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The Dirichlet problem for the *p*-fractional Laplace equation

Giampiero Palatucci

To Carlo Sbordone on the occasion of his 70th birthday, with admiration.

Abstract We deal with a class of equations driven by nonlocal, possibly degenerate, integro-differential operators of differentiability order $s \in (0,1)$ and summability growth $p \in (1,\infty)$, whose model is the fractional p-Laplacian operator with measurable coefficients. We review several recent results for the corresponding weak solutions/supersolutions, as comparison principles, a priori bounds, lower semicontinuity, boundedness, Hölder continuity up to the boundary, and many others. We then discuss the good definition of (s,p)-superharmonic functions, and the nonlocal counterpart of the Perron method in nonlinear Potential Theory, together with various related results. We briefly mention some basic results for the obstacle problem for nonlinear integro-differential equations. Finally, we present the connection amongst the fractional viscosity solutions, the weak solutions and the aforementioned (s,p)-superharmonic functions, together with other important results for this class of equations when involving general measure data, and a surprising fractional version of the Gehring lemma.

We sketch the corresponding proofs of some of the results presented here, by especially underlining the development of new fractional localization techniques and other recent tools. Various open problems are listed throughout the paper.

Keywords Quasilinear nonlocal operators · integro-differential operators · fractional Sobolev spaces · fractional Laplacian operator · nonlocal tail · Harnack inequalities · the Perron method · Caccioppoli inequalities · fractional viscosity solutions · comparison estimates · obstacle problem · measure data

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1 Introduction

In this note we are interested in a very general class of nonlinear nonlocal equations, which include as a particular case some fractional Laplacian-type equations; that is, those related to the operator $\mathcal{L}$ defined on suitable fractional Sobolev functions by

$$\mathcal{L}u(x) = P.V.\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\,\mathrm{d}y, \qquad x\in\mathbb{R}^n. \quad (1)$$

The nonlinear nonlocal operator $\mathcal{L}$ in the display above is driven by its symmetric *kernel* $K:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$, which is a measurable function of differentiability order $s\in(0,1)$ and summability exponent $p\in(1,\infty)$,

$$\Lambda^{-1}\leq K(x,y)|x-y|^{n+sp}\leq\Lambda \quad \text{for a.e. } x,y\in\mathbb{R}^n,$$

for some $\Lambda\geq 1$. We immediately refer to Section 2 for the precise assumptions on the involved quantities in the general framework. In order to simplify, one can just keep in mind the model case when the kernel $K=K(x,y)$ does coincide with the Gagliardo kernel $|x-y|^{-n-sp}$; that is, when the corresponding equation reduces to

$$(-\Delta)^s_p\, u = 0 \text{ in } \mathbb{R}^n,$$

where the symbol $(-\Delta)^s_p$ denotes the so-called *fractional p-Laplacian* operator, though in such a case the difficulties arising from having merely measurable coefficients disappear. The fractional p-Laplacian operator is a nonlocal version of the p-Laplacian operator $(-\Delta)_p$, which in recent years attracted extensive attention, not only since it appears in many models naturally arising from concrete applications in Biology, Finance, Physics, but even from a pure mathematical point of view because of the challenging difficulties due to its both nonlocal and nonlinear definition (see Section 1.1 below). However, despite its relatively short history, this problem has already evolved into an elaborate theory with several connections to other branches; the literature is too wide to attempt any comprehensive treatment in a single paper. We refer the interested reader to the papers [11–13, 28, 30, 72, 80, 90], the forthcoming book [58], and the references therein.

In these notes, we survey several results for the weak solutions to the Dirichlet problem for the nonlocal p-Laplace equation, lead by the operator $\mathcal{L}$ defined in (1). Clearly, it is not reasonable to condensate in a single paper all the results achieved in the (even recent) literature on the topic, so that we focus on the most relevant theorems together with the sketch of some of the corresponding proofs and/or significative related remarks, as well in particular on the main results achieved by the author in collaboration with A. Di Castro, J. Korvenpää, and T. Kuusi, in the papers [26, 27, 48–51], where new localization techniques to attack general nonlocal problems have been developed.

1.1 From the fractional Laplacian to nonlinear integro-differential operators with measurable coefficients

Now, a few observations about the equation we are considering. First of all, it is worth noticing that the main difficulty into the treatment of the operators $\mathcal{L}$ in (1) lies in their very definition, which combines the typical issues given by its *nonlocal* feature together with the ones given by its *nonlinear* growth behavior; also, further efforts are needed due to the presence of merely measurable coefficients in the kernel K. For this, some very important tools recently introduced in the nonlocal theory, as the by-now classic s-harmonic extension ([17]), the strong three-term commutators estimates to deduce the regularity of weak fractional harmonic maps ([24]), the pseudo-differential commutator compactness in [73–75], the energy density estimates in [31, 76, 83], and many other successful tricks seem not to be trivially adaptable to the nonlinear framework considered here. Moreover, increased difficulties are due to the non-Hilbertian structure of the involved fractional Sobolev spaces $W^{s,p}$ when $p \neq 2$. In spite of that, some related regularity results have been very recently achieved in this context, in [1, 5, 8–10, 20, 33–36, 49, 51, 57, 59, 60, 66, 68, 78, 79, 91, 92] and many others, where often a fundamental role to understand the nonlocality of the nonlinear operators $\mathcal{L}$ has been played by a special quantity, *the nonlocal tail* given by the following

Definition 1 [The nonlocal tail; [26, 27]]**.** Let $s \in (0,1)$ and $p \in (1,\infty)$. The *nonlocal tail* of a function u in the ball of radius $r > 0$ centered in $z \in \mathbb{R}^n$, is given by

$$\mathrm{Tail}(u;z,r) := \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} |u(x)|^{p-1} |x-z|^{-n-sp}\, \mathrm{d}x \right)^{\frac{1}{p-1}}.$$

The nonlocal tail has already proven to be a key-point in the proofs when a fine quantitative control of the long-range interactions, naturally arising when dealing with nonlocal operators as in (1), is needed. As mentioned before, this quantity has been already used in many recent results on the topic (see Section 2 for further details).

In clear accordance with Definition 1, for any $s \in (0,1)$ and any $p \in (1,\infty)$, we consider the corresponding *tail space*. We have the following

Definition 2 [The tail space; [49, 51, 57]]. For any $s \in (0,1)$ and any $p \in (1,\infty)$, the *tail space* $L^{p-1}_{sp}(\mathbb{R}^n)$ is given by

$$L^{p-1}_{sp}(\mathbb{R}^n) := \Big\{ u \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \mathrm{Tail}(u;0,1) < \infty \Big\}.$$

In particular, if $u \in L^{p-1}_{sp}(\mathbb{R}^n)$, then $\mathrm{Tail}(u;z,r) < \infty$ for all $z \in \mathbb{R}^n$ and $r \in (0,\infty)$. It is worth noticing that the two definitions above are very natural, by involving essentially only the leading parameters defining the nonlocal nonlinear operators; i. e., their differentiability order s and their summability exponent p, and not the special form of any particular elliptic or parabolic equations involving the fractional p-Laplacian. For this, these definition seem to be the natural quantities to be taken into account when working in the fractional framework; see Section 2 below for further details.

2 Preliminaries

It is convenient to fix some notation which will be used throughout the rest of the paper. Firstly, notice that we will follow the usual convention of denoting by c a general positive constant which will not necessarily be the same at different occurrences and which can also change from line to line. For the sake of readability, dependencies of the constants will be often omitted within the chains of estimates, therefore stated after the estimate.

As customary, we denote by

$$B_R(x_0) = B(x_0;R) := \Big\{ x \in \mathbb{R}^n : |x-x_0| < R \Big\}$$

the open ball centered in $x_0 \in \mathbb{R}^n$ with radius $R > 0$. When not important and clear from the context, we shall use the shorter notation $B_R := B(x_0;R)$. We denote by βB_R the concentric ball scaled by a factor $\beta > 0$, that is

$\beta B_R := B(x_0;\beta R)$. Moreover, if $f \in L^1(S)$ and the n-dimensional Lebesgue measure $|S|$ of the set $S \subseteq \mathbb{R}^n$ is finite and strictly positive, we write

$$(f)_S := \fint_S f(x)\,\mathrm{d}x = \frac{1}{|S|}\int_S f(x)\,\mathrm{d}x.$$

Let $k \in \mathbb{R}$, we denote by

$$w_+(x) := (u(x)-k)_+ = \max\{u(x)-k,0\},$$

and

$$w_-(x) := (u(x)-k)_- = (k-u(x))_+.$$

Clearly $w_+(x) \neq 0$ in the set $\big\{x \in S : u(x) > k\big\}$, and $w_-(x) \neq 0$ in the set $\big\{x \in S : u(x) < k\big\}$.

Let $\Omega \subseteq \mathbb{R}^n$. We now recall the class of integro-differential equations in which we are interested; that is,

$$\mathcal{L}u(x) = P.\,V.\!\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\,\mathrm{d}y = 0, \quad x \in \Omega. \tag{2}$$

The nonlocal operator $\mathcal{L}$ in the display above (being read a priori in the principal value sense) is driven by its *kernel* $K\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, which is a measurable function satisfying the following property:

$$\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda \quad \text{for a. e. } x,y \in \mathbb{R}^n, \tag{3}$$

for some $s \in (0,1)$, $p \in (1,\infty)$, $\Lambda \geq 1$. As already noticed, in the special case when $p=2$ and $\Lambda = 1$ we recover, up to a multiplicative constant, the well-known fractional Laplacian operator $(-\Delta)^s$. Notice also that the assumption on K can be weakened as follows

$$\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda \quad \text{for a. e. } x,y \in \mathbb{R}^n \text{ s. t. } |x-y| \leq 1,$$

$$0 \leq K(x,y)|x-y|^{n+\eta} \leq M \quad \text{for a. e. } x,y \in \mathbb{R}^n \text{ s. t. } |x-y| > 1,$$

for some s, p, Λ as above, $\eta > 0$ and $M \geq 1$, as seen, e. g., in the recent series of papers by M. Kassmann (see for instance the more general assumptions in the breakthrough paper [43]). In the same sake of generalizing, one can also consider the operator $\mathcal{L} = \mathcal{L}_\Phi$ defined by

$$\mathcal{L}_\Phi u(x) = P.\,V.\!\int_{\mathbb{R}^n} K(x,y)\,\Phi\big(u(x)-u(y)\big)\,\mathrm{d}y, \quad x \in \Omega,$$

where the real function Φ is assumed to be continuous, satisfying $\Phi(0) = 0$ together with the monotonicity property

$$\lambda^{-1}|t|^p \leq \Phi(t)t \leq \lambda|t|^p \quad \text{for every } t \in \mathbb{R} \setminus \{0\},$$

for some $\lambda > 1$, and some p as above (see, for instance, [57]).

Now, we recall the definition of weak supersolutions to nonlinear integro-differential equations driven by the operator $\mathcal{L}$ in (2). For this, we need first to recall the definition of *the nonlocal tail* $\mathrm{Tail}(u; z, r)$ *of a function* f *in the ball of radius* $r > 0$ *centered in* $z \in \mathbb{R}^n$, given in the introduction. For any function u initially defined in $L^{p-1}_{\rm loc}(\mathbb{R}^n)$,

$$\mathrm{Tail}(u; z, r) := \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} |u(x)|^{p-1} |x-z|^{-n-sp} \,\mathrm{d}x \right)^{\frac{1}{p-1}}. \tag{4}$$

We seize here the opportunity to mention the breakthrough paper [43] by M. Kassmann, where the revisitation of the classical Harnack inequalities in a new nonlocal form the needing of incorporating some precise fractional terms; as well as the paper [29] by S. Dipierro Et Al., where it is shown that in order to deal with the asymptotic behavior of the fractional perimeter functionals a precise nonlocal quantity has to be taken into account.

In accordance with the definition in (4), we recall the definition of the corresponding *tail space* $L^{p-1}_{sp}(\mathbb{R}^n)$,

$$\begin{aligned} L^{p-1}_{sp}(\mathbb{R}^n) &:= \Big\{ f \in L^{p-1}_{\rm loc}(\mathbb{R}^n) \;:\; \mathrm{Tail}(u; z, r) < \infty \quad \forall z \in \mathbb{R}^n, \forall r \in (0, \infty) \Big\} \\ &\equiv \Big\{ u \in L^{p-1}_{\rm loc}(\mathbb{R}^n) \;:\; \mathrm{Tail}(u; 0, 1) < \infty \Big\}; \end{aligned}$$

see also [51, Section 2] for further details. As expected, one can check that $L^\infty(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$, where we denoted by $W^{s,p}(\mathbb{R}^n)$ the usual fractional Sobolev space of order (s, p), defined by the norm

$$\begin{aligned} \|u\|_{W^{s,p}(\mathbb{R}^n)} &:= \|u\|_{L^p(\mathbb{R}^n)} + [u]_{W^{s,p}(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} |u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}. \end{aligned}$$

Similarly, one can define the fractional Sobolev spaces $W^{s,p}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$. By $W^{s,p}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\mathbb{R}^n)$. For the basic properties of these spaces and some related topics, we refer the reader to [28] and the references therein.

