John and uniform domains in generalized Siegel boundaries *

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Abstract

Given the pair of vector fields $X = \partial_x + |z|^{2m} y \partial_t$ and $Y = \partial_y - |z|^{2m} x \partial_t$, where $(x, y, t) = (z, t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, we give a condition on a bounded domain $\Omega \subset \mathbb{R}^3$ which ensures that Ω is an (ε, δ)-domain for the Carnot-Carathéodory metric. We also analyze the Ahlfors regularity of the natural surface measure induced on $\partial\Omega$ by the vector fields.

1. Introduction

In $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ we consider the vector fields

$$X = \partial_x + |z|^{2m} y \partial_t \qquad \text{and} \qquad Y = \partial_y - |z|^{2m} x \partial_t, \tag{1.1}$$

where $(x, y, t) = (z, t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and $m \in [1, +\infty[$ is a real parameter. The vector fields X and Y naturally arise as the real and imaginary part of the holomorphic vector field tangent to the boundary of the generalized Siegel domain $\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 > \frac{1}{2m+2}|z_1|^{2m+2}\}$.

We study the interaction of the Carnot-Carathéodory (CC) distance d induced by X and Y with the geometry of a surface embedded in $\mathbb{R}^3$. Namely, we give conditions on the boundary $\partial\Omega$ such that an open set $\Omega \subset \mathbb{R}^3$ is a John domain, a uniform domain and such that the natural surface measure induced on $\partial\Omega$ by X and Y is Ahlfors regular, see Definition 1.3.

John domains are also known as domains with the *twisted cone property*, see Definition 5.1. When the distance is induced by Hörmander vector fields in $\mathbb{R}^n$, several authors proved that a bounded John domain supports a global Sobolev-Poincaré inequality, see [Jer86, SC92, FLW96, GN96] and the discussion for a general metric space in [HK00]. The exterior twisted cone property is also relevant in classical potential theory because it implies the subelliptic Wiener criterion (see [NS87]).

Uniform domains are also known as (ε, δ)-domains, see Definition 6.1. They form a subset of John domains. In the global theory of Sobolev spaces for Hörmander vector fields, Garofalo and Nhieu proved in [GN98] that subelliptic Sobolev functions in a uniform domain can be extended to the whole space. In [DGN06] it is also shown that

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the trace of a Sobolev function in a uniform domain with Ahlfors regular boundary belongs to a suitable Besov space of the boundary. Also for this reason, we shall study the Ahlfors property very carefully. The trace problem was analyzed in [MM02] in the non-characteristic case and in a two-dimensional model. In [GMM], we study by a direct approach the trace problem at the boundary of the characteristic half plane $t > 0$ for vector fields of Martinet type $X = \partial_x$, $Y = \partial_y + |x|^\alpha \partial_t$ in $\mathbb{R}^3$.

In spite of the previous results, there are not many examples of John and uniform domains in Carnot-Carathéodory spaces. In fact, the subRiemannian case is more delicate than the Euclidean one because of the presence of *characteristic points*, i.e., points where the Hörmander vector fields are all tangent to the boundary. Such points make the construction of the inner cone more difficult. Sometimes the inner cone does not exist at all, even for analytic boundaries, see e.g. [MM05b, Theorem 1.2].

The only known elementary fact is that small CC-balls are John domains. To the authors' knowledge, it is not even clear whether the more restricted class of uniform domains is always nonempty, for any given family of Hörmander vector fields. In [MM05b] it is proved that C^2 domains in Carnot groups of step two are uniform. The case of cylindrically symmetric domains was already considered in [CG98] in the Heisenberg group, that is the model (1.1) with $m = 0$. In [MM04] and [MM05a], the authors studied the case of diagonal vector fields.

In this paper, we study uniform domains in $\mathbb{R}^3$ for the CC distance of the vector fields (1.1). Our sufficient condition for a domain to be uniform requires the boundary to be "flat" near characteristic points on the t-axis.

Let $\Omega \subset \mathbb{R}^3$ be an open set with C^∞ boundary. If both X and Y are tangent to $\partial\Omega$ at the point $p \in \partial\Omega$, then there is a neighborhood U_p of p such that $U_p \cap \partial\Omega$ is a graph of the form $t = \varphi(z)$. So we start from the following definition.

Let $A \subset \mathbb{R}^2$ be an open set and $\varphi \in C^\infty(A)$. We say that $\Sigma = \mathrm{gr}(\varphi) = \{(z, \varphi(z)) \in \mathbb{R}^3 : z \in A\}$ is an *m-admissible graph* if there exists a constant $C > 0$ such that for all $z \in A$

$$|D^3\varphi(z)| \le C|z|^{2m-1}, \quad |D^2\varphi(z)| \le C|z|^{2m} \quad \text{and} \quad |D\varphi(z)| \le C|z|^{2m+1}. \tag{1.2}$$

When $0 \notin A$, the three conditions (1.2) are trivially satisfied in a compact subset of A. The conditions are instead restrictive when $0 \in A$.

Definition 1.1. Let $m \in [1, +\infty[$. We say that a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary is m-admissible if for any characteristic point $p \in \partial\Omega$ there exists a neighborhood U_p of p in $\mathbb{R}^3$ such that $\partial\Omega \cap U_p$ is an m-admissible graph.

Our main result is the following:

Theorem 1.2. *Let $m \in \mathbb{N}$. Any m-admissible domain $\Omega \subset \mathbb{R}^3$ is uniform and, in particular, is a John domain in $(\mathbb{R}^3, d)$.*

In fact, our proof shows that admissible domains are also non-tangentially accessible (NTA). Concerning our requirements on the rate of growth (1.2) for the function φ, it is easy to check that any open set which agrees in a neighborhood of the origin with the epigraph $\{t > |z|^\alpha\}$ with $\alpha < 2m + 2$ is not a John domain.

On the other hand, let us consider the epigraph $\{t > -x^{2m+1}y\}$ of Example 3.1. All the points $(x, 0, 0)$ of the x-axis are characteristic points of the boundary. However, the

"order of degeneration" of such points is 2 when $x \neq 0$, while it is $2m+2$ when $x = 0$. The difficulty of our work in Section 5 is due to the fact that we need to construct a family of inner cones of *constant aperture* contained in $\{t > -x^{2m+1}y\}$ and with vertex at points arbitrarily close to the characteristic set. Furthermore, in order to prove the (ε, δ)-property, in Section 6 we also need to show that cones with close vertices have quantitatively close axes.

Theorem 1.2 is proved in Sections 5, 6 and 7. We first show that global (i.e., with $A = \mathbb{R}^2$) admissible graphs have the global cone property and then that they satisfy the (ε, δ)-condition. Finally, we deal with the case of bounded domains. The proofs rely on a precise description of the distance d, which will be discussed in Section 2, and on some preliminary results proved in Section 3.

The natural surface area on $\partial\Omega$ is the perimeter measure of Ω induced by the vector fields (1.1). This is the measure

$$\mu = \sqrt{\langle N, X\rangle^2 + \langle N, Y\rangle^2}\ \mathcal{H}^2 \llcorner \partial\Omega, \tag{1.3}$$

where N is the unit Euclidean normal to $\partial\Omega$, $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{R}^3$, and $\mathcal{H}^2 \llcorner \partial\Omega$ is the standard surface measure, i.e., the restriction of the 2-dimensional Hausdorff measure to $\partial\Omega$. This is a special case of the variational definition of perimeter measure in CC-spaces, see [GN98] and [MSC01]. For admissible domains, the measure μ is codimension 1 Ahlfors regular in the following sense.

Theorem 1.3. *Let $m \in \mathbb{N}$ and denote by $B(p, r)$ the CC-balls. For any m-admissible domain $\Omega \subset \mathbb{R}^3$ there exist constants $C > 0$ and $r_0 > 0$ such that for all $p \in \partial\Omega$ and $0 < r \leq r_0$*

$$C^{-1}\frac{|B(p,r)|}{r} \leq \mu(B(p,r)) \leq C\frac{|B(p,r)|}{r}. \tag{1.4}$$

Above, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^3$. This theorem is proved in Section 7 and relies on the delicate analysis of global admissible graphs tackled in Section 4. Our analysis will require the study of several situations, depending on how CC-balls intersect the graph near the characteristic set.

