathbb{C}}} + 2\sum_{j=0}^{i}(\langle f_k^j,\ u_k^{j+1}-u_k^j\rangle_{t_k^{j+1}} + \\
&+ 2\sum_{j=0}^{i}\langle f_k^j, w_k^{j+1}-w_k^j\rangle_{t_k^{j+1}} + 2\sum_{j=0}^{i}\langle \mathbb{C}\,\mathcal{E}u_k^{j+1} + \frac{1}{\tau_k}\mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \\
&- 2\sum_{j=0}^{i}\left\langle \frac{u_k^j-u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1}-w_k^j}{\tau_k} - \frac{w_k^j-w_k^{j-1}}{\tau_k}\right\rangle_H + 2\left\langle \frac{u_k^{i+1}-u_k^i}{\tau_k}, \frac{w_k^{i+1}-w_k^i}{\tau_k}\right\rangle_H - 2\left\langle \frac{u_k^1-u_k^0}{\tau_k}, \frac{w_k^1-w_k^0}{\tau_k}\right\rangle_H
\end{aligned}$$

We define the piecewise affine discrete approximations $u_k,\ v_k,\ w_k,\ z_k\colon [0,T] \to V$ for $t \in (t_k^j, t_k^{j+1}]$ by

$$u_k(t) := u_k^j + \frac{t-t_k^j}{\tau_k}(u_k^{j+1} - u_k^j), \tag{49}$$

$$w_k(t) := w_k^j + \frac{t-t_k^j}{\tau_k}(w_k^{j+1} - w_k^j), \tag{50}$$

$$v_k(t) := \frac{u_k^j-u_k^{j-1}}{\tau_k} + \frac{t-t_k^j}{\tau_k}\left(\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right) \tag{51}$$

$$z_n(t) := \frac{w_k^j-w_k^{j-1}}{\tau_k} + \frac{t-t_k^j}{\tau_k}\left(\frac{w_k^{j+1}-w_k^j}{\tau_k} - \frac{w_k^j-w_k^{j-1}}{\tau_k}\right), \tag{52}$$

and the piecewise constant discrete approximations $\tilde{u}_k,\ \tilde{w}_k,\ f_k\colon [0,T] \to V$ for $t \in (t_k^j, t_k^{j+1}]$ by

$$\tilde{u}_k(t) := u_k^{j+1}, \tilde{w}_k(t) := w_k^{j+1}, f_k(t) := f_k^j. \tag{53}$$

Rewriting (48) in terms of u_k, w_k, $\tilde{u}_k$, v_k, z_k we get the discrete energy balance for every $t \in (t_k^j, t_k^{j+1})$:

(54)

$$\|\dot{u}_k(t)\|_H^2 + \|\mathcal{E}u_k(t_k^{j+1})\|_{H_n^{\mathbb{C}}}^2 + \tau_k \int_0^{t_k^{j+1}} \|\dot{v}_k(\tau)\|_H^2 \, d\tau + \tau_k \int_0^{t_k^{j+1}} \|\mathcal{E}\dot{u}_k(\tau)\|_{H_n^{\mathbb{C}}}^2 \, d\tau +$$

$$+ \int_0^{t_k^{j+1}} \|\mathcal{E}\dot{u}_k(\tau)\|_{H_n^{\mathbb{B}}}^2 \, d\tau = \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}^2 + 2\int_0^{t_k^{j+1}} \langle f_k(\tau), \dot{u}_k(\tau)\rangle_{t_k^{j+1}} \, d\tau +$$

$$+ 2\int_0^{t_k^{j+1}} \langle f_k(\tau), \dot{w}_k(\tau)\rangle_{t_k^{j+1}} + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(\tau), \mathcal{E}\dot{w}_k(\tau)\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}_k(\tau), \mathcal{E}\dot{w}_k(\tau)\rangle_{H_n} \, d\tau +$$

$$- 2\int_0^{t_k^{j+1}} \langle \dot{u}_k(\tau), \dot{z}_k(\tau)\rangle_H \, d\tau + 2\langle \dot{u}_k(t), \dot{w}_k(t)\rangle_H - 2\langle u^1, \dot{w}_k(0)\rangle_H$$

Let $M_k := \sup_{t\in(0,T)} \|\dot{u}_k(t)\|_H$, $L_k := \sup_{t\in(0,T)} \|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}}$. By (54) we can give the estimate

$$M_k^2 + L_k^2 + \|\mathcal{E}\dot{u}_k\|_{L^2(0,T;H_n^{\mathbb{B}})}^2 \le a(M_k + L_k + \|\mathcal{E}\dot{u}_k\|_{L^2(0,T;H_n^{\mathbb{B}})}) + b, \tag{55}$$

where a and b are constants that depend only on $\|f\|_{L^2(0,T;V^*)}$, $\|w\|_{W^{1,2}(0,T;V)}$, $\|w\|_{W^{2,2}(0,T;H)}$, $\|u_1\|_H$ and on T. As a consequence we can deduce the following