We finally observe that, since we assume that the coefficients are merely measurable, the involved equation has to have a suitable weak formulation. For this, we recall the definitions of sub and supersolutions u to $\mathcal{L}u = 0$ in $\mathbb{R}^n$.

Definition 3 [Fractional weak supersolution; [26, 27, 49, 51]**].** Let $s \in (0, 1)$ and $p \in (1, \infty)$. A function $u \in W^{s,p}_{\rm loc}(\Omega)$ such that u_- belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$

is a *fractional weak p-supersolution* of (2) if

$$\begin{aligned}\langle \mathcal{L}u,\eta\rangle \equiv &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\eta(x)-\eta(y)\big)\,\mathrm{d}x\mathrm{d}y\\ &\geq 0 \end{aligned} \tag{5}$$

for every nonnegative $\eta \in C_0^\infty(\Omega)$.

Definition 4 [Fractional weak solution; [26, 27, 49, 51]]. Let $s \in (0,1)$ and $p \in (1,\infty)$. A function $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a *fractional weak p-subsolution* if $-u$ is a fractional weak p-supersolution.
A function u is a *fractional weak p-solution* if it is both fractional weak p-sub and supersolution.

In the rest of the paper, we suppress p from notation, by simply saying that u is a weak supersolution in Ω. Above, $\eta \in C_0^\infty(\Omega)$ can be replaced by $\eta \in W_0^{s,p}(D)$ with every $D \Subset \Omega$. Furthermore, it can be extended to a $W^{s,p}$-function in the whole $\mathbb{R}^n$ (see, e. g., Section 5 in [28]).

Finally, let us remark that we will assume that the kernel K is symmetric, which is not restrictive, in view of the weak formulation presented in Definition 3, since one may always define the corresponding symmetric kernel $K_{\rm sym}$ given by

$$K_{\rm sym}(x,y) := \frac{1}{2}\Big(K(x,y)+K(y,x)\Big).$$

Remark 1 It is worth noticing that, in accordance with the previous considerations, the summability assumption of u_- belonging to the tail space $L^{p-1}_{sp}(\mathbb{R}^n)$ is what one expects in the nonlocal framework considered here. This is one of the novelty with respect to the analog of the definition of supersolutions in the local case; i. e., when $s=1$, and it is necessary since here one has to use in a precise way Definition 1 in order to deal with the long-range interactions. For further motivations, see Section 2 in [51], and also, the regularity estimates in [26, 27, 48, 49, 56, 57].

Now, an important observation is in order. In Definition 3 it makes no difference to assume $u \in L^{p-1}_{sp}(\mathbb{R}^n)$ instead of $u_- \in L^{p-1}_{sp}(\mathbb{R}^n)$, as the next lemma implies.

Lemma 1 [Tail estimates; [51]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a weak supersolution in $B_{2r}(x_0)$. Then, for $c \equiv c(n,p,s)$,*

$$\begin{aligned}&\mathrm{Tail}(u;x_0,r)\\ &\quad\leq c\left(r^{\frac{sp-1-n}{p-1}}\,[u]_{W^{h,p-1}(B_r(x_0))} + r^{-\frac{n}{p-1}}\,\|u\|_{L^{p-1}(B_r(x_0))} + \mathrm{Tail}(u_-;x_0,r)\right)\end{aligned}$$

with

$$h = \max\left\{0, \frac{sp-1}{p-1}\right\} < s.$$

In particular, if u is a weak supersolution in an open set Ω, then $u \in L^{p-1}_{sp}(\mathbb{R}^n)$.

Proof Firstly, we write the weak formulation, for nonnegative $\phi \in C_0^\infty(B_{r/2}(x_0))$ such that $\phi \equiv 1$ in $B_{r/4}(x_0)$, with $0 \leq \phi \leq 1$ and $|\nabla\phi| \leq 8/r$. We have

$$\begin{aligned} 0 &\leq \int_{B_r(x_0)}\int_{B_r(x_0)} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\phi(x)-\phi(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n\setminus B_r(x_0)}\int_{B_{r/2}(x_0)} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\phi(x)K(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= I_1 + I_2. \end{aligned}$$

The first term can be easily estimated using $|\phi(x)-\phi(y)| \leq 8|x-y|/r$ to get

$$I_1 \leq \frac{c}{r^{\min\{sp,1\}}}\,[u]^{p-1}_{W^{h,p-1}(B_r(x_0))}$$

In order to estimate the second term, we can observe that

$$|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big) \leq 2^{p-1}\big(u_+^{p-1}(x) + u_-^{p-1}(y)\big) - u_+^{p-1}(y),$$

and thus

$$\begin{aligned} I_2 &\leq c\,r^{-sp}\,\|u\|^{p-1}_{L^{p-1}(B_r(x_0))} + c\,r^{n-sp}\mathrm{Tail}(u_-;x_0,r)^{p-1} \\ &\quad - \frac{r^{n-sp}}{c}\mathrm{Tail}(u;x_0,r)^{p-1}. \end{aligned}$$

By combining all the displays above, we obtain the desired estimate. The second statement plainly follows by an application of Hölder's Inequality. □

For related properties of the fractional weak solutions in the linear case without coefficients, when the operator in (2) does reduce to the usual fractional Laplacian operator $(-\Delta)^s$, we refer for instance to [72, 85, 86], and the references therein.

2.1 Basic results

We consider the following functional in $W^{s,p}(\mathbb{R}^n)$,

$$\mathcal{F}(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^p\,\mathrm{d}x\mathrm{d}y. \tag{6}$$

In view of the assumptions (3) on K, one can use the standard Direct Method to prove that there exists a unique p-minimizer of $\mathcal{F}$ over all $u \in W^{s,p}(\Omega)$ such that $u = g$ in $\mathbb{R}^n \setminus \Omega$ whose Tail is finite. Moreover, a p-minimizer u is a weak solution solution to problem (2) and vice versa, as stated in Theorem 1 below.

Let $g \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be the boundary value, we define

$$\mathcal{K}_g(\Omega) := \Big\{ u \in W^{s,p}(\Omega) \, : \, u = g \ \text{ a.e. on } \mathbb{R}^n \setminus \Omega \Big\};$$

i. e., the class where we are seeking solutions to the Dirichlet boundary value problem. Observe that we are assuming that g has bounded fractional Sobolev norm in a set Ω, and not necessarily in the whole $\mathbb{R}^n$ as in the literature before the aforementioned papers [26, 27, 48, 49, 57], where it has arises that the tail spaces L^{p-1}_{sp} are the optimal spaces to be considered for a wider generality.

Theorem 1 [Existence and uniqueness for fractional minimizers; [27, Theorem 2.3]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$, and let $g \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Then there exists a minimizer u of (6) over $\mathcal{K}_g$. Moreover, the solution is unique. Also, a function $u \in \mathcal{K}_g$ is a minimizer of (6) over $\mathcal{K}_g$ if and only if it is a weak solution to problem (2).*

Proof One can take any minimizing sequence $u_j \in \mathcal{K}_g$. Due to the assumptions on the kernel K, one can control the fractional seminorm of u_j, so that, one can find by pre-compactness in L^p (see, for instance, [28, Theorem 6.7]) a subsequence u_{j_k} converging pointwise a. e. to a function $u \in \mathcal{K}_g$. By Fatou's Lemma we deduce that u is actually a minimizer of (6) over $\mathcal{K}_g$. The uniqueness follows from the strict convexity of the functional.

Furthermore, the fact that u solves the corresponding Euler-Lagrange equation follows by perturbing $u \in \mathcal{K}_g$ with a test function in a standard way. Indeed, supposing that $u \in \mathcal{K}_g$ is a minimizer of (6) over $\mathcal{K}_g$, take any $\phi \in W^{s,p}_0(\Omega)$ and calculate formally

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u+t\phi)\Big|_{t=0} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{\mathrm{d}}{\mathrm{d}t}|u(x)-u(y)+t(\phi(x)-\phi(y))|^p\,\mathrm{d}x\mathrm{d}y\Big|_{t=0}\\ &= p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))\,\mathrm{d}x\mathrm{d}y\,.\end{aligned}$$

Since u is a minimizer, the term on the left is zero and hence $u \in \mathcal{K}_g$ is a weak solution to problem (2). For the converse, let $u \in \mathcal{K}_g$ be a weak solution to problem (2) and take $\phi = u - v \in W^{s,p}_0(\Omega)$, where $v \in \mathcal{K}_g$. Then, by Young's Inequality,

$$\begin{aligned}0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))\,\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p}\,\mathrm{d}x\mathrm{d}y\\ &\quad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^p\,\mathrm{d}x\mathrm{d}y \\
&\quad -\frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|v(x)-v(y)|^p\,\mathrm{d}x\mathrm{d}y\,,
\end{aligned}
$$

and hence u is a minimizer of (6) over $\mathcal{K}_g$. □

3 First fundamental estimates

In this section, we report some results from [26, 27, 49], to which we refer for a more complete presentation and for detailed proofs. We present some relevant estimates that, despite being established very recently, have been already extensively used and generalized in many results for equations involving the (nonlinear) fractional Laplacian and related nonlocal operators. The first of them states a natural extension of the well-known Caccioppoli inequality to the nonlocal framework, by showing that in such a case one can take into account the nonlocal tail, in order to detect deeper informations.

3.1 The Caccioppoli inequality with tail

We have the following

Theorem 2 [Caccioppoli estimate with tail; [27, Theorem 1.4]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a weak supersolution to (2). Then, for any $B_r \equiv B_r(z) \subset \Omega$ and any nonnegative $\varphi \in C_0^\infty(B_r)$, the following estimate holds true*

$$
\begin{aligned}
&\int_{B_r}\int_{B_r} K(x,y)|w_-(x)\varphi(x)-w_-(y)\varphi(y)|^p\,\mathrm{d}x\mathrm{d}y \\
&\qquad \leq c\int_{B_r}\int_{B_r} K(x,y)\big(\max\{w_-(x),w_-(y)\}\big)^p|\varphi(x)-\varphi(y)|^p\,\mathrm{d}x\mathrm{d}y \qquad (7)\\
&\qquad\quad +c\int_{B_r} w_-(x)\varphi^p(x)\,\mathrm{d}x\left(\sup_{y\,\in\,\mathrm{supp}\,\varphi}\int_{\mathbb{R}^n\setminus B_r} K(x,y)w_-^{p-1}(x)\,\mathrm{d}x\right),
\end{aligned}
$$

where $w_- := (u-k)_-$ for any $k \in \mathbb{R}$, K is any measurable kernel satisfying (3), and c depends only on p.

Remark 2 We underline that the estimate in (7) does hold by replacing w_- with $w_+ := (u-k)_+$ in the case when u is a fractional weak subsolution.

As mentioned before, in the nonlocal framework one has to take into account a suitable tail. Indeed, the second term in the right hand-side of (7) is controlled by the nonlocal tail given in Definition 1. To be more precise, we have

$$\left(\sup_{y\in B_r}\int_{\mathbb{R}^n\setminus B_r} K(x,y)w_\pm^{p-1}(x)\,\mathrm{d}x\right) \le cr^{-sp}\Big(\mathrm{Tail}(w_\pm;x_0,r)\Big)^{p-1};$$

see the proof of Lemma 5.1 in [27].

Remark 3 A fractional Caccioppoli-type inequality not taking into account the nonlocal tail was firstly implemented in the paper [71] to prove point-wise gradient bounds via Riesz potentials for solutions to general quasilinear equations; see in particular Section 2 there. Subsequently, other p-fractional Caccioppoli-type inequalities have been successfully used in different contexts to achieve important regularity results (see, for instance, [2, 19, 53–55, 69, 70]), and basic results for nonlocal fractional p-eigenvalues (see [34]): none of them still take into account the tail contribution. In the linear case when $p = 2$, we refer to [41] for similar estimates with additional nonlocal terms.

3.2 Boundedness up to the boundary

A first natural consequence of the Caccioppoli inequality with tail is the local boundedness of fractional weak subsolutions, as stated in the following

Theorem 3 [Local boundedness, with an interpolating estimate, [27, Theorem 1.1 and Remark 4.2]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a weak subsolution to (2) and let $B_r \equiv B_r(z) \subset \Omega$. Then the following estimate holds true*

$$\operatorname*{ess\,sup}_{B_{r/2}} u \le \delta\,\mathrm{Tail}(u_+;x_0,r/2) + c\,\delta^{-\gamma}\left(\fint_{B_r} u_+^p\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{8}$$

where $\mathrm{Tail}(\cdot)$ *is defined in* (1), $\gamma = (p-1)n/sp^2$, *the real parameter* $\delta \in (0,1]$, *and the constant c depends only on n, p, s, and Λ.*

To our knowledge, such a result was firstly proven in the paper [27] even in the linear case when $p = 2$; in the previous literature, the boundedness was assumed a priori, as for instance in fractional Hölder-type inequalities and other related results. Also, it is worth noticing that the parameter δ in (8) allows a precise *interpolation between the local and nonlocal terms*, which turned to be useful in some subsequent results, as we will see in some of the proofs sketched in the following sections.