The ball-box theorem for the distance d is proved in the first part of the paper. For any $(z, t), (\zeta, \tau) \in \mathbb{R}^3$, we define the function

$$\delta((z,t),(\zeta,\tau)) = |z-\zeta| + \min\left\{|v|^{\frac{1}{2m+2}}, \frac{|v|^{1/2}}{|z|^m}\right\}, \tag{1.5}$$

where $v = \tau - t + |z|^{2m}\omega(z,\zeta)$, and $\omega(z,\zeta) = x\eta - y\xi$ with $z = (x, y)$ and $\zeta = (\xi, \eta)$.

Theorem 1.4. *Let $m \in [1, +\infty[$. There is a constant $C > 1$ such that for all $p = (z,t), q = (\zeta,\tau) \in \mathbb{R}^3$*

$$C^{-1}\delta(p,q) \leq d(p,q) \leq C\delta(p,q). \tag{1.6}$$

This theorem is proved in Section 2. Our proof is completely self-contained and works for any $m \geq 1$, also noninteger. Note that when $m \in [1, +\infty[\setminus\mathbb{N}$, the well known ball-box theorem in [NSW85] cannot be applied, because the vector fields (1.1) are not smooth at $z = 0$.

2. Ball-box estimate

In this section, we prove Theorem 1.4 and, in Corollary 2.2 below, we rephrase it as a ball-box estimate.

An absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^3$ is *horizontal* for the vector fields (1.1), if it satisfies $\dot\gamma = \alpha(s)X(\gamma) + \beta(s)Y(\gamma)$ for a.e. $s \in [0,1]$. The length of γ is defined as

$$\text{length}(\gamma) = \int_0^1 |(\alpha(s), \beta(s))| ds.$$

Given points $(z,t), (\zeta,\tau) \in \mathbb{R}^3$, the CC distance $d((z,t),(\zeta,\tau))$ is defined as the infimum (the minimum, in fact) of the length of all absolutely continuous curves $\gamma : [0,1] \to \mathbb{R}^3$ connecting them.

We will use the following invariance properties of d. For all $(z,t), (\zeta,\tau) \in \mathbb{C}\times\mathbb{R}$, $s, \theta \in \mathbb{R}$, and $r > 0$ we have:

$$d((z,t),(\zeta,\tau)) = d(e^{i\theta}z, t), (e^{i\theta}\zeta,\tau)); \tag{2.1}$$

$$d((z,t),(\zeta,\tau)) = d((z,t+s),(\zeta,\tau+s)); \tag{2.2}$$

$$d\big((rz, r^{2m+2}t), (r\zeta, r^{2m+2}\tau)\big) = r d((z,t),(\zeta,\tau)). \tag{2.3}$$

We will also use the following elementary estimate, holding for any $x, y \in \mathbb{R}$ and $m \geq 1$:

$$C_m^{-1}(|x|^{m-1} + |y|^{m-1})\,\big||x| - |y|\big| \leq \big||x|^m - |y|^m\big| \leq C_m(|x|^{m-1} + |y|^{m-1})|y-x|. \tag{2.4}$$

Proof of Theorem 1.4 . Step 1. We claim that there exists a constant $C > 0$, depending on m, such that $\delta((z_0,t_0),(\zeta,\tau)) \leq C d((z_0,t_0),(\zeta,\tau))$ for all points $(z_0,t_0),(\zeta,\tau) \in \mathbb{R}^3$.

By (2.1)-(2.2), we can assume that $z_0 = (x_0, 0)$ with $x_0 \geq 0$ and $t_0 = 0$. In this case, we have $\omega(z_0,\zeta) = x_0\eta$, with $\zeta = (\xi,\eta)$, and the definition in (1.5) for δ reads, with $v = \tau + x_0^{2m+1}\eta$,

$$\delta((z_0,0),(\zeta,\tau)) = |z_0 - \zeta| + \min\Big\{\frac{|v|^{1/2}}{x_0^m}, |v|^{\frac{1}{2m+2}}\Big\}.$$

Let $\gamma = (z,t) : [0,T] \to \mathbb{R}^3$, $T > 0$, be a unit-speed horizontal curve connecting $(z_0,0)$ and (ζ,τ). We let $z = z(s) = (x(s), y(s)) = (x,y)$. From the unit-speed condition $|\dot z| \leq 1$, we deduce that

$$|z_0 - \zeta| = \Big|\int_0^T \dot z\, ds\Big| \leq T. \tag{2.5}$$

We estimate the quantity

$$v = \tau + x_0^{2m+1}\eta = \int_0^T \Big\{|z|^{2m} y\dot x + (x_0^{2m+1} - |z|^{2m}x)\dot y\Big\} ds.$$

We claim that there exists a constant $C > 0$ such that for all $s \in [0,T]$ we have

$$|z|^{2m}|y| + |x_0^{2m+1} - |z|^{2m}x| \leq C(x_0^{2m}s + s^{2m+1}). \tag{2.6}$$

The left-hand side is evaluated at $s \in [0,T]$. From $|z| \le x_0 + s$ and $|y| \le s$ we deduce that $|z|^{2m}|y| \le C(x_0^{2m}s + s^{2m+1})$. By the triangle inequality and (2.4), we have

$$\begin{aligned}\big|x_0^{2m+1} - |z|^{2m}x\big| &\le \big||z|^{2m} - x_0^{2m}\big||x| + x_0^{2m}|x - x_0| \\ &\le C_m(|z|^{2m-1} + x_0^{2m-1})\big||z| - x_0\big||x| + x_0^{2m}|x - x_0|.\end{aligned}$$

Using $|x| \le |z| \le x_0 + s$, $\big||z| - x_0\big| \le s$ and $|x - x_0| \le s$ we obtain $\big|x_0^{2m+1} - |z|^{2m}x\big| \le C(x_0^{2m}s + s^{2m+1})$. This finishes the proof of (2.6).

Now, (2.6) implies that $|\tau + x_0^{2m+1}\eta| \le C(x_0^{2m}T^2 + T^{2m+2})$, which is equivalent to

$$T \ge C^{-1}\min\Big\{\frac{|\tau + x_0^{2m+1}\eta|^{1/2}}{x_0^m}, |\tau + x_0^{2m+1}\eta|^{\frac{1}{2m+2}}\Big\}. \tag{2.7}$$

The inequalities (2.7) and (2.5) imply $\delta((z_0,0),(\zeta,\tau)) \le CT$ and minimizing on T we get the claim made in the Step 1.

Step 2. We claim that there exists a constant $C > 0$ such that $d((z,t),(z,\tau)) \le C\delta((z,t),(z,\tau))$ for all $z \in \mathbb{C}$ and $t, \tau \in \mathbb{R}$, i.e.,

$$d((z,t),(z,\tau)) \le C\min\Big\{|\tau - t|^{\frac{1}{2m+2}}, \frac{|\tau - t|^{1/2}}{|z|^m}\Big\}. \tag{2.8}$$

By (2.1)–(2.2), we can without loss of generality assume that $z = (x,0)$ with $x \ge 0$, $t = 0$ and $\tau \ge 0$.

For each $u \ge 0$ consider the unit-speed path $[0,4u] \ni s \mapsto \zeta(s) \in \mathbb{R}^2$ that linearly connects the points in the plane $(x,0)$, (x,u), $(x+u,u)$, $(x+u,0)$ and $(x,0)$. Let R_u be the square enclosed by ζ. The path ζ has length $4u$ and its unique absolutely continuous horizontal lift $s \mapsto \gamma(s) = (\zeta(s),\tau(s))$ satisfying $\tau(0) = 0$ has final point

$$\tau(4u) = \int_\zeta |\zeta|^{2m}(\eta d\xi - \xi d\eta) = 2(m+1)\int_{R_u}|\zeta|^{2m}d\xi d\eta \ge C_0^{-1}u^2(x^{2m} + u^{2m}). \tag{2.9}$$

We used Stokes' theorem with the counterclockwise orientation of ζ. The function $u \mapsto \int_{R_u}|\zeta|^{2m}d\xi d\eta$ is a strictly increasing bijection of $[0,+\infty[$ onto itself.