(56) $\mathcal{E}u_k(t)$ and $\mathcal{E}\tilde{u}_k(t)$ are bounded in H_n uniformly in t and k,

(57) $\dot{u}_k(t)$ and $v_k(t)$ are bounded in H uniformly in t and k

(58) $\mathcal{E}\dot{u}_k$ is bounded in $L^2(0,T;H_n)$ uniformly in k.

Notice also that $u^0 \in H$ implies that u_k is bounded in H uniformly in t and k. This together with (56) gives

(59) $u_k(t)$ is bounded in V uniformly in t and k.

Furthermore, using (49)-(51) and (53), we can rewrite (45) for all $t \in (t_k^j, t_k^{j+1})$ as

$$\langle \dot{v}_k(t), \phi\rangle_H + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}_k(t), \mathcal{E}\phi\rangle_{H_n} = \langle f_k(t), \phi\rangle_{t_k^{j+1}} \tag{60}$$

for every $\phi \in V_{t_k^{j+1}}$. The last equation leads us to write for all $t \in (t_k^j, t_k^{j+1})$

$$\|\dot{v}_k(t)\|_{t_k^{j+1}}^* \le \|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|\mathcal{E}\dot{u}_k(t)\|_{H_n^{\mathbb{B}}} + \|f_k(t)\|_{t_k^{j+1}}^*. \tag{61}$$

In particular fix $s \in [0,T)$, then for every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}_k(t)\|_s^* \, dt \le \int_{t_1}^{t_2} (\|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|\mathcal{E}\dot{u}_k(t)\|_{H_n^{\mathbb{B}}} + \|f_k(t)\|_s^*) \, dt. \tag{62}$$

Using (56)-(58), and (61), there exists a constant M such that eventually passing through a subsequence (depending on $s \in [0,T)$), if we call v a weak limit of v_k in $W^{1,2}(s,T;V_s^*)$, and $\tilde{g}$ a weak limit of $t \mapsto \|\mathcal{E}\dot{u}_k(t)\|_{H_n}$ in $L^2(0,T)$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}(t)\|_s^* \, dt \le M|t_2 - t_1| + \int_{t_1}^{t_2} (\tilde{g}(t) + \|f(t)\|_s^*) \, dt, \tag{63}$$

for every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$.

Now if we fix a dense set $D \subset [0,T]$ by using a diagonal argument we obtain a subsequence, not relabeled, such that

$$u_k \rightharpoonup u \text{ weakly in } W^{1,2}(0,T;V) \tag{64}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;H), \tag{65}$$

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s,T;V_s^*) \text{ for every } s \in D, \tag{66}$$

and

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}(t)\|_s^* \, dt \le M|t_2 - t_1| + \int_{t_1}^{t_2} (\tilde{g}(t) + \|f(t)\|_s^*) \, dt, \tag{67}$$

for every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$.

Moreover by using the continuity of the projection maps P_{st}, it follows that (66) and (67) become

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s,T;V_s^*) \text{ for every } s \in [0,T), \tag{68}$$

and

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}(t)\|_s^* \, dt \le M|t_2 - t_1| + \int_{t_1}^{t_2} (\tilde{g}(t) + \|f(t)\|_s^*) \, dt, \tag{69}$$

for every $s \in [0,T)$ and every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$. In particular

$$\|P_{st}\dot{v}(t)\|_s^* \le M + \tilde{g}(t) + \|f(t)\|_s^*, \tag{70}$$

for every $s \in [0,T)$ and a.e. $t > s$.

By (57) it is easy to see that in fact

$$u \in W^{1,\infty}(0,T;H), \tag{71}$$

and this with convergence (64) gives (28).