The result above can be generalized at some extents up to the boundary of Ω. For this, one has only to assume that the complement of Ω satisfies the following measure density condition: there exist $\delta_\Omega \in (0,1)$ and $r_0 > 0$ such that for every $x_0 \in \partial\Omega$

$$\inf_{0<r<r_0} \frac{|(\mathbb{R}^n \setminus \Omega) \cap B_r(x_0)|}{|B_r(x_0)|} \geq \delta_\Omega. \tag{9}$$

This requirement is in the same spirit of the classical nonlinear Potential Theory (see, e. g., [64]), and – as expected in view of the nonlocality of the involved equations – is translated into an information given on the complement of the set Ω. Also, it is worth noticing that this is an improvement with respect to the previous boundary regularity results in all the fractional literature when much stronger Lipschitz regularity or smoothness of the sets are usually assumed (see also forthcoming Remark 7).

Theorem 4 [Boundedness up to the boundary; [49, Theorem 5]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Suppose that the domain $\Omega \subset \mathbb{R}^n$ satisfies the measure density condition in* (9)*. Let $x_0 \in \partial\Omega$ and suppose that the boundary data g is essentially bounded close to x_0. Let u be a weak solution to* (2)*. Then u is essentially bounded close to x_0 as well.*

The result above is merely a plain application of a more general result for the fractional obstacle problem, which states that when the obstacle and the boundary values are bounded on the boundary, so is the solution to the obstacle problem. It just suffices to consider the special case when there is no obstacle at all; precisely, take $\Omega' \equiv \Omega$ and $h \equiv -\infty$ in forthcoming Theorem 6. We first need to introduce a suitable set of notation, and for this we recall here some of the introductory facts in [49], to which we refer for further details. Let $\Omega \Subset \Omega'$ be open bounded subsets of $\mathbb{R}^n$. Let $h\colon \mathbb{R}^n \to [-\infty,\infty)$ be an extended real-valued function, which is considered to be the obstacle, and let $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be the boundary values. We define

$$\mathcal{K}_{g,h}(\Omega,\Omega') := \Big\{ u \in W^{s,p}(\Omega') \,:\, u \geq h \ \text{ a.e. in } \Omega,\ u = g \ \text{ a.e. on } \mathbb{R}^n \setminus \Omega \Big\}.$$

The interpretation for the case $h \equiv -\infty$ is that

$$\mathcal{K}_g(\Omega,\Omega') \equiv \mathcal{K}_{g,-\infty}(\Omega,\Omega') := \Big\{ u \in W^{s,p}(\Omega') \,:\, u = g \ \text{ a.e. on } \mathbb{R}^n \setminus \Omega \Big\},$$

i. e., the class where we are seeking solutions to the Dirichlet boundary value problem. We observe that a natural assumption for any existence theory is that $\mathcal{K}_{g,h}(\Omega,\Omega')$ is a non-empty set; this is a property of functions g and h.

As well known, the obstacle problem can be reformulated as a standard problem in the theory of variational inequalities on Banach spaces, by seeking the energy minimizers in the set of suitable functions defined above. For this, by taking into account the nonlocality of the involved operators here, it is convenient to define a functional $\mathcal{A}\colon \mathcal{K}_{g,h}(\Omega,\Omega') \to [W^{s,p}(\Omega')]'$ given by

$$\mathcal{A}u(v) := \mathcal{A}_1u(v) + \mathcal{A}_2u(v)$$

for every $u \in \mathcal{K}_{g,h}(\Omega,\Omega')$ and $v \in W^{s,p}(\Omega')$, where

$$\mathcal{A}_1u(v) := \int_{\Omega'}\int_{\Omega'} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y$$

and

$$\mathcal{A}_2u(v) := 2\int_{\mathbb{R}^n\setminus\Omega'}\int_{\Omega} |u(x)-g(y)|^{p-2}\big(u(x)-g(y)\big)v(x)K(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The motivation for the definitions above is as follows. Assuming that $v \in W^{s,p}_0(\Omega)$, and $u \in W^{s,p}(\Omega')$ is such that $u = g$ on $\mathbb{R}^n \setminus \Omega'$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y\\
&\qquad= \int_{\Omega'}\int_{\Omega'} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y\\
&\qquad\quad+2\int_{\mathbb{R}^n\setminus\Omega'}\int_{\Omega} |u(x)-g(y)|^{p-2}\big(u(x)-g(y)\big)v(x)K(x,y)\,\mathrm{d}x\mathrm{d}y.\\
&\qquad\equiv A_1u(v) + \mathcal{A}_2u(v).
\end{aligned}$$

Remark 4 The functional $\mathcal{A}u$ really belongs to the dual of $W^{s,p}(\Omega')$. Indeed, by Hölder's Inequality we have

$$|\mathcal{A}_1u(v)| \leq c\,\|u\|^{p-1}_{W^{s,p}(\Omega')}\|v\|_{W^{s,p}(\Omega')}$$

Also, it holds

$$|\mathcal{A}_2u(v)| \leq c\,r^{-sp}\Big(\|u\|^{p-1}_{W^{s,p}(\Omega')} + \mathrm{Tail}(g;z,r)^{p-1}\Big)\|v\|_{W^{s,p}(\Omega')},$$

where $z \in \Omega$ and $r := \mathrm{dist}(\Omega,\partial\Omega') > 0$, and c depends on $n,p,s,\Lambda,\Omega,\Omega'$.

Now, we are ready to provide the natural definition of solutions to the obstacle problem in the general nonlinear nonlocal framework considered here. We have

Definition 5 [Solution to the obstacle problem; [49]**].** We say that $u \in \mathcal{K}_{g,h}(\Omega,\Omega')$ is *a solution to the obstacle problem* in $\mathcal{K}_{g,h}(\Omega,\Omega')$ if

$$\mathcal{A}u(v-u) \geq 0$$

whenever $v \in \mathcal{K}_{g,h}(\Omega,\Omega')$.

Below, we state the uniqueness of the solution to the obstacle problem and the fact that such a solution is a weak supersolution to (2). Also, under natural assumptions on the obstacle h, one can prove that the solution to the obstacle problem is fractional harmonic away from the contact set, in clear accordance with the classical results when $s = 1$. We have

Theorem 5 [Existence for solutions to the obstacle problem; [49, Theorem 1]]**.** *There exists a unique solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. Moreover, the solution to the obstacle problem is a weak supersolution to* (2) *in Ω.*

Corollary 1 [Basic properties of solutions to the obstacle problem; [49, Corollary 1]]**.** *Let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. If $B_r \subset \Omega$ is such that*

$$\operatorname*{ess\,inf}_{B_r}(u-h) > 0,$$

then u is a weak solution to (2) *in B_r. In particular, if u is lower semicontinuous and h is upper semicontinuous in Ω, then u is a weak solution to* (2) *in $\Omega_+ := \big\{x \in \Omega : u(x) > h(x)\big\}$.*

Remark 5 When solving the obstacle problem in $\mathcal{K}_{g,-\infty}(\Omega, \Omega')$, we obtain a unique weak solution to (2) in Ω having the boundary values $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ in $\mathbb{R}^n \setminus \Omega$.

Finally, we are ready to state the desired boundedness result up to the boundary.

Theorem 6 [Boundedness for solutions to the obstacle problem; [49, Theorem 5]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Suppose that $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. Let $x_0 \in \partial\Omega$ and suppose that*

$$\operatorname*{ess\,sup}_{B_r(x_0)} g + \operatorname*{ess\,sup}_{B_r(x_0)\cap\Omega} h < \infty \quad \textit{and} \quad \operatorname*{ess\,inf}_{B_r(x_0)} g > -\infty$$

for $r \in (0, r_0)$ with $r_0 := \operatorname{dist}(x_0, \partial\Omega')$. Then u is essentially bounded close to x_0.

4 Hölder (up to the boundary) and Harnack estimates

The results in [26,27] constitute an extension of classical results by De Giorgi-Nash-Moser to the nonlocal nonlinear framework investigated here.

4.1 Hölder continuity

We start with the Hölder result in Theorem 7 below which extends to the case $p \neq 2$ the result by Kassmann in [42], where a further boundedness assumption is also required, being now for free thanks to Theorem 3 stated in the previous section.

Theorem 7 [Hölder continuity; [27, Theorem 1.2]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be a weak solution to problem (2). Then u is locally Hölder continuous in Ω. In particular, there are positive constants α, $\alpha < sp/(p-1)$, and c, both depending only on $n, p, s, \lambda, \Lambda$, such that if $B_{2r}(x_0) \subset \Omega$, then*

$$\operatorname*{osc}_{B_\varrho(x_0)} u \leq c\left(\frac{\varrho}{r}\right)^{\alpha}\left(\mathrm{Tail}(u;x_0,r) + \left(\fint_{B_{2r}(x_0)} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right)$$

holds whenever $\varrho \in (0,r]$. The quantity $\mathrm{Tail}(u_-;x_0,R)$ is defined in (4).

For what concerns the proof, the following *logarithmic estimate* plays one of the key role.

Lemma 2 [Fractional logarithmic lemma; [27, Lemma 1.3]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be a weak supersolution to problem (2) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Then the following estimate holds for any $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$ and any $d > 0$,*

$$\int_{B_r}\int_{B_r} K(x,y)\left|\log\left(\frac{d+u(x)}{d+u(y)}\right)\right|^p \mathrm{d}x\mathrm{d}y$$
$$\leq c\,r^{n-sp}\left(d^{1-p}\left(\frac{r}{R}\right)^{sp}\Big(\mathrm{Tail}(u_-;x_0,R)\Big)^{p-1} + 1\right),$$

where $\mathrm{Tail}(u_-;x_0,R)$ is defined in (4), and c depends only on n, p, s, λ, Λ.

In analogy with respect to the boundedness results presented in the previous section, in the case when the set Ω does satisfy the measure density condition in (9), the weak solutions to the Dirichlet problem (2) enjoy Hölder continuity up to the boundary. Once again, such a result is a mere corollary of the more general result for the related obstacle problem.

Theorem 8 [Hölder continuity up to the boundary; [49, Theorem 6]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Suppose that the domain $\Omega \subset \mathbb{R}^n$ satisfies the measure density condition in (9). Let $x_0 \in \partial\Omega$ and suppose that the boundary data g is Hölder continuous close to x_0. Let u be a weak solution to (2). Then u is Hölder continuous close to x_0 as well.*

As said, the Hölder result above can be deduced by a more general theorem, see below, which states that the regularity of the solution to the obstacle problem inherits the Hölder continuity of the obstacle up to the boundary. The same happens in the case of continuity of the obstacle up to the boundary (see [49, Theorem 7]).

Theorem 9 [Hölder continuity up to the boundary for solutions to the obstacle problem; [49, Theorem 6]**].** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Suppose that u solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ and assume $x_0 \in \partial\Omega$ and $B_{2R}(x_0) \subset \Omega'$. If $g \in \mathcal{K}_{g,h}(\Omega,\Omega')$ is Hölder continuous in $B_R(x_0)$ and Ω satisfies (9) for all $r \leq R$, then u is Hölder continuous in $B_R(x_0)$ as well.*

Proof We may assume $x_0 = 0$ and $g(0) = 0$. Moreover, we may choose R_0 such that $\operatorname{osc}_{B_0} g \leq \operatorname{osc}_{B_0} u$ for $B_0 \equiv B_{R_0}(0)$ since otherwise we have nothing to prove, and we define $\omega_0 := 8\,(\operatorname{osc}_{B_0} u + \mathrm{Tail}(u;0,R_0))$. The proof of the Hölder continuity up to the boundary relies on a logarithmic estimate with tail ([49, Lemma 5]), which goes back to Lemma 2, obtained by a suitable choice of the test functions and by a careful estimates of the local and nonlocal energy contributions separately in the super and subquadratic cases. Such a logarithmic lemma can be subsequently extended to truncations of the solution to the obstacle problem, as follows: let $B_R \Subset \Omega'$, let $B_r \subset B_{R/2}$ be concentric balls and let

$$\infty > k_+ \geq \max\left\{\operatorname*{ess\,sup}_{B_R} g, \operatorname*{ess\,sup}_{B_R\cap\Omega} h\right\} \quad \text{and} \quad -\infty < k_- \leq \operatorname*{ess\,inf}_{B_R} g;$$

then the functions $w_\pm := \operatorname{ess\,sup}_{B_R}(u-k_\pm)_\pm - (u-k_\pm)_\pm + \varepsilon$ satisfy the following estimate

$$\int_{B_r}\int_{B_r}\left|\log\frac{w_\pm(x)}{w_\pm(y)}\right|^p K(x,y)\,\mathrm{d}x\mathrm{d}y \\ \leq c\, r^{n-sp}\left(1+\varepsilon^{1-p}\left(\frac{r}{R}\right)^{sp}\mathrm{Tail}((w_\pm)_-,x_0,R)^{p-1}\right)$$

for every $\varepsilon > 0$. Then, we combine the estimate in the display above with a fractional Poincaré-type inequality ([49, Lemma 7]) together with some estimates for the tail term thanks to an application of the Chebyshev inequality and in view of the result in Theorem 6. We arrive to prove the existence of τ_0, σ and θ depending only on n, p, s and δ_Ω, such that if

$$\operatorname*{osc}_{B_r(0)} u + \sigma\mathrm{Tail}(u;0,r) \leq \omega \quad \text{and} \quad \operatorname*{osc}_{B_r(0)} g \leq \frac{\omega}{8}$$

hold for a ball $B_r(0)$ and for $\omega > 0$, then

$$\operatorname*{osc}_{B_{\tau r}(0)} u + \sigma\mathrm{Tail}(u;0,\tau r) \leq (1-\theta)\omega$$

holds for every $\tau \in (0, \tau_0]$. Finally, as we can take $\tau \leq \tau_0$ such that

$$\operatorname*{osc}_{\tau^j B_0} g \leq (1-\theta)^j \frac{\omega_0}{8} \qquad \text{for every } j = 0, 1, \dots,$$

an iterative argument will give that u belongs to $C^{0,\alpha}(B_0)$. □

Now, a few observations are in order.

