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Submanifolds in Carnot groups

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Introduction

Recent years have witnessed an increasing interest towards Analysis and Geometry in Metric Spaces, in the perspective of generalizing to such structures classical methods and results. Many areas of research have therefore been investigated, such as Sobolev spaces [93, 94], the theory of quasiconformal maps [98] and typical subjects of Geometric Measure Theory such as currents [7] and rectifiable sets [8, 4, 5, 150, 104]; see also [90, 91, 92, 162, 11, 97, 13], and the references therein.

Carnot-Carathéodory spaces are a particular class of metric spaces in which these investigations have been carried out with prosperous results. Historically, the first items of this type appear in a 1909 work of C. Carathéodory [38], where a thermodynamic process is represented by a curve in $\mathbb{R}^n$ and the heat exchanged during it by the integral of a suitable 1-form θ along the same curve. The physicist J. Carnot proved the existence of states that are not connectable by means of adiabatic processes: in other words, by curves along which θ vanishes, that nowadays would be called *horizontal*. The problem of connecting points by means of horizontal curves, i.e. curves whose derivative lies in a proper subspace of the whole tangent bundle, was attacked by P. K. Rashevsky [151] and W. L. Chow [45]. They independently proved that a sufficient condition for connectivity is the distribution of subspaces Lie generating the whole tangent space at every point. This condition has subsequently played a key role in several branches of Mathematics (e.g. Nonholonomic Mechanics, subelliptic PDE's and Optimal Control Theory), under the different names of "total nonholonomicity", "Hörmander condition", "bracket generating condition" and "Chow condition". Let us remark that these results fit the ones by Carnot and Carathéodory showing that θ is integrable, i.e. $\theta = T\,dS$ for suitable functions S, T, which implies in particular that $\ker\theta$ does not Lie generate the whole tangent space.

A *Carnot-Carathéodory (CC) space* is an open subset $\Omega \subset \mathbb{R}^n$ (or, more generally, a manifold) endowed with a family $X = (X_1, \dots, X_m)$ of vector fields such that every two points $x, y \in \Omega$ can be joined, for some $T > 0$, by an absolutely continuous curve $\gamma : [0, T] \to \Omega$ such that

$$\dot\gamma(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) \quad \text{and} \quad |h(t)| \le 1 \quad \text{for a.e. } t\,.$$

We will call *subunit* such a curve and, according to the terminology in [90] and [145], we define the *Carnot-Carathéodory distance* between x and y to be

$$d_c(x,y) = \inf\{T \geq 0 : \text{ there exists a subunit curve } \gamma : [0,T] \to \mathbb{R}^n \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\}.$$

As we said earlier, Chow condition ensures connectivity of points by means of subunit curves, whence d_c is an actual finite distance. We stress here some peculiarly non-Riemannian features of d_c, such as non uniqueness of geodesics (even in small neighbourhoods), its anisotropic behaviour (there are directions along which $d_c \simeq |\cdot|^{1/j}$, $j > 1$ – see the Nagel-Stein-Wainger Ball-Box Theorem [142]) and the fact that the Hausdorff dimension is strictly bigger than the topological one.

Among CC spaces, a fundamental role is played by *Carnot groups*. These are finite dimensional, connected and simply connected Lie groups $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ of left invariant vector fields is *stratified*, i.e. it can be written as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\iota$$

for suitable subspaces $\mathfrak{g}_j$'s with the property that $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$ and $[\mathfrak{g}_1, \mathfrak{g}_\iota] = \{0\}$; the integer ι is called the *step* of $\mathbb{G}$. Such groups can be endowed with a natural CC structure given by a basis $X = (X_1, \ldots, X_m)$ of the first layer $\mathfrak{g}_1$. The importance of Carnot groups (also known as stratified groups) arose evident in [132], where it is proved that a suitable blow-up limit of a CC space at a generic point is a Carnot group. In other words, Carnot groups can be seen [21, 128] as the natural "tangent" spaces to CC spaces (exactly as Euclidean spaces are tangent to manifolds), and therefore can be considered as local models of general CC spaces. Moreover, they possess a rich enough structure for analytical and geometric investigations to be carried on: in particular, we have to mention the presence of a one-parameter family of group isomorphism, the so called *homogeneous dilations* $\delta_r, r > 0$. We recall that, in Carnot groups, the CC distance d_c is left invariant and homogeneous, i.e.

$$d_c(zx, zy) = d_c(x,y) \quad \text{and} \quad d_c(\delta_r\, x, \delta_r\, y) = r\, d_c(x,y) \qquad \text{for all } x, y, z \in \mathbb{G}, r > 0.$$

Anisotropicity is also evident in this setting, as

$$d_c(e, \exp(sX)) = C(X)|s|^{1/j} \quad \text{if } X \in \mathfrak{g}_j\,,$$

where e is the group identity. It is well known that the Hausdorff dimension of $\mathbb{G}$ is $Q := \sum_{j=1}^{\iota} j \dim \mathfrak{g}_j > n$. Beautiful accounts on CC spaces and Carnot groups, together with exaustive references, can also be found in [135] and [119].

Even before the formal introduction of CC spaces, their structure proved a key tool in several areas of research, such as hypoelliptic equations [102, 158], degenerate elliptic equations [28, 71, 72, 88, 70, 76, 108] and singular integrals [51]; see

also [142, 168, 164] and the more recent results [46, 84, 140, 141, 160, 34, 17, 37, 36]. It is worthwile to remind that Hörmander [102] proved the hypoellipticity of the sub-Laplacian operator

$$\Delta_X := \sum_{j=1}^{m} X_j^2$$

in case bracket generating condition holds. We should mention here also Sobolev spaces theory and its connections with Poincarè-type inequalities [103, 74, 32, 85, 111], the theory of quasiconformal mappings [107, 109] and a suitable differential calculus on CC spaces [59, 133, 125], but this list of subjects is surely incomplete. Moreover, many questions are still open, even among the fundamental ones: as an example, let us recall the problem of regularity of CC geodesics [99, 165, 166, 95, 115, 22, 2, 1, 134, 169, 29, 113]. We want to stress here that recently the importance of CC spaces has arisen evident as they have been used to formalize mathematical models of areas of the visual cortex [149, 50] and of ear's structure [152, 153].

The attempt to develope a Geometric Measure Theory (see [69, 163, 68, 67, 129, 61, 139, 3]) in CC spaces is more recent; the first result in this sense probably traces back to the proof of the isoperimetric inequality in the Heisenberg group [144]. About isoperimetric inequality we should mention also [32, 75] and [85]. An essential item of Geometric Measure Theory such as De Giorgi's notion of perimeter [62, 63, 64] has been extended in a natural way to CC spaces, by means of the so called *X-perimeter* (see [32, 23, 26, 77, 35, 54, 121, 123, 57, 101]): the X-perimeter of a measurable set $E \subset \Omega$ is defined as

$$\|\partial E\|_X := \sup\left\{ \int_E \mathrm{div}_X \varphi : \varphi \in \mathbf{C}_c^1(\Omega, \mathbb{R}^m), |\varphi| \leq 1 \right\},$$

where $\mathrm{div}_X \varphi = -\sum_{j=1}^m X_j^* \varphi_j$ and X_j^* is the formal adjoint operator to X_j. The X-perimeter measure has good natural properties, such as an integral representation [137] in case of sets with smooth boundary, or its $(Q-1)$-homogeneity in Carnot groups setting. More generally, it is also possible to give a good definition of functions of bounded X-variation [26, 32, 73, 77, 10], which fits the one given for functions in general metric spaces [131]. The theory of minimal surfaces [89] has been investigated [85, 56, 147, 41, 87, 42, 148, 43, 30, 31], and also differentiability of Lipschitz maps [145, 127, 39, 105, 170, 171, 40], fractal geometry [18, 16, 19], area and coarea formulae [77, 117, 118] and the isoperimetric problem [114, 55, 112, 156, 157, 136, 100, 44, 155] provided prosperous research themes. More recently, Bernstein type problems in the Heisenberg group have been attacked with different formulations [42, 86, 157, 60, 20, 58, 138]. However, basic techniques of classical Euclidean Geometry do not admit any counterpart in the CC settings, like Besicovitch covering theorem [154], while many others, like extension of Lipschitz maps between groups, are still open or only partially solved [124].

Another item which has been deeply analyzed is the possibility of giving good definitions of rectifiability [79, 80, 146, 52] and currents. The classical Federer's definition of rectifiability [69], given in terms of Lipschitz images of Euclidean spaces, does not suit the geometry of CC spaces, which in general are purely unrectifiable [161]. However, this problem can be amended by considering instead noncritical levels of functions whose horizontal derivatives are continuous [73, 79, 80, 48]: notice that rectifiable sets in this new sense can be highly irregular from the Euclidean viewpoint [106]. It is widely recognized that this notion of rectifiability fits quite well the nature of CC spaces: let us remind for instance that rectifiability properties of sets of finite X-perimeter have been proved [81, 47, 82, 9]. In general, however, a good theory of currents in these settings is far from being achieved [83, 159], expecially for high codimension and even for relatively "good" objects such as Euclidean surfaces. One of the main problems is that the behaviour of a surface seems to depend on the "position" of the tangent space with respect to the stratification. We recall in particular the notion of *characteristic* points, which received great attention [14, 42, 120, 122, 123] since they can be considered irregular points from the viewpoint of intrinsic geometry.

We should mention at this point the remarkable paper [79], where the problem of rectifiability of finite X-perimeter sets is considered in the setting of the *Heisenberg group* $\mathbb{H}^n$ (see [143, 164, 29] and the recent monograph [33]). The latter is the step 2 Carnot group with stratification $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, where

$$\mathfrak{h}_1 = \text{span}\,\{X_1, \dots, X_n, Y_1, \dots, Y_n\}\,, \qquad \mathfrak{h}_2 = \text{span}\,\{T\}$$

and the only nonvanishing commutator relations are given by $[X_j, Y_j] = -4T$. A set is called $\mathbb{H}$-*rectifiable* if contained, up to negligible sets, in a countable union of $\mathbb{H}$-*regular surfaces*, i.e. level sets of functions $f : \mathbb{H}^n \to \mathbb{R}$ such that $(X_1 f, \dots, Y_n f)$ is continuous and nonvanishing. In [79] it is proved that the X-perimeter measure (rather called $\mathbb{H}$-*perimeter*) of a set of finite $\mathbb{H}$-perimeter is concentrated on a rectifiable set (on which also a blow-up result holds), and moreover an implicit function theorem for $\mathbb{H}$-regular surfaces is given. More precisely, if the $\mathbb{H}$-regular surface S is the level set of a function f with $X_1 f \neq 0$, then there exists a unique *intrinsic parametrization*

$$\phi : \omega \subset V_1 \to \mathbb{R}$$

such that $S = \Phi(\omega)$. Here we have set ω to be a (relatively) open subset of the normal subgroup

$$V_1 := \exp\left(\text{span}\,\{X_2, \dots, Y_n, T\}\right) \equiv \mathbb{R}^{2n}\,,$$

and the map Φ is linked to ϕ via the formula

$$\Phi(A) := \exp(\phi(A) X_1)(A)\,, \qquad A \in \omega\,.$$

We will also say that S is the *intrinsic graph* of ϕ. This structure theorem will provide a crucial starting point for many of our discussions.

The title of the thesis is "Submanifolds in Carnot groups": we will in turn consider Euclidean or even intrinsic regular submanifolds, and we will carry on their analysis in the model setting of Carnot groups. In particular, our aim will be to examine their most basic properties from the viewpoint of Geometric Measure Theory, considerig for instance blow-up limits, perimeter measures, area formulae, parametrizations, minimal surface equations, etc. The original contributions of the author are illustrated in Chapters 2, 4 and 5, and are contained in the papers [12], [20], [126] and [25].

The structure of the book is the following. In Chapter 1 we state the main features about CC spaces and Carnot groups in particular. In Section 1.1 we recall the definition of Carnot-Carathéodory distance and the Chow-Rashevsky theorem, and then we pass to a brief analysis of functions with bounded X-variation and of sets of finite X-perimeter; conditions for the existence of X-perimeter minimizing sets are provided. Section 1.2 is entirely concerned with Carnot groups: after a brief introduction on Lie groups, we pass to the analysis of Carnot groups, with particular emphasis on their most relevant peculiarities, such as homogeneous dilations, graded coordinates and the structure of left invariant vector fields. Also, we will recall their basic metric properties and the classical technique of convolution in homogeneous groups.

Chapter 2 is devoted to the exposition of the results obtained in [126] in collaboration with V. Magnani. In Section 2.1 we state some definitions which will be crucial in the rest of the Chapter; we recall in particular the one of degree of a p-vector τ, which correspond to a sort of stratification of $\Lambda_p(\mathfrak{g})$ analogous to the one of the algebra $\mathfrak{g}$. This allows us to define, for any given p-dimensional submanifold S, its degree $d(S)$ as the maximum among the degrees of the tangent p-vectors $\tau_S(x)$ at $x \in S$. Similar notions of degree already appeared in [91] 0.6.B (see also [35]) and correspond to a sort of "pointwise" Hausdorff dimension of the surface. In Section 2.2 we prove (see Theorem 2.19) that the intrinsic blow-up limit (i.e., according to homogeneous dilations) of S exists at points x where $\tau_S(x)$ has maximum degree $d = d(S)$ and coincides with (a left translation of) a subgroup $\Pi_S(x)$. The technique used to obtain this result, which is probably one of the main contributions of [126], consists in foliating a neighbourhood of a point x with maximum degree by means of a certain family of curves $\gamma(\cdot, \lambda)$, $\lambda \in S^{p-1}$. These curves, up to higher order terms, turn out to be invariant under dilations, i.e. of the form $\gamma(t, \lambda) = x \cdot (\delta_t(y))$ for a suitable $y = y(\lambda) \in \Pi_S(x)$: see Lemma 2.15. The regularity we require for S is $\mathbf{C}^{1,1}$. As an immediate consequence (see Theorem 2.20 and Corollary 2.21), around a point with maximum degree the spherical d-dimensional Hausdorff measure $\mathcal{S}^d$ possesses a density with respect to a fixed Riemannian surface measure on S. Therefore, the

$\mathcal{S}^d$ measure of the set of points with maximum degree can be easily computed via the integral representation of Corollary 2.21. These observations are contained in Section 2.3, where we introduce the "natural" measure μ_S associated with S. An immediate question rising up is then the one of the d-negligibility of points with non-maximum degree, which we are able to prove, in Theorem 2.22, for any step 2 Carnot group. We also compare these results with other ones already known in literature. As an application, in Section 2.4 we analyse cases of submanifolds with topological dimension 2 in the Engel group $\mathbb{E}^4$.

Beginning with Chapter 3, for the rest of the book we focus our attention on the Heisenberg group $\mathbb{H}^n$; for computational convenience, rather than the CC metric d_c we will consider the equivalent distance d_∞ defined as

$$d_\infty\big(e, \exp(tT + \textstyle\sum_{j=1}^n x_j X_j + y_j Y_j)\big) := \max\{|(x,y)|_{\mathbb{R}^{2n}}, |t|^{1/2}\}\,,$$
$$d_\infty(P,Q) = d_\infty(e, P^{-1}Q) \qquad P, Q \in \mathbb{H}^n\,,$$

where e is the identity of the group. In Section 3.1 we recall some basic features of $\mathbb{H}^n$ and of the $\mathbb{H}$-perimeter measure in particular, and in the following Section 3.2 we introduce $\mathbf{C}^1_{\mathbb{H}}$ functions as those maps $f : \mathbb{H}^n \to \mathbb{R}$ such that the distribution $\nabla_{\mathbb{H}} f = (X_1 f, \dots, Y_n f)$ is represented by a continuous function. The main result of this Section is the Whitney-type extension Theorem 3.12 (see [79]), of which we give a complete proof. In Section 3.3 we define $\mathbb{H}$-regular surfaces as level sets of $\mathbf{C}^1_{\mathbb{H}}$ functions with nonvanishing horizontal gradient $\nabla_{\mathbb{H}}$, and we prove the already mentioned Implicit Function Theorem 3.16 of [79]. The last Section 3.4 contains a brief summary of the most important issues about rectifiability in the Heisenberg group. Almost all the material of Chapter 3 is taken from [79].

In Chapter 4 we show the results contained in [12] in collaboration with L. Ambrosio and F. Serra Cassano. In Section 4.1 we deepen the notion of intrinsic graph, introducing a suitable homogeneous structure on V_1. We utilize such a structure in the following Section 4.2 to define, for a fixed $\phi : \omega \to \mathbb{R}$, the concepts of W^ϕ-differentiability and uniform W^ϕ-differentiability (see Definition 4.9). These immediately yield the notion of the W^ϕ-differential of a function $\psi : \omega \to \mathbb{R}$, which is a continuous function from ω to $\mathbb{R}^{2n-1}$ in case of uniformly W^ϕ-differentiable functions. These notions of differentiability could sound quite strange (indeed, they depend of ϕ itself!), nevertheless they provide the key tool to characterize all the maps which parametrize $\mathbb{H}$-regular surfaces. In fact, in Section 4.3 we prove that a graph $S := Im\ \Phi$ is an $\mathbb{H}$-regular surface if and only if its parametrization ϕ is uniformly W^ϕ-differentiable (see Theorem 4.17); about this result we have to mention also [48]. Therefore the W^ϕ-differential $W^\phi\phi : \omega \to \mathbb{R}^{2n-1}$ is a continuous function and it is possible to prove (see Proposition 4.3) an area-type formula

$$c(n)\mathcal{S}^{Q-1}(S) = \int_\omega \sqrt{1 + |W^\phi\phi|^2}\, d\mathcal{L}^{2n}\,,$$

which is formally identical to the classical one for Euclidean graphs. This suggests the idea, also supported by a suitable formula for the horizontal normal, that the intrinsic gradient $W^\phi\phi$ is the correct counterpart of the Euclidean one. The importance of such a gradient will be evident throughout Chapters 4 and 5; see also [24]. Section 4.4 is devoted to the problem of characterizing those maps which are uniformly W^ϕ-differentiable. The main result of this Section, Theorem 4.22, shows that they are exactly those functions ϕ such that

$$(X_2\phi, \dots, X_n\phi, Y_1\phi - 2T(\phi^2), Y_2\phi, \dots, Y_n\phi)$$

is represented, in distributional sense, by a continuous function on ω (which turns out to coincide with $W^\phi\phi$) provided it is possible to approximate ϕ, locally uniformly together with its W^ϕ-differential, by means of smooth functions. An interesting application is Corollary 4.32, that furnishes an easy recipe to produce surfaces which are not Euclidean $\mathbf{C}^1$, but still $\mathbb{H}$-regular. We want to mention here that a key tool in the proof of Theorem 4.22 is provided by the exponential maps of W^ϕ (recall that in general ϕ lacks of regularity), which can be thought as those curves that are lifted, via Φ, to horizontal curves on S. The last Section 4.5 deals with the problem of finding a biLipschitz metric model space for $\mathbf{C}^1_{\mathbb{H}}$ surfaces in $\mathbb{H}^1$: in [52] this space was individuated in $(\mathbb{R}, |\cdot|) \times (\mathbb{R}, |\cdot|^{1/2})$ for $\mathbf{C}^1$ regular surfaces. In Theorem 4.35 (see also [25]) we show that this is no longer true for general $\mathbb{H}$-regular surfaces, in the sense that we find one of them which does not admit biLipschitz mappings with that space.

Chapter 5 contains the upshots of the paper [20] and is focused on minimal surfaces in $\mathbb{H}^n$ and the Bernstein problem in particular. In Section 5.1 we extend to CC spaces the classical method of calibrations, giving sufficient conditions for sets to be X-perimeter minimizing. Applications to meaningful situations are provided, also giving some flavour about regularity of minimal surfaces. In Section 5.2, starting from the area formula for intrinsic graphs, we derive suitable first and second variation formulae which will be of great use in what follows. We stress here that the minimal surface equation is formally analogous to the classical one and reads as

$$W^\phi \cdot \frac{W^\phi\phi}{\sqrt{1 + |W^\phi\phi|^2}} = 0 \qquad \text{on } \omega\,,$$

thus enforcing the idea that W^ϕ is the proper replacement of Euclidean gradient. Section 5.3 is therefore devoted to the study of the structure of entire solutions of this equation in $\mathbb{H}^1$, where it can be rewritten as the "double Burgers" equation $W^\phi(W^\phi\phi) = 0$. The main technical tool for this analysis is the study of the behaviour of ϕ along characteristics, i.e. integral lines of the vector field $W^\phi := Y_1 - 4\phi T$. Finally, in the last Section 5.4 we attack the Bernstein problem for intrinsic graphs in $\mathbb{H}^n$: more precisely, we observe that parametrizations of maximal

subgroups of $\mathbb{H}^n$ (or laterals of them) are trivial entire solutions of the minimal surface equation, and we ask whether there are different ones. A family of such solutions in the first Heisenberg group $\mathbb{H}^1$ was exhibited in [60], were it was also proved that these examples are not perimeter minimizing. In our main result, Theorem 5.23, we use the issues of Section 5.3 to show that any entire solution which does not parametrize a subgroup (or laterals) is not a minimizer of the $\mathbb{H}$-perimeter, but just a stationary point of the area functional. Conversely, a calibration argument immediately ensures that subgroups are actual minimizers. Using the well known classical results by Bombieri, De Giorgi and Giusti [27], for $n \geq 5$ we also provide solutions to the minimal surface equation in $\mathbb{H}^n$ that do not parametrize subgroups (see Subsection 5.4.2); as far as we know, the Bernstein problem for intrinsic graphs is still open for $n = 2, 3, 4$.

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Basic notation

$\Subset$	compactly contained
Δ	simmetric difference of sets
$\#A$	cardinality of a set A
$\oplus$	direct sum of vector spaces
$\circ$	composition of functions
$\mathbb{R}^n$	n-dimensional Euclidean space
∂_i	i-th vector of the standard basis of $\mathbb{R}^n$
$\partial_i f$	partial derivative of the function f along ∂_i
$\frac{\partial f}{\partial x}, \partial_x f, f_x$	partial derivative of f with respect to x
Ω	open set in $\mathbb{R}^n$
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\langle x, y \rangle$	standard Euclidean inner product of $x, y \in \mathbb{R}^n$
$\lvert x \rvert$	Euclidean norm of $x \in \mathbb{R}^n$
ω_k	Lebesgue measure of the unit ball in $\mathbb{R}^k$
χ_E	characteristic function of a measurable set $E \subset \mathbb{R}^n$
$\fint$	average integral
$f_\sharp \mu$	push-forward of the measure μ via f
$\mu \llcorner A$	restriction of a measure μ to a set A
$\dot{\gamma}$	time derivative of a curve γ
$\mathbb{G}$	a Carnot group
$\mathbb{H}^n$	n-th Heisenberg group
$\mathfrak{g}$	Lie algebra of $\mathbb{G}$
$\mathfrak{h}$	Lie algebra of $\mathbb{H}^n$
$\Lambda_p(\mathfrak{g})$	space of p-vectors of $\mathfrak{g}$
$[X, Y]$	commutator of vector fields $X, Y \in \mathfrak{g}$
$x \cdot y$	group product between $x, y \in \mathbb{G}$
ℓ_x	left translation by an element $x \in \mathbb{G}$
δ_r	homogeneous dilations in $\mathbb{G}$
$\star$	convolution on groups, see Subsection 1.2.7

TM, T_xM	tangent bundle to a manifold M and tangent space at x
HM, H_xM	horizontal subbundle to M and horizontal subspace at x
∇f	Euclidean gradient of f
Xf	gradient of f with respect to the vector fields $X_1, \dots, X_m$
$\nabla_{\mathbb{H}} f$	Heisenberg gradient of f
div	divergence
div_X	X-divergence
$\mathrm{div}_{\mathbb{H}}$	$\mathbb{H}$-divergence
spt f	support of f
$\mathbf{C}^k(\Omega)$	continuously k-differentiable real functions in Ω
$\mathbf{C}^k_c(\Omega)$	functions in $\mathbf{C}^k(\Omega)$ with compact support in Ω
$\mathbf{C}^1_{\mathbb{H}}(\Omega)$	continuously $\nabla_{\mathbb{H}}$-differentiable functions in Ω
BV_X	functions with bounded X-variation
$BV_{\mathbb{H}}$	functions with bounded $\mathbb{H}$-variation
$\|\partial E\|_X$	X-perimeter of E
$\|\partial E\|_{\mathbb{H}}$	$\mathbb{H}$-perimeter of E
d_c	Carnot-Carathéodory distance
$\|\cdot\|_\infty, d_\infty$	infinity norm and associated distance on $\mathbb{H}^n$, see (3.1)
$B(x,r)$	open Euclidean ball
$U(x,r)$	open sub-Riemannian ball (with respect to a fixed metric)
$\mathcal{H}^d, \mathcal{S}^d$	Euclidean d-dimensional Hausdorff and spherical Hausdorff measures
$\mathcal{H}^d_c, \mathcal{S}^d_c$	d-dimensional Hausdorff measures induced by d_c
$\mathcal{H}^d_\infty, \mathcal{S}^d_\infty$	d-dimensional Hausdorff measures on $\mathbb{H}^n$ induced by d_∞
$\mathcal{H}^d_\rho, \mathcal{S}^d_\rho$	d-dimensional Hausdorff measures induced by a distance ρ

Chapter 1

The Sub-Riemannian geometry of Carnot groups

This Chapter, which will provide the basic material used throughout the book, is devoted to the study of Carnot-Carathéodory (CC) spaces, and of Carnot groups in particular. The presentation will be self-contained: for a more detailed one we refer to [135] (for CC spaces) and to [119] (for Carnot groups), from which we will take most of the material. We refer to the Introduction for a motivational and historical summary of the subjects.

In Section 1.1 we provide a brief exposition of general features concerning CC spaces; we start, in Subsection 1.1.1, by recalling the definitions of subunitary curve and of CC metric, which is an actual distance provided Chow's connectivity condition (1.5) holds. Subsection 1.1.2 deals instead with the notion of X-perimeter: we introduce it as the total X-variation (Definition 1.4) of the characteristic function of a set E, and we define X-Caccioppoli sets as those with finite X-perimeter. For such sets a representation result for the X-perimeter holds (see Proposition 1.8) which allows us to introduce the horizontal normal ν_E; moreover, for sets with smooth boundary (see Theorem 1.9) this representation turns into an integral one, that furnishes also an explicit formula for ν_E. Finally, in Theorem 1.11, stated without proof, we give general sufficient conditions for the existence of perimeter minimizing sets.

Section 1.2, treating of Carnot groups, begins with some standard facts about Lie groups and algebras (Subsection 1.2.1): we underline in particular Theorem 1.15, which will ensure that Carnot groups $\mathbb{G}$ are diffeomorphic to some $\mathbb{R}^n$, and the Baker-Campbell-Hausdorff formula (1.19). Carnot groups are introduced, together with homogeneous dilations δ_r, in the following Subsection 1.2.2; then (Subsection 1.2.3) we will focus on properties of canonical representations of $\mathbb{G}$ by means of the so-called graded coordinates, i.e. exponential coordinates arising from an adapted

basis. Examples of graded coordinates are provided in Subsection 1.2.4 in the specific situations of Heisenberg $\mathbb{H}^n$ and Engel $\mathbb{E}^4$ groups. In Subsection 1.2.5 we make use of graded coordinates to study the properties of left invariant vector fields, showing that their components are homogeneous polynomials: this result, that will be crucial for several reasonings in Chapter 2, is contained in Proposition 1.24 and is based on the already mentioned Baker-Campbell-Hausdorff formula. The CC structure on $\mathbb{G}$ is introduced in Subsection 1.2.6: the CC metric turns out to be homogeneous, i.e. left invariant and dilation scaling (see Definition 1.27). Any two homogeneous distances are biLipschitz equivalent, hence the homogeneous Hausdorff dimension Q of $\mathbb{G}$ (with respect to any of them) is well defined, and the corresponding X-perimeter (rather called $\mathbb{G}$-perimeter) is $(Q-1)$-homogeneous with respect to dilations. Finally, in Subsection 1.2.7 we recall the classical technique of convolution on homogeneous groups.

1.1 Carnot-Carathéodory spaces

1.1.1 Carnot-Carathéodory distance

Let $X = (X_1, \dots, X_m)$ be a given family of Lipschitz continuous vector fields on $\mathbb{R}^n$

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i, \qquad j = 1, \dots, m$$

with $a_{ij} \in \mathrm{Lip}(\mathbb{R}^n)$ $(j = 1, \dots, m,\ i = 1, \dots, n)$. The subspace of $\mathbb{R}^n \equiv T_x\mathbb{R}^n$ generated by $X_1(x), \dots, X_m(x)$ is called *horizontal subspace* at the point x, and it will be denoted by $H_x\mathbb{R}^n$; the collection of all horizontal fibers $H_x\mathbb{R}^n$ forms the *horizontal subbundle* $H\mathbb{R}^n$ of $T\mathbb{R}^n$.

We call *subunit* a Lipschitz continuous curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ such that

$$\dot\gamma(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) \quad \text{and} \quad \sum_{j=1}^{m} h_j^2(t) \le 1 \quad \text{for a.e. } t \in [0, T], \tag{1.1}$$

with $h_1, \dots, h_m$ measurable coefficients.

Definition 1.1. We define the *Carnot-Carathéodory (CC) distance* between the points $x, y \in \mathbb{R}^n$ as

$$\begin{aligned} d_c(x, y) = \inf\{T \ge 0 :\ & \text{there exists a subunit path } \gamma : [0, T] \to \mathbb{R}^n \\ & \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned} \tag{1.2}$$

If the above set is empty we put $d_c(x, y) = +\infty$.

We will use the notation $U(x,r)$ to denote balls with respect to the CC distance. It is easy to recognize that if d_c is finite on $\mathbb{R}^n$, i.e. $d_c(x,y) < \infty$ for every $x, y \in \mathbb{R}^n$, it turns out to be a metric on $\mathbb{R}^n$: the metric space $(\mathbb{R}^n, d_c)$ is called Carnot-Carathéodory (CC) space (see, for instance, [91] and [134]). In particular we shall generally assume the following connectivity condition

$$d_c \text{ is finite and the identity map } (\mathbb{R}^n, d_c) \to (\mathbb{R}^n, |\cdot|) \text{ is a homeomorphism.} \quad (1.3)$$

There is a large variety of situations where condition (1.3) is satisfied; among them the most important are certainly the CC spaces satisfying Chow's condition, also called *Sub-Riemannian spaces*. Recall that, given two vector fields $Y_1, Y_2 \in \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we define the *commutator* $[Y_1, Y_2]$ as the $\mathbf{C}^\infty$ vector field given by $Y_1Y_2 - Y_2Y_1$ (as common in literature, we tacitly identify vector fields and first order operators); if $Y_1 = \sum_{i=1}^n a_i(x)\partial_i$ and $Y_2 = \sum_{i=1}^n b_i(x)\partial_i$, in coordinates $[Y_1, Y_2]$ is given by

$$[Y_1, Y_2](x) = \sum_{i,j=1}^{n} \left(a_j(x) \frac{\partial b_i}{\partial x_j}(x) - b_j(x) \frac{\partial a_i}{\partial x_j}(x) \right) \partial_i. \quad (1.4)$$

This product is antisymmetric ($[Y_1, Y_2] = -[Y_2, Y_1]$) and satisfies Jacobi's identity

$$[Y_1, [Y_2, Y_3]] + [Y_2, [Y_3, Y_1]] + [Y_3, [Y_1, Y_2]] = 0.$$

Therefore if the vector fields $X_1, \ldots, X_m$ are of class $\mathbf{C}^\infty$, they generate a Lie algebra $\mathfrak{L}(X_1, \ldots, X_m)$ (see Definition 1.13).

Definition 1.2. We say that the $\mathbf{C}^\infty$ vector fields $X_1, \ldots, X_m$ satisfy *Chow's condition* if

$$\text{rank } \mathfrak{L}(X_1, \ldots, X_m) = n \quad (1.5)$$

for all $x \in \mathbb{R}^n$.

The proof of the following well-known result can be found in [45, 151] and [110].

Theorem 1.3. *If the vector fields $X_1, \ldots, X_m$ satisfy Chow's condition, then the metric d_c verifies* (1.3)*. In particular, there is always a subunit path connecting any two points $x, y \in \mathbb{R}^n$ and the topology induced by d_c is the usual Euclidean one on $\mathbb{R}^n$.*

1.1.2 X-perimeter and X-Caccioppoli sets

Whenever Ω is an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is a measurable function we define its *horizontal gradient* Xf as

$$Xf = (X_1 f, \ldots, X_m f)$$

where the previous equality must be understood in distributional sense. If $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbf{C}^1_c(\Omega; \mathbb{R}^m)$ we put

$$\operatorname{div}_X \varphi := -\sum_{j=1}^{m} X_j^* \varphi_j; \tag{1.6}$$

here X_j^* is the adjoint operator of X_j in $L^2(\mathbb{R}^n)$ given by

$$X_j^* \psi(x) := -\sum_{i=1}^{n} \partial_i (a_{ij}\, \psi)(x).$$

Observe also that φ can be canonically identified with the section of the horizontal bundle given by $\sum_{j=1}^m \varphi_j X_j$; this identification is also one-to-one if $X_1, \dots, X_m$ are linearly independent.

Definition 1.4. Let Ω be an open subset of $\mathbb{R}^n$; we say that a function $f \in L^1(\Omega)$ belongs to the space $BV_X(\Omega)$ of functions with *bounded X-variation* if there exists a m-vector valued Radon measure $\mu = (\mu_1, \dots, \mu_m)$ on Ω such that

$$\int_\Omega f \operatorname{div}_X \varphi = -\sum_{j=1}^{m} \int_\Omega \varphi_j \, d\mu_j$$

for all $\varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^m)$.

It is not difficult to see that $f \in L^1(\Omega)$ is of bounded X-variation if and only if its *X-variation* in Ω

$$|Xf|(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div}_X \varphi : \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^m), |\varphi| \le 1 \right\}$$

is finite; moreover, we have $|Xf|(\Omega) = |\mu|(\Omega)$, where μ is as in Definition 1.4. Observe that, if f is regular, then $\mu_j = X_j f \, \mathcal{L}^n$

As in the Euclidean case, an important property of BV_X functions is the lower semicontinuity of the X-variation with respect to the L^1_{loc} convergence:

Proposition 1.5. *Let $f, f_h \in L^1(\Omega)$ be such that $f_h \to f$ in $L^1_{loc}(\Omega)$; then*

$$|Xf|(\Omega) \le \liminf_{h\to\infty} |Xf_h|(\Omega).$$

Proof. For any test function $\varphi \in \mathbf{C}^\infty_c(\Omega, \mathbb{R}^m)$ with $0 \le |\varphi| \le 1$ we have

$$\int_\Omega f \operatorname{div}_X \varphi = \lim_{h\to\infty} \int_\Omega f_h \operatorname{div}_X \varphi \le \liminf_{h\to\infty} |Xu_h|(\Omega)$$

and the thesis follows taking the supremum with respect to φ. □

An argument using Friedrichs regularization (see [77]) also gives the following approximation result:

Theorem 1.6. *A function $f \in L^1(\Omega)$ belongs to $BV_X(\Omega)$ if and only if there exists a sequence $\{f_h\}_h \subset \mathbf{C}^\infty(\Omega) \cap BV_X(\Omega)$ such that $f_h \to f$ in $L^1(\Omega)$ and*

$$\lim_{h\to\infty} |Xf_h|(\Omega) = \lim_{h\to\infty} \int_\Omega |Xf_h| = |Xf|(\Omega) < \infty.$$

Following the classical De Giorgi's approach to sets of finite perimeter (see [62] and [63]), we give the following

Definition 1.7. Given a measurable subset $E \subset \mathbb{R}^n$ we define the *X-perimeter measure* $\|\partial E\|_X(\Omega)$ of E in Ω as the total variation in Ω of the characteristic function χ_E, i.e.

$$\|\partial E\|_X(\Omega) := \sup\left\{ \int_E \mathrm{div}_X \varphi : \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^m), |\varphi| \le 1 \right\}. \tag{1.7}$$

We say that E is an *X-Caccioppoli set* in Ω if $\|\partial E\|_X(\Omega) < \infty$.

Riesz representation Theorem immediately gives the following

Proposition 1.8. *If E is an X-Caccioppoli set in Ω, then there exist a unique $\|\partial E\|_X$-measurable function $\nu_E : \Omega \to \mathbb{R}^m$ such that*

$$|\nu_E|_{\mathbb{R}^m} = 1 \quad \|\partial E\|_X\text{-a.e. in } \Omega$$
$$\int_E \mathrm{div}_X \varphi \, d\mathcal{L}^n = -\int_\Omega \langle \varphi, \nu_E \rangle_{\mathbb{R}^m} \, d\|\partial E\|_X \quad \text{for all } \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^m).$$

In the following we will call ν_E *horizontal inward normal to* E (see [77]).

Whenever E is an open subset with (Euclidean) Lipschitz boundary, one can give an integral representation for the X-perimeter measure:

Theorem 1.9. *Let $E \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\|\partial E\|_X(\Omega) = \int_{\partial E \cap \Omega} \left(\sum_{j=1}^m \langle X_j, \mathbf{n} \rangle^2 \right)^{1/2} d\mathcal{H}^{n-1}, \tag{1.8}$$

where $\mathbf{n}$ is the Euclidean unit inward normal to ∂E and the scalar products appearing in (1.8) *are the usual Euclidean ones. Moreover, one has the equality of measures*

$$X\chi_E = \nu_E \, \|\partial E\|_X = \big(\langle X_1, \mathbf{n} \rangle, \dots, \langle X_m, \mathbf{n} \rangle \big) \, \mathcal{H}^{n-1} \llcorner \partial E. \tag{1.9}$$

Proof. The proof we are going to present can be found in [135], Theorem 5.1.3.

First of all, we notice that by divergence Theorem

$$\begin{aligned}\int_E \mathrm{div}_X \varphi &= \int_E \sum_{i=1}^{n} \partial_i \left(\sum_{j=1}^{m} a_{ij}\varphi_j \right) \\ &= -\int_{\partial E} \sum_{i=1}^{n} \mathbf{n}_i \sum_{j=1}^{m} a_{ij}\varphi_j \, d\mathcal{H}^{n-1} = -\int_{\partial E} \langle \varphi, \nu \rangle_{\mathbb{R}^m} \, d\mathcal{H}^{n-1} \quad (1.10)\end{aligned}$$

for any $\varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^m)$ with $\|\varphi\|_\infty \leq 1$, where we have set

$$\nu := \big(\langle X_1, \mathbf{n} \rangle, \ldots, \langle X_m, \mathbf{n} \rangle \big) \in \mathbb{R}^m.$$

Thesis (1.9) immediately follows from (1.10).

Since $|\varphi| \leq 1$, it follows that

$$\|\partial E\|_X(\Omega) \leq \int_{\partial E \cap \Omega} |\nu| \, d\mathcal{H}^{n-1}; \quad (1.11)$$

and so (1.8) will follow in one stroke if we prove also the converse inequality in (1.11). Observe that the set

$$H := \{ x \in \partial E \cap \Omega : \mathbf{n}(x) \text{ exists and } \nu(x) \neq 0 \}$$

is $\mathcal{H}^{n-1}$-measurable and, since ∂E is Lipschitz, ν is $\mathcal{H}^{n-1}$-measurable on H. For fixed $\epsilon > 0$, by Lusin Theorem there exists a compact set $K_\epsilon \subset H$ such that $\mathcal{H}^{n-1}(H \setminus K_\epsilon) \leq \epsilon$ and ν is continuous on K_ϵ; therefore $\nu/|\nu| \neq 0$ is continuous on K_ϵ and so there exists $\tilde{\varphi} \in \mathbf{C}^0_c(\Omega, \mathbb{R}^m)$ such that

$$\tilde{\varphi} = \frac{\nu}{|\nu|} \text{ on } K_\epsilon \qquad \text{and} \qquad |\tilde{\varphi}| \leq 1 \text{ on } \Omega.$$

A classical regularization argument ensures the existence of a function $\varphi \in \mathbf{C}^1_c(\Omega)$ with $\|\varphi\|_\infty \leq 1$ and $\|\tilde{\varphi} - \varphi\|_\infty \leq \epsilon$; therefore

$$\begin{aligned}\|\partial E\|_X(\Omega) &\geq \int_E \mathrm{div}_X(-\varphi) = \int_{\partial E} \langle \varphi, \nu \rangle \, d\mathcal{H}^{n-1} \\ &= \int_{\partial E} \langle \varphi - \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1} + \int_{\partial E} \langle \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1}. \quad (1.12)\end{aligned}$$

We estimate the first term on the right hand side of (1.12) as follows

$$\int_{\partial E} \langle \varphi - \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1} \geq -\epsilon \mathcal{H}^{n-1}(\partial E) \, \|\nu\|_\infty \quad (1.13)$$

while, for the second term, one has

$$\int_{\partial E} \langle \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1} = \int_H |\nu| \, d\mathcal{H}^{n-1} - \int_{H \setminus K_\epsilon} |\nu| \, d\mathcal{H}^{n-1} + \int_{H \setminus K_\epsilon} \langle \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1}. \quad (1.14)$$

Since

$$\int_{H \setminus K_\epsilon} |\nu| \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(H \setminus K_\epsilon) \|\nu\|_\infty \leq \epsilon \, \|\nu\|_\infty \quad (1.15)$$

and

$$\int_{H \setminus K_\epsilon} \langle \tilde{\varphi}, \nu \rangle \, d\mathcal{H}^{n-1} \geq -\epsilon \|\nu\|_\infty \quad (1.16)$$

and taking into account that $\|\nu\|_\infty < \infty$, by putting together (1.12), (1.13), (1.14), (1.15) and (1.16) one obtains

$$\|\partial E\|_X(\Omega) \geq \int_{\partial E \cap \Omega} |\nu| \, d\mathcal{H}^{n-1} - \epsilon(2 + \mathcal{H}^{n-1}(\partial E)) \|\nu\|_\infty,$$

whence the thesis follows by letting $\epsilon \downarrow 0$. □

Definition 1.10. We will say that E is a minimizer for the X-perimeter in Ω if

$$\|\partial E\|_X(\Omega') \leq \|\partial F\|_X(\Omega')$$

for any open set $\Omega' \Subset \Omega$ and any measurable set $F \subset \mathbb{R}^n$ such that $E \Delta F \Subset \Omega'$.

The existence of perimeter minimizing set with given boundary condition has been proved in [85]. We give here the general result therein.

Theorem 1.11. *Suppose that the CC space $(\mathbb{R}^n, d_c)$ associated with the family $X = (X_1, \ldots, X_m)$ is such that for any set $\mathcal{U} \subset \mathbb{R}^n$, with diam $\mathcal{U} < \infty$, there exist constants $C_1, C_2 > 0$, $0 < R_0 \leq \infty$ and $A \geq 1$ such that for any $x_0 \in \mathcal{U}$ and $R \in]0, R_0[$ one has*

(H.1) *the Lebesgue measure $\mathcal{L}^n$ is doubling with respect to d_c, i.e. $\mathcal{L}^n(U(x_0, 2R)) \leq C_1 \mathcal{L}^n(U(x_0, R))$, where $U(x, r)$ denotes balls with respect to d_c;*

(H.2) *for any $f \in Lip(U(x_0, AR))$ and any $\lambda > 0$*

$$\mathcal{L}^n \left(\left\{ x \in U(x_0, R) : \left| f(x) - \fint_{U(x_0,R)} f \right| > \lambda \right\} \right) \leq C_2 \tfrac{R}{\lambda} \int_{U(x_0,AR)} |Xf|;$$

(H.3) *$(\mathbb{R}^n, d_c)$ is complete and is a length space, i.e. $d_c(x, y) = \inf \, l(\gamma)$, where the* inf *is taken on all continuous curves γ joining x to y, and $l(\gamma)$ denotes the length of γ (see [13]).*

Then for any open set $\Omega \subset \mathcal{U}$ with $diam(\Omega) < R_0/2$ and any X-Caccioppoli set $L \subset \mathbb{R}^n$ there exists an X-Caccioppoli set $E \subset \mathbb{R}^n$ such that $E\Delta L \subset \Omega$ which is perimeter minimizing, i.e.

$$\|\partial E\|_X(\mathbb{R}^n) \leq \|\partial F\|_X(\mathbb{R}^n)$$

for any $F \subset \mathbb{R}^n$ such that $F\Delta L \Subset \Omega$.

We want to stress here the fact that conditions (H.1), (H.2) and (H.3) are satisfied in a large class of CC spaces, e.g. whenever the fields $X_1, \dots, X_m$ are smooth and satisfy Chow condition (1.5): see also [103, 142] and [165].

1.2 Carnot groups

1.2.1 Lie groups and algebras

Before stating the definition of Carnot groups, we want to briefly recall some basic facts on Lie groups and algebras: a more complete description of these structures can be found in [167].

Definition 1.12. A *Lie group* $\mathbb{G}$ is a manifold endowed with the structure of differential group, i.e. a group where the maps

$$\mathbb{G} \times \mathbb{G} \ni (x, y) \longmapsto xy \in \mathbb{G}$$
$$\mathbb{G} \ni x \longmapsto x^{-1} \in \mathbb{G}$$

are of class $\mathbf{C}^\infty$.

We write e for the identity of the group, while for any $x \in \mathbb{G}$ we will denote with ℓ_x the *left translation* by x, i.e. the $\mathbf{C}^\infty$ map $y \longmapsto xy$.

Definition 1.13. A vector space $\mathfrak{g}$ is a *Lie algebra* if there is a bilinear and antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies Jacobi's identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Given two subalgebras $\mathfrak{a}, \mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ we will denote with $[\mathfrak{a}, \mathfrak{b}]$ the vector subspace generated by the elements of $\{[X, Y] : X \in \mathfrak{a},\ Y \in \mathfrak{b}\}$. We set $\mathfrak{g}^1 := \mathfrak{g}$ and, by induction, $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k]$, and will say that $\mathfrak{g}$ is *nilpotent of step ι* if $\mathfrak{g}^\iota \neq \{0\}$ and $\mathfrak{g}^{\iota+1} = \{0\}$.

One can check that the space $\Gamma(TM)$ of vector fields on a differential manifold M is a Lie algebra if endowed with the product $[X, Y] = XY - YX$ defined in (1.4).

Definition 1.14. A vector field $X \in \Gamma(T\mathbb{G})$ on a Lie group $\mathbb{G}$ is *left invariant* if for any $x \in \mathbb{G}$ one has

$$X(x) = \mathrm{d}\ell_x(X(e)).$$

It is not difficult to prove that X is left invariant if and only if

$$(Xf)(\ell_x y) = X(f \circ \ell_x)(y)$$

for any $f \in \mathbf{C}^\infty(\mathbb{G})$ and $x, y \in \mathbb{G}$. We will denote by $\mathfrak{g}$ the set of left invariant vector fields of $\Gamma(T\mathbb{G})$: since a commutator of left invariant fields is left invariant, $\mathfrak{g}$ is a Lie algebra. This algebra is canonically isomorphic to the tangent space $T_e\mathbb{G}$ at the identity via the isomorphism

$$T_e\mathbb{G} \ni v \longleftrightarrow X \in \mathfrak{g} \text{ such that } X(x) = \mathrm{d}\ell_x(v).$$

We will say that a Lie group $\mathbb{G}$ is nilpotent of step k if so is its associated Lie algebra $\mathfrak{g}$.

Given $x \in \mathbb{G}$ and $X \in \mathfrak{g}$ let us consider the curve γ_x^X solution of the Cauchy problem

$$\begin{cases} \dot\gamma_x^X(t) = X(\gamma_x^X(t)) \\ \gamma_x^X(0) = x. \end{cases} \tag{1.17}$$

The curve γ_x^X is defined for any $t \in \mathbb{R}$ (i.e. left invariant vector fields are complete): in fact, one has $\gamma_x^X(t+s) = \gamma_x^X(s) \cdot \gamma_x^X(t)$, and this formula allows to extend γ_x^X to all times $t \in \mathbb{R}$.

In the following we will set $\exp(X)(x) := \gamma_x^X(1)$, where γ_x^X is the solution to the problem (1.17); the *exponential map* $\exp : \mathfrak{g} \to \mathbb{G}$ is defined as

$$\exp(X) := \exp(X)(e).$$

Therefore one has $\exp(X)(x) = x\exp(X)$ and so $\exp(X)\big(\exp(Y)\big) = \exp(Y)\cdot\exp(X)$ for all $X, Y \in \mathfrak{g}$.

We recall the following basic result:

Theorem 1.15. *Let $\mathbb{G}$ be a nilpotent, connected and simply connected Lie group; then $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism.*

For $X, Y \in \mathfrak{g}$ let us define $C(X,Y) \in \mathfrak{g}$ via the formula $\exp(C(X,Y)) = \exp(X)\cdot\exp(Y)$; then it is possible to compute explicitly $C(X,Y)$ thanks to the *Baker-Campbell-Hausdorff formula*: for each multi-index of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_l)$ we define

$$|\alpha| := \alpha_1 + \cdots + \alpha_l$$
$$\alpha! := \alpha_1! \cdots \alpha_l!$$

and we will say that l is the length of α. If $\beta = (\beta_1, \dots, \beta_l)$ is another multi-index of length l such that $\alpha_l + \beta_l \geq 1$, and if $X, Y \in \mathfrak{g}$ we set

$$C_{\alpha\beta}(X,Y) := \begin{cases} (\operatorname{ad} X)^{\alpha_1}(\operatorname{ad} Y)^{\beta_1} \dots (\operatorname{ad} X)^{\alpha_l}(\operatorname{ad} Y)^{\beta_l - 1} Y & \text{if } \beta_l > 0 \\ (\operatorname{ad} X)^{\alpha_1}(\operatorname{ad} Y)^{\beta_1} \dots (\operatorname{ad} X)^{\alpha_l - 1} X & \text{if } \beta_l = 0. \end{cases} \tag{1.18}$$

We used the notation $(\operatorname{ad} X)(Y) := [X, Y]$, agreeing that $(\operatorname{ad} X)^0$ is the identity map. Then the Baker-Campbell-Hausdorff formula states that

$$C(X,Y) := \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_l) \\ \beta=(\beta_1,\dots,\beta_l) \\ \alpha_i+\beta_i \geq 1 \ \forall i}} \frac{1}{\alpha!\beta!|\alpha+\beta|} C_{\alpha\beta}(X,Y) \tag{1.19}$$

whenever the summation at the right hand side makes sense; in particular, (1.19) holds in nilpotent groups.

1.2.2 Carnot groups

Definition 1.16. We say that a Lie algebra $\mathfrak{g}$ is *stratified* if it admits linear subspaces $\mathfrak{g}_1, \dots, \mathfrak{g}_\iota$ such that

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_\iota \\ &\mathfrak{g}_k = [\mathfrak{g}_1, \mathfrak{g}_{k-1}] \quad \text{for } k = 2, \dots, \iota \text{ and } [\mathfrak{g}_1, \mathfrak{g}_\iota] = \{0\}. \end{aligned} \tag{1.20}$$

We will call *stratification* a decomposition of $\mathfrak{g}$ as in (1.20).

A group $\mathbb{G}$ is called stratified if its Lie algebra admits a stratification; if $\mathbb{G}$ is finite dimensional and stratified, then it is also nilpotent of step ι, where ι is the same integer appearing in (1.20).

Whenever we are in presence of a stratification, it is possible to define a one-parameter group $\{\delta_r\}$ of *dilations* of the algebra; for a fixed $r \geq 0$ we set $\delta_r X := r^k X$ if $X \in \mathfrak{g}_k$, and we extend this map to the whole $\mathfrak{g}$ by linearity. It is immediate to verify the following properties of dilations:

- $\delta_{rs} = \delta_r \circ \delta_s$;
- $\delta_r([X,Y]) = [\delta_r X, \delta_r Y]$;
- $\delta_r(C(X,Y)) = C(\delta_r X, \delta_r Y)$

for all $X, Y \in \mathfrak{g}$ and all $r, s > 0$. In the following, it will be sometimes convenient to agree that $\delta_r X = -\delta_{|r|} X$ for $r < 0$.

Definition 1.17. A *Carnot group* is a finite dimensional, connected, simply connected and stratified Lie group $\mathbb{G}$. If ι is as in Definition 1.16 we will say that $\mathbb{G}$ is a Carnot group of *step* ι; observe that such a group is also nilpotent of step ι.

One of the basic properties of Carnot groups is the fact that, thanks to Theorem 1.15, the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ turns out to be a diffeomorphism: therefore we can define a one-parameter group of automorphism of $\mathbb{G}$, which we still denote with $\{\delta_r\}_{r>0}$, via the formula $\delta_r := \exp_* \delta_r$, i.e.

$$\delta_r(x) = \exp\left(\delta_r(\exp^{-1}(x))\right).$$

From the properties of dilations in Lie algebras we immediately deduce the associated ones for dilations of Carnot groups:

- $\delta_{rs} = \delta_s \circ \delta_r$, indeed

$$\begin{aligned} \delta_{rs}(x) &= \exp\left(\delta_{rs} \exp^{-1}(x)\right) \\ &= \exp\left(\delta_r \delta_s \exp^{-1}(x)\right) \\ &= \exp\left(\delta_r \exp^{-1}(\exp \delta_s \exp^{-1}(x))\right) \\ &= \exp\left(\delta_r \exp^{-1}(\delta_s(x))\right) = \delta_r \delta_s(x) \end{aligned}$$

- $\delta_r(x \cdot y) = \delta_r(x) \cdot \delta_r(y)$, indeed

$$\begin{aligned} \delta_r(x \cdot y) &= \exp \delta_r \exp^{-1}(x \cdot y) \\ &= \exp \delta_r \left(C(\exp^{-1} x, \exp^{-1} y)\right) \\ &= \exp\left(C(\delta_r \exp^{-1} x, \delta_r \exp^{-1} y)\right) \\ &= \exp\left(\delta_r \exp^{-1}(x)\right) \cdot \exp\left(\delta_r \exp^{-1}(y)\right) = \delta_r(x) \cdot \delta_r(y). \end{aligned}$$

1.2.3 Graded coordinates

Very often it is convenient to study Carnot and, more generally, stratified groups in coordinates, through canonical representations which are called graded coordinates. Therefore let $X_1, \ldots, X_n$ be a basis of Lie algebra $\mathfrak{g}$ of left invariant vector fields; for given $X, Y \in \mathfrak{g}$ we will have $X = \sum_{j=1}^n x_j X_j$ and $Y = \sum_{j=1}^n y_j X_j$ for unique $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$.

Definition 1.18. A system of *exponential coordinates* associated with the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ is the map

$$\begin{aligned} F \;:\; & \mathbb{R}^n \longrightarrow \mathbb{G} \\ & x \longmapsto \exp\left(\sum_{j=1}^n x_j X_j\right). \end{aligned} \tag{1.21}$$

The group law we put on $\mathbb{R}^n$ is the one that makes F a group isomorphism, i.e.

$$x \cdot y = z \Longleftrightarrow \sum_{j=1}^{n} z_j X_j \, = \, C\left(\sum_{j=1}^{n} x_j X_j \, , \, \sum_{j=1}^{n} y_j X_j \right). \tag{1.22}$$

It is easy to check that, in this representation, the group identity is the origin 0 and that $x^{-1} = -x$ for all $x \in \mathbb{R}^n$. In this way $\mathbb{R}^n$, endowed with the group law (1.22), turns out to be a Lie group, whose Lie algebra is isomorphic to $\mathfrak{g}$; since both $\mathbb{G}$ and $\mathbb{R}^n$ are nilpotent, connected and simply connected, by Theorem 1.15 the map F in (1.21) is also a diffeomorphism.

Observe that, up to now, we have not used the fact that $\mathbb{G}$ is stratified: therefore let us consider a Carnot group $\mathbb{G}$ with stratified algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\iota$, and, for $k = 1, \ldots, \iota$, set $m_k :=$dim $\mathfrak{g}_k$, $n_k := m_1 + \cdots + m_k$ and $n_0 := 0$. We will say that a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ is *adapted* to the stratification if $X_{n_{k-1}+1}, \ldots, X_{n_k}$ is a basis of $\mathfrak{g}_k$ for each $k = 1, \ldots, \iota$.

Definition 1.19. A system of exponential coordinates $F : \mathbb{R}^n \to \mathbb{G}$ is a system of *graded coordinates* if it is associated with and adapted basis of $\mathfrak{g}$.

We will call *degree* of the coordinate x_j the unique positive integer d_j such that $n_{d_j - 1} < j \leq n_{d_j}$.

Therefore let $F : \mathbb{R}^n \to \mathbb{G}$ be a system of graded coordinates: for the sake of simplicity we will again denote with $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$ the homogeneous dilations read in coordinates, so that $\delta_r \circ F = F \circ \delta_r$. It is easy to check that, in this representation of the group, one has

$$\begin{aligned} \delta_r \quad &: \quad \mathbb{R}^n \to \mathbb{R}^n \\ &\qquad x \longmapsto (rx_1, \ldots, rx_{n_1}, r^2 x_{n_1+1}, \ldots, r^2 x_{n_2}, \ldots, r^\iota x_{n_{\iota-1}+1}, \ldots, r^\iota x_n) \end{aligned}$$

for $r \geq 0$.

1.2.4 Heisenberg and Engel groups

We give here the representation in graded coordinates of two well-known (and probably the most important ones) examples of Carnot groups, namely the Heisenberg and Engel group.

The n-th *Heisenberg group* $\mathbb{H}^n$ is the $2n + 1$-dimensional Carnot group with stratified algebra

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2;$$

here $\mathfrak{h}_1$ is $2n$-dimensional and generated by the vectors $X_1, \ldots, X_n, Y_1, \ldots, Y_2$, while dim $\mathfrak{h}_2 = 1$ and $\mathfrak{h}_2 =$ span $\{T\}$. The only nonvanishing commutation relationships among the generators are

$$[X_j, Y_j] = -4T$$

for all $j = 1, \dots, n$, and so $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$ is the center of the algebra.

Since $\mathfrak{h}$ is nilpotent of step 2, Baker-Campbell-Hausdorff formula (1.19) reduces to

$$C(X, Y) = X + Y + \frac{1}{2}[X, Y]$$

and so

$$C(X,Y) = \sum_{j=1}^{n}(x_j + x'_j)X_j + \sum_{j=1}^{n}(y_j + y'_j)Y_j + \sum_{j=1}^{n}\big(t_j + t'_j + 2\langle x'_j y_j\rangle - 2\langle x_j y'_j\rangle\big)X_j$$

provided $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ are such that

$$X = \sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{n} y_j Y_j + t\,T \quad \text{and} \quad Y = \sum_{j=1}^{n} x'_j X_j + \sum_{j=1}^{n} y'_j Y_j + t'T.$$

Therefore, through graded coordinates associated with the adapted basis $X_1, \dots, X_n, Y_1, \dots, Y_2, T$, it is possible to represent $\mathbb{H}^n$ as $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ t + t' + 2\langle x', y\rangle - 2\langle x, y'\rangle \end{pmatrix}$$

Observe that the group identity is 0 and that, for $r > 0$, homogeneous dilations are given by $\delta_r(x, y, t) = (rx, ry, r^2 t)$.

Let us compute the explicit representation of the left invariant vector fields X_j, Y_j, T: recall that a left invariant vector field X satisfies $X(g) = d\ell_g(X(e))$ for any $g \in \mathbb{G}$. If $\partial_1, \dots, \partial_{2n+1}$ denotes the standard basis of vectors in $\mathbb{R}^{2n+1}$ we have $X_j(0) = \partial_j, Y_j(0) = \partial_{j+n}$ and $T(0) = \partial_{2n+1}$; since

$$d\ell_{(x,y,t)}(0) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 2y & -2x & 1 \end{pmatrix},$$

where I is the $n \times n$ identity matrix, one can compute that

$$\begin{aligned} X_j(x, y, t) &= d\ell_{(x,y,t)}(\partial_j) = \partial_j + 2y_j\,\partial_{2n+1} \\ Y_j(x, y, t) &= d\ell_{(x,y,t)}(\partial_{j+n}) = \partial_{j+n} - 2x_j\,\partial_{2n+1} \\ T(x, y, t) &= d\ell_{(x,y,t)}(\partial_{2n+1}) = \partial_{2n+1}. \end{aligned}$$

In what follows, we will always deal with the Heisenberg group $\mathbb{H}^n$ using this representation.