Let $\overline{u}$ be the unique number such that $\tau(4\overline{u}) = \tau$. By the definition of the distance d and by (2.9), we have

$$\begin{aligned}d((x,0,0),(x,0,\tau)) &\le 4\overline{u} = \min\big\{4u > 0 : \tau(4u) \ge \tau\big\} \\ &\le \min\big\{4u > 0 : \tau \le C_0^{-1}u^2(x^{2m} + u^{2m})\big\} \\ &\le C\min\Big\{\frac{\tau^{1/2}}{x^m}, \tau^{\frac{1}{2m+2}}\Big\}.\end{aligned}$$

This concludes the proof of the Step 2.

Step 3. We claim that the inequality $d((z,t),(\zeta,\tau)) \le C\delta((z,t),(\zeta,\tau))$ holds for all points $(z,t),(\zeta,\tau) \in \mathbb{C}\times\mathbb{R}$.

We preliminarily observe that, given $(u,v) = w \in \mathbb{C}$, for any point $(z,t) = (x,y,t)$ we have

$$e^{uX+vY}(z,t) = \Big(z + w, t + \omega(w,z)\int_0^1 |z + sw|^{2m}ds\Big),$$

where $\omega(w,z) = uy - vx$ and $e^{uX+vY}(z,t)$ denotes the value at time 1 of the integral curve of $uX + vY$ starting from (z,t) at time 0.

By the triangle inequality, it follows that

$$d((z,t),(\zeta,\tau)) \le d\big((z,t), e^{(\xi-x)X+(\eta-y)Y}(z,t)\big) + d\big(e^{(\xi-x)X+(\eta-y)Y}(z,t),(\zeta,\tau)\big).$$

In the last distance, the points are one above each other and so, by (2.8), we get

$$d((z,t),(\zeta,\tau)) \le C\Big(|z-\zeta| + \min\Big\{|t-\tau+\lambda|^{\frac{1}{2m+2}}, |t-\tau+\lambda|^{\frac12}/|\zeta|^m\Big\}\Big), \tag{2.10}$$

where

$$\lambda = \omega(\zeta,z)\int_0^1 |z+s(\zeta-z)|^{2m} ds.$$

We used $\omega(\zeta - z, z) = \omega(\zeta, z)$.

In order to prove the claim in the Step 3, we have to show that the right-hand side in (2.10) is less than $C\delta((z,t),(\zeta,\tau))$. By (2.1)–(2.2), it is enough to prove this estimate in the case $z = (x,0)$ with $x \ge 0$ and $t = 0$. In this case, the distance δ is

$$\delta((z,0),(\zeta,\tau)) = |z-\zeta| + \min\left\{|v|^{\frac{1}{2m+2}}, \frac{|v|^{1/2}}{x^m}\right\}, \qquad v = \tau + x^{2m+1}\eta.$$

We distinguish two cases:

Case G1: $|v|^{\frac{1}{2m+2}} \le |v|^{\frac12}/x^m$, i.e., $x \le |v|^{\frac{1}{2m+2}}$;

Case G2: $|v|^{\frac{1}{2m+2}} \ge |v|^{\frac12}/x^m$, i.e., $x \ge |v|^{\frac{1}{2m+2}}$.

Case G1. When $z = (x,0)$ and $t = 0$ we have $\omega(\zeta,z) = -x\eta$ and, see (2.10),

$$|t-\tau+\lambda| = \Big|\tau + x\eta\int_0^1 |z+s(\zeta-z)|^{2m} ds\Big|.$$

We claim that in the Case G1 we have

$$\Big|\tau + x\eta\int_0^1 |z+s(\zeta-z)|^{2m} ds\Big|^{\frac{1}{2m+2}} \le C\Big(|\zeta-z| + |v|^{\frac{1}{2m+2}}\Big). \tag{2.11}$$

This and (2.10) finish the proof of the the Step 3 in the Case G1, because the right-hand side in (2.11) is $C\delta((z,0),(\zeta,\tau))$.

We prove (2.11). By the triangle inequality, we have

$$\begin{aligned}
\Big|\tau+x\eta\int_0^1 |z+s(\zeta-z)|^{2m} ds\Big| &\le \Big||v| + x|\eta|\Big| - x^{2m} + \int_0^1 |z+s(\zeta-z)|^{2m} ds\Big| \\
&= \Big||v| + x\,|\eta|\Big|\int_0^1\int_0^s \frac{d}{d\varrho}|z+\varrho(\zeta-z)|^{2m} d\varrho\, ds\Big| \\
&= \Big||v| + 2mx\,|\eta|\Big|\int_0^1\int_0^s |z+\varrho(\zeta-z)|^{2m-2}\big\{\langle z,\zeta-z\rangle + \varrho|\zeta-z|^2\big\} d\varrho\, ds\Big| \\
&\le C(|v| + \Theta),
\end{aligned} \tag{2.12}$$

where we let

$$\Theta = x|\eta|(x+|\zeta-z|)^{2m-2}(x|\xi-x|+|\zeta-z|^2). \tag{2.13}$$

By the Hölder inequality and by the Case G1 we have

$$\Theta \le Cx|\zeta-z|(x+|\zeta-z|)^{2m} \le C(x+|\zeta-z|)^{2m+2} \le C(|v|+|\zeta-z|^{2m+2}).$$

This and (2.12) finish the proof of (2.11).

Case G2. In this case we have $x \ge |v|^{\frac{1}{2m+2}}$, and thus

$$\delta((z,0),(\zeta,\tau)) = |z-\zeta| + \frac{|v|^{1/2}}{x^m}, \qquad v = \tau + x^{2m+1}\eta.$$

We distinguish the following three subcases:

$$|\xi - x| \le \frac{1}{2}x; \tag{G2a}$$

$$\max\{|\xi|,|\eta|\} \le \frac{1}{2}x; \tag{G2b}$$

$$\max\{|\xi|,|\eta|\} \ge \frac{1}{2}x \quad \text{and} \quad |\xi-x| \ge \frac{1}{2}x. \tag{G2c}$$

In the Case G2a, the quantity Θ in (2.13) can be estimated as follows

$$\Theta \le Cx|\eta|(x+|\eta|)^{2m-2}(x|x-\xi|+\eta^2)$$

and from $x \le x+|\eta| \le C|\zeta|$ we deduce that

$$\frac{1}{|\zeta|^{2m}}(|v|+\Theta) \le C\Big(\frac{|v|}{x^{2m}} + \frac{x|\eta|}{(x+|\eta|)^2}(x|x-\xi|+\eta^2)\Big) \le C\Big(\frac{|v|}{x^{2m}} + |\zeta-z|^2\Big).$$

This along with (2.12) and (2.10) finish the proof of the Step 3 in the Case G2a.

In the Case G2b, the quantities x, $x+|\xi|$, $x+|\eta|$, $|\xi-x|$ are mutually comparable with absolute constants and therefore, also using the Hölder inequality, the quantity Θ in (2.13) can be estimated as follows

$$\Theta \le Cx^{2m+1}|\eta| \le C|\zeta-z|^{2m+2}.$$

On the other hand, we have

$$|v|^{\frac{1}{2m+2}} \le C\Big(x + \frac{|v|^{1/2}}{x^m}\Big) \le C\Big(|\zeta-z| + \frac{|v|^{1/2}}{x^m}\Big).$$

These two inequalities imply, via (2.12), that the claim (2.11) holds also in the Case G2b.