Remark 6 We notice that one has to assume that the datum g belongs to $\mathcal{K}_{g,h}(\Omega, \Omega')$, since otherwise the solution may be discontinuous on every boundary point, as one can see by taking $\Omega = B_1(0)$, $\Omega' = B_2(0)$, and (s, p) such that $sp < 1$. It plainly follows that the characteristic function χ_Ω solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$ with constant functions $g \equiv 0$ and $h \equiv 1$. Indeed, $\chi_\Omega \in W^{s,p}(\Omega')$, and one can check that it is a weak supersolution. As a consequence, by recalling Proposition 1 in [49], which claims that the solution to the obstacle problem is the smallest supersolution above the obstacle, the function χ_Ω is the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. See in particular Example 1 in [49].

Remark 7 Boundary regularity for nonlocal equations driven by singular, possibly degenerate, operators as in (1) seems to be a difficult problem in a general nonlinear framework under natural assumptions on the involved quantities. The situation simplifies considerably in the linear case when $p = 2$; see for instance the survey [80] and the references therein. Coming back to the nonlinear case, to our knowledge, the solely nonlocal result of global Hölder regularity has been obtained very recently in the interesting paper [59], where the authors deal with the non-homogeneous equation, in the special case when the operator $\mathcal{L}$ in (2) does coincide with the nonlinear fractional Laplacian $(-\Delta)^s_p$, by considering exclusively zero Dirichlet boundary data, and by assuming $C^{1,1}$-regularity up to the boundary for the domain Ω. The proofs there are indeed strongly based on the construction of suitable barriers near $\partial\Omega$, by relying on the fact that the function $x \mapsto x^s_+$ is an explicit solution in the half-space. For this, one cannot expect to plainly extend such a strategy in the general framework considered here, in view of the presence of merely measurable coefficients in (1). In [49], nonzero boundary Dirichlet data can be chosen, and the domain Ω has to satisfy only the natural measure density condition given in (9). Consequently, new proofs have been developed there in order to extend up to the boundary part of the results in [26, 27] together with a careful handling of the tail-type contributions, as sketched above.

We conclude this section by briefly commenting about some open problems that arise in the nonlinear fractional obstacle framework.

Open Problem 1. A first natural open problem concerns the optimal regularity for the solutions to the nonlinear nonlocal obstacle problem. We recall that for the classical obstacle problem, when $\mathcal{L}$ coincides with the Laplacian operator, the solutions are known to be in $C^{1,1}$. The intuition behind this regularity result is that in the contact set one has $-\Delta u = -\Delta h$, while where $u > h$ one has $-\Delta u = 0$; since the Laplacian jumps from $-\Delta h$ to 0 across the free boundary, the second derivatives of u must have a discontinuity... and thus $C^{1,1}$ is the maximum regularity class that can be expected. Surprisingly, when $\mathcal{L} \equiv (-\Delta)^s$, despite the previous local argument does suggest that the solutions u belong to C^{2s}, the optimal regularity is $C^{1,s}$; that is, the regularity exponent is higher than the order of the equation. In the nonlinear nonlocal framework presented here, starting from the Hölder regularity proven in [49], one still expects higher regularity results. Notwithstanding, in view of the interplay between the local and nonlocal contributions, and without having the possibility to rely on all the linear tools mentioned in Section 1.1, it is not completely clear what the optimal exponent could be as the nonlinear growth does take its part. For preliminary very important results in this direction, it is worth mentioning the recent paper [15], where optimal regularity results of the solution to the obstacle problem, and of the free boundary near regular points, have been achieved for integro-differential operators as in (2) in the linear case when $p = 2$. We refer also to the fundamental results achieved for the case of the pure fractional Laplacian operator in [16,88] and the references therein.

Open Problem 2. Another interesting open problem concerns the regularity in a generic point of the free boundary, which is known to be analytic in the case of the Laplacian, except on a well-defined set of singular points, and smooth in the case of the fractional Laplacian.

Open Problem 3. A natural goal is the investigation of the related parabolic version of the nonlinear nonlocal obstacle problem (see the important papers [4, 14] for what concerns regularity results for the parabolic version of the obstacle problem involving the pure fractional Laplacian $(-\Delta)^s$), as it is inspired in the so-called optimal stopping problem with deadline, by corresponding to the American option pricing problem with expiration at some given time. An extension in the nonlinear setting presented here could be quite important as it would essentially describe a situation which also takes into account the interactions coming from far together with a natural inhomogeneity. Accordingly with the optimal stopping problem model, a starting point in such an investigation could be the special case when the obstacle h coincides with the boundary value g.

4.2 Nonlocal Harnack inequalities

Combining Theorem 2 together with the nonlocal logarithmic Lemma 2, one can prove that both the p-minimizers and weak solutions enjoy oscillation estimates, which yield some natural Harnack estimates with tail, as the following nonlocal weak Harnack inequality presented in Theorem 10 below, and the nonlocal Harnack inequality presented in forthcoming Theorem 11.

Theorem 10 [Nonlocal weak Harnack inequality; [26, Theorem 1.2]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a weak supersolution to (2) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Let*

$$\bar{t} := \begin{cases} \frac{(p-1)n}{n-sp}, & 1 < p < \frac{n}{s}, \\ +\infty, & p \geq \frac{n}{s}. \end{cases}$$

Then the following estimate holds for any $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$ and for any $t < \bar{t}$

$$\left(\fint_{B_r} u^t \,\mathrm{d}x \right)^{\frac{1}{t}} \leq c \operatorname*{ess\,inf}_{B_{2r}} u + c \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; x_0, R),$$

where $\mathrm{Tail}(\cdot)$ *is defined in* (4)*, and the constant c depends only on n, p, s, and Λ.*

Notice that the case $p \geq \frac{n}{s}$ was not treated in the proof of the weak Harnack with tail in [26], but one may deduce the result in this case by straightforward modifications of the proof there.

The way how the nonlocal tail is handled is one of the key-points in the proof of the following nonlocal Harnack estimates. We have

Theorem 11 [Nonlocal Harnack inequality; [26, Theorem 1.1]]. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a weak solution to (2) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Then the following estimate holds for any $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$,*

$$\sup B_r u \leq c \inf B_r u + c \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; x_0, R), \tag{10}$$

where $\mathrm{Tail}(\cdot)$ *is defined in* (4)*, and the constants c depend only on n, p, s and on the structural constants λ and Λ defined in* (3)*.*

As expected, the contribution given by the nonlocal tail has to be taken into account, and once again the result in (10) is analogous to the classical local case if the function u is nonnegative in the whole $\mathbb{R}^n$. Moreover, in the proofs of the theorems above, one could take trace of the dependence of the constants on s, in order to rewrite the tail term quantity so that it disappears when $s \nearrow 1$, in the same spirit of the by-now classic papers by Kassmann [42, 43].

Remark 8 A straightforward adaptation of the proofs of Theorems 10-11 in [26] allow to take into account (bounded) zero order terms in equation (2), by also extending to the general nonlinear fractional framework the results for the pure fractional Laplacian operators proven in [89]. For further observations, and an application to nonlocal semilinear equations in the plane in order to achieve one-dimensional symmetry results, we refer to [39].

Open Problem 4. No much is known on nonlocal boundary Harnack results.

Open Problem 5. We expect similar regularity estimates for the parabolic counterpart, by following the techniques presented in the papers [26, 27] with the needed modifications.

4.3 Further related results

We now seize the opportunity to mention some related important results. In the paper [23], some of the results presented in the previous sections have been generalized to functions belonging to a fractional counterpart of the so-called De Giorgi classes, providing the nonlocal counterpart of the celebrated results by M. Giaquinta and E. Giusti for minima of local functionals with p-growth. In [8], higher differentiability properties of weak solutions have been proven: if $sp > p-1$, then the gradient of solutions belongs to $W^{\sigma,p}$ for some $\sigma > 0$. Finally, in the very recent paper [9], an explicit Hölder exponent for solutions to the correspondent non-homogeneous equation with data in L^q, for $q > N/(sp)$ and $p \geq 2$, is provided, being proven to be sharp in a restricted range of differentiability and growth exponents.

Open Problem 6. For what concerns the higher regularity, it should be remarked that in view of the local theory one would expect $C^{1,\alpha}$-regularity from weak solutions, at least for a certain range of s and p. To our knowledge, this is still an open problem.

5 The p-fractional Dirichlet problem with arbitrary boundary data in general, possibly irregular, domains

The Perron method (also known as the PWB method, after Perron, Wiener, and Brelot) is a consolidated method introduced at the beginning of the last century in order to solve the Dirichlet problem for the Laplace equation in a given open set Ω with arbitrary boundary data g; i. e.,

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u = g & \text{on the boundary of } \Omega, \end{cases} \tag{11}$$

when $\mathcal{L} = \Delta$. Roughly speaking, the Perron method works by finding the least superharmonic function with boundary values above the given values g. Under the assumption $g \in H^1(\Omega)$, the so-called Perron solution coincides with the desired Dirichlet energy solution. However, energy methods do not work for general g, and this is precisely the motivation of the Perron method. The method essentially works for many other partial differential equations whenever a comparison principle is available and appropriate barriers can be constructed to assume the boundary conditions. Thus, perhaps surprisingly, it turns out that the method extends to the case when the Laplacian operator in (11) is replaced by the p-Laplacian operator $(-\Delta_p)$ (see e.g. [38]) or even by more general nonlinear operators. Consequently, the Perron method has become a fundamental tool in nonlinear Potential Theory, as well as in the study of several branches of Mathematics and Mathematical Physics when problems as in (11), and the corresponding variational formulations arising from different contexts. The nonlinear Potential Theory covers a classical field having grown a lot during the last three decades from the necessity to understand better properties of supersolutions, potentials and obstacles. Much has been written about this topic and the connection with the theory of degenerate elliptic equations; we refer the reader to the exhaustive book [40] by J. Heinonen, T. Kilpeläinen and O. Martio, and to the useful lecture notes [65] by P. Lindqvist.

The main subject of the present section is to present the nonlocal counterpart of the Perron method, when one considers the operator (1) in problem (11). For this, one needs a good definition of superharmonic functions in such a framework, the (s, p)-superharmonic functions, whose name emphasizes the (s, p)-order of the involved Gagliardo kernel; see [51]. The (s, p)-superharmonic functions are very much connected to fractional weak supersolutions, which by the definition belong locally to the Sobolev space $W^{s,p}$ (see Section 2). Consequently, in Section 5.1 below, we will recall very general results for the weak supersolutions u to (2), as e. g. the natural comparison principle given in Lemma 3 which takes into account what happens outside Ω, the lower semicontinuity of u (see Theorem 12), the fact that the minimum of two supersolutions is a supersolution as well (see Theorem 13), the pointwise convergence of sequences of supersolutions (Theorem 14). Clearly, all these results well known in Nonlinear Potential Theory are expected, but further efforts and a somewhat new approach to the corresponding proofs are needed due to the nonlocal nonlinear framework considered here.

5.1 Main properties of the fractional supersolutions

In order to prove all the forthcoming results in the present section, one needs to perform careful computations on the strongly nonlocal form of the operators in (2). Hence, it could be important to understand how to modify the classical techniques to deal with nonlocal integro-differential energies, in particular to manage the contributions coming from far. For this, the results for fractional weak supersolutions presented here are of some general and independent use, other than necessary to introduce the (s,p)-harmonic functions and the related Perron method.

First, a comparison principle for weak sub- and supersolution, which typically constitutes a powerful tool, playing a fundamental role in the whole Partial Differential Equations theory.