Now we want to show

$$\dot{u}(t) = v(t) \text{ in } H \text{ for a.e. } t \in [0,T]. \tag{72}$$

First of all for $t \in (t_k^j, t_k^{j+1})$ we have $\dot{u}_k(t) = v_k(t_k^{j+1})$ and so

$$\|\dot{u}_k(t) - v_k(t)\|_{t_k^{j+1}}^* = \|v_k(t_k^{j+1}) - v_k(t)\|_{t_k^{j+1}}^* \le \int_{t_k^j}^{t_k^{j+1}} \|\dot{v}_k(\tau)\|_{t_k^{j+1}}^* \, d\tau \le \tau_k^{\frac{1}{2}} C, \tag{73}$$

where C is a uniform bound on the L^2 norm of the right-hand side of (61). Then for all $s < t$ we have $\|\dot{u}_k(t) - v_k(t)\|_s^* \le \tau_k^{\frac{1}{2}} C$, and this together with (65) implies $\dot{u}_k \rightharpoonup v$ weakly in $L^2(s,T;V_s^*)$ for any $s \in [0,T)$. But also $\dot{u}_k \rightharpoonup \dot{u}$ weakly in $L^2(0,T;H)$ by (64). So $v(t) = \dot{u}(t)$ in V_s^*, for every $s \in [0,T)$ and for a.e. $t \in (s,T)$. Since $v(t)$ and $\dot{u}(t)$ belong to H, and H is embedded in V_s^* for every $s \in [0,T]$, we finally get that $v(t) = \dot{u}(t)$ as elements of H for a.e. $t \in [0,T]$. This together with (64) and (69) allows to conclude that

$$u \in W^{2,2}(t,T;V_s^*) \text{ for every } s,t \in [0,T] \text{ with } s < t. \tag{74}$$

Let $g(t) := M + \tilde{g}(t) + \|f(t)\|_t^*$ then by (70) we have

$$\|P_{st}\ddot{u}(t)\|_s^* \le g(t), \tag{75}$$

for every $s \in [0,T)$ and for a.e. $t > s$, and we obtain (31).

Now we investigate the convergence of the constant piecewise interpolated $\tilde{u}_k$. Since by (64) u_k are Lipschitz with values in H uniformly in k, as before we get that

$$\tilde{u}_k \rightharpoonup u \text{ weakly in } L^2(0,T;H), \tag{76}$$

and since by (56) $\mathcal{E}\tilde{u}_k$ is bounded in $L^2(0,T;H_n)$, we also obtain that up to subsequences

$$\tilde{u}_k \rightharpoonup u \text{ weakly in } L^2(0,T;V) \tag{77}$$

Furthermore, note that $\tilde{u}_k(t-\tau_k) - \tilde{w}_k(t-\tau_k) \in V_t$ for every $t \in [0,T]$, and by (64) we can write for every $t \in (t_k^j, t_k^{j+1}]$

$$\|u_k(t) - \tilde{u}_k(t-\tau_k)\|_V = \|u_k(t) - u_k(t_k^j)\|_V \le \int_{t_k^j}^{t_k^{j+1}} \|\dot{u}_k(\tau)\|_V \, d\tau \le \tau_k^{1/2} C,$$

where C is a constant independent on k. This means that, by using also $w \in W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$, we can write

$$\tilde{u}_k(\cdot - \tau_k) - \tilde{w}_k(\cdot - \tau_k) \rightharpoonup u - w \text{ weakly in } L^2(0,T;V). \tag{78}$$

Since the linear subspace $\{v \in L^2(0,T;V) \mid v(t) \in V_t \text{ for a.e. } t \in [0,T]\}$ is strongly closed in $L^2(0,T;V)$, it is also weakly closed in $L^2(0,T;V)$. Therefore $u(t) \in V_t + w(t)$ for a.e. $t \in [0,T]$. Moreover for every $t \in (0,T]$ there exists an increasing sequence $t_i \in [0,T]$ converging to t such that $u(t_i) - w(t_i) \in V_{t_i}$ for every i. Thanks to (64) we know that $t \mapsto u(t) - w(t)$ is a strongly continuous map with values in V, and we obtain $u(t) - w(t) \in V_t$ for every $t \in (0,T]$. Together with the initial condition $u(0) = u^0 \in V_0$ we obtain (29). Moreover thanks to (74) and (75) we are in position to apply Lemma 2.14 to the function $u - w$ and hence to deduce that for a.e. $t \in [0,T]$

$$\frac{\dot{u}(t+h) - \dot{u}(t)}{h} \rightharpoonup \ddot{u}(t) \text{ weakly in } V_t^* \text{ as } h \to 0^+. \tag{79}$$

Now we want to show that (33) holds for a.e. $t \in [0,T]$ for every $\phi \in V_t$. We claim that there exists a negligible set $W \subset [0,T]$ such that for $s \in D$ and for all $\phi \in V_s$, we have

$$\langle \ddot{u}(t), \phi \rangle_s + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot{u}(t), \mathcal{E}\phi \rangle_{H_n} = \langle f(t), \phi \rangle_s, \tag{80}$$

for every $t \in (s,T] \setminus W$.