In the Case G2c, we have $x \le C|\zeta|$ and from (2.13) we estimate

$$\Theta \le C|\zeta||\eta|(|\zeta|+|\zeta-z|)^{2m-2}(|\zeta||\zeta-z|+|\zeta-z|^2) \le C|\zeta|^{2m}|\zeta-z|^2,$$

and we conclude that

$$\frac{1}{|\zeta|^m}(\Theta+|v|)^{1/2} \le C\Big(|\zeta-z| + \frac{|v|^{1/2}}{x^m}\Big).$$

The proof is concluded also in this case. □

Remark 2.1. The argument of the proof of Theorem 1.4 shows in fact the global equivalence

$$|(u,v)| \le d((z,t), e^{uX+vY}(z,t)) \le C_0|(u,v)|, \qquad u,v,t \in \mathbb{R}, \quad z \in \mathbb{R}^2. \tag{2.14}$$

Next we describe the d-balls as suitable boxes. For any $\beta > 0$ we define the weighted norm of $u = (u_1, u_2, u_3) \in \mathbb{R}^3$

$$\|u\|_{1,1,\beta} = \max\{|u_1|, |u_2|, |u_3|^{1/\beta}\},$$

and for any $p = (z,t) = (x,y,t) \in \mathbb{R}^3$ and $r > 0$ we define the boxes

$$\begin{aligned}
\mathrm{Box}_I(p,r) &= \Big\{ \big(x + u_1, y + u_2, t + |z|^{2m}(u_3 + yu_1 - xu_2)\big) : \|u\|_{1,1,2} < r \Big\}, \\
\mathrm{Box}_J(p,r) &= \Big\{ \big(x + u_1, y + u_2, t + u_3 + |z|^{2m}(yu_1 - xu_2)\big) : \|u\|_{1,1,2m+2} < r \Big\}.
\end{aligned}$$

Corollary 2.2. *Let $m \in [1, +\infty[$. For any $\alpha > 0$ there exist constants $b_1, b_2, \delta_0 > 0$ such that for all $p = (z,t) \in \mathbb{R}^3$ and $r > 0$ we have:*
(i) if $|z| \ge \alpha r$, then

$$\mathrm{Box}_I(p, \delta_0 r) \subset B(p,r) \subset \mathrm{Box}_I(p, b_1 r); \tag{2.15}$$

(ii) if $r \ge \alpha |z|$, then

$$\mathrm{Box}_J(p, \delta_0 r) \subset B(p,r) \subset \mathrm{Box}_J(p, b_2 r). \tag{2.16}$$

Proof. Step 1. We claim that for a suitable $\delta_0 > 0$ we have

$$\mathrm{Box}_I(p, \delta_0 r) \cup \mathrm{Box}_J(p, \delta_0 r) \subset B(p,r) \subset \mathrm{Box}_I(p, r/\delta_0) \cup \mathrm{Box}_J(p, r/\delta_0).$$

To prove these inclusions, we observe that, letting $v = \tau - t + |z|^{2m}(x\eta - y\xi)$,

$$(\zeta,\tau) \in \mathrm{Box}_I(p,r) \quad \Leftrightarrow \quad \max\left\{ |\xi - x|, |\eta - y|, \frac{|v|^{1/2}}{|z|^m} \right\} < r, \tag{2.17}$$

$$(\zeta,\tau) \in \mathrm{Box}_J(p,r) \quad \Leftrightarrow \quad \max\left\{ |\xi - x|, |\eta - y|, |v|^{\frac{1}{2m+2}} \right\} < r. \tag{2.18}$$

Thus the point (ζ,τ) belongs to the union of the boxes if and only if $|\xi - x| < r$, $|\eta - y| < r$ and

$$\min\left\{ |v|^{\frac{1}{2m+2}}, \frac{|v|^{\frac{1}{2}}}{|z|^m} \right\} < r.$$

Now the claim follows from Theorem 1.4. We also proved both the inclusions in the left-hand side of (2.15) and (2.16).

Step 2. We prove the inclusion in the right-hand side of (2.15). Let $|z| \ge \alpha r$ and let $(\zeta,\tau) \in B(p,r)$. By the Step 1 we know that $(\zeta,\tau) \in (B_I \cup B_J)(p, r/\delta_0)$. Then, we are left to show that

$$|z| \ge \alpha r \quad \text{and} \quad \min\left\{ |v|^{\frac{1}{2m+2}}, \frac{|v|^{1/2}}{|z|^m} \right\} < r/\delta_0 \quad \Rightarrow \quad \frac{|v|^{1/2}}{|z|^m} < b_1 r.$$

If the minimum is $|v|^{1/2}/|z|^m$, there is nothing to prove. Otherwise we have $|v|^{1/(2m+2)} < r/\delta_0$, i.e., $|v|^{1/2} < (r/\delta_0)^{m+1}$. This and $|z| \geq \alpha r$ imply

$$\frac{|v|^{1/2}}{|z|^m} \leq \frac{|v|^{1/2}}{(\alpha r)^m} \leq \frac{(r/\delta_0)^{m+1}}{(\alpha r)^m} = \frac{1}{\alpha^m \delta_0^{m+1}} r.$$

Step 3. We prove the inclusion (2.16). Arguing as in the Step 2, it suffices to prove that

$$r \geq \alpha |z| \quad \text{and} \quad \min\Big\{ |v|^{\frac{1}{2m+2}}, \frac{|v|^{1/2}}{|z|^m} \Big\} < \frac{r}{\delta_0} \quad \Rightarrow \quad |v|^{\frac{1}{2m+2}} \leq b_2 r.$$

If the minimum is $|v|^{\frac{1}{2m+2}}$, there is nothing to prove. Otherwise we have

$$\frac{r}{\delta_0} \geq \frac{|v|^{1/2}}{|z|^m} \geq |v|^{1/2} \Big(\frac{\alpha}{r}\Big)^m,$$

that is equivalent to $|v|^{1/2} \leq r^{m+1}/\delta_0 \alpha^m$. This is the claim. □

3. Geometry of admissible graphs

Let $\varphi \in C^\infty(\mathbb{R}^2)$ be a smooth function satisfying the flatness conditions (1.2) at any point $z \in \mathbb{R}^2$. A defining function for the graph of φ is the function $F \in C^\infty(\mathbb{R}^3)$ given by $F(z,t) = \varphi(z) - t$. The derivatives

$$\begin{aligned} XF(z,t) &= XF(z) = \varphi_x(z) - |z|^{2m} y, \\ YF(z,t) &= YF(z) = \varphi_y(z) + |z|^{2m} x \end{aligned}$$

do not depend on t and we let $ZF(z) = (XF(z), YF(z))$. A point $(z, \varphi(z)) \in \Sigma = \mathrm{gr}(\varphi)$ is characteristic if and only if $ZF(z) = 0$. By (1.2), the function ZF satisfies

$$|ZF(z)| \leq C|z|^{2m+1}, \quad z \in \mathbb{R}^2. \tag{3.1}$$

Example 3.1. Let $m \in \mathbb{N}$. The graph of the function $\varphi(z) = -x^{2m+1} y$ is m-admissible and each point of the x-axis is characteristic.

The next proposition describes the restriction of the distance d to an admissible graph.

Lemma 3.2. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ satisfy the conditions* (1.2)*. Then there exist a constant $C_0 > 0$ such that for all $p = (z, \varphi(z)), q = (\zeta, \varphi(\zeta)) \in \Sigma$*

$$C_0^{-1} d(p,q) \leq |\zeta - z| + \Big| \frac{\varphi(\zeta) - \varphi(z)}{\mu_{z,\zeta}^{2m}} + \omega(z,\zeta) \Big|^{\frac{1}{2}} \leq C_0 d(p,q), \tag{3.2}$$

where $\mu_{z,\zeta} = \max\{|z|, |\zeta|\}$ and $\omega(z,\zeta) = x\eta - y\xi$.

Proof. Without loss of generality, we prove the lemma in the case $|z| \geq |\zeta|$. We claim that, letting $v = \varphi(\zeta) - \varphi(z) + |z|^{2m} \omega(z,\zeta)$, we have

$$\frac{|v|^{1/2}}{|z|^m} \leq C(|\zeta - z| + |v|^{\frac{1}{m+2}}) \tag{3.3}$$

Taking (3.3) for granted and starting from Theorem 1.4, by (3.3) it follows that

$$d(p,q) \simeq |\zeta - z| + \min\Big\{ |v|^{\frac{1}{m+2}} + \frac{|v|^{1/2}}{|z|^m} \Big\} \simeq |\zeta - z| + \frac{|v|^{1/2}}{|z|^m},$$

and this is (3.2).