Lemma 3 [Comparison Principle; [51, Lemma 6]**].** *Let $s \in (0,1)$ and $p \in (1,\infty)$. Let $\Omega \Subset \Omega'$ be bounded open subsets of $\mathbb{R}^n$. Let $u \in W^{s,p}(\Omega')$ be a weak supersolution to (2) in Ω, and let $v \in W^{s,p}(\Omega')$ be a weak subsolution to (2) in Ω such that $u \geq v$ almost everywhere in $\mathbb{R}^n \setminus \Omega$. Then $u \geq v$ almost everywhere in Ω as well.*

Proof Consider the function $\eta := (u-v)_-$. Notice that η is a nonnegative function in $W^{s,p}_0(\Omega)$. For this, we can use it as a test function in (5) for both $u, v \in W^{s,p}(\Omega')$ and, by summing up, we get

$$
\begin{aligned}
0 \leq & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\eta(x)-\eta(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \qquad (12)\\
& -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)\big(\eta(x)-\eta(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

It is now convenient to split the integrals above by partitioning the whole $\mathbb{R}^n$ into separate sets comparing the values of u with those of v, so that, from (12) we get

$$
\begin{aligned}
0 \leq & \int_{\{u<v\}}\int_{\{u<v\}} \big(L(u(x),u(y)) - L(v(x),v(y))\big)\big(\eta(x)-\eta(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13)\\
& + \int_{\{u\geq v\}}\int_{\{u<v\}} \big(L(u(x),u(y)) - L(v(x),v(y))\big)\eta(x)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
& - \int_{\{u<v\}}\int_{\{u\geq v\}} \big(L(u(x),u(y)) - L(v(x),v(y))\big)\eta(y)K(x,y)\,\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

where we denoted for shortness

$$
L(a,b) := |a-b|^{p-2}(a-b), \qquad a,b \in \mathbb{R}. \tag{14}
$$

The goal is now to prove that the right-hand side of the inequality above is nonpositive. In view of the very definition of η, we can estimate the three terms in (13) as follows

$$
\begin{aligned}
[...] \leq & - \int_{\{u<v\}} \int_{\{u<v\}} \big(L(u(x), u(y)) - L(v(x), v(y))\big) \\
& \qquad\qquad \times \big(u(x) - u(y) - v(x) + v(y)\big) K(x,y)\, \mathrm{d}x\mathrm{d}y \\
& + \int_{\{u\geq v\}} \int_{\{u<v\}} \big(L(v(x), v(y)) - L(v(x), v(y))\big) \eta(x) K(x,y)\, \mathrm{d}x\mathrm{d}y \\
& - \int_{\{u<v\}} \int_{\{u\geq v\}} \big(L(v(x), v(y)) - L(v(x), v(y))\big) \eta(y) K(x,y)\, \mathrm{d}x\mathrm{d}y \\
\leq & \; 0.
\end{aligned} \tag{15}
$$

By combining (15) with (13), we deduce that all the terms in (13) have to be equal to 0, which implies $\eta = 0$ almost everywhere in $\{u < v\}$, in turn giving the desired result. □

In particular, since the weak subsolutions and supersolutions belong locally to $W^{s,p}$, we get the following comparison principle.

Corollary 2 [Comparison Principle; [51, Corollary 2]]**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$, and let $D \Subset \Omega$. Let u be a weak supersolution to (2) in Ω, and let v be a weak subsolution to (2) in Ω such that $u \geq v$ almost everywhere in $\mathbb{R}^n \setminus D$. Then $u \geq v$ almost everywhere in D.*

Now, we give an expected lower semicontinuity result for the weak supersolutions, which, as in the classic local setting, is a fundamental object to provide other important topological tools in nonlinear Potential Theory.

Theorem 12 [Lower semicontinuity of supersolutions; [51, Theorem 9]]**.** *Let u be a weak supersolution in Ω. Then*

$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y) \qquad \text{for a. e. } x \in \Omega.$$

In particular, u has a lower semicontinuous representative.

The proof of Theorem 12 relies in the supremum estimates given by Theorem 3 performing there a careful choice of the interpolation parameter δ in (8) between the local contributions and the nonlocal ones. This is a relevant difference with respect to the classical nonlinear Potential Theory, where on the contrary the lower semicontinuity is a straight consequence of weak Harnack estimates (see for instance [40, Theorem 3.51 and 3.63]). While, in the purely fractional Laplacian case when $p = 2$, the proof of the same result is simply based on a characterization of supersolutions somewhat similar to the super mean value formula for classical superharmonic functions (see, e.g., [82, Proposition A4]), which is not available in our general nonlinear nonlocal framework due to the presence of possible irregular coefficients in the kernel K in (1). We refer to [51] for further details.

We now state the following basic result which concerns the minimum of two fractional weak supersolutions.

Theorem 13 [The minimum of two supersolutions; [50, Theorem 1.1]]**.** *Suppose that u and v are fractional weak supersolutions in Ω. Then the function $w := \min\{u, v\}$ is a fractional weak supersolution in Ω as well.*

In contrast with respect to the classic local case of the p-Laplace equation (that is, when $s = 1$; see for instance Theorem 3.23 in [40]), here the proof that the function $w := \min\{u, v\}$ is a weak supersolution tangles up significantly in the nonlocality of the involved operators $\mathcal{L}$. We refer to [49, 50] for further details.

The class of uniformly globally bounded weak supersolutions is closed with respect to the pointwise convergence.

Theorem 14 [Convergence of sequences of supersolutions; [51, Theorem 10]]**.** *Let $s \in (0, 1)$, $p \in (1, \infty)$, and let $g \in L^{p-1}_{sp}(\mathbb{R}^n)$ and $h \in L^{p-1}_{sp}(\mathbb{R}^n)$ be such that $h \leq g$ in $\mathbb{R}^n$. Let $\{u_j\}$ be a sequence of weak supersolutions in Ω such that $h \leq u_j \leq g$ almost everywhere in $\mathbb{R}^n$ and u_j is uniformly locally essentially bounded from above in Ω. Suppose that u_j converges to a function u pointwise almost everywhere as $j \to \infty$. Then u is a weak supersolution in Ω as well.*

If the sequence is increasing, we do not have to assume any boundedness from above.

Corollary 3 [Convergence of increasing sequences of supersolutions; [51, Corollary 3]]**.** *Let $s \in (0, 1)$ and $p \in (1, \infty)$. Let $\{u_j\}$ be an increasing sequence of weak supersolutions in Ω such that u_j converges to a function $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ pointwise almost everywhere in $\mathbb{R}^n$ as $j \to \infty$. Then u is a weak supersolution in Ω as well.*

Proof For any $M > 0$, denote by $u_M := \min\{u, M\}$ and $u_{M,j} := \min\{u_j, M\}$, which is a weak supersolution by Theorem 13. Then $\{u_{M,j}\}_j$ is a sequence satisfying the assumptions of Theorem 14 converging pointwise almost everywhere to u_M, and consequently u_M is a weak supersolution in Ω. Let $\eta \in C^\infty_0(\Omega)$ be a nonnegative test function. Let L be defined by (14), since

$$|L(u_M(x), u_M(y))| \leq |u(x) - u(y)|^{p-1}$$

for every $M > 0$ and every $x, y \in \mathbb{R}^n$, where $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$, we can let $M \to \infty$ to obtain by the dominated convergence theorem that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L(u(x), u(y))\big(\eta(x) - \eta(y)\big)K(x, y)\,{\rm d}x{\rm d}y \geq 0.$$

We conclude that u is a weak supersolution in Ω. $\square$

5.2 The (s,p)-superharmonic functions: definition and basic properties

We are now in the position to introduce the (s,p)-*superharmonic functions*, as in the recent paper [51]. The (s,p)-superharmonic functions constitute the nonlocal counterpart of the p-superharmonic functions considered in the important paper [64], and in the subsequent literature. As expected, in view of the nonlocality of the involved operators $\mathcal{L}$, this new definition requires to take into account the nonlocal tail in (4), in the form of the suitable tail space $L^{p-1}_{sp}(\mathbb{R}^n)$. This is in clear accordance with the theory encountered in all the aforementioned papers, when nonlocal operators have to be dealt with in bounded domains.

Although some of the following results are well-known in the linear nonlocal case; i. e., when $\mathcal{L}$ reduces to the pure fractional Laplacian operator $(-\Delta)^s$, all the corresponding proofs are new even in this case. As mentioned, since we actually deal with very general operators with measurable coefficients, one has to change the approach to the problem. We have the following

Definition 6 [(s,p)-superharmonic functions; [51]]. Let $s \in (0,1)$ and $p \in (1,\infty)$. We say that a function $u\colon \mathbb{R}^n \to [-\infty,\infty]$ is an (s,p)-*superharmonic function* in an open set Ω if it satisfies the following four assumptions:

(i) $u < +\infty$ almost everywhere and $u > -\infty$ everywhere in Ω,
(ii) u is lower semicontinuous (l. s. c.) in Ω,
(iii) u satisfies the comparison in Ω against solutions bounded from above; that is, if $D \Subset \Omega$ is an open set and $v \in C(\overline{D})$ is a weak solution in D such that $v_+ \in L^\infty(\mathbb{R}^n)$ and $u \geq v$ on ∂D and almost everywhere on $\mathbb{R}^n \setminus D$, then $u \geq v$ in D,
(iv) u_- belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$.

We say that a function u is (s,p)-*subharmonic* in Ω if $-u$ is (s,p)-superharmonic in Ω; and when both u and $-u$ are (s,p)-superharmonic, we say that u is (s,p)-*harmonic*.

Remark 9 An (s,p)-superharmonic function is locally bounded from below in Ω as the lower semicontinuous function attains its minimum on compact sets and it cannot be $-\infty$ by the definition.

Remark 10 From the definition it is immediately seen that the pointwise minimum of two (s,p)-superharmonic functions is (s,p)-superharmonic as well.

Remark 11 In forthcoming Corollary 4 we will show that a function u is (s,p)-harmonic in Ω if and only if u is a continuous weak solution in Ω. The connection with fractional viscosity solutions is postponed to Section 6.1.

Remark 12 In the case $p = 2$ and $K(x, y) = |x - y|^{-n-2s}$, the Riesz kernel $u(x) = |x|^{2s-n}$ is an $(s, 2)$-superharmonic function in $\mathbb{R}^n$, but it is not a weak supersolution. It is the integrability $W^{s,2}_{\mathrm{loc}}$ that fails.

The next theorem describes the basic fundamental properties of (s, p)-superharmonic functions.

Theorem 15 [Basic properties of (s, p)-superharmonic functions; [51, Theorems 11 and 13, Corollary 6, Lemmata 11 and 14]]**.** *Let $s \in (0, 1)$ and $p \in (1, \infty)$. Suppose that u is (s, p)-superharmonic in an open set Ω. Then it has the following properties:*

(i) **Pointwise behavior**.

$$u(x) = \liminf_{y \to x} u(y) = \operatorname{ess}\liminf_{y \to x} u(y) \quad \textit{for every } x \in \Omega.$$

(ii) **Summability**. *For*

$$\bar{t} := \begin{cases} \frac{(p-1)n}{n-sp}, & 1 < p < \frac{n}{s}, \\ +\infty, & p \geq \frac{n}{s}, \end{cases} \qquad \bar{q} := \min\left\{ \frac{n(p-1)}{n-s}, p \right\},$$

and $h \in (0, s)$, $t \in (0, \bar{t})$ and $q \in (0, \bar{q})$, $u \in W^{h,q}_{\mathrm{loc}}(\Omega) \cap L^t_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$.

(iii) **Comparison**. *If $D \Subset \Omega$ is an open set and $v \in C(\overline{D})$ is a weak solution in D such that $u \geq v$ on ∂D and almost everywhere on $\mathbb{R}^n \setminus D$, then $u \geq v$ in D.*

(iv) **Connection to weak supersolutions**. *If u is locally bounded in Ω or $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$, then it is a weak supersolution in Ω.*

We can show that the (s, p)-superharmonic functions can be also approximated by continuous weak supersolutions in regular sets.

Lemma 4 [Approximation of (s, p)-superharmonic functions; [51, Lemma 9]]**.** *Let $s \in (0, 1)$ and $p \in (1, \infty)$. Let u be an (s, p)-superharmonic function in Ω and let $D \Subset \Omega$ be an open set such that $\mathbb{R}^n \setminus D$ satisfies the measure density condition (9). Then there is an increasing sequence $\{u_j\}$, $u_j \in C(\overline{D})$, of weak supersolutions in D converging to u pointwise in $\mathbb{R}^n$.*

Proof Let U be an open set satisfying $D \Subset U \Subset \Omega$, which is possible by Urysohn's Lemma. By a suitable approximation result (see Lemma 8 in [51]) there is an increasing sequence of smooth functions $\{\psi_j\}$, $\psi_j \in C^\infty(\overline{U})$, converging to u pointwise in U. For each j, define

$$g_j(x) := \begin{cases} \psi_j(x), & x \in U, \\ \min\{j, u(x)\}, & x \in \mathbb{R}^n \setminus U. \end{cases}$$

Clearly $g_j \in W^{s,p}(U) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ by smoothness of ψ_j and the fact that $u_- \in L^{p-1}_{sp}(\mathbb{R}^n)$. Now we can solve the obstacle problem using the functions g_j as obstacles to obtain solutions $u_j \in \mathcal{K}_{g_j,g_j}(D,U)$, $j = 1, 2, \dots$, so that u_j is continuous in $\overline{D}$ by [51, Theorem 7], whose result is a straight consequence of Theorem 9, and a weak supersolution in D by Theorem 5. To see that $\{u_j\}$ is an increasing sequence, denote by $A_j := D \cap \{u_j > g_j\}$. Since u_j is a weak solution in A_j by Corollary 1 and clearly $u_{j+1} \geq u_j$ in $\mathbb{R}^n \setminus A_j$, the comparison principle (Lemma 3) implies that $u_{j+1} \geq u_j$. Similarly, $u_j \leq u$ by Definition 6(iii). Since g_j converges pointwise to u, we must also have that

$$\lim_{j\to\infty} u_j(x) = u(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

This finishes the proof. □

With Lemma 4 in force, one can prove the connection to weak supersolutions, as it follows.