To prove the claim, first we fix $s \in D$ and $\phi \in V_s$. Using (60) we have for a.e. $t > s$

$$\langle \dot{v}_k(t), \phi \rangle_s + \langle \mathbb{C}\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot{u}_k(t), \mathcal{E}\phi \rangle_{H_n} = \langle f_k(t), \phi \rangle_s. \tag{81}$$

Hence we have also

$$\int_s^T (\langle \dot{v}_k(t), \phi \rangle_s + \langle \mathbb{C}\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot{u}_k(t), \mathcal{E}\phi \rangle_{H_n} - \langle f_k(t), \phi \rangle_s) \, dt = 0. \tag{82}$$

By construction $f_k \to f$ strongly in $L^2(0,T;V^*)$. We know that $\dot{v}_k \rightharpoonup \dot{v}$ weakly in $L^2(s,T;V_s^*)$ by (125). Since $\dot{u} = v$ in $W^{1,2}(s,T;V_s^*)$, we also have that $\ddot{u} = \dot{v}$ in $L^2(s,T;V_s^*)$. Using also (64) and (77), we can pass to the limit in (82) to have

$$\int_s^T (\langle \ddot{u}(t), \phi \rangle_s + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot{u}(t), \mathcal{E}\phi \rangle_{H_n} - \langle f(t), \phi \rangle_s) \, dt = 0. \tag{83}$$

By (81) we deduce that the integrand in (83) is zero for a.e. $t > s$. Since V_s is separable, the set N_s of $t > s$ for which (81) does not hold can be taken independent of ϕ. We set W to be the union over $s \in D$ of the sets N_s, so that W also has measure zero. It follows that for every $s \in D$ and for every $t \in (s,T] \setminus W$ (80) holds, and this shows the claim.

Using Lemma 2.14 it follows that for a.e. t and for every $\phi \in V_t$, there exist $s_i \nearrow t$ with $s_i \in D$ and $\phi_i \in V_{s_i}$, such that $\phi_i \to \phi$ strongly in V_t. Now note that if t belongs also to $(0,T] \setminus W$, by our previous claim we have

$$\begin{aligned}(84)\qquad &\langle \ddot u(t), \phi_i\rangle_t + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi_i\rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot u(t), \mathcal{E}\phi_i\rangle_{H_n} - \langle f(t), \phi_i\rangle_t \\ &= \langle \ddot u(t), \phi_i\rangle_{s_i} + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi_i\rangle_{H_n} + \langle \mathbb{B}\mathcal{E}\dot u, \mathcal{E}\phi_i\rangle_{H_n} - \langle f(t), \phi_i\rangle_{s_i} = 0.\end{aligned}$$

The convergence of the ϕ_i to ϕ gives (33).

Since by construction

$$(85)\qquad f_k \to f \text{ strongly in } L^2(0,T;V^*),$$

$$(86)\qquad w_k \to w \text{ strongly in } W^{1,2}(0,T;H^1(\Omega)^n),$$

$$(87)\qquad \dot w_k \to \dot w \text{ strongly in } H \text{ for every } t \in [0,T],$$

$$(88)\qquad \dot z_k \to L^2(0,T;H) \text{ strongly in } L^2(0,T;H),$$

using also (64) and (77), passing to the limit as $k \to \infty$ in (54), we obtain (39) by lower semicontinuity.

To prove (34), it is equivalent to show that there exists a set $N \subset [0,T]$ of measure zero such that for every $t_i \in [0,T] \setminus N$ with $t_i \to 0$, we have

$$(89)\qquad \dot u(t_i) \to u^1 \text{ strongly in } H$$

(by Remark 2.12 $\dot u(t)$ is a well defined element in H for every $t \in [0,T]$). By (28) and (39) we have

$$(90)\qquad \|\dot u(t)\|_H^2 + \|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 \le \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}^2 + o(1), \quad \text{as } t \to 0^+$$

for every $t \in [0,T] \setminus N$ where $N \subset [0,T]$ is a set of measure zero. Now let (t_i) be such that $t_i \in [0,T] \setminus N$ and $t_i \to 0$. By Remark 2.12, we already know that

$$(91)\qquad \dot u(t) \rightharpoonup \dot u(0) \quad \text{weakly in } H, \text{ as } t \to 0^+.$$