The Engel group $\mathbb{E}^4$ is the Carnot group associated with the stratified algebra

$$\mathfrak{e} = \mathfrak{e}_1 \oplus \mathfrak{e}_2 \oplus \mathfrak{e}_3$$

where $\mathfrak{e}_1 = \text{span}\,\{X_1, X_2\}$, $\mathfrak{e}_2 = \text{span}\,\{X_3\}$ and $\mathfrak{e}_3 = \text{span}\,\{X_4\}$. The only nonvanishing commutation relationships among the generators are given by

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = [X_2, X_3] = X_4;$$

since $\mathbb{E}^4$ is 3-nilpotent, for all $X, Y \in \mathfrak{e}$ Baker-Campbell-Hausdorff formula becomes

$$[X, Y] = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]].$$

Proceeding as in the Heisenberg group case, we can represent explicitly $\mathbb{E}^4$ by means of graded coordinates associated with the adapted basis X_1, X_2, X_3, X_4; in this way we have $\mathbb{E}^4 \equiv (\mathbb{R}^4, \cdot)$ and the group law is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \\ x_4 + x_4' + \frac{1}{2}\big[(x_1 x_3' - x_3 x_1') + (x_2 x_3' - x_3 x_2')\big] + \\ \quad + \frac{1}{12}(x_1 - x_1' + x_2 - x_2')(x_1 x_2' - x_2 x_1') \end{pmatrix}.$$

Again 0 is the identity element of the group and homogeneous dilations are given by $\delta_r(x_1, x_2, x_3, x_4) = (rx_1, rx_2, r^2 x_3, r^3 x_4)$. Our basis X_1, X_2, X_3, X_4 is given in coordinates by

$$\begin{aligned} X_1(x_1, x_2, x_3, x_4) &= \partial_1 - \tfrac{x_2}{2}\partial_3 - \left(\tfrac{x_3}{2} + \tfrac{x_2}{12}(x_1 + x_2)\right)\partial_4 \\ X_2(x_1, x_2, x_3, x_4) &= \partial_2 + \tfrac{x_1}{2}\partial_3 - \left(\tfrac{x_3}{2} - \tfrac{x_1}{12}(x_1 + x_2)\right)\partial_4 \\ X_3(x_1, x_2, x_3, x_4) &= \partial_3 + \tfrac{1}{2}(x_1 + x_2)\partial_4 \\ X_4(x_1, x_2, x_3, x_4) &= \partial_4. \end{aligned}$$

Another possible representation of $\mathbb{E}^4$ is given by the adapted basis Y_1, Y_2, Y_4, Y_4 and the relations

$$[Y_1, Y_2] = Y_3, \qquad [Y_1, Y_3] = Y_4, \qquad [Y_2, Y_3] = 0,$$

which correspond to the change of basis $Y_1 = (X_1 + X_2)/2$, $Y_2 = (Y_1 - Y_2)/2$, $Y_3 = -2X_3$, $Y_4 = -4X_4$. In the associated graded coordinates the group law reads

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \cdot \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} y_1 + y_1' \\ y_2 + y_2' \\ y_3 + y_3' + \frac{1}{2}(y_1 y_2' - y_2 y_1') \\ y_4 + y_4' + \frac{1}{2}(y_1 y_3' - y_3 y_1') + \frac{1}{12}(y_1 - y_1')(y_1 y_2' - y_2 y_1') \end{pmatrix},$$

group dilations are $\delta_r(y_1, y_2, y_3, y_4) = (ry_1, ry_2, r^2y_3, r^3y_4)$ and left invariant vector fields are generated by the basis

$$\begin{aligned}
Y_1(y_1, y_2, y_3, y_4) &= \partial_1 - \tfrac{y_2}{2}\partial_3 - \left(\tfrac{y_3}{2} + \tfrac{y_1 y_2}{12}\right)\partial_4 \\
Y_2(y_1, y_2, y_3, y_4) &= \partial_2 + \tfrac{y_1}{2}\partial_3 + \tfrac{y_1^2}{12}\partial_4 \\
Y_3(y_1, y_2, y_3, y_4) &= \partial_3 + \tfrac{y_1}{2}\partial_4 \\
Y_4(y_1, y_2, y_3, y_4) &= \partial_4.
\end{aligned}$$

1.2.5 Left invariant vector fields

Let $\mathbb{G}$ be a Carnot group and $F : \mathbb{R}^n \to \mathbb{G}$ a system of graded coordinates associated with the adapted basis $X_1, \dots, X_n$.

Definition 1.20. A function $P : \mathbb{G} \to \mathbb{R}$ is a *polynomial* on $\mathbb{G}$ if the composition $P \circ F$ is a polynomial function on $\mathbb{R}^n$.

We observe that the definition of polynomial is well posed: indeed, if G is another system of graded coordinates, then $F^{-1} \circ G : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map (basically, it is a change of basis of $\mathfrak{g}$), and therefore $P \circ F$ is a polynomial function if and only if so is $P \circ G = (P \circ F) \circ (F^{-1} \circ G)$.

Let $\pi^j : \mathbb{R}^n \to \mathbb{R}$ be the canonical projection on the j-th coordinate; for the sake of simplicity we will denote with π^j also the map $\pi^j \circ F^{-1} : \mathbb{G} \to \mathbb{R}$. Finally, for a given n-multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers we set

$$\begin{aligned}
\pi^\alpha \quad : \quad & \mathbb{G} \to \mathbb{R} \\
& x \longmapsto \prod_{j=1}^{n} \big(\pi^j(x)\big)^{\alpha_j}.
\end{aligned}$$

Any such a π^α is a polynomial on $\mathbb{G}$, and it is easy to check that any polynomial on $\mathbb{G}$ can be written as a finite linear combination of the π^α's. We will call homogeneous degree of π^α the integer $\deg_H(\pi^\alpha) := \sum_{j=1}^n d_j \alpha_j$.

Definition 1.21. The *homogeneous degree* of a polynomial $P = \sum_\alpha c_\alpha \pi^\alpha$ on $\mathbb{G}$ is the integer

$$\deg_H(P) := \max\{\deg_H(\pi^\alpha) : c_\alpha \neq 0\}.$$

For example, the polynomial $xy^2 - t^2$ in the Heisenberg group $\mathbb{H}^1$ has homogeneous degree 4.

Proposition 1.22. *The homogeneous degree of a polynomial P does not depend on the choice of graded coordinates.*

Proof. Let $F : \mathbb{R}^n_x \to \mathbb{G}$ and $G : \mathbb{R}^n_y \to \mathbb{G}$ be two systems of graded coordinates, related respectively to the basis $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ adapted to the stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\iota$. Let A be the $n \times n$ matrix associated with the change of basis $X \to Y$, i.e. such that

$$Y_j = \sum_{i=1}^{n} A^i_j X_i.$$

Therefore we have

$$F^{-1} \circ G(y) = \left(\textstyle\sum_{j=1}^n A^1_j y^j, \dots, \sum_{j=1}^n A^n_j y^j\right)$$

and, as the two basis are adapted, we have $A^i_j \neq 0$ only if $n_{d_j - 1} < i \leq n_{d_j}$, whence A is of the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{pmatrix} \tag{1.23}$$

where A_j denotes an $m_j \times m_j$ matrix, while the 0's denote null matrices of the proper size.

To obtain our thesis it will be sufficient to prove that for any α the map $\pi^\alpha \circ G : \mathbb{R}^n_y \to \mathbb{R}$ has the same homogeneous degree of the polynomial $(\pi_\alpha \circ F)(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We have

$$\pi^\alpha \circ G = (\pi^\alpha \circ F) \circ (F^{-1} \circ G) = \prod_{i=1}^{n} \left(\textstyle\sum_{j=1}^n A^i_j y^j\right)^{\alpha_i}.$$

Since A is invertible, none of its columns is null and so for any $i \in \{1, \dots, n\}$ there is j_i such that $A^i_{j_i} \neq 0$. As $A^i_j = 0$ if $d_j \neq d_i$ one has

$$\deg_H \left(\textstyle\sum_{j=1}^n A^i_j y^j\right)^{\alpha_i} = d_i \alpha_i$$

where the homogeneous degree is computed according to the coordinates G. Finally we have

$$\begin{aligned} \deg_H(\pi^\alpha \circ F) &= \sum_{i=1}^{n} d_i \alpha_i = \sum_{i=1}^{n} \deg_H \Big(\sum_{j=1}^{n} A^i_j y^j\Big)^{\alpha_i} \\ &= \deg_H \left(\prod_{i=1}^{n} \Big(\sum_{j=1}^{n} A^i_j y^j\Big)^{\alpha_i}\right) \\ &= \deg_H(\pi^\alpha \circ G)\,. \end{aligned}$$

□

Definition 1.23. A polynomial $P : \mathbb{G} \to \mathbb{R}$ is *homogeneous* of degree $d > 0$ if $P(\delta_r\, x) = r^\alpha P(x)$ for all $x \in \mathbb{G}$ and all $r > 0$.

For example, the polynomial xy^3+t^2 on the Heisenberg group $\mathbb{H}^1$ is homogeneous of degree 4. It is not difficult to check that a polynomial P is homogeneous of degree d if and only if it is a linear combination of polynomials π^α with $\deg_H \pi^\alpha = d$.

In graded coordinates, the left translation ℓ_x by an element $x \in \mathbb{G}$ can be written as

$$\ell_x(y) = F^{-1}(F(x) \cdot F(y)) = \big(P_1(x,y), \dots, P_n(x,y)\big) \tag{1.24}$$

where the maps $P_j(x,y)$ are polynomials which can be derived from the Baker-Campbell-Hausdorff formula. It is not difficult to prove that they are homogeneous polynomials of degree d_j, in fact

$$\begin{aligned} r^{d_j} P_j(x,y) &= \pi^j \circ \delta_r \left(P_1(x,y), \dots, P_n(x,y)\right) = \pi^j \circ \delta_r \left(F^{-1}(Fx \cdot Fy)\right) \\ &= \pi^j \circ F^{-1} \left(\delta_r(Fx) \cdot \delta_r(Fy)\right) = \pi^j \circ F^{-1} \left(F(\delta_r x) \cdot F(\delta_r y)\right) = P_j(\delta_r x, \delta_r y) \end{aligned}$$

where we have set π^j to be the map $x \mapsto x_j$.

Our next step will be to derive properties of the representation in graded coordinates of the adapted basis $X_1, \dots, X_n$; we collect them in the following

Proposition 1.24. *Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^n$ through graded coordinates associated with an adapted basis $X_1, \dots, X_n$; let $\{\partial_i\}_{i=1,\dots,n}$ be the standard basis of vectors of $\mathbb{R}^n$ and set $X_j(x) := \sum_{i=1}^n a_{ij}(x)\partial_i$. Then*

(i) $a_{ij}(x) = \dfrac{\partial P_i(x,\cdot)}{\partial y_j}\bigg|_{y=0}$ *is a homogeneous polynomial of degree $d_i - d_j$;*

(ii) $X_j(x) = \partial_j + \displaystyle\sum_{i:d_i>d_j} a_{ij}(x)\partial_i = \partial_j + \sum_{i=n_{d_j}+1}^{n} a_{ij}(x)\partial_l$;

(iii) $a_{ij}(x)$ *depends only on the coordinates x_r with $d_r < d_i$.*

In particular, $a_{ij}(x) = a_{ij}(x, \dots, x_{i-1})$.

Proof. As usual, we identify vector fields and first order operators; by left invariance and the fact that

$$X_j f(0) = \frac{d}{dt} f(\exp\, tX_j)\bigg|_{t=0} = \partial_j f(0)$$

for any smooth f, one has

$$X_j f(x) = \partial_j (f \circ \ell_x)(0) = \sum_{i=1}^{n} \partial_j \ell_x^i(0)\, \partial_i f(x),$$

where ℓ_x^i denotes the i-th component of ℓ_x. From (1.24) we deduce

$$a_{ij}(x) = \partial_j \ell_x^i(0) = \partial_{y_j} P_i(x, \cdot)(0).$$

By the homogeneity of P_i one gets

$$r^{d_i} a_{ij}(x) = r^{d_i} \partial_{y_j} P_i(x,y)\big|_{y=0} = \partial_{y_j} P_i(\delta_r x, \delta_r y)\big|_{y=0} = r^{d_j} \partial_{y_j} P_i(x,y)\big|_{y=0} = r^{d_j} a_{ij}(\delta_r x)$$

and so the a_{ij}'s are homogeneous polynomials of degree $d_i - d_j$. This implies that $a_{ij} \equiv 0$ if $d_j > d_i$; moreover, since a 0-homogeneous polynomial is constant, we have

$$X_j(x) = \sum_{d_i = d_j} c_{ij} \partial_i + \sum_{i : d_i > d_j} a_{ij}(x) \partial_i$$

for suitable constants c_{ij}: since $X_j(0) \partial_j$ one must have $c_{ij} = \delta_{ij}$ and so

$$X_j(x) = \partial_j + \sum_{i = n_{d_j+1}}^{n} a_{ij}(x) \partial_i. \tag{1.25}$$

Since each a_{ij} is homogeneous of degree $d_i - d_j$, the coordinates x_r with $d_r > d_i - d_j$ cannot appear in the polynomial structure of a_{ij}; therefore a_{ij} cannot depend on the coordinates x_r with $d_r \geq d_i$, i.e.

$$a_{ij}(x) = a_{ij}(x_1, \dots, x_{n_{d_i - 1}}).$$

In particular, one has $a_{ij}(x) = a_{ij}(x_1, \dots, x_{i-1})$. □

1.2.6 Carnot-Carathéodory and homogeneous metrics

Let $\mathbb{G}$ be a Carnot group, which we consider represented by $(\mathbb{R}^n, \cdot)$ through a system of graded coordinates associated with a basis adapted to the stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Let $m := m_1 = \dim \mathfrak{g}_1$ and let $X = (X_1, \dots, X_m)$ be a basis of $\mathfrak{g}_1$: the stratification assumption ensures that $\mathfrak{g}_1$ Lie generates the whole algebra, whence the family X satisfies Chow's condition (1.5) inducing a CC metric d_c on $\mathbb{R}^n$. As we did for general CC spaces, we will also use the notations $H\mathbb{G}$ and $H_x\mathbb{G}$ to denote $\mathfrak{g}_1$ and $\mathfrak{g}_1(x)$ respectively.

The presence of a stratification induces many "good" properties of d_c, with respect to both left translations and omogeneous dilations, which are collected in the following Proposition 1.25. According to the subsequent Definition 1.27, we will say that d_c is a homogeneous distance.

Proposition 1.25. *For any $x, y, z \in \mathbb{R}^n$ and any $r > 0$ we have*

(i) $d_c(z \cdot x, z \cdot y) = d_c(x, y)$;

(ii) $d_c(\delta_r x, \delta_r y) = r d_c(x, y)$.

Proof. Part (i) of the thesis follows from the fact that $\gamma : [0, T] \to \mathbb{R}^n$ is a subunit path from x to y if and only if $\tilde{\gamma} := \ell_z \circ \gamma$ is a subunit path from zx to zy. In fact, if $\dot{\gamma}(t) = \sum_{j=1}^m h_j X_j(\gamma(t))$ then

$$\begin{aligned}
\dot{\tilde{\gamma}}(t) &= d\ell_z(\gamma(t))[\dot{\gamma}(t)] \\
&= d\ell_z(\gamma(t)) \left[\sum_{j=1}^m h_j(t) X_j(\gamma(t)) \right] \\
&= \sum_{j=1}^m h_j(t) d\ell_z(\gamma(t)) X_j(\gamma(t)) = \sum_{j=1}^m h_j(t) X_j(z \cdot \gamma(t)) \\
&= \sum_{j=1}^m h_j(t) X_j(\tilde{\gamma}(t)),
\end{aligned}$$

where $d\ell_z$ denotes the differential of the left translation by z.

As for (ii), it will be sufficient to prove that a path $\gamma : [0, T] \to \mathbb{R}^n$ from x to y is subunit if and only if so is the curve $\gamma_r : [0, rT] \to \mathbb{R}^n$, joining $\delta_r x$ and $\delta_r y$, defined by $\gamma_r(t) := \delta_r(\gamma(t/r))$. One has

$$\dot{\gamma}(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) = \sum_{l=1}^n \left(\sum_{j=1}^m h_j(t) a_{lj}(\gamma(t)) \right) \partial_l.$$

Since $d_j = 1$ for all $j = 1, \dots, m$, by Proposition 1.24 all the a_{lj}'s appearing in the sum are $(d_l - 1)$-homogeneous and so

$$\begin{aligned}
\dot{\gamma}_r(t) &= \sum_{l=1}^n r^{d_l - 1} \left(\sum_{j=1}^m h_j(t/r) a_{lj}(\gamma(t/r)) \right) \partial_l \\
&= \sum_{l=1}^n \left(\sum_{j=1}^m h_j(t/r) a_{lj}(\gamma_r(t)) \right) \partial_l \\
&= \sum_{j=1}^m h_j(t/r) X_j(\gamma_r(t)). \qquad (1.26)
\end{aligned}$$

Part (ii) follows in one stroke. □

Corollary 1.26. *Let $Y \in \mathfrak{g}_j$, then the CC distance behaves like $|\cdot|^{1/j}$ along Y; more precisely,*

$$d_c(x, \exp(sY)(x)) = C(Y)|s|^{1/j} \qquad \text{for any } x \in \mathbb{G}, s \in \mathbb{R}.$$

Definition 1.27. We say that a metric ρ on a Carnot group $\mathbb{G}$ is an *homogeneous distance* if

(*i*) $\rho(x,y) = \rho(z\cdot x, z\cdot y)$ and

(*ii*) $\rho(\delta_r x, \delta_r y) = r\rho(x,y)$

for all $x, y, z \in \mathbb{G}$ and all $\rho > 0$.

Notice that the thesis of Corollary 1.26 holds for general homogeneous distances, and not only for the CC one.

Apart from d_c, another important example of homogeneous distance is given by the distance d_∞ defined as

$$d_\infty(x,y) := \|y^{-1}x\|_\infty\,,$$

where the *infinity norm* $\|x\|_\infty$ of a point $x = (p_1,\dots,p_\iota) \in \mathbb{R}^n = \mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_\iota}$ (we use graded coordinates) is given by

$$\|x\|_\infty := \max\{\,\epsilon_k |p_k|^{1/k}_{\mathbb{R}^{m_k}} : k = 1,\dots,\iota\}.$$

Here $\epsilon_1 = 1$ and the ϵ_k's are suitable positive constants which depends on the group structure and are chosen in order to make d_∞ a distance: see also [81], Theorem 5.1. In particular, in the Heisenberg group $\mathbb{H}^n$ we will often use the distance d_∞ arising from the norm

$$\|(x,y,t)\|_\infty := \max\{|(x,y)|_{\mathbb{R}^{2n}}, |t|^{1/2}\},$$

where we used the coordinates of Section 1.2.4.

It is not difficult to check that any two homogeneous distances are biLipschitz equivalent; the integer $Q := \sum_{j=1}^k j \dim \mathfrak{g}_j$ is called *homogeneous dimension* of $\mathbb{G}$, and it coincides with the Hausdorff dimension of the group with respect to any homogeneous metric ρ. We will denote with $\mathcal{H}^d_\rho$ and $\mathcal{S}^d_\rho$, respectively, the d-dimensional Hausdorff and spherical Hausdorff measures associated with ρ (see [69]). It is straightforward to check that

$$\mathcal{H}^d_\rho(x\cdot E) = \mathcal{H}^d_\rho(E) \quad \text{and} \quad \mathcal{H}^d_\rho(\delta_r E) = r^d \mathcal{H}^d_\rho(E)$$

for any measurable $E \subset \mathbb{G}$ and any $x \in \mathbb{G}, r > 0$; moreover, the same formulae hold for $\mathcal{S}^d_\rho$. If we represent $\mathbb{G}$ as $\mathbb{R}^n$ via graded coordinates, then the Lebesgue measure $\mathcal{L}^n$ is the Haar measure of $\mathbb{G}$ and is both left- and right-invariant:

$$\mathcal{L}^n(x\cdot E) = \mathcal{L}^n(E\cdot x) = \mathcal{L}^n(E),$$

whence

$$\mathcal{L}^n(U(x,r)) = r^Q \mathcal{L}^n(U(x,1)) = \mathcal{L}^n(U(0,1)),$$

where $U(x,r)$ denotes the ball with respect to a fixed homogeneous metric. If not specified, integration on $\mathbb{G}$ or on open subsets of $\mathbb{G}$ will be always understood with respect to this measure.

The X-perimeter measure of a measurable set $E \subset \mathbb{G}$, defined as in Section 1.1.2 according to the family X, will be referred to as the $\mathbb{G}$-*perimeter measure* $\|\partial E\|_{\mathbb{G}}$ of E; from its definition it is easy to prove that

$$\|\partial(x \cdot E)\|_{\mathbb{G}}(x \cdot \Omega) = \|\partial E\|_{\mathbb{G}}(\Omega) \quad \text{and} \quad \|\partial(\delta_r E)\|_{\mathbb{G}}(\delta_r \Omega) = r^{Q-1}\|\partial E\|_{\mathbb{G}}(\Omega)$$

for any $x \in \mathbb{G}$, for any open set $\Omega \subset \mathbb{G}$ and any $r > 0$.

1.2.7 Convolution on groups

We want to briefly recall the classical technique of intrinsic convolution in homogeneous groups (see [73]). Let $\mathbb{G}$ be a Carnot group and let $\zeta \in \mathbf{C}_c^\infty(\mathbb{G})$ be such that

$$0 \le \zeta \le 1, \quad \int_{\mathbb{G}} \zeta = 1, \quad \zeta(x^{-1}) = \zeta(x) \quad \text{and} \quad \text{spt } \zeta \subset U(0,1), \tag{1.27}$$

where $U(x,r)$ denote balls of $\mathbb{G}$ with respect to a fixed homogeneous metric. Let us denote

$$\zeta_\epsilon(x) := \epsilon^{-Q}\zeta\left(\delta_{1/\epsilon}(x)\right), \qquad x \in \mathbb{G}\,; \tag{1.28}$$

$$(\zeta_\epsilon \star f)(x) := \int_{\mathbb{G}} \zeta_\epsilon(y)\, f(y^{-1} \cdot x) d\mathcal{L}^n(y) = \int_{\mathbb{G}} \zeta_\epsilon(x \cdot y^{-1})\, f(y) d\mathcal{L}^n(y)\,. \tag{1.29}$$

Then the following results hold

Proposition 1.28. *We have*

(i) if $f \in L^p(\mathbb{G})$, $1 \le p < \infty$, then $\zeta_\epsilon \star f \in \mathbf{C}^\infty(\mathbb{G})$ and $\zeta_\epsilon \star f \to f$ in $L^p(\mathbb{G})$ as $\epsilon \to 0$;

(ii) spt $\zeta_\epsilon \star f \subset U(0,\epsilon) \cdot$ spt$f$*;*

(iii) $X(\zeta_\epsilon \star f) = \zeta_\epsilon \star (Xf)$ for any $f \in \mathbf{C}^1(\mathbb{G})$ and each $X \in \mathfrak{g}$;

(iv) $\int_{\mathbb{G}} (\zeta_\epsilon \star f)\, g = \int_{\mathbb{G}} (\zeta_\epsilon \star g)\, f$ for every $f \in L^1(\mathbb{G})$, $g \in L^\infty(\mathbb{G})$;

(v) if $f \in \mathbf{C}^0(\Omega)$ for a suitable open set $\Omega \subset \mathbb{G}$ then $\zeta_\epsilon \star f \to f$ uniformly on compact subsets of Ω as $\epsilon \to 0$.

The statements of Proposition 1.28 can be easily proved with standard arguments. For the sake of completeness we show point (iii), where the key tool is the left invariance of X, in fact

$$\begin{aligned} X(\zeta_\epsilon \star f)(z) &= X\left(\int_{\mathbb{G}} \zeta_\epsilon(y)\, f(y^{-1}\cdot x) d\mathcal{L}^n(y)\right)_{|x=z} \\ &= \int_{\mathbb{G}} X\left(\zeta_\epsilon(y)\, f(y^{-1}\cdot z)\right) d\mathcal{L}^n(y) \\ &= \int_{\mathbb{G}} \zeta_\epsilon(y)\, (Xf)(y^{-1}\cdot z) d\mathcal{L}^n(y) \\ &= \zeta_\epsilon \star (Xf)(z). \end{aligned}$$

However, it is possible to improve this result:

Proposition 1.29. *Let $f : \mathbb{G} \to \mathbb{R}$ a continuous function and $X \in \mathfrak{g}$ be such that the distributional derivative Xf is represented by a continuous function on $\mathbb{G}$; then one has*

$$X(\zeta_\epsilon \star f) = \zeta_\epsilon \star (Xf).$$

Proof. Since $\zeta_\epsilon \star f$ is of class $\mathbf{C}^\infty$, it will be sufficient to prove that for any $g \in \mathbf{C}_c^\infty(\mathbb{G})$ one has

$$\langle X(\zeta_\epsilon \star f), g\rangle = \langle \zeta_\epsilon \star (Xf), g\rangle,$$

where for $u, v : \mathbb{G} \to \mathbb{R}$ we use the common notation

$$\langle u, v\rangle := \int_{\mathbb{G}} uv.$$

Using Proposition 1.28 (iii), (iv) and thanks to the following Lemma 1.30 one has

$$\begin{aligned} \langle X(\zeta_\epsilon \star f), g\rangle &= -\langle \zeta_\epsilon \star f, Xg\rangle = -\langle f, \zeta_\epsilon \star Xg\rangle \\ &= -\langle f, X(\zeta_\epsilon \star g)\rangle = \langle Xf, \zeta_\epsilon \star g\rangle = \langle \zeta_\epsilon \star (Xf), g\rangle. \end{aligned} \tag{1.30}$$

□

Lemma 1.30. *Any left invariant vector field $X \in \mathfrak{g}$ is self-adjoint, i.e.*

$$\int_{\mathbb{G}} v\, Xu = -\int_{\mathbb{G}} u\, Xv$$

for any $u, v \in \mathbf{C}_c^\infty(\mathbb{G})$.

Proof. By the invariance of the Lebesgue measure, the integral

$$\int_{\mathbb{G}} u(xa)v(xa)\, d\mathcal{L}^n(x)$$

does not depend on $a \in \mathbb{G}$. Taking $a = \exp(tX)$ and differentiating at $t = 0$ one gets

$$\int_{\mathbb{G}} (v\, Xu + u\, Xv) = 0.$$

□

Chapter 2

Measure of submanifolds in Carnot groups

In this Chapter we will focus our attention on how a submanifold of a Carnot group $\mathbb{G}$ inherits its sub-Riemannian geometry from a stratified group equipped with its Carnot-Carathéodory distance. Our aim is finding the sub-Riemannian measure "naturally" associated with a submanifold. For hypersurfaces, this measure is exactly the $\mathbb{G}$-perimeter, which is widely acknowledged as the appropriate measure in connection with intrinsic regular hypersurfaces, trace theorems, isoperimetric inequalities, the Dirichlet problem for sub-Laplacians, minimal surfaces, and more. Here we address the reader to some relevant papers [32, 35, 47, 54, 57, 75, 85, 83, 123, 137] and the references therein.

Our question is: what is the natural replacement of the $\mathbb{G}$-perimeter for submanifolds of higher codimension? Clearly, once the Hausdorff dimension of the submanifold is known, the natural candidate should be the corresponding Hausdorff measure: more precisely, the spherical one, see also [79, 81, 120]. However, this measure is not manageable, since it is not clear whether it is lower semicontinuous with respect to the Hausdorff convergence of sets and so it cannot be used in minimization problems. In general, lower semicontinuity of Hausdorff measures in metric spaces is a delicate problem, see [7]. It is then convenient to find an equivalent measure, that can be represented as the supremum among a suitable family of linear functionals, in analogy with the classical theory of currents.

Our strategy will be to exhibit a natural number d, which will coincide with the Hausdorff dimension of the submanifold, and a measure μ_S that is "naturally" associated to it, in the sense that it will coincide with the d-dimensional spherical Hausdorff measure of the surface. Moreover, this number d is the same conjectured by Gromov [91]: see also Remark 2.3. The measure μ_S possesses a density with respect to any Riemannian surface measure on S, providing an integral representa-

tion of μ_S. We stress however that our result is not complete, since we are able to characterize only a "big" portion of S and not the whole of it (see Section 2.3 for more details). All the results contained in this Chapter have been obtained in [126] in collaboration with V.Magnani.

We then start, in Section 2.1, by illustrating some preliminary material. More precisely, in Subsection 2.1.1 we give a stratification of the space $\Lambda_p\mathfrak{g}$ of p-vectors, which allows us to define, for any given p-vector τ and any integer r, the projection of τ with degree r (see Definition 2.1); the degree of τ will then be the maximum r such that the r-projection of τ is not zero. For any fixed p-dimensional $\mathbf{C}^{1,1}$ submanifold S we set its degree $d = d(S)$ to be the maximum among the degrees $D_s(x)$ of the tangent p-vectors $\tau_S(x)$ for $x \in S$: this number will be exactly the one we were looking for. Subsection 2.1.2 contains a purely algebraic result, Lemma 2.5, that will be crucial in Lemma 2.14.

The main result of Section 2.2 is Theorem 2.19, where we prove that the intrinsic blow-up of S, i.e. the limit (with respect to the Hausdorff convergence of sets) as $r \to 0$ of $\delta_{1/r}(x^{-1} \cdot S)$, does exist at points x with maximum degree, i.e. those points where the degree of $\tau_S(x)$ is equal to d. Moreover, this limit is a subgroup $\Pi_S(x)$ which is associated with the p-vector given by the d-projection of $\tau_S(x)$: indeed, the latter turns out to be simple, and Lemma 2.14 ensures that it is a subgroup. The proof of the blow-up results is quite technical: first of all, thanks to Lemma 2.6 we are able to conveniently fix a basis of T_xS and one of $\mathfrak{g}$, and we utilize the latter to make all computations in the associated graded coordinates. After that, in Lemma 2.15 we make use of our basis of T_xS to foliate the submanifold with curves that are "almost homogeneous", thus obtaining our blow-up result. More precisely, we are able to recover a neighbourhood of S as the image of a map $\gamma : [0, t_0] \times L \to S$ with the property that

$$\gamma(t, \lambda) = x \cdot \delta_t(y + O(t))$$

where $y = y(\lambda) \in \Pi_S(x)$. Here L is a compact subset of $\mathbb{R}^{p-1}$, diffeomorphic to S^{p-1}, which will be specified during the proof; we stress however that it is just a family of parameters, whose structure we will not care about.

Thanks to Theorem 2.19, in Section 2.3 we finally obtain our desired "natural" measure: first of all, in Theorem 2.20 we compute the limit

$$\lim_{r \to 0} \frac{\sigma_{\tilde{g}}(S \cap U(x, r))}{r^d} =: q(x)\,,$$

where x is a point with maximum degree and $\sigma_{\tilde{g}}$ is the p-dimensional surface measure arising from a Riemannian metric $\tilde{g}$ on $\mathbb{G}$. A standard result about differentiation of measure will then provide the required measure

$$\mu_S := q\, \sigma_{\tilde{g}} \llcorner S_d = \mathcal{S}^d_\rho \llcorner S_d\,,$$

where S_d is the (open) set of points of S with maximum degree; let us stress that the density q depends uniquely on the fixed homogeneous distance ρ and on the d-projection of $\tau_S(x)$. We conjecture however that $\mathcal{S}^d_\rho(S \setminus S_d) = 0$ and so that $\mu_S = \mathcal{S}^d_\rho \llcorner S$: we are able to prove this result for the step 2 case (Theorem 2.22), while it is an open problem for the general case. Before Theorem 2.22, we also compare our results with the existing literature.

Finally, in Section 2.4, as an application we study the case of 2-dimensional submanifolds of the Engel group $\mathbb{E}^4$, providing examples of surfaces of degree 3,4,5 and the nonexistence of submanifolds with other degrees.

2.1 Preliminaries

2.1.1 Some linear algebra

Let $\mathbb{G}$ be a fixed Carnot group with topological dimension n, whose Lie algebra admits the stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\iota \,; \tag{2.1}$$

as in Chapter 1, we will denote homogeneous dilations by δ_r and with ρ a fixed homogeneous distance, while open balls of radius $r > 0$ and centered at x with respect to ρ will be denoted by $U(x, r)$. By $\mathcal{H}^d_\rho$ and $\mathcal{S}^d_\rho$ we will mean, respectively, the d-dimensional Hausdorff and spherical Hausdorff measures associated with ρ.

In the sequel, whenever $X_1, \dots, X_n$ is an adapted basis of $\mathfrak{g}$, we will frequently alternate the two notations

$$(X_1, \dots, X_n) = (X^1_1, \dots X^1_{m_1}, X^2_1, \dots, X^2_{m_2}, \dots, X^\iota_1, \dots, X^\iota_{m_\iota});$$

observe that $X^k_1, \dots, X^k_{m_k}$ is a basis of the layer $\mathfrak{g}_k$ for every $k = 1, \dots, \iota$. We recall also that by d_j we denote the *degree* of X_j, i.e. the unique integer k such that $X_j \in \mathfrak{g}_k$.

Let

$$X_J := X_{j_1} \wedge \cdots \wedge X_{j_p}$$

be a simple p-vector of $\Lambda_p\mathfrak{g}$, where $J = (j_1, j_2, \dots, j_p)$ and $1 \le j_1 < j_2 < \cdots < j_p \le n$. The degree of X_J is the integer d_J defined by the sum $d_{j_1} + \cdots + d_{j_p}$.

Definition 2.1. Let $\tau \in \Lambda_p(\mathfrak{g})$ be a simple p-vector and let $1 \le r \le Q$ be a natural number. Let $\tau = \sum_J \tau_J \, X_J$ be represented with respect to the fixed adapted basis $(X_1, \dots, X_n)$. The projection of τ with degree r is defined as

$$\tau^r := \sum_{d_J = r} \tau_J \, X_J. \tag{2.2}$$

The *degree* of τ is defined as the integer

$$d(\tau) = \max\{r \in \mathbb{N} : \tau^r \neq 0\}.$$

Notice that the degree of a p-vector is independent from the adapted basis we have chosen.

In the sequel, we will fix a *graded metric* g on $\mathbb{G}$, namely, a left invariant Riemannian metric on $\mathbb{G}$ such that the subspaces $\mathfrak{g}_k$ are orthogonal. It is easy to observe that all left invariant Riemannian metrics such that $(X_1, \ldots, X_n)$ is an orthonormal basis are graded metrics and the family of X_J's forms an orthonormal basis of $\Lambda_p(\mathfrak{g})$ with respect to the induced metric. The norm induced by g on $\Lambda_p(\mathfrak{g})$ will be simply denoted by $|\cdot|_g$. When an adapted basis $(X_1, \ldots, X_n)$ is also orthonormal with respect to the fixed graded metric g is called *graded basis*.

The next definition introduces the metric factor associated with a simple p-vector. Notice that this definition generalizes the notion of metric factor first introduced in [120].

Definition 2.2. Let $\mathfrak{g}$ be a Carnot algebra equipped with a graded metric g and a homogeneous distance ρ. Let τ be a simple p-vector of $\Lambda_p(\mathfrak{g})$. We define $\mathcal{L}(\tau)$ as the unique subspace associated with τ. The *metric factor* of τ with respect to g is defined by

$$\theta(\tau) = \mathcal{H}^p\left(F^{-1}\big(\exp\big(\mathcal{L}(\tau)\big) \cap U_1\big)\right), \tag{2.3}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{G}$ is a system of graded coordinates with respect to an adapted orthonormal basis $(X_1, \ldots, X_n)$. The p-dimensional Hausdorff measure with respect to the Euclidean norm of $\mathbb{R}^n$ has been denoted by $\mathcal{H}^p$ and U_1 is the open unit ball (with respect to the fixed homogeneous distance ρ) centered at e.

In the sequel, also an arbitrary auxiliary Riemannian metric $\tilde{g}$ will be given. We define $\tau_S(x)$ as the unit tangent p-vector to a $\mathbf{C}^1$ submanifold S at $x \in S$ with respect to the metric $\tilde{g}$, i.e. $|\tau_S(x)|_{\tilde{g}} = 1$; here p is the topological dimension of S. The *degree* of x is defined as

$$d_S(x) = d(\tau_S(x)) \tag{2.4}$$

and the *degree* of S is $d(S) = \max_{x\in S} d_S(x)$. We will say that $x \in S$ has *maximum degree* if $d_S(x) = d(S)$.

It is not difficult to check that these definitions are independent from the fixed adapted basis $X_1, \ldots, X_n$: they depend just on the tangent subbundle TS and the grading of $\mathfrak{g}$, namely only on the "geometric" position of the points with respect to the grading (2.1). According to (2.2), we define $\tau_S^d(x)$ as the part of $\tau_S(x)$ with maximum degree $d = d(S)$, namely,

$$\tau_S^d(x) = \big(\tau_S(x)\big)^d. \tag{2.5}$$

Remark 2.3. As an interesting point to be investigated, we emphasize the correspondence between $d_S(x)$, $d(S)$ and the numbers $D'(x)$, $D_H(S)$ introduced by Gromov in 0.6.B of [91], where he also indicates how, for a smooth manifold, $D_H(S)$ must correspond to the Hausdorff dimension of S.

Definition 2.4. Let $x \in S$ be a point of maximum degree. Then we define

$$\Pi_S(x) := \{y \in \mathbb{G} : y = \exp(v) \text{ with } v \in \mathfrak{g} \text{ and } v \wedge \tau_S^d(x) = 0\}\,.$$

We will see in Lemma 2.14 that $\Pi_S(x)$ is a subgroup of $\mathbb{G}$. Notice that, with the notation of Definition 2.2, we have $\Pi_S(x) = \exp(\mathcal{L}(\tau_S^d(x)))$ and

$$\theta(\tau_S^d(x)) = \mathcal{H}^d(\Pi_S(x) \cap U_1),$$

where we have understood the identification of $\mathbb{G}$ with $\mathbb{R}^n$ via the graded coordinates of Definition 2.2.

2.1.2 An algebraic Lemma

Let $X_1, \dots, X_n$ be an adapted basis of $\mathfrak{g}$; in what follows we will represent $\mathbb{G}$ by means of the associated system of graded coordinates $F : \mathbb{R}^n \to \mathbb{G}$, according to which homogeneous dilations can be read as

$$\delta_r(x) = (rx_1, \dots, r^{d_j}x_j, \dots, r^{\iota}x_n) \qquad \text{for every } r > 0.$$

For $X, Y \in \mathfrak{g}$, the vector $C(X, Y) \in \mathfrak{g}$ will be defined as in the Baker-Campbell-Hausdorff formula (1.19). As in (1.24), we define the families of homogeneous polynomials $P = (P_1, \dots, P_n)$ and $Q = (Q_1, \dots, Q_n)$ via the formula

$$x \cdot y = P(x, y) = x + y + Q(x, y). \tag{2.6}$$

Remember that, by Proposition 1.24, one has

$$P_i(\delta_r(x), \delta_r(y)) = r^{d_i}\, P_i(x, y) \quad \text{and} \quad Q_i(\delta_r(x), \delta_r(y)) = r^{d_i}\, Q_i(x, y)\,. \tag{2.7}$$

and

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\, \partial_i = \sum_{i=1}^{n} \frac{\partial P_i}{\partial y_j}(x, 0)\, \partial_i = \quad \partial_j + \sum_{d_i > d_j} \frac{\partial Q_i}{\partial y_j}(x, 0)\, \partial_i\,, \tag{2.8}$$

where each a_{ij} is a homogeneous polynomial of degree $d_i - d_j$. From the homogeneity property (2.7) one gets

$$\begin{cases} Q_1 = \cdots = Q_{m_1} = 0 \\ Q_i(x, y) = Q_i\left(\sum_{d_j < i} x_j\, e_j, \sum_{d_j < i} y_j\, e_j\right) \quad \text{if } d_i > 1, \end{cases} \tag{2.9}$$

where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$.

We now present a result which will be crucial for the proof of Lemma 2.15.

Lemma 2.5. *Let $J \subset \{1, 2, \dots, n\}$ be such that $\mathcal{F} = \operatorname{span}\{X_j : j \in J\}$ is a subalgebra of $\mathfrak{g}$, where $(X_1, \dots, X_n)$ is an adapted basis of $\mathfrak{g}$. Then, for every index $i \notin J$, the polynomial $Q_i(x, y)$ lies in the ideal generated by $\{x_l, y_l : l \notin J\}$; namely, we have*

$$Q_i(x, y) = \sum_{l \notin J,\, d_l < d_i} \left(x_l R_{il}(x, y) + y_l S_{il}(x, y)\right), \tag{2.10}$$

where R_{il}, S_{il} are homogeneous polynomials of degree $d_i - d_l$.

Proof. Let us fix $x, y \in \mathbb{R}^n$ and consider

$$X := \sum_{j=1}^{n} x_j X_j, \qquad Y := \sum_{j=1}^{n} y_j X_j.$$

By definition of $C(X, Y)$ and Baker-Campbell-Hausdorff formula (1.19), for any $X, Y \in \mathfrak{g}$ we have

$$C(X, Y) = \sum_{j=1}^{n} P_j(x, y)\, X_j\,.$$

Therefore, defining $\pi_i : \mathfrak{g} \to \mathbb{R}$ as the function which associates to every vector its X_i's coefficient, we clearly have $P_i(x, y) = \pi_i\big(C(X, Y)\big)$. Thus, formulae (1.19) and (2.6) yield

$$Q_i(x, y) = \sum_{l=1}^{\iota} \frac{(-1)^{l+1}}{l} \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_l)\\ \beta=(\beta_1,\dots,\beta_l)\\ \alpha_i+\beta_i\geq 1\ \forall i}} \frac{1}{\alpha!\beta!|\alpha+\beta|} \pi_i(C_{\alpha\beta}(X, Y)) - x_i - y_i.$$

Observe that $C_{\alpha\beta}(X, Y)$, which is defined in (1.18), is a commutator of X and Y, whose length is equal to $|\alpha + \beta|$; as the sum of commutator with length 1 gives $X + Y$ we get

$$Q_i(x, y) = \sum_{l=1}^{\iota} \frac{(-1)^{l+1}}{l} \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_l)\\ \beta=(\beta_1,\dots,\beta_l)\\ \alpha_i+\beta_i\geq 1\ \forall i\\ |\alpha+\beta|\geq 2}} \frac{1}{\alpha!\beta!|\alpha+\beta|} \pi_i(C_{\alpha\beta}(X, Y)).$$

When the commutator $C_{\alpha\beta}(X, Y)$ has length $h \geq 2$, we can decompose it into the sum of commutators of the vector fields $\{x_l X_l, y_l X_l : 1 \leq l \leq n\}$. Let us focus our attention on an individual addend of this sum and consider its projection π_i. Clearly, this addend is a commutator of length h. If this term is a commutator containing an element of the family $\{x_l X_l, y_l X_l : l \notin J\}$, then its projection π_i will be a multiple

of x_l or y_l for some $l \notin J$, i.e. the projection π_i of this term is a polynomial of the ideal

$$\{x_l, y_l : l \notin J\}.$$

On the other hand, if in the fixed commutator only elements of $\{x_l X_l, y_l X_l : l \in J\}$ appear, then it belongs to $\mathcal{F}$. In view of our hypothesis, we have $\mathcal{F} \cap \text{span}\{X_i\} = \{0\}$, hence its projection through π_i vanishes. This fact along with (2.9) proves that $Q_i(x, y)$ has the form (2.10). □

2.2 Blow-up at points of maximum degree

Lemma 2.6. *Let S be a p-dimensional submanifold of class $\mathbf{C}^1$ and let $x \in S$ be a point of maximum degree. Then we can find*

- *a graded basis $X_1, \dots, X_n$ of $\mathfrak{g}$;*
- *a neighbourhood U of x;*
- *a basis $v_1(y), \dots, v_p(y)$ of $T_y S$ for all $y \in U$*

such that writing $v_j(y) = \sum_{i=1}^n C_{ij}(y) X_i(y)$, we have

$$C(y) := (C_{ij}(y))_{\substack{i=1,\dots,n \\ j=1,\dots,p}} = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} & 0 & \cdots & 0 \\ O_1(y) & * & \cdots & * \\ \hline 0 & Id_{\alpha_2} & \cdots & 0 \\ 0 & O_2(y) & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & Id_{\alpha_\iota} \\ 0 & 0 & \cdots & O_\iota(y) \end{array}\right] \tag{2.11}$$

where α_k are integers satisfying $0 \le \alpha_k \le m_k$ and $\alpha_1 + \cdots + \alpha_\iota = p$. The $(m_k - \alpha_k) \times \alpha_k$-matrix valued continuous functions O_k vanish at x and $*$ denotes a continuous bounded matrix valued function.

Proof. Observing that since the degree of a point in S is invariant under left translations, it is not restrictive to assume that x coincides with the unit element e of $\mathbb{G}$.

Step 1. Here we wish to find the graded basis $(X_1, \dots, X_n)$ of $\mathfrak{g}$ and the basis $v_1, \dots, v_p$ of $T_e S$ required in the statement of the lemma and that satisfy (2.11) when $y = e$. Let us fix a basis $(t_1, \dots, t_p)$ of $T_e S$ and use the same notation to denote the corresponding basis of left invariant vector fields of $\mathfrak{g}$. We denote by π_k the canonical projection of $\mathfrak{g}$ onto V_k. Let $0 \le \alpha_\iota \le m_\iota$ be the dimension of the subspace spanned by

$$\pi_\iota(t_1), \dots, \pi_\iota(t_p).$$

Taking linear combinations of the t_j's, we can suppose that the first α_ι projected vectors $\{\pi_\iota(t_j) : 1 \le j \le \alpha_\iota\}$ form an orthonormal set of V_ι with respect to the fixed graded metric g. Then we set

$$X_j^\iota := \pi_\iota(t_j) \in V_\iota \quad \text{and} \quad v_j^\iota := t_j \in T_eS\,,$$

whenever $1 \le j \le \alpha_\iota$. Adding proper linear combinations of these t_j to the remaining vectors of the basis, we can assume that $\{t_j^{\iota-1} := t_{j+\alpha_\iota}\}_{1\le j\le p-\alpha_\iota}$ are linearly independent and that

$$\pi_\iota(t_j^{\iota-1}) = 0 \qquad \text{whenever} \quad j = 1, \dots, p-\alpha_\iota.$$

Now consider the $p-\alpha_\iota$ vectors

$$\pi_{\iota-1}(t_1^{\iota-1}), \dots, \pi_{\iota-1}(t_{p-\alpha_\iota}^{\iota-1})$$

and let $0 \le \alpha_{\iota-1} \le m_{\iota-1}$ be the rank of the subspace of $V_{\iota-1}$ generated by these vectors. Taking linear combinations of $t_j^{\iota-1}$, we can suppose that $\pi_{\iota-1}(t_j^{\iota-1})$ with $j = 1, \dots, \alpha_{\iota-1}$ form an orthonormal set of $V_{\iota-1}$ and that defining

$$t_j^{\iota-2} := t_{j+\alpha_{\iota-1}}^{\iota-1} \qquad \text{for } 1 \le j \le p-\alpha_\iota-\alpha_{\iota-1}$$

we have

$$\pi_{\iota-1}(t_j^{\iota-2}) = 0 \quad \text{whenever} \quad j = 1, \dots, p-\alpha_\iota-\alpha_{\iota-1}.$$

Then we set

$$X_j^{\iota-1} := \pi_{\iota-1}(t_j^{\iota-1}) \in V_{\iota-1} \quad \text{and} \quad v_j^{\iota-1} := t_j^{\iota-1} \in T_eS\,.$$

for every $j = 1, \dots, \alpha_{\iota-1}$. Repeating this argument in analogous way, we obtain integers α_k with $0 \le \alpha_k \le m_k$ for every $k = 1, \dots, \iota$ and vectors

$$X_j^k \in V_k\,, \qquad v_j^k \in T_eS, \quad \text{where} \quad k = 1, \dots, \iota \quad \text{and} \quad j = 1, \dots, \alpha_k.$$

Notice that $\alpha_1 + \cdots + \alpha_\iota = p$ and that

$$(v_1^1, \dots, v_{\alpha_1}^1, \dots, v_1^\iota, \dots, v_{\alpha_\iota}^\iota) \tag{2.12}$$

is a basis of T_eS. We complete the X_j^k's to a graded basis

$$(X_1^1, \dots X_{m_1}^1, X_1^2, \dots, X_{m_2}^2, \dots, X_1^\iota, \dots, X_{m_\iota}^\iota)$$

of $\mathfrak{g}$, that will be also denoted by $(X_1, \dots, X_n)$. It is convenient to relabel the basis (2.12) as $(v_1, \dots, v_p)$, hence we write $v_j = \sum_{i=1}^n C_{ij} X_i$ obtaining

$$C := (C_{ij}) = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} & * & \cdots & * \\ 0 & * & \cdots & * \\ \hline 0 & Id_{\alpha_2} & \cdots & * \\ 0 & 0 & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & Id_{\alpha_\iota} \\ 0 & 0 & \cdots & 0 \end{array}\right].$$

Performing suitable linear combinations of v_j's, we can assume that

$$C = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \hline 0 & Id_{\alpha_2} & \cdots & 0 \\ 0 & 0 & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & Id_{\alpha_\iota} \\ 0 & 0 & \cdots & 0 \end{array}\right]. \tag{2.13}$$

Step 2. The basis $(v_1, \dots, v_p)$ of T_eS can be extended to a frame of continuous vector fields $(v_1(y), \dots, v_p(y))$ on S defined in neighborhood U of e. Thanks to the previous step, defining $v_j(y) = \sum_{i=1}^n C_{ij}(y) X_i(y)$ we have

$$C(y) := (C_{ij}(y)) = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} + o(1) & o(1) & \cdots & o(1) \\ o(1) & * & \cdots & * \\ \hline o(1) & Id_{\alpha_2} + o(1) & \cdots & o(1) \\ o(1) & o(1) & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline o(1) & o(1) & \cdots & Id_{\alpha_\iota} + o(1) \\ o(1) & o(1) & \cdots & o(1) \end{array}\right]$$

where $o(1)$ denotes a matrix-valued continuous function vanishing at e. Observing that $Id_{\alpha_k} + o(1)$ are still invertible for every y in a smaller neighbourhood $U' \subset U$

of e, we can replace the v_j's with linear combinations to get

$$C(y) = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} + o(1) & 0 & \cdots & 0 \\ o(1) & * & \cdots & * \\ \hline 0 & Id_{\alpha_2} + o(1) & \cdots & 0 \\ o(1) & o(1) & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & Id_{\alpha_\iota} + o(1) \\ o(1) & o(1) & \cdots & o(1) \end{array}\right].$$

The same argument leads us to define a new frame with matrix

$$C(y) = \left[\begin{array}{c|c|c|c} Id_{\alpha_1} & 0 & \cdots & 0 \\ O_1(y) & * & \cdots & * \\ \hline 0 & Id_{\alpha_2} & \cdots & 0 \\ o(1) & O_2(y) & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & Id_{\alpha_\iota} \\ o(1) & o(1) & \cdots & O_\iota(y) \end{array}\right], \tag{2.14}$$

where O_j have the same properties as in the statement of the present lemma. To finish the proof, it remains to show that all $o(1)$'s of (2.14) are actually null matrix functions. Here we use the fact that the submanifold has maximum degree at e. Notice that the simple p-vector

$$v_1(y) \wedge \cdots \wedge v_p(y) = \sum_J a_J(y) X_J(y)$$

is proportional to the unit (according to the Riemannian metric $\tilde{g}$) tangent vector $\tau_S(y)$. In addition, if $J = (j_1, \ldots, j_p)$, then $a_J(y)$ is the determinant of the $p \times p$ submatrix obtained taking the j_1-th, j_2-th, $\ldots$, j_{p-1}-th and j_p-th row of $C(y)$. From (2.13) we immediately conclude that $d_S(e) = \alpha_1 + 2\alpha_2 + \cdots + \iota\alpha_\iota$. Finally, whenever one entry of some $o(1)$ does not vanish, it is possible to find some J_0 such that $d_{J_0} > \alpha_1 + 2\alpha_2 + \cdots + \iota\alpha_\iota$ and $a_{J_0}(y) \neq 0$. This would imply $d_S(y) > d_S(e)$, contradicting the assumption that $d_S(e) = \max_{y \in U'} d_S(y)$. □

Remark 2.7. It is easy to interpret the statement and the proof of Lemma 2.6 in the case some α_k vanishes. Clearly, the α_k columns in (2.11) intersecting I_{α_k} and the corresponding vectors v_j^k disappear.

Remark 2.8. When S is of class $\mathbf{C}^r$ the v_j's of the previous lemma are of class $\mathbf{C}^{r-1}$: in fact, the linear transformations performed in the proof of Lemma 2.6 are of class $\mathbf{C}^{r-1}$.

The previous Lemma 2.6 allows us to state the following definitions.

Definition 2.9. Let S be a $\mathbf{C}^1$ smooth submanifold and let $x \in S$ be a point of maximum degree. Then we can define the *degree* $\sigma : \{1, \dots, p\} \longrightarrow \mathbb{N}$ induced by S at x as

$$\sigma_j = k \qquad \text{if} \qquad \sum_{s=1}^{k-1} \alpha_s < j \leq \sum_{s=1}^{k} \alpha_s,$$

where the α_k's are defined in Lemma 2.6.

Definition 2.10. Let S be a $\mathbf{C}^1$ smooth submanifold and let $x \in S$ be a point of maximum degree. Then we will denote by

$$(X_1^1, \dots, X_{m_1}^1, \dots, X_1^\iota, \dots, X_{m_\iota}^\iota) \quad \text{and} \quad (v_1^1, \dots, v_{\alpha_1}^1, \dots, v_1^\iota, \dots, v_{\alpha_\iota}^\iota)$$

the frames on $\mathbb{G}$ and on a neighbourhood U of z in S, respectively, which satisfy the conditions of Lemma 2.6. We will also denote these frames by

$$(X_1, \dots, X_n) \qquad \text{and} \qquad (v_1, \dots, v_p)\,.$$

Corollary 2.11. *Let S be a $\mathbf{C}^1$ smooth submanifold with $x \in S$ satisfying $d_S(x) = d(S)$. Then $\tau_S^d(x)$ is a simple p-vector which is proportional to*

$$X_1^1 \wedge \cdots \wedge X_{\alpha_1}^1 \wedge \cdots \wedge X_1^\iota \wedge \cdots \wedge X_{\alpha_\iota}^\iota\,,$$

and we also have

$$\Pi_S(x) = \exp\bigl(\operatorname{span}\{X_1^1, \dots, X_{\alpha_1}^1, \dots, X_1^\iota, \dots, X_{\alpha_\iota}^\iota\}\bigr)$$

Proof. By expression (2.11), τ_S is clearly proportional to

$$X_1^1 \wedge \cdots \wedge X_{\alpha_1}^1 \wedge \cdots \wedge X_1^\iota \wedge \cdots \wedge X_{\alpha_\iota}^\iota + R\,. \tag{2.15}$$

where R is a linear combination of simple p-vectors with degree less than $d(X_1^1 \wedge \cdots \wedge X_{\alpha_\iota}^\iota)$. Then $d = d(X_1^1 \wedge \cdots \wedge X_{\alpha_\iota}^\iota) = \alpha_1 + 2\alpha_2 + \cdots + \iota\alpha_\iota$ and $\tau_S^d(x)$ is proportional to $X_1^1 \wedge \cdots \wedge X_{\alpha_\iota}^\iota$. □

Definition 2.12. We will denote by

$$(X_1^1, \dots, X_{\alpha_1}^1, \dots, X_1^\iota, \dots, X_{\alpha_\iota}^\iota) \tag{2.16}$$

the frame of Corollary 2.11, arising from Lemma 2.6, and by

$$\pi_S(x) : \mathbb{G} \longrightarrow \Pi_S(x) \tag{2.17}$$

the corresponding canonical projection.

Corollary 2.13. *Let $e \in S$ be such that $d_S(e) = d(S)$. Let us embed S into $\mathbb{R}^n$ by the system of graded coordinates F induced by $\{X_j^k\}_{k=1,\dots,\iota,\, j=1,\dots,m_k}$. Then there exists a function*

$$\begin{aligned}\phi \;:\; & A \subset \mathbb{R}^p \longrightarrow \mathbb{R}^{n-p} \\ & x = (x_1^1, \dots, x_{\alpha_1}^1, \dots, x_{\alpha_\iota}^\iota) \longmapsto (\phi_{\alpha_1+1}^1, \dots, \phi_{m_1}^1, \dots,\ \phi_{\alpha_\iota+1}^\iota, \dots, \phi_{m_\iota}^\iota)(x),\end{aligned}$$

defined on an open neighbourhood $A \subset \mathbb{R}^p$ of 0, such that $\phi(0) = 0$ and $S \supset \Phi(A)$, where $\Phi : A \to \mathbb{R}^n$ is the mapping defined by

$$x \mapsto \big(x_1^1, \dots, x_{\alpha_1}^1, \phi_{\alpha_1+1}^1(x), \dots, \phi_{m_1}^1(x), \dots, x_1^\iota, \dots, x_{\alpha_\iota}^\iota, \phi_{\alpha_\iota+1}^\iota(x), \dots, \phi_{m_\iota}^\iota(x)\big). \quad (2.18)$$

Moreover, Φ satisfies $\nabla\Phi(0) = C(0)$, with C given by Lemma 2.6.

Proof. Representing $\pi_S(x)$ with respect to our graded coordinates, we obtain

$$\begin{aligned}\pi_S(x) \;:\; & \mathbb{R}^n \to \mathbb{R}^p \\ & x \longmapsto (x_1^1, \dots, x_{\alpha_1}^1, \dots, x_1^\iota, \dots, x_{\alpha_\iota}^\iota)\,.\end{aligned}$$

Taking its restriction

$$\begin{aligned}\pi \;:\; & S \to \mathbb{R}^p \\ & x \longmapsto (x_1^1, \dots, x_{\alpha_1}^1, \dots, x_1^\iota, \dots, x_{\alpha_\iota}^\iota)\,,\end{aligned}$$

we wish to prove that π is invertible near 0, i.e. that $d\pi(0) : T_0S \to \mathbb{R}^p$ is onto. According to (2.11) and the fact that π is the restriction of a linear mapping, it follows that $d\pi(v_j^k(0)) = \partial_{x_j^k}$ for every $k = 1, \dots, \iota$ and $j = 1, \dots, \alpha_k$. This implies the existence of $\Phi = \pi_{|U}^{-1}$ having the representation (2.18), hence one can easily check that $d\pi(\partial_{x_j^k}\Phi\,(0)) = \partial_{x_j^k}$ also holds for every $k = 1, \dots, \iota$ and $j = 1, \dots, \alpha_k$. As a consequence, invertibility of $d\pi(0) : T_0S \to \mathbb{R}^p$ gives $v_j^k(0) = \partial_{x_j^k}\Phi\,(0)$. It follows that each column of $\nabla\Phi(0)$ equals the corresponding one of $C(0)$, i.e. that $\nabla\Phi(0) = C(0)$. □

From now on, we will assume that S is a $\mathbf{C}^{1,1}$ submanifold of $\mathbb{G}$.

Lemma 2.14. *Let $x \in S$ be such that $d_S(x) = d(S)$. Then $\Pi_S(x)$ is a subgroup.*

Proof. Posing $d := d(S)$, due to Corollary 2.11, $\tau_S^d(x)$ is proportional to the simple p-vector

$$X_1^1 \wedge \cdots \wedge X_{\alpha_1}^1 \wedge \cdots \wedge X_1^\iota \wedge \cdots \wedge X_{\alpha_\iota}^\iota\,.$$

We define $\mathcal{F}$ as the space of linear combinations of vectors $\{X_j^k\}_{j=1,\dots,\alpha_k}^{k=1,\dots,\iota}$. It suffices to prove that each bracket $[X_j^k, X_i^l]$ lies in $\mathcal{F}$ for every $1 \le k, l \le \iota$, $1 \le j \le \alpha_k$ and

$1 \leq i \leq \alpha_l$: this implies that $\mathcal{F}$ is a subalgebra and so that $\Pi_S(x) = \exp(\mathcal{F})$ is a subgroup.

Taking into account Remark 2.8, we can find Lipschitz functions φ_r and ψ_s, which vanish at x whenever $d_r = k$ or $d_s = l$, such that

$$v_j^k = X_j^k + \sum_{d_r \leq k} \varphi_r \, X_r \quad \text{and} \quad v_i^l = X_i^l + \sum_{d_s \leq l} \psi_s \, X_s.$$

For a.e. y belonging to a neighbourhood U of x, we have

$$\begin{aligned}
[v_j^k, v_i^l] &= \Big[X_j^k + \sum_{d_r \leq k} \varphi_r \, X_r, \;\; X_i^l + \sum_{d_s \leq l} \psi_s \, X_s\Big] \\
&= [X_j^k, X_i^l] + \sum_{d_r \leq k} \varphi_r \, [X_r, X_i^l] + \sum_{d_s \leq l} \psi_s \, [X_j^k, X_s] + \sum_{d_r \leq k, d_s \leq l} \varphi_r \, \psi_s \, [X_r, X_s] \\
&+ \sum_{d_s \leq l} (X_j^k \psi_s) X_s - \sum_{d_r \leq k} (X_i^l \varphi_r) X_r \qquad (2.19) \\
&+ \sum_{d_r \leq k, d_s \leq l} \Big(\varphi_r \, (X_r \psi_s) \, X_s \; - \; \psi_s \, (X_s \varphi_r) \, X_r\Big).
\end{aligned}$$

By Frobenius theorem we know that this vector is tangent to S, i.e. it is a linear combination of $v_1^1, \ldots, v_{\alpha_\iota}^\iota$ and lies in $V_1 \oplus \cdots \oplus V_{k+l}$, hence Lemma 2.6 implies that it must be of the form

$$[v_j^k, v_i^l] = \sum_{\sigma_r \leq k+l} a_r v_r.$$

Projecting both sides of the previous identity onto V_{k+l} and taking into account equation (2.19) we obtain

$$\begin{aligned}
&[X_j^k, X_i^l] + \sum_{d_r = k} \varphi_r \, [X_r, X_i^l] + \sum_{d_s = l} \psi_s \, [X_j^k, X_s] + \sum_{d_r = k, d_s = l} \varphi_r \, \psi_s \, [X_r, X_s] \\
= \; &\sum_{\sigma_r = k+l} a_r \, \pi_{k+l}(v_r).
\end{aligned}$$

From (2.11) the projections $\pi_{k+l}\big(v_r(y)\big)$ converge to a linear combination of vectors X_i^{k+l} as y goes to x, where $1 \leq i \leq \alpha_{k+l}$. We can find a sequence of points (y_ν) contained in U, where $[v_j^k, v_i^l]$ is defined and $y_\nu \to x$ as $\nu \to \infty$. Then the coefficients a_r are defined on y_ν and up to extracting subsequences it is not restrictive assuming that $a_r(y_\nu)$, which is bounded since S is $\mathbf{C}^{1,1}$, converges for every r such that $\sigma_r \leq k+l$. Thus, restricting the previous equality on the set $\{y_\nu\}$ and taking the limit as $\nu \to \infty$, it follows that $[X_j^k, X_i^l]$ is a linear combination of $\{X_i^{k+l}\}_{1 \leq i \leq \alpha_{k+l}}$. This ends the proof. □

Let us consider the parameters $\lambda = (\lambda_1^1, \dots, \lambda_{\alpha_1}^1, \dots, \lambda_1^\iota, \dots, \lambda_{\alpha_\iota}^\iota) \in \mathbb{R}^p$ and a point $e \in S$ with $d_S(e) = d(S)$. We aim to study properties of solution $\gamma(t,\lambda)$ of the Cauchy problem

$$\begin{cases} \partial_t \gamma(t,\lambda) = \displaystyle\sum_{\substack{k=1,\dots,\iota \\ j=1,\dots,\alpha_k}} \lambda_j^k \, v_j^k\big(\gamma(t,\lambda)\big)\, t^{k-1} \\ \gamma(0,\lambda) = 0\,, \end{cases} \tag{2.20}$$

where the vector fields v_j^k are defined in Lemma 2.6 with $x = e$. Notice that for every compact set $L \subset \mathbb{R}^p$, there exists a positive number $t_0 = t_0(L)$ such that $\gamma(\cdot,\lambda)$ is defined on $[0,t_0]$ for every $\lambda \in L$.