Proof of Theorem 15(iv). Let $D \Subset \Omega$ be an open set such that $\mathbb{R}^n \setminus D$ satisfies the measure density condition (9). Then by Lemma 4 there is an increasing sequence $\{u_j\}$ of weak supersolutions in D converging to u pointwise in $\mathbb{R}^n$ such that each u_j is continuous in $\overline{D}$. Since each u_j satisfies $u_1 \leq u_j \leq u$ with $u_1, u \in L^{p-1}_{sp}(\mathbb{R}^n)$ and u is bounded from above in D, u is a weak supersolution in D by Theorem 14. Finally, because of the arbitrariness of the set $D \Subset \Omega$, we can deduce that the function u is a weak supersolution in Ω, as desired. □

On the other hand, lower semicontinuous representatives of weak supersolutions are (s,p)-superharmonic.

Theorem 16 [L.s.c. supersolutions are (s,p)-superharmonic; [51, Theorem 12]]. *Let u be a lower semicontinuous weak supersolution in Ω satisfying*

$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y) \qquad \text{for every } x \in \Omega. \tag{16}$$

Then u is an (s,p)-superharmonic function in Ω.

Proof According to the definition of u, by Lemma 1 and by the fact that weak supersolutions are locally essentially bounded from below (see [51, Lemma 3]), together with (16), we have that (i–ii) and (iv) of Definition 6 hold. Thus it remains to check that u satisfies the comparison given in Definition 6(iii). For this, take $D \Subset \Omega$ and a weak solution v in D such that $v \in C(\overline{D})$, $v \leq u$ almost everywhere in $\mathbb{R}^n \setminus D$ and $v \leq u$ on ∂D. For any $\varepsilon > 0$ define $v_\varepsilon := v - \varepsilon$ and consider the set $K_\varepsilon = \{v_\varepsilon \geq u\} \cap \overline{D}$. Notice that by construction the set K_ε is compact and $K_\varepsilon \cap \partial D = \emptyset$. Thus, it suffices to prove that $K_\varepsilon = \emptyset$. This is now a plain consequence of the comparison principle. Indeed, one can find an open set D_1 such that $K_\varepsilon \subset D_1 \Subset D$. Moreover, $v_\varepsilon \leq u$ in $\mathbb{R}^n \setminus D_1$ almost everywhere and thus Corollary 2 yields $u \geq v_\varepsilon$ almost everywhere in D_1. In particular, $u \geq v - \varepsilon$ almost everywhere in D. To obtain an inequality that holds everywhere in D, fix $x \in D$. Then there exists $r > 0$ such that $B_r(x) \subset D$ and

$$u(x) \geq \operatorname*{ess\,inf}_{B_r(x)} u - \varepsilon \geq \inf_{B_r(x)} v - 2\,\varepsilon \geq v(x) - 3\,\varepsilon,$$

by (16) and continuity of v. Since $\varepsilon > 0$ and $x \in D$ were arbitrary, we have $u \geq v$ in D. □

From Theorem 15(iv) and Theorem 16 we see that a function is a continuous weak solution in Ω if and only if it is both (s,p)-superharmonic and (s,p)-subharmonic in Ω.

Corollary 4 *Let $s \in (0,1)$ and $p \in (1,\infty)$. A function u is (s,p)-harmonic in Ω if and only if u is a continuous weak solution in Ω.*

Proof of Theorem 15(i). Fix $x \in \Omega$ and denote by $\lambda := \operatorname{ess\,lim\,inf}_{y\to x} u(y)$. Then

$$\lambda \geq \liminf_{y\to x} u(y) \geq u(x)$$

by the lower semicontinuity of u. To prove the reverse inequality, pick $t < \lambda$. Then there exists $r > 0$ such that $B_r(x) \subset \Omega$ and $u \geq t$ almost everywhere in $B_r(x)$. By Lemma 10 in [51]), the (s,p)-superharmonic function

$$v := \min\{u, t\} - t$$

is identically 0 in $B_r(x)$. In particular, $u(x) \geq t$ and the claim follows by arbitrariness of $t < \lambda$. □

Proof of Theorem 15(ii). The summability result for (s,p)-superharmonic functions can be basically found in [57], where it is given for equations involving nonnegative source terms (see forthcoming Section 6.3), but the proof is identical in the case of weak supersolutions. The needed information is just that the weak supersolutions belong locally to $W^{s,p}$. This can be checked by showing that the positive part of an (s,p)-superharmonic function also belongs to the Tail space; see Theorem 14 in [51]. □

Remark 13 We conclude this section with the following observation. In Definition 6(iii) it is demanded that the comparison functions are globally bounded from above. Then, it is reasonably asking how would the definition change if one removes such an assumption. In other words, if the solution is allowed to have too wild nonlocal contributions, would this be able to break the comparison? The answer is negative. Indeed, the lemma below shows that one can remove the boundedness assumption $v_+ \in L^\infty(\mathbb{R}^n)$ in the definition of (s,p)-superharmonic functions and still get the same class of functions. This is in fact Theorem 15(iii).

Lemma 5 [Unbounded comparison; [51, Lemma 14]]**.** *Let u be an (s,p)-superharmonic function in Ω. Then it satisfies the following unbounded comparison statement:*

u satisfies the comparison in Ω against solutions, that is, if $D \Subset \Omega$ is an open set and $v \in C(\overline{D})$ is a weak solution in D such that $u \geq v$ on ∂D and almost everywhere on $\mathbb{R}^n \setminus D$, then $u \geq v$ in D.

Proof Let u be an (s,p)-superharmonic function in Ω. We will show that then it also satisfies (iii'). To this end, take $D \Subset \Omega$ and v as in (iii'). Let $\varepsilon > 0$. Due to lower semicontinuity of $u - v$ and the boundary condition, the set $K_\varepsilon := \{u \leq v - \varepsilon\} \cap D$ is a compact set of D. Therefore we find open sets D_1, D_2 such that $K_\varepsilon \subset D_1 \Subset D_2 \Subset D$ and $\mathbb{R}^n \setminus D_2$ satisfies the measure density condition (9). Truncate v as $u_k := \min\{v - \varepsilon, k\}$. Applying a stability result for sequence of functions converging to continuous weak solution (see [51, Lemma 13], with $\Omega \equiv D$ and $D \equiv D_2$ there) we find a sequence of continuous weak solutions $\{v_k\}$ in D_2 such that $v_k \to v - \varepsilon$ in D_2. The convergence is uniform in $\overline{D}_1$. Therefore, there is large enough k such that $|v_k - v| \leq 2\varepsilon$ on $\overline{D}_1$. Moreover, by the comparison principle (Lemma 3), $v_k \leq v$ in $\mathbb{R}^n$. Since $u > v_k - \varepsilon$ on ∂D_1 and almost everywhere in $\mathbb{R}^n \setminus D_1$ by the definition of K_ε, we have by Definition 6(iii) that $u \geq v_k - \varepsilon \geq v - 3\varepsilon$ in D_1, and thus we also have that $u \geq v - 3\varepsilon$ in the whole D, because in $D \setminus K_\varepsilon$ we have $u > v - \varepsilon$. Since this holds for an arbitrary positive ε, we have that (iii') holds, completing the proof. □

For further interesting properties of the (s,p)-superharmonic functions, as for instance a more general version of the comparison principle, we refer to Section 4 in [51].

5.3 The fractional Perron Method

Let us come back to the celebrated Perron method. First of all, the main difference with respect to the local case is that for nonlocal equations the Dirichlet condition has to be taken in the whole complement $\mathbb{R}^n \setminus \Omega$ of the domain, instead of only on the boundary $\partial\Omega$. This comes from the very definition of the fractional operators in (2), and it is strictly related to the natural nonlocality of those operators, and the fact that the behavior of a function outside the set Ω does affect the problem in the whole space (and particularly on the boundary of Ω), which is indeed one of the main feature why those operators naturally arise in many contexts. On the other hand, such a nonlocal feature is also one of the main difficulties to be handled when dealing with fractional operators. For this, some sophisticated tools and techniques have been recently developed to treat the nonlocality, and to achieve many fundamental results for nonlocal equations.

As said before, for the nonlocal Perron method the (s,p)-superharmonic and (s,p)-subharmonic functions (presented in the previous section) are the building blocks. We are thus in a position to introduce this concept. As well as in the classical local framework, in order to solve the boundary value problem, we have to construct two classes of functions leading to the upper Perron solution and the lower Perron solution.

Definition 7 [Perron solutions; [51]]. Let $s \in (0,1)$ and $p \in (1,\infty)$, and let Ω be an open set. Assume that $g \in L^{p-1}_{sp}(\mathbb{R}^n)$. The *upper class* $\mathcal{U}_g$ of g consists of all functions u such that

(i) u is (s,p)-superharmonic in Ω,
(ii) u is bounded from below in Ω,
(iii) $\liminf_{\Omega \ni y \to x} u(y) \geq \operatorname{ess}\limsup_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y)$ for all $x \in \partial\Omega$,
(iv) $u = g$ almost everywhere in $\mathbb{R}^n \setminus \Omega$.

The lower class is $\mathcal{L}_g := \{u \,:\, -u \in \mathcal{U}_{-g}\}$. The function $\overline{H}_g := \inf\{u : u \in \mathcal{U}_g\}$ is the *upper Perron solution* with boundary datum g in Ω, where the infimum is taken pointwise in Ω, and $\underline{H}_g := \sup\{u : u \in \mathcal{L}_g\}$ is the *lower Perron solution* with boundary datum g in Ω.

A few important observations are in order.

Remark 14 Notice that when g is continuous in a vicinity of the boundary of Ω, we can replace $\operatorname{ess}\limsup_{y\to x} g(y)$ with $g(x)$ in Definition 7(iii) above.

Remark 15 We could also consider more general Perron solutions by dropping the conditions (ii)–(iii) in Definition 7 above. However, in such a case it does not seem easy to exclude the possibility that the corresponding upper Perron solution is identically $-\infty$ in Ω even for simple boundary value functions such as constants.

In the case of the fractional Laplacian, we have the Poisson formula for the solution u in a unit ball with boundary values g as

$$u(x) = c_{n,s}\left(1-|x|^2\right)^s \int_{\mathbb{R}^n \setminus B_1(0)} g(y)\left(|y|^2-1\right)^{-s}|x-y|^{-n}\,\mathrm{d}y,$$

for every $x \in B_1(0)$; see e. g. [43], and also [72, 86] for related applications, and [32] for explicit computations. Using the Poisson formula one can consider some examples in the unit ball.

Example 1 Taking the function $g(x) = \left||x|^2-1\right|^{s-1}$, $g \in L^1_{2s}(\mathbb{R}^n)$, as boundary values in the Poisson formula above, the integral does not converge. This example suggests that in this case $\overline{H}_g \equiv \underline{H}_g \equiv +\infty$ in $B_1(0)$. The example tells that if the boundary values g merely belong to $L^1_{2s}(\mathbb{R}^n)$, one cannot, in general, expect to find reasonable solutions.

Example 2 Let us consider the previous example with g reflected to the negative side in the half space; i. e.,

$$g(x) := \begin{cases} \left||x|^2 - 1\right|^{s-1}, & x_n > 0, \\ 0, & x_n = 0, \\ -\left||x|^2 - 1\right|^{s-1}, & x_n < 0. \end{cases}$$

Then the "solution" via Poisson formula, for $x \in B_1$, is

$$u(x) = \begin{cases} +\infty & x_n > 0, \\ 0, & x_n = 0, \\ -\infty, & x_n < 0, \end{cases}$$

which is suggesting that we should now have $\overline{H}_g \equiv +\infty$ and $\underline{H}_g \equiv -\infty$ in $B_1(0)$. In view of this example it is reasonable to conjecture that the resolutivity fails in the class $L^1_{2s}(\mathbb{R}^n)$.

In accordance with the classical Perron theory, one can prove that the upper and lower *nonlocal* Perron solutions act in the expected order (see Lemma 6 below), and that the boundedness of the boundary values assures that the nonlocal Perron classes are non-empty (see forthcoming Lemma 7).