Moreover by (68) together with (72) we can write for a.e. $t \in [0,T]$

$$(92)\qquad \dot u(t) - \dot u(0) = \int_0^t \ddot u(\tau)\, d\tau = \lim_{k\to\infty} \int_0^t \dot v_k(\tau)\, d\tau = \lim_{k\to\infty} v_k(t) - u^1 \text{ in } V_0^*,$$

hence by choosing t such that (up to subsequences) $v_k(t) \to \dot u(t)$ in V_0^* (which is possible by (68)), we deduce that $\dot u(0) = u^1$ in V_0^*. Since both $\dot u(0)$ and u^1 are elements of H, and H is embedded in V_0^*, we deduce $\dot u(0) = u^1$ in H. Therefore the convergence (91) becomes

$$\dot u(t_i) \rightharpoonup u^1 \quad \text{weakly in } H.$$

This means that (89) is equivalent to

$$(93)\qquad \limsup_{i\to\infty} \|\dot u(t_i)\|_H^2 \le \|u^1\|_H^2.$$

Since by (28) $t \mapsto \mathcal{E}u(t)$ is strongly continuous in H_n we clearly have $\mathcal{E}u(t_i) \to \mathcal{E}u(0)$ strongly in H_n. By using the convergence (64) and a similar argument to (92) we deduce that $u(0) = u^0$, hence that $\mathcal{E}u(t_i) \to \mathcal{E}u^0$ strongly in H_n. Therefore, by using t_i in inequality (90) and passing to the limit as $i \to \infty$, we deduce exactly (93) and hence also (89).

□

Proposition 3.4. *Let u be the function given by Theorem 4.2, then u satisfies condition (32) and the energy balance (38) for every t Lebesgue point of $\|\dot{u}(\cdot)\|_H^2$.*

Proof. Let w be the Dirichlet boundary condition considered in Definition 3.1. We note that for every $h \in (0,T)$ and for a.e. $t \in [0,T]$ the functions $u(t) - u(t-h) - (w(t) - w(t-h)) \in V_t$. Hence if we define $z(t) := u(t) - w(t)$ we can test equation (33) with $\frac{z(t)-z(t-h)}{h}$ and integrate on (h,T) to get

$$
\begin{aligned}
&\int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_h^t \left\langle \mathbb{C}\,\mathcal{E}u(\tau), \frac{\mathcal{E}z(\tau)-\mathcal{E}z(\tau-h)}{h} \right\rangle_{H_n} d\tau \\
&\quad + \int_h^t \left\langle \mathbb{B}\,\mathcal{E}\dot{u}(\tau), \frac{\mathcal{E}z(\tau)-\mathcal{E}z(\tau-h)}{h} \right\rangle_{H_n} d\tau \\
&\quad = \int_h^t \left\langle f(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_h^t \left\langle F(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\Theta d\tau.
\end{aligned} \tag{94}
$$

Since $u \in W^{1,2}(0,T;V)$ and $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;V)$, we take the limit as $h \to 0^+$ on both side of the previous equality

$$
\begin{aligned}
&\lim_{h\to 0^+} \int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_0^t \langle \mathbb{C}\,\mathcal{E}u(\tau), \mathcal{E}\dot{z}(\tau)\rangle_{H_n} d\tau \\
&\quad + \int_0^t \langle \mathbb{B}\,\mathcal{E}\dot{u}(\tau), \mathcal{E}\dot{z}(\tau)\rangle_{H_n} d\tau = \int_0^t \langle f(\tau), \dot{z}(t)\rangle_\tau d\tau + \int_0^t \langle F(\tau), \dot{z}(\tau)\rangle_\Theta d\tau
\end{aligned} \tag{95}
$$

In order to compute the limit of the first term in the left hand-side of (95), we use Lemma 2.16 to write

$$
\begin{aligned}
\int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau &= \left\langle \dot{u}(t), \frac{z(t)-z(t-h)}{h} \right\rangle_H - \left\langle \dot{u}(h), \frac{z(h)-z(0)}{h} \right\rangle_H \\
&\quad - \int_h^t \left\langle \dot{u}(\tau), \frac{\dot{z}(\tau)-\dot{z}(\tau-h)}{h} \right\rangle_H d\tau
\end{aligned} \tag{96}
$$

Now, since t is a Lebesgue point of $\|\dot{u}(\cdot)\|_H^2$ and u satisfies the initial condition (34) we have

$$
\lim_{h\to 0^+} \left\langle \dot{u}(t), \frac{u(t)-u(t-h)}{h} \right\rangle_H = \|\dot{u}(t)\|_H^2 \text{ and } \lim_{h\to 0^+} \left\langle \dot{u}(h), \frac{u(h)-u(0)}{h} \right\rangle_H = \|u^1\|_H^2.
$$