The next lemma gives crucial estimates on the coordinates of $\gamma(\cdot,\lambda)$. Notice that graded coordinates arising from the corresponding graded basis $(X_1,\dots,X_n)$ will be understood.

Lemma 2.15. *Let $\gamma(\cdot,\lambda)$ be the solution of (2.20). Then for every $k = 1,\dots,\iota$ and every $j = 1,\dots,m_k$ there exist homogeneous polynomials g_j^k of degree k such that*

(i) $g_j^1 \equiv 0$ *for any* $j = 1,\dots,m_k$*;*

(ii) $g_j^k(\lambda) = g_j^k(\lambda_1^1,\dots,\lambda_{\alpha_1}^1,\dots,\lambda_1^{k-1},\dots,\lambda_{\alpha_{k-1}}^{k-1})$ *when* $k > 1$*;*

(iii) $g_j^k(0) = 0$*;*

(iv) the estimates

$$\gamma_j^k(t,\lambda) = \begin{cases} \Big[\frac{\lambda_j^k}{k} + g_j^k(\lambda_1^1,\dots,\lambda_{\alpha_{k-1}}^{k-1})\Big] t^k + O(t^{k+1}), & j = 1,\dots,\alpha_k \\ O(t^{k+1}), & j = \alpha_k{+}1,\dots,m_k \end{cases} \tag{2.21}$$

hold for every $\lambda \in L$ and every $t \in [0,t_0]$.

Proof. From (2.8) and Proposition 1.24 (i), we have $X_s = \sum_{i=1}^n a_{is}\,\partial_i$ where

$$X_{is}(x) = \begin{cases} \delta_{is} & \text{if} \quad d_i \le d_s \\ u_{is}(x_1^1,\dots,x_{m_1}^1,\dots,x_1^{d_i-1},\dots,x_{m_{d_i}-1}^{d_i-1}) & \text{if} \quad d_i > d_s \end{cases} \tag{2.22}$$

and u_{is} is a homogeneous polynomial satisfying $u_{is}(\delta_r(x)) = r^{d_i-d_s} u_{is}(x)$. Setting

$$\tilde\lambda = \tilde\lambda(t) = (\lambda_1^1,\dots,\lambda_{\alpha_1}^1,\lambda_1^2 t,\dots,\lambda_{\alpha_2}^2 t,\dots,\lambda_1^\iota t^{\iota-1},\dots,\lambda_{\alpha_\iota}^\iota t^{\iota-1}) \in \mathbb{R}^p$$

and taking into account the expression of v_j given in Lemma 2.6, we can write the Cauchy problem (2.20) as

$$\partial_t \gamma(t,\lambda) = \sum_{r=1}^p v_r\big(\gamma(t,\lambda)\big)\,\tilde\lambda_r(t) = \sum_{r=1}^p \sum_{s=1}^n C_{sr}\big(\gamma(t,\lambda)\big)\, X_s\big(\gamma(t,\lambda)\big)\tilde\lambda_r(t)\,, \tag{2.23}$$

where $C(\cdot)$ is given by Lemma 2.6. Now we fix $\lambda \in L$ and write for simplicity γ in place of $\gamma(\cdot, \lambda)$. The coordinates of γ will be also denoted as follows

$$(\gamma_1^1, \dots, \gamma_{m_1}^1, \dots, \gamma_1^\iota, \dots, \gamma_{m_\iota}^\iota).$$

Step 1. We start proving (2.21) for the coordinates of γ belonging to the first layer, i.e.

$$\begin{cases} \gamma_j^1(t) = \lambda_j^1 t & \text{if } 1 \le j \le \alpha_1 \\ \gamma_j^1(t) = O(t^2) & \text{if } \alpha_1 + 1 \le j \le m_1 \,. \end{cases} \tag{2.24}$$

In view of (2.23), we get

$$\dot\gamma_j^1 = \sum_{r=1}^{p} \sum_{s=1}^{n} C_{sr}(\gamma)\, a_{js}(\gamma)\, \tilde\lambda_r.$$

For $1 \le j \le \alpha_1$ we have $1 = d_j \le d_s$, then (2.22) imply that $a_{js} = \delta_{js}$, whence

$$\dot\gamma_j^1 = \sum_{r=1}^{p} C_{jr}(\gamma)\tilde\lambda_r = \tilde\lambda_j = \lambda_j^1,$$

where the second equality follows from (2.11), which implies $C_{jr}(x) = \delta_{jr}$. This shows the first equality of (2.24).

Now we consider the case $\alpha_1 + 1 \le j \le m_1$. Due to (2.22) and $1 = d_j \le d_s$, we have

$$\dot\gamma_j^1 = \sum_{r=1}^{p} C_{jr}(\gamma)\tilde\lambda_r = \sum_{\sigma_r = 1} C_{jr}(\gamma)\tilde\lambda_r + \sum_{\sigma_r \ge 2} C_{jr}(\gamma)\tilde\lambda_r \,. \tag{2.25}$$

From (2.11), we have $C_{jr}(y) = o(1)$ whenever $\sigma_r = 1$, hence $C_{jr}(\gamma(t)) = o(t)$. From the same formula, we deduce that $C_{jr}(x)$ is bounded whenever $\sigma_r \ge 2$, and for the same indices r we also have $\tilde\lambda_r = O(t)$, hence the second addend of (2.25) is equal to $O(t)$. We have shown that $\dot\gamma_j^1 = O(t)$ for every $\alpha_1 + 1 \le j \le m$, therefore the second equality of (2.24) is proved.

Step 2. We will prove (2.21) by induction on $k = 1, \dots, \iota$. The previous step yields these estimates for $k = 1$. Let us fix $k \ge 2$ and suppose that (2.21) holds for all integers less than or equal to $k - 1$; we wish to prove (2.21) for components of γ with degree k and for any fixed $1 \le j \le m_k$. We denote by i the unique integer between 1 and n such that $X_i = X_j^k$ and accordingly we have $\gamma_i = \gamma_j^k$, where $d_i = k$. Taking into account (2.22) and that C_{sr} vanishes when $d_s > \sigma_r$, it follows that

$$\dot\gamma_i \;=\; \sum_{r=1}^{p} \sum_{s=1}^{n} a_{is}(\gamma) C_{sr}(\gamma)\tilde\lambda_r = \sum_{\substack{1 \le r \le p \\ d_s \le d_i \\ d_s \le \sigma_r}} a_{is}(\gamma) C_{sr}(\gamma)\tilde\lambda_r \,. \tag{2.26}$$

We split this sum into three addends

$$\dot\gamma_j^k = \dot\gamma_i = \sum_{\substack{1\le r\le p\\ d_i\le\sigma_r}} C_{ir}(\gamma)\tilde\lambda_r + \sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s=\sigma_r}} a_{is}(\gamma)C_{sr}(\gamma)\tilde\lambda_r + \sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s<\sigma_r}} a_{is}(\gamma)C_{sr}(\gamma)\tilde\lambda_r\,. \tag{2.27}$$

Step 3. We first consider the case $1\le j\le\alpha_k$: (2.11) implies that $C_{ir}(x)=\delta_{ir}$, therefore the first term of (2.27) coincides with $\tilde\lambda_i(t)=\lambda_j^k t^{k-1}$. For the remaining terms, our inductive hypothesis yields

$$\gamma_s^l(t,\lambda) = \begin{cases} \big(\lambda_s^l/l + g_s^l(\lambda_1^1,\dots,\lambda_{\alpha_{l-1}}^{l-1}) + O(t)\big)\,t^l & \text{if } 1\le s\le\alpha_l\\ O(t)\,t^l & \text{if } \alpha_l+1\le s\le m_l \end{cases} \tag{2.28}$$

whenever $l\le k-1$, where g_s^l is a homogeneous polynomial of degree l. Due to (2.22), a_{is} are homogeneous polynomials of degree $d_i-d_s=k-d_s>0$, then applying (2.28), we achieve

$$a_{is}(\gamma_1^1,\dots,\gamma_{m_{k-1}}^{k-1}) = \big(N_{is}(\lambda_1^1,\dots,\lambda_{\alpha_{k-1}}^{k-1}) + O(t)\big)\,t^{k-d_s} \tag{2.29}$$

whenever $d_s\le d_i=k$ and $u_{is}=\delta_{is}$ if $d_s=k$. Notice that N_{is} are homogeneous polynomials of degree $k-d_s$ since it is a composition of the homogeneous polynomial a_{is} and of the homogeneous polynomials $\lambda_s^l/l+g_s^l(\lambda_1^1,\dots,\lambda_{\alpha_{l-1}}^{l-1})$ with degree l.

Let us focus our attention on the second addend of (2.27). By definition of $\tilde\lambda$, we have $\tilde\lambda_r=\lambda_{l(r)}^{\sigma_r}t^{\sigma_r-1}$, for some $1\le l(r)\le\alpha_{\sigma_r}$, hence this second term equals

$$\begin{aligned} &\sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s=\sigma_r}} \big[C_{sr}(0)+O(t)\big]\big[N_{is}(\lambda_1^1,\dots\lambda_{\alpha_{k-1}}^{k-1})t^{k-d_s}+O(t^{k-d_s+1})\big]\lambda_{l(r)}^{\sigma_r}t^{\sigma_r-1}\\ &= \sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s=\sigma_r}} C_{sr}(0)\,N_{is}(\lambda_1^1,\dots\lambda_{\alpha_{k-1}}^{k-1})\,\lambda_{l(r)}^{d_s}\,t^{k-1}+O(t^k)\\ &= \tilde N_i(\lambda_1^1,\dots,\lambda_{\alpha_{k-1}}^{k-1})\,t^{k-1}+O(t^k), \end{aligned}$$

where $\tilde N_i$ is a homogeneous polynomial of degree $k=d_i$. From (2.29) and taking into account the definition of $\tilde\lambda_r$, the last term of (2.27) can be written as follows

$$\begin{aligned} &\sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s<\sigma_r}} C_{sr}(\gamma(t))\big[N_{is}(\lambda_1^1,\dots,\lambda_{\alpha_{k-1}}^{k-1})t^{k-d_s}+O(t^{k-d_s+1})\big]O(t^{\sigma_r-1})\\ &= \sum_{\substack{1\le r\le p\\ d_s<d_i\\ d_s<\sigma_r}} O(t^{k-d_s+\sigma_r-1}) = O(t^k)\,. \end{aligned}$$

Summing up the results obtained for the three addends of (2.27), we have shown that

$$\dot\gamma_j^k(t) = (\lambda_j^k + \tilde N_i(\lambda_1^1, \dots, \lambda_{\alpha_{k-1}}^{k-1}))t^{k-1} + O(t^k)$$

whence the first part of (2.21) follows.

Step 4. Finally, we consider the case $\alpha_k + 1 \le j \le m_k$. In this case we decompose (2.26) into the following two addends

$$\dot\gamma_i \;=\; \sum_{\substack{1\le r\le p\\ k\le\sigma_r}} C_{ir}(\gamma)\ \tilde\lambda_r \;+\; \sum_{\substack{1\le r\le p\\ d_s<k\\ d_s\le\sigma_r}} a_{is}(\gamma)C_{sr}(\gamma)\tilde\lambda_r\,. \tag{2.30}$$

The first term of (2.30) can be written as

$$\sum_{\substack{1\le r\le p\\ k\le\sigma_r}} C_{ir}(\gamma)\tilde\lambda_r \;=\; \sum_{\substack{1\le r\le p\\ k=\sigma_r}} C_{ir}(\gamma)\tilde\lambda_r + \sum_{\substack{1\le r\le p\\ k<\sigma_r}} C_{ir}(\gamma)\tilde\lambda_r.$$

From (2.11), the Lipschitz function $C_{ir}(x)$ vanishes at zero when $\alpha_k + 1 \le j \le m_k$ and $d_i = \sigma_r$, then $C_{ir}(\gamma(t)) = O(t)$ and

$$\sum_{\substack{1\le r\le p\\ k\le\sigma_r}} C_{ir}(\gamma)\tilde\lambda_r = \sum_{\substack{1\le r\le p\\ k=\sigma_r}} O(t)\ t^{k-1} + \sum_{\substack{1\le r\le p\\ k<\sigma_r}} O(1)\ t^{\sigma_r-1} = O(t^k). \tag{2.31}$$

Let us now consider the second term of (2.30). According to (2.29), we know that $a_{is}(\gamma(t)) = O(t^{k-d_s})$. Unfortunately, this estimate is not enough for our purposes, as one can check observing that $\tilde\lambda_r = O(t^{\sigma_r-1})$ and $C_{sr} = O(1)$ for some of s, r. To improve the estimate on a_{is} we will use Lemma 2.14, according to which the subspace spanned by

$$\left(X_1^1, \dots, X_{\alpha_1}^1, \dots, X_1^\iota, \dots, X_{\alpha_\iota}^\iota\right)$$

is a subalgebra. Then we define

$$\mathcal{F} = \operatorname{span}\{X_s^k \mid 1 \le k \le \iota\,,\ 1 \le s \le \alpha_k\}$$

along with the set J, that is given by the condition

$$\mathcal{F} = \operatorname{span}\{X_j : j \in J\}.$$

We first notice that $i \notin J$, due to our assumption $\alpha_k + 1 \le j \le m_k$. This will allow us to apply Lemma 2.5, according to which we have

$$P_i(x,y) = x_i + y_i + Q_i(x,y) = x_i + y_i + \sum_{l\notin J,\ d_l<k} \left(x_l R_{il}(x,y) + y_l S_{il}(x,y)\right).$$

As a result, assuming that $s \in J$, we obtain the key formula

$$a_{is}(x) = \frac{\partial P_i}{\partial y_s}(x,0) = \sum_{l \notin J,\, d_l < k} x_l \, \frac{\partial R_{il}}{\partial y_s}(x,0)\,,$$

where $\partial_{y_s} R_{il}(x,0)$ is a homogeneous polynomial of degree $k - d_s - d_l$. By both inductive hypothesis and definition of J, we get

$$\gamma_l(t) = O(t^{d_l+1})\,,$$

for every $l \notin J$ such that $d_l < k$. By these estimates, we achieve

$$a_{is}(\gamma(t)) = \sum_{l \notin J,\, d_l < k} \gamma_l(t)\, \frac{\partial R_{il}}{\partial y_s}(\gamma(t),0) = \sum_{l \notin J,\, d_l < k} O(t^{d_l+1}) O(t^{k-d_s-d_l}) = O(t^{k+1-d_s})\,.$$

Then it is convenient to split the second term of (2.30) as follows

$$\sum_{\substack{r=1,\dots,p\\ d_s<k\\ d_s\le\sigma_r}} a_{is}(\gamma)\, C_{sr}(\gamma)\tilde{\lambda}_r = \sum_{\substack{r=1,\dots,p\\ d_s<k\\ d_s\le\sigma_r\\ s\in J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r + \sum_{\substack{r=1,\dots,p\\ d_s<k\\ d_s\le\sigma_r\\ s\notin J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r\,, \tag{2.32}$$

where the first addend of the previous decomposition can be estimated as

$$\sum_{\substack{1\le r\le p\\ d_s<k\\ d_s\le\sigma_r\\ s\in J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r \;=\; \sum_{\substack{1\le r\le p\\ d_s<k\\ d_s\le\sigma_r\\ s\in J}} O(t^{k+1-d_s})\, O(1)\, O(t^{\sigma_r-1}) \;=\; O(t^k)\,. \tag{2.33}$$

Finally, we consider the second addend of (2.32), writing it as the following sum

$$\sum_{\substack{1\le r\le p\\ d_s<k\\ d_s\le\sigma_r\\ s\notin J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r = \sum_{\substack{1\le r\le p\\ d_s<k\\ d_s=\sigma_r\\ s\notin J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r + \sum_{\substack{1\le r\le p\\ d_s<k\\ d_s<\sigma_r\\ s\notin J}} a_{is}(\gamma)\, C_{sr}(\gamma)\, \tilde{\lambda}_r\,. \tag{2.34}$$

The first term of (2.34) can be written as

$$\sum_{\substack{1\le r\le p\\ d_s<k\\ d_s=\sigma_r\\ s\notin J}} O(t^{k-d_s})\, O(t)\, O(t^{\sigma_r-1}) = O(t^k)\,,$$

where we have used the fact that $C_{sr}(x) = O(|x|)$ when $d_s = \sigma_r$ and $s \notin J$, according to (2.11). The second term of (2.34) corresponds to the sum

$$\sum_{\substack{1\le r\le p\\ d_s<k\\ d_s<\sigma_r\\ s\notin J}} O(t^{k-d_s})\, O(1)\, O(t^{\sigma_r-1}) = O(t^k)\,.$$

As a result, the second term of (2.32) is also equal to some $O(t^k)$, hence thanks to (2.33) we get that the second term of (2.30) is $O(t^k)$. Thus, taking into account (2.30) and (2.31) we achieve $\dot\gamma(t) = O(t^k)$, which proves the second part of (2.21) and ends the proof. □

Remark 2.16. Analysing the proof of Lemma 2.15, it is easy to realize that the functions $O(t^{k+1})$ appearing in the statement of Lemma 2.15 can be estimated by t^{k+1}, uniformly with respect to λ varying in a compact set: more precisely, there exists a constant $M > 0$ such that

$$\begin{aligned} &\Big|\gamma_j^k(t,\lambda) - \big[\lambda_j^k/k + g_j^k(\lambda_1^1,\dots,\lambda_{\alpha_{k-1}}^{k-1})\big]t^k\Big| \le Mt^{k+1} && \text{if } 1 \le j \le \alpha_k \\ &|\gamma_j^k(t,\lambda)| \le Mt^{k+1} && \text{if } \alpha_k + 1 \le j \le m_k. \end{aligned} \tag{2.35}$$

for all λ belonging to a compact set L and every $t < t_0$.

Our next step will be to prove that our curves $\gamma(\cdot,\lambda)$ do cover a neighbourhood of a point with maximum degree. To do this, we fix graded coordinates with respect to the basis (X_j^k) and consider the diffeomorphism $G : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ arising from Lemma 2.15 and that can be associated with any point of maximum degree in a $\mathbf{C}^{1,1}$ smooth submanifold: precisely, we set

$$G_i(\lambda) := \lambda_i/\sigma_i + g_i(\lambda_1,\dots,\lambda_{\sum_{s=1}^{\sigma_i-1}\alpha_s}), \tag{2.36}$$

where $(g_1,\dots,g_p) = (g_1^1,\dots,g_{\alpha_1}^1,\dots,g_1^\iota,\dots,g_{\alpha_\iota}^\iota)$ and g_j^k are given by Lemma 2.15. Then $G(0) = 0$ and by explicit computation of the inverse function, the definition (2.36) implies global invertibility of G.

Remark 2.17. The diffeomorphism G also permits us to state Lemma 2.15 as follows

$$\gamma(t,\lambda) = \delta_t\big(G(\lambda) + O(t)\big) \in \mathbb{R}^n, \tag{2.37}$$

where $G(\lambda)$ belongs to $\mathbb{R}^p \times \{0\}$, precisely, it lies in the p-dimensional subspace $\Pi_S(x)$ with respect to the associated graded coordinates.

We will denote by $c(t,\lambda)$ the projection of $\gamma(t,\lambda)$ on $\Pi_S(x)$, namely

$$c(t,\lambda) = \pi_S(x)\big(\gamma(t,\lambda)\big), \tag{2.38}$$

where $\pi_S(x)$ is as in (2.17) and graded coordinates arising from (2.16) are understood. In the sequel, the estimates

$$c_i(t,\lambda) = G_i(\lambda)t^{\sigma_i} + O(t^{\sigma_i+1}) \tag{2.39}$$

will be used. They follow from Lemma 2.15 and the definitions of c and G.

Lemma 2.18. *There exists $t_0 > 0$ such that for every $t_1 \in]0, t_0[$, there exists a neighbourhood V of 0 such that*

$$V \cap S \subset \left\{\gamma(t,\lambda) : \lambda \in G^{-1}(S^{p-1}) \text{ and } 0 \leq t < t_1\right\}.$$

Proof. We fix $t_0 = t_0(L) > 0$ as in Lemma 2.15, where we have chosen $L = G^{-1}(S^{p-1})$. Let $t_1 \in]0, t_0[$ be arbitrarily fixed. Taking into account Corollary 2.13, it suffices to prove that the set $\{c(t,\lambda) : \lambda \in L, 0 \leq t < t_1\}$ covers a neighbourhood of 0 in $\mathbb{R}^p$. For each $t \in]0, t_1[$, we define the "projected dilations" $\Delta_t = \pi_S(x) \circ \delta_t$ corresponding to the following diffeomorphisms of $\mathbb{R}^p$

$$\Delta_t(y_1, \ldots, y_p) = (t^{\sigma_1} y_1, \ldots, t^{\sigma_i} y_i, \ldots, t^{\sigma_p} y_p).$$

Now we can rewrite (2.39) as

$$c(t,\lambda) = \Delta_t\big(G(\lambda) + O(t)\big), \tag{2.40}$$

where $O(t)$ is uniform with respect to λ varying in $G^{-1}(S^{p-1})$, according to Remark 2.16. Then we define the mapping

$$\begin{array}{rcl} L_t & : & S^{p-1} \to \mathbb{R}^p \\ & & u \longmapsto \Delta_{1/t}\big(c(t, G^{-1}(u))\big) \end{array}$$

and (2.40) implies

$$L_t(u) = u + O(t).$$

As a consequence, $L_t \to Id_{S^{p-1}}$ as $t \to 0$, uniformly with respect to u varying in S^{p-1}. Then, possibly considering a smaller t_0, for any $0 < \tau < t_1$ we have $L_\tau(S^{p-1}) \cap B_{1/2} = \emptyset$ and L_τ is homotopic to $Id_{S^{p-1}}$ in $\mathbb{R}^p \setminus \{A\}$ for all $A \in B_{1/2}$. In particular, since $Id_{S^{p-1}}$ is not homotopic to a constant, L_τ is not homotopic to a constant in $\mathbb{R}^p \setminus \{A\}$ for all $A \in B_{1/2}$.

Now, we are in the position to prove that

$$\left\{c(t,\lambda) : \lambda \in G^{-1}(S^{p-1}) \text{ and } 0 \leq t < \tau\right\}$$

covers the open neighbourhood of 0 in $\mathbb{R}^p$ given by $\Delta_\tau(B_{1/2} \cap \Pi_S(e))$ that leads us to the conclusion. By contradiction, if this were not true, then we could find a point $A \in B_{1/2}$ such that $A \neq \Delta_{1/\tau}(c_\lambda(t))$ for all $\lambda \in G^{-1}(S^{p-1})$ and $0 \leq t < \tau$, but then

$$\begin{array}{rcl} H & : & [0,\tau] \times S^{p-1} \to \mathbb{R}^p \setminus \{A\} \\ & & (s,u) \longmapsto \Delta_{1/\tau}\big(c\big(s, G^{-1}(u)\big)\big) \end{array}$$

would provide a homotopy in $\mathbb{R}^p \setminus \{A\}$ between the constant 0 and L_τ, which cannot exist. □

As an important consequence of Lemma 2.15, we can finally obtain the main result of this Section.

Theorem 2.19. *Let S be a $\mathbf{C}^{1,1}$ smooth submanifold of $\mathbb{G}$ and let $x \in S$ be a point of maximum degree. Then for every $R > 0$ we have*

$$\delta_{1/r}(x^{-1}S) \cap \overline{U_R} \to \Pi_S(x) \cap \overline{U_R} \qquad \text{as } r \to 0^+ \tag{2.41}$$

with respect to the Hausdorff distance; moreover, $\Pi_S(x)$ is a subgroup of $\mathbb{G}$.

Proof. We first notice that $\Pi_S(x)$ is a subgroup of $\mathbb{G}$, due to Lemma 2.14. Setting $S_{x,r} := \delta_{1/r}(x^{-1}S)$, it is sufficient to prove (see [13], Proposition 4.5.5) that $S_{x,r} \cap \overline{U_R}$ converges to $\Pi \cap \overline{U_R}$ in the Kuratowski sense, i.e. that

(i) if $y = \lim_{n\to\infty} y_n$ for some sequence $\{y_n\}$ such that $y_n \in S_{x,r_n} \cap \overline{U_R}$ and $r_n \to 0$, then $y \in \Pi_S(x) \cap \overline{U_R}$;

(ii) if $y \in \Pi_S(x) \cap \overline{U_R}$, then there are $y_r \in S_{x,r} \cap \overline{U_R}$ such that $y_r \to y$.

It is not restrictive assuming that $x = e$.

To prove (i), we set $z_n = \delta_{r_n}(y_n) \in S \cap \overline{U_{r_n R}}$. From (2.37), we can find $t_1 > 0$ arbitrarily small such that

$$\inf_{\substack{u \in S^{p-1} \\ 0<t<t_1}} |u + O(t)| > 0, \tag{2.42}$$

where $|\cdot|$ is the Euclidean norm and $O(t)$ is defined in (2.37). Then for n sufficiently large and taking $t_1 < t_0$, Lemma 2.18 yields a sequence $\{\tau_n\} \subset]0, t_1[$ and $\lambda_n \in G^{-1}(S^{p-1})$ such that $\gamma(\tau_n, \lambda_n) = \delta_{r_n} y_n$. Due to (2.37), we achieve

$$\delta_{\tau_n/r_n}\left(G(\lambda_n) + O(\tau_n)\right) = y_n\,,$$

hence (2.42) implies that τ_n/r_n is bounded. Up to subsequences, we can assume that $G(\lambda_n) \to \zeta$ and $\tau_n/r_n \to s$, then $y_n \to \delta_s \zeta = y$. From Remark 2.17, we know that $G(\lambda) \in \Pi_S(x)$ with respect to our graded coordinates, hence $y \in \Pi_S(x)$.

To prove (ii), we choose $y \in \Pi_S(x) \cap \overline{U_R}$ and set $\lambda = G^{-1}(y)$. By Lemma 2.15 there exists $r_0 > 0$ depending on the compact set $G^{-1}(\Pi_S(x) \cap \overline{U_R})$ such that the solution $r \to \gamma(r, \lambda')$ of (2.20) is defined on $[0, r_0]$ for every $\lambda' \in G^{-1}(\Pi_S(x) \cap \overline{U_R})$. Clearly, $\gamma(r, \lambda') \in S$, then (2.37) implies that

$$\delta_{1/r}\big(S\big) \ni y_r = \delta_{1/r}\big(\gamma(r, \lambda)\big) \longrightarrow G(\lambda) = y\,.$$

This ends the proof. □

2.3 Measure of submanifolds in Carnot groups

In the following Theorem 2.20 we will denote by $\sigma_{\tilde{g}}$ the Riemannian p-dimensional surface measure with respect to an arbitrary metric $\tilde{g}$. We also stress that the right hand side of (2.43) is effectively dependent on $\tilde{g}$ like the left hand one, because so is the definition itself of metric factor θ, and in particular the p-dimensional measure $\mathcal{H}^p$ appearing in (2.3) of Definition 2.2.

Theorem 2.20. *Let S be a $\mathbf{C}^{1,1}$ smooth p-dimensional submanifold of degree $d = d(S)$ and let $x \in S$ be of the same degree. Then we have*

$$\lim_{r\downarrow 0} \frac{\sigma_{\tilde{g}}(S \cap U(x,r))}{r^d} = \frac{\theta(\tau_S^d(x))}{|\tau_S^d(x)|_g}\,. \tag{2.43}$$

Proof. Without loss of generality we assume that x is the identity element e and identify $\mathbb{G}$ with $\mathbb{R}^n$ through graded coordinates centered at 0 with respect to X_j^k. According to Corollary 2.13, we parametrize S by the $\mathbf{C}^{1,1}$ function $\varphi : A \subset \Pi_S(e) \to \mathbb{R}^{n-p}$, such that S is the image of

$$\begin{aligned} \Phi \;:\; & A \subset \Pi_S(e) \longrightarrow \mathbb{R}^n \\ & y \mapsto (y_1^1, \dots, y_{\alpha_1}^1, \phi_{\alpha_1+1}^1(y), \dots, \phi_{m_1}^1(y), \dots, y_1^\iota, \dots, y_{\alpha_\iota}^\iota, \phi_{\alpha_\iota+1}^\iota(y), \dots, \phi_{m_\iota}^\iota(y)). \end{aligned}$$

For any sufficiently small $r > 0$, we have

$$\begin{aligned} \lim_{r\downarrow 0} \frac{\sigma_{\tilde{g}}(S \cap U_r)}{r^d} &= \frac{1}{r^d} \int_{\Phi^{-1}(U_r)} J_{\tilde{g}}\Phi(y)\, dy \\ &= \int_{\Delta_{1/r}(\Phi^{-1}(U_r))} J_{\tilde{g}}\Phi(\Delta_r(y)) dy\,, \end{aligned} \tag{2.44}$$

where $\Delta_r = \delta_{r|\Pi_S(e)}$ and its jacobian is exactly equal to r^d. Notice that the set $\Delta_{1/r}(\Phi^{-1}(U_r)) = (\delta_{1/r} \circ \Phi \circ \Delta_r)^{-1}(U_1)$ contains exactly those elements $y \in \Pi_S(e)$ such that

$$\left(y_1^1, \dots, y_{\alpha_1}^1, \frac{\phi_{\alpha_1+1}^1(\Delta_r y)}{r}, \dots, \frac{\phi_{m_1}^1(\Delta_r y)}{r}, \dots, y_1^\iota, \dots, y_{\alpha_\iota}^\iota, \frac{\phi_{\alpha_\iota+1}^\iota(\Delta_r y)}{r^\iota}, \dots, \frac{\phi_{m_\iota}^\iota(\Delta_r y)}{r^\iota}\right)$$

belongs to U_1 and that

$$\Delta_{1/r}(\Phi^{-1}(U_r)) = \pi_S(e)(S_{0,r} \cap U_1),$$

where $\pi_S(e)$ is the projection onto $\Pi_S(e)$ with respect to graded coordinates, i.e. the mapping

$$\mathbb{R}^n \ni (z_1^1, \dots, z_{m_1}^1, \dots, z_1^\iota, \dots, z_{m_\iota}^\iota) \longmapsto (z_1^1, \dots, z_{\alpha_1}^1, \dots, z_1^\iota, \dots, z_{\alpha_\iota}^\iota) \in \Pi_S(e).$$

For the sake of simplicity, we will write π instead of $\pi_S(e)$. By continuity of π, for every $\epsilon > 0$ we can find a neighbourhood $\mathcal{N} \subset \mathbb{R}^n$ of $\Pi_S(e) \cap \overline{U_r}$ such that $\pi(\mathcal{N}) \subset \Pi_S(e) \cap U_{1+\epsilon}$; by Theorem 2.19 and the definition of Hausdorff convergence, for sufficiently small r we have $S_{0,r} \cap \overline{U_1} \subset \mathcal{N}$ and so

$$\Delta_{1/r}(\Phi^{-1}(U_r)) \subset \pi(S_{0,r} \cap \overline{U_1}) \subset \Pi_S(e) \cap U_{1+\epsilon}. \tag{2.45}$$

If we also prove that

$$\Pi_S(e) \cap U_{1-\epsilon} \subset \Delta_{1/r}(\Phi^{-1}(U_r)) \tag{2.46}$$

for small r, we will have $\chi_{\delta_{1/r}(\Phi^{-1}(U_r))} \to \chi_{\Pi_S(e)\cap U_1}$ in $L^1(\Pi_S(e))$. This fact and (2.44) imply that

$$\lim_{r\downarrow 0} \frac{\sigma_{\tilde{g}}(S \cap U_r)}{r^d} = J_{\tilde{g}}\Phi(0)\, \mathcal{H}^p(\Pi_S(e) \cap U_1) = J_{\tilde{g}}\Phi(0)\, \theta(\tau_S^d(0)).$$

By Corollary 2.13 we know that $\nabla\Phi(0) = C(0)$, where C is given by Lemma 2.6; therefore $J_{\tilde{g}}\Phi(0)$ must coincide with the Jacobian of the matrix $C(0)$, i.e. with $|v_1(0) \wedge \cdots \wedge v_p(0)|_{\tilde{g}}$. By virtue of Corollary 2.11, we have

$$|\tau_S^d(e)|_g = \left| \frac{X_1^1 \wedge \cdots \wedge X_{\alpha_1}^\iota \wedge \cdots \wedge X_1^\iota \wedge \cdots \wedge X_{\alpha_\iota}^\iota}{|v_1(0) \wedge \cdots \wedge v_p(0)|_{\tilde{g}}} \right|_g = \frac{1}{|v_1(0) \wedge \cdots \wedge v_p(0)|_{\tilde{g}}}.$$

Finally, it remains to prove (2.46). We fix

$$y = (y_1, \ldots, y_p) = (y_1^1, \ldots, y_{\alpha_1}^1, \ldots, y_{\alpha_\iota}^\iota) \in \Pi_S(e) \cap U_{1-\epsilon}$$

and set $z := \delta_r(y) \in U_{(1-\epsilon)r}$. Let $t_0 > 0$ be as in Lemma 2.18 and consider $t_1 \in]0, t_0[$ to be chosen later. By the same lemma, for every $r > 0$ sufficiently small there exist $\lambda \in G^{-1}(S^{p-1})$ and $t \in [0, t_1[$ such that $\Phi(z) = \gamma(t, \lambda)$. Since $|G(\lambda)| = 1$, we can find $1 \le i \le p$ such that $|G_i(\lambda)| \ge 1/\sqrt{p}$. Notice that

$$\pi_S(e)(\Phi(z)) = z = \pi_S(e)(\gamma(t, \lambda)) = c(t, \lambda), \tag{2.47}$$

then (2.39) implies

$$Mt^{\sigma_i+1} \ge |G_i(\lambda)|t^{\sigma_i} - |z_i| \ge t^{\sigma_i}/\sqrt{p} - |y_i| r^{\sigma_i},$$

where $M > 0$ is given in Remark 2.16 with $L = G^{-1}(S^{p-1})$. It follows that

$$(1/\sqrt{p} - Mt_1)t^{\sigma_i} \le (1/\sqrt{p} - Mt)t^{\sigma_i} \le |y_i|\, r^{\sigma_i}\,.$$

Now, we can choose $t_1 > 0$ such that $1/\sqrt{p} - Mt_1 \ge \epsilon > 0$, getting a constant $N > 0$ depending only on p, $|y|$ and M such that

$$t \le N\, r\,. \tag{2.48}$$

Taking into account (2.47) and the explicit estimates of (2.35), we get some $1 \le k \le \iota$ and $\alpha_j + 1 \le j \le m_j$ such that

$$|c_i(t,\lambda)| = |\gamma_j^k(t,z)| = |\phi_j^k(z)| \le Mt^{k+1}\,,$$

where we notice that $k = \sigma_i$. By (2.48), the previous estimate yield

$$|\phi_j^k(\delta_r y)| = |\phi_j^k(z)| \le \tilde{M} r^{k+1}\,, \tag{2.49}$$

where $\tilde{M} = MN^{k+1}$. Estimate (2.49) has been obtained with $\tilde{M}$ independent from $r > 0$ sufficiently small. Therefore

$$\left(y_1^1, \dots, y_{\alpha_1}^1, \frac{\phi_{\alpha_1+1}^1(\delta_r y)}{r}, \dots, \frac{\phi_{m_1}^1(\delta_r y)}{r}, \dots, y_1^\iota, \dots, y_{\alpha_\iota}^\iota, \frac{\phi_{\alpha_\iota+1}^\iota(\delta_r y)}{r^\iota}, \dots, \frac{\phi_{m_\iota}^\iota(\delta_r y)}{r^\iota}\right)$$

belongs to U_1 definitely as r goes to zero, namely, $y \in \Delta_{1/r}\Phi^{-1}(U_r)$ for $r > 0$ small enough. We observe that N linearly depends on $|y|$ and is independent from $r > 0$, then the constant $\tilde{M}$ in (2.49) can be fixed independently from y varying in the bounded set $\Pi_S(e) \cap U_{1-\epsilon}$, whence (2.46) follows. □

Let S and d be as in Theorem 2.20; for $i = 1, \dots, d$ we set

$$S_i := \{x \in S : d_S(x) = i\}.$$

Then, using Theorem 2.20 and standard theorems on differentiation of measures (see [69]), it is immediate to deduce the following

Corollary 2.21. *Suppose that S is a $\mathbf{C}^{1,1}$ submanifold of degree d; then*

$$\mathcal{S}_\rho^d(S_d) = \int_{S_d} \frac{|\tau_S^d(x)|_g}{\theta(\tau_S^d(x))}\, d\sigma_{\tilde{g}}(x)\,. \tag{2.50}$$

In particular, if $\mathcal{S}_\rho^d$-almost every point has maximum degree d, i.e. if

$$\mathcal{S}_\rho^d(S \setminus S_d) = 0\,, \tag{2.51}$$

one has that

$$\mathcal{S}_\rho^d(S) = \int_S \frac{|\tau_S^d(x)|_g}{\theta(\tau_S^d(x))}\, d\sigma_{\tilde{g}}(x)\,. \tag{2.52}$$

and S has Hausdorff dimension d.

In formula (2.52) we used the facts that $|\tau_S^d(x)|_g = 0$ on $S \setminus S_d$ and that metric factors are uniformly bounded from below.

Corollary 2.21 shows that $\mathcal{S}_\rho^d$ is positive and finite on open bounded sets of the submanifold and yields the "natural" sub-Riemannian measure on Σ

$$\mu_S = \mathcal{S}_\rho^d \llcorner S = \frac{|\tau_S^d(\cdot)|_g}{\theta(\tau_S^d(\cdot))}\, \sigma_{\tilde{g}} \llcorner S\,. \tag{2.53}$$

Also the equivalent measure

$$\widetilde{\mu}_S := |\tau_S^d(\cdot)|_g\, \sigma_{\tilde{g}} \llcorner S \tag{2.54}$$

can be considered a natural one, with the further property that it does not depend on the metric $\tilde{g}$. In fact, parametrizing a piece of S by a mapping $\Psi : U \longrightarrow \mathbb{G}$, we have

$$\begin{aligned}
\widetilde{\mu}_S\big(\Psi(U)\big) &= \int_{\Psi(U)} |\tau_S^d(x)|_g \, d\sigma_{\tilde{g}}(x) \\
&= \int_{\Psi(U)} \frac{\Big|\big[(\partial_{x_1}\Psi \wedge \cdots \wedge \partial_{x_p}\Psi)(\Psi^{-1}(x))\big]^d\Big|_g}{\big|(\partial_{x_1}\Psi \wedge \cdots \wedge \partial_{x_p}\Psi)(\Psi^{-1}(x))\big|_{\tilde{g}}}\, d\sigma_{\tilde{g}}(x) \\
&= \int_U \Big|\big(\partial_{x_1}\Psi \wedge \cdots \wedge \partial_{x_p}\Psi\big)^d\Big|_g \, d\mathcal{L}^p\,,
\end{aligned} \tag{2.55}$$

where we used classical area formula and the fact that

$$\tau_S(x) = \frac{\partial_{x_1}\Psi \wedge \cdots \wedge \partial_{x_p}\Psi}{|\partial_{x_1}\Psi \wedge \cdots \wedge \partial_{x_p}\Psi|_{\tilde{g}}}(\Psi^{-1}(x))\,.$$

Integral formula (2.55) can be seen as an area-type formula where the jacobian is projected on vectors of fixed degree.

It is possible to prove that the restrictive hypothesis (2.51) holds true in many interesting cases, namely when

- S is a p-dimensional Legendrian submanifold in the Heisenberg groups $\mathbb{H}^n$, i.e. $T_xS \subset H_x\mathbb{H}^n$ for any $x \in S$ (in this case one must have $p \leq n$ and it is easy to check that $d = p$, see [83]);
- S is a p-dimensional non-Legendrian submanifold in the Heisenberg groups $\mathbb{H}^n$ (in this case $d = p + 1$, see [83] and [122]);
- S is a codimension 1 hypersurface of a Carnot group $\mathbb{G}$, where we have $d = Q - 1$ (see [120]);

- S is a "non-horizontal" submanifold in a Carnot group $\mathbb{G}$, i.e. $d = Q - k$, where k is the topological codimension os S (see [121]).

Observe however that, for general submanifolds in a Carnot group, the non-horizontality condition is quite restrictive: for example, it cannot hold when the codimension k is too large (namely, when $k > m_1$). Presently, we are not able to prove the validity of (2.51) in the general case; however, one could expect (possibly requiring more regularity on S) not only that it holds true, but in fact that

$$\mathcal{S}^i_\rho(S_i) < \infty \quad \text{for all } i = 1, \dots, d. \tag{2.56}$$

Indeed, this is exactly what happens in step 2 Carnot groups:

Theorem 2.22. *Let S be a $\mathbf{C}^{1,1}$ submanifold of degree d of a step 2 Carnot group $\mathbb{G}$; then* (2.56) *holds. In particular, also formula* (2.52) *holds and the Hausdorff dimension of S is d.*

Proof. In view of Corollary 2.21, it will be sufficient to prove (2.56). By [121], Theorem 1.3, we know that there exist two real constants $c_1, c_2 > 0$ such that

$$0 < c_1 \le \liminf_{r \to 0^+} \frac{\sigma_{\tilde{g}}(S \cap U(x,r))}{r^i} \le \limsup_{r \to 0^+} \frac{\sigma_{\tilde{g}}(S \cap U(x,r))}{r^i} \le c_2$$

for any $x \in S_i$; therefore one has

$$c_1 \mathcal{S}^i_\rho(S_i) \le \sigma_{\tilde{g}}(S_i) \le \sigma_{\tilde{g}}(S) < \infty$$

and this is sufficient to conclude. □

2.4 Some examples in the Engel group

As an application, in this section we wish to present examples of 2-dimensional submanifolds of all possible degrees in the Engel group $\mathbb{E}^4$.

It will be convenient, more than using graded coordinates, to represent $\mathbb{E}^4$ as $\mathbb{R}^4$ equipped with the vector fields $X_j = \sum_{j=1}^4 a_{ij}(x)\partial_i$, with

$$A(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & x_1^2/2 & x_1 & 1 \end{pmatrix},$$

where $x = (x_1, x_2, x_3, x_4)$; observe that $d_1 = d_2 = 1$, $d_3 = 2$ and $d_4 = 3$.

Let $\Psi : U \longrightarrow \mathbb{R}^4$ be the parametrization of a 2-dimensional submanifold S, where U is an open subset of $\mathbb{R}^2$. We set $(u_1, u_2) \in U \subset \mathbb{R}^2$ and consider $\Psi_{u_i} = \sum_{j=1}^4 \Psi^j_{u_i}\, \partial_j$. Taking into account that

$$A(x)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & x_1^2/2 & -x_1 & 1 \end{pmatrix}$$

and that

$$\partial_j = \sum_{k=1}^{4} (A(x)^{-1})_{kj} X_k \,, \tag{2.57}$$

we obtain

$$\Psi_{u_i} = \Psi^1_{u_i} X_1 + \Psi^2_{u_i} X_2 + \left(\Psi^3_{u_i} - \Psi^1\, \Psi^2_{u_i}\right) X_3 + \left(\Psi^4_{u_i} - \Psi^1\, \Psi^3_{u_i} + \frac{(\Psi^1)^2}{2}\, \Psi^2_{u_i}\right) X_4 \,.$$

It follows that

$$\begin{aligned} \Psi_{u_1} \wedge \Psi_{u_2} = {} & \Psi^{12}_u X_1 \wedge X_2 + \left(\Psi^{13}_u - \Psi^1 \Psi^{12}_u\right) X_1 \wedge X_3 + \\ & \left(\Psi^{14}_u - \Psi^1\, \Psi^{13}_u + \frac{(\Psi^1)^2}{2}\, \Psi^{12}_u\right) X_1 \wedge X_4 + \Psi^{23}_u X_2 \wedge X_3 + \\ & \left(\Psi^{24}_u - \Psi^1\, \Psi^{23}_u\right) X_2 \wedge X_4 + \left(\Psi^{34}_u + \frac{(\Psi^1)^2}{2} \Psi^{23}_u - \Psi^1 \Psi^{24}_u\right) X_3 \wedge X_4 \end{aligned} \tag{2.58}$$

where we have set

$$\Psi^{ij}_u = \det \begin{pmatrix} \Psi^i_{u_1} & \Psi^i_{u_2} \\ \Psi^j_{u_1} & \Psi^j_{u_2} \end{pmatrix} .$$

In the sequel, we will use (2.58) to obtain nontrivial examples of 2-dimensional submanifolds with different degrees in $\mathbb{E}^4$.

Remark 2.23. Recall that 2-dimensional submanifolds of degree 2 in $\mathbb{E}^4$ cannot exist, due to non-integrability of the horizontal distribution $\mathrm{span}\{X_1, X_2\}$.

The next Example wants to give a rather general method to obtain nontrivial examples of 2-dimensional submanifolds of degree 3. Clearly, the submanifold $\{(0, x_2, x_3, 0\}$ is the simplest example, as one can check using (2.58).

Example 2.24. Having degree 3 means that the first order fully non-linear conditions

$$\begin{cases} \Psi^{34}_u + \frac{(\Psi^1)^2}{2}\Psi^{23}_u - \Psi^1\Psi^{24}_u = 0 \\ \Psi^{24}_u - \Psi^1\,\Psi^{23}_u = 0 \\ \Psi^{14}_u - \Psi^1\,\Psi^{13}_u + \frac{(\Psi^1)^2}{2}\,\Psi^{12}_u = 0 \end{cases} \tag{2.59}$$

must hold. By elementary properties of determinants, one can realize that the previous system is equivalent to requiring that

$$\nabla\Psi^3 - \Psi^1\nabla\Psi^2 \quad \text{is parallel to} \quad \nabla\Psi^4 - \frac{(\Psi^1)^2}{2}\,\nabla\Psi^2\,, \tag{2.60}$$

$$\nabla\Psi^2 \quad \text{is parallel to} \quad \nabla\Psi^4 - \Psi^1\,\nabla\Psi^3\,, \tag{2.61}$$

$$\nabla\Psi^1 \quad \text{is parallel to} \quad \nabla\Psi^4 - \Psi^1\,\nabla\Psi^3 + \frac{(\Psi^1)^2}{2}\nabla\Psi^2\,. \tag{2.62}$$

We restrict our search to submanifolds with $\Psi^1(u_1,u_2) = u_1$ and $\Psi^{23}_u \neq 0$ on U. This implies that $\nabla\Psi^2 \neq 0$ and so (2.61) is equivalent to the existence of a function $\lambda : U \to \mathbb{R}$ such that

$$\nabla\Psi^4 - u_1\,\nabla\Psi^3 = \lambda\,\nabla\Psi^2\,.$$

Imposing the further assumptions $\lambda(u) = -u_1^2/2$ it follows that

$$\nabla\Psi^4 = -\frac{u_1^2}{2}\nabla\Psi^2 + u_1\,\nabla\Psi^3\,, \tag{2.63}$$

whence also (2.62) is satisfied; since

$$u_1(\nabla\Psi^3 - u_1\nabla\Psi^2) = \nabla\Psi^4 - \frac{u_1^2}{2}\,\nabla\Psi^2\,,$$

it follows that also (2.60) is satisfied, namely, the system (2.59) holds whenever we are able to find Ψ^4 satisfying (2.63). Clearly, we have an ample choice of families of functions Ψ^2, Ψ^3, Ψ^4 satisfying (2.63). We choose the injective embedding of $\mathbb{R}^2$ into $\mathbb{R}^4$ defined by

$$\Psi(u_1,u_2) = \begin{pmatrix} u_1 \\ u_1 + e^{u_2} \\ u_1 e^{u_2} + \frac{u_1^2}{2} \\ \frac{u_1^3}{6} + \frac{u_1^2}{2}e^{u_2} \end{pmatrix}.$$

One can check that $d_S(\Psi(u)) = 3$ for every $u \in \mathbb{R}^2$, where $S = \Psi(\mathbb{R}^2)$. Here the part of τ_S with maximum degree is

$$\tau_S^3\big(\Psi(u_1,u_2)\big) = -\frac{e^{u_2}}{\sqrt{\big(1+\frac{u_1^2}{2}\big)^2(1+e^{2u_2})}}\; X_2 \wedge X_3$$

and due to Corollary 2.21, the spherical Hausdorff measure of bounded portions of S is positive and finite.

It is clear that submanifolds of higher degree are easier to be contructed.

Example 2.25. Let us consider

$$\Psi(u_1,u_2)=\left(u_1,u_2,\frac{u_2^2}{2},\frac{u_2^2}{2}\right).$$

Then we have

$$\begin{array}{lll}\Psi_u^{12}=1, & \Psi_u^{13}=u_2, & \Psi_u^{14}=u_2\\ \Psi_u^{23}=0, & \Psi_u^{24}=0, & \Psi_u^{34}=0.\end{array}$$

By (2.58) we have

$$\Psi_{u_1}\wedge\Psi_{u_2}=X_1\wedge X_2+(u_2-u_1)\,X_1\wedge X_3+\left(u_2-u_1u_2+\frac{u_1^2}{2}\right)X_1\wedge X_4. \quad (2.64)$$

Recall that S_i is the subset of points in S with degree equal to i. With this notation we have

$$\begin{aligned}S_4&=\left\{\Psi(u_1,u_2):u_2\in]0,2[\right\}\cup\left\{\Psi(u_1,u_2):u_2\in\mathbb{R}\setminus[0,2],|u_2-u_1|^2\neq u_2^2-2u_2\right\}\\ S_3&=\left\{\Psi\left(u_2+\sigma\sqrt{u_2^2-2u_2},u_2\right)\;\middle|\;\sigma\in\{1,-1\}\quad\text{and}\quad u_2\in\mathbb{R}\setminus[0,2]\right\}\\ S_2&=\left\{\Psi(0,0),\Psi(2,2)\right\}.\end{aligned}$$

We will check that the curves

$$\mathbb{R}\setminus[0,2]\ni u_2\longmapsto\gamma(u_2)=\Phi\big(u_2+\sigma\sqrt{u_2^2-2u_2},u_2\big)$$

with $\sigma\in\{1,-1\}$ have degree constantly equal to 2. Due to (2.57), we achieve

$$\dot\gamma=\dot\gamma^1X_1+\dot\gamma^2X_2+\left(\dot\gamma^3-\gamma^1\,\dot\gamma^2\right)X_3+\left(\dot\gamma^4-\gamma^1\,\dot\gamma^3+\frac{(\gamma^1)^2}{2}\,\dot\gamma^2\right)X_4\,,$$

where one can check that

$$\left(\dot\gamma^4-\gamma^1\,\dot\gamma^3+\frac{(\gamma^1)^2}{2}\,\dot\gamma^2\right)=0\quad\text{and}\quad\left(\dot\gamma^3-\gamma^1\,\dot\gamma^2\right)=-\sigma\sqrt{u_2^2-2u_2}\neq0\,. \quad (2.65)$$

It follows that S_3 is the union of two curves with degree constantly equal to 2. Applying (2.52) we get that $\mathcal{S}^2_\rho\llcorner S_3$ is positive and finite on bounded open pieces of S_3, hence $\mathcal{S}^4_\rho(S_3)=0$. In particular, we have proved that

$$\mathcal{S}^4_\rho(S\setminus S_4)=0,$$

then the Hausdorff dimension of S is 4 and furthermore $\mathcal{S}^4_\rho\llcorner S$ is positive and finite on open bounded pieces of S. Clearly, (2.52) holds.

Example 2.26. Using (2.58) one can check that 2-dimensional submanifolds given by

$$\Psi(u_1, u_2) = \begin{pmatrix} 0 \\ \Psi^2(u_1, u_2) \\ \Psi^3(u_1, u_2) \\ \Psi^4(u_1, u_2) \end{pmatrix}$$

with $\Psi_u^{34} \neq 0$ have degree $5 = Q - k$, where $Q = 7$ is the homogeneous dimension of $\mathbb{E}^4$ and $k = 2$ is the codimension of S. Notice that these submanifolds are then non-horizontal.

Remark 2.27. Let us consider S as in Example 2.25. It is easy to check that the thesis of Theorem 2.19 does not hold, indeed

$$\delta_{1/r} S \cap \overline{U_R} \longrightarrow P \cap \overline{U_R}$$

where

$$P = \{(x_1, 0, 0, x_4) \mid x_4 \geq 0\}.$$

Clearly, P cannot be a subgroup of $\mathbb{E}^4$, since all p-dimensional subgroups of Carnot groups are homeomorphic to $\mathbb{R}^p$, see [167]. This fact may occur since the origin in S has not maximum degree, as one can check in Example 2.25.

Chapter 3

Elements of Geometric Measure Theory in the Heisenberg group

Starting with this Chapter, in almost all the rest of the book we will concentrate our attention on the most important example of non Euclidean Carnot group, namely the Heisenberg group $\mathbb{H}^n$. In particular, we will summarize the principal results of Geometric Measure Theory in this setting, taking great part of the material from [79]. Exhaustive introductions to Heisenberg groups can be found also in [164] and in the recent book [33].

Section 3.1 contains a brief presentation of $\mathbb{H}^n$, on which from now on we will fix a system of graded coordinates, as a CC space; rather than on the CC distance d_c, we will make use of the equivalent homogeneous distance d_∞ defined in (3.1) and of the associated Hausdorff and spherical Hausdorff measures $\mathcal{H}^m_\infty$ and $\mathcal{S}^m_\infty$. Following the approach of Section 1.1, we will define the $\mathbb{H}$-perimeter of a measurable set E: some comparisons between this notion and the Euclidean one are provided in Proposition 3.7 and in Example 3.8, while Theorem 3.9 allows us to introduce the horizontal normal ν_E.

Section 3.2 is concerned with $\mathbf{C}^1_{\mathbb{H}}$ functions, i.e. those continuous real functions on $\mathbb{H}^n$ whose horizontal derivatives are represented, in distributional sense, by continuous functions. This definition goes back to Folland and Stein [73]. Lemma 3.11 contains an estimate on horizontal difference quotients of $\mathbf{C}^1_{\mathbb{H}}$ functions which will be crucial in the proof of Theorem 4.17, while the main result of the Section is Whitney Extension Theorem 3.12: its proof was sketched in [79], here we give a complete one.

In Section 3.3 we introduce one of the main objects of the book, namely the $\mathbb{H}$-regular surfaces. The notion of regular surface is related to a notion of rectifiability in metric spaces which goes back to Federer (see [69] 3.2.14). It has been used by Ambrosio and Kirchheim [7, 8] in the framework of a theory of currents in metric spaces (as for rectifiability in metric spaces, see for instance [104, 150], the monograph [129]

and the references therein). According to this notion, a "good" surface in a metric space should be the image of an open subset of an Euclidean space via a Lipschitz map. Unfortunately, such a viewpoint does not fit the geometry of the Heisenberg group, that indeed would be, according with this definition, purely unrectifiable (see [8]). On the other hand, in the Euclidean setting $\mathbb{R}^n$, a $\mathbf{C}^1$-hypersurface can be equivalently viewed as the level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a concept was easily transposed in [79] to the Heisenberg group by means of $\mathbf{C}^1_{\mathbb{H}}$-functions: we will consequently define $\mathbb{H}$-regular surfaces as noncritical level sets of $\mathbf{C}^1_{\mathbb{H}}$ functions. These surfaces can have an extremely bad behaviour from the Euclidean viewpoint, nevertheless they turn out to be regular with respect to the intrinsic geometry, thus constituting the natural counterpart of $\mathbf{C}^1$ surfaces in a classical setting. See also [32, 104, 109, 91, 78, 85, 54, 7, 8, 79, 146, 137, 80, 81]. In Definition 3.15 we state the notion of intrinsic graph already mentioned in the Introduction, and in the main result of the Section, Theorem 3.16, we prove that $\mathbb{H}$-regular surfaces are locally intrinsic graphs: again the proof of this fact, which is given with several simplifications at some technical points, is taken from [79]. We mention also the recent paper [82], were the notion of $\mathbb{H}$-*Lipschitz* surface is introduced, together with the one of $\mathbb{H}$-Lipschitz graph.

Finally, in Section 3.4 we summarize (without proofs) the results of the latter paper concerning rectifiability of sets E with finite $\mathbb{H}$-perimeter. More precisely, we will introduce the $\mathbb{H}$-reduced boundary $\partial^*_{\mathbb{H}} E$, on which a blow-up result holds (Theorem 3.20). This set, up to $\mathcal{H}^{Q-1}$-negligible sets, is contained (Theorem 3.22) in a countable union of $\mathbb{H}$-regular surfaces. Observe that all these results apply to $\mathbb{H}$-regular surfaces; in particular, the blow-up result is consistent with Theorem 2.19 for $\mathbf{C}^{1,1}$ hypersurfaces.

3.1 The Heisenberg group

As in Section 1.2.4, the Heisenberg group $\mathbb{H}^n$ will be always identified with $\mathbb{R}^{2n+1} = \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t$ with group law

$$P \cdot Q = (x + x', y + y', t + t' + 2\langle x', y\rangle_{\mathbb{R}^n} - 2\langle x, y'\rangle_{\mathbb{R}^n}),$$

where we denote with $P = (x, y, t)$ and $Q = (x', y', t')$ elements of $\mathbb{H}^n$; observe that 0 is the identity of the group and that $(x, y, t)^{-1} = (-x, -y, -t)$. We will use the notation ℓ_P to denote the left translation by an element P.

The Lie algebra $\mathfrak{h}$ of left invariant vector fields is generated by

$$X_j = \partial_{x_j} + 2y_j\partial_t, \qquad Y_j = \partial_{y_j} - 2x_j\partial_t, \qquad T = \partial_t;$$

for $n + 1 \le j \le 2n$ we will often use the notation $X_j := Y_{j-n}$. In this way, $\mathfrak{h}$ is endowed with the stratification $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 =$span $\{X_1, \dots, X_{2n}\}$ and $\mathfrak{h}_2 =$span

$\{T\}$ and where the only nonvanishing commutation relationships are $[X_j, Y_j] = -4T, j = 1, \dots, n$. For $r > 0$ the homogeneous dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are defined as

$$\delta_r(x, y, t) = (rx, ry, r^2 t).$$

For $P = (x, y, t) \in \mathbb{H}^n$ set $\|P\|_\infty := \max\{|(x, y)|_{\mathbb{R}^{2n}}, |t|^{1/2}\}$; then for any $P, Q \in \mathbb{H}^n$ the function

$$d_\infty(P, Q) := \|P^{-1} \cdot Q\|_\infty = \|Q^{-1} \cdot P\|_\infty \tag{3.1}$$

is a homogeneous distance on $\mathbb{H}^n$. In particular

$$d_\infty(\ell_P Q, \ell_P Q') = d_\infty(Q, Q') \quad \text{and} \quad d_\infty(\delta_r Q, \delta_r Q') = r\, d_\infty(Q, Q') \tag{3.2}$$

for any $P, Q, Q' \in \mathbb{H}^n$; moreover, for any bounded subset Ω of $\mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that

$$c_1(\Omega)|P - Q|_{\mathbb{R}^{2n+1}} \le d_\infty(P, Q) \le c_2(\Omega)|P - Q|^{1/2}_{\mathbb{R}^{2n+1}} \tag{3.3}$$

for $P, Q \in \Omega$. In particular, the topologies defined by d_∞ and by the Euclidean distance coincide on $\mathbb{H}^n$. From now on, $U(P, r)$ will be the open ball with centre P and radius r with respect to the distance d_∞. We notice that $U(P, r)$ is a Euclidean Lipschitz domain in $\mathbb{R}^{2n+1}$.

There is a natural measure on $\mathbb{H}^n$ which is given by the Lebesgue measure $d\mathcal{L}^{2n+1} = dx\, dy\, dt$ on $\mathbb{R}^{2n+1}$. This measure is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^n$ then $|E|$ is its Lebesgue measure.

Definition 3.1. (see [69]) We shall denote by $\mathcal{H}^m$ the m-dimensional *Hausdorff measure* obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, and by $\mathcal{H}^m_\infty$ the m-dimensional Hausdorff measure obtained from the distance d_∞ in $\mathbb{H}^n$. Analogously, $\mathcal{S}^m$ and $\mathcal{S}^m_\infty$ will denote the corresponding spherical Hausdorff measures.

Remark 3.2. We stress that, because the topologies defined by d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n + 1$. On the contrary the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $Q = 2n + 2$ (see [132] and [143]). Moreover, one has (see also [79], Theorem 2.18)

$$\mathcal{L}^{2n+1} = \frac{2\omega_{2n}}{\omega_{2n+2}} \mathcal{S}^Q_\infty = \frac{2\omega_{2n}}{\mathcal{H}^Q_\infty(U(0, 1))} \mathcal{H}^Q_\infty.$$

Here and in the following we adopt the standard notation $\omega_k := \mathcal{L}^k(B(0, 1))$, where $B(0, 1)$ is the unit Euclidean ball in $\mathbb{R}^k$.