Lemma 6 [The Perron solutions are in order; [51, Lemma 17]**].** *The Perron solutions $\overline{H}_g$ and $\underline{H}_g$ satisfy $\overline{H}_g \geq \underline{H}_g$ in $\mathbb{R}^n$.*

Proof If $\mathcal{U}_g$ or $\mathcal{L}_g$ is empty, there is nothing to prove since $\overline{H}_g \equiv +\infty$ or $\underline{H}_g \equiv -\infty$, respectively. Assume then that the classes are non-empty, and take $u \in \mathcal{U}_g$ and $v \in \mathcal{L}_g$. Then

$$\liminf_{\Omega \ni y \to x} u(y) \geq \operatorname*{ess\,lim\,sup}_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y) \geq \operatorname*{ess\,lim\,inf}_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y) \geq \limsup_{\Omega \ni y \to x} v(y)$$

for every $x \in \partial\Omega$ by Definition 7(iii). Both sides of the inequality above cannot be simultaneously $-\infty$ or $+\infty$, again according to Definition 7(ii). Moreover, since $u = g = v$ almost everywhere in $\mathbb{R}^n \setminus \Omega$, we have $u \geq v$ in Ω by the comparison principle. Finally, taking the infimum over $\{u \in \mathcal{U}_g\}$ and the supremum over $\{v \in \mathcal{L}_g\}$ finishes the proof. □

Lemma 7 [The Perron classes are non-empty; [51, Lemma 18]**].** *If $g \in L^{p-1}_{sp}(\mathbb{R}^n)$ is bounded from above, then the class $\mathcal{U}_g$ is nonempty.*

Proof Let $\sup_{\mathbb{R}^n} g \leq M < \infty$ and take $u := M\chi_\Omega + g\chi_{\mathbb{R}^n \setminus \Omega}$. Then clearly u satisfies the properties (ii-iv) of Definition 7. In order to obtain the property (i), we first have that $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$, and testing against a nonnegative test function $\eta \in C_0^\infty(\Omega)$ gives

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L(u(x), u(y))\big(\eta(x) - \eta(y)\big)K(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= 2 \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} L(M, g(y)) \eta(x) K(x,y) \,\mathrm{d}x\mathrm{d}y \geq 0.$$

Thus u is a weak supersolution in Ω, and further (s,p)-superharmonic in Ω by Theorem 16. $\square$

Now, we are ready to state the nonlocal counterpart of the fundamental alternative theorem for the classical nonlinear Potential Theory.

Theorem 17 [The nonlocal Perron theorem; [51, Theorem 2]**].** *The Perron solutions $\overline{H}_g$ and $\underline{H}_g$ can be either identically $+\infty$ in Ω, identically $-\infty$ in Ω, or (s,p)-harmonic in Ω, respectively.*

We conclude this section by investigating the Perron resolutivity in the nonlocal framework. Firstly, collecting some of the tools presented before, together with the continuity results up to the boundary for the nonlocal obstacle problem (see Theorem 9 in [49]), it is rather straightforward to prove a basic existence and regularity result for the solution to the nonlocal Dirichlet boundary value problem, under suitable assumptions on the boundary values and the domain Ω.

Theorem 18 [Existence of solutions; [51, Theorem 17]**].** *Let $\Omega \Subset \Omega'$ be bounded open sets, and assume that $\mathbb{R}^n \setminus \Omega$ satisfies the measure density condition (9). Suppose that $g \in C(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Then there is a weak solution in Ω, which is continuous in Ω' and has boundary values g on $\mathbb{R}^n \setminus \Omega$. Such a solution is unique.*

Secondly, if there is a solution to the nonlocal Dirichlet problem then it is necessarily the nonlocal Perron solution (see Lemma 8 below). In particular, this is the case under the natural hypothesis of Theorem 18.

Lemma 8 [Continuous weak solutions are Perron solutions; [51, Lemma 19]**].** *Assume that $h \in C(\overline{\Omega})$ is a weak solution in Ω such that*

$$\lim_{\Omega \ni y \to x} h(y) = g(x) \quad \text{for every } x \in \partial\Omega \qquad \text{and} \qquad h = g \quad \text{a. e. in } \mathbb{R}^n \setminus \Omega$$

for some $g \in C(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ with $\Omega' \Supset \Omega$. Then $\overline{H}_g = h = \underline{H}_g$.

Proof The situation is symmetric, so we only need to prove the result for $\overline{H}_g$. We have $h \geq \overline{H}_g$ since $h \in \mathcal{U}_g$. To obtain the reverse inequality, let $u \in \mathcal{U}_g$. Then for every $\varepsilon > 0$ there exists an open set $D \Subset \Omega$ such that $u + \varepsilon > h$ in $\mathbb{R}^n \setminus D$. Consequently, $u + \varepsilon \geq h$ in D since $u + \varepsilon$ is (s,p)-superharmonic in Ω. Letting $\varepsilon \to 0$ we obtain that $u \geq h$, and taking the infimum over $\mathcal{U}_g$ yields $\overline{H}_g \geq h$. $\square$

Remark 16 It is worth mentioning the strictly related paper [63], where E. Lindgren and P. Lindqvist they deal with a general class of fractional Laplace equations with bounded boundary data, in the case when the operator $\mathcal{L}$ in (1) does reduce to the pure fractional p-Laplacian $(-\Delta)^s_p$ without coefficients. This very relevant paper contains several important results, as a fractional Perron method and a Wiener resolutivity theorem, together with the subsequent classification of the regular points, in such a nonlinear fractional framework. We suggest the interested reader to compare those results together with the ones presented here.

Open Problem 7. It seems natural to provide a necessary and sufficient condition for the geometry of the boundary at a point $z \in \partial\Omega$ such that if g is "continuous at z" then so are $\overline{H}_g$ and $\underline{H}_g$. To our knowledge, the nonlocal counterpart of the Wiener criterion is still an open problem.

6 Viscosity solutions, Measure data problems, and further related results

We now briefly present some very important related results for the considered nonlinear integro-differential equation (2). We start by presenting the connection with the nonlocal viscosity solutions.

6.1 Viscosity solutions

Notice that in all the previous sections we have dealt with two main notions of nonlocal solutions to the equation in (2): the weak solutions given by Definition 3, which naturally arise as minimizers of the Gagliardo seminorm; and the (s,p)-harmonic functions as in Definition 6, which are defined via comparison with weak solutions, naturally arising for example in the Perron method presented before. A third very important notion of solution is that based on the pointwise evaluation of the principal value appearing in (1): the *viscosity solutions* to (2). In order to provide the good notion of viscosity solutions, one can observe that for exponents in the range $p \leq \frac{2}{2-s}$, a more restricted class of test functions is needed. Indeed, in such a range the operator is singular, in the sense that it is not well defined even on smooth functions. For example, defining

$$u(x) = \begin{cases} |x|^2, & x \in B_1, \\ 1, & x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

which is smooth close to origin, we have that the principal value $\mathcal{L}u(0)$ is finite if and only if $p > \frac{2}{2-s}$. Thus, when x_0 is an isolated critical point, essentially we would like to test viscosity solutions by merely using functions of the type $|x-x_0|^\beta$. However, we need some flexibility in the choice of test functions and this motivates the definition of the space C^2_β below, which in particular contains monomials like $|x-x_0|^\beta$ plus suitable perturbations.

Now we are ready to introduce some notation. We denote the set of critical points of a differentiable function u, and the distance from the critical points by

$$N_u := \{x \in \Omega : \nabla u(x) = 0\}, \quad d_u(x) := \operatorname{dist}(x, N_u),$$

respectively. Let $D \subset \Omega$ be an open set.

Definition 8 We denote the class of C^2-functions whose gradient and Hessian are controlled by d_u as

$$C^2_\beta(D) := \left\{ u \in C^2(D) : \sup_{x \in D} \left(\frac{\min\{d_u(x), 1\}^{\beta-1}}{|\nabla u(x)|} + \frac{|D^2 u(x)|}{d_u(x)^{\beta-2}} \right) < \infty \right\}.$$

The supremum in the definition is denoted by $\|\cdot\|_{C^2_\beta(D)}$.

Notice that, in particular, when $\beta \geq 2$, the function $\phi(x) = |x|^\beta$ is in the class C^2_β.

Definition 9 [(s,p)-viscosity solutions; [47]]. We say that a function $u\colon \mathbb{R}^n \to [-\infty, \infty]$ is an (s,p)-*viscosity supersolution* in Ω if it satisfies the following four assumptions.

(i) $u < +\infty$ almost everywhere in $\mathbb{R}^n$, and $u > -\infty$ everywhere in Ω.
(ii) u is lower semicontinuous in Ω.
(iii) If $\phi \in C^2(B_r(x_0))$ for some $B_r(x_0) \subset \Omega$ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in $B_r(x_0)$, and one of the following holds
  (a) $p > \frac{2}{2-s}$ or $\nabla\phi(x_0) \neq 0$,
  (b) $1 < p \leq \frac{2}{2-s}$, $\nabla\phi(x_0) = 0$ such that x_0 is an isolated critical point of ϕ, and $\phi \in C^2_\beta(B_r(x_0))$ for some $\beta > \frac{sp}{p-1}$,
  then $\mathcal{L}\phi_r(x_0) \geq 0$, where

$$\phi_r(x) = \begin{cases} \phi(x), & x \in B_r(x_0), \\ u(x), & x \in \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

(iv) u_- belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$.

A function u is an (s,p)-*viscosity subsolution* in Ω if $-u$ is an (s,p)-viscosity supersolution. Moreover, u is an (s,p)-*viscosity solution* in Ω if it is both an (s,p)-viscosity supersolution and a subsolution.

A natural question immediately arises: do all those notions of solutions coincide (under suitable assumptions)? First of all, notice that the weak solutions, as well as potential-theoretic (s,p)-harmonic functions, are well-defined for very general, merely measurable kernels, since there is a natural weak formulation behind as soon as the kernel $K(\cdot,\cdot)$ is symmetric. However, in order to obtain the equivalence between different notions of solutions, one is forced to assume that K is translation invariant. This is in fact a necessity, as already explained in [87].

The answer to the question above is positive. Indeed, by using the recent results in [48] and [51], presented in the previous sections, one can show that solutions defined via comparison and viscosity solutions are exactly the same for a precise class $\mathrm{Ker}(\Lambda)$ of kernels, where Λ is measuring the ellipticity; see Definition 10 below. This has been proven by J. Korvempää, T. Kuusi and E. Lindgren in [47]. We thus give the description of those suitable kernels.

Definition 10 [The Ker($\boldsymbol{\Lambda}$)-kernels; [47, Section 2]]. We say that the kernel $K\colon \mathbb{R}^n \times \mathbb{R}^n \to (0,\infty]$ belongs to $\mathrm{Ker}(\Lambda)$, if it satisfies the following properties:

(i) **Symmetry.** $K(x,y) = K(y,x)$ for all $x,y \in \mathbb{R}^n$.
(ii) **Translation invariance.** $K(x+z,y+z) = K(x,y)$ for all $x,y,z \in \mathbb{R}^n$, $x \neq y$.
(iii) **Growth condition.** $\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda$ for all $x,y \in \mathbb{R}^n$, $x \neq y$.
(iv) **Continuity.** The map $x \mapsto K(x,y)$ is continuous in $\mathbb{R}^n \setminus \{y\}$.

Above $\Lambda \geq 1$ is a constant.

Remark 17 Notice that, by symmetry, the function $y \mapsto K(x,y)$ is continuous as well in $\mathbb{R}^n \setminus \{x\}$ if $K \in \mathrm{Ker}(\Lambda)$. Also, we immediately notice that the required properties apply in particular in the case of the fractional p-Laplace equation; that is, when $K = K(x,y)$ reduces to $|x-y|^{-n-sp}$.

We have the following important results.

Theorem 19 [Viscosity vs harmonic; [47, Theorem 1.1]]. *Suppose that the kernel K belongs to* $\mathrm{Ker}(\Lambda)$*. Then, a function u is (s,p)-superharmonic in Ω if and only if it is an (s,p)-viscosity supersolution in Ω.*

In the case when the supersolutions are bounded or belong to the right Sobolev space, we have the desired equivalence of all the notions of solutions.

Theorem 20 [Equivalence of nonlocal supersolutions; [47, Theorem 1.2]]. *Suppose that the kernel K belongs to* $\mathrm{Ker}(\Lambda)$*. Assume that u is locally bounded from above in Ω or $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$. Then the following statements are equivalent:*

(1) u is the lower semicontinuous representative of a weak supersolution in Ω.
(2) u is (s,p)-superharmonic in Ω.
(3) u is an (s,p)-viscosity supersolution in Ω.

In particular, thanks to the theorem above one get that a continuous bounded energy solution is a viscosity solution. Moreover, if a weak solution is trapped between two functions that are regular enough, then the principal value in (1) is well-defined and zero, as stated in Proposition 3.1 in [47]. As a matter of fact, this result and the two theorems above assert that if a lower semicontinuous supersolution touches a smooth function from above, then the principal value exists at that point and is nonnegative.