Moreover using the identity $\langle \dot{u}(\tau), \dot{u}(\tau) - \dot{u}(\tau-h)\rangle_H = -\frac{1}{2}\|\dot{u}(\tau-h)\|_H^2 + \frac{1}{2}\|\dot{u}(\tau)\|_H^2 + \frac{1}{2}\|\dot{u}(\tau)-\dot{u}(\tau-h)\|_H^2$, we can write

$$
\begin{aligned}
\int_h^t \left\langle \dot{u}(\tau), \frac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h} \right\rangle_H d\tau &= \frac{1}{2h}\int_t^{t+h} \|\dot{u}(\tau)\|_H^2 \, d\tau - \frac{1}{2h}\int_0^h \|\dot{u}(\tau)\|_H^2 \, d\tau \\
&\quad + \frac{1}{2h}\int_h^t \|\dot{u}(\tau)-\dot{u}(\tau-h)\|_H^2 \, d\tau.
\end{aligned} \tag{97}
$$

Again using condition (34) and the fact that t is a Lebesgue point of $\|\dot u(\cdot)\|_H^2$, we can write

$$\lim_{h\to 0^+}\int_h^t \left\langle \dot u(\tau), \frac{\dot u(\tau)-\dot u(\tau-h)}{h}\right\rangle_H d\tau = \frac{1}{2}\|\dot u(t)\|_H^2 - \frac{1}{2}\|u^1\|_H^2 + \lim_{h\to 0^+}\frac{1}{2}\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h}\, d\tau, \tag{98}$$

By using the regularity assumption $w \in W^{2,2}(0,T;H)$ a simple calculation leads to

$$\lim_{h\to 0^+}\int_h^t \left\langle \ddot u(\tau), \frac{w(\tau)-w(\tau-h)}{h}\right\rangle_\tau d\tau = \langle \dot u(t), \dot w(t)\rangle_H - \langle u^1, \dot w(0)\rangle_H - \int_0^t \langle \dot u(\tau), \ddot w(\tau)\rangle_H\, d\tau. \tag{99}$$

Putting together (97) with (99), and using the fact that $z = u - w$, we can take the limit on both sides of(96) to get

$$\begin{aligned}\lim_{h\to 0^+}\int_h^t \left\langle \ddot u(\tau), \frac{z(\tau)-z(\tau-h)}{h}\right\rangle_\tau d\tau &= \frac{1}{2}\|\dot u(t)\|_H^2 - \frac{1}{2}\|u^1\|_H^2 \\ + \lim_{h\to 0^+}\frac{1}{2}\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h}\, d\tau &+ \frac{1}{2}\langle \dot u(t), \dot w(t)\rangle_H \\ -\frac{1}{2}\langle u^1, \dot w(0)\rangle_H &+ \frac{1}{2}\int_0^t \langle \dot u(\tau), \ddot w(\tau)\rangle_H\, d\tau.\end{aligned} \tag{100}$$

Putting together (95) with (100) we obtain for every Lebesgue point t of $\|\dot u(\cdot)\|_H^2$

(101)

$$\begin{aligned}\frac{1}{2}\|\dot u(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \int_0^t \|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 d\tau - \lim_{h\to 0^+}\frac{1}{2}\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h} d\tau \\ = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t).\end{aligned}$$

But since we already know that u satisfies the energy inequality (39), we immediately conclude that both condition (32) and the energy balance (38) hold. □

Remark 3.5. At each time t we expect that any energy increment for the solution $u(t)$, is due to the external forces. By identity (101), which is true for every u satisfying (28)-(31) and (33), the requirement (32) i.e.

$$\lim_{h\to 0^+}\frac{1}{2}\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h} d\tau = 0,$$

is natural in term of energy balance: indeed, the sum of the kinetic energy plus elastic energy plus viscous energy at time t cannot exceed the amount of energy at time zero plus the work done by the external forces in the time interval $[0,t]$.

Remark 3.6. Since $\dot u(t) - \dot w(t) \in V_t$ for a.e. $t \in [0,T]$, we can use it as test function in (33) and integrate on $(0,t)$ to obtain

$$\int_0^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau\, d\tau + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \int_0^t \|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2\, d\tau = \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}^2 + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t).$$

Comparing this last identity with the energy balance (38), we have for a.e. $t \in (0,T)$ that

$$\frac{1}{2}\|\dot u(t)\|_H^2 - \frac{1}{2}\|u^1\|_H^2 = \int_0^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \, d\tau,$$

and since $\tau \mapsto \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \in L^1(0,T)$ we deduce that $\|\dot u(\cdot)\|_H^2 \in W^{1,1}(0,T)$.

Putting together Theorem 3.2 and Proposition 3.4, we deduce the existence of a solution u to the damped system of elastodynamics. Moreover using (101) we can also obtain the uniqueness of weak solutions considered in Definition 3.1. This is the content of the next theorem.