Translation invariance and homogeneity under dilations of Hausdorff measures follow directly from (3.2), more precisely we have

Proposition 3.3. *Let $\Omega \subseteq \mathbb{H}^n$, $P \in \mathbb{H}^n$ and $m,r \in [0,\infty)$. Then*

$$\mathcal{H}^m_\infty(\ell_P\Omega) = \mathcal{H}^m_\infty(\Omega) \quad \textit{and} \quad \mathcal{H}^m_\infty(\delta_r(\Omega)) = r^m\mathcal{H}^m_\infty(\Omega).$$

The same statements hold for $\mathcal{S}^m_\infty$.

For the sake of completness, we recall that the Carnot-Carathéodory metric d_c on $\mathbb{H}^n$ is defined as in Section 1.2.6 starting from the family $X_1,\dots,X_n,Y_1,\dots,Y_n$; it is not difficult to check that also d_c is a homogeneous metric and so

Proposition 3.4. *The Carnot-Carathéodory distance d_c is (globally) equivalent to the distance d_∞.*

We shall denote with $U_c(P,r)$ the open balls for d_c and with $\mathcal{H}^m_c, \mathcal{S}^m_c$ the associated Hausdorff and spherical Hausdorff measures.

We will identify vector fields and associated first order differential operators; thus the vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called horizontal vector bundle $H\mathbb{H}^n$ according to the notation of Gromov (see [91] and [109]), that is a vector subbundle of $T\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $H\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section φ of $H\mathbb{H}^n$ can be identified with a map $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}^n$ the horizontal fiber is denoted as $H_P\mathbb{H}^n$ and each fiber can be endowed with the scalar product $\langle\cdot,\cdot\rangle_P$ and the associated norm $|\cdot|_P$ that make the vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ orthonormal, hence we shall also identify a section of $H\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame. In this way, a section φ will be identified with the function $\varphi = (\varphi_1,\dots,\varphi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$ such that $\varphi = \sum_{j=1}^{2n} \varphi_j X_j$. As it is common in Riemannian geometry, when dealing with two sections φ and φ' whose argument is not explicitly written, we shall drop the index P in the scalar product writing $\langle\varphi,\varphi'\rangle$ for $\langle\varphi(P),\varphi'(P)\rangle_P$. The same convention shall be adopted for the norm.

If Ω is an open subset of $\mathbb{H}^n$ and $k \geq 0$ is a non negative integer, the symbols $\mathbf{C}^k(\Omega)$, $\mathbf{C}^\infty(\Omega)$ denote the usual (Euclidean) spaces of real valued continuously differentiable functions. We denote by $\mathbf{C}^k(\Omega, H\mathbb{H}^n)$ the set of all $\mathbf{C}^k$-sections of $H\mathbb{H}^n$ where the $\mathbf{C}^k$ regularity is understood as regularity between smooth manifolds. The notions of $\mathbf{C}^k_c(\Omega, H\mathbb{H}^n)$, $\mathbf{C}^\infty(\Omega, H\mathbb{H}^n)$ and $\mathbf{C}^\infty_c(\Omega, H\mathbb{H}^n)$ are defined analogously.

Definition 3.5. If Ω is an open subset of $\mathbb{H}^n$ and $\varphi = (\varphi_1,\dots,\varphi_{2n}) \in \mathbf{C}^1(\Omega, H\mathbb{H}^n)$ we define the *horizontal divergence* of φ as

$$\operatorname{div}_{\mathbb{H}}\varphi := \sum_{j=1}^{n} X_j\varphi_j + Y_j\varphi_{n+j}. \tag{3.4}$$

Observe that, since $X_j^* = -X_j,\ j = 1, \dots, 2n$, the horizontal divergence of φ coincides with the divergence $\mathrm{div}_X \varphi$ (see (1.6)) with $X = (X_1, \dots, X_{2n})$.

Finally, let us recall some of the definitions and results already presented, in a more general setting, in Section 1.1.2.

Definition 3.6. The $\mathbb{H}$-*perimeter* of $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$\|\partial E\|_{\mathbb{H}}(\Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \ : \varphi \in \mathbf{C}_c^1(\Omega, H\mathbb{H}^n),\ |\varphi(P)|_P \le 1 \ \forall P \in \mathbb{H}^n \right\}$$

We say that E is an $\mathbb{H}$-*Caccioppoli* set in Ω if $\|\partial E\|_{\mathbb{H}}(\Omega) < \infty$.

In the same way, and according to Section 1.1.2, one can define the space $BV_{\mathbb{H}}(\Omega)$ and the $\mathbb{H}$-variation of a L^1 function f.

Using Theorem 1.9 it is not difficult to show the following

Proposition 3.7. *If E is a Euclidean Lipschitz domain, then*

$$\|\partial E\|_{\mathbb{H}} = \sqrt{\textstyle\sum_{j=1}^{2n} \langle X_j, \mathbf{n}\rangle^2_{\mathbb{R}^{2n+1}}}\ \mathcal{H}^{2n} \llcorner \partial E,$$

where $\mathbf{n}$ is a Euclidean unit normal to ∂E. Moreover, any Euclidean Caccioppoli set in $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ is an $\mathbb{H}$-Caccioppoli set and the $\mathbb{H}$-perimeter measure $\|\partial E\|_{\mathbb{H}}$ is absolutely continuous with respect to the Euclidean surface measure on ∂E.

It is easy to show that Proposition 3.7 is strict, in the sense that there are $\mathbb{H}$-Caccioppoli sets that are not Caccioppoli sets in $\mathbb{R}^{2n+1}$; consider in fact the following

Example 3.8. Let $\{r_k\}$ be a strictly decreasing sequence of positive real numbers such that

$$\sum_{k\in\mathbb{N}} r_k^2 = \infty \qquad \text{and} \qquad \sum_{k\in\mathbb{N}} r_k^3 < \infty$$

and set

$$E_k := \{P \in \mathbb{H}^1 : r_{2k+1} \le \|P\|_\infty \le r_{2k}\} \qquad \text{and} \qquad E := \textstyle\bigcup_{k\in\mathbb{N}} E_k.$$

For any open neighbourhood of the origin Ω there is k_0 sufficiently large such that $\cup_{k\ge k_0} E_k \subset \Omega$ and so

$$\|\partial E\|_{Eucl}(\Omega) \ge \sum_{k\ge k_0} \mathcal{H}^2(\partial E_k) \simeq \sum_{k\ge k_0} [r_{2k}^2 + r_{2k+1}^2] = \infty,$$

i.e. E is not a Euclidean Caccioppoli set. On the other hand, taking into account Proposition 3.7, it will be sufficient to prove that

$$\sum_{k=1}^{\infty} \int_{\partial E_k} \sqrt{\textstyle\sum_{j=1}^{2n} \langle X_j, \mathbf{n}\rangle}\ d\mathcal{H}^2 < \infty$$

in order to obtain E being an $\mathbb{H}$-Caccioppoli set. An explicit computation gives

$$\int\limits_{\{\|(x,y,t)\|_\infty = r_k\}} \sqrt{\textstyle\sum_{j=1}^{2n} \langle X_j, \mathbf{n}\rangle}\, d\mathcal{H}^2 = \int\limits_{\{|(x,y)\leq r_k|\}} |(x,y)|\, dx\, dy \;+\; \int\limits_{\{|(x,y)|=r_k, |t|\leq r_k^2\}} d\mathcal{H}^2 \simeq r_k^3.$$

For $\mathbb{H}$-Caccioppoli sets the following divergence-type theorem holds (see [79])

Theorem 3.9. *Suppose that $\|\partial E\|_{\mathbb{H}}(\Omega) < \infty$; then there exists a $\|\partial E\|_{\mathbb{H}}$-measurable section ν_E of $H\mathbb{H}^n$ such that*

$$\begin{aligned} &|\nu_E(P)|_P = 1 \qquad \text{for } \|\partial E\|_{\mathbb{H}}\text{-a.e. } P \in \mathbb{H}^n; \\ &-\int_E \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} = \int_{\mathbb{H}^n} \langle \nu_E, \varphi\rangle \, d\|\partial E\|_{\mathbb{H}} \qquad \forall \varphi \in \mathbf{C}^1_c(\Omega; H\mathbb{H}^n). \end{aligned}$$

Here, the measurability of ν_E is meant in the sense that its coordinates $\nu_1, \dots, \nu_{2n}$ are $\|\partial E\|_{\mathbb{H}}$-measurable functions.

The function ν_E can be interpreted $\|\partial E\|_{\mathbb{H}}$-almost everywhere as a generalized "horizontal" inward normal to the set E.

Finally, as in Definition 1.10, we say that a set E is $\mathbb{H}$-*perimeter minimizing* in Ω if

$$\|\partial E\|_{\mathbb{H}}(\Omega) \leq \|\partial F\|_{\mathbb{H}}(\Omega)$$

for any measurable set $F \subset \mathbb{H}^n$ such that $E \Delta F \Subset \Omega$.

3.2 $\mathbf{C}^1_{\mathbb{H}}$ functions and Whitney Extension Theorem

Definition 3.10. We shall denote by $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions f in Ω such that the distributional derivative

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f). \tag{3.5}$$

is represented by a $\mathbf{C}^0$ section of $H\mathbb{H}^n$. Moreover, we shall denote by $\mathbf{C}^k_{\mathbb{H}}(\Omega, H\mathbb{H}^n)$ the set of all sections φ of $H\mathbb{H}^n$ whose canonical coordinates φ_j belong to $\mathbf{C}^k_{\mathbb{H}}(\Omega)$ for $j = 1, \dots, 2n$.

We stress that the inclusion $\mathbf{C}^1(\Omega) \subset \mathbf{C}^1_{\mathbb{H}}(\Omega)$ is strict; see for example [79], Remark 5.9. It is not difficult to prove (e.g. using an intrinsic convolution argument) that $\mathbf{C}^1_{\mathbb{H}}$ functions are Lipschitz continuous with respect to the distance d_∞.

We introduce the following notation: let $P = (x, y, t) \in \mathbb{H}^n$ and $P_0 \in \mathbb{H}^n$ be given, then we set

$$\pi_{P_0}(P) := \sum_{j=1}^{n} x_j X_j(P_0) + \sum_{j=1}^{n} y_j Y_j(P_0). \tag{3.6}$$

Observe that the map $P_0 \longmapsto \pi_{P_0}(P)$ is a smooth section of $H\mathbb{H}^n$ and so for $k : \mathbb{H}^n \to H\mathbb{H}^n \equiv \mathbb{R}^{2n}$ the scalar product of sections $\langle k(P_0), \pi_{P_0}(P)\rangle$ is well defined.

The following Lemma 3.11 will be a key tool in the proof of Theorem 4.17.

Lemma 3.11. *Let $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r_0))$. Then there exists a $C = C(P, r_0)$ such that, for each $Q \in U(P, r_0/2)$, $r \in]0, r_0/4[$ and $Q' \in U(Q, r)$ we have*

$$\frac{|f(Q') - f(Q) - \langle \nabla_{\mathbb{H}} f(Q), \pi_Q(Q^{-1}Q')\rangle|}{d_\infty(Q, Q')} \leq C \left\| \nabla_{\mathbb{H}} f - \nabla_{\mathbb{H}} f(Q) \right\|_{L^\infty(U(Q, 2d_\infty(Q,Q')))}.$$

Proof. Let us define

$$g(Q') := f(Q') - \langle \nabla_{\mathbb{H}} f(Q), \pi_Q(Q^{-1}Q')\rangle$$

and notice that $\nabla_{\mathbb{H}} g = \nabla_{\mathbb{H}} f - \nabla_{\mathbb{H}} f(Q)$. Since a Morrey type inequality

$$|g(Q') - g(Q)| \leq Cr \left(\fint_{U(Q,r)} |\nabla_{\mathbb{H}} g|^p \right)^{1/p} \quad \text{for all } Q' \in U(Q, r)$$

holds for a certain $C > 0$ and for $p \geq 1$ (see [116]), we have

$$\begin{aligned}
& \frac{|f(Q') - f(Q) - \langle \nabla_{\mathbb{H}} f(Q), \pi_Q(Q^{-1}Q')\rangle_Q|}{d_\infty(Q, Q')} \\
= \; & \frac{|g(Q') - g(Q)|}{d_\infty(Q, Q')} \\
\leq \; & 2C \left(\fint_{U(Q, 2d_\infty(Q,Q'))} |\nabla_{\mathbb{H}} g|^p \right)^{1/p} \\
= \; & 2C \left(\fint_{U(Q, 2d_\infty(Q,Q'))} |\nabla_{\mathbb{H}} f - \nabla_{\mathbb{H}}(Q)|^p \right)^{1/p},
\end{aligned}$$

whence the thesis follows. □

We end this Section by presenting Whitney's extension Theorem 3.12 for $\mathbf{C}^1_{\mathbb{H}}$ functions: we present here the proof given in [79], Theorem 6.8, which in turn closely follows the one in Euclidean spaces as can be found in Section 6.5 of [67].

Theorem 3.12. [Whitney Extension Theorem] *Let $F \subset \mathbb{H}^n$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to H\mathbb{H}^n$ be two continuous functions. We set*

$$R(Q,P) := \frac{f(Q) - f(P) - \langle k(P), \pi_P(P^{-1} \cdot Q)\rangle}{d_\infty(P,Q)}$$

and, if $K \subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(Q,P)| : P, Q \in K,\ 0 < d_\infty(P,Q) < \delta\}.$$

If $\rho_K(\delta) \to 0$ as $\delta \to 0$ for every compact set $K \subset F$, then there exist $\tilde{f} : \mathbb{H}^n \to \mathbb{R}$, $\tilde{f} \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ such that $\tilde{f}_{|F} \equiv f$ and $\nabla_{\mathbb{H}} \tilde{f}_{|F} \equiv k$.

Proof. Step 1. Let U be the open set $\mathbb{H}^n \setminus F$, and set

$$r(P) := \frac{1}{20} \min\{1, d_\infty(P,F)\}, \qquad P \in \mathbb{H}^n$$

where we have set $d_\infty(P,F) := \inf\{d_\infty(P,Q) : Q \in F\}$. By Vitali's covering theorem (see e.g.[3], Teorema 2.1.6) there exist a countable set $C \subset U$ such that

$$U = \bigcup_{P \in C} U(P, 5r(P))$$

and all the balls $U(P, 5r(P))$ are pairwise disjoint. For any $Q \in U$ we set

$$C_Q := \{P \in C : U(Q, r(Q)) \cap U(P, r(P)) \neq \emptyset\}.$$

Step 2. Let us prove that $\#C_Q \leq (129)^{2n+2}$ and $1/3 \leq r(Q)/r(P) \leq 3$ for any $P \in C_Q$. In fact, if $P \in C_Q$ one has

$$|r(P) - r(Q)| \leq \frac{1}{20} d_\infty(P,Q) \leq \frac{1}{20}(10r(P) + 10r(Q)) = \frac{1}{2}(r(P) + r(Q)).$$

Hence $r(P) \leq 3r(Q)$ and $r(Q) \leq 3r(P)$, whence the upper and lower bounds on $r(Q)/r(P)$ follow.

In addition, we have

$$d_\infty(P,Q) + r(P) \leq 10(r(P) + r(Q)) + r(P) \leq 43 r(Q)$$

and so $U(P, r(P)) \subset U(Q, r(Q))$. Since the balls $\{U(P, r(P)) : P \in C_Q\}$ are disjoint and contained in $U(Q, 43r(Q))$ and $r(P) \leq r(Q)/3$ we have

$$\#C_Q\, \mathcal{L}^{2n+1}(U(0,1)) \left(\frac{r(Q)}{3}\right)^{2n+2} \leq \mathcal{L}^{2n+1}(U(0,1))\, (43 r(Q))^{2n+2}$$

whence the claim $\#C_Q \le (129)^{2n+2}$.

Step 3. Now let $\mu : \mathbb{R} \to \mathbb{R}$ be a smooth nonincreasing function such that

$$0 \le \mu \le 1, \qquad \mu(t) = 1 \text{ for } t \le 1, \qquad \mu(t) = 0 \text{ for } t \ge 2^{3/4}.$$

For any $P \in C$ define

$$g_P(Q) := \mu\left(\frac{d_K(P,Q)}{5r(P)}\right);$$

here d_K is the regularized distance defined by $d_K(P', P'') := \|P'^{-1} \cdot P''\|_K$, where $\|\cdot\|_K$ is the homogeneous gauge

$$\|(x,y,t)\|_K = \left((|x|^2+|y|^2)^2+t^2\right)^{1/4}.$$

Being a homogeneous distance, d_K is globally equivalent to d_∞ and in particular one has

$$d_\infty(P', P'') \le d_K(P', P'') \le 2^{1/4} d_\infty(P', P'').$$

It follows that $g_P \in \mathbf{C}^\infty(\mathbb{H}^n)$, $0 \le g_P \le 1$ and

$$\begin{aligned} &g_P \equiv 1 \quad \text{on } U(P, 5r(P)) \\ &g_P \equiv 0 \quad \text{on } \mathbb{H}^n \setminus U(P, 10r(P)). \end{aligned} \tag{3.7}$$

Moreover there is a constant $M > 0$ such that $|X_j g_P| \le M/r(P)$ for all $j = 1, \dots, 2n$; it follows that $|X_j g_P(Q)| \le 3M/r(Q)$ if $P \in C_Q$. Observing that, thanks to (3.7), $g_P(Q) = 0$ if $P \notin C_Q$, one has

$$|X_j g_P(Q)| \le 3M/r(Q) \qquad \text{for all } Q \in \mathbb{H}^n, j = 1, \dots, 2n. \tag{3.8}$$

Define $\sigma(Q) = \sum_{P \in C} g_P(Q)$, $Q \in \mathbb{H}^n$; again by (3.7), one obtains that $g_P \equiv 0$ on $U(Q, 10r(Q))$ whenever $P \notin C_Q$, and so

$$\sigma(Q') = \sum_{P \in C_Q} g_P(Q') \qquad \text{if } Q' \in U(Q, 10r(Q)).$$

Observe that $\sigma \ge 1$ on U; in fact, for any $Q \in U$ there exists $\overline{P}$ such that $Q \in U(\overline{P}, 5r(\overline{P}))$, whence $\sigma(Q) \ge g_{\overline{P}}(Q) = 1$. Moreover, since $\#C_Q < (129)^{2n+2}$ and because of (3.8), we have $\sigma \in \mathbf{C}^\infty(U)$ and there is a constant $M' > 0$ such that

$$|X_j \sigma(Q)| \le \frac{M'}{r(Q)} \qquad \text{for all } Q \in U, j = 1, \dots, 2n.$$

Now we define a partition of the unity subordinate to the covering $\{U(P, 10r(P)) : P \in C)\}$ as

$$v_P(Q) := \frac{g_P(Q)}{\sigma(Q)}.$$

Notice that $v_P \in \mathbf{C}^\infty$ and $X_j v_P = \frac{X_j g_P}{\sigma} - \frac{g_P X_j \sigma}{\sigma^2}$ and so there exists $M'' > 0$ such that

$$\sum_{P\in C} v_P(Q) = 1, \quad \sum_{P\in C} X_j v_P(Q) = 0 \quad \text{and } |\nabla_{\mathbb{H}} v_P(Q)| \leq \frac{M''}{r(Q)} \tag{3.9}$$

for any $Q \in U$.

Step 4. For any $P \in C$ choose $Q_P \in F$ such that $d_\infty(P, Q_P) = d_\infty(P, F)$ and define $\tilde{f} : \mathbb{H}^n \to \mathbb{R}$ as follows:

$$\tilde{f}(Q) := \begin{cases} f(Q) & \text{if } Q \in F \\ \sum_{P\in C} v_P(Q)[f(Q_P) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1} \cdot Q)\rangle] & \text{if } Q \in U. \end{cases}$$

Notice that $\tilde{f} \in \mathbf{C}^\infty(U)$ and that

$$\nabla_{\mathbb{H}} \tilde{f}(Q) = \sum_{P\in C} \left\{ \left[f(Q_P) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1} \cdot Q)\rangle \right] \nabla_{\mathbb{H}} v_P(Q) + v_P(Q) k(Q_P) \right\}$$

on U.

Step 5. We claim that $\nabla_{\mathbb{H}} \tilde{f} \equiv k$ on F. In fact, let $Q \in F$ and set H to be the compact $F \cap \overline{U(Q,1)}$. Define

$$\begin{aligned} \psi(\delta) := & \sup \{ |R(P, P')| : P, P' \in H, 0 < d_\infty(P, P') \leq \delta \} \\ & + \sup \{ |k(P) - k(P')| : P, P' \in H, d_\infty(P, P') \leq \delta \}. \end{aligned}$$

By the continuity of k on F and the hypothesis $\rho_H(\delta) \to 0$, we have

$$\psi(\delta) \to 0 \qquad \text{as } \delta \to 0. \tag{3.10}$$

If $Q' \in H$ one has

$$\begin{aligned} \left| \tilde{f}(Q') - \tilde{f}(Q) - \langle k(Q), \pi_Q(Q^{-1} \cdot Q')\rangle \right| &= \left| f(Q') - f(Q) - \langle k(Q), \pi_Q(Q^{-1} \cdot Q')\rangle \right| \\ &= |R(Q', Q)||\pi_Q(Q^{-1} \cdot Q')| \\ &\leq \psi(|d_\infty(Q, Q')|)|d_\infty(Q, Q')| \end{aligned} \tag{3.11}$$

and $|k(Q') - k(Q)| \leq \psi(|d_\infty(Q, Q')|)$.

Instead, if $Q' \in U$ one has

$$\begin{aligned}
&\left|\tilde f(Q') - \tilde f(Q) - \langle k(Q), \pi_Q(Q^{-1}\cdot Q')\rangle\right| \\
= &\left|\tilde f(Q') - f(Q) - \langle k(Q), \pi_Q(Q^{-1}\cdot Q')\rangle\right| \\
\le &\sum_{P\in C_{Q'}} v_P(Q')\big|f(Q_P) - f(Q) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1}\cdot Q')\rangle - \langle k(Q), \pi_Q(Q^{-1}\cdot Q')\rangle\big| \\
\le &\sum_{P\in C_{Q'}} v_P(Q')\big|f(Q_P) - f(Q) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1}\cdot Q)\rangle\big| + \\
&+ \sum_{P\in C_{Q'}} v_P(Q')\big|\langle k(Q_P) - k(Q), \pi_{Q_P}(Q^{-1}\cdot Q')\rangle\big|; \qquad (3.12)
\end{aligned}$$

if moreover one supposes $d_\infty(Q', Q) \le 1/6$, then $r(Q') \le d_\infty(Q', Q)/20$ and then for any $P \in C_{Q'}$ we obtain

$$\begin{aligned}
d_\infty(Q, Q_P) &\le d_\infty(Q, P) + d_\infty(P, Q_P) \le 2d_\infty(Q, P) \\
&\le 2(d_\infty(Q, Q') + d_\infty(Q', P)) \le 2(d_\infty(Q', Q) + 10(r(Q') + r(P))) \\
&\le 2(d_\infty(Q', Q) + 40r(Q')) \\
&\le 6\, d_\infty(Q', Q).
\end{aligned}$$

Therefore by (3.12) and Step 2 we get

$$\left|\tilde f(Q') - \tilde f(Q) - \langle k(Q), \pi_Q(Q^{-1}\cdot Q')\rangle\right| \le L\psi(6|d_\infty(Q', Q)|)|d_\infty(Q', Q)|$$

which, together with (3.11), gives

$$\left|\tilde f(Q') - \tilde f(Q) - \langle k(Q), \pi_Q(Q^{-1}\cdot Q')\rangle\right| = o(|d_\infty(Q', Q)|),$$

whence our claim follows.

Step 6. We conclude by proving that $\tilde f \in \mathbf{C}^1_{\mathbb{H}}$. We fix $Q \in F$ and $Q' \in \mathbb{H}^n$ with $d_\infty(Q, Q') \le 1/6$. If $Q' \in F$ then

$$|\nabla_{\mathbb{H}}\tilde f(Q') - \nabla_{\mathbb{H}}\tilde f(Q)| = |k(Q') - k(Q)| \le \psi(d_\infty(Q', Q))$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is defined as in Step 5 and depends only on H, i.e. on Q and F. If $Q' \in U$ we choose $\overline{Q} \in F$ such that $d_\infty(Q', \overline{Q}) = d_\infty(Q', F)$, whence

$$\begin{aligned}
|\nabla_{\mathbb{H}}\tilde f(Q') - \nabla_{\mathbb{H}}\tilde f(Q)| &= |\nabla_{\mathbb{H}}\tilde f(Q') - k(Q)| \\
&\le |\nabla_{\mathbb{H}}\tilde f(Q') - k(\overline{Q})| + |k(\overline{Q}) - k(Q)| \\
&\le |\nabla_{\mathbb{H}}\tilde f(Q') - k(\overline{Q})| + \psi(2d_\infty(Q, Q')) \qquad (3.13)
\end{aligned}$$

where in the last inequality we used the fact that

$$d_\infty(Q,\overline{Q}) \le d_\infty(Q,Q') + d_\infty(Q',\overline{Q}) \le 2d_\infty(Q,Q').$$

Thus we have to estimate the first addend in the right hand side of (3.13); recalling (3.9) we get

$$\begin{aligned}
&|\nabla_{\mathbb{H}}\tilde{f}(Q') - k(\overline{Q})| \\
= &\left|\sum_{P\in C_{Q'}} \big[f(Q_P) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1}\cdot Q')\rangle\big]\nabla_{\mathbb{H}} v_P(Q') + \big[k(Q_P) - k(\overline{Q})\big]v_P(Q')\right| \\
\le &\left|\sum_{P\in C_{Q'}} \big[f(Q_P) - f(\overline{Q}) + \langle k(Q_P), \pi_{Q_P}(Q_P^{-1}\cdot \overline{Q})\rangle\big]\nabla_{\mathbb{H}} v_P(Q')\right| + \\
&+\left|\sum_{P\in C_{Q'}} \langle k(Q_P) - k(\overline{Q}), \pi_{Q_P}(\overline{Q}^{-1}\cdot Q')\rangle\nabla_{\mathbb{H}} v_P(Q')\right| + \\
&+\left|\sum_{P\in C_{Q'}} \big[k(Q_P) - k(\overline{Q})\big]v_P(Q')\right| \\
\le &\frac{M''}{r(Q')}\sum_{P\in C_{Q'}} \psi(d_\infty(\overline{Q},Q_P))d_\infty(\overline{Q},Q_P) + \\
&+\frac{M''}{r(Q')}\sum_{P\in C_{Q'}} \psi(d_\infty(\overline{Q},Q_P))d_\infty(Q',\overline{Q}) + \sum_{P\in C_{Q'}} \psi(d_\infty(\overline{Q},Q_P)) \qquad (3.14)
\end{aligned}$$

where, in the last inequality, the estimate on the first summation comes from an argument analogous to the one in (3.11). Since $d_\infty(Q',\overline{Q}) \le d_\infty(Q',Q) \le 1/6$, one has $r(Q') = d_\infty(Q',\overline{Q})/20 \le 1/120$ and so

$$r(P) \le 3r(Q') \le 1/40 < 1/20$$

for all $P \in C_{Q'}$, whence $r(P) = d_\infty(P,Q_P)/20$ for such a P. Therefore

$$\begin{aligned}
d_\infty(\overline{Q},Q_P) &\le d_\infty(\overline{Q},Q') + d_\infty(Q',P) + d_\infty(P,Q_P) \\
&\le 20r(Q') + 10(r(Q') + r(P)) + 20r(P) \\
&\le 120r(Q') = 6d_\infty(Q',\overline{Q}) \\
&\le d_\infty(Q',Q) \qquad (3.15)
\end{aligned}$$

holds for any $P \in C_{Q'}$. Combining (3.15) with (3.14) we obtain we get

$$|\nabla_{\mathbb{H}}\tilde{f}(Q') - k(\overline{Q})| \le \tilde{M}\psi(6d_\infty(Q',Q))$$

which, together with (3.13), gives

$$|\nabla_{\mathbb{H}}\tilde{f}(Q') - \nabla_{\mathbb{H}}\tilde{f}(Q)| \leq \tilde{M}'\psi(6d_\infty(Q', Q)).$$

and this completes the proof. □

3.3 $\mathbb{H}$-regular surfaces and Implicit Function Theorem

Definition 3.13. We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular hypersurface* if for every $P \in S$ there exist an open ball $U(P, r)$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r))$ such that $\nabla_{\mathbb{H}} f \neq 0$ and

$$S \cap U(P, r) = \{Q \in U(P, r) : f(Q) = 0\}.$$

We will denote with $\nu_S(P)$ the *horizontal normal* to S at a point $P \in S$, i.e. the unit vector

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|_P}.$$

We will see later (see Corollay 3.17) that ν_S is continuous and well defined, i.e. it does not depend on the particular choice of f.

If $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface and $P \in S$, we define the *tangent group* $T^g_{\mathbb{H}}S(P)$ to S at P as

$$T^g_{\mathbb{H}}S(P) := \{Q \in \mathbb{H}^n : \langle \nabla_{\mathbb{H}}(f \circ \ell_P)(0), \pi_0(Q)\rangle = 0\},$$

where f is any $\mathbf{C}^1_{\mathbb{H}}$ function defining S near P. Again, this definition does not depend on the choice of f (one could also define $T^g_{\mathbb{H}}S(P)$ as the set $\{Q \in \mathbb{H}^n : \langle \nu_{P^{-1}\cdot S}(0), \pi_0(Q)\rangle = 0\}$), and it is easy to check that $T^g_{\mathbb{H}}S(P)$ is a maximal subgroup of $\mathbb{H}^n$. The *tangent plane* to S at P is then the lateral

$$T_{\mathbb{H}}S(P) := P \cdot T^g_{\mathbb{H}}S(P).$$

Remark 3.14. We stress the fact that the classes of Euclidean regular hypersurfaces and $\mathbb{H}$-regular surfaces are disjoint. In fact, it is not difficult to check that

$$S := \{(x, y, t) \in \mathbb{H}^1 \ : \ f(x, y, t) = x - \sqrt{x^4 + y^4 + t^2} = 0\}$$

is $\mathbb{H}$-regular in a neighbourhood of 0 (in fact $f \in \mathbf{C}^1_{\mathbb{H}}$ and $X_1 f(0) = 1$) but not $\mathbf{C}^1$ regular at the origin. One could produce even worse situations: for example, in [106] an $\mathbb{H}$-regular surface of Eucliden Hausdorff dimension 2.5 is provided.

On the other hand, the Euclidean plane $O := \{(x, y, t) \in \mathbb{H}^1 : t = 0\}$ is Euclidean regular but not $\mathbb{H}$-regular at the origin: this can be easily proved observing that $O \setminus \{0\}$ is $\mathbb{H}$-regular and its horizontal normal

$$\nu_{O\setminus\{0\}} = \frac{(y, -x)}{\sqrt{x^2 + y^2}}$$

cannot be extended continuously at the origin. However, it is straightforward that every Euclidean $\mathbf{C}^1$ surface S is also $\mathbb{H}$-regular provided it has no characteristic point, (a point P is said *characteristic* if the Euclidean tangent plane at S coincides with the horizontal fiber $H_P\mathbb{H}^n$).

The main result of this Section, Theorem 3.16, is an Implicit Function Theorem for $\mathbb{H}$-regular surfaces: as in the Euclidean setting we can (locally) see $\mathbf{C}^1$ regular surfaces as graphs of $\mathbf{C}^1$ functions defined on a hyperplane, in the Heisenberg group $\mathbb{H}$-regular surfaces are (locally and in an intrinsic sense) "graphs" of functions (whose regularity will be studied in Chapter 4). Here the role of Euclidean hyperplanes (i.e. of maximal subgroups of $\mathbb{R}^n$) is played by sets of the type

$$V_w = \left\{ Q \in \mathbb{H}^n : \left\langle \sum_{j=1}^{2n} w_j X_j(0), \pi_0(Q) \right\rangle = 0 \right\} \tag{3.16}$$

for some $w \in \mathbb{R}^{2n}$: observe that the V_w's constitute all the maximal subgroups of $\mathbb{H}^n$ and that, for an $\mathbb{H}$-regular surface, one has $T^g_{\mathbb{H}} S(P) = V_{\nu_{P^{-1}\cdot S}(0)}$.

In what follows we will focus our attention on intrinsic graphs over the hyperplane

$$V_1 := V_{(1,0,\dots,0)} = \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\};$$

this will not be restrictive, see also Remark 4.7. We can identify V_1 with $\mathbb{R}^{2n}$ through the map

$$\begin{aligned} \iota \quad : \quad & \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau \longrightarrow V_1 \subset \mathbb{H}^n \\ & (\eta, \tau) \longmapsto (0, \eta, \tau) \end{aligned} \tag{3.17}$$

if $n = 1$, while we set

$$\begin{aligned} \iota \quad : \quad & \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_{v=(v_2,\dots,v_n,v_{n+2},\dots,v_{2n})} \times \mathbb{R}_\tau \longrightarrow V_1 \subset \mathbb{H}^n \\ & (\eta, v, \tau) \longmapsto (0, v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau) \end{aligned} \tag{3.18}$$

if $n \geq 2$. We stress the strange choice for the enumeration of the components of v, which however is justified by the structure of ι. Finally, for $s \in \mathbb{R}$ we use the notation $se_1 := (s, 0, \dots, 0) \in \mathbb{H}^n$.

Definition 3.15. Let ω be an open subset of $\mathbb{R}^{2n}$, and let ϕ be a real function defined on ω. The *intrinsic X_1-graph* of ϕ is the map

$$\begin{aligned} \Phi \ &: \ \omega \to \mathbb{H}^n \\ &\quad A \longmapsto \iota(A) \cdot \phi(A) e_1. \end{aligned} \tag{3.19}$$

In the following, we will make no distinction between an intrinsic X_1-graph and its image, saying that $\Phi(\omega)$ is the intrinsic graph of ϕ (also, we will often omit the X_1-prefix). In coordinates, we have

$$\Phi(\eta, v, \tau) = \big(\phi(\eta, v, \tau), v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau + 2\eta\phi(\eta, v, \tau)\big) \tag{3.20}$$

if $n \geq 2$, and a similar formula for $n = 1$.

One could also interpret the notion of intrinsic X_1-graph in this way (see Figure 3.1): start from the point $\iota(A) \in V_1 \subset \mathbb{H}^n$ and follow the flux of the field X_1 (which is a sort of "normal direction" to V_1) for a time $\phi(A)$, then the point one reaches is exactly $\Phi(A)$. Observe that this is exactly what happens for Euclidean graphs: one starts from a point of the hyperplane and follows the flux of the normal for a length given by the function itself, thus reaching the graph.

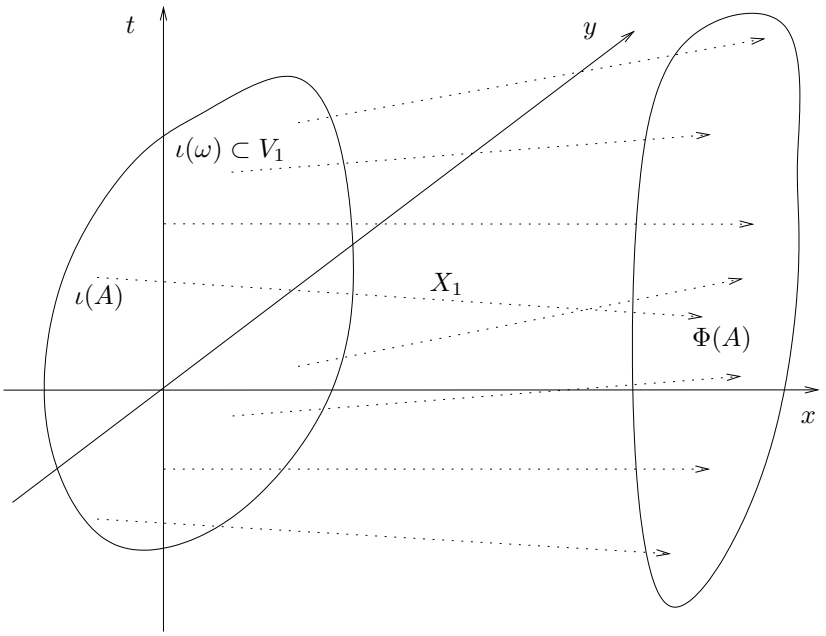

Figure 3.1: Intrinsic graphs.

Notice that a point $P = (x, y, t) \in \mathbb{H}^n$ can be written in a unique way in the form $\iota(A) \cdot se_1$ for some $A \in \mathbb{R}^{2n}$, $s \in \mathbb{R}$ which can be easily computed since $s = x_1$

and $A = (\eta, v, \tau)$ (a similar formula holds in the case $n = 1$) with

$$\eta = y_1, \quad v = (x_2, \dots, x_n, y_2, \dots, y_n), \quad \tau = t - 2x_1 y_1. \tag{3.21}$$

We will write $\pi_1(P)$ to denote the "projection" of P on $\mathbb{R}^{2n} \equiv V_1$ defined, according to (3.21), by

$$\pi_1(x, y, t) := \big(y_1, (x_2, \dots, x_n, y_2, \dots, y_n), t - 2x_1 y_1\big). \tag{3.22}$$

We presently have all the tools to state the main result of this Section, which has been proved by Franchi, Serapioni and Serra Cassano in [79], Theorem 6.5:

Theorem 3.16. [Implicit Function Theorem] *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f(0) > 0$ and $f(0) = 0$. Let*

$$E := \{P \in \Omega : f(P) < 0\} \quad \text{and} \quad S := \{P \in \Omega : f(P) = 0\};$$

then there exist $\delta, h > 0$ such that, if we put $\omega := \,]-\delta, \delta[^{2n-1} \times \,]-\delta^2, \delta^2[\,\subset \mathbb{R}^{2n}$, $J := \{se_1 \in \mathbb{H}^n : s \in]-h, h[\,\}$ and $\mathcal{U} := \iota(\omega) \cdot J$, we have $\mathcal{U} \Subset \Omega$ and

E has finite $\mathbb{H}$-perimeter in $\mathcal{U}$;
$\partial E \cap \mathcal{U} = S \cap \mathcal{U}$;
$\|\partial E\|_{\mathbb{H}} \llcorner \mathcal{U}$ is concentrated on S and $\nu_E = \nu_S$ $\|\partial E\|_{\mathbb{H}}$-a.e. on $\mathcal{U}$.

Moreover there exists a unique continuous function $\phi : \omega \to]-h, h[$ such that $S \cap \mathcal{U}$ is the X_1-graph of ϕ, and the $\mathbb{H}$-perimeter has the integral representation

$$\|\partial E\|_{\mathbb{H}}(\mathcal{U}) = \int_\omega \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\Phi(A))\, d\mathcal{L}^{2n}(A), \tag{3.23}$$

where Φ depends on ϕ as in (3.19).

Proof. We divide the proof of the Theorem in several steps.

Step 1. We start by proving the existence of the continuous parametrization ϕ. Choose $\delta, h > 0$ small enough to have $X_1 f > 0$ on $\overline{\mathcal{U}} \subset \Omega$, where $\mathcal{U}, \omega$ and J are defined as in the statement of the Theorem; take a convolution kernel $\zeta \in \mathbf{C}^\infty_c(\mathbb{H}^n)$ satsfying (1.27) and, as in (1.29), set

$$f_\epsilon(P) := (\zeta_\epsilon \star f)(P). \tag{3.24}$$

By Propositions 1.28 and 1.29, the maps f_ϵ are smooth and for any $j = 1, \dots, n$ one has

$$X_j f_\epsilon = (\zeta_\epsilon \star X_j f) \to X_j f, \qquad Y_j f_\epsilon = (\zeta_\epsilon \star Y_j f) \to Y_j f \qquad \text{as } \epsilon \to 0$$

uniformly on $\overline{\mathcal{U}}$. In particular, for any $A \in \omega$ the map $s \mapsto f_\epsilon(\iota(A) \cdot se_1)$ is differentiable in $]-h, h[$ and an easy computation gives

$$\frac{d}{ds} f_\epsilon(\iota(A) \cdot se_1) = (X_1 f_\epsilon)(\iota(A) \cdot se_1)$$

which converges to $(X_1 f)(\iota(A) \cdot se_1)$ uniformly in s. Therefore also $s \mapsto f(\iota(A) \cdot se_1)$ is differentiable for $|s| < h$ with

$$\frac{d}{ds} f(\iota(A) \cdot se_1) = (X_1 f)(\iota(A) \cdot se_1) > 0. \tag{3.25}$$

Since $f(\iota(A) \cdot se_1) = 0$ for $A = 0$ and $s = 0$ we have $f(\iota(0) \cdot (-he_1)) < 0 < f(\iota(0) \cdot he_1)$, and by continuity (choosing a smaller δ if necessary) one has

$$f(\iota(A) \cdot (-h\, e_1)) < 0 < f(\iota(A) \cdot he_1)$$

for any $A \in \omega$. The existence of an $s \in]-h, h[$ with $f(\iota(A) \cdot se_1) = 0$ then follows from a continuity argument, while its uniqueness is a consequence of (3.25): this gives the implicitely defined function $\phi : \omega \to \mathbb{R}$.
In order to show that ϕ is continuous, it is sufficient to prove that, if $A^k \in \omega$ are such that $A^k \to A \in \omega$ as $k \to \infty$, then there is a subsequence A^{k_l} such that $\phi(A^{k_l}) \to \phi(A)$. But one can easily find a subsequence such that $\phi(A^{k_l}) \to s_0 \in [-h, h]$, and so by the continuity of f and ι we have

$$0 = f(\iota(A^{k_l}) \cdot \phi(A^{k_l}) e_1) \to f(\iota(A) \cdot s_0 e_1),$$

whence $s_0 = \phi(A)$ and the claim is proved.

Step 2. Let us prove that $\partial E \cap \mathcal{U} = S \cap \mathcal{U}$. The continuity of f immediately yields that $\partial E \subset S$; on the other hand, for any given $P = (x, y, t) \in S \cap \mathcal{U}$ let us set $P = \iota(A) \cdot x_1 e_1$, where $A = \pi_1(P) \in \mathbb{R}^2$. As in Step 1, the function $s \mapsto f(\iota(A) \cdot se_1)$ is strictly increasing and vanishes for $s = x_1 = \phi(A)$, then there is a sequence $s_k \uparrow x_1$ such that

$$f(\iota(A) \cdot s_k e_1) < 0$$

for all k. Since $\iota(A) \cdot s_k e_1 \to P$ we infer $P \in \partial E$.

Step 3. We want to prove now that E has finite $\mathbb{H}$-perimeter in $\mathcal{U}$; this will be done, thanks to Proposition 1.5, by constructing a sequence $\{h_\epsilon\}_\epsilon \subset BV_{\mathbb{H}}(\mathcal{U})$ with equibounded $\mathbb{H}$-variation and such that $h_\epsilon \to \chi_E$ in $L^1(\mathcal{U})$.

Again let f_ϵ be defined as in (3.24) and consider the maps

$$\begin{aligned} g, g_\epsilon \ : \ & \overline{\omega} \times [-h, h] \longrightarrow \mathbb{R} \\ & g(A, s) := f(\iota(A) \cdot se_1) \\ & g_\epsilon(A, s) := f_\epsilon(\iota(A) \cdot se_1). \end{aligned}$$

As before, one has $\frac{\partial g_\epsilon}{\partial s}(A,s) \to (X_1 f)(\iota(A)\cdot se_1)$ uniformly on $\overline{\omega}\times[-h,h]$; therefore there exist constants $\mu, \epsilon_0 > 0$ such that

$$g_\epsilon(\cdot,-h) < 0 < g_\epsilon(\cdot,h) \text{ on } \overline{\omega} \qquad \text{and} \qquad \frac{\partial g_\epsilon}{\partial s} \geq \mu \text{ on } \overline{\omega}\times[-h,h]$$

for any $0<\epsilon<\epsilon_0$, and applying the classical implicit function theorem we obtain smooth functions $\phi_\epsilon : \overline{\omega} \to]-h,h[$ such that $g_\epsilon(A,\phi_\epsilon(A)) = f_\epsilon(\iota(A)\cdot\phi_\epsilon(A)e_1) \equiv 0$. Then for $0<\epsilon<\epsilon_0$ we set

$$E_\epsilon := \{P\in\mathcal{U} : P = \iota(A)\cdot se_1 \text{ for some } A\in\omega, -h < s < \phi_\epsilon(A)\}$$

and $h_\epsilon := \chi_{E_\epsilon}$; observe also that E_ϵ coincides with

$$\big\{\iota(A)\cdot se_1 \in \mathcal{U} : (A,s)\in\omega\times]-h,h[\,,\ g_\epsilon(A,s)<0\big\} = \{P\in\mathcal{U} : f_\epsilon(P)<0\}.$$

We start by proving that $h_\epsilon \to \chi_E$ in $L^1(\mathcal{U})$ as $\epsilon\to 0$: by Lebesgue convergence theorem it will be sufficient to show that $\chi_{E_\epsilon}\to\chi_E$ pointwise a.e. Observe that, since $f_\epsilon \to f$, if $P\in E$ (i.e. $f(P)<0$) for small ϵ one has $f_\epsilon(P)<0$, whence $\chi_{E_\epsilon}(P) = 1 = \chi_E(P)$ definitively; the same argument can be applied whenever $f(P)>0$ (obtaining $\chi_{E_\epsilon}(P)=0=\chi_E(P)$ definitively) and so it will be enough to prove that

$$|\{P\in\mathcal{U} : f(P)=0\}| = |S\cap\mathcal{U}| = 0.$$

Setting $\tilde{S}_n := \{P\in\mathcal{U} : P = \iota(A)\cdot se_1 \text{ for } A\in\omega \text{ and } |\phi(A)-s|<1/n\}$ and observing that the Jacobian matrix of the map

$$\mathbb{R}^{2n+1} \supset \omega\times]-h,h[\ni (A,s) \longmapsto (\iota(A)\cdot se_1)\in\mathbb{H}^n\equiv\mathbb{R}^{2n+1}$$

has determinant equal to 1, we obtain that $|\tilde{S}_n| \leq 2|\omega|/n$, whence

$$|S\cap\mathcal{U}| = |\bigcap_n \tilde{S}_n| = 0.$$

Let us show now that the functions h_ϵ have equibounded $\mathbb{H}$-variation in $\mathcal{U}$, i.e. that the sets E_ϵ have equibounded $\mathbb{H}$-perimeter in $\mathcal{U}$. Notice that ∂E_ϵ is Euclidean regular and so for any $\varphi\in\mathbf{C}^1_c(\mathcal{U},H\mathbb{H}^n)$ with $|\varphi|\leq 1$ we have

$$\begin{aligned}\int_{\mathcal{U}} h_\epsilon \mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} &= \int_{\mathcal{U}\cap E_\epsilon} h_\epsilon \mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} \\ &= \int_{\mathcal{U}\cap\partial E_\epsilon} \langle\varphi, \mathbf{n}^\epsilon_{\mathbb{H}}\rangle\, d\mathcal{H}^{2n} \leq \int_{\mathcal{U}\cap\partial E_\epsilon} |\mathbf{n}^\epsilon_{\mathbb{H}}|\, d\mathcal{H}^{2n} \qquad (3.26)\end{aligned}$$

and so it is sufficient to give a bound, independent of ϵ, on the right hand side of (3.26); in the previous formula, for $P \in \partial E_\epsilon$ we have set $\mathbf{n}^\epsilon_{\mathbb{H}}(P)$ to be the section of $H\mathbb{H}^n$ given by

$$\big(\langle \mathbf{n}^\epsilon(P), X_1(P)\rangle_{\mathbb{R}^{2n+1}}, \dots, \langle \mathbf{n}^\epsilon(P), X_{2n}(P)\rangle_{\mathbb{R}^{2n+1}} \big),$$

where $n^\epsilon(P)$ is the Euclidean unit normal to ∂E_ϵ at P. Observe that (3.26) could have been deduced also directly from Theorem 1.9.

Remember that a parametrization of $\mathcal{U} \cap \partial E_\epsilon$ is given by

$$\begin{aligned} \Phi_\epsilon \ : \ & \omega \longrightarrow \mathbb{H}^n \\ & (\eta, v, \tau) \longmapsto (\phi_\epsilon(\eta, v, \tau), v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau + 2\eta\phi_\epsilon(\eta, v, \tau)); \end{aligned}$$

from now on we suppose $n \geq 2$, since the case $n = 1$ is completely analogous. By area formula (see [69]) we infer

$$\int_{\mathcal{U}\cap\partial E_\epsilon} |\mathbf{n}^\epsilon_{\mathbb{H}}| \, d\mathcal{H}^{2n} = \int_\omega |\mathbf{n}^\epsilon_{\mathbb{H}} \circ \Phi_\epsilon| \, J\Phi_\epsilon \, d\mathcal{L}^{2n} \tag{3.27}$$

where $J\Phi_\epsilon$ is the Jacobian of $\nabla\Phi_\epsilon$. Explicitly, the Jacobi matrix $\nabla\Phi_\epsilon$ is

$$\begin{pmatrix} \partial_\eta\phi_\epsilon & \partial_{v_2}\phi_\epsilon & \cdots & \partial_{v_n}\phi_\epsilon & \partial_{v_{n+2}}\phi_\epsilon & \cdots & \partial_{v_{2n}}\phi_\epsilon & \partial_\tau\phi_\epsilon \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 2\phi_\epsilon + 2\eta\partial_\eta\phi_\epsilon & 2\eta\partial_{v_2}\phi_\epsilon & \cdots & 2\eta\partial_{v_n}\phi_\epsilon & 2\eta\partial_{v_{n+2}}\phi_\epsilon & \cdots & 2\eta\partial_{v_{2n}}\phi_\epsilon & 1 + 2\eta\partial_\tau\phi_\epsilon \end{pmatrix}$$

and $J\Phi^2_\epsilon$ is the sum of the squares of all the deteminants of $2n \times 2n$ minors of $\nabla J\Phi_\epsilon$; a direct computation gives

$$J{\Phi_\epsilon}^2 = (1 + 2\eta\partial_\tau\phi_\epsilon)^2 + \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} (\partial_{v_j}\phi_\epsilon)^2 + (\partial_\eta\phi_\epsilon - 2\phi_\epsilon\partial_\tau\phi_\epsilon)^2 + (\partial_\tau\phi_\epsilon)^2.$$

Notice that $\mathcal{U} \cap \partial E_\epsilon$ can be seen also as the zero level of the regular map f'_ϵ

$$(x, y, t) \xmapsto{f'_\epsilon} x_1 - \phi_\epsilon(\pi_1(x, y, t)) = x_1 - \phi_\epsilon(y_1, (x_2, \dots, x_n, y_2, \dots, y_n), t - 2x_1y_1)$$

and so the Euclidean unit normal $\mathbf{n}^\epsilon(x,y,t)$ to ∂E_ϵ is given by $\frac{\nabla f'_\epsilon}{|\nabla f'_\epsilon|}$. By explicit computation one gets

$$\begin{aligned}\nabla f'_\epsilon(x,y,t) = \big(&1+2\eta\partial_\tau\phi_\epsilon,\ -\partial_{v_2}\phi_\epsilon,\dots,\ -\partial_{v_n}\phi_\epsilon,\\ &-\partial_\eta\phi_\epsilon+2\phi\,\partial_\tau\phi_\epsilon,\ -\partial_{v_{n+2}}\phi_\epsilon,\dots,\ -\partial_{v_{2n}}\phi_\epsilon,\ -\partial_\tau\phi_\epsilon\big)(\pi_1(x,y,t)),\end{aligned}$$

for all $(x,y,t)\in\mathcal{U}\cap\partial E_\epsilon$, where we have used the fact that $z_1=\phi_\epsilon(\pi_1(x,y,t))$ there. Observe that $|\nabla f'_\epsilon\circ\Phi_\epsilon| = J\Phi_\epsilon$ and so equation (3.27) becomes

$$\begin{aligned}&\int_{\mathcal{U}\cap\partial E_\epsilon}|\mathbf{n}^\epsilon_{\mathbb{H}}|\,d\mathcal{H}^{2n}=\int_\omega\left(\sum_{j=1}^{2n}\langle\nabla f'_\epsilon\circ\Phi_\epsilon,X_j\rangle^2\right)^{1/2}d\mathcal{L}^{2n}\quad=\\ &\int_\omega\Big[1+(-\partial_{v_2}\phi_\epsilon-2v_{n+2}\partial_\tau\phi_\epsilon)^2+\cdots+(-\partial_{v_n}\phi_\epsilon-2v_{2n}\partial_\tau\phi_\epsilon)^2+(-\partial_\eta\phi_\epsilon+4\phi_\epsilon\partial_\tau\phi_\epsilon)^2\\ &\quad+(-\partial_{v_{n+2}}\phi_\epsilon+2v_2\partial_\tau\phi_\epsilon)^2+\cdots+(-\partial_{v_{2n}}\phi_\epsilon+2v_n\partial_\tau\phi_\epsilon)^2\Big]^{1/2}d\mathcal{L}^{2n}.\end{aligned}\tag{3.28}$$

By differentiating the equation

$$0\equiv f_\epsilon(\phi_\epsilon(\eta,v,\tau),v_2,\dots,v_n,\eta,v_{n+2},\dots,v_{2n},\tau+2\eta\phi_\epsilon(\eta,v,\tau))$$

one obtains

$$\begin{aligned}&\partial_{v_j}\phi_\epsilon=-\frac{\partial_{x_j}f_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\quad\text{and}\quad\partial_{v_{j+n}}\phi_\epsilon=-\frac{\partial_{y_j}f_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\,,\qquad j=2,\dots,n\\ &\partial_\eta\phi_\epsilon=-\frac{\partial_{y_1}f_\epsilon+2\phi_\epsilon\,\partial_tf_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\\ &\partial_t\phi_\epsilon=-\frac{\partial_tf_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\end{aligned}\tag{3.29}$$

which, substituted into (3.28), give

$$\int_{\mathcal{U}\cap\partial E_\epsilon}|\mathbf{n}^\epsilon_{\mathbb{H}}|\,d\mathcal{H}^{2n}=\int_\omega\frac{|\nabla_{\mathbb{H}}f_\epsilon|}{|X_1f_\epsilon|}\circ\Phi_\epsilon\,d\mathcal{L}^{2n}.\tag{3.30}$$

If we show that $\phi_\epsilon\to\phi$ uniformly on $\overline{\omega}$, the right hand side of (3.30) will automatically converge to

$$\int_\omega\frac{|\nabla_{\mathbb{H}}f|}{|X_1f|}\circ\Phi\,d\mathcal{L}^{2n}<\infty$$

(where Φ is as in (3.19)) and this is enough to prove our goal, i.e. that the functions h_ϵ have equibounded $\mathbb{H}$-variation.

Suppose on the contrary that there are $\sigma > 0$, $\epsilon_k \to 0$ and $A^k \in \overline{\omega}$ such that $|\phi_{\epsilon_k}(A^k) - \phi(A^k)| \geq \sigma$. By compactness we can suppose that $A^k \to A \in \overline{\omega}$ and $\phi_{\epsilon_k}(A^k) \to s_0$ as $k \to \infty$; it follows that $|\phi(A) - s_0| \geq \sigma$ but, on the other hand, the uniform convergence of f_ϵ to f implies

$$0 = \phi_{\epsilon_k}(\iota(A^k) \cdot f_{\epsilon_k}(A^k)e_1) \to f(A \cdot s_0 e_1),$$

whence the contradiction $s_0 = \phi(A)$.

Step 4. We are now in order to prove the area type formula (3.23). Arguing as in Step 3, for any $\varphi \in \mathbf{C}^1_c(\mathcal{U}, H\mathbb{H}^n), |\varphi| \leq 1$ one has

$$\begin{aligned}\int_{\mathcal{U}} \chi_E \operatorname{div}_{\mathbb{H}} \varphi &= \lim_{\epsilon \to 0} \int_{\mathcal{U}} h_\epsilon \operatorname{div}_{\mathbb{H}} \varphi \\ &= \lim_{\epsilon \to 0} \int_\omega \frac{\langle \varphi, \nabla_{\mathbb{H}} f_\epsilon \rangle}{|X_1 f_\epsilon|} \circ \Phi_\epsilon \, d\mathcal{L}^{2n} = \int_\omega \frac{\langle \varphi, \nabla_{\mathbb{H}} f \rangle}{|X_1 f|} \circ \Phi \, d\mathcal{L}^{2n}, \end{aligned} \tag{3.31}$$

where in the last equality we used Lebesgue convergence theorem. Taking the supremum with respect to φ we obtain (3.23).

Notice that taking the supremum in (3.31) on $\varphi \in \mathbf{C}^1_c(\mathcal{V}, H\mathbb{H}^n), |\varphi| \leq 1$, where $\mathcal{V} \Subset \mathcal{U}$ is an open set, it is straightforward to prove that

$$\|\partial E\|_{\mathbb{H}}(\mathcal{V}) = \int_{\Phi^{-1}(\mathcal{V})} \frac{|\nabla_{\mathbb{H}} f|}{|X_1 f|} \circ \Phi \, d\mathcal{L}^{2n},$$

i.e. that

$$\|\partial E\|_{\mathbb{H}} \llcorner \mathcal{U} = \frac{|\nabla_{\mathbb{H}} f|}{|X_1 f|} \; \Phi_\sharp\big(\mathcal{L}^{2n} \llcorner \omega\big). \tag{3.32}$$

It follows that $\|\partial E\|_{\mathbb{H}} \llcorner \mathcal{U}$ is concentrated on S.

Step 5. We are only left to prove that $\nu_E = -\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}$ $\|\partial E\|_{\mathbb{H}}$-a.e. on $S \cap \mathcal{U}$. By Theorem 3.9, (3.31) and (3.32), for any $\varphi \in \mathbf{C}^1_c(\mathcal{U}, H\mathbb{H}^n), |\varphi| \leq 1$ we have

$$\begin{aligned}\int_\omega \langle \varphi \circ \Phi, \nu_E \circ \Phi \rangle \frac{|\nabla_{\mathbb{H}} f \circ \Phi|}{|X_1 f \circ \Phi|} \, d\mathcal{L}^{2n} &= \int_{\mathcal{U}} \langle \varphi, \nu_E \rangle \, d\|\partial E\|_{\mathbb{H}} \\ &= -\int_E \operatorname{div}_{\mathbb{H}} \varphi = -\int_\omega \frac{\langle \varphi \circ \Phi, \nabla_{\mathbb{H}} f \circ \Phi \rangle}{|X_1 f \circ \Phi|} \, d\mathcal{L}^{2n}, \end{aligned}$$

whence $\nu_E \circ \Phi = -\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|} \circ \Phi$ $\mathcal{L}^{2n}$-a.e. on ω, i.e.

$$\nu_E = -\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|} = \nu_S \qquad \|\partial E\|_{\mathbb{H}}\text{-a.e. on } S \cap \mathcal{U}.$$

□

Corollary 3.17. *The horizontal normal to an $\mathbb{H}$-regular surface S is well defined, i.e. it does not depend on the choice of the defining function f.*

3.4 Rectifiability in the Heisenberg group

In this Section we collect, without proof, the most relevant results contained in [79] which have not been presented in previous Sections; observe that, more generally, many of them have been established also for step 2 Carnot groups (see [81]).

In the spirit of De Giorgi's approach to rectifiability for sets of finite perimeter (see e.g. [65]) we start by defining the $\mathbb{H}$-*reduced boundary* $\partial^*_{\mathbb{H}}E$ of an $\mathbb{H}$-Caccioppoli set E as the set of points $P \in \mathbb{H}^n$ such that

(a) $\|\partial E\|_{\mathbb{H}}(U(P,r)) > 0 \quad$ for all $r > 0$,

(b) there exists $\lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d\|\partial E\|_{\mathbb{H}} \qquad$ and

(c) $\left|\lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d\|\partial E\|_{\mathbb{H}}\right| = 1,$

where ν_E is the horizontal inward normal to E of Theorem 3.9.

Remark 3.18. Notice that, thanks to Theorem 3.16 and using the notations therein, for an $\mathbb{H}$-regular surface S one has $S = \partial^*_{\mathbb{H}}E$.

We have the following

Lemma 3.19 (Lemma 7.3 in [79])**.** *If E is an $\mathbb{H}$-Caccioppoli set, then*

$$\lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d\|\partial E\|_{\mathbb{H}} = \nu_E(P) \qquad \text{for } \|\partial E\|_{\mathbb{H}}\text{-a.e. } P.$$

This implies, in particular, that $\|\partial E\|_{\mathbb{H}}$-a.e. point $P \in \mathbb{H}^n$ belongs to $\partial^*_{\mathbb{H}}E$; moreover, up to re-defining ν_E on a $\|\partial E\|_{\mathbb{H}}$-negligible set, we are allowed to suppose that

$$\nu_E(P) = \lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d\|\partial E\|_{\mathbb{H}}$$

for any $P \in \partial^*_{\mathbb{H}}E$.

The first key result for rectifiability, exactly as in De Giorgi's program, is a blow-up theorem for $\mathbb{H}$-Caccioppoli sets at points of the $\mathbb{H}$-reduced boundary. More precisely, for any $P \in \mathbb{H}^n$ we define

$$E_{r,P_0} := \delta_{1/r}(\ell_{P_0^{-1}}E) = \{P \in \mathbb{H}^n : P_0 \cdot \delta_r(P) \in E\}$$

and for $\nu \in H_{P_0}\mathbb{H}^n$ let us introduce the halfspaces $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ "orthogonal" to ν as

$$S^+_{\mathbb{H}}(\nu) := \{P \in \mathbb{H}^n : \langle \pi_{P_0}(P), \nu\rangle \geq 0\}$$
$$S^-_{\mathbb{H}}(\nu) := \{P \in \mathbb{H}^n : \langle \pi_{P_0}(P), \nu\rangle \leq 0\}.$$

The common topological boundary of $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ is the maximal subgroup V_ν (see (3.16)), which we will also denote by $T^g_{\mathbb{H}}(\nu) = \{P \in \mathbb{H}^n : \langle \pi_{P_0}(P), \nu\rangle = 0\}$. We then have

Theorem 3.20 (Theorem 4.1 in [79]). *Let E be an $\mathbb{H}$-Caccioppoli set and let $P_0 \in \partial^*_{\mathbb{H}} E$; then*

$$\lim_{r\to 0} \chi_{E_{r,P_0}} = \chi_{S^+_{\mathbb{H}}(\nu_E(P_0))} \qquad \text{in } L^1_{loc}(\mathbb{H}^n) \tag{3.33}$$

and

$$\begin{aligned} \lim_{r\to 0} \|\partial E_{r,P_0}\|_{\mathbb{H}}(U(0,R)) &= \|\partial S^+_{\mathbb{H}}(\nu_E(P_0))\|_{\mathbb{H}}(U(0,R)) \\ &= \mathcal{L}^{2n}\big(T^g_{\mathbb{H}}(\nu_E(P_0)) \cap U(0,R)\big) = 2\omega_{2n-1}R^{2n+1} \end{aligned} \tag{3.34}$$

for any $R > 0$.