Notice also that in the definition of viscosity supersolutions no integrability or differentiability assumptions are required. Thus, in view of Theorems 19-20, we may directly apply Theorem 15, together with all the results presented in Section 5 to obtain many useful result for viscosity supersolutions, which are far from being obvious starting from the very definition.

Remark 18 In [62] the local Hölder regularity for viscosity solutions is studied, inspired by the methods and results in [42, 87].

Remark 19 In the liner case when $p = 2$ there is a vast literature about viscosity solutions for integro-differential equations, and in many cases the obtained regularity is optimal; see for instance [18, 81] and the references therein.

6.2 Improved differentiability of solutions to integro-differential equations

A classical theorem, which is basically an implication of the so-called *Gehring lemma* [37], asserts that solution to uniformly elliptic equations in divergence form
$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } \Omega,$$
where the matrix of coefficients $A = A(\cdot)$ is supposed to have measurable entries, and to be bounded and elliptic, does belong to the space $W^{1,2+\epsilon}$ for some $\epsilon > 0$. This is a nontrivial result, since the variational formulation of the problem above only gives us a solution in $W^{1,2}$. Thus, the result provides an improvement in the integrability of u from $W^{1,2}$ to $W^{1,2+\epsilon}$, and this constitutes a fundamental tool in modern nonlinear analysis, with crucial implications in several different fields, ranging from nonlinear elliptic and parabolic equations to the Calculus of Variations, from quasi-conformal geometry to stability issues. The ultimate essence of the Gehring lemma does not simply deal with the solution to the equations above, but more generally it relies on basic self-improving properties of certain reverse Hölder type inequalities; see [37].

In the (linear) fractional framwork, where the leading operator is the one considered here in (2) in the case when $p = 2$, it turns out that the solution u belongs to the space $W^{s+\epsilon,2+\epsilon}$; see the relevant paper by Kuusi, Mingione and Sire [56], and in particular Theorem 1.1 there. The surprising part of such a result is that there is an improvement of differentiability: not only the power of integrability is improved from 2 to $2+\epsilon$, but also the order of differentiability is improved from s to $s+\epsilon$. For this, a new fractional version of the Gehring lemma valid for general fractional Sobolev functions, and not only for solutions to nonlocal equations, has been proved. Rather than holding for functions, this new version of the Gehring lemma does hold for what is called "dual pairs", as in the following

Definition 11 Let $s \in (0,1)$, and let $u \in W^{s,2}(\mathbb{R}^n)$, and let $\eta \in (0, s/2)$. Define the function

$$U(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{s+\eta}},$$

whenever $x \neq y$ and the measure

$$\mu(A) := \int_A \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n-2\eta}},$$

whenever $A \subset \mathbb{R}^n \times \mathbb{R}^n$ is a measurable subset. The couple (μ, U) is called a *dual pair* generated by the function u.

The idea is now the following: the problem of proving self-improving properties for a function $\in W^{s,2}$ in $\mathbb{R}^n$ is lifted in $\mathbb{R}^n \times \mathbb{R}^n$; it i then proved a higher integrability result for U with respect to the measure μ. Essentially, this is a higher integrability result for the dual pair (μ, U). This eventually implies the higher differentiability of u.

Theorem 21 [Fractional Gehring lemma, [56, Theorem 1.3]]**.** *Let $u \in W^{s,2}(\mathbb{R}^n)$ for $s \in (0,1)$, and let (μ, U) be the dual pair generated by u in the sense of* Definition 11. *Assume that the following reverse Hölder-type inequality with tail holds for every $\sigma \in (0,1)$ and for every ball $B \subset \mathbb{R}^n$,*

$$\left(\fint_{B\times B} U^2\,\mathrm{d}\mu\right)^{\frac{1}{2}} \leq \frac{c(\sigma)}{\sigma\eta^{1/q-1/2}}\left(\fint_{2B\times 2B} U^q\,\mathrm{d}\mu\right)^{\frac{1}{q}} \tag{17}$$

$$+\frac{\sigma}{\eta^{1/q-1/2}}\sum_{k=2}^{\infty} 2^{-k(s-\eta)}\left(\fint_{2^kB\times 2^kB} U^q\,\mathrm{d}\mu\right)^{\frac{1}{q}}, \tag{18}$$

where $q \in (1,2)$ is a fixed exponent. Then there exists a positive number $\epsilon \in (0, 1-s)$, depending only on $s, \eta, q, c(\sigma)$, such that $U \in L^{2+\epsilon}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n; \mu)$ and $u \in W^{s+\epsilon,2+\epsilon}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover. the following inequality holds whenever $B \subset \mathbb{R}^n$, for a constant $c \equiv c(N, s, \epsilon, c(\sigma), q)$,

$$\left(\fint_{B\times B} U^{2+\epsilon}\,\mathrm{d}\mu\right)^{\frac{1}{2+\epsilon}} \leq c\sum_{k=2}^{\infty} 2^{-k(s-\eta)}\left(\fint_{2^kB\times 2^kB} U^2\,\mathrm{d}\mu\right)^{\frac{1}{2}}.$$

Notice that the main point in the preceding theorem is that it is not asserted the higher integrability of any function U satisfying the reverse Hölder inequality (17), but only on diagonals balls $B \times B \subset \mathbb{R}^n \times \mathbb{R}^n$. Basically, with Theorem 21 it is asserted the higher integrability of U in $L^{2+\epsilon}(\mu)$ provided that (μ, U) is a dual pair, and this is the crucial point allowing to recover the missing information on non-diagonal balls. Once the theorem is proved, one can recover the desired higher differentiability of solutions to the equations, or more generally of functions satisfying the correspondent Caccioppoli inequality with tail.

Remark 20 The results in [56] are still valid with no important modifications in the nonlinear fractional case when $p > 2$. In this range of validity ($p \geq 2$), a different approach can be find in the very relevant paper by A. Schikorra [84] via a robust nonlocal nonlinear commutator estimate concerning the transfer of derivatives onto test functions. Apart from the higher regularity for solutions to nonlocal equations, as a further application, one can also deduce that sequences of uniformly bounded n/s-harmonic maps do converge strongly outside at most finitely many points.

6.3 Measure data problems

We conclude this paper with a glimpse on the regularity theory for the inhomogeneous counterpart,

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{19}$$

which has been recently settled in [57] in a general setting, including also the case when the source term f is merely a measure. This remarkable paper includes a quite comprehensive existence, regularity and potential theory for solutions to nonlinear, possibly degenerate, integro-differential equations with measure data. The correspondent results are partly based on the quantitative estimates (involving the nonlocal tail contribution) established in [26,27], and they basically constitute the nonlocal analogue of the fundamental ones available in the local degenerate case after the work of L. Boccardo and T. Gallouët [6,7] and T. Kilpeläinen and J. Malý [46]. Recalling the assumptions in Section 2, and assuming that the datum f is a signed Borel measure with finite total mass, one can define a particular class of solution to (19), the class of the so-called *Solutions Obtained as Limits of Approximations* (SOLA) as the collection of distributional solutions to (19), which coincide with g almost everywhere in the complement of Ω, and which can be constructed via an approximation procedure; we refer to [57, Definition 2] for more details. The existence and several properties of those solutions to (19) have been proven.

Firstly, under slightly more general assumptions than the ones described above, one can prove that there exists a SOLA u to (19) such that

$$u \in W^{h,q}(\Omega) \text{ for } h \in (0,s) \text{ and } q \in \left[q_* := \max\Big\{1, p-1\Big\}, \min\left\{\frac{N(p-1)}{N-s}, p\right\}\right); \tag{20}$$

see [57, Theorem 1.1]. Then, the authors are able to prove other important results in the fractional nonlinear potential theory, whose main role is played by the truncated Wolff potential of the measure μ; that is,

$$\boldsymbol{W}^{\mu}_{s,p}(x_0, r) = \int_0^r \left[\frac{|\mu|(B_\rho(x_0))}{\rho^{N-sp}}\right]^{\frac{1}{p-1}} \frac{\mathrm{d}\rho}{\rho},$$

for all $x_0 \in \Omega$ and $r > 0$. From [57, Theorem 1.2] we have that if

$$\boldsymbol{W}^{\mu}_{s,p}(x_0, r) < \infty \text{ for some } B_r(x_0) \subset \Omega, \tag{21}$$

then x_0 is a Lebesgue point of the solution u, and the following pointwise estimate does hold,

$$|u(x_0)| \leq c\, \boldsymbol{W}^{\mu}_{s,p}(x_0, r) + c\left(\fint_{B_r(x_0)} |u|^{q_*}\,\mathrm{d}x\right)^{\frac{1}{q_*}} + c\,\mathrm{Tail}(u; x_0, r), \tag{22}$$

where the exponent q_* is defined in (20). Notice that the nonlocal tail contribution defined in (4) is again necessary in order to encode the long-range interactions appearing when dealing with nonlocal operators.

Pointwise estimates via potentials imply local Calderón-Zygmund-type estimates; this is a consequence of the fact that the behavior of the Wolff potentials in rearrangement invariant function spaces, and in particular in Lebesgue spaces, is known; see [21]. Thus, one can prove for instance that

$$\mu \in L^q_{\mathrm{loc}}(\Omega) \Longrightarrow u \in L^{\frac{nq(p-1)}{n-spq}}_{\mathrm{loc}}(\Omega)$$

in the case when $sp < N/q$ and $q > 1$; see [57, Corollary 1.1].

The estimate in (22) is sharp in describing the pointwise behaviour of the SOLA in the sense that the Wolff potential appearing on the right-hand side there cannot be replaced by any other potential. Indeed, if the measure μ is nonnegative, then, under the assumption in (21), one also has the following potential lower bound (see [57, Theorem 1.3]),

$$\boldsymbol{W}^{\mu}_{s,p}(x_0, r/8) \leq c\, u(x_0) + c\,\mathrm{Tail}(u_-; x_0, r/2).$$

All these results above are related with oscillation bounds for the solutions. For this, one needs to introduce *the global excess functional* $E(f;x_0,r)$ of a function f in $L^{q_*}_{\rm loc}(\mathbb{R}^n)$ whose Tail is finite by

$$E(f;x_0,r) := \left(\fint_{B_r(x_0)} |f-(f)_{x_0,r}|^{q_*}\,dx\right)^{1/q_*} + \mathrm{Tail}\big(f-(f)_{x_0,r};x_0,r\big),$$

Above $(f)_{x_0,r}$ stands for the average of f over $B_r(x_0)$. We have that a SOLA to (19) satisfies

$$\int_0^r E(u;x_0,\rho)\,\frac{d\rho}{\rho} + |u(x_0)-(u)_{r,x_0}| \le c\,\boldsymbol{W}^{\mu}_{s,p}(x_0,r) + c\,E(u;x_0,r)$$

when $B_r(x_0) \subset \Omega$; see [57, Theorem 1.4]. This estimate allows one to find sharp criteria for the local continuity of u as

$$\boldsymbol{W}^{\mu}_{s,p}(\cdot,r) \underset{r\to 0}{\longrightarrow} 0 \ \text{ locally uniformly} \Longrightarrow u \in C^0_{\rm loc}(\Omega).$$

Notice that, in the case $sp<n$, if μ belongs to the Lorentz space $L^{\frac{N}{sp},\frac{1}{p-1}}$ locally in Ω, then $\boldsymbol{W}^{\mu}_{s,p}(\cdot,r)\to 0$ locally uniformly. Moreover, as the last corollary, continuity of u is inferred once the measure μ satisfies a density condition of the type

$$|\mu|(B_r(x_0)) \le c\, r^{n-p}h(r) \quad \text{with} \quad \int_0^1 \big[h(\rho)\big]^{1/(p-1)}\frac{d\rho}{\rho} < \infty$$

for balls $B_r(x_0)\subset\mathbb{R}^n$; see [57, Corollary 1.3]. This is the borderline nonlocal counterpart of the classic results in [61] and [44].

Finally, in the case when $\mu=0$, a nonlocal Campanato theorem describes a global excess decay. Namely, whenever $0,\rho\le r\le R$, the following inequality does hold for an exponent $\alpha\in(0,sp/q_*)$.

$$E(u;x_0,\rho) \le c\left(\frac{\rho}{r}\right)^{\alpha}\left(\left(\frac{r}{R}\right)^{sp/q_*}E(u;x_0,R) + \int_r^R \left(\frac{r}{\sigma}\right)^{sp/q_*}E(u;x_0,\sigma)\frac{d\sigma}{\sigma}\right);$$

see Theorem 1.6 in [57].

Open Problem 8. The existence of solutions is widely studied issue in the case of local equations. Typical classes of solutions, other than the ones obtained via limiting approximation (SOLA), are renormalized solutions, and entropy solutions. In the case of nonnegative measures all these classes do coincide with the superharmonic solutions, as proven in [45]. The uniqueness in measure data problems is still a major open problem. Also in the case of the nonlocal equations, the uniqueness of solutions is an open problem.

Open Problem 9. An interesting open problem is whether it is possible to extend those results in [57] to the case when the operator (1) involves more general kernels, which can be, for example, unbounded away from the diagonal $x = y$; even in the case $p = 2$, the shape of fundamental solutions is generally not well-understood.

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