We prove (3.3). From the elementary inequality $a^{1/2}b^{1/2} \le C(a^{\frac{2m+1}{2m+2}} b^{\frac{1}{2m+2}} + b)$ for $a, b \ge 0$, we obtain

$$\begin{aligned}
\frac{|v|^{1/2}}{|z|^m} &= |z|^{1/2} \Big| \frac{\varphi(\zeta) - \varphi(z)}{|z|^{2m+1}} + \frac{\omega(z,\zeta)}{|z|} \Big|^{1/2} \\
&\le C \Big\{ |z|^{\frac{2m+1}{2m+2}} \Big| \frac{\varphi(\zeta) - \varphi(z)}{|z|^{2m+1}} + \frac{\omega(z,\zeta)}{|z|} \Big|^{\frac{1}{2m+2}} + \Big| \frac{\varphi(\zeta) - \varphi(z)}{|z|^{2m+1}} + \frac{\omega(z,\zeta)}{|z|} \Big| \Big\} \\
&\le C \Big\{ |v|^{\frac{1}{2m+2}} + \frac{|\varphi(\zeta) - \varphi(z)|}{|z|^{2m+1}} + \frac{|\omega(z,\zeta)|}{|z|} \Big\} \\
&\le C \Big\{ |v|^{\frac{1}{2m+2}} + |z - \zeta| \Big\},
\end{aligned}$$

because by (1.2) and $|\zeta| \le |z|$ we have

$$|\varphi(\zeta) - \varphi(z)| \le C(|z|^{2m+1} + |\zeta|^{2m+1})|\zeta - z| \le C|z|^{2m+1}|\zeta - z|,$$

and, moreover, $|\omega(z,\zeta)| = |\omega(z, \zeta - z)| \le |z||\zeta - z|$. □

In the next propositions, we discuss other consequences of the conditions (1.2).

Proposition 3.3. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ satisfy* (1.2)*. For all $z, \zeta \in \mathbb{R}^2$ we have*

$$\varphi(\zeta) - \varphi(z) + \mu_{\zeta,z}^{2m} \omega(z,\zeta) = \langle ZF(z), \zeta - z \rangle + \mu_{z,\zeta}^{2m} O(|\zeta - z|^2), \tag{3.4}$$

where $\mu_{z,\zeta} = \max\{|z|, |\zeta|\}$ and the remainder satisfies $|O(|\zeta - z|^2)| \le C|z - \zeta|^2$ for a constant $C > 0$.

Proof. Expanding φ at the second order at a point $z \in \mathbb{R}^2$, we obtain for any $\zeta \in \mathbb{R}^2$

$$\varphi(\zeta) - \varphi(z) + |z|^{2m} \omega(z,\zeta) = \langle ZF(z), \zeta - z \rangle + \mu_{z,\zeta}^{2m} O(|\zeta - z|^2). \tag{3.5}$$

By (1.2), the remainder satisfies the uniform estimate $|O(|\zeta - z|^2)| \le C|\zeta - z|^2$ for all $z, \zeta \in \mathbb{C}$. If $\mu_{z,\zeta} = |z|$, this is our claim (3.4).

If $|z| \le |\zeta|$, starting from (3.5) it suffices to use the estimates $|\omega(z,\zeta)| \le |\zeta||\zeta - z|$ and

$$||\zeta|^{2m} - |z|^{2m}| \le C\mu_{\zeta,z}^{2m-1} |\zeta - z| \le C|\zeta|^{2m-1}|\zeta - z|.$$

and the proof is concluded. □

Next we get a Taylor expansion of $ZF(\zeta)$ with a remainder $O(|\zeta - z|^2)$. This is the only point where we use the assumption $|D^3\varphi(z)| \le C|z|^{2m-1}$.

Let $z, \zeta \in \mathbb{R}^2$ be points with $|\zeta| \le 2|z|$. There exist points $z' = z'(z,\zeta)$ and $z'' = z''(z,\zeta)$ in the line segment $[z,\zeta]$ such that:

$$\begin{aligned}
\varphi_x(\zeta) &= \varphi_x(z) + \langle D\varphi_x(z'), \zeta - z \rangle + |z|^{2m-1} O(|\zeta - z|^2), \\
\varphi_y(\zeta) &= \varphi_y(z) + \langle D\varphi_y(z''), \zeta - z \rangle + |z|^{2m-1} O(|\zeta - z|^2).
\end{aligned} \tag{3.6}$$

On the other hand, we have

$$|\zeta|^{2m}\eta = |z|^{2m}y + |z|^{2m-2}\big\langle (2mxy, |z|^2 + 2my^2), \zeta - z\big\rangle + |z|^{2m-1}O(|\zeta - z|^2),$$
$$|\zeta|^{2m}\xi = |z|^{2m}x + |z|^{2m-2}\big\langle (|z|^2 + 2mx^2, 2mxy), \zeta - z\big\rangle + |z|^{2m-1}O(|\zeta - z|^2).$$

With these estimates, we have proved the following:

Proposition 3.4. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ satisfy* (1.2). *For any $z, \zeta \in \mathbb{R}^2$ with $|\zeta| \le 2|z|$ we have*

$$ZF(\zeta) = ZF(z) + M(z,\zeta)(\zeta - z) + |z|^{2m-1}O(|\zeta - z|^2)$$

where $ZF = \begin{bmatrix} XF \\ YF \end{bmatrix}$, $\zeta - z = \begin{bmatrix} \xi - x \\ \eta - y \end{bmatrix}$ and $M = M(z,\zeta)$ is the 2×2 matrix

$$M = \begin{bmatrix} \varphi_{xx}(z') - 2m|z|^{2m-2}xy & \varphi_{xy}(z') - |z|^{2m-2}(|z|^2 + 2my^2) \\ \varphi_{xy}(z'') + |z|^{2m-2}(|z|^2 + 2mx^2) & \varphi_{yy}(z'') + 2m|z|^{2m-2}xy \end{bmatrix}. \tag{3.7}$$

If $\varphi = 0$, then $\det M = (1+2m)|z|^{4m}$. In this case, the matrix M is nonsingular for all $z \neq 0$. Example 3.1 shows that, for some admissible functions, nonsingularity may fail also at points $z \neq 0$. However, we are able to show that the matrix $M(z,\zeta)$ has always rank at least one and that it satisfies the following quantitative nondegeneration property. This property is needed to get a Ahlfors lower bound in the noncharacteristic case, the Case 1c in next section.

Proposition 3.5. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ satisfy* (1.2). *There exist constants $C_2 > 1$, $\varepsilon_0, \varepsilon_2 \in]0,1[$ such that for all $z \neq 0$ there is a unit vector $u \in \mathbb{S}^1 \subset \mathbb{R}^2$ such that for all $r \in]0, \varepsilon_0|z|[$ we have*

$$|M(z,\zeta)(\zeta - z)| \ge C_2^{-1}|z|^{2m}r \tag{3.8}$$

for all $\zeta \in B_{\mathrm{Euc}}\big(z + \frac{r}{2}u, \varepsilon_2 r\big) \subset B_{\mathrm{Euc}}(z,r) \subset \mathbb{R}^2$.