Remark 3.21. Notice that, in the case of an $\mathbb{H}$-regular surface S, the blow up limit of E at a point $P_0 \in S$ (where E is as in 3.16) is exactly the halfspace $S^+_{\mathbb{H}}(\nu_S(P_0))$ whose boundary is the tangent group $T^g_{\mathbb{H}}S(P_0)$ to S at P_0.

Analogously to the classical Euclidean case, we say that a set $\Gamma \subset \mathbb{H}^n$ is $\mathbb{H}$-*rectifiable* if

$$\Gamma \subset N \cup \bigcup_{j=1}^{\infty} K_j \tag{3.35}$$

where $\mathcal{H}^{Q-1}_\infty(N) = 0$ and each K_j is a compact subset of an $\mathbb{H}$-regular surface S_j. We then have

Theorem 3.22 (Theorem 7.1 in [79]). *If $E \subset \mathbb{H}^n$ is an $\mathbb{H}$-Caccioppoli set, then its $\mathbb{H}$-reduced boundary $\partial^*_{\mathbb{H}} E$ is $\mathbb{H}$-rectifiable. More precisely, it is possible to find a decomposition*

$$\partial^*_{\mathbb{H}} E = N \cup \bigcup_{j=1}^{\infty} K_j$$

such that $\mathcal{H}^{Q-1}_\infty(N) = 0$ and each K_j is a compact subset of an $\mathbb{H}$-regular surface S_j with the property that

$$\nu_E(P) = \nu_{S_j}(P) \quad \text{for all } P \in K_j.$$

Finally, one has

$$\|\partial E\|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}}\, \mathcal{S}^{Q-1}_\infty \llcorner \partial^*_{\mathbb{H}} E. \tag{3.36}$$

As usual in the literature, one can also define the *measure theoretic boundary* $\partial_{*\mathbb{H}}E$ of E as the set of points $P \in \mathbb{H}^n$ such that

$$\limsup_{r\to 0} \frac{|E \cap U(P,r)|}{|U(P,r)|} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{|U(P,r) \setminus E|}{|U(P,r)|} > 0.$$

It is not difficult to prove that for an $\mathbb{H}$-Caccioppoli set E we have

$$\partial^*_{\mathbb{H}}E \subset \partial_{*\mathbb{H}}E \subset \partial E;$$

moreover, one has $\mathcal{H}^{Q-1}_{\infty}(\partial_{*\mathbb{H}}E \setminus \partial^*_{\mathbb{H}}E) = 0$. Finally, the following result also holds

Theorem 3.23 (Corollary 7.6 in [79])**.** *If E is an $\mathbb{H}$-Caccioppoli set, then*

$$\|\partial E\|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}}\, \mathcal{S}^{Q-1}_{\infty} \llcorner \partial_{*\mathbb{H}}E$$

and the following divergence formula holds

$$-\int_E \operatorname{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial_{*\mathbb{H}}E} \langle \nu_E, \varphi \rangle \, d\mathcal{S}^{Q-1}_{\infty}.$$

Chapter 4

Intrinsic parametrization of $\mathbb{H}$-regular surfaces

The main aim of this Chapter is to give necessary and sufficient conditions for maps $\phi : V_1 \to \mathbb{R}$ to parametrize $\mathbb{H}$-regular surfaces, in the sense of the X_1-graphs introduced in Section 3.3. These conditions turn out to be of crucial importance in the study of several features regarding $\mathbb{H}$-regular surfaces (regularity of the parametrizations, rectifiability, etc.), allowing for example the explicit exhibition of non Euclidean $\mathbb{H}$-regular surfaces. We will also investigate area-type formulae for $\mathbb{H}$-regular X_1-graphs, thus paving the way for classical questions of Geometric Measure Theory, such as Minimal Surfaces or the Bernstein problem (see also Chapter 5). Similar items have been studied also in [48].

All the results of this Chapter are quite technical and will be illustrated in the following brief overview. We want to stress in particular the importance of the operator W^ϕ, which seems to be the correct intrinsic replacement of Euclidean gradient for $\mathbf{C}^1$ surfaces: we will see how they share several common features. Regarding the operator W^ϕ, we should address the reader also to the recent paper [24]. All the results contained in this Chapter, except for the ones of Section 4.5, have been obtained in [12] in collaboration with L. Ambrosio and F. Serra Cassano. Theorem 4.33 is due to Cole and Pauls [52], while Remark 4.34 and Theorem 4.35 are results contained in a joint work with F. Bigolin [25].

We then begin with Section 4.1, where we deepen the study of implicit graphs; in particular, we endow $\mathbb{R}^{2n} \equiv V_1$ with the homogeneous structure inherited from $\mathbb{H}^n$, thus defining the group law $\diamond$, the left invariant vector fields $\widetilde{X}_j, \widetilde{Y}_j, \widetilde{T}$, the homogeneous dilations $\delta_r^\diamond$ and the $\diamond$-linear functionals on $\mathbb{R}^{2n}$. Through Proposition 4.3 and Corollary 4.5 we provide an integral formula for the $\mathcal{S}_\infty^{Q-1}$ measure of an $\mathbb{H}$-regular surface S, in terms of (derivatives of) its intrinsic parametrization only. This formula will be extensively used in the rest of the book. With Remark 4.7 we also

show that it is not restrictive to consider X_1-graphs rather than general X_j-graphs.

In Section 4.2 we provide the basic tools for the analysis of parametrizations of $\mathbb{H}$-regular surfaces. Namely, for any fixed continuous function $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ we introduce the (quasi-)distance d_ϕ on ω and the concepts of W^ϕ-differentiability and uniform W^ϕ-differentiability for functions $\psi : \omega \to \mathbb{R}$, see (4.12) and Definition 4.9. When ϕ parametrizes an $\mathbb{H}$-regular surface S, it turns out that d_ϕ is equivalent to the restriction of d_∞ to S, i.e. to the pull back $\Phi_\sharp^{-1} d_\infty$. The notion of W^ϕ-differentiability, a sort of intrinsic differentiability taking into account d_ϕ (and so ϕ itself) and the homogeneous structure $(\mathbb{R}^{2n}, \diamond, \delta_r^\diamond)$, carries on the concept of the W^ϕ-differential of ψ. The latter is a function $W^\phi\psi : \omega \to \mathbb{R}^{2n-1}$ turning out to be continuous in case of uniform W^ϕ-differentiability (see Proposition 4.14). In the regular case $\phi, \psi \in \mathbf{C}^1(\omega)$ one can prove that

$$W^\phi\psi = (\widetilde{X}_2\psi, \dots, \widetilde{X}_n\psi, \widetilde{Y}_1\psi - 4\phi\widetilde{T}\psi, \widetilde{Y}_2\psi, \dots, \widetilde{Y}_n\psi)\,;$$

this quite technical result is proved in Theorem 4.16.

The main item of Section 4.3 is Theorem 4.17, where we prove that a map ϕ parametrizes an $\mathbb{H}$-regular surface $S = \Phi(\omega)$ if and only if ϕ is uniformly W^ϕ-differentiable. Moreover, we get two explicit formulae for the horizontal normal (4.36) and for the $\mathcal{S}^{Q-1}$ measure of S (4.37), which are consistent with Proposition 4.3. These two formulae suggest that the intrinsic gradient $W^\phi\phi$ is the correct counterpart of Euclidean gradients for classical graphs, since both of them can be obtained by formally substituting the classical gradient with $W^\phi\phi$. We also remark that intrinsic regular parametrizations have continuous intrinsic gradient, exactly like parametrizations of regular $\mathbf{C}^1$ surfaces have continuous gradient. The proof of Theorem 4.17 is quite technical and makes use of Lemma 3.11 and Whitney Extension Theorem 3.12. As a byproduct, we obtain that parametrizations of $\mathbb{H}$-regular surfaces are $1/2$ Hölder continuous from the Euclidean viewpoint (a fact already known [106]) and, in fact, also a bit more regular (see Corollary 4.20).

In Section 4.4 we characterize uniformly W^ϕ-differentiable functions ϕ (i.e. parametrizations of $\mathbb{H}$-regular surfaces) by means of equivalent conditions. The main result in this sense is Theorem 4.22, where we prove that such ϕ's are exactly those for which

$$(\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \widetilde{Y}_1\phi - 2\widetilde{T}(\phi^2), \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi)$$

coincides, in distributional sense, with a continuous function (and, *a posteriori*, with $W^\phi\phi$) and it is possible to find a family $\{\phi_\epsilon\} \subset \mathbf{C}^\infty(\omega)$ such that $\phi_\epsilon \to \phi$ and $W^{\phi_\epsilon}\phi_\epsilon \to W^\phi\phi$ locally uniformly in ω. The proof of this fact is similar to the one of Theorem 4.16: the main technical obstacle is the absence of a good definition of integral lines for the vector field $\widetilde{Y}_1 - 4\phi\widetilde{T}$, which however can be bypassed thanks to a suitable notion of exponential maps. As an application, Corollary 4.32 furnishes a recipe to easily construct $\mathbb{H}$-regular surfaces that are not Euclidean $\mathbf{C}^1$.

Finally, in Section 4.5 we restrict our attention to the problem of finding a model metric space for $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$. This was identified [52] in the space $(\mathbb{R}, |\cdot|) \times (\mathbb{R}, |\cdot|^{1/2})$ for $\mathbf{C}^1$ surfaces, while this result is no longer true for general $\mathbb{H}$-regular ones. We are in fact able (Theorem 4.35) to exhibit, by means of Corollary 4.32, an $\mathbb{H}$-regular surface S such that there are no Lipschitz maps from S into that space with Lipschitz continuous inverse map.

4.1 More on intrinsic graphs

Let us introduce some subspaces of the Lie algebra $\mathfrak{h}$ associated with $\mathbb{H}^n$ (here $\widehat{X}_j$ means that in an enumeration we omit X_j):

$$\begin{aligned}
&\mathfrak{o} := \mathfrak{h}_1 = \operatorname{span}\{X_1, \dots, X_{2n}\};\\
&\mathfrak{v}_j := \operatorname{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}, T\} \quad (1 \le j \le 2n);\\
&\mathfrak{o}_j := \operatorname{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}\} \quad (1 \le j \le 2n);\\
&\mathfrak{l}_j := \operatorname{span}\{X_j\} \quad (1 \le j \le 2n);\\
&\mathfrak{z} := \mathfrak{h}_2 = \operatorname{span}\{T\}
\end{aligned}$$

and let $\pi_{\mathfrak{o}}, \pi_{\mathfrak{v}_j}, \pi_{\mathfrak{o}_j}, \pi_{\mathfrak{l}_j}, \pi_{\mathfrak{z}}$ be the projections of $\mathfrak{h}_n$ onto $\mathfrak{o}, \mathfrak{v}_j, \mathfrak{o}_j, \mathfrak{l}_j$ and $\mathfrak{z}$ respectively. Define the following subsets of $\mathbb{H}^n$:

$$\begin{aligned}
&O := \exp(\mathfrak{o}) = \{P \in \mathbb{H}^n : p_{2n+1} = 0\};\\
&V_j := \exp(\mathfrak{v}_j) = \{P \in \mathbb{H}^n : p_j = 0\};\\
&O_j := \exp(\mathfrak{o}_j) = O \cap V_j = \{P \in \mathbb{H}^n : p_j = p_{2n+1} = 0\};\\
&L_j := \exp(\mathfrak{l}_j) = \{P \in \mathbb{H}^n : p_i = 0 \ \forall i \ne j\};\\
&Z := \exp(\mathfrak{z}) = \{P \in \mathbb{H}^n : p_1 = \dots = p_{2n} = 0\}.
\end{aligned}$$

and let $\pi_O, \pi_{V_j}, \pi_{O_j}, \pi_{L_j}$ and π_Z be the maps defined by $\exp \circ \pi_{\mathfrak{o}} \circ \exp^{-1}, \exp \circ \pi_{\mathfrak{v}_j} \circ \exp^{-1}$ and so on; we will refer to them as orthogonal projections of $\mathbb{H}^n$ on O, V_j, O_j, L_j and Z. Observe that V_j coincides with the maximal subgroup V_{e_j} according to (3.16), where e_j are the vectors of the canonical basis of $\mathbb{R}^{2n+1}$.

The following properties of these projections are straightforward:

Proposition 4.1. *For any $P, Q \in \mathbb{H}^n$ we have*

$$\begin{aligned}
&\pi_{O_1}(P) = \pi_O \circ \pi_{V_1}(P) = \pi_{V_1} \circ \pi_O(P)\\
&\pi_{O_1}(P \cdot Q) = \pi_{O_1}(\pi_{O_1}(P) \cdot \pi_{O_1}(Q))\\
&\pi_Z(P \cdot Q) = \pi_Z(P) \cdot \pi_Z(Q) \cdot \pi_Z(\pi_O(P) \cdot \pi_O(Q))\\
&\|\pi_M(P)\|_\infty \le \|P\|_\infty \quad \forall M \in \{O, O_1, V_1, L_1, Z\}\,.
\end{aligned}$$

Let us observe that Z is the center of the group, and that only Z, L_j and V_j are subgroups; O_j is a subgroup only if $n = 1$ (because in this case it coincides with L_j), while O is never a subgroup. We agree to denote with αe_j the point $\exp(\alpha X_j) \in L_j$; then for each $P = (p_1, \dots, p_{2n+1}) \in \mathbb{H}^n$ there is a unique way to write it in the form $P_{V_j} \cdot P_{L_j}$ for points $P_{V_j} \in V_j, P_{L_j} \in L_j$: it is sufficient to take $P_{L_j} = p_j e_j$ and $P_{V_j} = P \cdot P_{L_j}^{-1} \in V_j$.

Recalling the definition of the diffeomorphism $\iota : \mathbb{R}^{2n} \to V_1$ given in (3.17) and (3.18) we can endow $\mathbb{R}^{2n}$ with the group law $\diamond$ induced by ι, i.e.

$$A \diamond B := \iota^{-1}(\iota(A) \cdot \iota(B)) \quad A, B \in \mathbb{R}^{2n}. \tag{4.1}$$

We will use $\ell_A^\diamond$ to denote the left translation by A in $\mathbb{R}^{2n}$. Explicitly, if $n > 1$ and $A = (\eta, v, \tau), B = (\eta', v', \tau') \in \mathbb{R}^{2n}$ we have

$$A \diamond B = (\eta + \eta', v + v', \tau + \tau' + \sigma(v, v'))$$

where

$$\sigma(v, v') = 2 \sum_{j=2}^{n} (v_{n+j} v'_j - v_j v'_{n+j}) \tag{4.2}$$

if $v = (v_2, \dots, v_n, v_{n+2}, \dots v_{2n})$, $v' = (v'_2, \dots, v'_n, v'_{n+2}, \dots v'_{2n})$. Instead if $n = 1$ and $A = (\eta, \tau), B = (\eta', \tau') \in \mathbb{R}^2$ we simply have

$$A \diamond B = (\eta + \eta', \tau + \tau').$$

Notice that in both cases the induced group structure is the one arising from direct product $\mathbb{R} \times \mathbb{R}$ if $n = 1$ and $\mathbb{R} \times \mathbb{H}^{n-1}$ if $n > 1$, via the identification $\mathbb{R}^{2n} = \mathbb{R}_\eta \times (\mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau) = \mathbb{R} \times \mathbb{H}^{n-1}$.

Moreover, since V_1 is closed under group dilations, for $r > 0$ we can define the family of induced intrinsic dilations

$$\delta_r^\diamond(A) := \iota^{-1}(\delta_r(\iota(A)) \quad \in \mathbb{R}^{2n}; \tag{4.3}$$

which can be written explicitly as

$$\begin{aligned} &\delta_r^\diamond(\eta, v, \tau) = (r\eta, rv, r^2\tau) && \text{for } n \geq 2 \\ &\delta_r^\diamond(\eta, \tau) = (r\eta, r^2\tau) && \text{for } n = 1. \end{aligned}$$

Therefore, $(\mathbb{R}^{2n}, \diamond, \delta_r^\diamond)$ turns out to be a homogeneous group in the sense of Folland and Stein ([73]), and ι is a group isomorphism. We define a $\diamond$-*linear functional* $L : \mathbb{R}^{2n} \to \mathbb{R}$ as a homomorphism which is also homogeneous of degree 1 with respect to the dilations, i.e. $L \circ \delta_r^\diamond = rL$. The following Proposition comes from Proposition 5.4 in [79]:

Proposition 4.2. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}$ be a $\diamond$-linear functional; then there is a unique vector $w_L \in \mathbb{R}^{2n-1}$ such that $L(A) = \langle A, w_L \rangle$, where we write*

$$\begin{aligned} &\langle A, w_L \rangle = \eta w_{Ln+1} + \sum_{j=2, j\neq n+1}^{2n} v_j w_{Lj} \quad \text{if } n \geq 2, w_L = (w_{L2}, \dots, w_{L2n}) \text{ and } A = (\eta, v, \tau) \\ &\langle A, w_L \rangle = \eta w_{L2} \qquad\qquad\qquad\qquad\quad \text{if } n = 1, w_L = w_{L2} \text{ and } A = (\eta, \tau)\,. \end{aligned}$$

Conversely, through the previous formulae we can associate to each $w \in \mathbb{R}^{2n-1}$ a unique $\diamond$-linear functional L_w.

Observe that the choice of the enumeration of the components of w_L has been made in order to be coherent with the one made for the components of v and with the fact that η is the $(n+1)$-th coordinate of $\iota(A)$.

For $n > 1$, the tangent space to V_1 is generated by the restrictions of $X_2, \dots, X_n$, $Y_1, \dots, Y_n, T$, and so we can define the vector fields $\widetilde{X}_2 \dots, \widetilde{X}_n, \widetilde{Y}_1, \dots, \widetilde{Y}_n$ and $\widetilde{T}$ on $\mathbb{R}^{2n}$ given by $\widetilde{X}_j := (\iota^{-1})_* X_j$ and $\widetilde{Y}_j := (\iota^{-1})_* Y_j, \widetilde{T} := (\iota^{-1})_* T$. In coordinates, they can be written as

$$\begin{aligned} \widetilde{X}_j(\eta, v, \tau) &= \frac{\partial}{\partial v_j} + 2v_{j+n}\frac{\partial}{\partial \tau} \quad \text{for } j = 2, \dots, n \\ \widetilde{Y}_1(\eta, v, \tau) &= \frac{\partial}{\partial \eta} \\ \widetilde{Y}_j(\eta, v, \tau) &= \frac{\partial}{\partial v_{j+n}} - 2v_j\frac{\partial}{\partial \tau} \quad \text{for } j = 2, \dots, n \\ \widetilde{T}(\eta, v, \tau) &= \frac{\partial}{\partial \tau}\,. \end{aligned} \tag{4.4}$$

For $n+1 \leq j \leq 2n$ we will also use the notation $\widetilde{X}_j := \widetilde{Y}_{j-n}$.
If $n = 1$ the tangent space to V_1 is generated by Y_1 and T, and as before we can define

$$\begin{aligned} \widetilde{Y}_1(\eta, \tau) &:= (\iota^{-1})_* Y_1 = \frac{\partial}{\partial \eta} \\ \widetilde{T}(\eta, \tau) &:= (\iota^{-1})_* T = \frac{\partial}{\partial \tau} \end{aligned} \tag{4.5}$$

and it could happen that we will write $\widetilde{X}_2$ instead of $\widetilde{Y}_1$. It follows from the definition that $\widetilde{X}_j, \widetilde{Y}_j, \widetilde{T}$ are $\diamond$-left-invariant.

With these notations, let us provide an improvement of Theorem 3.16:

Proposition 4.3. *Under the same assumptions as in Theorem 3.16, let us consider the distribution*

$$\mathfrak{B}\phi := \widetilde{Y}_1\phi - 2\,\widetilde{T}(\phi^2) = \frac{\partial \phi}{\partial \eta} - 2\frac{\partial \phi^2}{\partial \tau}$$

on $\omega =]-\delta,\delta[^{2n-1}\times]-\delta^2,\delta^2[$, *where* ϕ *and* δ *are given by the same Theorem 3.16. Then, if* $n>1$ *we have*

$$\widetilde{X}_j\phi = -\frac{X_jf}{X_1f}\circ\Phi, \quad \widetilde{Y}_j\phi = -\frac{Y_jf}{X_1f}\circ\Phi, \quad \mathfrak{B}\phi = -\frac{Y_1f}{X_1f}\circ\Phi \tag{4.6}$$

for $j=2,\dots,n$, *where the equalities must be understood in distributional sense on* ω. *Moreover, the* $\mathbb{H}$*-perimeter has the integral representation*

$$\|\partial E\|_{\mathbb{H}}(\mathcal{U}) = c(n)\mathcal{S}^{Q-1}_{\infty}\llcorner(S\cap\mathcal{U}) = \int_{\omega}\sqrt{1+|\mathfrak{B}\phi|^2+\textstyle\sum_{j=2}^{n}\left[|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2\right]}\,d\mathcal{L}^{2n}\,, \tag{4.7}$$

where we have set $c(n) := \frac{2\omega_{2n-1}}{\omega_{2n+1}}$. *If* $n=1$ *we have simply*

$$\|\partial E\|_{\mathbb{H}}(\mathcal{U}) = c(1)\mathcal{S}^{Q-1}_{\infty}\llcorner(S\cap\mathcal{U}) = \int_{\omega}\sqrt{1+|\mathfrak{B}\phi|^2}\,d\mathcal{L}^2. \tag{4.8}$$

Proof. We will give the proof only for the case $n\geq 2$; the adaptation to $n=1$ does not present difficulties.

Arguing as in Step 1 of the proof of Theorem 3.16, we can suppose that there exists a family of functions $f_\epsilon:\mathcal{U}\to\mathbb{R}$ such that $f_\epsilon\in\mathbf{C}^1(\overline{\mathcal{U}})$, $X_1f_\epsilon>0$ on $\mathcal{U}$ and

$$X_jf_\epsilon\to X_jf, \quad Y_jf_\epsilon\to Y_jf_\epsilon \quad \text{uniformly on } \mathcal{U} \quad (j=1,\dots,n).$$

Now, following Step 3 of the same proof, we obtain the existence (for ϵ_0 small enough and h as in Theorem 3.16) of functions $\phi_\epsilon\in\mathbf{C}^1(\omega,]-h,h[)$ $(0<\epsilon<\epsilon_0)$ such that

$$\begin{aligned} &f_\epsilon(\iota(A)\cdot\phi_\epsilon(A)e_1)=0 \quad \text{for all } A\in\omega\\ &\phi_\epsilon\to\phi \quad \text{uniformly on } \omega \text{ for } \epsilon\to 0. \end{aligned}$$

Using formulae (3.29), for $j=2,\dots,n$ we get

$$\begin{aligned} \widetilde{X}_j\phi_\epsilon &= -\frac{X_jf_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\\ \widetilde{Y}_j\phi_\epsilon &= -\frac{Y_jf_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\\ \mathfrak{B}\phi_\epsilon &= \frac{\partial\phi_\epsilon}{\partial\eta}-2\frac{\partial\phi_\epsilon^2}{\partial\tau} = \frac{\partial\phi_\epsilon}{\partial\eta}-4\phi_\epsilon\frac{\partial\phi_\epsilon}{\partial\tau} = -\frac{Y_1f_\epsilon}{X_1f_\epsilon}\circ\Phi_\epsilon\,, \end{aligned}$$

where as usual Φ_ϵ is the map $A\longmapsto\iota(A)\cdot\phi_\epsilon(A)e_1$; this immediately implies (4.6). The integral representation (4.7) follows from the area type formula (3.23), together with (4.6) and (3.36). □

Remark 4.4. The operator $\mathfrak{B}$ is known in the literature as Burgers' operator: see for example [66], section 3.4.

Corollary 4.5. *Let Ω be an open subset of $\mathbb{H}^n$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f > 0$ on $S := \{f = 0\}$. Suppose that S is intrinsically parametrized by a real continuous function ϕ defined on an open set $\omega \subset \mathbb{R}^{2n}$ (i.e. $S := \Phi(\omega)$, where as usual $\Phi(A) := \iota(A) \cdot \phi(A)e_1$), and let $E := \{f < 0\}$. Then for each Borel set $F \subset \Omega$ we have*

$$\|\partial E\|_{\mathbb{H}}(F) = c(n)\, \mathcal{S}^{Q-1}_{\infty}(F \cap S) = \int\limits_{\Phi^{-1}(F)} \sqrt{1+|\mathfrak{B}\phi|^2+\textstyle\sum_{j=2}^n \left[|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2\right]}\, d\mathcal{L}^{2n} \quad (4.9)$$

if $n \geq 2$, and

$$\|\partial E\|_{\mathbb{H}}(F) = c(1)\, \mathcal{S}^{Q-1}_{\infty}(F \cap S) = \int_{\Phi^{-1}(F)} \sqrt{1+|\mathfrak{B}\phi|^2}\, d\mathcal{L}^2 \quad (4.10)$$

if $n = 1$.

Proof. Again we give the proof only for the case $n \geq 2$. Let $\mu := \pi_{1\sharp}(\|\partial E\|_{\mathbb{H}})$, where $\pi_{1\sharp}$ is the usual push-forward of measures through the map defined in (3.22). Observe that, as $\pi_1 \equiv \Phi^{-1}$ on S and $\|\partial E\|_{\mathbb{H}}$ is concentrated on S, we have

$$\|\partial E\|_{\mathbb{H}}(F) = \mu(\Phi^{-1}(F \cap S)).$$

Therefore by Proposition 4.3 there are locally (i.e. for each $A \in \omega$) rectangles I such that $\mu_{|I} = \sqrt{1+|\mathfrak{B}\phi|^2+\sum_{j=2}^n \left[|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2\right]}\, \mathcal{L}^{2n}$. The class of these rectangles is sufficiently rich to apply the measure coincidence criterion (see for instance [6], Theorem 1.8), and so $\mu = \sqrt{1+|\mathfrak{B}\phi|^2+\sum_{j=2}^n \left[|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2\right]}\mathcal{L}^{2n}$ on all ω, whence

$$\begin{aligned} \|\partial E\|_{\mathbb{H}}(F) &= \mu(\Phi^{-1}(F \cap S)) \\ &= \int_{\Phi^{-1}(F\cap S)} \sqrt{1+|\mathfrak{B}\phi|^2+\textstyle\sum_{j=2}^n [|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2]}\, d\mathcal{L}^{2n}, \end{aligned}$$

which is the thesis. □

More generally, after fixing an identification $\iota_j : \mathbb{R}^{2n} \to V_j$, for $j = 2, \ldots, 2n$ we can define X_j-graphs as those subsets S of $\mathbb{H}^n$ for which there exists a function $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ such that $S = \{\iota_j(A) \cdot \phi(A)e_j : A \in \omega\}$.

A general definition of intrinsic graph in $\mathbb{H}^n$, which applies also to surfaces with topological codimension bigger than 1, is given in [83]. In particular this notion is stable with respect to left translations of the group; more precisely, from Proposition 3.11 in [83] we infer

Proposition 4.6. *Let $S \subset \mathbb{H}^n$ be an X_j-graph, i.e. $S = \{\Phi(A) := \iota_j(A) \cdot \phi(A)e_j : A \in \omega\}$. Let $P = (p_1, \dots p_{2n+1}) \in \mathbb{H}^n$, $P = P_{V_j} \cdot P_{L_j}$ with $P_{L_j} = p_j e_j \in L_j$ and $P_{V_j} \in V_j$. Then the translated set $\ell_P S$ still is an X_j-graph; more precisely, if we define*

$$\begin{aligned} \sigma_P \ &: \ \mathbb{R}^{2n} \to \mathbb{R}^{2n} \\ &\quad A \longmapsto \iota_j^{-1}(P \cdot \iota_j(A) \cdot P_{L_j}^{-1}) = \iota_j^{-1}(P) \diamond A \diamond \iota_j^{-1}(P_{L_j}^{-1}), \end{aligned}$$

we have

$$\ell_P S = \{\Phi'(A) := \iota_j(A) \cdot \phi'(A)e_j : A \in \omega'\},$$

where $\omega' := \sigma_P(\omega)$ and $\phi' : \omega' \to \mathbb{R}$ is defined by

$$\phi'(A) = p_j + \phi(\sigma_{P^{-1}}(A)).$$

In addition we have $\Phi' = \ell_P \circ \Phi \circ \sigma_{P^{-1}}$.

Remark 4.7. In Theorem 3.16, and more generally in all related results, we made a precise choice, i.e. to consider only regular hypersurfaces that are zero sets of functions $f \in \mathbf{C}^1_{\mathbb{H}}$ with $X_1 f > 0$. This fact, somehow, makes X_1 a "privileged" direction: for example, observe that such surfaces turn out to be X_1-graphs, i.e. functions on V_1, and that we translate points of V_1 by an element with all the coordinates null except the first one. One can prove that this is not restrictive; the key tool in this sense are the so-called "horizontal rotations" introduced in [120], section 2.1.

Suppose in fact that, in an open set $\Omega \subset \mathbb{H}^n$, $X_k f > 0$ for some $2 \le k \le 2n$. Let us consider a second Heisenberg group, which we denote $\mathbb{H}^{n\prime}$: all the objects related to this second group will be denoted with the apex $'$, such as the algebra $\mathfrak{h}'$, the vector fields X'_j, Y'_j, T', the subgroup V'_1, the map $\iota' : V'_1 \to \mathbb{R}^{2n}$, etc. If $k \le n$ we define a Lie algebras isomporphism $l : \mathfrak{h} \to \mathfrak{h}'$ given by the extension by linearity of

$$\begin{aligned} &l(X_k) = X'_1, \quad l(Y_k) = Y'_1, \quad l(X_1) = X'_k, \quad l(Y_1) = Y'_k \\ &l(V) = V' \quad \text{if } V \in \operatorname{span}\{X_1, Y_1, X_k, Y_k\}^{\perp}. \end{aligned}$$

In the other case $k \ge n+1$, i.e. $Y_{k-n} > 0$, we define l by extending

$$\begin{aligned} &l(Y_{k-n}) = X'_1, \quad l(X_{k-n}) = -Y'_1, \quad l(X_1) = X'_k, \quad l(Y_1) = Y'_k \\ &l(V) = V' \quad \text{if } V \in \operatorname{span}\{X_1, Y_1, X_{k-n}, Y_{k-n}\}^{\perp}. \end{aligned}$$

It follows that $L := \exp' \circ l \circ \exp^{-1}$ is a group isomorphism and a global diffeomorphism between $\mathbb{H}^n$ and $\mathbb{H}^{n\prime}$.

Let $f' := f \circ L^{-1}$; as $X'_1 f' = X_k f > 0$ we have that there is an open set $\omega \subset \mathbb{R}^{2n}$ and a map $\phi : \omega \to \mathbb{R}$ such that $S' := \{f' = 0\} \cap \Omega' = \Phi'(\omega)$, where $\Omega' := L(\Omega)$

and $\Phi'(A) := \iota'(A) \cdot \phi(A)e_1$. Let $\iota_k := L^{-1} \circ \iota'$, which identifies V_k and $\mathbb{R}^{2n}$, and for $A \in \omega$ define $\Phi(A) := L^{-1}(\Phi'(A)) = \iota_k(A) \cdot \phi(A)e_k$: it is immediate to see that $S := \{f = 0\} \cap \Omega = \Phi(\omega)$.

Also, we can easily extend the results of Theorem 3.16, Proposition 4.3 and Corollary 4.5. In particular, the distributional equality (4.6) becomes

$$\widetilde{X}_j\phi = -\frac{(l^{-1}X'_j)f}{X_k f} \circ \Phi, \quad \widetilde{Y}_j\phi = -\frac{(l^{-1}Y'_j)f}{X_k f} \circ \Phi, \quad \mathfrak{B}\phi = -\frac{(l^{-1}Y'_1)}{X_k f} \circ \Phi.$$

4.2 Graph distance and W^ϕ-differentiability

From now on $\phi : \omega \to \mathbb{R}$ will be a fixed continuous function defined on an open, connected and bounded set $\omega \subset \mathbb{R}^{2n}$; we will denote with W^ϕ the family of first-order operators $(W_2^\phi, \dots W_{2n}^\phi)$ (the reasons of the enumeration from 2 will be clear later) defined for $n \geq 2$ by

$$W_j^\phi := \begin{cases} \widetilde{X}_j = \dfrac{\partial}{\partial v_j} + 2v_{j+n}\dfrac{\partial}{\partial \tau} & \text{if } 2 \leq j \leq n \\ \widetilde{Y}_1 - 4\phi\widetilde{T} = \dfrac{\partial}{\partial \eta} - 4\phi\dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \widetilde{Y}_{j-n} = \dfrac{\partial}{\partial v_j} - 2v_{j-n}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \leq j \leq 2n, \end{cases} \tag{4.11}$$

while for $n = 1$ we put $W^\phi = W_2^\phi := \widetilde{Y}_1 - 4\phi\widetilde{T} = \frac{\partial}{\partial \eta} - 4\phi\frac{\partial}{\partial \tau}$.

As usual, by Φ we will denote the function $\omega \ni A \mapsto \iota(A) \cdot \phi(A)e_1 \in \mathbb{H}^n$, whose explicit expression is given by (3.20). The *graph distance* between $A, B \in \omega$ is defined by

$$d_\phi(A, B) := \|\pi_{O_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty + \|\pi_Z(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty \tag{4.12}$$

which is equivalent to $\|\pi_{V_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty$. Explicitly, for $n \geq 2$ and $A = (\eta, v, \tau)$, $B = (\eta', v', \tau')$ we have

$$d_\phi(A, B) = |(\eta', v') - (\eta, v)| + |\tau' - \tau + 2(\phi(B) + \phi(A))(\eta' - \eta) + \sigma(v', v)|^{1/2}$$

where $\sigma(v', v)$ has been defined in (4.2); if $n = 1$ and $A = (\eta, \tau), B' = (\eta', \tau')$ we have

$$d_\phi(A, B) = |\eta' - \eta| + |\tau' - \tau + 2(\phi(B) + \phi(A))(\eta' - \eta)|^{1/2}.$$

With this definition we are able to prove the following

Proposition 4.8. *If there is an $L > 0$ such that*

$$|\phi(A) - \phi(B)| \leq L\, d_\phi(A, B) \tag{4.13}$$

for all $A, B \in \omega$, then the quantity d_ϕ in (4.12) is a quasimetric on ω, i.e.

(i) $d_\phi(A, B) = 0 \Leftrightarrow A = B$;

(ii) $d_\phi(A, B) = d_\phi(B, A)$;

(iii) there exists $q > 1$ such that $d_\phi(A, B) \leq q \left[d_\phi(A, C) + d_\phi(C, B) \right]$

for all $A, B, C \in \omega$.

Proof. The assertions in (i) and (ii) are straightforward, while for (iii) we use the inequality

$$d_\infty(\Phi(A), \Phi(B)) \leq |\phi(A) - \phi(B)| + d_\phi(A, B)$$

to achieve

$$\begin{aligned} d_\phi(A, B) &\leq 2 d_\infty(\Phi(A), \Phi(B)) \\ &\leq 2[\, d_\infty(\Phi(A), \Phi(C)) + d_\infty(\Phi(C), \Phi(B))\,] \\ &\leq 2[\, |\phi(A) - \phi(C)| + d_\phi(A, C) + |\phi(C) - \phi(B)| + d_\phi(C, B)\,] \\ &\leq 2(L+1)[\, d_\phi(A, C) + d_\phi(C, B)\,]. \end{aligned}$$

□

Let us observe that if ϕ satisfies the condition (4.13), then it is locally 1/2-Hölder continuous in the Euclidean sense, i.e. for all compact set $K \subset \omega$ there exist an $L' = L'(K) > 0$ such that

$$|\phi(B) - \phi(A)| \leq L' |B - A|^{1/2} \tag{4.14}$$

for all $A, B \in K$. First, let us observe that for any $P \in \mathbb{H}^n, \alpha \in \mathbb{R}$

$$\begin{aligned} \|\pi_Z(P \cdot \alpha e_1)\|_\infty &\leq \|\pi_Z(P)\|_\infty + \sqrt{2}|\alpha|^{1/2} \|\pi_{V_1}(P)\|_\infty^{1/2} \\ \|\pi_Z(\alpha e_1 \cdot P)\|_\infty &\leq \|\pi_Z(P)\|_\infty + \sqrt{2}|\alpha|^{1/2} \|\pi_{V_1}(P)\|_\infty^{1/2}. \end{aligned}$$

Now let $M := \sup_K |\phi|, \Delta := \sup_{A \in K} |A|$ and, for the sake of simplicity, $\phi := \phi(A), \phi' := \phi(B)$; then

$$\begin{aligned} & \tfrac{1}{L} |\phi(B) - \phi(A)| \leq d_\phi(B, A) \\ = \;& \|\pi_{O_1}(-\phi e_1 \cdot \iota(A)^{-1} \cdot \iota(B) \cdot \phi' e_1)\|_\infty + \|\pi_Z(-\phi e_1 \cdot \iota(A)^{-1} \cdot \iota(B) \cdot \phi' e_1)\|_\infty \\ \leq \;& |B - A| + \|\pi_Z(\iota(A)^{-1} \cdot \iota(B) \cdot \phi' e_1)\|_\infty + \sqrt{2M} \|\pi_{V_1}(\iota(A)^{-1} \cdot \iota(B) \cdot \phi' e_1)\|_\infty^{1/2} \\ \leq \;& (2\sqrt{\Delta} + \sqrt{2M}) |B - A|^{1/2} + \|\pi_Z(\iota(A)^{-1} \iota(B))\|_\infty + \sqrt{2M} \|\pi_{V_1}(\iota(A)^{-1} \iota(B))\|_\infty^{1/2} \\ \leq \;& (2\sqrt{\Delta} + 2\sqrt{2M} + C(K)) |B - A|^{1/2}. \end{aligned} \tag{4.15}$$

where in the last passage we used (3.3).

If $n \geq 2$ and $A = (\eta, v, \tau) \in \mathbb{R}^{2n}$ and $r > 0$ are given, we define

$$I_r(A) := \left\{(\eta', v', \tau') \in \mathbb{R}^{2n} : |(\eta', v') - (\eta, v)| < r, |\tau' - \tau| < r\right\},$$

while if $n = 1$ and $A = (\eta, \tau)$ we put

$$I_r(A) := \left\{(\eta', \tau') \in \mathbb{R}^2 : |\eta' - \eta| < r, |\tau' - \tau| < r\right\}.$$

Now we have all the tools to state our notion of W^ϕ-differentiability:

Definition 4.9. Let $A \in \omega$ and $\psi : \omega \to \mathbb{R}$ be given.

(*i*) We say that ψ is *W^ϕ-differentiable* at A if there is a $\diamond$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$\lim_{B \to A} \frac{|\psi(B) - \psi(A) - L(A^{-1} \diamond B)|}{d_\phi(A, B)} = 0. \tag{4.16}$$

(*ii*) We say that ψ is *uniformly W^ϕ-differentiable* at A if there is a $\diamond$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\lim_{r \downarrow 0} M_\phi(\psi, A, L, r) = 0$, where

$$M_\phi(\psi, A, L, r) := \sup_{\substack{B, B' \in I_r(A) \\ B \neq B'}} \left\{ \frac{|\psi(B') - \psi(B) - L(B^{-1} \diamond B')|}{d_\phi(B, B')} \right\}. \tag{4.17}$$

Let us observe that, if ψ is uniformly W^ϕ-differentiable at A, then it is also W^ϕ-differentiable at A, as (4.16) is satisfied with the same functional L in (4.17).

Remark 4.10. If ψ is W^ϕ-differentiable at A, then it is continuous at A. Indeed, if $L \in \mathbb{R}^{2n-1}$ is such that (4.16) holds and w_L is as in Proposition 4.2, then for any $B \in \omega$

$$\psi(B) - \psi(A) = \frac{\psi(B) - \psi(A) - \langle w_L, A^{-1} \diamond B\rangle}{d_\phi(A, B)} \cdot d_\phi(A, B) + \langle w_L, A^{-1} \diamond B\rangle$$

and we deduce the continuity of ψ at A from the W^ϕ-differentiability at A together with the fact that $d_\phi(A, B)$ is bounded near A.

Remark 4.11. We stress the fact that if $\psi : \omega \to \mathbb{R}$ is uniformly W^ϕ-differentiable at $A \in \omega$, then ψ is Lipschitz continuous (between the quasimetric spaces (ω, d_ϕ) and $(\mathbb{R}, d_{eucl})$) in a neighbourhood of A; in fact there exist $C, r > 0$ such that

$$\frac{|\psi(B) - \psi(A) - L(A^{-1} \diamond B)|}{d_\phi(A, B)} \leq C$$

for all $B \in I_r(A)$, whence

$$|\psi(B) - \psi(A)| \leq |\langle w_L, A^{-1} \diamond B\rangle| + C d_\phi(A, B) \leq (|w_L| + C) d_\phi(A, B).$$

We will denote by $d_{W^\phi}\psi(A)$ the $\diamond$-linear functional L such that (4.16) holds; we will call the vector w_L the W^ϕ-*differential* of ψ at A, and we will denote it by $W^\phi\psi(A)$, writing $[W^\phi\psi(A)]_j$ for w_{Lj}, $j = 2, \dots, 2n$. These definitions are well posed because of the following

Lemma 4.12. *Let $\phi, \psi : \omega \to \mathbb{R}$ be such that ψ is W^ϕ-differentiable at $A \in \omega$, and let L be a $\diamond$-linear functional such that* (4.16) *holds; then L is unique.*

Proof. We have to prove that, if $w, w' \in \mathbb{R}^{2n-1}$ are W^ϕ-differentials of ψ at A, then $w = w'$. We will give the proof only for the case $n \geq 2$, as it can be easily adapted for $n = 1$. Therefore let $A = (\eta, v, \tau)$: it is easy to prove that

$$\lim_{B=(\eta',v',\tau')\to A} \frac{\langle w - w', (\eta' - \eta, v' - v)\rangle}{d_\phi(A, B)} = 0. \tag{4.18}$$

Let

$$\begin{aligned}\mathcal{A} &= \{B = (\eta', v', \tau') \in \omega : d_\phi(A, B) = |(\eta' - \eta, v' - v)|\} \\ &= \{B = (\eta', v', \tau') \in \omega : \pi_Z(\Phi'^{-1} \cdot \Phi) = 0\} \\ &= \{B = (\eta', v', \tau') \in \omega : \tau' = \tau - 2(\phi' + \phi)(\eta' - \eta) - \sigma(v', v)\}\end{aligned}$$

where, here and in the following, we write Φ', Φ, ϕ' and ϕ instead of $\Phi(B)$, $\Phi(A)$, $\phi(B)$ and $\phi(A)$ respectively. Let $\delta_2 > 0$ be such that $I := \overline{I_{\delta_2}(A)} \subset \omega$; we want to prove that there exists a $\delta_1 > 0$ with the property that for all (η', v') with $|(\eta' - \eta, v' - v)| \leq \delta_1$ there is a $\tau' \in [\tau - \delta_2, \tau + \delta_2]$ such that $(\eta', v', \tau') \in \mathcal{A}$, i.e.

$$\tau' = \tau - 2(\phi' + \phi)(\eta' - \eta) - \sigma(v', v)\,.$$

Being ϕ continuous we can suppose that $|\phi| \leq M$ on I; then, for each (η', v') with $|(\eta' - \eta, v' - v)| \leq \delta_1$, the functions

$$\gamma_{(\eta',v')}(\tau') := \tau - 2(\phi(\eta', v', \tau') + \phi(A))(\eta' - \eta) - \sigma(v', v)$$

map the closed interval $[\tau - \delta_2, \tau + \delta_2]$ into itself provided δ_1 is sufficiently small. In fact

$$\begin{aligned}&|\gamma_{(\eta',v')}(\tau') - \tau| \\ =\ &\left|2(\phi' + \phi)(\eta' - \eta) + 2\textstyle\sum_{j=2}^n (v_j v'_{n+j} - v_{n+j} v'_j)\right| \\ =\ &\left|2(\phi' + \phi)(\eta' - \eta) + 2\textstyle\sum_{j=2}^n (v_j (v'_{n+j} - v_{n+j}) - v_{n+j}(v'_j - v_j))\right| \\ \leq\ &2M\delta_1 + 2|v|\delta_1 \end{aligned} \tag{4.19}$$

so it is sufficient to choose δ_1 such that $(2M + 2|v|)\delta_1 \leq \delta_2$. The fixed point theorem guarantees that $\gamma_{(\eta',v')}$ has a fixed point $\tau'(\eta', v')$ if $|(\eta' - \eta, v' - v)| \leq \delta_1$, so that $(\eta', v', \tau'(\eta', v')) \in \mathcal{A}$, i.e.

$$d_\phi((\eta', v', \tau'(\eta', v')), (\eta, v, \tau)) = |(\eta' - \eta, v' - v)|\,.$$

Moreover, it is not difficult to prove that $\tau'(\eta', v') \to \tau$ if $(\eta', v') \to (\eta, v)$ (it is sufficient to use the very same estimate as in (4.19)). Now, for each fixed $j = 2, \dots 2n$, we can easily construct a sequence $B^h = (\eta^h, v^h, \tau^h) \in \mathcal{A}$ such that

- $B^h \to A$;
- $\eta^h \equiv \eta$, $v_i^h \equiv v_i \; \forall\, i \neq j$ and $d_\phi(B^h, A) = v_j^h - v_j > 0 \quad$ if $j \neq n+1$;
- $v^h \equiv v$ and $d_\phi(B^h, A) = \eta^h - \eta > 0 \quad$ if $j = n+1$.

By (4.18) we obtain

$$0 = \lim_{h\to\infty} \frac{\langle w - w', (\eta^h - e, v^h - v)\rangle}{d_\phi(B^h, A)} = w_j - w'_j\,,$$

whence $w = w'$. □

Remark 4.13. Let $A \in \omega$ and $P := \Phi(A)$. With the same notations of Proposition 4.6, set $\sigma_{P^{-1}}(B) := \iota^{-1}(P^{-1} \cdot \iota(B) \cdot P_{L_1})$ and $\omega' := \sigma_{P^{-1}}(\omega)$. Let $\alpha\Theta$ denote the element $(0, \dots, 0, \alpha) \in \mathbb{R}^{2n}$ and define

$$\begin{aligned} \phi' :\;& \omega' \longrightarrow \mathbb{R} \\ & B = (\eta', v', \tau') \longmapsto \phi(\sigma_P(B)) - \phi(A); \end{aligned}$$

then $\Phi'(\omega') = \ell_{P^{-1}}(\Phi(\omega))$, where as usual $\Phi'(B) = \iota(B) \cdot \phi'(B) e_1$.
It is not difficult to show that a function ψ is W^ϕ-differentiable (resp. uniformly W^ϕ-differentiable) at $B \in \omega$ if and only if $\psi \circ \sigma_P$ is $W^{\phi'}$-differentiable (resp. uniformly $W^{\phi'}$-differentiable) at $\sigma_{P^{-1}}(B) \in \omega'$: the key observation is that

$$d_\phi(B, B') = d_{\phi'}(\sigma_{P^{-1}}(B), \sigma_{P^{-1}}(B')).$$

The following Proposition shows that uniformly W^ϕ-differentiable functions have continuous W^ϕ-differentials:

Proposition 4.14. *Let $\phi, \psi : \omega \to \mathbb{R}$ be two continuous functions; suppose that there exists an $\overline{A} \in \omega$ such that ψ is uniformly W^ϕ-differentiable at $\overline{A}$ and that ψ is W^ϕ-differentiable in an open neighbourhood $\mathcal{U}$ of $\overline{A}$. Then $W^\phi : \mathcal{U} \to \mathbb{R}^{2n-1}$ is continuous at $\overline{A}$.*

Proof. As usual we give the proof only for $n \geq 2$. Suppose that the thesis is not true; then there exist $\delta > 0$ and a sequence $\{A^j\} \subset \mathcal{U}$ such that $A^j \to \overline{A}$ and

$$|W^\phi\psi(A^j) - W^\phi\psi(\overline{A})| \geq 3\delta.$$

By the uniform W^ϕ-differentiability of ψ at $\overline{A}$ we can find an open rectangle I centered at $\overline{A}$ such that

$$\sup_{\substack{A,B\in I\\ A=(\eta,v,\tau)\neq B=(\eta',v',\tau')}} \left\{ \frac{|\psi(B)-\psi(A)-\langle W^\phi\psi(\overline{A}),(\eta'-\eta,v'-v)\rangle|}{d_\phi(B,A)} \right\} \leq \delta. \tag{4.20}$$

There is no loss of generality if we suppose that $A^j=(\eta^j,v^j,\tau^j)\in I$ for all j; then, using the W^ϕ-differentiability of ψ at A^j and reasoning as in Lemma 4.12, we can find a sequence of points $B^j=(\eta'^j,v'^j,\tau'^j)\in I$ such that

$$\frac{|\psi(B^j)-\psi(A^j)-\langle W^\phi\psi(A^j),(\eta'^j-\eta^j,v'^j-v^j)\rangle|}{d_\phi(B^j,A^j)}\leq\delta; \tag{4.21}$$

$$d_\phi(B^j,A^j)=|(\eta'^j-\eta^j,v'^j-v^j)|; \tag{4.22}$$

$$\text{the vectors } (\eta'^j-\eta^j,v'^j-v^j) \text{ and } (W^\phi\psi(A^j)-W^\phi\psi(\overline{A})) \text{ are parallel.} \tag{4.23}$$

Observe that (4.22) and (4.23) imply that

$$\begin{aligned}|\langle W^\phi\psi(A^j)-W^\phi\psi(\overline{A}),(\eta'^j-\eta^j,v'^j-v^j)\rangle|&\\ =|W^\phi\psi(A^j)-W^\phi\psi(\overline{A})|d_\phi(B^j,A^j)\ &\geq\ 3\delta d_\phi(B^j,A^j)\,.\end{aligned}$$

Then, using also (4.21), we get

$$\begin{aligned}\frac{|\psi(B^j)-\psi(A^j)-\langle W^\phi\psi(\overline{A}),(\eta'^j-\eta^j,v'^j-v^j)\rangle|}{d_\phi(B^j,A^j)}&\\ \geq\ \frac{|\langle W^\phi\psi(A^j)-W^\phi\psi(\overline{A}),(\eta'^j-\eta^j,v'^j-v^j)\rangle|}{d_\phi(B^j,A^j)}&+\\ -\frac{|\psi(B^j)-\psi(A^j)-\langle W^\phi\psi(A^j),(\eta'^j-\eta^j,v'^j-v^j)\rangle|}{d_\phi(B^j,A^j)}&\\ \geq\ \frac{3\delta d_\phi(B^j,A^j)-\delta d_\phi(B^j,A^j)}{d_\phi(B^j,A^j)}\geq 2\delta&\end{aligned}$$

which contradicts (4.20). □

It is not clear whether the converse is true, i.e. if W^ϕ-differentiability in an open neighbourhood and continuity of the W^ϕ-differential imply uniform W^ϕ-differentiability. Observe that this is true when we consider the classical notion of differentiability in Euclidean spaces.

Recalling how we defined the family W^ϕ of the $2n$-1 first-order operators W^ϕ_j (that, as usual, we identify with the associated vector fields), the following Proposition explains why we call the vector w_L (with L as in (4.16)) the W^ϕ-differential of ψ: the fact is that the j-th component of this vector is (at least for regular maps) the derivative of ψ in the W^ϕ_j-direction:

Proposition 4.15. *Let $\phi, \psi : \omega \to \mathbb{R}$ be continuous functions such that ψ is W^ϕ-differentiable at a point $A = (\eta, v, \tau) \in \omega$ (respectively $A = (\eta, \tau)$ if $n = 1$). For $j = 2, \dots, 2n$ let $\gamma^j : [-\delta, \delta] \to \omega$ be a $\mathbf{C}^1$-integral curve of the vector field W_j^ϕ with $\gamma^j(0) = A$ and such that the map*

$$[-\delta, \delta] \ni s \longmapsto \phi(\gamma^j(s)) \in \mathbb{R}$$

is of class $\mathbf{C}^1$. Then we have

$$\lim_{s \to 0} \frac{\psi(\gamma^j(s)) - \psi(\gamma^j(0))}{s} = \left[W^\phi \psi(A)\right]_j . \tag{4.24}$$

Proof. Again we accomplish the proof only for $n \geq 2$. Let us fix the following notation: if $\gamma^j(s) = (\eta(s), v(s), \tau(s))$ we set

$$\begin{array}{ll} \gamma_i^j(s) := v_i(s) & \text{for } 2 \leq i \leq 2n, i \neq n+1 \\ \gamma_{n+1}^j(s) := \eta(s) & \\ \gamma_{2n+1}^j(s) := \tau(s) & \end{array}$$

For $j \neq n+1$ the thesis is obvious: indeed we must have $\gamma^j(s) = A \diamond \exp(s\widetilde{X}_j)$ i.e. $\iota(\gamma^j(s)) = \iota(A) \cdot \exp(sX_j)$, and so

$$d_\phi(A, \gamma^j(s)) = |s| ,$$

which gives immediately (4.24) as a consequence of the W^ϕ-differentiability and the fact that

$$\gamma_i^j(s) \equiv v_i \text{ for } i \notin \{j, 2n+1\}, \qquad \gamma_j^j(s) = v_j + s.$$

For $j = n+1$ we have

$$\begin{cases} \gamma_i^{n+1}(s) = v_i & \text{if} \quad i \neq n+1, 2n+1 \\ \gamma_{n+1}^{n+1}(s) = \eta + s & \\ \gamma_{2n+1}^{n+1}(s) = \tau - 4 \int_0^s \phi(\gamma^{n+1}(r)) dr & \end{cases} \tag{4.25}$$

and so

$$\begin{aligned} & d_\phi(\gamma^{n+1}(s), \gamma^{n+1}(0)) \\ = \;& |s| + \left| -4 \int_0^s \phi(\gamma^{n+1}(r)) dr + 2[\phi(\gamma^{n+1}(s)) + \phi(A)] s \right|^{1/2} \\ = \;& |s| \left(1 + \frac{1}{|s|} \left| -4 \int_0^s \phi(\gamma^{n+1}(r)) dr + 2[\phi(\gamma^{n+1}(s)) + \phi(A)] s \right|^{1/2} \right) \\ =: \;& |s| \left(1 + \frac{1}{|s|} |\Delta(s)|^{1/2} \right) . \end{aligned}$$

One has $|\Delta(s)| \le Cs^2$ for a certain $C > 0$, indeed

$$\begin{aligned}\Delta(s) &= -4\int_0^s \phi(\gamma^{n+1}(r))dr + 2[\phi(\gamma^{n+1}(s)) + \phi(A)]s \\ &= -4\int_0^s [\phi(\gamma^{n+1}(r)) - \phi(A)]dr + 2[\phi(\gamma^{n+1}(s)) - \phi(A)]s \\ &= O(s^2)\,; \end{aligned} \tag{4.26}$$

it follows that $d_\phi(\gamma^{n+1}(s), A) \le (1 + \sqrt{C})|s|$ and so

$$\begin{aligned}&\frac{|\,\psi(\gamma^{n+1}(s)) - \psi(\gamma^{n+1}(0)) - [W^\phi\psi(A)]_{n+1}s\,|}{|s|} \\ &\qquad \le\ (1+\sqrt{C})\frac{|\,\psi(\gamma^{n+1}(s)) - \psi(A) - L_{W^\phi\psi(A)}(A^{-1}\diamond\gamma^{n+1}(s))\,|}{d_\phi(\gamma^{n+1}(s), A)}.\end{aligned}$$

By letting $s \to 0$ and using the W^ϕ-differentiability of ψ at A we obtain the thesis (4.24). □

The following result shows that the class of ϕ, ψ such that ψ is W^ϕ-differentiable (in fact, uniformly W^ϕ-differentiable) is not empty, and gives an explicit formula for the differential $W^\phi\psi$ of smooth functions.

Theorem 4.16. *Let $\phi, \psi \in \mathbf{C}^1(\omega)$; then ψ is uniformly W^ϕ-differentiable at A for all $A \in \omega$ and*

$$W^\phi\psi(A) = \left(\widetilde{X}_2\psi, \dots, \widetilde{X}_n\psi, \widetilde{Y}_1\psi - 4\phi\widetilde{T}\psi, \widetilde{Y}_2\psi, \dots, \widetilde{Y}_n\psi\right)(A)$$

for all $A \in \omega$. In particular, $W^\phi\psi : \omega \to \mathbb{R}^{2n-1}$ is continuous.

Proof. Let us fix $A = (\overline{\eta}, \overline{v}, \overline{\tau}) \in \omega$ ($A = (\overline{\eta}, \overline{\tau})$ if $n = 1$) and set

$$w(A) := \left(\widetilde{X}_2\psi, \dots, \widetilde{X}_n\psi, \widetilde{Y}_1\psi - 4\phi\widetilde{T}\psi, \widetilde{Y}_2\psi, \dots, \widetilde{Y}_n\psi\right)(A) \in \mathbb{R}^{2n-1}$$

if $n \ge 2$, while for $n = 1$ we set

$$w(A) := \widetilde{Y}_1\psi(A) - 4\phi(A)\,\widetilde{T}\psi(A) = \frac{\partial\psi}{\partial\eta}(A) - 4\phi(A)\frac{\partial\psi}{\partial\tau}(A)\,.$$

According to the notation of Definition 4.9, we have to prove that

$$\lim_{r\to 0} M_\phi(\psi, A, w(A), r) = 0. \tag{4.27}$$

Therefore let $B, B' \in \omega$ be sufficiently close to A (in a way we are going to specify later), and for $n \geq 2$ let $\overline{X}, \overline{W}$ be the $\mathbf{C}^1$ vector fields given by

$$\overline{X} := \sum_{j=2, j\neq n+1}^{2n} (v'_j - v_j)\widetilde{X}_j, \qquad \overline{W} := \frac{\partial}{\partial \eta} - 4\phi \frac{\partial}{\partial \tau}.$$

Let us set

$$\begin{aligned} B^* &:= \exp(\overline{X})(B) \\ &= B \diamond (0, (v'_2 - v_2, \dots, v'_n - v_n, v'_{n+2} - v_{n+2}, \dots, v'_{2n} - v_{2n}), 0) \\ &= (\eta, v', \tau - \sigma(v', v)) \\ B'' &:= \exp((\eta' - \eta)\overline{W})(B^*) = (\eta', v', \tau'') \quad \text{(for a certain } \tau''); \end{aligned}$$

observe that B^* and B'' are well defined if $B, B' \in I_{\delta_0}(A)$ for a sufficiently small δ_0. For $n = 1$, $\overline{X}$ is not defined and we set $B^* = B$ and $B'' := \exp((\eta' - \eta)\overline{W})(B) = (\eta', \tau'')$.
As ψ is of class $\mathbf{C}^1$ we have

$$\begin{aligned} & \psi(B') - \psi(B) \\ = \ & [\psi(B') - \psi(B'')] + [\psi(B'') - \psi(B^*)] + [\psi(B^*) - \psi(B)] \\ = \ & [\psi(B') - \psi(B'')] + \int_0^{\eta' - \eta} (\overline{W}\psi)\big(\exp(s\overline{W})(B^*)\big)\, ds + \\ & + \int_0^1 \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} (v'_j - v_j)(\widetilde{X}_j\psi)\big(\exp(s\overline{X})(B)\big)\, ds \\ = \ & [\psi(B') - \psi(B'')] + \sum_{j=2, j\neq n+1}^{2n} (v'_j - v_j)\widetilde{X}_j\psi(A) + \\ & + (\eta' - \eta)\overline{W}\psi(A) + o(|(\eta' - \eta, v' - v)|) \\ = \ & [\psi(B') - \psi(B'')] + \langle w(A), (\eta' - \eta, v' - v)\rangle + o(d_\phi(B', B)). \end{aligned} \tag{4.28}$$

For $n = 1$ the same calculation leads to

$$\psi(B') - \psi(B) = [\psi(B') - \psi(B'')] + w(A)(\eta' - \eta) + o(d_\phi(B', B)).$$

Therefore it is sufficient to prove that $\psi(B') - \psi(B'') = o(d_\phi(B', B))$. We have

$$\frac{|\psi(B') - \psi(B'')|}{d_\phi(B', B)} \leq \omega_\psi(\delta_0) \cdot \frac{|\tau' - \tau''|^{1/2}}{d_\phi(B', B)} \tag{4.29}$$

where

$$\omega_\psi(\delta) := \sup\left\{\frac{|\psi(A') - \psi(A'')|}{|A' - A''|^{1/2}} : A' \neq A'' \in I_\delta(A)\right\}, \tag{4.30}$$

and where we know that $\omega_\psi(\delta) \to 0$ as $\delta \downarrow 0$ because ψ is $\mathbf{C}^1$. So we have to prove that $|\tau' - \tau''|^{1/2}/d_\phi(B', B)$ is bounded in a proper neighbourhood of A. Observe that

$$\begin{aligned}
& |\tau' - \tau''| \\
= \;& \left|\tau' - \tau + \sigma(v', v) + 4\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*))\, ds\right| \\
\leq \;& |\tau' - \tau + 2(\phi(B') + \phi(B))(\eta' - \eta) + \sigma(v', v)| + \\
& + 2\left|2\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*))\, ds - (\phi(B') + \phi(B))(\eta' - \eta)\right| \\
\leq \;& d_\phi(B', B)^2 + 2\,|\phi(B') - \phi(B'')||\eta' - \eta| + 2\,|\phi(B) - \phi(B^*)||\eta' - \eta| + \\
& + 2\left|2\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*)\, ds - [\phi(B'') + \phi(B^*)](\eta' - \eta)\right| \\
=: \;& d_\phi(B', B)^2 + R_1(B', B) + R_2(B', B) + R_3(B', B).
\end{aligned} \tag{4.31}$$

For the case $n = 1$ we arrive to (4.31) along the same lines (it is sufficient to follow the same steps "erasing" the term $\sigma(v', v)$).