Theorem 3.7 (Existence and uniqueness)**.** *Under hypothesis of Theorem 3.2 there exists a unique solution u considered in Definition 3.1. Moreover u satisfies the energy balance*

$$\begin{aligned}\frac{1}{2}\|\dot u(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \frac{1}{2}\int_0^t \frac{1}{2}\|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 \, d\tau = \\ = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t)\end{aligned} \tag{102}$$

for a.e. $t \in [0,T]$.

Proof. The existence of a solution satisfying (102) is simply a consequence of Theorem 3.2 and Proposition 3.4.

To show uniqueness, we notice that (28)-(33) are all preserved under linear combinations. Therefore, the difference v between two solutions is a solution with Dirichlet and Neumann homogeneous conditions, with forcing term $f = 0$, and satisfying $v(0) = 0$ and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot v(t)\|_h^2 \, dt = 0.$$

Moreover using the same argument as in Proposition 3.4, since (32) holds for v, we have

$$\frac{1}{2}\|\dot u(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \frac{1}{2}\int_0^t \frac{1}{2}\|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 \, d\tau = 0,$$

for a.e. $t \in [0,T]$. Therefore $\dot v(t) = 0$ a.e. on $[0,T]$. Since $v \in W^{1,\infty}(0,T;H)$ and $v(0) = 0$, we conclude $v(t) = 0$ a.e. on $[0,T]$. □

Finally, one can also prove that the energy balance (38) holds for every $t \in [0,T]$ and that the map $t \mapsto \dot u(t)$ is strongly continuous in H. For the proof of this result we refer to [8, Lemma 3.10].

Proposition 3.8. *Under the assumptions of Theorem 3.2 , let u be the weak solution of the damped wave equation considered in Definition (3.1), with initial conditions $u(0) = u^0$ and (34) . Then $t \mapsto \dot u(t)$ is continuous from $[0,T]$ to H and the energy balance (38) holds for every $t \in [0,T]$.*

4. The undamped system of elastodynamics

In this section we study weak solutions of the undamped system of elastodynamics

$$\ddot u(t) - \operatorname{div}[\mathbb{C}\,\mathcal{E}u(t)] = f(t). \tag{103}$$

As for the damped case we give the definition of weak solution.

Definition 4.1. Assume (a), (b), (c) and (d). With the notation introduced in Section 2, let $f \in W^{1,2}(0,T;V^*)$, let $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;H^1(\Omega)^n)$ and let $F \in W^{1,2}(0,T;N_\Theta)$ where Θ is the function relative to the crack set Γ given by Theorem 2.5 . We say that u is a weak solution of (1) on the time dependent domain $t \mapsto \Omega \smallsetminus \Gamma(t)$ with Dirichlet boundary condition $w(t)$ on $\partial_D\Omega$, Neumann boundary condition $F(t)$ on $\partial_N\Omega$, and homogeneous Neumann boundary condition on $\Gamma(t)$, if

(104) $u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H)$.

(105) For every $t \in [0,T]$ $u(t) - w(t) \in V_t$.

(106) For every $s \in [0,T)$ $u \in W^{2,2}(s,T;V_s^*)$ and

(107) $\|P_{st}\ddot{u}(t)\|_s^* \le g(t)$ for a.e. $t \in (s,T)$, for some $g \in L^2(0,T)$.

For a.e. $t \in [0,T]$

(108) $\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} - \langle F(t), Tr(\phi)\rangle_\Theta = \langle f(t), \phi\rangle_t,$ for every $\phi \in V_t$

where $\ddot{u}(t)$ is the one given by Definition 2.15.

Given $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, since $t \mapsto u(t)$ is strongly continuous in H the initial value for u is well defined as element of H. Moreover we are able to prescribe the initial conditions respectively for $\mathcal{E}u(0)$ and $\dot{u}(0)$ by asking

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\mathcal{E}u(t) - \mathcal{E}u^0\|_{H_n}^2 \, dt = 0, \tag{109}$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot{u}(t) - u^1\|_H^2 \, dt = 0. \tag{110}$$

Since in this case $\dot{u}(t)$ is in general only an element of H, we need to consider also a weakened formulation of the work due to the Neumann boundary conditions. More precisely, the term appearing in the work due to the boundary forces $\mathcal{W}_{bdry}(u;t_1,t_2)$, which in the damped case read as

$$\int_{t_1}^{t_2} \langle F(t), \dot{u}(t)\rangle_\Theta \, dt,$$

becomes

$$\langle F(t_2), u(t_2)\rangle_\Theta - \langle F(t_1), u(t_1)\rangle_\Theta - \int_{t_1}^{t_2} \langle \dot{F}(t), u(t)\rangle_\Theta \, dt, \tag{111}$$

for every time interval $[t_1,t_2] \subset [0,T]$.