Proof. Denote by e_1, e_2 the coordinate versors of $\mathbb{R}^2$. Then, letting $M = M(z,\zeta)$, we have

$$\begin{aligned} |Me_1| + |Me_2| &\ge \Big|\varphi_{xy}(z'') + |z|^{2m-2}(|z|^2 + 2mx^2)\Big| + \Big|\varphi_{xy}(z') - |z|^{2m-2}(|z|^2 + 2my^2)\Big| \\ &\ge (2m+2)|z|^{2m} - |\varphi_{xy}(z') - \varphi_{xy}(z'')|. \end{aligned}$$

From (1.2) and $|z' - z''| \le |\zeta - z| \le \varepsilon_0|z|$, we deduce that, if ε_0 is conveniently small, then we have the inequality $|\varphi_{xy}(z') - \varphi_{xy}(z'')| \le |z|^{2m}$. This implies that $|Me_1| + |Me_2| \ge 2m|z|^{2m}$. Thus, given $z \neq 0$, at least one of the choices $u = e_1$ or $u = e_2$ ensures that $|M(z,\zeta)u| \ge |z|^{2m}$ for all ζ such that $|\zeta - z| < \varepsilon_0|z|$. Therefore, for any v with $|v| \le 1$ and $\varepsilon_2 > 0$ we have

$$|M(z,\zeta)(u + \varepsilon_2 v)| \ge |z|^{2m} - \varepsilon_2|M(z,\zeta)| \ge |z|^{2m} - \varepsilon_2 C|z|^{2m}$$

where $|M(z,\zeta)|$ denotes the operatorial norm, which under our assumptions satisfies $|M(z,\zeta)| \le C|z|^{2m}$. Thus, taking ε_2 small enough we get a lower estimate with $\frac{1}{2}|z|^{2m}$.

The claim (3.8) follows by multiplying the last inequality by $r/2$. □

4. Ahlfors property for entire admissible graphs

In this section we prove Theorem 1.3 in the case when the boundary of the domain is an entire admissible graph. The case of a bounded domain is in Section 7. A discussion of the problem in a translation invariant setting of step two is contained in [CG06].

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $\pi(z,t) = z$ and denote by Σ the graph of a function $\varphi \in C^\infty(\mathbb{R}^2)$ satisfying (1.2). By the area formula, the measure μ defined in (1.3) satisfies, for any $p \in \Sigma$ and $r > 0$,

$$\mu(B(p,r) \cap \Sigma) = \int_{\pi(B(p,r)\cap\Sigma)} |ZF(\zeta)| d\xi d\eta. \tag{4.1}$$

The integration domain can be estimated using Lemma 3.2. For any $z \in \mathbb{R}^2$ and $r > 0$ we define the "disks"

$$D(z,r) = \{\zeta \in \mathbb{R}^2 : |\zeta - z| \le r,\ |\varphi(\zeta) - \varphi(z) + \mu_{\zeta,z}^{2m}\omega(z,\zeta)| \le \mu_{\zeta,z}^{2m} r^2\}, \tag{4.2}$$

where $\mu_{\zeta,z} = \max\{|z|, |\zeta|\}$. By Lemma 3.2, there exists a constant $C_0 > 0$ such that

$$D(z, r/C_0) \subset \pi(B(p,r) \cap \Sigma) \subset D(z, C_0 r), \quad \text{for all } z \in \mathbb{C} \text{ and } r \in]0, +\infty[. \tag{4.3}$$

The Lebesgue measure of the ball $B(p,r)$, $p = (z,t) \in \mathbb{R}^3$, can be computed using Corollary 2.2. Let $\varepsilon_0 \in]0,1[$ be the constant given by Proposition 3.5. From (2.15)–(2.17) and (2.16)–(2.18), using Fubini-Tonelli Theorem we obtain

$$\begin{aligned} |B(p,r)| &\simeq |z|^{2m} r^4, \quad \text{if} \quad r \le \varepsilon_0 |z|, \\ |B(p,r)| &\simeq r^{2m+4}, \quad \text{if} \quad r \ge \varepsilon_0 |z|. \end{aligned}$$

The equivalence constants depend on the parameter ε_0.

4.1. Proof in the Case 1: $r \le \varepsilon_0 |z|$.

We claim that for any point $p = (z, \varphi(z)) \in \Sigma$ and $r > 0$ such that $r \le \varepsilon_0 |z|$ we have:

$$\mu(B(p,r) \cap \Sigma) \simeq |z|^{2m} r^3. \tag{4.4}$$

Observe that in Case 1 we have the obvious equivalence $\frac{1}{2}|z| \le |\zeta| \le \frac{3}{2}|z|$. Let $\beta > 0$ be a parameter that will be fixed after (4.10). We distinguish two subcases:

Case 1c: $|z|^{2m} r \ge \beta |ZF(z)|$. This is the characteristic case.

Case 1nc: $|z|^{2m} r \le \beta |ZF(z)|$. This is the non-characteristic case.

In the Case 1c, points are in a quantitative way near the characteristic set of Σ, where $|ZF(z)| = 0$.

Case 1c – upper bound. We start from the elementary inclusion $\pi(B(p,r) \cap \Sigma) \subset \{\zeta \in \mathbb{R}^2 : |\zeta - z| \le r\}$. Thus, using the expansion (3.4) and the trivial estimate $|M(\zeta - z)| \le C|z|^{2m}|\zeta - z|$ we obtain

$$\begin{aligned} \int_{|\zeta - z| \le r} |ZF(\zeta)| d\xi d\eta &= \int_{|\zeta - z| \le r} \Big| ZF(z) + M(\zeta - z) + |z|^{2m-1} O(|\zeta - z|^2) \Big| d\xi d\eta \\ &\le C \int_{|\zeta - z| \le r} \Big(\frac{1}{\beta} |z|^{2m} r + C|z|^{2m}|\zeta - z| + C|z|^{2m-1}|\zeta - z|^2 \Big) d\xi d\eta \\ &\le C\Big(\frac{1}{\beta} + C \Big) |z|^{2m} r^3. \end{aligned}$$

We also used $r \le \varepsilon_0 |z|$ to estimate the third term.

Case 1c – lower bound. We claim that there exist constants $\varepsilon_1 > 0$ and $\beta > 0$ such that

$$\{\zeta \in \mathbb{R}^2 : |\zeta - z| \le \varepsilon_1 r/C_0\} \subset \pi(B(p,r) \cap \Sigma). \tag{4.5}$$

The constant C_0 is the one given by Lemma 3.2.

In view of the expansion (3.4), the set $D(z, r/C_0)$ introduced in (4.2) satisfies

$$D(z,r/C_0) = \big\{\zeta \in \mathbb{R}^2 : |\zeta - z| \le r/C_0,\, |\langle ZF(z), \zeta - z\rangle + \mu_{z,\zeta}^{2m} O(|\zeta - z|^2)| \le C_0^{-2} \mu_{z,\zeta}^{2m} r^2\big\},$$

where, for some absolute constant C_1, we have $|O(|\zeta - z|^2)| \le C_1 |\zeta - z|^2 \le C_1 C_0^{-2} \varepsilon_1^2 r^2$, provided that $|\zeta - z| \le \varepsilon_1 r/C_0$. If ε_1 satisfies

$$C_1 \varepsilon_1^2 \le \frac{1}{2}, \tag{4.6}$$

then we have the inclusion

$$\big\{\zeta \in \mathbb{R}^2 : |\zeta - z| \le \varepsilon_1 r/C_0,\, |\langle ZF(z), \zeta - z\rangle| \le \frac{1}{2} C_0^{-2} \mu_{z,\zeta}^{2m} r^2\big\} \subset D(z, r/C_0). \tag{4.7}$$

By the Case 1c, we have

$$|\langle ZF(z), \zeta - z\rangle| \le \frac{1}{\beta} |z|^{2m} r |\zeta - z| \le \frac{1}{\beta} \varepsilon_1 C_0^{-1} \mu_{z,\zeta}^{2m} r^2.$$

Thus, if ε_1 and β are such that

$$\frac{\varepsilon_1 C_0^{-1}}{\beta} \le \frac{C_0^{-2}}{2}, \tag{4.8}$$

the inclusion (4.5) holds, as we claimed.