Now we want to prove that there exist $C_1, C_2 > 0$ such that

$$R_3(B', B) \leq C_1|\eta' - \eta|^2 \tag{4.32}$$

$$R_2(B', B) \leq C_2 d_\phi(B', B)^2 \tag{4.33}$$

for all $B', B \in I_{\delta_0}(A)$, and that for all $\epsilon > 0$ there is a $\delta_\epsilon \in]0, \delta_0]$ such that

$$R_1(B', B) \leq |\eta' - \eta|^2 + \epsilon|\tau' - \tau''| \tag{4.34}$$

for all $B', B \in I_{\delta_\epsilon}(A)$. These estimates are sufficient to conclude: in fact, choosing $\epsilon := 1/2$ and using (4.31), (4.32), (4.34) and (4.33), we get

$$|\tau' - \tau''| \leq d_\phi(B', B)^2 + C_1|\eta' - \eta|^2 + |\eta' - \eta|^2 + |\tau' - \tau''|/2 + C_2 d_\phi(B', B)^2$$

whence

$$|\tau' - \tau''|^{1/2} \leq C_3 d_\phi(B, B')$$

which is the thesis.

For $s \in [-\delta_0, \delta_0]$ we can define

$$g(s) := 2\int_0^s \phi(\exp(r\overline{W})(B^*))\, dr - \big[\phi\big(\exp(s\overline{W})(B^*)\big) + \phi(B^*)\big]\, s; \tag{4.35}$$

as in (4.26) one can prove that there is a $C_1 > 0$ such that

$$|g(s)| \le C_1 s^2 \quad \text{for all } s \in [-\delta_0, \delta_0]\,,$$

so that (4.32) follows with $s = \eta' - \eta$.

If ω_ϕ is as in (4.30) (with ϕ instead of ψ), then

$$\begin{aligned} R_1(B, B') &\le 2\omega_\phi(\delta)|\tau' - \tau''|^{1/2}|\eta' - \eta| \\ &\le |\eta' - \eta|^2 + \omega_\phi(\delta)^2|\tau' - \tau''|. \end{aligned}$$

Since ϕ is $\mathbf{C}^1$, $\omega_\phi(\delta) \to 0$ for $\delta \downarrow 0$, and so for all $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that for all $\delta \in]0, \delta_\epsilon]$ we have $\omega_\phi(\delta)^2 \le \epsilon$, whence (4.34) follows.

Finally,(4.33) follows from $R_2(B, B') = 0$ if $n = 1$, and from

$$\begin{aligned} R_2(B, B') &= |\eta' - \eta||\phi(B) - \phi(B^*)| \\ &= |\eta' - \eta| \left| \sum_{j=2,\, j\neq n+1}^{2n} (v'_j - v_j)(w(A)_j + o(1)) \right| \\ &\le 2C_2|\eta' - \eta||v' - v| \le C_2 d_\phi(B', B)^2 \end{aligned}$$

if $n \ge 2$. □

4.3 $\mathbb{H}$-regular graphs and W^ϕ-differentiability

In this section we are going to characterize $\mathbb{H}$-regular graphs in terms of the uniform W^ϕ-differentiability of their parametrizations. In the sequel, for a given function f of class $\mathbf{C}^1_{\mathbb{H}}$ on an open set $\Omega \subset \mathbb{H}^n$ it will be convenient to write

$$\widehat{\nabla_{\mathbb{H}}} f := (X_2 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \quad \in \mathbf{C}^0(\Omega, \mathbb{R}^{2n-1}).$$

The main theorem of the section is the following

Theorem 4.17. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function and let $\Phi : \omega \to \mathbb{H}^n$ be the function defined by*

$$\Phi(A) := \iota(A) \cdot \phi(A) e_1.$$

Let $S := \Phi(\omega)$. Then the following conditions are equivalent:

(i) S is an $\mathbb{H}$-regular surface and $\nu_{S,1}(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu_{S,1}(P), \dots, \nu_{S,2n}(P))$ denotes the horizontal normal to S at a point $P \in S$;

(ii) ϕ is uniformly W^ϕ-differentiable at any $A \in \omega$.

Moreover, for all $P \in S$ we have

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1+|W^\phi \phi|^2}}, \frac{W^\phi \phi}{\sqrt{1+|W^\phi \phi|^2}} \right) (\Phi^{-1}(P)) \in \mathbb{R} \times \mathbb{R}^{2n-1}. \tag{4.36}$$

and

$$c(n)\, \mathcal{S}^{Q-1}_\infty(S) = \int_\omega \sqrt{1+|W^\phi \phi(A)|^2}\, d\mathcal{L}^{2n}(A). \tag{4.37}$$

Proof. We will give the proof only for $n \geq 2$, since the generalization to $n = 1$ is immediate.

Step 1. Let us begin with the proof of the implication $(i) \Rightarrow (ii)$. Let $P = \Phi(A) \in S$, where $A = (\eta, v, \tau) \in \omega$; then there exist an $r_0 > 0$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r_0))$ such that

$$\begin{aligned} &S \cap U(P, r_0) = \{Q \in U(P, r_0) : f(Q) = 0\} \\ &\nabla_{\mathbb{H}} f(Q) = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)(Q) \neq 0 \text{ for all } Q \in U(P, r_0). \end{aligned}$$

As $\nu_S(Q) = -\nabla_{\mathbb{H}} f(Q)/|\nabla_{\mathbb{H}} f(Q)|$, by hypothesis we have that

$$X_1 f(Q) > 0 \text{ for all } Q \in S \cap U(P, r_0). \tag{4.38}$$

Moreover without loss of generality we can suppose that

$$A = (\eta, v, \tau) = (0, 0, 0) \text{ and } P = \Phi(0, 0, 0) = 0. \tag{4.39}$$

Indeed, if this is not the case, let us consider $S' := \ell_{P^{-1}}(S) = \Phi'(\omega')$, where we use the same notations of Remark 4.13. We have that $S' \cap U(0, r_0)$ is an $\mathbb{H}$-regular surface because it is the zero set of the function $f' = f \circ \ell_P$, and by left invariance $X_1 f'(Q) = X_1 f(P \cdot Q) > 0$ for all $Q \in U(0, r_0)$. Finally (again by Remark 4.13), ϕ' (which is equal to $\phi \circ \sigma_P$ up to an additive constant) is uniformly $W^{\phi'}$-differentiable if and only if ϕ is uniformly W^ϕ-differentiable.

By the uniqueness of the parametrization provided by the Implicit Function Theorem we can assume that there is a $\overline{\delta} > 0$ such that $\overline{I_\delta} := \overline{I_{\overline{\delta}}(0,0,0)} \Subset \omega$ and

$$f(\Phi(B)) = 0 \text{ for all } B \in \overline{I_{\overline{\delta}}}. \tag{4.40}$$

With the assumptions in (4.39), by the continuity of Φ for each $r \in]0, r_0/4[$ there is a $0 < \delta_r < r$ such that

$$\Phi(I_{\delta_r}(0,0,0)) \subset U(0, r). \tag{4.41}$$

For each $B=(\eta,v,\tau), B'=(\eta',v',\tau')\in I_{\delta_r}(0)$, with δ_r sufficiently small, we get, by applying Lemma 3.11 to f with $P=0$, $Q=\Phi(B)$, $Q'=\Phi(B')$, that

$$\begin{aligned}
& |\langle \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi_{\Phi(B)}(\Phi(B)^{-1}\Phi(B'))\rangle| \\
= \;& |f(\Phi(B'))-f(\Phi(B))+\langle\, \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi_{\Phi(B)}(\Phi(B)^{-1}\Phi(B'))\,\rangle| \\
\le \;& C_1\, R(\delta_r)\, d_\infty(\Phi(B'),\,\Phi(B)) \\
\le \;& C_2\, R(\delta_r)\, \big[\|\pi_{L_1}(\Phi(B)^{-1}\Phi(B'))\|_\infty + \|\pi_{O_1}(\Phi(B)^{-1}\Phi(B'))\|_\infty + \\
& + \|\pi_Z(\Phi(B)^{-1}\Phi(B'))\|_\infty\big] \\
\le \;& C_2\, R(\delta_r)\, [\,|\phi(B')-\phi(B)|+d_\phi(B,B')\,] \qquad (4.42)
\end{aligned}$$

where C_1 is given by Lemma 3.11 and

$$R(\delta):=\sup\big\{\|\nabla_{\mathbb{H}} f(\cdot)-\nabla_{\mathbb{H}} f(P')\|_{L^\infty(U(P',2d_\infty(P',P'')))} : P',P''\in\Phi(I_\delta(0,0)).\big\}$$

By the uniform continuity of $\nabla_{\mathbb{H}} f:\overline{U(0,r_0/2)}\to H\mathbb{H}^n$ we have

$$\lim_{r\downarrow 0} R(\delta_r)=0. \qquad (4.43)$$

Therefore, (4.42) and (4.38) imply

$$\begin{aligned}
& \left|\phi(B')-\phi(B)+\frac{\langle\widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\,(\eta'-\eta,v'-v)\rangle}{X_1 f(\Phi(B))}\right| \\
= \;& \frac{\big|\langle \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi_{\Phi(B)}(\Phi(B)^{-1}\Phi(B'))\rangle\big|}{X_1 f(\Phi(B))} \\
\le \;& \Big[\inf_{U(0,r_0)} X_1 f\Big]^{-1} C_2 R(\delta_r)\,\big[|\phi(B')-\phi(B)|+d_\phi(B,B')\big] \qquad (4.44)
\end{aligned}$$

for any $B,B'\in I_{\delta_r}$. By (4.43) we can suppose

$$\frac{C_2}{\inf_{U(0,r_0)} X_1 f} R(\delta_{\overline{r}})\le\frac{1}{2}$$

for a certain $\overline{r}\in]0,r_0/4[$, and so

$$\begin{aligned}
|\phi(B')-\phi(B)| \;\le\; & \left|\phi(B')-\phi(B)+\frac{\langle\widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\,(\eta'-\eta,v'-v)\rangle}{X_1 f(\Phi(B))}\right| + \\
& + \left|\frac{\langle\widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\,(\eta'-\eta,v'-v)\rangle}{X_1 f(\Phi(B))}\right| \\
\le \;& \big[|\phi(B')-\phi(B)|+d_\phi(B,B')\big]/2 + C_3|(\eta'-\eta,v'-v)|
\end{aligned}$$

for each $B, B' \in I_{\delta_{\overline{r}}}$. Therefore there exists a constant $C_4 > 0$ such that

$$|\phi(B') - \phi(B)| \leq C_4 d_\phi(B, B'). \tag{4.45}$$

Putting together (4.44) and (4.45) we get that there is a $C_5 > 0$ for which

$$\left| \phi(B') - \phi(B) + \frac{\langle \widehat{\nabla}_{\mathbb{H}}(\Phi(B)),\, (\eta' - \eta, v' - v) \rangle}{X_1 f(\Phi(B))} \right| \leq C_5\, R(\delta_r)\, d_\phi(B, B') \tag{4.46}$$

and so

$$\frac{\left| \phi(B') - \phi(B) + \left\langle \dfrac{\widehat{\nabla}_{\mathbb{H}} f(0)}{X_1 f(0)},\, (\eta' - \eta, v' - v) \right\rangle \right|}{d_\phi(B, B')} \leq C_5 R(\delta_r) + \sup_{I_{\delta_r}(0)} \left| \frac{\widehat{\nabla}_{\mathbb{H}} f(\Phi(\cdot))}{X_1 f(\Phi(\cdot))} - \frac{\widehat{\nabla}_{\mathbb{H}} f(0)}{X_1 f(0)} \right|$$

for each $B, B' \in I_{\delta_r}(0)$ with $r \leq \overline{r}$. Thanks to (4.43) and the fact that f is of class $\mathbf{C}^1_{\mathbb{H}}$ we get that

$$\lim_{r \downarrow 0} \; M_\phi\big(\phi, 0, \tfrac{\widehat{\nabla}_{\mathbb{H}} f(0)}{X_1 f(0)}, \delta_r\big) = 0\,,$$

i.e. ϕ is uniformly W^ϕ-differentiable at 0 and

$$W^\phi \phi(0) = -\frac{\widehat{\nabla}_{\mathbb{H}} f}{X_1 f}(0). \tag{4.47}$$

More generally, one has

$$W^\phi \phi(\Phi^{-1}(P)) = -\frac{\widehat{\nabla}_{\mathbb{H}} f}{X_1 f}(P)\,,$$

from which (4.36) immediately follows; therefore the implication $(i) \Rightarrow (ii)$ is completely proved.

Step 2. Now we have to prove the converse implication $(ii) \Rightarrow (i)$. Let $A = (\eta, v, \tau) \in \omega$ and $P = \Phi(A) \in S$. We have to find $r_0 > 0$ and a $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r_0))$ such that

$$S \cap U(P, r_0) = \{Q \in U(0, r_0) : f(Q) = 0\} \tag{4.48}$$

$$X_1 f(Q) > 0 \text{ for all } Q \in U(P, r_0). \tag{4.49}$$

Let δ_1 be such that $I_{\delta_1}(A) \Subset \omega$; as $\Phi : \omega \to S$ is a homeomorphism we can suppose that

$$S \cap \overline{\mathcal{U}} = \Phi\big(\overline{I_{\delta_1}(A)}\big)$$

for a certain open bounded neighbourhood $\mathcal{U}$ of P. Let $F := S \cap \overline{\mathcal{U}}$ and $g : F \to \mathbb{R}$ be defined by $g(Q) := 0$. Define

$$\begin{aligned} k \;:\; & F \longrightarrow H\mathbb{H}^n \equiv \mathbb{R}^{2n} \\ & Q \longmapsto (1, -W^\phi\phi(\Phi^{-1}(Q))) \end{aligned}$$

We start by proving that, thanks to Whitney's extension Theorem 3.12, there is a function $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n, \mathbb{R})$ such that

$$f \equiv g \equiv 0 \quad \text{on } F \tag{4.50}$$

$$\nabla_{\mathbb{H}} f(Q) = k(Q) = \big(1, -W^\phi\phi(\Phi^{-1}(Q))\big) \quad \text{for all } Q \in F. \tag{4.51}$$

Consider a compact subset K of F; for $Q, Q' \in K$ and $\delta > 0$ let

$$R(Q,Q') := \frac{g(Q') - g(Q) - \langle\, k(Q), \pi_Q(Q^{-1}Q')\,\rangle}{d_\infty(Q,Q')} = -\frac{\langle\, k(Q), \pi_Q(Q^{-1}Q')\,\rangle}{d_\infty(Q,Q')}$$

$$\rho_K(\delta) := \sup\left\{|R(Q,Q')| : Q, Q' \in K, 0 < d_\infty(Q,Q') < \delta\right\}.$$

In order to apply Whitney's Theorem (which will provide the desired f) we have only to show that

$$\lim_{\delta \downarrow 0} \rho_K(\delta) = 0. \tag{4.52}$$

Let us suppose that the converse is true, i.e. that there is an $\epsilon_0 > 0$ such that for all $h \in \mathbb{N}$ there are

$$\begin{aligned} & Q^h = \Phi(B^h), \quad Q^{h\prime} = \Phi(B^{h\prime}) \in K\,, \\ & B^h = (\eta^h, v^h, \tau^h), \quad B^{h\prime} = (\eta^{h\prime}, v^{h\prime}, \tau^{h\prime}) \end{aligned}$$

for which

$$0 < d_\infty(Q^h, Q^{h\prime}) < 1/h \tag{4.53}$$

$$\epsilon_0 \le |R(Q^h, Q^{h\prime})| \le \frac{|\phi^{h\prime} - \phi^h - \langle\, W^\phi\phi(B^h), (\eta^{h\prime} - \eta^h, v^{h\prime} - v^h)\,\rangle|}{d_\phi(B^h, B^{h\prime})} \tag{4.54}$$

where as usual we denoted by $\phi^{h\prime}, \phi^h$ the quantities $\phi(B^{h\prime})$ and $\phi(B^h)$ respectively. In (4.54) we used the fact that $d_\infty(\Phi(B), \Phi(B')) \ge d_\phi(B, B')$; this estimate, together with (4.53), implies that $d_\phi(B^h, B^{h\prime})) \le 1/h$ and so

$$|(\eta^{h\prime} - \eta^h, v^{h\prime} - v^h)| \le 1/h \tag{4.55}$$

$$|\tau^{h\prime} - \tau^h + 2(\phi^{h\prime} + \phi^h)(\eta^{h\prime} - \eta^h) + \sigma(v^{h\prime}, v^h)| \le 1/h^2. \tag{4.56}$$

Setting $M := \sup_K |\phi|$ and $N := \sup_K |(\eta, v)|$ we get

$$\begin{array}{rcll} |\tau^{h\prime} - \tau^h| & \leq & 1/h^2 + 2|\phi^{h\prime} + \phi^h||\eta^{h\prime} - \eta^h| + 2|\sigma(v'^h, v^h)| & (*) \\ & \leq & 1/h^2 + 4M|\eta^{h\prime} - \eta^h| + 2N|v^{h\prime} - v^h| & (**) \\ & \leq & C/h & \end{array} \tag{4.57}$$

where $C := 1 + 4M + 2N > 0$ depends only on K. In $(*)$ we used that $\sigma(v^{h\prime}, v^h) = 2\sum_{j=2}^n [v^{h\prime}_{n+j}(v^h_j - v^{h\prime}_j) - v^{h\prime}_j(v^h_{n+j} - v^{h\prime}_{n+j})]$, while (4.55) justifies $(**)$. Since K is compact, up to subsequences there is a $B = (\eta', v', \tau') \in \overline{I_{\delta_1}(A)} \supset K$ such that

$$\lim_{h\to\infty} B^h = \lim_{h\to\infty} B^{h\prime} = B.$$

In particular $B^h, B^{h\prime} \in \overline{I_{r(h)}(B)}$ (where $r(h) \to 0$ as $r \to 0$), and by (4.54) and the continuity of the W^ϕ-differential one has

$$0 < \epsilon'_0 \leq M_\phi(\phi, B, W^\phi\phi(B), r(h))$$

for any h, which contradicts the fact that ϕ is uniformly W^ϕ-differentiable at $B \in I_{\delta_1}(A)$. This is sufficient to apply Whitney's Extension Theorem, and so we get the existence of an $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n, \mathbb{R})$ for which (4.50) and (4.51) hold.

The proof of the implication $(ii) \Rightarrow (i)$ will be complete if we show the validity of (4.48) and (4.49) for a certain r_0. Let $S' := \{Q \in \mathbb{H}^n : f(Q) = 0, \nabla_{\mathbb{H}} f(Q) \neq 0\}$; as we have already said, we can suppose that $P = 0$ and $A = 0$. Since $0 \in S \cap \mathcal{U} \subset S'$, one has

$$f(0) = 0 \quad \text{and} \quad \nabla_{\mathbb{H}} f(0) = (1, -W^\phi\phi(0))$$

and by the Implicit Function Theorem there are an open neighbourhood $\mathcal{U}'$ of 0 and a continuous function $\phi' : \overline{I_{\delta'}(0)} \to \mathbb{R}$ such that

$$\begin{array}{rcl} \Phi' & : & \overline{I_{\delta'}(0)} \to S' \cap \overline{\mathcal{U}'} \\ & & B \longmapsto \iota(B) \cdot \phi'(B) e_1 \end{array}$$

is a homeomorphism. Therefore $\Phi'^{-1}(S' \cap \mathcal{U}')$ is an open subset of $\overline{I_{\delta'}(0)}$ which contains 0, and so there exists a $\delta'' \in]0, \delta'[$ for which $I_{\delta''}(0) \subset \Phi'^{-1}(S' \cap \mathcal{U}')$; by the uniqueness of the parametrization we get that $\Phi' \equiv \Phi$ on $I_{\delta''}(0)$. Now, let $\mathcal{U}''$ and $\mathcal{U}'''$ be open neighbourhoods of 0 in $\mathbb{H}^n$ such that

$$S \cap \mathcal{U}'' = \Phi(I_{\delta''}(0)) = \Phi'(I_{\delta''}(0)) = S' \cap \mathcal{U}''' \tag{4.58}$$

and let $r_0 > 0$ be such that $U(0, r_0) \subset \mathcal{U}'' \cap \mathcal{U}'''$. Then by (4.58) we get $U(0, r_0) \cap S = U(0, r_0) \cap S'$, from which (4.48) and (4.49) follow.

Finally, the area type formula (4.37) follows from Corollary 4.5 after finding a global f (that is given only locally), which can be done by a standard argument involving a partition of the unity. This completes the proof of the Theorem. □

Corollary 4.18. *With the same notations of Theorem 4.17, suppose that $S := \Phi(\omega)$ is $\mathbb{H}$-regular; then $\phi : (\omega, d_\phi) \to \mathbb{R}$ is locally Lipschitz continuous.*

Proof. The thesis follows from Theorem 4.17 and Remark 4.11. □

Now we want to establish some Hölder continuity properties for uniformly W^ϕ-differentiable functions on ω and therefore for parametrizations of $\mathbb{H}$-regular graphs; in particular we want to improve the Hölder continuity obtained in (4.14). More precisely we have the following

Proposition 4.19. *Let $\phi : \omega \to \mathbb{R}$ be uniformly W^ϕ-differentiable at $A \in \omega$. Then there is an $r_0 > 0$ such that $I_{r_0}(A) \Subset \omega$ and*

$$\lim_{r \downarrow 0} \quad \sup \left\{ \frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} : B, B' \in I_{r_0}(A), 0 < |B - B'| < r \right\} = 0.$$

Proof. Again we treat only the case $n \geq 2$.

If $B = (\eta, v, \tau)$ and $B' = (\eta', v', \tau')$ let us set

$$R(\delta) := \sup \left\{ \frac{|\phi(B') - \phi(B) - \langle\, W^\phi \phi(A), (\eta' - \eta, v' - v)\,\rangle|}{d_\phi(B', B)} : B' \neq B \in I_\delta(A) \right\};$$

by the uniform W^ϕ-differentiability of ϕ at A we know that $\lim_{\delta \downarrow 0} R(\delta) = 0$. In particular there is an $r_0 > 0$ such that ϕ is Lipschitz continuous between $(I_{r_0}(A), d_\phi)$ and $\mathbb{R}$, i.e. (4.13) holds. Then by (4.14) (see the steps that lead to (4.15)) there is a $C_1 > 0$ such that

$$d_\phi(B', B) \leq C_1 |B' - B|^{1/2} \quad \text{for all } B', B \in I_{r_0}(A). \tag{4.59}$$

But if $B' \neq B \in I_r(A)$, $0 < r < r_0$, we have

$$\begin{aligned} \frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} \quad &\leq \quad \frac{|\phi(B') - \phi(B) - \langle W^\phi \phi(A), (\eta' - \eta, v' - v)\rangle|}{d_\phi(B', B)} \cdot \frac{d_\phi(B', B)}{|B' - B|^{1/2}} + \\ &\quad + |W^\phi \phi(A)| \frac{|(\eta' - \eta, v' - v)|}{|B' - B|^{1/2}} \\ &\leq \quad C_1 R(r) + C_2 |W^\phi \phi(A)| r^{1/2} \longrightarrow 0 \qquad \text{for } r \downarrow 0. \end{aligned}$$

This completes the proof. □

From Proposition 4.19 and a standard compactness argument we get the following

Corollary 4.20. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function and consider the related $\Phi : \omega \to \mathbb{H}^n$. Let $S := \Phi(\omega)$ and suppose that S is an $\mathbb{H}$-regular surface with $\nu_{S,1}(P) < 0$ for all $P \in S$; then for each $\omega' \Subset I$ we have*

$$\lim_{r \downarrow 0} \quad \sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in \omega', 0 < |A - B| < r \right\} = 0.$$

Finally, we stress an interesting approximation property for the parametrizations of $\mathbb{H}$-regular graphs:

Proposition 4.21. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function which is uniformly W^ϕ-differentiable at any $A \in \omega$; then for any $A \in \omega$ there is a $\delta = \delta(A) > 0$, with $I_\delta(A) \Subset \omega$, and a family $\{\phi_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^\infty(\overline{I_\delta(A)}, \mathbb{R})$ such that*

$$\phi_\epsilon \to \phi \text{ and } W^{\phi_\epsilon}\phi_\epsilon \to W^\phi \phi \qquad \text{uniformly on } \overline{I_\delta(A)}.$$

Proof. Arguing as in the proof of Theorem 4.17 we can suppose that $A = 0$, $\Phi(0) = 0$ and

$$S \cap U(0, r) = \{P \in U(0, r) : f(P) = 0\}$$

for certain $r > 0$ and $f \in \mathbf{C}^1_{\mathbb{H}}(U(0, r))$ such that $f \circ \Phi \equiv 0$ on $I_\delta(A)$, with δ sufficiently small. Moreover, arguing as in the proof of the Implicit Function Theorem 3.16, we can suppose that, for a certain $0 < r' < r$ (and possibly considering a smaller δ), there are two families $\{f_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^\infty(\overline{U(0, r')})$ and $\{\phi_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^\infty(\overline{I_\delta(A)})$ such that

$$\begin{aligned} &f_\epsilon \to f \text{ and } \nabla_{\mathbb{H}} f_\epsilon \to \nabla_{\mathbb{H}} f && \text{uniformly on } \overline{U(0, r')} \\ &\phi_\epsilon \to \phi \text{ and } -\frac{\widehat{\nabla_{\mathbb{H}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon \to -\frac{\widehat{\nabla_{\mathbb{H}}} f}{X_1 f} \circ \Phi = W^\phi \phi && \text{uniformly on } \overline{I_\delta(A)} \end{aligned}$$

where $\Phi_\epsilon(A) := \iota(A) \cdot \phi_\epsilon(A) e_1$ is such that $f_\epsilon \circ \Phi_\epsilon = 0$; indeed the set $S_\epsilon := \{P \in \overline{U(0, r')} : f_\epsilon(P) = 0\} \supset \Phi_\epsilon(\overline{I_\delta(A)})$ is a (Euclidean) $\mathbf{C}^1$-surface, and then its parametrization ϕ_ϵ is uniformly W_{ϕ_ϵ}-differentiable with

$$W_{\phi_\epsilon}\phi_\epsilon = -\frac{\widehat{\nabla_{\mathbb{H}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon \,,$$

from which the thesis follows. □

4.4 Characterization of the uniform W^ϕ-differentiability and applications

The main result we are going to prove in this section is the following

Theorem 4.22. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then the following conditions are equivalent:*

(i) ϕ is uniformly W^ϕ-differentiable at A for each $A \in \omega$;

(ii) there exist a $w \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$ such that, in distributional sense,

$$\begin{array}{ll} w = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi) & \text{if } n \geq 2 \\ w = \mathfrak{B}\phi & \text{if } n = 1 \end{array}$$

and there is a family $\{\phi_\epsilon\}_{\epsilon>0} \subset \mathbf{C}^\infty(\omega)$ such that, for any open $\omega' \Subset \omega$, we have

$$\phi_\epsilon \to \phi \text{ and } W^{\phi_\epsilon}\phi_\epsilon \to w \text{ uniformly on } \omega'. \tag{4.60}$$

Moreover, $w = W^\phi\phi$ on ω and

$$\lim_{r\to 0^+} \sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in \omega', 0 < |A - B| < r \right\} = 0. \tag{4.61}$$

for each $\omega' \Subset \omega$.

Remark 4.23. Suppose $n = 1$ and $w \equiv 0$, then the functions $\phi : \omega \to \mathbb{R}$ satisfying condition (ii) of Theorem 4.22 are entropy solutions of Burgers' scalar conservation law in classical sense. Indeed by performing the change of variables

$$\mathbb{R}^2 = \mathbb{R}_x \times R_t \ni (x, t) \longmapsto (t, -4x) \in \mathbb{R}^2 = \mathbb{R}_\eta \times R_\tau$$

the Burgers' operator $\mathfrak{B}$ can be represented in classical way with respect to the variables (x, t) as

$$\mathfrak{B}u = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x}$$

if $u = u(x, t) \in \mathbf{C}^1(\omega^*)$ and $\omega^* \subset \mathbb{R}^2$ is a fixed open set (see [66], chapter III, section 3). In this case condition (ii) of Theorem 4.22 reads as the existence of a function $u : \omega^* \to \mathbb{R}$ and of a family $\{u_\varepsilon\}_\varepsilon \subset \mathbf{C}^\infty(\omega^*)$ such that

$$u_\epsilon \to u \quad \text{and} \quad \mathfrak{B}u_\epsilon \to 0 \qquad \text{uniformly on } \omega' \tag{4.62}$$

for any open $\omega' \Subset \omega^*$. Let us assume now $\omega^* = (a, b) \times (-\delta, \delta)$ and let $g(x) := u(x, 0)$ if $x \in (a, b)$. We claim that u is an *entropy solution* of the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial u^2}{\partial x} = 0 & \text{in } (a, b) \times (0, \delta) \\ u = g & \text{on } (a, b) \times \{t = 0\}, \end{cases}$$

More precisely, by definition (see [66], chapter XI, section 11.4.3), we have to prove that

$$u \in \mathbf{C}^0([0,\delta), L^1_{loc}(a,b)) \cap L^\infty_{loc}(\omega^*)\,; \tag{4.63}$$

$$u(\cdot,t) \to g \text{ in } L^1_{loc}(a,b) \quad \text{as } t \to 0^+\,; \tag{4.64}$$

$$\int_{\omega^*} \left[e(u)\frac{\partial v}{\partial t} + d(u)\frac{\partial v}{\partial x} \right] dxdt \geq 0 \tag{4.65}$$

for each $v \in \mathbf{C}^1_c(\omega^*)$, $v \geq 0$ and for each entropy/entropy flux pair (e,d), i.e. two smooth functions $e, d : \mathbb{R} \to \mathbb{R}$ such that e is convex and $e'(u)u = d'(u)\,\forall u \in \mathbb{R}$. Then (4.63) and (4.64) follow at once because $u \in \mathbf{C}^0(\omega^*)$. As $u_\varepsilon \in \mathbf{C}^1(\omega^*)$

$$\frac{\partial(e(u_\epsilon))}{\partial t} + \frac{\partial(d(u_\epsilon))}{\partial x} = w_\epsilon\, e'(u_\epsilon) \text{ in } \omega^* \tag{4.66}$$

pointwise, with $w_\epsilon = \mathfrak{B}u_\epsilon$ and, by (4.62), $w_\epsilon \to 0$ uniformly in ω'. Therefore multiplying both sides of (4.66) for a given $v \in \mathbf{C}^1_c(\omega^*)$, integrating by parts and taking the limit as $\epsilon \to 0^+$ we get (4.65) too (actually with an equality, so with no entropy production).

Remark 4.24. Let $n \geq 2$ and let assume that $\phi : \omega \to \mathbb{R}$ satisfies condition (ii) of Theorem 4.22 with $w \equiv 0$ in an open connected set $\omega \subset \mathbb{R}^{2n}$; then ϕ is constant in ω. Indeed for a fixed $A_0 \in \omega$ let $B = B(A_0, r_0) \subset \omega$ be a Euclidean ball centered at A_0 with radius $r_0 > 0$ and, for a fixed $\eta \in \mathbb{R}$, let

$$B_\eta := \{(v,\tau) \in \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau : (\eta, v, \tau) \in B\}, \quad \phi_\eta(v,\tau) := \phi(\eta, v, \tau) \text{ if } (v,\tau) \in B_\eta\,.$$

The open set $B_\eta \subset \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau \equiv \mathbb{H}^{n-1}$ is connected and

$$\widetilde{X}_j \phi_\eta = \widetilde{Y}_j \phi_\eta = 0 \text{ in } B_\eta \quad (j = 2, \dots, n)\,,$$

in distributional sense; therefore we get

$$\phi(\eta, v, \tau) \equiv \phi(\eta) \quad \forall (\eta, v, \tau) \in B\,. \tag{4.67}$$

In fact a Poincaré inequality holds in $(\mathbb{H}^{n-1}, d_c)$ with respect to the horizontal gradient $\nabla_{\mathbb{H}} := (\widetilde{X}_2, \dots, \widetilde{X}_n, \widetilde{Y}_2, \dots, \widetilde{Y}_n)$ (see, for instance, [94], Proposition 11.17) and then there exists a constant $c > 0$ such

$$\int_{U_c(P,r)} |\phi_\eta - \phi_{\eta,U_c}|\, d\mathcal{L}^{2n-1} \leq\ c\, r \int_{U_c(P,r)} |\nabla_{\mathbb{H}} \phi_\eta|\, d\mathcal{L}^{2n-1}$$

for every $P \in \mathbb{H}^{n-1}$, $r > 0$ such that $U_c(P,r) := \{Q \in \mathbb{H}^{n-1} : d_c(P,Q) < r\} \subset B_\eta$ and

$$\phi_{\eta,U_c} := \fint_{U_c(P,r)} \phi_\eta\, d\mathcal{L}^{2n-1}\,.$$

On the other hand by (4.67) we infer

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} = 0 \text{ in } B$$

in distributional sense. Thus ϕ is constant in $B(A_0, r_0)$ for all $A_0 \in \omega$ for suitable $r_0 > 0$. As ϕ is continuous and ω is connected we can conclude that ϕ actually is constant in the whole ω.

Observe that the same statement fails when $n = 1$: see e.g. Example 5.8

In order to prove Theorem 4.22 we will need some further notation and preliminary results.

Let $\phi : \omega \to \mathbb{R}$ be a continuous function, and suppose that for all $A \in \omega$ there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$ there exists a map

$$\begin{array}{rcl} \gamma_j & : & [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega \\ & & (s, B) \longmapsto \gamma_j^B(s) \end{array}$$

such that $\gamma_j^B \in \mathbf{C}^1([-\delta_2, \delta_2], \mathbb{R}^{2n})$ for each $B \in \overline{I_{\delta_2}(A)}$ and, with the usual identification between vector fields and differential operators,

$$\text{(E.1)} \quad \begin{cases} \dot\gamma_j^B = W_j^\phi \circ \gamma_j^B = \begin{cases} \widetilde{X}_j \circ \gamma_j^B & \text{if } j \neq n+1 \\ \partial_\eta - 4(\phi \circ \gamma_{n+1}^B)\partial_\tau & \text{if } j = n+1 \end{cases} \\ \gamma_j^B(0) = B; \end{cases}$$

(E.2) there is a suitable continuous function $w_j : \omega \to \mathbb{R}$ (depending only on ϕ) such that

$$\phi(\gamma_j^B(s)) - \phi(\gamma_j^B(0)) = \int_0^s w_j(\gamma_j^B(r))\, dr$$

for each $s \in [-\delta_2, \delta_2]$.

We will call the $\{\gamma_j\}$ a family of *exponential maps of* W^ϕ at A; we will write $\exp_A(sW_j^\phi)(B) := \gamma_j^B(s)$. Notice that here we are not asking these maps to be continuous in the parameter B: see also Remark 4.34.

Remark 4.25. Notice that if the exponential maps of W^ϕ at A exist, then the map

$$[-\delta_2, \delta_2] \ni s \longmapsto \phi(\exp_A(sW_j^\phi)(B))$$

is of class $\mathbf{C}^1$ for each $j = 2, \dots, 2n$ and each $B \in I_{\delta_2}(A)$.

Remark 4.26. Observe that, because of the left invariance of the fields $\widetilde{X}_j$, for $j \neq n$ one must have

$$\exp_A(sW_j^\phi)(B) = B \diamond \iota^{-1}(\exp sX_j) = B \diamond \iota^{-1}(s\, e_j). \tag{4.68}$$

Moreover, if there exist exponential maps of W^ϕ at A (in particular there are w_j as in $(E.2)$), then for any $\lambda = (\lambda_2, \dots, \lambda_n, \lambda_{n+2}, \dots, \lambda_{2n}) \in \mathbb{R}^{2n-2}$ there exists also an exponential map for the field $\sum \lambda_j W_j^\phi$, i.e. there are two continuous maps $\gamma_\lambda : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$ (with, possibly, a $\delta_2 > 0$ smaller than the one in (E.1), depending on λ) and $w_\lambda : \omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\dot\gamma_\lambda(s, B) = \sum \lambda_j\, W_j^\phi(\gamma_\lambda(s, B)) \\
&\gamma_\lambda(0, B) = B \\
&\phi(\gamma_\lambda(s, B)) - \phi(\gamma_\lambda(0, B)) = \int_0^s w_\lambda(\gamma(r, B))\, dr
\end{aligned}$$

In fact, it is sufficient to take $\gamma_\lambda(s, B) := B \diamond (0, s\lambda, 0)$ and $w_\lambda := \sum \lambda_j w_j$.

The following Lemma provides sufficient conditions to guarantee the existence of exponential maps of W^ϕ.

Lemma 4.27. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function and suppose that*

(i) there exists $w \in \mathbf{C}^0(\omega)$ such that

$$\begin{aligned}
&w = (w_2, \dots, w_{2n}) = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{X}_{n+2}\phi, \dots, \widetilde{X}_{2n}\phi) && \text{if } n \geq 2 \\
&w = \mathfrak{B}\phi && \text{if } n = 1
\end{aligned}$$

in distributional sense;

(ii) there is a family of functions $\{\phi_\epsilon\}_{\epsilon>0} \subset \mathbf{C}^\infty(\omega, \mathbb{R})$ such that

$$\phi_\epsilon \to \phi,\ W^{\phi_\epsilon}\phi_\epsilon \to w \quad \text{uniformly on } \overline{\omega'}$$

for any $\omega' \Subset \omega$.

Then for each $A \in \omega$ there are $0 < \delta_2 < \delta_1$ such that, for any $j = 2, \dots, 2n$ and all $(s, B) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$, there exists $\exp_A(sW_j^\phi)(B) \in \overline{I_{\delta_1}(A)} \Subset \omega$; moreover,

$$w_j(B) = \frac{d}{ds}\phi\big(\exp_A(sW_j^\phi)(B)\big)_{|s=0}.$$

Proof. Again we can suppose $n \geq 2$, as for $n = 1$ the proof can easily be derived.

There is no problem if $j \neq n+1$; in fact by (4.68) it is sufficient to set

$$\exp_A(sW_j^\phi)(B) := B \diamond \exp(s\widetilde{X}_j)$$

which is defined on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$ for a sufficiently small δ_2 with values in $\overline{I_{\delta_1(A^0)}} \Subset \omega$. Then (E.1) is fulfilled by construction and (E.2) comes from the continuity of ϕ and the fact that $w_j = \widetilde{X}_j\phi$ in distributional sense.

For $j = n+1$ and $\epsilon > 0$ consider the Cauchy problem

$$\begin{cases} \dot{\gamma}_\epsilon(s, B) = \partial_\eta - 4\phi_\epsilon(\gamma_\epsilon(s, B))\partial_\tau = W_{n+1}^{\phi_\epsilon}(\gamma_\epsilon(s, B)) \\ \gamma_\epsilon(0, B) = B \end{cases}$$

which has a solution $\gamma_\epsilon : [-\delta_2(\epsilon), \delta_2(\epsilon)] \times \overline{I_{\delta_2(\epsilon)}(A)} \to \overline{I_{\delta_1}(A)}$. By Peano's estimate on the existence time for solutions of ordinary differential equations we obtain that $\delta_2(\epsilon)$ can be taken greater than $C/\|\phi_\epsilon\|_{L^\infty(I_{\delta_1}(A))}$ (where the constant C depends only on δ_1), and so we get a $\delta_2 > 0$ such that $\delta_2(\epsilon) \geq \delta_2$ for all ϵ.
Now, for each fixed $B \in \overline{I_{\delta_2}(A)}$ the functions $\gamma_\epsilon(\cdot, B)$ are uniformly continuous on $[-\delta_2, \delta_2]$, and by Ascoli-Arzelá's Theorem we get a sequence $\{\epsilon_h\}_h$ such that $\epsilon_h \to 0$ and $\gamma_{\epsilon_h}(\cdot, B) \to \gamma(\cdot, B)$ uniformly on $[-\delta_2, \delta_2]$. Remembering that

$$\gamma_{\epsilon_h}(s, B) = B + \int_0^s \left[\frac{\partial}{\partial\eta} - 4\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s, B))\frac{\partial}{\partial\tau}\right] ds$$
$$\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s, A)) - \phi_{\epsilon_h}(\gamma_{\epsilon_h}(0, B)) = \int_0^s W_{n+1}^{\phi_{\epsilon_h}}\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s, B))\, ds$$

and for $j \to \infty$ we get (all the involved convergences are uniform)

$$\gamma(s, B) = B + \int_0^s \left[\frac{\partial}{\partial\eta} - 4\phi(\gamma(s, B))\frac{\partial}{\partial\tau}\right] ds$$
$$\phi(\gamma(s, B)) - \phi(\gamma(0, B)) = \int_0^s w_{n+1}(\gamma(s, B))\, ds$$

i.e. (E.1) and (E.2) holds. □

As in Euclidean spaces the gradient of a function is the vector composed by the derivatives along the exponentials of the vectors of the canonical basis, we will prove, in the following theorem, that the W^ϕ-differential is the vector made by the derivatives along the exponentials of W^ϕ.

Theorem 4.28. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function such that, for a certain $A \in \omega$, the following conditions are fulfilled:*

(i) there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$ there exist a family of exponential maps

$$\exp_A(sW_j^\phi) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)}.$$

(ii) for each $\omega' \Subset \omega$

$$\lim_{r \to 0^+} \sup \left\{ \frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} : B', B \in \overline{\omega'},\ 0 < |B' - B| \le r \right\} = 0.$$

Then ϕ is uniformly W^ϕ-differentiable at A and

$$[(W^\phi \phi)(A)]_j = \frac{d}{ds} \phi\big(\exp_A(sW_j^\phi)(A)\big)_{|s=0}.$$

Proof. For $n \ge 2$ let $A = (\overline{\eta}, \overline{v}, \overline{\tau}), B = (\eta, v, \tau), B' = (\eta', v', \tau') \in \omega$, while for $n = 1$ $A = (\overline{\eta}, \overline{\tau}), B = (\eta, \tau), B' = (\eta', \tau') \in \omega$, and let $w = (w_2, \dots, w_{2n})$ be as in (E.2). We have to prove that

$$\lim_{\delta \to 0} M_\phi(\phi, A, w(A), \delta) = 0 \tag{4.69}$$

where M_ϕ is defined as in (4.17).

The proof is exactly the same as in Theorem 4.16: at first, for $n > 1$, we define the vector field $\overline{X} := \sum_{j=2, j \neq n+1}^{2n} (v_j' - v_j) W_j^\phi = \sum_{j=2, j \neq n+1}^{2n} (v_j' - v_j) \widetilde{X}_j$, and then we set

$$\begin{aligned} B^* &:= \exp_A(\overline{X})(B) \\ &= B \diamond (0, (v_2' - v_2, \dots, v_n' - v_n, v_{n+2}' - v_{n+2}, \dots, v_{2n}' - v_{2n}), 0) \\ &= (\eta, v', \tau - \sigma(v', v)). \end{aligned}$$

If $n = 1$, $\overline{X}$ is not defined and we set $B^* := B$.

The main obstacle is that in general we cannot integrate along the vector field W_{n+1}^ϕ, i.e. we cannot define $B'' := \exp\big((\eta' - \eta)(\frac{\partial}{\partial \eta} - 4\phi \frac{\partial}{\partial \tau})\big)(B^*)$; however, this problem can be solved using the existence of exponential maps, more precisely by posing

$$B'' := \exp_A((\eta' - \eta) W_{n+1}^\phi)(B^*) = \begin{array}{ll} (\eta', v', \tau'') & \text{if } n \ge 2 \\ (\eta', \tau'') & \text{if } n = 1 \end{array}$$

for a certain τ''. Therefore, we can rewrite (4.28) as

$$\begin{aligned}
\phi(B')-\phi(B) &= [\phi(B')-\phi(B'')]+[\phi(B'')-\phi(B^*)]+[\phi(B^*)-\phi(B)]\\
&= [\phi(B')-\phi(B'')]+\int_0^{\eta'-\eta} w_{n+1}(\exp_A(sW^\phi_{n+1})(B^*))\,ds+\\
&\quad+\int_0^1 \sum_{j=2,\,j\neq n+1}^{2n}(v'_j-v_j)w_j(\exp_A(s\overline{X})(B)) \quad (*)\\
&= [\phi(B')-\phi(B'')]+\sum_{j=2,\,j\neq n+1}^{2n}(v'_j-v_j)w_j(A)+\\
&\quad+(\eta'-\eta)w_{n+1}(A)+o(|(\eta'-\eta,v'-v)|)\\
&= [\phi(B')-\phi(B'')]+\langle w(A),\,(\eta'-\eta,v'-v)\rangle+o(d_\phi(B',B))
\end{aligned}$$

if $n\geq 2$, and as

$$\phi(B')-\phi(B)=[\phi(B')-\phi(B'')]+w(A)(\eta'-\eta)+o(d_\phi(B',B))$$

if $n=1$. Observe that in the passage signed with $(*)$ we have used the continuity of the w_j at A. Reasoning as in (4.29) and (4.30), the keypoint is again to prove that the quantity $|\tau'-\tau''|^{1/2}/d_\phi(B',B'')$ is bounded in a neighbourhood of A, and rewriting (4.31) we obtain

$$\begin{aligned}
&|\tau'-\tau''|\\
=\;&\left|\tau'-\tau+\sigma(v',v)+4\int_0^{\eta'-\eta}\phi(\exp_A(sW^\phi_{n+1})(B^*))\,ds\right|\\
\leq\;&|\tau'-\tau+2(\phi(B')+\phi(B))(\eta'-\eta)+\sigma(v',v)|+\\
&+2\left|2\int_0^{\eta'-\eta}\phi(\exp_A(sW^\phi_{n+1})(B^*))\,ds-(\phi(B')+\phi(B))(\eta'-\eta)\right|\\
\leq\;&d_\phi(B',B)^2+2\,|\phi(B')-\phi(B'')||\eta'-\eta|+2\,|\phi(B)-\phi(B^*)||\eta'-\eta|+\\
&+2\left|2\int_0^{\eta'-\eta}\phi(\exp_A(sW^\phi_{n+1})(B^*)\,ds-[\phi(B'')+\phi(B^*)](\eta'-\eta)\right|\\
=:\;&d_\phi(B',B)^2+R_1(B',B)+R_2(B',B)+R_3(B',B) \qquad (4.70)
\end{aligned}$$

for $n\geq 2$; for $n=1$ simply don't consider the term $\sigma(v',v)$. Therefore we have once again to prove (4.32), (4.33), (4.34); this can be done following exactly the same line as in the proof of Theorem 4.16 and using (E.1) and (E.2): the only thing one must pay attention to is to write $\exp_A(\cdot\, W^\phi_{n+1})$ instead of $\exp(\cdot\,\overline{W})$ in (4.35). □

We are now in order to give the proof of Theorem 4.22.

Proof (of Theorem 4.22). We will accomplish the proof only for $n \geq 2$, because as usual the generalization to $n = 1$ is immediate.

Step 1. Let us begin with the proof of the implication $(i) \Rightarrow (ii)$. The statement in (4.61) follows from Theorem 4.17 and Corollary 4.20. By Proposition 4.21 we get that for each $B \in \omega$ there is a $\delta(B) > 0$ (with $I_{\delta(B)}(B) \Subset \omega$) and a family of $\mathbf{C}^\infty$ functions $\{\phi_{\epsilon,B} : \overline{I_{\delta(B)}(B)} \to \mathbb{R}\}_{0<\epsilon<1}$ such that

$$\phi_{\epsilon,B} \to \phi \quad \text{and} \quad W_{\phi_{\epsilon,B}}\phi_{\epsilon,B} \to W^\phi\phi \qquad \text{uniformly on } \overline{I_{\delta(B)}(B)}. \tag{4.71}$$

As $\mathcal{F} := \{I_{\delta(B)}(B) : B \in \omega\}$ is an open covering of ω we can associate a partition of the unity $\{\theta_i : i \in \mathbb{N}\}$ which is subordinate to it, i.e.

$$\theta_i \in \mathbf{C}^\infty_c(\omega), 0 \leq \theta_i \leq 1 \text{ on } \omega \text{ for all } i \tag{4.72}$$

$$\{\text{spt } \theta_i\}_{i\in\mathbb{N}} \text{ form a locally finite covering of } \omega, \text{ and for all } i \in \mathbb{N} \text{ there is an } I_i := I_{\delta(B(i))}(B(i)) \in \mathcal{F} \text{ such that spt } \theta_i \subset I_i \tag{4.73}$$

$$\textstyle\sum_{i=1}^\infty \theta_i \equiv 1 \text{ on } \omega. \tag{4.74}$$

Let $\phi_{\epsilon,i} := \phi_{\epsilon,B(i)} : \mathbb{R}^{2n} \to \mathbb{R}$ where from now on, if necessary, we use the convention of extending functions by letting them vanish outside their domain. Let $\phi_\epsilon := \sum_{i=1}^\infty \theta_i\, \phi_{\epsilon,i}$; by construction $\phi_\epsilon \in \mathbf{C}^\infty(\omega)$ and

$$\begin{aligned}
\frac{\partial\phi_\epsilon}{\partial\eta} &= \sum_{i=1}^\infty \left(\frac{\partial\theta_i}{\partial\eta}\phi_{\epsilon,i} + \theta_i\frac{\partial\phi_{\epsilon,i}}{\partial\eta}\right)\\
\frac{\partial\phi_\epsilon}{\partial v_j} &= \sum_{i=1}^\infty \left(\frac{\partial\theta_i}{\partial v_j}\phi_{\epsilon,i} + \theta_i\frac{\partial\phi_{\epsilon,i}}{\partial v_j}\right) \qquad (n \geq 2)\\
\frac{\partial\phi_\epsilon}{\partial\tau} &= \sum_{i=1}^\infty \left(\frac{\partial\theta_i}{\partial\tau}\phi_{\epsilon,i} + \theta_i\frac{\partial\phi_{\epsilon,i}}{\partial\tau}\right).
\end{aligned}$$

In particular

$$W^{\phi_\epsilon}\phi_\epsilon = \sum_{i=1}^\infty \left(\phi_{\epsilon,i}W^{\phi_\epsilon}\theta_i + \theta_i W^{\phi_\epsilon}\phi_{\epsilon,i}\right) \qquad \text{on } \omega.$$

We have to show that (4.60) holds for any fixed $\omega' \Subset \omega$; by (4.73) there is only a finite number of index $i_1, \ldots, i_k$ such that $\overline{\omega'} \cap \text{spt } \theta_{i_h} \neq \emptyset$, and $\overline{\omega'} \subset \cup_{h=1}^k \text{spt } \theta_{i_h}$. Then

$$\phi_\epsilon = \sum_{h=1}^k \theta_{i_h}\phi_{\epsilon,i_h} \quad \text{and} \quad \phi = \sum_{h=1}^k \theta_{i_h}\phi \qquad \text{on } \overline{\omega'} \tag{4.75}$$

$$W^{\phi_\epsilon}\phi_\epsilon = \sum_{h=1}^k (\phi_{\epsilon,i_h}W^{\phi_\epsilon}\theta_{i_h} + \theta_{i_h}W^{\phi_\epsilon}\phi_{\epsilon,i_h}) \qquad \text{on } \overline{\omega'}. \tag{4.76}$$

Equations (4.75) and (4.76), together with (4.71), give

$$\phi_\epsilon \to \phi \tag{4.77}$$

$$W^{\phi_\epsilon}\phi_\epsilon \to \sum_{h=1}^{k}(\phi W^\phi_{i_h} + \theta_{i_h} W^\phi \phi) =: w \tag{4.78}$$

uniformly on $\overline{\omega'}$, where we put

$$W^\phi_{i_h} := \left(\widetilde{X}_2\theta_{i_h}, \dots, \widetilde{X}_n\theta_{i_h}, \frac{\partial\theta_{i_h}}{\partial\eta} - 4\phi\frac{\partial\theta_{i_h}}{\partial\tau}, \widetilde{Y}_2\theta_{i_h}, \dots, \widetilde{Y}_n\theta_{i_h}\right).$$

Observing that $\sum_{h=1}^k \phi\, W^\phi_{i_h} = 0$ we get that $w = W^\phi\phi \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$ and

$$w = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi)$$

in distributional sense.

Step 2. The reverse implication $(ii) \Rightarrow (i)$ follows from Lemma 4.27 and Theorem 4.28. The hypothesis (ii) of Theorem 4.28 (i.e. the assertion in (4.61)) is satisfied because of the following Theorem 4.30: the key observation is that, thanks to the uniform convergence of ϕ_ϵ and $W^{\phi_\epsilon}\phi_\epsilon$, we can estimate $\|\phi_\epsilon\|_{L^\infty(\omega'')}$ and $\|W^{\phi_\epsilon}\phi_\epsilon\|_{L^\infty(\omega'')}$ uniformly in ϵ for any $\omega'' \Subset \omega$. Moreover, the uniform convergence of $W^{\phi_\epsilon}\phi_\epsilon$ allows us to choose a modulus of continuity for $W^{\phi_\epsilon}\phi_\epsilon$ which is independent of ϵ. Therefore there is a function $\alpha :]0, +\infty[\to \mathbb{R}$, which does not depend on ϵ, such that $\lim_{r\to 0}\alpha(r) = 0$ and

$$\sup\left\{\frac{|\phi_\epsilon(B') - \phi_\epsilon(B)|}{|B' - B|^{1/2}} : B', B \in \omega', 0 < |B' - B| \le r\right\} \le \alpha(r).$$

which implies (4.61). □

Theorem 4.29. *Let $I \subset \mathbb{R}^{2n}$ be a rectangle and let $\phi \in \mathbf{C}^1(I)$ be such that $W^\phi\phi = (w_2, \dots, w_{2n}) \in \mathbf{C}^0(I, \mathbb{R}^{2n-1})$, i.e.*

$$\begin{cases} \widetilde{X}_j\phi = w_j,\ \widetilde{Y}_j\phi = w_{j+n} \quad \text{for all } j = 2, \dots, n \\ \dfrac{\partial\phi}{\partial\eta} - 4\phi\dfrac{\partial\phi}{\partial\tau} = w_{n+1}. \end{cases}$$

Then for any rectangle $I' \Subset I$ there exists a function $\alpha :]0, +\infty[\to [0, +\infty[$, which depends only on I'', $\|\phi\|_{L^\infty(I'')}$, $\|W^\phi\phi\|_{L^\infty(I'')}$ and on the modulus of continuity of w_{n+1} on I'' (where I'' is any open rectangle satisfying $I' \Subset I'' \Subset I$), such that $\lim_{r\to 0}\ \alpha(r) = 0$ and

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in I', 0 < |A - B| \le r\right\} \le \alpha(r). \tag{4.79}$$

Proof. As usual we suppose $n \geq 2$, since the proof can be easily adapted to the case $n = 1$. We start by setting

$$K := \sup_{A \in I''} |A|, \quad M := \|\phi\|_{L^\infty(I'')} \quad \text{and} \quad N := \|W^\phi \phi\|_{L^\infty(I'')},$$

and let β be the modulus of continuity of w_{n+1} on I'', i.e. an increasing function $]0, +\infty[\ni r \to \beta(r) \in \mathbb{R}^+$ such that $|w_{n+1}(A) - w_{n+1}(B)| \leq \beta(|A - B|)$ for all $A, B \in I''$ and $\lim_{r \to 0} \beta(r) = 0$. We divide the proof in several steps.

Step 1. Let us fix another rectangle $J \subset \mathbb{R}^{2n}$ such that $I' \Subset J \Subset I''$, and let us introduce the following notation: for $A = (\eta, v, \tau) \in J$ we define γ_A as the curve solution of the Cauchy problem

$$\begin{cases} \dot{\gamma_A}(t) = \dfrac{\partial}{\partial \eta} - 4\phi(\gamma_A(t)) \dfrac{\partial}{\partial \tau} \\ \gamma_A(\eta) = A \end{cases}$$

Standard considerations on ordinary differential equations ensure that γ_A belongs to $\mathbf{C}^1([\eta - \epsilon, \eta + \epsilon], I'')$ for a certain $\epsilon > 0$ which does not depend on A; moreover, we can choose ϵ so that $\gamma_A([\eta - \epsilon, \eta + \epsilon]) \subset J$ for all $A \in I'$. Let $\gamma_A(t) = (\eta + t, v, \tau_A(t))$, then

$$\frac{d^2}{dt^2} \tau_{A_0}(t) = \frac{d}{dt}[-4\phi(\gamma_{A_0}(t))] = -4w_{n+1}(\gamma_{A_0}(t)). \tag{4.80}$$

Step 2. Set $\delta(r) := \max\{r^{1/4}, \beta(Er^{1/4})^{1/2}\}$, where $E > 0$ is a constant which will be specified later; we start by proving that $\alpha'(r) \leq \delta(r) + 2N^{1/2}\delta(r) + Nr^{1/2}$ for r "sufficiently small" (in a way we are going to specify, but depending on K, M, N and β only), where we have set

$$\alpha'(r) := \sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, v, \tau), B = (\eta', v, \tau') \in I', 0 < |A - B| \leq r \right\}.$$

Suppose on the contrary that there exist $A = (\eta, v, \tau), B = (\eta', v, \tau') \in I'$ such that $r := |A - B|$ is "sufficiently small" and

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} > \delta + 2N^{1/2}\delta + Nr^{1/2},$$

where from now on we will write δ instead of $\delta(|A - B|)$. We observe explicitly that by definition of $\delta(r)$ we have $\delta' := \delta(|\tau - \tau'|) \leq \delta$ and so

$$\begin{aligned} \frac{\beta\big(|\tau - \tau'| + 8M|\tau - \tau'|^{1/2}/\delta\big)}{\delta^2} &\leq \frac{\beta\big(|\tau - \tau'| + 8M|\tau - \tau'|^{1/2}/\delta'\big)}{\delta'^2} \\ &\leq \frac{\beta\big(|\tau - \tau'| + 8M|\tau - \tau'|^{1/4}\big)}{\delta'^2} \\ &\leq 1 \end{aligned} \tag{4.81}$$

provided $E > 0$ is such that $|\tau - \tau'| + 8M|\tau - \tau'|^{1/4} \le E|\tau - \tau'|^{1/4}$. Let $C := (\eta, v, \tau') \in I'$; as $|A - C|^{1/2} = |\tau - \tau'|^{1/2}$ and $|C - B|^{1/2} = |\eta - \eta'|^{1/2}$ we have

$$\begin{aligned}\delta + 2N^{1/2}\delta + Nr^{1/2} &\le \frac{|\phi(A) - \phi(B)|}{|\eta - \eta'|^{1/2} + |\tau - \tau'|^{1/2}} \\ &\le \frac{|\phi(A) - \phi(C)|}{|\tau - \tau'|^{1/2}} + \frac{|\phi(C) - \phi(B)|}{|\eta - \eta'|^{1/2}} =: R_1 + R_2.\end{aligned}$$

Thereforeone must have $R_1 \ge \delta$ or $R_2 \ge 2N^{1/2}\delta + Nr^{1/2}$.

Step 3. We want to prove that the first case cannot occur; indeed, we will prove that

$$\frac{|\phi(A) - \phi(C)|}{|\tau - \tau'|^{1/2}} \le \delta$$

for $A, B \in J$ (not for I' only!). We can suppose that $\tau > \tau'$ (for the other case it is sufficient to exchange the roles of A and C). Consider γ_A and γ_C; thanks to (4.80) we have, for $t \in [\eta - \epsilon, \eta + \epsilon]$

$$\begin{aligned}&\tau_A(t) - \tau_C(t) \\ &= \tau - \tau' + \int_\eta^t \left[\dot\tau_A(\eta) - \dot\tau_C(\eta) + \int_\eta^s \big[\ddot\tau_A(\sigma) - \ddot\tau_C(\sigma)\big] d\sigma\right] ds \\ &= \tau - \tau' - 4(t-\eta)\big[\phi(A) - \phi(C)\big] - 4\int_\eta^t\int_\eta^s \big[w_{n+1}(\gamma_A(\sigma)) - w_{n+1}(\gamma_C(\sigma))\big] d\sigma\, ds \\ &\le \tau - \tau' - 4(t-\eta)\big[\phi(A) - \phi(C)\big] + 2(t-\eta)^2\beta\big(|\tau - \tau'| + 8M|t - \eta|\big), \qquad (4.82)\end{aligned}$$

where in the last inequality we used the fact that

$$\begin{aligned}|\gamma_A(\sigma) - \gamma_C(\sigma)| &\le |\gamma_A(\eta) - \gamma_C(\eta)| + |\sigma - \eta|\,(\|\dot\tau_A\|_\infty + \|\dot\tau_C\|_\infty) \\ &\le |\tau - \tau'| + 8M|t - \eta|.\end{aligned}$$

We substitute in (4.82) the value

$$t := \begin{array}{ll} \eta + (\tau - \tau')^{1/2}/\delta & \text{if } \phi(A) - \phi(C) > 0 \\ \eta - (\tau - \tau')^{1/2}/\delta & \text{otherwise;}\end{array}$$

if $|\tau - \tau'|$ is "sufficiently small", $\gamma_A(t)$ and $\gamma_C(t) \in I''$ are well defined (it is sufficient to take $\epsilon \ge (\tau - \tau')^{1/4} \ge (\tau - \tau')^{1/2}/\delta = |t - \eta|$) and from (4.81), (4.82) and $R_1 \ge \delta$ we get (in both cases)

$$\begin{aligned}&\tau_A(t) - \tau_C(t) \\ &\le (\tau - \tau') - 4(\tau - \tau') + 2(\tau - \tau')\beta\big(|\tau - \tau'| + 8M|\tau - \tau'|^{1/2}/\delta\big)/\delta^2 \\ &\le -(\tau - \tau') \;<\; 0. \qquad (4.83)\end{aligned}$$

This leads to a contradiction: in fact τ_A and τ_C are solutions to the same Cauchy problem

$$\dot{\tau}(s) = -4\phi(s, v, \tau(s))$$

with initial data $\tau(\eta) = \tau, \tau'$ respectively. The contradiction is given by the fact that two such solutions cannot meet, while $\tau_A(\eta) - \tau_C(\eta) > 0$ and $\tau_A(t) - \tau_C(t) < 0$.