The following is the main result.

Theorem 4.2. *Assume (a), (b), (c) and (d). Let f, w and F be as in Definition 4.1. Then, given two initial conditions $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, there exists a solution u of* (1) *with initial conditions* (109) *and* (110)*. Moreover u satisfies the energy inequality*

$$\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 \le \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t) \tag{112}$$

for a.e. $t \in [0,T]$.

Proof. Since the argument is similar to the one given for Theorem 4.2, we simply give a sketch of the proof.

For $k \in \mathbb{N}$, we set $\tau_k := T/k$ and $t_k^j := j\tau_k$. For $j = 1, 2, \ldots, k$ we define $f_k^j \in V^*$ and $w_k^j \in H^1(\Omega)^n$ by

$$f_k^j := f(t_k^j),\ w_k^j := w(t_k^j), \tag{113}$$

using that $f \in W^{1,2}(0,T;V^*)$ and $w \in W^{1,2}(0,T;H^1(\Omega)^n)$, so f and w are well defined elements of V^* and $H^1(\Omega)^n$, respectively, for every $t \in [0,T]$. Inductively we define u_k^j for $j = -1, 0, \ldots, k$ by the following

$$u_k^0 := u^0,\ u_k^{-1} := u^0 - \tau_k u^1; \tag{114}$$

then, for $j = 0, 1, \ldots, k-1$, the function u_k^{j+1} is the minimizer in $V_{t_k^{j+1}} + w_k^j$ of

$$u \mapsto \left\| \frac{u - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k} \right\|_H^2 + \|\mathcal{E}u\|_{H_n^{\mathbb{C}}}^2 - 2\langle f_k^j, u\rangle_{t_k^{j+1}}. \tag{115}$$

Then if we define u_k, $\tilde{u}_k$, w_k, $\tilde{w}_k$, v_k, and z_k as in (49)-(53), then proceeding exactly as in 3.2, we deduce the following bounds

$$\mathcal{E}u_k(t) \text{ and } \mathcal{E}\tilde{u}_k(t) \text{ are bounded in } H_n \text{ uniformly in } t \text{ and } k, \tag{116}$$

$$\dot{u}_k(t) \text{ and } v_k(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } k \tag{117}$$

$$u_k(t) \text{ is bounded in } V \text{ uniformly in } t \text{ and } k. \tag{118}$$

Furthermore, using the Euler equation for u_k^{j+1} we can write for all $t \in (t_k^j, t_k^{j+1})$

$$\langle \dot{v}_k(t), \phi\rangle_H + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi\rangle_{H_n} = \langle f_k(t), \phi\rangle_{t_k^{j+1}} \tag{119}$$

for every $\phi \in V_{t_k^{j+1}}$. The last equation leads us to write for all $t \in (t_k^j, t_k^{j+1})$

$$\|\dot{v}_k(t)\|_{t_k^{j+1}}^* \le \|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|f_k(t)\|_{t_k^{j+1}}^*. \tag{120}$$

In particular fix $s \in [0,T)$, then for every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}_k(t)\|_s^*\, dt \le \int_{t_1}^{t_2} (\|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|f_k(t)\|_s^*)\, dt. \tag{121}$$

Again following exactly 3.2 we deduce that up to subsequences

$$u_k \rightharpoonup u,\ \text{weakly in } W^{1,2}(0,T;H), \tag{122}$$

$$\tilde{u}_k \rightharpoonup u,\ \text{weakly in } L^2(0,T;V) \tag{123}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;H), \tag{124}$$

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s,T;V_s^*). \tag{125}$$

Moreover $u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H)$, $\dot{u}(t) = v(t)$ a.e. on $[0,T]$, and

$$\|P_{st}\ddot{u}(t)\|_t^* \le M + \|f(t)\|_s^*, \tag{126}$$

for every $s \in [0,T)$ and for a.e. $t > s$.

Now the proof that u is a solution, and that satisfies the energy inequality (112) proceeds as in the damped case. Finally, it remains to prove that u satisfies the initial conditions (109) (110). It is enough to show that there exists a set $N \subset [0,T]$ of measure zero such that for every $t_i \in [0,T] \smallsetminus N$ with $t_i \to 0$, we have

$$\dot{u}(t_i) \to u^1 \text{ strongly in } H, \tag{127}$$

and

$$\mathcal{E}u(t_i) \to \mathcal{E}u^0 \text{ strongly in } H_n. \quad (128)$$

Again this can be achieved following a similar argument to the damped case. □

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