Next we use Proposition 3.5 to estimate $|ZF(\zeta)|$ from below at all points $\zeta \in B_{\mathrm{Euc}}\big(z + \varrho u/2, \varepsilon_2 \varrho\big)$, where $\varrho = \varepsilon_1 C_0^{-1} r$ and $u \in \mathbb{S}^1$ is such that (3.8) holds. Namely, we have

$$|M(\zeta - z)| \ge C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r,$$

where $M = M(z, \zeta)$ is the matrix (3.7). For such a point ζ we have

$$\begin{aligned} |ZF(\zeta)| &= |ZF(z) + M(\zeta - z) + |z|^{2m-1} O(|\zeta - z|^2)| \\ &\ge C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r - |ZF(z)| - |z|^{2m-1} O(|\zeta - z|^2) \\ &\ge C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r - \frac{1}{\beta} |z|^{2m} r - C_1 |z|^{2m-1} \varepsilon_1^2 r^2 \\ &\ge C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r - \frac{1}{\beta} |z|^{2m} r - C_1 |z|^{2m} \varepsilon_1^2 r \ge \frac{1}{2} C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r, \end{aligned} \tag{4.9}$$

provided that

$$\frac{1}{\beta} \le \frac{1}{4} C_2^{-1} \varepsilon_1 C_0^{-1} \qquad \text{and} \qquad C_1 \varepsilon_1^2 \le \frac{1}{4} C_2^{-1} \varepsilon_1 C_0^{-1}. \tag{4.10}$$

We may choose $\varepsilon_1 > 0$ such that the inequalities in the right-hand side of (4.10) and in (4.6) both hold. Then we fix $\beta > 0$ such that the inequalities in the left-hand side of (4.10) and in (4.8) both hold.

By (4.5) and (4.9), we finally obtain

$$\int_{\pi(B(p,r)\cap\Sigma)} |ZF(\zeta)|d\xi d\eta \geq C_3^{-1}|z|^{2m}r^3,$$

where, after fixing ε_1 and β, the constant C_3 is absolute.

Case 1nc – upper bound. In order to evaluate from above the integral in (4.1), we start from the estimates $|M(\zeta - z)| \leq C|z|^{2m}|\zeta - z| \leq C|z|^{2m}r \leq C\beta|ZF(z)|$, where C is an absolute constant. Here we used the fact that $|\zeta - z| < r$ for all points ζ in the integration set. Therefore, the weight in the integral (4.1) satisfies

$$|ZF(\zeta)| = \big|ZF(z) + M(\zeta - z) + |z|^{2m-1}O(|\zeta - z|^2)\big| \leq C(1+\beta)|ZF(z)|. \tag{4.11}$$

In order to get the required estimate, the obvious inclusion $\pi(B(p,r)\cap\Sigma) \subset B_{\mathrm{Eucl}}(x,r)$ does not suffice. We need the stronger condition (3.2), which tells that for some absolute constant $C_0 > 0$ we have $\pi(B(p,r)\cap\Sigma) \subset D(z, C_0 r)$. Using the definition (4.2) of $D(z, C_0 r)$, the expansion (3.4) and also using $|z| \simeq |\zeta|$, that follows from Case 1, we discover that

$$\Big|\Big\langle \frac{ZF(z)}{|ZF(z)|}, \zeta - z\Big\rangle\Big| \leq C\frac{|z|^{2m}}{|ZF(z)|}r^2, \qquad \text{for all } \zeta \in D(z, C_0 r).$$

This tells that the projection of the set $D(z, C_0 r)$ along the unit direction $\frac{ZF(z)}{|ZF(z)}$ has size $\frac{|z|^{2m}}{|ZF(z)|}r^2$. Therefore the Lebesgue measure of $\pi(B(p,r)\cap\Sigma)$ satisfies the inequalities

$$|\pi(B(p,r)\cap\Sigma)| \leq |D(z, C_0 r)| \leq C\frac{|z|^{2m}}{|ZF(z)|}r^3, \tag{4.12}$$

for an absolute constant $C > 0$. Ultimately, from (4.11) and (4.12), we obtain the upper-bound:

$$\mu(B(p,r)\cap\Sigma) = \int_{\pi(B(p,r)\cap\Sigma)} |ZF(\zeta)|d\xi d\eta \leq C(1+\beta)|z|^{2m}r^3.$$

Case 1nc – lower bound. Observe first that (4.7) holds also in this case. Then, under our choice of ε_1 and β, we have

$$\Big\{\zeta \in \mathbb{R}^2 : |\zeta - z| \leq \varepsilon_1 C_0^{-1} r \ \text{ and } |\langle ZF(z), \zeta - z\rangle| \leq \frac{1}{2}\mu_{z,\zeta}^{2m} C_0^{-2} r^2\Big\} \subset \pi(B(p,r)\cap\Sigma).$$

Let $\varepsilon_3 \leq \varepsilon_1$ be a small positive constant to be fixed below. Then we have

$$\Big\{\zeta \in \mathbb{R}^2 : |\zeta - z| \leq \varepsilon_3 C_0^{-1} r \ \text{ and } |\langle ZF(z), \zeta - z\rangle| \leq |z|^{2m} C^{-1} r^2\Big\} \subset \pi(B(p,r)\cap\Sigma),$$

The set in the left-hand side has size $\frac{|z|^{2m}}{|ZF(z)|}r^2$ along the unit direction $ZF(z)/|ZF(z)|$. Then we have the estimate from below for the Lebesgue measure of the integration set.

$$|\pi(B(p,r)\cap\Sigma)| \geq C\frac{|z|^{2m}r^3}{|ZF(z)|}. \tag{4.13}$$

To conclude the proof, we get a lower estimate for the function in the integral.

$$\begin{aligned} |ZF(\zeta)| &= \big|ZF(z) + M(\zeta - z) + |z|^{2m-1}O(|\zeta - z|^2)\big| \geq |ZF(z)| - C_4|z|^{2m}|\zeta - z| \\ &\geq |ZF(z)| - C_4|z|^{2m}\varepsilon_3 C_0^{-1} r \\ &\geq |ZF(z)| - C_4\varepsilon_3 C_0^{-1}\beta|ZF(z)| \geq \frac{1}{2}|ZF(z)|, \end{aligned} \tag{4.14}$$

provided that ε_3 is so small that $C_4\beta\varepsilon_3 C_0^{-1} \leq \frac{1}{2}$. The lower bound follows from (4.13) and (4.14). This ends the proof of (4.4).

4.2. Proof in the Case 2: $r \geq \varepsilon_0|z|$.

We claim that for any point $p = (z, \varphi(z)) \in \Sigma$ and $r > 0$ such that $r \geq \varepsilon_0|z|$ we have:

$$\mu(B(p, r) \cap \Sigma) \simeq r^{2m+3}. \tag{4.15}$$

We can without loss of generality assume that $\varphi(0) = 0$. By Lemma 3.2 and $|\varphi(z)| \leq C|z|^{2m+2}$, there exists a constant $C_5 > 1$ such that

$$C_5^{-1}|z| \leq d((0,0), (z, \varphi(z)) \leq C_5|z|. \tag{4.16}$$

Case 2 – upper bound. For $z \in \mathbb{R}^2$ and $r \geq \varepsilon_0|z|$, we have the inclusions

$$B((z, \varphi(z)), r) \subset B((0,0), C_5|z| + r) \subset B\big((0,0), (\varepsilon_0^{-1} + C_5)r\big).$$

Thus, we obtain

$$\int_{\pi(B(p,r)\cap\Sigma)} |ZF(\zeta)| d\xi d\eta \leq \int_{|\zeta|\leq Cr} C|\zeta|^{2m+1} d\xi d\eta \leq Cr^{2m+3}.$$

To check the lower bound, we distinguish two cases.

Case 2a: $\varepsilon_0|z| \leq r \leq 2C_5|z|$. It suffices to start from inclusion $B((z, \varphi(z)), r) \supset B((z, \varphi(z)), \varepsilon_0|z|)$. Then the perimeter measure of the smaller ball can be estimated as in Case 1. To conclude observe that $|z| \simeq r$.

Case 2b: $2C_5|z| \leq r < \infty$. In such case we have

$$B\big(0, r/(2C_5^2)\big) \cap \Sigma \subset B(p, r) \cap \Sigma. \tag{4.17}$$

Indeed, for any $q = (\zeta, \varphi(\zeta)) \in B(0, r/(2C_5^2))$ we have $|\zeta| \leq r/(2C_5)$ and

$$d(p, q) \leq d(0, p) + d(0, q) \leq C_5|z| + C_5|\zeta| \leq r,$$

as claimed.