Step 4. Now let us examine the second case $R_2 \geq 2N^{1/2}\delta + Nr^{1/2}$; we can suppose that $\eta' < \eta$ (otherwise it is sufficient to exchange the roles of B and C). Consider γ_B; again, the point $D := \gamma_B(\eta) = (\eta, v, \tau'') \in J$ is well defined for $\eta - \eta'$ "sufficiently small", and

$$|\phi(B) - \phi(D)| = \left| \int_{\eta'}^{\eta} w_{n+1}(\gamma_B(t))dt \right| \leq N|\eta - \eta'|; \tag{4.84}$$

moreover

$$|\tau'' - \tau'| = \left|4 \int_{\eta'}^{\eta} \phi(\gamma_B(t))\, dt\right| \leq 4N|\eta - \eta'|. \tag{4.85}$$

Then for $|\eta' - \eta|$ "sufficiently small" (and precisely when $N|\eta - \eta'|^{1/2} \leq |\eta - \eta'|^{1/4} \leq \delta$) we obtain

$$\begin{aligned} |\phi(C) - \phi(D)| &\geq |\phi(C) - \phi(B)| - |\phi(B) - \phi(D)| \\ &\geq \left[\, 2N^{1/2}\delta + Nr^{1/2} - N|\eta - \eta'|^{1/2} \,\right] |\eta - \eta'|^{1/2} \\ &\geq 2N^{1/2}\delta|\eta - \eta'|^{1/2} \;\geq\; \delta|\tau'' - \tau'|^{1/2} \end{aligned} \tag{4.86}$$

so that we are in the first case again (with the couple $C, D \in J$ instead of A, C) which we have seen is not possible. This proves that $\lim_{r\to 0} \alpha'(r) = 0$, and that we are able to control α' with only K, M, N and β. Observe that what we said up to now, properly translated in the notation we use when $n = 1$, gives directly the thesis for the case $n = 1$.

Step 5. For the general case, let $A = (\eta, v, \tau), B = (\eta', v', \tau') \in I$, and set

$$A^* := A \diamond (0, v' - v, 0) = (\eta, v', \tau + \sigma(v, v')).$$

We can see A^* also as $\exp(\sum_{j=2, j\neq n+1}^{2n}(v'_j - v_j)W_j^{\phi})(A)$ and so

$$\begin{aligned} |\phi(A) - \phi(A^*)| &\leq \left| \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} \int_0^1 (v'_j - v_j) W_j^{\phi} \phi\big(\exp(t \textstyle\sum_{j=2, j\neq n+1}^{2n}(v'_j - v_j)W_j^{\phi})(A)\big) dt \right| \\ &\leq N|v' - v| \leq N|A - B|. \end{aligned}$$

As $\left|\sigma(v, v')\right| = \left|2\sum_{j=2}^{n}[v_{n+j}(v'_j - v_j) - v_j(v'_{n+j} - v_{n+j})]\right| \leq 2K|A - B|$ we get

$$\begin{aligned} |A^* - B| &\leq |\eta' - \eta| + |\tau' - \tau| + |\sigma(v, v')| \\ &\leq (2K + 2)|A - B| \end{aligned}$$

and so

$$\begin{aligned}\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} &\leq \frac{|\phi(A)-\phi(A^*)|}{|A-B|^{1/2}}+\frac{|\phi(A^*)-\phi(B)|}{|A-B|^{1/2}}\\ &\leq N|A-B|^{1/2}+(2K+2)\frac{|\phi(A^*)-\phi(B)|}{|A^*-B|^{1/2}}\\ &\leq N|A-B|^{1/2}+(2K+2)\alpha'(|A^*-B|^{1/2})\\ &\leq N|A-B|^{1/2}+(2K+2)\alpha'([(K+2)|A-B|]^{1/2}).\end{aligned}$$

Step 6. The proof is accomplished for r "sufficiently small" only; however, this is sufficient to conclude. □

By a standard compactness argument we get the following

Theorem 4.30. *Let $\phi \in \mathbf{C}^1(\omega)$ and set $W^\phi\phi = (w_2, \dots, w_{2n}) \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$. Then for all $\omega' \Subset \omega$ there exists a function $\alpha :]0, +\infty[\to [0, +\infty[$, which depends only on $\omega', \|\phi\|_{L^\infty(\omega'')}$ (where ω'' is any open set such that $\omega' \Subset \omega'' \Subset \omega$), $\|W^\phi\phi\|_{L^\infty(\omega'')}$ and on the modulus of continuity of w_{n+1} on ω'', such that $\lim_{r\to 0}\alpha(r) = 0$ and*

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} : A, B \in \omega', 0 < |A-B| \leq r\right\} \leq \alpha(r). \tag{4.87}$$

We end this section with two applications of Theorem 4.22; the first one is a negative answer to the problem of a good parametrization of $\mathbb{H}$-regular hypersurfaces. Indeed a natural question arising is the (local) Lipschitz continuity of $\phi : \omega \subset (\mathbb{R}^{2n}, \varrho) \to \mathbb{R}$, where ϱ denotes the restriction distance of d_∞ to $V_1 \equiv \mathbb{R}^{2n}$. More precisely we investigate the case $n = 1$, when ϱ concides with the so-called *parabolic distance* on $\mathbb{R}_\eta \times \mathbb{R}_\tau$ defined by

$$\varrho((\eta,\tau),(\eta',\tau')) = |\eta'-\eta| + |\tau'-\tau|^{1/2}.$$

Corollary 4.31. *There exist a functions $\phi : \omega \to \mathbb{R}$ which parametrizes an $\mathbb{H}$-regular surface $S = \Phi(\omega) \subset \mathbb{H}^1$ and for which there is no constant $L > 0$ such that*

$$|\phi(\eta',\tau') - \phi(\eta,\tau)| \leq L(|\eta-\eta'| + |\tau-\tau'|^{1/2}) \quad \text{for all } (\eta,\tau),(\eta',\tau') \in \omega.$$

In particular, $\Phi : (\omega, \varrho) \to \mathbb{H}^1$ is not Lipschitz continuous.

Proof. We argue by contradiction. Whithout loss of generality we can assume that $\omega = (a, b) \times (c, d)$: it follows that for each $\tau \in (c, d)$ the function $\phi(\cdot, \tau)$ is Lipschitz continuous in (a, b), and so for any $\tau \in (c, d)$ there exists the distributional derivative $\frac{\partial\phi}{\partial\eta}(\cdot,\tau) \in L^\infty(a, b)$ with $\|\frac{\partial\phi}{\partial\eta}(\cdot,\tau)\|_{L^\infty(a,b)} \leq L$ for all $\tau \in (c, d)$. In particular there

exists the distributional derivative $\frac{\partial\phi}{\partial\eta} \in L^\infty(\omega)$ on all ω. By Theorem 4.22 we know that

$$\mathfrak{B}\phi = \frac{\partial\phi}{\partial\eta} - 2\frac{\partial\phi^2}{\partial\tau} \in \mathbf{C}^0(\omega)$$

in distributional sense, thus $\frac{\partial\phi^2}{\partial\tau} \in L^\infty_{loc}(\omega)$. It follows that $\phi^2 \in \mathrm{Lip}_{loc}(\omega)$.

We claim that S is Euclidean 2-rectifiable. Indeed there is no loss of generality in supposing that actually $\phi^2 \in \mathrm{Lip}(\omega)$, i.e. $|\phi^2(A) - \phi^2(B)| \le M|B - A|$ for some $M > 0$ and all $A, B \in \omega$. Then for $h \in \mathbb{N}$ set

$$\begin{aligned}\omega_h^+ &:= \{A \in \omega : \phi(A) > 1/h\}\\ \omega_h^- &:= \{A \in \omega : \phi(A) < -1/h\}\\ \omega_0 &:= \{A \in \omega : \phi(A) = 0\}\end{aligned}$$

and observe that, when $A, B \in \omega_h^+$ or $A, B \in \omega_h^-$, we have

$$\begin{aligned}2|\phi(A) - \phi(B)|/h \;&\le\; |\phi(A) - \phi(B)| \cdot |\phi(A) + \phi(B)|\\ &=\; |\phi^2(A) - \phi^2(B)| \le M|B - A|,\end{aligned}$$

i.e. $\phi_{|\omega_h^\pm}$ is Lipschitz continuous; extending it to $\phi_h^\pm : \omega \to \mathbb{R}$ (with the same Lipschitz constant) and defining $\Phi_h^\pm$ in the usual way, we get that $\Phi(\omega_h^\pm) \subset \Phi_h^\pm(\omega)$ is Euclidean 2-rectifiable. Observing that $\Phi(\omega_0) \subset V_1$, we get that also

$$\Phi(\omega) \subset \Phi(\omega_0) \cup \bigcup_h \Phi(\omega_h^+) \cup \bigcup_h \Phi(\omega_h^+)$$

is Euclidean 2-rectifiable. On the other hand there are $\mathbb{H}$-regular surfaces $S = \Phi(\omega) \subset \mathbb{H}^1$ which are not Euclidean 2-rectifiable (see [106], Theorem 3.1), that gives a contradiction. □

A second interesting corollary of Theorem 4.22 provides a simple way to exihibit $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$ which are not Euclidean regular.

Corollary 4.32. *Let $\phi : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which depends only on τ, i.e. $\phi = \phi(\tau) : I \to \mathbb{R}$ for a certain open (and possibly unbounded) interval $I \subset \mathbb{R}$, and suppose that $\phi^2 : I \to \mathbb{R}^+$ is of class $\mathbf{C}^1$. Then ϕ is uniformly W^ϕ-differentiable at A for every $A \in \omega$ and*

$$W^\phi\phi(A) = -2(\phi^2)'(A).$$

In particular, $W^\phi\phi$ is continuous and ϕ parametrizes an $\mathbb{H}$-regular surface in $\mathbb{H}^1$.

Proof. Thanks to Theorem 4.22, it is sufficient to find a family $\{\phi_\epsilon\}_\epsilon$ such that (4.60) holds. The family we are going to consider is of the form $\phi_\epsilon = \phi_\epsilon(\tau) := (\phi^2+\delta_\epsilon^2)^{1/2}\cdot g_\epsilon$, where δ_ϵ and g_ϵ are to be found; the key idea is to construct g_ϵ such that $g_\epsilon \to$ sign ϕ and g'_ϵ is "controlled", in a way we are going to specify; then our thesis becomes

$$\phi_\epsilon \to \phi \quad \text{and} \quad (\phi_\epsilon^2)' \to (\phi^2)' \quad \text{uniformly on } J \tag{4.88}$$

for each $J \Subset I$.

We recall the following general fact: let D, E two closed subsets of I such that $d(D,E) := \inf\{|a-b| : a \in D, b \in E\} \geq C > 0$; then there exists a $g \in \mathbf{C}^\infty(I,[-1,1])$ such that $g_{|D} \equiv 1, g_{|E} \equiv -1$ and $\|g'\|_\infty \leq 4/C$.

Now let us set

$$\alpha(r) := \sup\left\{\frac{|\phi(\tau')-\phi(\tau)|}{|\tau'-\tau|^{1/2}} : \tau', \tau \in J,\ 0 < |\tau'-\tau| \leq r\right\},$$

and suppose that $\alpha(r) \to 0$ as $r \to 0^+$: then if we set $\delta_\epsilon := \alpha(\epsilon)\epsilon^{1/2}/2$ we have $\lim_{\epsilon\to 0}\delta_\epsilon = 0$. For each ϵ let

$$D_\epsilon := \{\tau : \phi(\tau) \geq \delta_\epsilon\} \cap J \quad \text{and} \quad E_\epsilon := \{\tau : \phi(\tau) \leq -\delta_\epsilon\} \cap J;$$

by construction $d(D_\epsilon, E_\epsilon) \geq \epsilon$ and so there exists a $g_\epsilon \in \mathbf{C}^\infty(I,[-1,1])$ with

$$g_\epsilon \equiv 1 \text{ on } D_\epsilon, \quad g_\epsilon \equiv -1 \text{ on } E_\epsilon \quad \text{and} \quad \|g'_\epsilon\|_\infty \leq 4/\epsilon = \alpha(\epsilon)^2/\delta_\epsilon^2.$$

As we said earlier, set $\phi_\epsilon := (\phi^2+\delta_\epsilon^2)^{1/2}g_\epsilon$; it is easy to prove that $\phi_\epsilon \to \phi$ uniformly on J and

$$\begin{aligned}
2\|(\phi_\epsilon^2)' - (\phi^2)'\|_{L^\infty(J)} &\leq 4\|g_\epsilon g'_\epsilon(\phi^2+\delta_\epsilon^2)\|_{L^\infty(J)} + 2\|(g_\epsilon^2-1)(\phi^2)'\|_{L^\infty(J)} \\
&\leq 4\|g_\epsilon g'_\epsilon(\phi^2+\delta_\epsilon^2)\|_{L^\infty(J\setminus(D_\epsilon\cup E_\epsilon))} + 4\|(\phi^2)'\|_{L^\infty(J\setminus(D_\epsilon\cup E_\epsilon))} \\
&\leq 8\frac{\alpha(\epsilon)^2}{\delta_\epsilon^2}\delta_\epsilon^2 + 4\|(\phi^2)'\|_{L^\infty(J\cap\{|\phi|\leq\delta_\epsilon\})} \longrightarrow 0
\end{aligned}$$

for $\epsilon \to 0^+$; in the last passage we used the implication $\phi(\tau) = 0 \Rightarrow (\phi^2)'(\tau) = 0$, and so $\|(\phi^2)'\|_{L^\infty(J\cap\{|\phi|\leq\delta_\epsilon\})} \to 0$ because of the continuity of $(\phi^2)'$.

Let us remark that ϕ_ϵ actually depends on J; however, if we consider a sequence $\{J^n\}_{n\in\mathbb{N}}$ of closed intervals such that $J^n \subset J^{n+1}$ and $J^n \uparrow]\alpha,\beta[$, we get sequences $\{\phi_\epsilon^n\}_\epsilon$ for each n, and one can conclude with a diagonal argument.

Finally, we have to prove that $\alpha(r) \to 0$ as $r \to 0$. Suppose that the converse is true; then there exist $\sigma > 0$ and $a_h, b_h \in J$ such that

$$|\phi(a_h) - \phi(b_h)| > 2\sigma|a_h - b_h|^{1/2} \quad \text{and} \quad |a_h - b_h| \to 0. \tag{4.89}$$

We can suppose that $\phi(a_h)$ and $\phi(b_h)$ have the same sign (i.e. $\phi(a_h)\phi(b_h) \geq 0$); in fact, if this is not the case, by the continuity of ϕ there is a $c_h \in]a_h, b_h[$ such that $\phi(c_h) = 0$, and we can suppose that $c_h \in J$ (because there is no loss of generality supposing that J is an interval). As

$$2\sigma < \frac{|\phi(a_h) - \phi(b_h)|}{|a_h - b_h|^{1/2}} \leq \frac{|\phi(a_h) - \phi(c_h)|}{|a_h - c_h|^{1/2}} + \frac{|\phi(c_h) - \phi(b_h)|}{|c_h - b_h|^{1/2}}$$

there exists a $d_h \in \{a_h, b_h\}$ such that $|\phi(c_h) - \phi(d_h)| > \sigma |c_h - d_h|^{1/2}$. Therefore (possibly considering c_h and d_h instead of a_h and b_h) we can assume that a_h and b_h satisfy (4.89) (possibly with σ instead of 2σ) and that $\phi(a_h)$ and $\phi(b_h)$ have the same sign.

As J is compact, we can suppose (up to subsequences) that there is a $\overline{\tau} \in J$ such that $a_h \to \overline{\tau}$ and $b_h \to \overline{\tau}$. It is not possible that $\phi(\overline{\tau}) \neq 0$: in fact, ϕ is of class $\mathbf{C}^1$ in the open set $\{\tau : \phi(\tau) \neq 0\}$ (it is easy to show that here $\phi' = (\phi^2)'/2\phi$) that would imply the boundedness of the quantities $|\phi(a_h) - \phi(b_h)|/|a_h - b_h|$ for h sufficiently large, which is in contradiction with (4.89). Therefore $\phi(\overline{\tau}) = 0$ and so one must have $(\phi^2)'(\overline{\tau}) = 0$. As $\phi(a_h)$ and $\phi(b_h)$ have the same sign, we have $|\phi(a_h) - \phi(b_h)| \leq |\phi(a_h) + \phi(b_h)|$ and so

$$\begin{aligned} \sigma^2 &< \left(\frac{|\phi(a_h) - \phi(b_h)|}{|a_h - b_h|^{1/2}} \right)^2 \\ &\leq \left(\frac{|\phi(a_h) - \phi(b_h)|}{|a_h - b_h|^{1/2}} \right) \left(\frac{|\phi(a_h) + \phi(b_h)|}{|a_h - b_h|^{1/2}} \right) \\ &= \frac{|\phi(a_h)^2 - \phi(b_h)^2|}{|a_h - b_h|} = (\phi^2)'(\tau_h) \end{aligned}$$

for a certain τ_h contained in the interval between a_h and b_h. Therefore $\tau_h \to \overline{\tau}$ and so $(\phi^2)'(\overline{\tau}) \geq \sigma$ by the continuity of $(\phi^2)'$, which is a contradiction. □

4.5 BiLipschitz parametrization of hypersurfaces in $\mathbb{H}^1$

In the spirit of Federer's definition of rectifiable sets (see [69]), a natural question is the one of finding a model metric space for $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$ with no characteristic points (see Remark 3.14), i.e. a metric space (M, ϱ) which (locally) parametrizes any $\mathbb{H}$-regular surface S. It turns out that the natural candidate is $\mathbb{R}^2$ with the so-called "parabolic" distance

$$\varrho\big((x,z),(x',z')\big) := |x - x'| + |z - z'|^{1/2};$$

this space can be naturally identified with the subgroup $V_1 \subset \mathbb{H}^1$ endowed with the restriction of d_∞. The following result, of which we give a slightly different proof [15], is due to Cole and Pauls [52].

Theorem 4.33. *Let S be a $\mathbf{C}^1$ surface; then for any non characteristic point $P \in S$ there is a Lipschitz continuous mapping*

$$\psi : (\mathcal{A}, \varrho) \longrightarrow (\mathcal{U}, d_\infty)\,,$$

from an open set $\mathcal{A} \subset \mathbb{R}^2$ to a neighbourhood $\mathcal{U}$ of P in S, with Lipschitz inverse map ψ^{-1}.

Proof. As usual, it is not restrictive to suppose that $P = 0$; moreover, since any sufficiently small neighbourhood $\mathcal{U}$ of P in S can be intrinsically parametrized through a $\mathbf{C}^1$ map $\phi : \omega \subset \mathbb{R}^2 \to \mathbb{R}$, and since

$$\Phi : (\omega, d_\phi) \longrightarrow (\mathcal{U}, d_\infty)$$

is biLipschitz (see Corollary 4.18), our problem is equivalent to that of finding a biLipschitz mapping

$$\psi : (\mathcal{A}, \varrho) \longrightarrow (\omega, d_\phi).$$

We claim that the map

$$\psi(x,z) := \exp(xW^\phi)(0,z) = (x, \tau(x,z)) = \big(x, z - 4\textstyle\int_0^x \phi(s, \tau(s,z))\, ds\big)$$

satisfies our requests.

Step 1. We start by proving that ψ is Lipschitz continuous, i.e. that

$$|\tau(x,z) - \tau(x',z') + 2(\phi + \phi')(x - x')| \preceq |x - x'|^2 + |z - z'| \tag{4.90}$$

where, here and in the following, we denote $\phi := \phi(\psi(x,z)), \phi' := \phi(\psi(x',z'))$ and we write $\preceq$ whenever an inequality $\leq$ holds up to a multiplicative constant. The left hand side of (4.90) can be split as

$$\begin{aligned} &|\tau(x,z) - \tau(x',z') + 2(\phi + \phi')(x - x')| \\ &\quad \leq \tfrac{1}{2}\big\{\big|\tau(x,z) - \tau(x',z) + 4\phi(x - x')\big| + \big|\tau(x,z') - \tau(x',z') + 4\phi'(x - x')\big| \\ &\qquad + \big|\tau(x,z) - \tau(x,z')\big| + \big|\tau(x',z) - \tau(x',z')\big|\big\}. \end{aligned} \tag{4.91}$$

The first and second addend in (4.91) can be estimated in a similar way:

$$\begin{aligned} & |\tau(x,z)-\tau(x',z)+4\phi(x-x')| \\ = \;& 4\left|\int_{x'}^{x}\big[-\phi(s,\tau(s,z))+\phi(x,\tau(x,z))\big]ds\right| \qquad (*) \\ \preceq \;& \int_{x'}^{x}\big[|x-s|+|\tau(x,z)-\tau(s,z)|\big]ds \\ \leq \;& \int_{x'}^{x}\left[|x-s|+\int_{s}^{x}|\phi(r,\tau(r,z))|\,dr\right]ds \\ \preceq \;& |x-x'|^2, \end{aligned}$$

where, in the step marked by $(*)$, we used the fact that ϕ is Lipschitz.

Therefore, it will be sufficient to estimate the third and fourth addend in (4.91); more precisely, we need an estimate

$$|\tau(x,z)-\tau(x,z')| \preceq |z-z'| \tag{4.92}$$

uniformly in x. We also observe that, in order for ψ to be Lipschitz, (4.92) is also necessary, since the left hand side is (part of the square of) the distance between $\psi(x,z)$ and $\psi(x,z')$, while the right hand one is (the square of) the distance between (x,z) and (x,z'). By the Lipschitz continuity of ϕ, one has

$$\begin{aligned} |\tau(x,z)-\tau(x,z')| \;&\leq\; |z-z'|+4\int_0^x|\phi(s,\tau(s,z))-\phi(s,\tau(s,z'))|\,ds \\ &\preceq\; |z-z'|+\int_0^x|\tau(s,z)-\tau(s,z')|\,ds \end{aligned} \tag{4.93}$$

and (4.92) follows thanks to Gronwall's lemma.

Step 2. The inverse map of ψ is

$$\psi^{-1}(\eta,\tau)=\big(\eta,\tau-4\textstyle\int_\eta^0\phi(s,h_{\eta,\tau}(s))ds\big)=:(\eta,z(\eta,\tau)),$$

where $h_{\eta,\tau}$ solves the Cauchy problem

$$\begin{cases} \dot h_{\eta,\tau}(s)=-4\phi(s,h_{\eta,\tau}(s)) \\ h_{\eta,\tau}(\eta)=\tau\,. \end{cases}$$

Notice also that $(0,z(\eta,\tau))=\exp(-\eta W^\phi)(\eta,\tau)$.

For the Lipschitz continuity of ψ^{-1} it will be sufficient to show that the inequality

$$|z(\eta,\tau)-z(\eta',\tau')| \preceq d_\phi((\eta,\tau),(\eta',\tau'))^2 \tag{4.94}$$

holds in a neighbourhood of 0. Notice that when $\eta = \eta'$ one can use the Lipschitz continuity of ϕ exactly as in (4.93), obtaining

$$\begin{aligned} & |z(\eta,\tau) - z(\eta,\tau')| \;=\; |h_{\eta,\tau}(0) - h_{\eta,\tau'}(0)| \\ = \;& \left|\tau - 4\int_\eta^0 \phi(s, h_{\eta,\tau}(s))ds - \tau' - 4\int_\eta^0 \phi(s, h_{\eta,\tau'}(s))ds\right| \\ \preceq \;& |\tau - \tau'| + \int_\eta^0 |h_{\eta,\tau}(s) - h_{\eta,\tau'}(s)|ds\,. \end{aligned}$$

By Gronwall's lemma we conclude

$$|z(\eta,\tau) - z(\eta,\tau')| \preceq |\tau - \tau'|\,. \tag{4.95}$$

For the general case, as in Theorem 4.16 one can set

$$(\eta',\tau'') := \exp((\eta'-\eta)W^\phi)(\eta,\tau) \tag{4.96}$$

and, as in the proof of the same Theorem, one has

$$|\tau' - \tau''| \preceq d_\phi((\eta,\tau),(\eta',\tau'))^2\,. \tag{4.97}$$

Observing that, by construction, $z(\eta,\tau) = z(\eta',\tau'')$, we obtain the thesis (4.94) by combining (4.95), (4.96) and (4.97). □

Remark 4.34. When ϕ is just uniformly W^ϕ-differentiable, one could be tempted to follow the same line of Theorem 4.33 by using the exponential maps of Section 4.4 and define

$$\psi(x,z) := (x, \exp_0(xW^\phi)(0,z)).$$

Beside the problems given by the non-uniqueness of this exponential map, it is not difficult to check that such a ψ is in general not continuous: consider in fact the function

$$\phi(\eta,\tau) := \begin{cases} \frac{\tau^\alpha}{4(1-\alpha)} & \text{if } \tau \geq 0 \\ 0 & \text{if } \tau < 0\,. \end{cases} \tag{4.98}$$

For $\frac{1}{2} < \alpha < 1$, the X_1-graph of ϕ is an $\mathbb{H}$-regular surface because of Corollary 4.32 and it is not difficult to check that the only possible definition of exponential maps provides

$$\exp_0(xW^\phi)(0,z)) = \begin{cases} (x, (z^{1-\alpha} - x)^{\frac{1}{1-\alpha}}) & \text{if } x \leq 0 \text{ and } z > 0 \\ (x,z) & \text{if } x \leq 0 \text{ and } z < 0 \end{cases} \tag{4.99}$$

which is not continuous since

$$\lim_{z\to 0^+} \exp_0(xW^\phi)(0,z)) = (x, |x|^{\frac{1}{1-\alpha}}) \neq (x,0) = \lim_{z\to 0^-} \exp_0(xW^\phi)(0,z))$$

for any $x < 0$.

The following result shows that the statement of Theorem 4.33 fails for general $\mathbb{H}$-surfaces:

Theorem 4.35. *Let S be the $\mathbb{H}$-regular surface given by the X_1-graph of the map ϕ in (4.98) with $\frac{1}{2} < \alpha < 1$, and suppose that*

$$\psi : (\mathcal{A}, \varrho) \longrightarrow (\mathcal{U}, d_\infty)$$

is a Lipschitz continuous and surjective map from an open set $\mathcal{A} \subset \mathbb{R}^2$ to a neighbourhood $\mathcal{U}$ of 0 in S. Then ψ is not a homeomorphism; in particular, it cannot be biLipschitz.

Proof. Step 1. For any fixed z the curve $\gamma_z := \psi(\cdot, z) : \mathbb{R} \to \mathbb{H}^1$ is Lipschitz continuous; in particular (see [145]) it must be horizontal, i.e. absolutely continuous and such that $\dot\gamma_z \in H_{\gamma_z}\mathbb{H}^1$ almost everywhere. Since γ_z lies on S, it must be contained in (a piece of) an integral curve of the vector field

$$Y_1 + (W^\phi \phi \circ \Phi^{-1}) X_1 \,,$$

which is (up to a normalization) the unique vector field which is both horizontal and tangent to S. Since

$$(\Phi^{-1})_*(Y_1 + (W^\phi \phi \circ \Phi^{-1}) X_1) = \partial_\eta - 4\phi\partial_\tau = W^\phi \,,$$

it follows that $\gamma_z \circ \Phi^{-1}$ is (a piece of) an integral curve of W^ϕ in $\mathbb{R}^2$.

Let us investigate the qualitative behaviour of the integral curves of W^ϕ. If one of these curves lies in the upper half-plane $\{\tau > 0\}$ (where we have uniqueness for solutions of the associated ODE) at a certain time x, then its second τ coordinate is decreasing, so it must lie in the upper (open) half-plane also before x; however, after x, it must reach the zero level in a finite time, and it is not difficult to prove that it must stay at 0 after that. In the lower half-plane $\{\tau < 0\}$ we have again uniqueness of solutions and the curves are straight lines parallel to the η axis; therefore, according to (4.99) we can divide the integral curves of W^ϕ into two families (see also Figure 4.1):

(a) for $w \in \mathbb{R}$, the curves

$$c_w^+(x) = \begin{cases} (x, (w-x)^{\frac{1}{1-\alpha}}) & \text{if } x \le w \\ (x, 0) & \text{if } x \ge w; \end{cases}$$

(b) for $\zeta \le 0$, the curves $c_\zeta^-(x) = (x, \zeta)$.

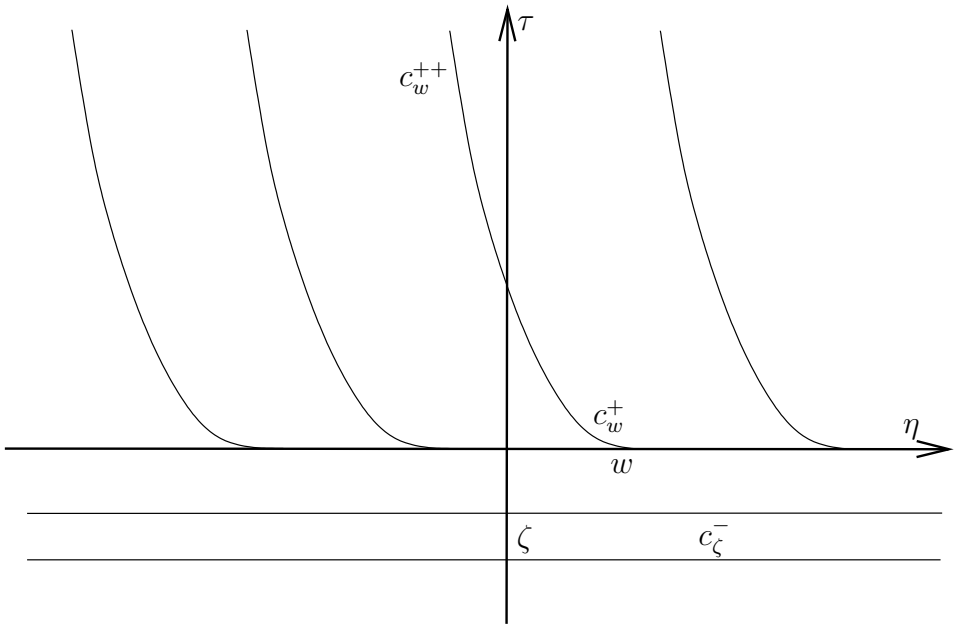

Figure 4.1: Exponential lines of W^ϕ for ϕ as in (4.98).

Notice that for curves c_w^+ the parameter w denotes the point where they touch the η axis, i.e. $(w,0)$; we will also write c_w^{++} to denote the restriction of c_w^+ to $]-\infty, w]$. The upper (closed) halfplane is connected by means of c_0^- and of paths of type c_w^+.

Step 2. It will not be restrictive to suppose $\psi(0,0) = 0 \in S$ and $\mathcal{U} = \Phi\big(]-\delta, \delta[^2\big)$ for some $\delta > 0$. For the sake of simplicity let us write ψ also for the $(\varrho\text{-}d_\phi)$-Lipschitz induced map $\Phi^{-1}\circ\psi : \mathcal{A} \to]-\delta, \delta[^2$, which is surjective and such that $\psi(0,0) = (0,0)$; suppose by contradiction that it is also a homeomorphism. Then the set

$$L := \psi^{-1}\{(0,\tau) : \tau \in [0, \delta/2]\}$$

is a compact subset of $\mathcal{A}$, and so for sufficiently small $r > 0$ one has that

$$\{(x+h, z) : (x,z) \in L, -r \le h \le r\} \subset \mathcal{A}\,. \tag{4.100}$$

Let us set

$$r_+ := \sup\{x > 0 : \psi(x,0) \in \mathbb{R} \times \{0\}\} \ge 0$$
$$r_- := \inf\{x < 0 : \psi(x,0) \in \mathbb{R} \times \{0\}\} \le 0\,.$$

Step 3. First of all, we prove that we cannot have $r_+ = r_- = 0$; indeed, this would imply that

$$\{\psi(x,0) : x > 0\} \subset \text{Im } c_0^{++} \setminus \{0\} \qquad \text{and} \qquad \{\psi(x,0) : x < 0\} \subset \text{Im } c_0^{++} \setminus \{0\}\,,$$

and by continuity we obtain

$$\{\psi(x,0) : x > 0\} \cap \{\psi(x,0) : x < 0\} \neq \emptyset$$

i.e. ψ is not injective, a contradiction.

Step 4. Since $r_+ \neq r_-$, one of them is nonzero: by substituting ψ, if necessary, with the map $\tilde{\psi}(x,t) := \psi(-x,t)$ we can suppose that $r_+ > 0$. It is not difficult to prove that

$$\{\psi(x,0) : 0 \leq x \leq r_+\} \subset \mathbb{R} \times \{0\},$$

for otherwise the curve $\psi(\cdot,0)|_{[0,r_+]}$ would leave the η axis $\mathbb{R} \times \{0\}$ and then return on it after some time, which can be done only by covering forward and then backward a piece of some c_w^{++}, and contradicting in particular the injectivity of ψ. Choose therefore $r \in]0, r_+[$ such that (4.100) holds, and set $A := \psi(r,0) = (\overline{\eta}, 0)$; by continuity one must have

$$\begin{array}{ll} [0,\overline{\eta}] \times \{0\} \subset \{\psi(x,0) : 0 \leq x < r\} & \text{if } \overline{\eta} > 0 \\ [\overline{\eta},0] \times \{0\} \subset \{\psi(x,0) : 0 \leq x < r\} & \text{if } \overline{\eta} < 0. \end{array} \tag{4.101}$$

Since $A \neq 0$ (i.e. $\overline{\eta} \neq 0$) we easily find an $\epsilon > 0$ such that

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset,$$

where (see Figure 4.2)

$$\mathcal{V}_1 := \bigcup_{0<w<\epsilon} \text{Im } c_w^{++} \qquad \text{and} \qquad \mathcal{V}_2 := \bigcup_{\overline{\eta}-\epsilon<w<\overline{\eta}+\epsilon} \text{Im } c_w^{++} \ni A\,.$$

Notice that $A \in \mathcal{V}_2$, since $A \in \text{Im } c_{\overline{\eta}}^{++}$. Now, in order to join a point $A_1 \in \mathcal{V}_1$ with a point $A_2 \in \mathcal{V}_2$ by following only exponential lines of W^ϕ, one must cover the whole segment I, where $I := [\epsilon, \overline{\eta} - \epsilon] \times \{0\}$ in case $\overline{\eta} > 0$ and $I := [\overline{\eta} + \epsilon, 0] \times \{0\}$ in case $\overline{\eta} < 0$.

Therefore set $(x_\tau, z_\tau) := \psi^{-1}(0,\tau)$, and notice that

$$\lim_{\tau \to 0} \psi(x_\tau + r, z_\tau) = \psi(r,0) = A.$$

For sufficiently small $\tau > 0$ the curve $\psi(\cdot, z_\tau)$ goes from $A_1 := (0,\tau) \in \mathcal{V}_1$ to the point $A_2 := \psi(x_\tau + r, z_\tau)$ following only exponentials of W^ϕ; moreover, A_2 must belong to $\mathcal{V}_2$. This implies that $I \subset \text{Im } \psi(\cdot, z_\tau)$; since (see (4.101)) we have also $I \subset \text{Im } \psi(\cdot, 0)$, this would contradict the injectivity of ψ in case we were able to find a sufficiently small τ such that $z_\tau \neq 0$. If this were not possible, there would exist $\lambda > 0$ such that $\psi^{-1}(0,\tau) = (x_\tau, 0)$ for any $\tau \in [0,\lambda]$, i.e.

$$\{0\} \times [0,\lambda] \subset \text{Im } \psi(\cdot, 0)\,.$$

Therefore, the segment $\{0\} \times [0,\lambda]$ would be of finite length, which is not possible since it is not contained in an exponential curve of W^ϕ. □

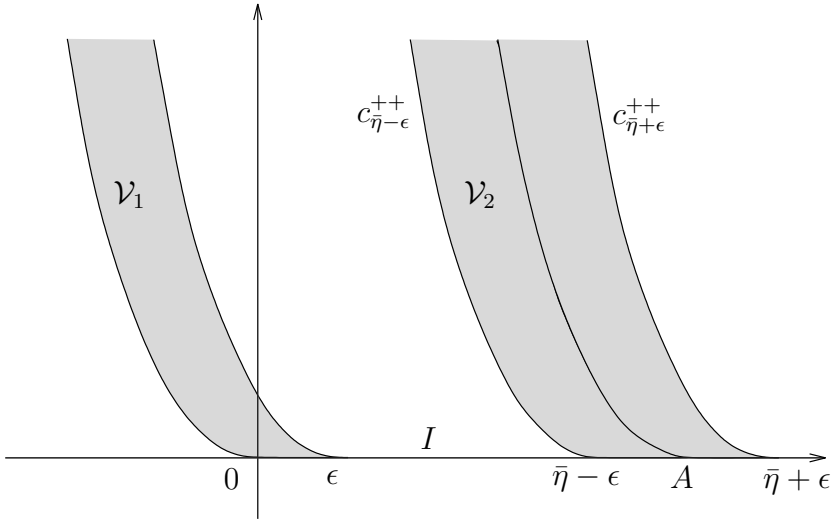

Figure 4.2: The sets $\mathcal{V}_1$ and $\mathcal{V}_2$ and the interval I.

We end this Section by remembering that, as far as we know, the analogous of Theorem 4.33 in $\mathbb{H}^n, n \geq 2$ is still an open problem even for smooth ($\mathbf{C}^\infty$) hypersurfaces; the natural candidate metric space in this case seems to be $\mathbb{R} \times \mathbb{H}^{n-1}$.

Chapter 5

The Bernstein problem in Heisenberg groups and calibrations

In this final Chapter we investigate a question that, although under different formulations, has recently received an increasing attention: namely, the Bernstein problem in the Heisenberg group, see [42, 86, 157, 41, 60, 20, 58, 138]. Recall that the classical Bernstein problem consists in finding entire functions $\psi : \mathbb{R}^m \to \mathbb{R}$ solving the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = 0 \tag{5.1}$$

and that are not affine, i.e. functions parametrizing hyperplanes or, which is the same, (translations of) maximal subgroups of $\mathbb{R}^{m+1}$. It is well known that this problem has been completely solved thanks to many contributions (see [89] for an interesting historical survey). Here we summarize these celebrated results in the following

Theorem 5.1. *Every smooth function $\psi : \mathbb{R}^m \to \mathbb{R}$ solving* (5.1) *must be an affine function if $m \leq 7$. If $m \geq 8$ there are analytic solutions which are not affine functions.*

We will then compute the minimal surface equation (5.17) for intrinsic graphs and we will observe that maps parametrizing (laterals of) maximal subgroups (the so called vertical hyperplanes) are entire solutions of the equation. In analogy with the classical case, our formulation of the Bernstein problem in the Heisenberg group $\mathbb{H}^n$ will consist in looking for solutions of the minimal surface equation which are

not vertical hyperplanes. We will exhibit such solutions in the cases $n = 1$ (where, however, hyperplanes are the only minimizers) and $n \geq 5$, while the case $n = 2, 3, 4$ are still open. In the discussion, we will also extend to CC spaces the classical calibration argument [96, 3], providing sufficient conditions for measurable sets to be X-perimeter minimizing. This result has been suggested by L. Ambrosio, while all the other ones have been obtained in [20] in collaboration with V. Barone Adesi and F. Serra Cassano.

In Section 5.1 we state (Theorem 5.2) the calibration argument for CC spaces, which is refined in Theorem 5.3 for the Carnot groups setting. Applications of these results are also exhibited, showing the minimality in significant cases, in Examples 5.5, 5.6, 5.7 and 5.8. We particularly stress the last two ones, where, respectively, we analyse the case of t-graphs in $\mathbb{H}^1$ and we show that in general X-perimeter minimizers are not smooth (see also [148]).

In Section 5.2 we derive first and second variation formulae for intrinsic graphs of class $\mathbf{C}^2$, therefore obtaining the minimal surface equation (5.17) and the second variation formula (5.26) which will be of use in our main result about the Bernstein problem in $\mathbb{H}^1$, Theorem 5.23. Similar formulae have been obtained also in [56, 59, 133, 100, 101]. We stress that again the minimal surface equation (5.17) can be obtained by formally substituting classical gradient in (5.1) with the operator W^ϕ.

In Section 5.3 we restrict to the case of the first Heisenberg group $\mathbb{H}^1$ and study the structure of entire solutions of the minimal surface equation. Up to a change of coordinates, in $\mathbb{H}^1$ this turns out to be equivalent to the "double Burgers" equation $(\partial_t + u\partial_x)^2 u = 0$ in $\mathbb{R}^2$. The key observation for the analysis of solutions u is that they must be linear along characteristic lines, i.e. integral curves of the vector field $\partial_t + u\partial_x$. Starting from this fact we are able (Theorem 5.9) to implicitly characterize such functions only in terms of their value B and derivative A at time 0, with some restrictions on A and B too. An existence result (Theorem 5.19) for entire solution is provided together with some example of them.

Last Section 5.4 deals with the Bernstein problem in the Heisenberg group. In Subsection 5.4.1 we restrict to the $\mathbb{H}^1$ case, where it is known [60] that counterexamples exist. Our main result, Theorem 5.23, states that hyperplanes are the unique entire $\mathbf{C}^2$ solutions to the Bernstein problem provided $\mathbb{H}$-perimeter minimization is assumed. Indeed, for any other solution we can exhibit a family of competitors with strictly negative second variation of area, thus proving that it is not a minimizer. In this approach we will heavily use the second variation formula (5.26) and the structure Theorem 5.9. We stress that this phenomenon is quite unexpected, since in the classical case a calibration argument ensures that any solution to (5.1) is actually a minimizer. In Subsection 5.4.2 we analyse the Bernstein problem in $\mathbb{H}^n$, $n \geq 2$: as we already said, we are able to provide counterexamples when $n \geq 5$, while the cases $n = 2, 3, 4$ are still open. Some of the results of the present Chapter

have been generalized in [58]. It was shown recently [138] that Theorem 5.23 fails when the $\mathbf{C}^2$ assumption on the function is dropped: namely, there exists entire minimizing solutions (in a weak sense) that are not $\mathbf{C}^2$ regular.

5.1 A calibration method for the X-perimeter and applications

The following result is a refinement of one due to L. Ambrosio and extends the classical calibration method (see e.g. [96, 3]) giving sufficient conditions for a Borel set $E \subset \mathbb{R}^n$ to be minimizer of X-perimeter (as in Definition 1.10).

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $X_1, \dots, X_m$ be a family of Lipschitz continuous vector fields in Ω and let E be a set of locally finite X-perimeter in Ω. Suppose there are two sequences $(\Omega_h)_h$ and $(\nu_h)_h$, $h \in \mathbb{N}$, such that*

(i) $\Omega_h \subset \Omega$ *is open,* $\Omega_h \Subset \Omega_{h+1}$, $\Omega_h \uparrow \Omega$;

(ii) $\nu_h \in \mathbf{C}^1(\Omega; \mathbb{R}^m)$, $|\nu_h(x)|_{\mathbb{R}^m} \leq 1$ *for all* $x \in \Omega$ *and any* $h \in \mathbb{N}$;

(iii) $\operatorname{div}_X \nu_h = 0$ *in* Ω_h *for each* h;

(iv) $\nu_h(x) \to \nu_E(x)$ $\|\partial E\|_X$*-a.e.* $x \in \Omega$.

Then E is a minimizer for the X-perimeter in Ω.

Proof. Fix an open set $\Omega' \Subset \Omega$ and a measurable set $F \subset \mathbb{R}^n$ such that $E \Delta F \Subset \Omega'$. Let Ω'' be another open set with $E \Delta F \Subset \Omega'' \Subset \Omega'$. Let $\overline{h}$ and $\psi \in \mathbf{C}^1_c(\Omega')$ be such that $\Omega' \subset \Omega_{\overline{h}}$, $0 \leq \psi \leq 1$ and

$$\Omega'' \Subset \{\psi = 1\} \Subset \Omega' \Subset \Omega\,. \tag{5.2}$$

Now notice that for each $h > \overline{h}$

$$\int_\Omega \langle \psi \nu_h, \nu_E \rangle_{\mathbb{R}^m} \, d\|\partial E\|_X = \int_\Omega \langle \psi \nu_h, \nu_F \rangle_{\mathbb{R}^m} \, d\|\partial F\|_X \tag{5.3}$$

Indeed by (5.2) and (*iii*)

$$\begin{aligned} &\int_\Omega \langle \psi \nu_h, \nu_E \rangle_{\mathbb{R}^m} \, d\|\partial E\|_X - \int_\Omega \langle \psi \nu_h, \nu_F \rangle_{\mathbb{R}^m} \, d\|\partial F\|_X \\ =\ & -\int_{\Omega'} (\chi_E - \chi_F) \operatorname{div}_X (\psi \nu_h) \, d\mathcal{L}^n \\ =\ & -\int_{\Omega''} (\chi_E - \chi_F) \operatorname{div}_X \nu_h \, d\mathcal{L}^n \ =\ 0 \end{aligned}$$

By (5.3)

$$\|\partial F\|_X(\Omega') \geq \left| \int_\Omega \langle \psi \nu_h, \nu_F \rangle_{\mathbb{R}^m} \, d\|\partial F\|_X \right| = \left| \int_\Omega \psi \langle \nu_h, \nu_E \rangle_{\mathbb{R}^m} d\|\partial E\|_X \right| .$$

By (ii) and (iv) and thanks to Lebesgue convergence theorem, as $h \to \infty$ we get

$$\|\partial F\|_X(\Omega') \geq \int_{\Omega'} \psi \, d\|\partial E\|_X \geq \|\partial E\|_X(\Omega''). \tag{5.4}$$

We obtain the thesis by increasing $\Omega'' \uparrow \Omega'$. □

In Carnot groups one can refine Theorem 5.2 as follows:

Theorem 5.3. *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group. Let E, Ω be respectively a measurable and open set of $\mathbb{R}^n$, and denote by $\nu_E : \Omega \to \mathbb{R}^m$ the horizontal inward normal to E in Ω. Suppose that*

(i) E has locally finite X-perimeter in Ω;

(ii) $\operatorname{div}_X \nu_E = 0$ in Ω in distributional sense;

(iii) there exists an open set $\widetilde{\Omega} \subset \Omega$ such that $\|\partial E\|_X(\Omega \setminus \widetilde{\Omega}) = 0$ and $\nu_E \in \mathbf{C}^0(\widetilde{\Omega})$.

Then E is a minimizer of the X-perimeter in Ω.

Proof. Let ζ_ϵ be the family of mollifiers introduced in Proposition 1.28 and set $\overline{\nu} : \mathbb{R}^n \to \mathbb{R}^m$ to be defined by $\overline{\nu} \equiv \nu$ in Ω, $\overline{\nu} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Let us define

$$\nu_\epsilon(x) := (\zeta_\epsilon \star \overline{\nu})(x) = \big((\zeta_\epsilon \star \overline{\nu}_1)(x), \dots, (\zeta_\epsilon \star \overline{\nu}_m)(x)\big), \qquad x \in \mathbb{R}^n .$$

Let us begin to prove that for a fixed open set $\Omega' \Subset \Omega$

$$\int_\Omega \psi \operatorname{div}_X \nu_\epsilon \, d\mathcal{L}^n = 0 \tag{5.5}$$

for every $\psi \in \mathbf{C}^\infty_c(\Omega')$ and $0 < \epsilon < \frac{dist(\Omega', \mathbb{R}^n \setminus \Omega)}{2}$. Since $\psi_\epsilon := \zeta_\epsilon \star \psi \in \mathbf{C}^\infty_c(\Omega)$ and the vector fields X_j's are self-adjoint, by Proposition 1.28 we can integrate by parts getting

$$\int_\Omega \psi \operatorname{div}_X \nu_\epsilon \, d\mathcal{L}^n = - \int_\Omega \sum_{j=1}^m \langle \nu, X_j \psi_\epsilon \rangle_{\mathbb{R}^m} \, d\mathcal{L}^n = 0 .$$

From (5.5) we get

$$\operatorname{div}_X \nu_\epsilon = 0 \qquad \text{in } \Omega' \tag{5.6}$$

for every open set $\Omega' \Subset \Omega$ provided $0 < \epsilon < \frac{dist(\Omega', \mathbb{R}^n \setminus \Omega)}{2}$.

Let $(\Omega_h)_h$ be a sequence of open subsets of Ω verifying (i) of Theorem 5.2. Then by (5.6) there exists a sequence $\epsilon_h \to 0$ such that the maps $\nu_h := \nu_{\epsilon_h}$ satisfy the assumptions of Theorem 5.2: indeed (i)-(iii) therein are immediately satisfied, while by (iii) and Proposition 1.28 we get that $\nu_h \to \nu$ uniformly on compact subsets of $\widetilde{\Omega}$, whence (iv) of Theorem 5.2 follows. □

Remark 5.4. Notice that, through the calibration argument 5.3, one can prove that every Euclidean subgraph parametrized by an entire solution of (5.1) is a minimizer for the classical perimeter.

We have now all the tools to state some results about minimizers of the X-perimeter in CC spaces: for all of them our calibration results will be crucial.

Example 5.5 (Hypersurfaces with constant horizontal normal)**.** Let X be a family of Lipschitz continuous vector fields $X_1, \dots, X_m$ on $\mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a set of locally finite X-perimeter in an open set $\Omega \subset \mathbb{R}^n$ which admits a constant inward horizontal normal ν_E in Ω, i.e.

$$\nu_E \equiv \nu_0 \quad \|\partial E\|_X\text{-a.e. in } \Omega$$

for a suitable constant vector $\nu_0 \in \mathbb{R}^m$. Then, thanks to Theorem 5.2, it is straightforward to check that E is a minimizer for the X-perimeter.

Observe that many interesting questions, such as regularity and rectifiability, are open even in this quite simple class of sets: see e.g. Example 5.8.

Example 5.6 (t-graphs in $\mathbb{H}^1$)**.** Let $\mathbb{G} = \mathbb{H}^1 \equiv \mathbb{R}^3$ and $\psi \in \mathbf{C}^2(\omega)$ for a suitable open set $\omega \subset \mathbb{R}^2$, and let E be defined by

$$E := \{(x, y, t) \in \mathbb{H}^1 : t < \psi(x, y)\}.$$

Let $\Omega := \omega \times \mathbb{R} \subset \mathbb{H}^1$, $S = \partial E \cap \Omega$ and set

$$C(S) = \{(x, y, t) \in \Omega : \ \psi_x(x, y) - 2y = \psi_y(x, y) + 2x = 0\}$$

to be the set of so-called *characteristic points* of S, i.e. those points $P \in S$ such that $T_P S = H_P \mathbb{H}^1$. Then $C(S)$ is closed in Ω and it was proved in [14] that $\mathcal{H}^2(C(S)) = 0$. On the other hand $\|\partial E\|_{\mathbb{H}} \ll \mathcal{H}^2 \llcorner S$ by virtue of Proposition 3.7, and so

$$\|\partial E\|_{\mathbb{H}}(\Omega \setminus \widetilde{\Omega}) = 0 \tag{5.7}$$

where $\widetilde{\Omega} := \Omega \setminus C(S)$. A simple calculation shows the horizontal normal $\nu_E(x, y, t)$ is

$$\nu_E(x, y, t) = -\frac{\nabla_{\mathbb{H}} f(x, y, t)}{|\nabla_{\mathbb{H}} f(x, y, t)|} = N(x, y) = (N_1(x, y), N_2(x, y)) \tag{5.8}$$

for each $(x,y,t) \in S \setminus C(S)$, where $f(x,y,t) := t - \psi(x,y)$ if $(x,y,t) \in \Omega$ and

$$N(x,y) := \frac{(-\psi_x(x,y)+2y, -\psi_y(x,y)-2x)}{\sqrt{(-\psi_x(x,y)+2y)^2 + (\psi_y(x,y)+2x)^2}} \quad (x,y,t) \in \widetilde{\Omega}\,.$$

The minimal surface equation has been studied in [147, 86] and [42] when $C(S) = \emptyset$ and it simply reads as

$$\mathrm{div}_{\mathbb{H}} \nu_E = \mathrm{div}\ N = \frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} = 0 \quad \text{in} \quad \omega. \tag{5.9}$$

In particular, whenever (5.9) is satisfied pointwise, we can apply Theorem 5.3 obtaining that E is a minimizer for the $\mathbb{H}$-perimeter measure in Ω.

Very recently the more delicate case $C(S) \neq \emptyset$ has been studied in [157] and [43]. In particular, in [43] it has been proved that (5.9) holds in weak sense, i.e.

$$\int_\omega \langle N, \nabla\zeta\rangle_{\mathbb{R}^2}\, d\mathcal{L}^2 = 0 \qquad \forall \zeta \in \mathbf{C}^1_c(\omega)\,, \tag{5.10}$$

iff ψ is a minimizer of the area functional in $\mathbb{H}^1$ for Euclidean t-graph. When $n \geq 2$, if ϕ is a classic solution of (5.9) in $\Omega \setminus C(S)$, then it also satisfies (5.10) (see [43], Corollary F), while counterexamples are provided when $n = 1$ (see [43], section 7).

We can get a strong result by exploiting Theorem 5.3: in fact, if (5.10) holds, by (5.7) and (5.8) we obtain that E is a minimizer for $\mathbb{H}$-perimeter in Ω. In particular E minimizes the $\mathbb{H}$-perimeter not only among sets whose boundary is a Euclidean t-graph, but in a very much larger class of competitors.

Eventually let us stress our technique applies to the case studied in [157], Theorem 5.3. Indeed in our setting $\omega = \mathbb{R}^2$, $\psi(x,y) = 2xy + ay + b$, and

$$N(x,y) = \left(0, \frac{4x-a}{|4x-a|}\right), \qquad (x,y,t) \in \widetilde{\Omega} = \{(x,y,t) : x \neq a/4\}$$

being a, $b \in \mathbb{R}$ fixed constants. On the other hand, a simple calculation shows that (5.10) holds, whence E is a minimizer of the $\mathbb{H}$-perimeter in $\Omega = \mathbb{R}^3$.

Example 5.7. In the Heisenberg group $\mathbb{H}^1$ let E be the set defined by

$$E := \{\iota(\eta,\tau)\cdot se_1 \,:\, (\eta,\tau) \in \mathbb{R}^2, s < \phi(\eta,\tau)\},$$

where we choose $\phi(\eta,\tau) := -\frac{\alpha\eta\tau}{1+2\alpha\eta^2}$ for a fixed constants $\alpha > 0$. This family has been extensively studied in [60], where it was proved that $S = \partial E$ is an entire X_1-graph which is not minimizing for the $\mathbb{H}$-perimeter measure in the whole $\mathbb{H}^1$. Let

us stress the difference with Example 5.6: here in fact S is not a minimizer for $\mathbb{H}$-perimeter measure though it satisfies the intrinsic minimal surface equation (5.17) on all $\mathbb{R}^2$.

On the other hand we can prove it is a minimizer in $\Omega = \mathbb{R}^3 \setminus \{y = 0\}$: indeed with a simple calculation we get

$$\nu_E(x,y,t) := -\frac{\nabla_{\mathbb{H}} f(x,y,t)}{|\nabla_{\mathbb{H}} f(x,y,t)|} = \frac{y}{|y|}\left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right),$$

where $f(x,y,t) := x + \alpha\, yt$. Moreover it easy to see that $\nu_E \in \mathbf{C}^\infty(\Omega)$ and

$$\mathrm{div}_{\mathbb{H}} \nu_E = 0 \quad \text{in} \quad \Omega\,.$$

Therefore applying Theorem 5.3 we obtain the thesis. It is still not known whether S is $\mathbb{H}$-perimeter minimizing in a neighbourhood of a point $(0,y,0)$.

Example 5.8 (Nonsmooth minimal surfaces in $\mathbb{H}^1$)**.** We provide a way to produce minimizers of the $\mathbb{H}$-perimeter in $\mathbb{H}^1$ whose regularity is not better than (Euclidean) Lipschitz. Examples with this regularity are also provided in [43] for minimal Euclidean t-graphs and very recently S. Pauls informed us of a work in progress on this subject.

Our key idea is to construct a "not too regular" parametrization $\phi : \omega \to \mathbb{R}$ such that $W^\phi \phi = 0$ on an open set $\omega \subset \mathbb{R}^2_{\eta,\tau}$: indeed this property ensures that the horizontal normal to the surface is constant $\nu \equiv X_1$, and we conclude by calibrating with a constant section $\nu \equiv X_1$.

We will prove later that for a Lipschitz map ϕ the distribution $\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} - 2\frac{\partial(\phi^2)}{\partial \tau}$ is represented by the L^∞_{loc} function $(\partial_\eta - 4\phi\partial_\tau)\phi$: therefore the required condition is equivalent to ϕ being constant along the integral curves of the vector field W^ϕ, i.e. to these integral curves being straight lines. Notice that, using the notations of Section 5.3, this is equivalent to look for (local) solutions of (5.29) with initial conditions $A \equiv 0$.

We then start by fixing a Lipschitz function $\beta : \mathbb{R} \to \mathbb{R}$, with $L :=\mathrm{Lip}\ \beta < +\infty$, which will give the "initial value" of ϕ in the sense that we look for a ϕ such that $\phi(0,\cdot) = \beta$ (β is simply the counterpart of the function B of Section 5.3). Fix a point $(\eta,\tau) \in \mathbb{R}^2$, consider the integral curve of W^ϕ passing through it and let $(0,t)$ be the point in which this line meets the τ-axis: the condition of ϕ being constant along this line then becomes $-4\phi(\eta,\tau) = -4\beta(t) = \frac{\tau - t}{\eta}$, i.e.

$$\tau = t - 4\eta\beta(t). \tag{5.11}$$

Consider the Lipschitz continuous map

$$\begin{aligned} F \;:\; & \mathbb{R}^2_{x,t} \to \mathbb{R}^2_{\eta,\tau} \\ & (x,t) \longmapsto (x, t - 4x\beta(t))\,; \end{aligned}$$

F plays the role of the F of Section 5.3, and the variable t the one of c. Observe that $F^{-1}(\eta,\tau)$ is well defined when $|\eta| < 1/4L$: this is an easy consequence of

$$\|F(x,t_1) - F(x,t_2)\| = |t_1 - 4x\beta(t_1) - t_2 + 4x\beta(t_2)| \geq (1 - 4L|x|)|t_1 - t_2|.$$

If we put $F^{-1}(\eta,\tau) =: (\eta, t(\eta,\tau))$ it turns out that condition (5.11) is equivalent to define $\phi(\eta,\tau) := \beta(t(\eta,\tau))$, where from now on we suppose

$$(\eta,\tau) \in \omega := \left]-\tfrac{1}{4L}, \tfrac{1}{4L}\right[\times \mathbb{R};$$

observe that ϕ has the same (Lipschitz or better) regularity of β (but no more since $\phi(0,\tau) = \beta(\tau)$).

Let us verify that $(\partial_\eta - 4\phi\partial_\tau)\phi \equiv 0$: as

$$\nabla F(x,t) = \begin{pmatrix} 1 & 0 \\ -4\beta(t) & 1 - 4x\beta'(t) \end{pmatrix}$$

holds almost everywhere, one must have

$$\nabla F^{-1}(\eta,\tau) = \big(\nabla F(F^{-1}(\eta,\tau))\big)^{-1} = \begin{pmatrix} 1 & 0 \\ \dfrac{4\beta(t(\eta,\tau))}{1 - 4\eta\beta'(t(\eta,\tau))} & \dfrac{1}{1 - 4\eta\beta'(t(\eta,\tau))} \end{pmatrix}$$

a.e., and so

$$\begin{aligned} & (\partial_\eta - 4\phi\partial_\tau)\phi(\eta,\tau) = \big(\partial_\eta - 4\beta(t(\eta,\tau))\partial_\tau\big)\beta(t(\eta,\tau)) \\ = \;& \beta'(t(\eta,\tau))\frac{\partial t(\eta,\tau)}{\partial \eta} - 4\beta(t(\eta,\tau))\beta'(t(\eta,\tau))\frac{\partial t(\eta,\tau)}{\partial \tau} \\ = \;& \beta'(t(\eta,\tau))\frac{4\beta(t(\eta,\tau))}{1 - 4\eta\beta(t(\eta,\tau))} - \frac{4\beta(t(\eta,\tau))\beta'(t(\eta,\tau))}{1 - 4\eta\beta'(t(\eta,\tau))} = 0\,. \end{aligned}$$

Therefore we are only left to prove that $\mathfrak{B}\phi = (\partial_\eta - 4\phi\partial_\tau)\phi$ in distributional sense. In this perspective, it will be sufficient to show that the distribution $\frac{\partial(\phi^2)}{\partial\tau}$ is represented by the function $2\phi\,\partial_\tau\phi$: this in turn is true since ϕ^2 is locally Lipschitz continuous, whence the pointwise partial derivative

$$\begin{aligned} \frac{\partial(\phi^2)}{\partial\tau}(\eta,\tau) &= \lim_{\sigma\to\tau} \frac{\phi(\eta,\sigma)^2 - \phi(\eta,\tau)^2}{\sigma - \tau} \\ &= \lim_{\sigma\to\tau} (\phi(\eta,\sigma) + \phi(\eta,\tau)) \cdot \frac{\phi(\eta,\sigma) - \phi(\eta,\tau)}{\sigma - \tau} \;=\; 2\phi(\eta,\tau)\frac{\partial\phi}{\partial\tau}(\eta,\tau) \end{aligned}$$

exists almost everywhere in ω and coincides with $\frac{\partial(\phi^2)}{\partial\tau}$ in distributional sense.

We stress that all the maps $\phi : \omega \to \mathbb{R}$ arising from the previous discussion effectively parametrize a $\mathbf{C}^1_{\mathbb{H}}$ surface; in fact by Theorem 4.22 it is sufficient to find $\mathbf{C}^\infty$ functions $\phi_\epsilon : \omega \to \mathbb{R}$ such that

$$\begin{array}{ll} \phi_\epsilon \to \phi & \text{locally uniformly on } \omega \\ W^{\phi_\epsilon}\phi_\epsilon \to 0 & \text{locally uniformly on } \omega \end{array}$$

as $\epsilon \to 0$. Fix then (e.g. mollifying β) a sequence $\beta_\epsilon \in \mathbf{C}^\infty$ such that $\operatorname{Lip} \beta_\epsilon \leq L$ and $\beta_\epsilon \to \beta$ locally uniformly in $\mathbb{R}$, and consider the maps ϕ_ϵ arising from the previous discussion but considering β_ϵ instead of β. By construction we have $W^{\phi_\epsilon}\phi_\epsilon \equiv 0$; moreover, ϕ_ϵ are well defined on all ω (since $\operatorname{Lip} \beta_\epsilon \leq L$) and it is not difficult to check that they converge locally uniformly to ϕ. Observe that if β is not $\mathbf{C}^1$, then the surface parametrized by ϕ cannot be of class $\mathbf{C}^1$, since its intersection with the plane $\{y = 0\}$ is the line $\{(\beta(t), 0, t) : t \in \mathbb{R}\}$ which is not $\mathbf{C}^1$.

For instance, let us put $\beta(t) = |t|$: it is not difficult to compute that the associated parametrization is

$$\begin{array}{rcl} \phi & : &]-1/4, 1/4[\times \mathbb{R} \to \mathbb{R} \\ & & (\eta, \tau) \longmapsto \begin{cases} \dfrac{\tau}{1-4\eta} & \text{if } \tau \geq 0 \\ -\dfrac{\tau}{1+4\eta} & \text{if } \tau < 0. \end{cases} \end{array}$$

The surface parametrized by this ϕ is then perimeter minimizing of class $\mathbf{C}^1_{\mathbb{H}}$ but not $\mathbf{C}^1$.

5.2 First and second variation of the area functional for intrinsic graphs

In this section we want to obtain first and second variation formulae of the area functional for intrinsic graphs; similar formulae have been obtained also in [56, 59, 133, 100, 101]. We will study in Section 5.3 the structure of all entire stationary points (i.e. those functions with vanishing first variation), while a proper second variation formula (see (5.26)) will be crucial in the study of the Bernstein problem in $\mathbb{H}^1$ (see Section 5.4.1).

5.2.1 First variation of the area

Let us fix a $\mathbf{C}^1$ map $\phi : \omega \to \mathbb{R}$, where ω is an open subset of $\mathbb{R}^{2n}$, and put

$$E_\phi := \{\iota(A) \cdot se_1 \in \mathbb{H}^n : A \in \omega \text{ and } s < \phi(A)\} \subset C_{X_1}(\omega) \tag{5.12}$$

where we $C_{X_1}(\omega)$ is the cylinder of base $\iota(\omega)$ along X_1 defined by

$$C_{X_1}(\omega) := \iota(\omega) \cdot \{se_1 \in \mathbb{H}^n : s \in \mathbb{R}\};$$

observe that $C_{X_1}(\omega)$ is an open neighbourhood of $S := \Phi(\omega)$, where as usual Φ is the map $A \mapsto \iota(A) \cdot \phi(A)e_1$.

Let us assume that E_ϕ is a minimizer for the $\mathbb{H}$-perimeter in $C_{X_1}(\omega)$, fix $\psi \in \mathbf{C}_c^\infty(\omega)$ and set $\phi_s := \phi + s\psi$; we can therefore consider the class of competitors E_{ϕ_s}, which are defined as in (5.12) (observe that $E \Delta E_{\phi_s} \Subset C_{X_1}(\omega)$), and set

$$g(s) := \|\partial E_{\phi_s}\|_{\mathbb{H}}(C_{X_1}(\omega)) = \int_\omega \sqrt{1 + |W^{\phi_s}\phi_s|^2} d\mathcal{L}^{2n}. \tag{5.13}$$

The fact that $g(s) \geq g(0)$ for all $s \in \mathbb{R}$ implies that $g'(0) = 0$. It is not difficult to check that

$$(W_{n+1}^\phi)^* \psi = -W_{n+1}^\phi \psi + 4\psi \widetilde{T}\phi \qquad \text{for all } \psi \in \mathbf{C}^\infty,$$

whence

$$\begin{aligned} W_{n+1}^{\phi_s}\phi_s &= \widetilde{Y}_1\phi + s\widetilde{Y}_1\psi - 4(\phi + s\psi)(\widetilde{T}\phi + s\widetilde{T}\psi) \\ &= W_{n+1}^\phi \phi - s\left(W_{n+1}^\phi\right)^* \psi - 4s^2\psi\widetilde{T}\psi \end{aligned}$$

and so

$$g(s) = \int_\omega \left[1 + \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} (\widetilde{X}_j\phi + s\widetilde{X}_j\psi)^2 + \left(W_{n+1}^\phi\phi - s(W_{n+1}^\phi)^*\psi - 4s^2\psi\widetilde{T}\psi\right)^2\right]^{1/2} d\mathcal{L}^{2n}. \tag{5.14}$$

From now on we will write just $\sum_j$ to mean the sum on indices $j = 2, \dots, 2n$ with $j \neq n+1$; when $n = 1$ the previous formula and the following ones are to be understood by "erasing" all sums of this type.

Starting from (5.14) it is not difficult to compute

$$g'(s) = \int_\omega \frac{\sum_j \widetilde{X}_j\phi_s\, \widetilde{X}_j\psi + W_{n+1}^{\phi_s}\phi_s \left(-(W_{n+1}^\phi)^*\psi - 8s\psi\widetilde{T}\psi\right)}{\sqrt{1 + |W^{\phi_s}\phi_s|^2}} d\mathcal{L}^{2n} \tag{5.15}$$

and in particular

$$g'(0) = \int_\omega \frac{\sum_j \widetilde{X}_j\phi\, \widetilde{X}_j\psi - W_{n+1}^\phi\phi\,(W_{n+1}^\phi)^*\psi}{\sqrt{1 + |W^\phi\phi|^2}} d\mathcal{L}^{2n} \tag{5.16}$$

The Euler equation for stationary points of the area functional is then

$$W^\phi \cdot \frac{W^\phi\phi}{\sqrt{1 + |W^\phi\phi|^2}} = 0 \qquad \text{on } \omega, \tag{5.17}$$

where the previous equality must be understood in distributional sense.

5.2.2 Second variation of the area

If $\phi \in \mathbf{C}^1$, from (5.15) we can compute

$$g''(s) = \int_\omega \frac{1}{1+|W^{\phi_s}\phi_s|^2}\Bigg\{\sqrt{1+|W^{\phi_s}\phi_s|^2}\times \\ \times\left[\sum_j (\widetilde{X}_j\psi)^2 + \left((W^\phi_{n+1})^*\psi + 8s\psi\widetilde{T}\psi\right)^2 - 8\psi\widetilde{T}\psi W^{\phi_s}_{n+1}\phi_s\right] + \\ -\left[\frac{\left[\sum_j \widetilde{X}_j\phi_s\,\widetilde{X}_j\psi + W^{\phi_s}_{n+1}\phi_s\left(-(W^\phi_{n+1})^*\psi - 8s\psi\widetilde{T}\psi\right)\right]^2}{\sqrt{1+|W^{\phi_s}\phi_s|^2}}\right]\Bigg\} d\mathcal{L}^{2n} \quad (5.18)$$

and so

$$g''(0) = \int_\omega \frac{(1+|W^\phi\phi|^2)\left[|W^{\phi^*}\psi|^2 - 8\psi\widetilde{T}\psi W^\phi_{n+1}\phi\right] - \left(W^\phi\phi\cdot W^{\phi^*}\psi\right)^2}{\left[1+|W^\phi\phi|^2\right]^{3/2}} d\mathcal{L}^{2n}, \quad (5.19)$$

where we put

$$W^{\phi^*}\psi := \left(\widetilde{X}^*_2\psi,\dots,\widetilde{X}^*_n\psi,(W^\phi_{n+1})^*\psi,\widetilde{X}^*_{n+2}\psi,\dots,\widetilde{X}^*_{2n}\psi\right) \quad \text{if } n\geq 2$$
$$W^{\phi^*}\psi := (W^\phi_2)^*\psi \quad \text{if } n=1;$$

the fact that E_ϕ is a minimizer implies that $g''(0)\geq 0$ for all $\psi\in\mathbf{C}^1_c(\omega)$.

Notice that when $n=1$ formula (5.19) for the second variation reads as

$$g''(0) = \int_\omega \frac{|W^{\phi^*}\psi|^2 - 8\psi\widetilde{T}\psi W^\phi\phi(1+|W^\phi\phi|^2)}{\left[1+|W^\phi\phi|^2\right]^{3/2}} d\mathcal{L}^2; \quad (5.20)$$

in particular when $W^\phi\phi\equiv 0$ one has $g''(0)\geq 0$ for all $\mathbf{C}^1_c(\omega)$. If we suppose $\phi\in\mathbf{C}^2$ we can further exploit (5.20) as

$$\begin{aligned} g''(0) &= \int_\omega \frac{|W^{\phi^*}\psi|^2 - 4\widetilde{T}(\psi^2)W^\phi\phi(1+|W^\phi\phi|^2)}{\left[1+|W^\phi\phi|^2\right]^{3/2}} d\mathcal{L}^2 \\ &= \int_\omega \left[\frac{|W^{\phi^*}\psi|^2}{\left[1+|W^\phi\phi|^2\right]^{3/2}} + 4\psi^2\widetilde{T}\left(\frac{W^\phi\phi}{\left[1+|W^\phi\phi|^2\right]^{1/2}}\right)\right] d\mathcal{L}^2. \end{aligned} \quad (5.21)$$

We will see in Section 5.3 that if $n=1$ and ϕ is a stationary point of the area functional, i.e. if ϕ solves (5.17), then

$$(W^\phi)^2\phi = 0 \quad (5.22)$$

and thanks to this the first term of (5.21) becomes, integrating by parts,

$$\begin{aligned}\int_\omega \frac{|W^{\phi^*}\psi|^2\, d\mathcal{L}^2}{[1+|W^\phi\phi|^2]^{3/2}} &= \int_\omega \psi W^\phi\left(\frac{W^{\phi^*}\psi}{[1+|W^\phi\phi|^2]^{3/2}}\right) d\mathcal{L}^2 \\ &= \int_\omega \psi \frac{W^\phi W^{\phi^*}\psi}{[1+|W^\phi\phi|^2]^{3/2}} d\mathcal{L}^2. \end{aligned} \tag{5.23}$$

Since

$$\begin{aligned}(W^\phi W^{\phi^*} - W^{\phi^*}W^\phi)\psi &= W^\phi\big(-W^\phi + 4\widetilde{T}\phi\ \mathrm{Id}\big)\psi - \big(-W^\phi + 4\widetilde{T}\phi\ \mathrm{Id}\big)W^\phi\psi \\ &= 4\psi\, W^\phi\widetilde{T}\phi\end{aligned}$$

we can rewrite (5.23) as

$$\begin{aligned}\int_\omega \frac{|W^{\phi^*}\psi|^2}{[1+|W^\phi\phi|^2]^{3/2}} d\mathcal{L}^2 &= \int_\omega \psi \frac{W^{\phi^*}W^\phi\psi + 4\psi\, W^\phi\widetilde{T}\phi}{[1+|W^\phi\phi|^2]^{3/2}} d\mathcal{L}^2 \\ &= \int_\omega \left[\frac{(W^\phi\psi)^2}{[1+|W^\phi\phi|^2]^{3/2}} + 4\psi^2\frac{W^\phi\widetilde{T}\phi}{[1+|W^\phi\phi|^2]^{3/2}}\right] d\mathcal{L}^2\,, \end{aligned} \tag{5.24}$$

where we used (5.22) again. Therefore (5.21) becomes

$$\begin{aligned} g''(0) &= \int_\omega \left\{ \frac{(W^\phi\psi)^2}{[1+|W^\phi\phi|^2]^{3/2}} + 4\psi^2\left[\frac{W^\phi\widetilde{T}\phi}{[1+|W^\phi\phi|^2]^{3/2}} + \widetilde{T}\left(\frac{W^\phi\phi}{[1+|W^\phi\phi|^2]^{1/2}}\right)\right]\right\} d\mathcal{L}^2 \\ &= \int_\omega \left\{ \frac{(W^\phi\psi)^2}{[1+|W^\phi\phi|^2]^{3/2}} + \right. \\ &\qquad \left. + 4\psi^2\left[\frac{W^\phi\widetilde{T}\phi}{[1+|W^\phi\phi|^2]^{3/2}} + \frac{[1+|W^\phi\phi|^2]\widetilde{T}W^\phi\phi - |W^\phi\phi|^2\widetilde{T}W^\phi\psi}{[1+|W^\phi\phi|^2]^{3/2}}\right]\right\} d\mathcal{L}^2 \\ &= \int_\omega \frac{(W^\phi\psi)^2 + 4\psi^2\,[W^\phi\widetilde{T}\phi + \widetilde{T}W^\phi\phi]}{[1+|W^\phi\phi|^2]^{3/2}} d\mathcal{L}^2 \end{aligned} \tag{5.25}$$

Finally, one has

$$W^\phi\widetilde{T}\phi = \phi_{\eta\tau} - 4\phi\phi_{\tau\tau} = \widetilde{T}W^\phi\phi + 4(\widetilde{T}\phi)^2$$

and so from (5.25) we can also write

$$g''(0) = \int_\omega \frac{(W^\phi\psi)^2 + 8\psi^2\,[\widetilde{T}W^\phi\phi + 2(\widetilde{T}\phi)^2]}{[1+|W^\phi\phi|^2]^{3/2}} d\mathcal{L}^2. \tag{5.26}$$

Equation (5.26) will be crucial in the proof of Theorem 5.23.

5.3 Entire solutions of the minimal surface equation in $\mathbb{H}^1$

In this section we will give a characterization (see Corollary 5.20) of all the entire $\mathbf{C}^2$ solutions $\phi : \mathbb{R}^2_{\eta,\tau} \to \mathbb{R}$ of the minimal surface equation for intrinsic graphs in $\mathbb{H}^1$, i.e. of

$$W^\phi \left(\frac{W^\phi \phi}{\sqrt{1 + |W^\phi \phi|^2}} \right) = 0 \qquad \text{in } \mathbb{R}^2; \tag{5.27}$$

this result will provide the key tool to attack the Bernstein problem in $\mathbb{H}^1$. Observe that (5.27) can be written as

$$0 = \frac{(W^\phi)^2 \phi \ \sqrt{1 + |W^\phi \phi|^2} \ - \ W^\phi \phi \ \frac{W^\phi \phi \cdot (W^\phi)^2 \phi}{\sqrt{1+|W^\phi \phi|^2}}}{1 + |W^\phi \phi|^2} = \frac{(W^\phi)^2 \phi}{(1 + |W^\phi \phi|^2)^{3/2}}$$

which means that ϕ is a solution of (5.27) if and only if it solves

$$(W^\phi)^2 \, \phi = 0 \qquad \text{in } \mathbb{R}^2. \tag{5.28}$$

Notice that (5.28) is equivalent to a "double" Burgers' equation: in fact by performing the change of variables

$$\begin{aligned} G \ : \ & \mathbb{R}^2_{x,t} \to \mathbb{R}^2_{\eta,\tau} \\ & (x,t) \longmapsto (t, -4x), \end{aligned}$$

setting $u(x,t) := (\phi \circ G)(x,t) = \phi(t,-4x)$ and defining L_u to be the operator

$$(L_u v)(x,t) = (v_t + u \, v_x)(x,t) \qquad (v \in \mathbf{C}^1(\mathbb{R}^2)),$$

we get

$$(L_u(L_u u))(x,t) = ((W^\phi)^2 \phi)(t,-4x).$$

This means that we can restrict to consider the $\mathbf{C}^2$ solutions u of the "double" Burgers' equation

$$L_u^2 u = 0 \qquad \text{in } \mathbb{R}^2 \tag{5.29}$$

(recall that $L_u u = 0$ is the classical Burgers' equation, see [66]). We will focus our attention on the problem (5.29) rather than (5.27) or (5.28).

5.3.1 Characteristic curves for entire solutions of $L_u^2 u = 0$

Suppose u is an entire $\mathbf{C}^2$ solution of (5.29) and let us consider the characteristic curves (see [66]) of the equation $L_u v = 0$, i.e., for any fixed $c \in \mathbb{R}$, the maximal solution $x = x(c, \cdot) : I_c \to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} \dot{x}(c,t) = u(x(c,t),t) \\ x(c,0) = c. \end{cases} \tag{5.30}$$

From (5.29) one gets $\frac{d}{dt} L_u u(x(c,t),t) = 0$ and so

$$L_u u(x(c,t),t) = A(c) \qquad \text{for all } t \in I_c\,.$$

Since

$$\frac{d}{dt} u(x(c,t),t) = \big(u_t(x(c,t),t) + u_x(x(c,t),t)\,\dot{x}(c,t)\big) = L_u u(x(c,t),t) = A(c)$$

we obtain

$$u(x(c,t),t) = A(c)t + B(c) \qquad \text{for all } t \in I_c, \tag{5.31}$$

where we have set $B(c) := u(c,0)$. Equation (5.31), together with (5.30), gives

$$x(c,t) = \frac{A(c)}{2} t^2 + B(c)t + c;$$

in particular, $I_c = \mathbb{R}$. We have therefore the following

Theorem 5.9. *Let u be an entire $\mathbf{C}^2$ solution of (5.29) and for $c, t \in \mathbb{R}$ set*

$$x(c,t) := \frac{A(c)}{2} t^2 + B(c)t + c\,,$$

where $A(c) := L_u u(c,0)$ and $B(c) := u(c,0)$. Then for all c, t we have

(i) $u(x(c,t),t) = A(c)t + B(c)$;

(ii) $L_u u(x(c,t),t) = A(c)$;

(iii) $x(\cdot,t)$ *is strictly increasing for any fixed time t;*

(iv) for all $c \in \mathbb{R}$ we have either $A'(c) = B'(c) = 0$ or $B'(c)^2 < 2A'(c)$.

In particular, the family of characteristics $x(c,\cdot)$ are parabolas which do not intersect.

Proof. We have already proved (i) and (ii); for (iii), it will be sufficient to prove that, for every t,

$$x(c,t) \neq x(c',t) \qquad \text{if } c \neq c'; \tag{5.32}$$

in fact, were (iii) false, we could find $c < c'$ and t' such that $x(c,t') \geq x(c',t')$, but since the characteristics are continuous and $x(c,0) = c < c' = x(c',0)$ we could find a t between 0 and t' such that (5.32) does not hold.

Arguing by contradiction, let us assume that (5.32) does not hold for some $c \neq c'$ and t; observe that from (i) and (ii) one has

$$\begin{aligned} &A(c) = L_u u(x(c,t),t) = A(c') \\ &A(c)t + B(c) = u(x(c,t),t) = A(c')t + B(c') \end{aligned}$$

whence $c = x(c,t) - \frac{A(c)}{2}t^2 - B(c)t = c'$, which is a contradiction.

Notice that (iii) implies that

$$\frac{\partial x}{\partial c}(c,t) = \frac{A'(c)}{2}t^2 + B'(c)t + 1 \geq 0$$

for all c, t, and this in turn implies $B'(c)^2 \leq 2A'(c)$. Observe in particular that $A'(c) \geq 0$ and $\frac{\partial x}{\partial c}(c,t) \geq 0$. In order to prove (iv), suppose by contradiction that for a certain c we have $B'(c)^2 = 2A'(c) \neq 0$. Differentiating (i) with respect to c one gets

$$\frac{\partial u}{\partial c}(x(c,t),t) = \frac{A'(c)t + B'(c)}{\frac{\partial x}{\partial c}(c,t)} = \frac{A'(c)\big(t + \frac{B'(c)}{A'(c)}\big)}{A'(c)\big(t + \frac{B'(c)}{A'(c)}\big)^2} = \frac{1}{t + \frac{B'(c)}{A'(c)}} \qquad \text{for all } t$$

which contradicts the hypothesis $u \in \mathbf{C}^2(\mathbb{R}^2)$. □

Remark 5.10. Observe that if u is a $\mathbf{C}^2$ solution of

$$\begin{cases} L_u^2 u = 0 \\ u(x,0) = B(x) \\ L_u u(x,0) \equiv A \in \mathbb{R} \end{cases}$$

then one must have also $B(x) \equiv B(0) = B$. In particular, Theorem 5.9 (i) implies that $u(x,t) = At + B$.

Remark 5.11. Following the same proof of Theorem 5.9 (i), it is possible to prove that if u is a $\mathbf{C}^1$ solution of the Burgers' equation

$$L_u u = u_t + u u_x \equiv k$$

for a suitable constant $k \in \mathbb{R}$, then $B = u(\cdot, 0)$ must be constant.

It is not difficult to extend the proof of Theorem 5.9 and get the following

Theorem 5.12. *Let Ω be an open set of $\mathbb{R}^2_{x,t}$ such that $\{(x,0) : x \in \mathbb{R}\} \subset \Omega$, let $u \in \mathbf{C}^2(\Omega)$ be a solution of*

$$L_u^2 u = 0 \qquad \text{in } \Omega\,, \tag{5.33}$$

and let $A(c), B(c)$ and $x(c,t)$ be as in Theorem 5.9. Suppose moreover that the set $\{(x(c,t),t) : c,t \in \mathbb{R}\}$ is contained in Ω. Then the statements (i)-(iv) of Theorem 5.9 still hold.

From Theorem 5.12 we get the following uniqueness result for the "double" Burgers' equation (see also [53], Chap V, Section 7, and [130]).

Theorem 5.13. *Let $u_0 \in \mathbf{C}^2(\mathbb{R}), u_1 \in \mathbf{C}^1(\mathbb{R})$ be given functions and set $A := u_0, B := u_1 + u_0 u_0'$. Let $x(c,t) := A(c)t^2/2 + B(c)t + c$ and set*

$$\Omega = \{(x(c,t),t) : c,t \in \mathbb{R}\}. \tag{5.34}$$

Then there is at most one solution $u \in \mathbf{C}^2(\Omega)$ of the problem

$$\begin{cases} L_u^2 u = 0 & \text{in } \Omega \\ u(x,0) = u_0(x) & \forall\, x \in \mathbb{R} \\ u_t(x,0) = u_1(x) & \forall\, x \in \mathbb{R}. \end{cases} \tag{5.35}$$

Proof. By Theorem 5.12 any solution $u \in \mathbf{C}^2(\Omega)$ of (5.35) has to satisfy

$$u(x(c,t),t) = A(c)t + B(c);$$

however, hypothesis (5.34) ensures that for all $(x,t) \in \Omega$ we can find a c such that $x = x(c,t)$. This proves that u is uniquely determined in Ω by A and B, i.e. by u_0 and u_1. □

Corollary 5.14. *Let $u_0, u_1, A, B, x(t,c)$ and Ω be as in Theorem 5.13, and suppose moreover that for all $c \in \mathbb{R}$ we have $A'(c) = B'(c) = 0$ or $B'(c)^2 < A'(c)$. Then*

(i) Ω is an open neigbourhood of the x-axis $\{(x,0) : x \in \mathbb{R}\}$;

(ii) there is at most one solution $u \in \mathbf{C}^2(\Omega)$ of the problem (5.35).

Proof. Observe that the map

$$\begin{aligned} F \;:\; & \mathbb{R}^2 \to \mathbb{R}^2 \\ & (c,t) \longmapsto (x(c,t),t) \end{aligned}$$

is regular and one-to-one; in particular, it is an open map and (i) follows. This means that condition (5.34) of Theorem 5.13 is automatically fulfilled, and so (ii) must hold too. □

Corollary 5.15. *Under the same assumptions as in Theorem 5.9 let us denote* $l_1 := \lim_{c\to+\infty} A(c)$ *(respectively* $l_2 := \lim_{c\to-\infty} A(c)$*). Then for any fixed* $t \in \mathbb{R}$ *we can conclude*

$$\lim_{c\to+\infty} x(c,t) = +\infty \quad (\textit{resp.} \quad \lim_{c\to-\infty} x(c,t) = -\infty) \tag{5.36}$$

if either $l_1 \in \mathbb{R}$ *(resp.* $l_2 \in \mathbb{R}$*), or* $l_1 = +\infty$ *(resp.* $l_2 = -\infty$*) and one of the following conditions is satisfied:*

$$\liminf_{c\to+\infty} \frac{A(c)}{c} = 0 \quad \left(\textit{resp.} \ \liminf_{c\to-\infty} \frac{A(c)}{c} = 0\right) \tag{5.37}$$

$$\limsup_{c\to+\infty} \frac{A(c)}{c} = +\infty \quad \left(\textit{resp.} \ \limsup_{c\to+\infty} \frac{A(c)}{c} = -\infty\right) \tag{5.38}$$

$$\liminf_{c\to+\infty} \left|\frac{B(c)}{\sqrt{cA(c)}}\right| < \sqrt{2} \quad \left(\textit{resp.} \ \liminf_{c\to-\infty} \left|\frac{B(c)}{\sqrt{cA(c)}}\right| < \sqrt{2}\right). \tag{5.39}$$

In particular, when $\lim_{c\to+\infty} x(c,t) = +\infty$ *and* $\lim_{c\to-\infty} x(c,t) = -\infty$ *we have that* $x(\cdot,t) : \mathbb{R} \to \mathbb{R}$ *is a homeomorphism and* $\Omega := \{(x(c,t),t) : c,t \in \mathbb{R}\} = \mathbb{R}^2$.

Proof. Observe that for fixed $t \in \mathbb{R}$ and $c \neq 0$ one can write

$$x(c,t) = \sqrt{|c|}\left[\frac{1}{2}\left(\frac{A(c)-A(0)}{\sqrt{|c|}} + \frac{A(0)}{\sqrt{|c|}}\right)t^2 + \left(\frac{B(c)-B(0)}{\sqrt{|c|}} + \frac{B(0)}{\sqrt{|c|}}\right)t + \sqrt{c}\right]. \tag{5.40}$$

Being A increasing there exist

$$m_1 := \lim_{c\to+\infty} (A(c)-A(0)) \qquad \big(\text{resp. } m_2 := \lim_{c\to-\infty} (A(c)-A(0))\big)$$

with $-\infty \le m_2 \le 0 \le m_1 \le +\infty$. Notice also that, using Theorem 5.9 (iv), one can get

$$|B(c)-B(0)| \le \int_0^c |B'(s)|\,ds \le \sqrt{2}\int_0^c \sqrt{A'(s)}\,ds \le \sqrt{2\,|c|\,|A(c)-A(0)|} \tag{5.41}$$

and this allows us to conclude when $l_1 \in \mathbb{R}$ (resp. $l_2 \in \mathbb{R}$), since in this case we have $m_1 \in \mathbb{R}$ (resp. $m_2 \in \mathbb{R}$) and so $x(c,t) \approx c$ for large (resp. small) c.

Instead, when $l_1 = +\infty$, for large c we can write

$$\begin{aligned} x(c,t) \ = \sqrt{c(A(c)-A(0))}\Bigg[&\frac{1}{2}\left(\sqrt{\frac{A(c)-A(0)}{c}} + \frac{A(0)}{\sqrt{c(A(c)-A(0))}}\right)t^2 + \\ &+ \left(\frac{B(c)-B(0)}{\sqrt{c(A(c)-A(0))}} + \frac{B(0)}{\sqrt{c(A(c)-A(0))}}\right)t + \sqrt{\frac{c}{A(c)-A(0)}}\Bigg] \end{aligned} \tag{5.42}$$

whence (using (5.41) again) $\limsup_{c\to\infty} x(c,t) = +\infty$ in case (5.37) or (5.38) hold; however, this implies (5.36) since $x(\cdot,t)$ is increasing. When $c \to -\infty$ we have instead

$$x(c,t) = \sqrt{c(A(c)-A(0))}\left[-\frac{1}{2}\left(\sqrt{\frac{A(c)-A(0)}{c}} + \frac{A(0)}{\sqrt{c(A(c)-A(0))}}\right)t^2 + \right.$$
$$\left. + \left(\frac{B(c)-B(0)}{\sqrt{c(A(c)-A(0))}} + \frac{B(0)}{\sqrt{c(A(c)-A(0))}}\right)t - \sqrt{\frac{c}{A(c)-A(0)}}\right]$$

and analogously we conclude $\liminf_{c\to-\infty} x(c,t) = -\infty$, which is sufficient.

Instead if (5.39) holds together with $l_1 = +\infty$, we have a sequence $c_h \to +\infty$ such that

$$\frac{B'(c_h)^2}{2A'(c_h)} \le (1-\epsilon)c_h \qquad \forall h; \tag{5.43}$$

observe that the parabola $x(c_h,\cdot)$ reaches its minimum at $t = -\frac{B(c_h)}{A(c_h)}$ and so

$$x(c_h,t) \ge x(c_h, -\tfrac{B(c_h)}{A(c_h)}) = c_h - \frac{B'(c_h)^2}{2A'(c_h)} \ge \epsilon c_h \overset{h\to\infty}{\longrightarrow} +\infty$$

which, together with the fact that $x(\cdot,t)$ is increasing, proves (5.36) when $c \to +\infty$. It is a little more complicated to prove the thesis when $l_2 = -\infty$ and $c \to -\infty$; however, as in (5.43) we get a sequence $c_h \to -\infty$ such that

$$-\frac{B'(c_h)^2}{2A'(c_h)} \le \epsilon c_h - c_h \qquad \forall h$$

and so

$$\begin{aligned} x(c_h,t) &= \frac{A(c_h)}{2}\left(t + \frac{B(c_h)}{A(c_h)}\right)^2 + \left(c_h - \frac{B'(c_h)^2}{2A'(c_h)}\right) \\ &\le \frac{A(c_h)}{2}\left(t + \frac{B(c_h)}{A(c_h)}\right)^2 + \epsilon c_h \end{aligned}$$

which allows us to conclude since $A(c) \to -\infty$ as $c \to -\infty$. □

Example 5.16. Set $A(c) := c/2$ and $B(c) := -c$; then it is easy to check that the family of characteristic curves for the related problem (5.29) are

$$x(c,t) = (t-2)^2 c/4.$$

Notice that $x(c,2) \equiv 0$, i.e. the thesis of Corollary 5.15 does not hold; here in fact (5.39) is not fulfilled since

$$\lim_{c\to\pm\infty} \frac{B(c)}{\sqrt{cA(c)}} = \sqrt{2}.$$

Moreover, taking into account Theorem 5.9, a global $\mathbf{C}^2$ solution u of (5.29), with $u(x,0) = -x$ and $L_u u(x,0) = x/2$, cannot exist.

Example 5.17. Let $A(c) = c$ and $B(c) = \sqrt{2(1+c^2)}$, and let us consider the associated family of characteristic parabolas

$$x(c,t) = \frac{c}{2}t^2 + \sqrt{2(1+c^2)}t + c.$$

Then for fixed t we have

$$\frac{\partial x}{\partial c}(c,t) = \frac{t^2}{2} + \frac{\sqrt{2}\,c}{\sqrt{1+c^2}}t + 1$$

which is (strictly) positive for any c: in particular, the family the characteristics cannot intersect, and in fact one has

$$B'(c)^2 = 2\frac{c^2}{1+c^2} < 2 = 2A'(c).$$

Observe also that

$$\lim_{c\to\pm\infty} \left| \frac{B(c)}{\sqrt{cA(c)}} \right| = \sqrt{2}.$$

If we set $F(c,t) := (x(c,t),t)$ it is easy to see that the image $F(\mathbb{R}^2)$ is the open set

$$\Omega := \mathbb{R}^2 \setminus \left(\{(x,\sqrt{2}) : x \le 0\} \cup \{(x,-\sqrt{2}) : x \ge 0\} \right).$$

Indeed a simple calculation gives $F^{-1}(x,t) = (c(x,t),t)$ where

$$c(x,t) = \begin{cases} \dfrac{x(1+t^2/2) - \sqrt{2}|t|\sqrt{x^2 + (1-t^2/2)^2}}{(1-t^2/2)^2} & \text{if } |t| \neq \sqrt{2} \\ \dfrac{x^2-4}{4x} \qquad \text{if} \quad t = \sqrt{2}, x > 0 \quad \text{or} \quad t = -\sqrt{2}, x < 0. \end{cases}$$

We will see that $u(x,t) := A(c(x,t))t + B(c(x,t))$ is the unique solution of (5.33) in Ω such that $L_u u(x,0) = A(x)$ and $u(x,0) = B(x)$.

Example 5.18. If we require $B \equiv 0$, then the solution to the "double" Burgers' equation with initial data A, B is defined everywhere for any $\mathbf{C}^2$ increasing function A. Obviously, even if it is possible to characterize it intrinsically as in Theorem 5.9 (i), in general it is not possible to give an explicit formula for the solution.

5.3.2 Existence of entire solutions

In the following theorem we provide an existence and uniqueness result for the equation (5.29).

Theorem 5.19. *Let $A, B \in \mathbf{C}^2(\mathbb{R})$ and for $c, t \in \mathbb{R}$ set*

$$\begin{aligned} &x(c,t) := \frac{A(c)}{2}t^2 + B(c)t + c \\ &F : \mathbb{R}^2 \ni (c,t) \longmapsto (x(c,t),t) \in \mathbb{R} \\ &\Omega := F(\mathbb{R}^2) = \{(x(c,t),t) : c,t \in \mathbb{R}\} \end{aligned}$$

and suppose that

$$\text{for all } c \in \mathbb{R} \text{ one has either } A'(c) = B'(c) = 0 \text{ or } B'(c)^2 < 2A'(c). \tag{5.44}$$

Then

(i) F is $\mathbf{C}^2$ regular and one-to-one and, in particular, Ω is open;

(ii) if $F^{-1}(x,t) := (c(x,t),t)$, $(x,t) \in \Omega$, then $u(x,t) := A(c(x,t))t + B(c(x,t))$ is the unique $\mathbf{C}^2$ solution of $L_u^2 u = 0$ in Ω satisfying $L_u u(x,0) = A(x)$ and $u(x,0) = B(x)$.

Proof. We begin by proving that the $\mathbf{C}^2$ map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one. By construction it is enough to prove that for any fixed t the map $x(\cdot,t)$ is strictly increasing, and this is an easy consequence of (5.44) which implies that

$$\frac{\partial x}{\partial c}(t,c) = \frac{A'(c)}{2}t^2 + B'(c) + 1 > 0$$

for any c. Being one-to-one and continuous, F is also an open map, i.e. $\Omega \subset \mathbb{R}^2$ is open, and (i) is proved.

For (ii), observe that the Jacobian matrix of F is given by

$$JF(c,t) = \begin{pmatrix} \frac{A'(c)}{2}t^2 + B'(c) + 1 & A(c)t + B(c) \\ 0 & 1 \end{pmatrix}$$

and so the Inverse Function Theorem implies that the Jacobian matrix of F^{-1} is

$$\begin{aligned} JF^{-1}(x,t) &= \big(JF(F^{-1}(x,t))\big)^{-1} \\ &= \frac{1}{\frac{A'(c(x,t))}{2}t^2 + B'(c(x,t)) + 1} \begin{pmatrix} 1 & -A(c(x,t))t - B(c(x,t)) \\ 0 & \frac{A'(c(x,t))}{2}t^2 + B'(c(x,t)) + 1 \end{pmatrix}. \end{aligned}$$

Thus

$$\frac{\partial c}{\partial x}(x,t) = \frac{1}{\frac{A'(c(x,t))}{2}t^2 + B'(c(x,t)) + 1} \tag{5.45}$$

$$\frac{\partial c}{\partial t}(x,t) = -\frac{A(c(x,t))t + B(c(x,t))}{\frac{A'(c)}{2}t^2 + B'(c) + 1} \tag{5.46}$$

and so one can compute

$$\begin{aligned} L_u u(x,t) &= \big[A'(c(x,t))t + B'(c(x,t))\big]\frac{\partial c}{\partial t}(x,t) + A(c(x,t)) + \\ &\quad + \big[A'(c(x,t))t + B'(c(x,t))\big]\big[A(c(x,t))t + B(c(x,t))\big]\frac{\partial c}{\partial x}(x,t) \\ &= A(c(x,t)) \end{aligned}$$

and

$$L_u^2 u(x,t) = A'(c(x,t))\frac{\partial c}{\partial t}(x,t) + \big[A(c(x,t))t + B(c(x,t))\big]A'(c(x,t))\frac{\partial c}{\partial x}(x,t) = 0.$$

Therefore u is a solution of the given problem, and the proof is completed since uniqueness follows from Theorem 5.13. □

Corollary 5.20. *Suppose that $A, B \in \mathbf{C}^2(\mathbb{R})$ and that $u : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathbf{C}^2$ entire solution of the problem*

$$\begin{cases} L_u^2 u = 0 \\ u(x,0) = B(x) \\ L_u u(x,0) = A(x) \end{cases}$$

Let $\Omega, c(x,t)$ be as in Theorem 5.19; then

$$u(x,t) = A(c(x,t))t + B(c(x,t)) \qquad \text{for all } (x,t) \in \Omega$$

and u is the unique solution in Ω of the same problem.

5.3.3 Examples of entire solutions of $L_u^2 u = 0$

Example 5.21. Let $A(c) = \alpha c$ ($\alpha > 0$) and $B \equiv 0$, then it is easy to see that in this case $\Omega = \mathbb{R}^2$; since $c(x,t) = \frac{2x}{2+\alpha t^2}$, the required solution of (5.29) is given by

$$u(x,t) = \frac{2\alpha x t}{2 + \alpha t^2}.$$

These solutions correspond to the maps $\phi_{\alpha'}(\eta,\tau) = -\frac{\alpha'\eta\tau}{1+2\alpha'\eta^2}$ (where $\alpha' := \alpha/4$) solutions of $(W^\phi)^2\phi = 0$ (see also Example 5.7); it is not difficult to notice that the

surfaces parametrized by $\phi_{\alpha'}$ corresponds to $\{(x, y, t \in \mathbb{H}^1 : x = -\alpha' yt)\}$, which are deeply studied in [60]: in particular (see Theorem 1.2 therein) it is proved that they are not $\mathbb{H}$-perimeter minimizing (see also Theorem 5.23).

Example 5.22. Let $B \equiv 0$ and choose a bounded, not constant and strictly increasing $A \in \mathbf{C}^2$; then, if Ω and $c(x,t)$ are as in Theorem 5.19, by Corollary 5.15 we have $\Omega = \mathbb{R}^2$ and that $u(x,t) := A(c(x,t))t + B(c(x,t))$ is the unique entire solution of (5.29); moreover, $L_u u(x,t) = A(c(x,t))$ is bounded.

Observe that an analogous situation cannot occur in the Euclidean case: in fact (see [89], Theorem 17.5), any smooth global solution ψ of the classical minimal surface equation with $\|\nabla\psi\|_{L^\infty} < \infty$ must be linear. Here, instead, it happens that the map ϕ, which arises from the u of this construction, solves (5.27), is not linear (and, in particular, not of type (5.48), see Section 5.4) but is such that $\|W^\phi\phi\|_{L^\infty} < \infty$.

5.4 The Bernstein problem in $\mathbb{H}^n$

Let us recall the minimal surface equation for minimal $\mathbb{H}$-graphs in $\mathbb{H}^n$

$$W^\phi \cdot \left(\frac{W^\phi \phi}{\sqrt{1 + |W^\phi \phi|^2}} \right) = 0, \tag{5.47}$$

where $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ is of class $\mathbf{C}^2$. Observe that the "affine" functions given by

$$\phi(\eta, v, \tau) = c + \langle (\eta, v), w \rangle_{\mathbb{R}^{2n-1}} \tag{5.48}$$

for $c \in \mathbb{R}, w \in \mathbb{R}^{2n-1}$ (the previous formula has to be read as $\phi(\eta, \tau) = c + \eta w$ when $n = 1$) are trivial solutions of (5.47), and that they parametrize the so called "vertical hyperplanes", i.e. (right-translations of) maximal subgroups of $\mathbb{H}^n$ (see also (3.16)): it follows that these hypersurfaces are stationary points of the area functional, and a calibration argument implies that they are also minimizers since they have constant horizontal normal (see Example 5.5). These considerations suggest that the right counterpart of the classical Bernstein problem in the Heisenberg setting is

Bernstein problem for X_1-graphs in $\mathbb{H}^n$: are there entire solutions $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ of the minimal surface equation (5.47) which cannot be written as in (5.48)?

As we will see, again the answer seems to depend on the dimension n of the space; however, new and unexpected phenomena arise, e.g. the fact that we have solutions to (5.1) which are not area minimizing.

5.4.1 The Bernstein problem in $\mathbb{H}^1$

We have seen in Section 5.3 that for $n = 1$ there exist solutions of (5.47) which cannot be written as in (5.48); see for instance Examples 5.21 and 5.22. We already pointed out that every solution of the classic minimal surface equation (5.1) parametrizes (the boundary of) a global minimizer; in $\mathbb{H}^1$ instead a new phenomenon occurs, in the sense that there are entire solutions of the intrinsic minimal surface equation (5.47) which parametrize a surface which is not a minimizer. Anyway, whenever the surface is $\mathbb{H}$-perimeter minimizing in $\mathbb{H}^1$ it has to be a vertical plane: more precisely, we have the following

Theorem 5.23 (Minimizers vs. stationary entire X_1-graphs)**.** *Suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathbf{C}^2$ and define $S, E \subset \mathbb{H}^1$ to be respectively the X_1-graph and the X_1-subgraph induced by ϕ, i.e.*

$$S := \{\Phi(\eta,\tau) := \iota(\eta,\tau)\cdot\phi(\eta,\tau)e_1 : (\eta,\tau)\in\mathbb{R}^2\}$$
$$E := \{\iota(\eta,\tau)\cdot se_1 : (\eta,\tau)\in\mathbb{R}^2, s < \phi(\eta,\tau)\}.$$

Let us suppose E is a minimizer for the $\mathbb{H}$-perimeter measure in $\mathbb{H}^1$; then S is a vertical plane, i.e. $\phi(\eta,\tau) = w\eta + c$ for all $(\eta,\tau)\in\mathbb{R}^2$ for some constants $w, c \in \mathbb{R}$.

Proof. Step 1. First of all, we want to rewrite the second variation formula (5.26) in the coordinates c, t introduced in Section 3. Therefore let G be defined by

$$\begin{aligned} G \;:\; & \mathbb{R}^2_{x,t} \to \mathbb{R}^2_{\eta,\tau} \\ & (x,t) \longmapsto (t, -4x) \end{aligned}$$

and set

$$A(x) := (W^\phi\phi\circ G)(x,0), \qquad B(x) := (\phi\circ G)(x,0);$$

in particular, $\phi\circ G$ is an entire solution of (5.29). As in Section 5.3 we set $x(c,t) := \frac{A(c)}{2}t^2 + B(c)t + c$ and

$$\begin{aligned} F \;:\; & \mathbb{R}^2_{c,t} \to \mathbb{R}^2_{x,t} \\ & (c,t) \longmapsto (x(c,t), t) \end{aligned}$$

Therefore, if we define

$$\Omega := F(\mathbb{R}^2) \subset \mathbb{R}^2_{x,t},$$
$$F^* := G\circ F,$$
$$\Omega^* := F^*(\mathbb{R}^2) = G(\Omega) \subset \mathbb{R}^2_{\eta,\tau}$$

and $c : \Omega \to \mathbb{R}$ through the formula $F^{-1}(x,t) = (c(x,t), t)$, thanks to Theorem 5.12 one gets

- for any $c \in \mathbb{R}$ we have either $A'(c) = B'(c) = 0$ or $B'(c)^2 < 2A'(c)$;
- F^* is a $\mathbf{C}^2$ diffeomorphism between $\mathbb{R}^2_{c,t}$ and Ω^*. Moreover, Ω and Ω^* are open neighbourhood of the lines $\{t = 0\}$ and $\{\eta = 0\}$ respectively.

It is not difficult to prove that for all $(\eta, \tau) \in \Omega^*$ one has

$$\phi(\eta,\tau) = A(c(-\tau/4,\eta))\eta + B(c(-\tau/4,\eta))) = \frac{\partial x}{\partial t}(F^{*-1}(\eta,\tau)); \tag{5.49}$$

$$W^\phi\phi(\eta,\tau) = A(c(-\tau/4,\eta)). \tag{5.50}$$

and taking into account that

$$\frac{\partial c}{\partial x}(x,t) = \frac{1}{\frac{\partial x}{\partial c}(F^{-1}(c,t))} = \frac{1}{\frac{A'(c(x,t))}{2}t^2 + B'(c(x,t)) + 1} \tag{5.51}$$

$$\frac{\partial c}{\partial t}(x,t) = -\frac{\frac{\partial x}{\partial t}(F^{-1}(c,t))}{\frac{\partial x}{\partial c}(F^{-1}(c,t))} = -\frac{A(c(x,t))t + B(c(x,t))}{\frac{A'(c(x,t))}{2}t^2 + B'(c(x,t)) + 1} \tag{5.52}$$

for all $x, t \in \Omega$, we get for all $(\eta, \tau) \in \Omega^*$ that

$$\begin{aligned}
\widetilde{T}W^\phi\phi + 2(\widetilde{T}\phi)^2 &= -\frac{1}{4}\frac{A'(c(F^{*-1}(\eta,\tau)))}{\frac{\partial x}{\partial c}(F^{*-1}(\eta,\tau))} + \\
&\quad +2\left[\frac{1}{4}\frac{A'(c(F^{*-1}(\eta,\tau))) + B'(c(F^{*-1}(\eta,\tau)))}{\frac{\partial x}{\partial c}(F^{*-1}(\eta,\tau))}\right]^2 \\
&= \frac{-2A'(c)\frac{\partial x}{\partial c} + (\frac{\partial^2 x}{\partial c \partial t})^2}{8\,(\frac{\partial x}{\partial c})^2}(F^{*-1}(\eta,\tau)).
\end{aligned} \tag{5.53}$$

Observe that for any $(c, t) \in \mathbb{R}^2$ we have

$$\begin{aligned}
\frac{-2A'(c)\frac{\partial x}{\partial c}(c,t) + (\frac{\partial^2 x}{\partial c \partial t}(c,t))^2}{8\,(\frac{\partial x}{\partial c}(c,t))^2} &= \frac{-2A'(c)(\frac{A'(c)}{2}t^2 + B'(c)t + 1) + (A'(c)t + B'(c))^2}{8\,(\frac{A'(c)}{2}t^2 + B'(c)t + 1)^2} \\
&= \frac{-2A'(c) + B'(c)^2}{8\,(\frac{A'(c)}{2}t^2 + B'(c)t + 1)^2} \ \leq 0
\end{aligned} \tag{5.54}$$

and notice that the correspondance

$$\mathbf{C}^1_c(\mathbb{R}^2_{c,t}) \ni \zeta \longleftrightarrow \psi := \zeta \circ F^{*-1} \in \mathbf{C}^1_c(\Omega^*)$$

is bijective and

$$\begin{aligned}(W^\phi\psi)(F^*(c,t)) &= \frac{\partial\psi}{\partial\eta}(F^*(c,t)) - 4\phi(F^*(c,t))\frac{\partial\psi}{\partial\tau}(F^*(c,t)) \\ &= \frac{\partial\psi}{\partial\eta}(F^*(c,t)) - 4\frac{\partial x}{\partial t}(c,t)\frac{\partial\psi}{\partial\tau}(F^*(c,t)) \\ &= \frac{\partial\zeta}{\partial t}(c,t). \end{aligned} \tag{5.55}$$

Since

$$\begin{aligned}\det\, JF^*(c,t) &= \det\, JG(F(c,t))\det\, JF(c,t) \\ &= 4(\tfrac{A'(c)}{2}t^2 + B'(c)t + 1) \;> 0\end{aligned}$$

a change of variable and equations (5.26), (5.53), (5.54) and (5.55) give

$$\begin{aligned}g''(0) &= \int_{\Omega^*} \frac{(W^\phi\psi)^2 + 8\psi^2\,[\widetilde{T}W^\phi\phi + 2(\widetilde{T}\phi)^2]}{[1+|W^\phi\phi|^2]^{3/2}}\, d\eta d\tau \\ &= 4\int_{\mathbb{R}^2} \frac{(\frac{\partial\zeta}{\partial t})^2 + \zeta^2\frac{-2A'(c)+B'(c)^2}{(\frac{A'(c)}{2}t^2+B'(c)t+1)^2}}{[1+A(c)^2]^{3/2}} \left[\frac{A'(c)}{2}t^2 + B'(c)t + 1\right]\, dc\, dt \\ &= 4\int_{\mathbb{R}^2} \left[\left(\frac{\partial\zeta}{\partial t}\right)^2 u + \zeta^2 v\right]\, dc\, dt \end{aligned} \tag{5.56}$$

where g is as in (5.13) and we have set $\zeta := \psi\circ F^*$ and

$$u(c,t) := \frac{\frac{A'(c)}{2}t^2 + B'(c)t + 1}{[1+A(c)^2]^{3/2}}$$

$$v(c,t) := \frac{B'(c)^2 - 2A'(c)}{[1+A(c)^2]^{3/2}[\frac{A'(c)}{2}t^2 + B'(c)t + 1]}\,.$$

The fact that ϕ parametrizes a minimizer implies that $g''(0)\geq 0$ for all $\psi\in\mathbf{C}^1_c(\Omega^*)$; since $F^*:\mathbb{R}^2\to\Omega^*$ is a $\mathbf{C}^2$ diffeomorphism we deduce that

$$\int_{\mathbb{R}^2}\left[\left(\frac{\partial\zeta}{\partial t}\right)^2 u + \zeta^2 v\right]\, dc\, dt \geq 0 \qquad \forall\zeta\in\mathbf{C}^1_c(\mathbb{R}^2). \tag{5.57}$$

Step 2. It is easy to see that our thesis on ϕ is equivalent to A and B being constant, i.e. to $A'=B'\equiv 0$. Suppose by contradiction that there exist a $c_0\in\mathbb{R}$ such that this does not hold, then by Theorem 5.12 we have $b^2<2a$, where $b :=$

$B'(c_0)$ and $a := A'(c_0) > 0$. We want to use the second variation formula (5.57) to obtain simpler conditions, namely inequalities on certain one-dimensional integrals involving a and b (see equation (5.62)).
Fix therefore a function $\zeta \in \mathbf{C}^1_c(\mathbb{R}^2)$ and set

$$\zeta_\epsilon(c,t) := \frac{1}{\sqrt{\epsilon}}\zeta\big(c_0 + \tfrac{c-c_0}{\epsilon}, t\big);$$

by (5.57) we get

$$0 \le \int_{\mathbb{R}^2} \left(\frac{\partial \zeta_\epsilon}{\partial t}\right)^2 u \; dc\, dt + \int_{\mathbb{R}^2} \zeta_\epsilon^2 v \; dc\, dt =: I_\epsilon + II_\epsilon \tag{5.58}$$

Observe that

$$\begin{aligned} I_\epsilon &= \frac{1}{\epsilon}\int_{\mathbb{R}^2} \left(\frac{\partial \zeta}{\partial t}\big(c_0 + \tfrac{c-c_0}{\epsilon}, t\big)\right)^2 u(c,t) \; dc\, dt \\ &= \int_{\mathbb{R}^2} \left(\frac{\partial \zeta}{\partial t}(u,t)\right)^2 u(c_0 + \epsilon(u-c_0), t) \; du\, dt \end{aligned}$$

and by Lebesgue convergence theorem one obtains

$$\lim_{\epsilon\to 0} I_\epsilon = \int_{\mathbb{R}^2} \left(\frac{\partial \zeta}{\partial t}(c,t)\right)^2 u(c_0,t) \; dc\, dt. \tag{5.59}$$

Analogously one gets

$$\lim_{\epsilon\to 0} II_\epsilon = \int_{\mathbb{R}^2} \zeta(c,t)^2 v(c_0,t) \; dc\, dt. \tag{5.60}$$

Combining (5.58), (5.59) and (5.60) we obtain

$$\int_{\mathbb{R}^2} \left(\frac{\partial \zeta}{\partial t}(c,t)\right)^2 h(t) \; dc\, dt \ge (2a - b^2) \int_{\mathbb{R}^2} \zeta(c,t)^2 \frac{1}{h(t)} \; dc\, dt \tag{5.61}$$

for all $\zeta \in \mathbf{C}^1_c(\mathbb{R}^2)$, where we have put

$$h(t) := \frac{a}{2}t^2 + bt + 1.$$

By standard arguments (taking for example $\zeta(c,t)$ of the form $\zeta_1(c)\zeta_2(t)$) we can infer the one-dimensional inequalities

$$\int_{\mathbb{R}} \zeta'^2 h \; dt \ge (2a - b^2) \int_{\mathbb{R}} \zeta^2 \frac{1}{h} \; dt \qquad \text{for all } \zeta \in \mathbf{C}^1_c(\mathbb{R}). \tag{5.62}$$

Step 3. We will follow here the technique used in [60] to provide a counterexample to (5.62), which will give a contradiction. For $\epsilon > 0$ fix $\chi_\epsilon \in \mathbf{C}^1_c(\mathbb{R})$ such that

$$\begin{aligned} &0 \le \chi_\epsilon \le 1 \\ &\chi_\epsilon \equiv 1 \text{ on } \left(-\tfrac{1}{\epsilon}, \tfrac{1}{\epsilon}\right), \quad \text{spt } \chi_\epsilon \Subset \left(-\tfrac{2}{\epsilon}, \tfrac{2}{\epsilon}\right) \\ &|\chi'_\epsilon| \le C\epsilon, \quad C > 0 \text{ independent of } \epsilon \end{aligned}$$

and set

$$\zeta_\epsilon(t) := \frac{\chi_\epsilon(t)}{\sqrt{h(t)}}.$$

Equation (5.62) becomes then

$$\int_{\mathbb{R}} \zeta_\epsilon'^2 h \, dt \ge (2a - b^2) \int_{\mathbb{R}} \zeta_\epsilon^2 \frac{1}{h} \, dt \tag{5.63}$$

and observe that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} \zeta_\epsilon^2 \frac{1}{h} \, dt = \int_{\mathbb{R}} \frac{dt}{\left(\frac{a}{2}t^2 + bt + 1\right)^2}. \tag{5.64}$$

As for the left hand side of (5.63), we have

$$\int_{\mathbb{R}} \zeta_\epsilon'^2 h \, dt = \int_{\mathbb{R}} \left(\frac{\chi'_\epsilon}{\sqrt{h}} - \frac{\chi_\epsilon h'}{2h^{3/2}} \right)^2 h \, dt = \int_{\mathbb{R}} \chi_\epsilon'^2 \, dt - \int_{\mathbb{R}} \frac{\chi_\epsilon \chi'_\epsilon h'}{h} \, dt + \frac{1}{4} \int_{\mathbb{R}} \chi_\epsilon^2 \frac{h'^2}{h^2} \, dt; \tag{5.65}$$

an integration by parts gives

$$\int_{\mathbb{R}} \chi_\epsilon \chi'_\epsilon \frac{h'}{h} \, dt = -\frac{1}{2} \int_{\mathbb{R}} \chi_\epsilon^2 \frac{h''}{h} \, dt + \frac{1}{2} \int_{\mathbb{R}} \chi_\epsilon^2 \frac{h'^2}{h^2} \, dt$$

whence (5.65) rewrites as

$$\int_{\mathbb{R}} \zeta_\epsilon'^2 h \, dt = \int_{\mathbb{R}} \chi_\epsilon'^2 \, dt + \frac{1}{2} \int_{\mathbb{R}} \chi_\epsilon^2 \frac{h''}{h} \, dt - \frac{1}{4} \int_{\mathbb{R}} \chi_\epsilon^2 \frac{h'^2}{h^2} \, dt \, .$$

Finally, by Lebesgue convergence theorem we infer

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} \zeta_\epsilon'^2 h \, dt = \frac{1}{2} \int_{\mathbb{R}} \frac{h''}{h} \, dt - \frac{1}{4} \int_{\mathbb{R}} \frac{h'^2}{h^2} \, dt = \frac{1}{4} \int_{\mathbb{R}} \frac{h''}{h} \, dt \tag{5.66}$$

where, in the last equality, we integrated by parts again.

From (5.63), (5.64) and (5.66) we obtain therefore

$$\frac{1}{4} \int_{\mathbb{R}} \frac{a \, dt}{\frac{a}{2}t^2 + bt + 1} \ge (2a - b^2) \int_{\mathbb{R}} \frac{dt}{\left(\frac{a}{2}t^2 + bt + 1\right)^2}. \tag{5.67}$$

Since for $\alpha > 0$ we have

$$\int_{\mathbb{R}} \frac{dt}{1+\alpha t^2} = \frac{\pi}{\sqrt{\alpha}} \quad \text{and} \quad \int_{\mathbb{R}} \frac{dt}{(1+\alpha t^2)^2} = \frac{\pi}{2\sqrt{\alpha}}$$

and observing that

$$\int_{\mathbb{R}} \frac{dt}{\left(\frac{a}{2}t^2+bt+1\right)^m} = \left(\frac{2a}{2a-b^2}\right)^m \int_{\mathbb{R}} \frac{dt}{(1+\alpha t^2)^m} \qquad m = 1, 2$$

with $\alpha := \frac{a^2}{2a-b^2}$, by (5.67) we obtain

$$\frac{a}{4}\frac{2a}{2a-b^2}\pi\frac{\sqrt{2a-b^2}}{a} \geq (2a-b^2)\frac{4a^2}{(2a-b^2)^2}\frac{\pi}{2}\frac{\sqrt{2a-b^2}}{a}$$

which reduces to $1/2 \geq 2$ (recall that $a > 0$), which gives a contradiction.

Step 4. We have proved that A and B are constant functions, and this in turn implies that $\Omega^* = \mathbb{R}^2$ and $\phi(\eta, \tau) = A\eta + B$. This completes the proof of the Theorem. □

5.4.2 The Bernstein problem in $\mathbb{H}^n$ for $n \geq 2$

Let us exploit equation the minimal surface equation (5.47) and write it as

$$\sum_{j=2}^{n} \widetilde{X}_j \left(\frac{\widetilde{X}_j\phi}{\sqrt{1+|W^\phi\phi|^2}}\right) + W^\phi_{n+1}\left(\frac{W^\phi_{n+1}\phi}{\sqrt{1+|W^\phi\phi|^2}}\right) + \sum_{j=2}^{n} \widetilde{Y}_j \left(\frac{\widetilde{Y}_j\phi}{\sqrt{1+|W^\phi\phi|^2}}\right) = 0 \tag{5.68}$$

where $\phi : \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau \to \mathbb{R}$ is of class $\mathbf{C}^2$. Notice that, if one looks for solutions ϕ which do not depend on the τ variable, i.e. such that $\phi(\eta, v, \tau) = \psi(\eta, v)$ for some $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$, equation (5.68) rewrites as the classic minimal surface equation (5.1). This observation allows us to easily construct a counterexample to the Bernstein problem for X_1-graphs in $\mathbb{H}^n$ when $n \geq 5$; in fact in this case we have $2n-1 \geq 9$ and Theorem 5.1 provides a function $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$ which solves (5.1) and is not affine, i.e. the related $\phi(\eta, v, \tau) = \psi(\eta, v)$ solves (5.68) and cannot be written as in (5.48).

We also notice that X_1-graphs of such τ-independent functions $\phi(\eta, v, \tau) = \psi(\eta, v)$ (where again ψ solves (5.1)) are actually minimizers of the $\mathbb{H}$-perimeter; in fact it is easy to check that the smooth section $\nu : \mathbb{H}^n \to H\mathbb{H}^n$ defined by

$$\begin{aligned} \nu(x, y, t) &= \left(-\frac{1}{\sqrt{1+|W^\phi\phi|^2}}, \frac{W^\phi\phi}{\sqrt{1+|W^\phi\phi|^2}}\right)(\eta, v, 0) \\ &= \left(-\frac{1}{\sqrt{1+|\nabla\psi|^2}}, \frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right)(\eta, v), \end{aligned}$$

where we put $\eta := y_1$ and $v := (x_2, \dots, x_n, y_2, \dots, y_n)$, is a calibration for the graph of ϕ according to Theorem 5.3, i.e.

- $\operatorname{div}_X \nu = 0$;
- $|\nu(P)| = 1$ for all $P \in \mathbb{H}^n$;
- ν coincides with the horizontal inward normal to the X_1-graph of ϕ (see Theorem 4.17).

Observe that in this argument it was essential the non-dependance of ϕ on the vertical variable τ: as we have seen in Section 5.4.1, in general it is not true that an entire solution of (5.47) parametrizes a minimizer.

The Bernstein problem for intrinsic graphs in $\mathbb{H}^n$, as far as we know, is still open for $n = 2, 3, 4$; observe that any possible negative answer must effectively depend on the variable τ, or the previous argument leading to the classic Bernstein equation could apply, contradicting Theorem 5.1.

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