By (4.17), it is enough to prove the lower-bound estimate in the case $z = 0$. For $\varrho > 0$, we calculate by Stokes' theorem the following integral on the curve $\gamma_\varrho(s) = \varrho e^{-is}$, with $s \in [0, 2\pi]$,

$$\begin{aligned} \int_{\gamma_\varrho} (XF d\xi + YF d\eta) &= \int_{|\zeta|<\varrho} \Big(-\partial_\eta(\partial_\xi\varphi(\zeta) - |\zeta|^{2m}\eta) + \partial_\xi(\partial_\eta\varphi(\zeta) + |\zeta|^{2m}\xi)\Big) d\xi d\eta \\ &= (2 + 2m) \int_{|\zeta|<\varrho} |\zeta|^{2m} d\xi d\eta = C^{-1}\varrho^{2m+2}, \end{aligned}$$

that implies $\int_{|\zeta|=\varrho} |ZF(\zeta)| d\mathcal{H}^1(\zeta) \geq C^{-1}\varrho^{2m+2}$. Thus we get the estimate

$$\int_{|\zeta|<r} |ZF(\zeta)| d\xi d\eta = \int_0^r \int_{|\zeta|=\varrho} |ZF(\zeta)| d\mathcal{H}^1(\zeta) d\varrho \geq C^{-1} r^{2m+3}.$$

This inequality ends the proof of (4.15) and thus of Theorem 1.3 in the case of entire admissible graphs.

5. Cone property for admissible entire epigraphs

We recall the definition of a John domain, specialized to the metric space $(\mathbb{R}^3, d)$.

Definition 5.1. A bounded open set $\Omega \subset \mathbb{R}^3$ is a λ-John domain, with $\lambda > 0$, if there exists a point $p_0 \in \Omega$ such that for all $p \in \Omega$ there is a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = p$, $\gamma(1) = p_0$, and

$$B\big(\gamma(t), \lambda \operatorname{diam}(\gamma|_{[0,t]})\big) \subset \Omega \quad \text{for all } t \in]0,1[. \tag{5.1}$$

A curve γ satisfying (5.1) is called a *John curve in Ω with parameter $\lambda > 0$*.

In our definition, the curve γ is not required to be rectifiable. By the results of [MS79], Definition 5.1 is equivalent to the more standard one with rectifiable curves and with diameters replaced by lengths. John domains are also known as domains with the twisted interior cone property.

In this section, we consider an unbounded domain of the epigraph type $\Omega = \operatorname{epi}(\varphi) = \{(z,t) \in \mathbb{R}^3 : t > \varphi(z)\}$, where $\varphi \in C^\infty(\mathbb{R}^2)$ is an m-admissible function, and we construct a nontrivial John curve starting from any point $p = (z,t) \in \Omega$. The case of a bounded domain is discussed in Section 7.

For any point $z \in \mathbb{R}^2$ with $|ZF(z)| \neq 0$ there exists a unit vector $(u,v) \in \mathbb{S}^1 \subset \mathbb{R}^2$ such that

$$-(uX + vY)F(z) > \frac{1}{2}|ZF(z)|, \tag{5.2}$$

where $F(z,t) = \varphi(z) - t$. Our John curve starting from $(z,t) \in \operatorname{epi}(\varphi)$ is the integral curve of $-(uX + vY)$ for times $s \in [0, \bar{s}]$, where the time $\bar{s} = \bar{s}(z)$ is

$$\bar{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}}, \tag{5.3}$$

and $\varepsilon_0 > 0$ is a suitable constant. For $s \geq \bar{s}$, the John curve is an integral curve of $\partial/\partial t$. This piece of curve is nonrectifiable. When $(z, \varphi(z)) \in \operatorname{gr}(\varphi)$ is a characteristic point of $\operatorname{gr}(\varphi)$, i.e. $|ZF(z)| = 0$, we have $\bar{s} = 0$ and the first piece of the curve disappears.

By the rotational invariance (2.1) of the metric d, in (5.2) we can assume that $u = 1$ and $v = 0$.

Theorem 5.2. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ be a function satisfying* (1.2)*. There exist constants $\varepsilon_0 > 0$ and $\lambda > 0$ such that for any $z \in \mathbb{R}^2$ with*

$$-XF(z) > \frac{1}{2}|ZF(z)|, \tag{5.4}$$

the curve $\gamma : [0,\infty) \to \mathbb{R}^3$

$$\gamma(s) = \begin{cases} e^{sX}(z,t), & \text{if} \quad 0 \le s \le \bar{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}} \\ \gamma(\bar{s}) + (0, s-\bar{s}), & \text{if} \quad \bar{s} \le s < \infty \end{cases}$$

is a John curve in $\mathrm{epi}(\varphi)$ *with parameter* λ *starting from* $(z,t) \in \mathrm{epi}(\varphi)$.

Let $\lambda > 0$ be the parameter of our John curves. In the proof of the theorem and in the following sections, we denote by σ_λ any constant of the form $C\lambda^\beta$, where C is an absolute constant and $\beta > 0$ is a positive power.

Proof. Without loss of generality, we prove the claim for $t = \varphi(z)$, i.e., we construct a John curve in the epigraph of φ starting from a boundary point. In the following we let $z_s = z + se_1$. When $s \in [0,\bar{s}]$, an explicit formula for $\gamma(s)$ is

$$\gamma(s) = e^{sX}(z,\varphi(z)) = \Big(z_s, \varphi(z) + y\int_0^s |z_\varrho|^{2m} d\varrho\Big).$$

The definition in (5.3) for $\bar{s}$ implies that $\bar{s} \le \varepsilon_0 C|z|$, where $C > 0$ is the constant appearing in (3.1). We can choose $\varepsilon_0 > 0$ such that $\varepsilon_0 C < \frac{1}{2}$. Then, for any $s \in [0,\bar{s}]$ we have

$$\frac{|z|}{2} \le |z_s| \le \frac{3}{2}|z|. \tag{5.5}$$

Further conditions on ε_0 will be required below.

Step 1. We claim that for any sufficiently small $\lambda > 0$ we have $B(\gamma(s),\lambda s) \subset \mathrm{epi}(\varphi)$ for all $0 < s \le \bar{s}$.

From $s \le \bar{s} \le \frac{1}{2}|z|$ and Corollary 2.2, we have the inclusion $B(\gamma(s),\lambda s) \subset \mathrm{Box}_I(\gamma(s),\sigma_\lambda s)$ for any $0 < \lambda \le 1$, where $\sigma_\lambda = b_1\lambda$ and b_1 is the constant given by Corollary 2.2. So our claim is implied by $\mathrm{Box}_I(\gamma(s),\sigma_\lambda s) \subset \mathrm{epi}(\varphi)$. We have $p \in \mathrm{Box}_I(\gamma(s),\sigma_\lambda s)$ if and only if

$$p = \Big(z_s + v, \varphi(z) + y\int_0^s |z_\varrho|^{2m} d\varrho + |z_s|^{2m}\big(u_3 + \omega(v,z_s)\big)\Big),$$

with $\|u\|_{1,1,2} \le \sigma_\lambda s$ and $u = (u_1,u_2,u_3) = (v,u_3)$. Then the claim in Step 1 is implied by the inequality

$$\varphi(z_s + v) - \varphi(z) < y\int_0^s |z_\varrho|^{2m} d\varrho + |z_s|^{2m}\big(u_3 + \omega(v,z_s)\big) =: H, \tag{5.6}$$

for $\|u\|_{1,1,2} < \sigma_\lambda s$ and $s \le \bar{s}$. The left-hand side of (5.6) can be expanded using (3.5):

$$\varphi(z_s + v) - \varphi(z) = \langle ZF(z), v_s\rangle + |z|^{2m}\omega(v_s,z) + \mu_{z,z_s}^{2m} O(|v_s|^2).$$

By (5.4), $|z_s| \le C|z|$, and $u_1 + s \ge 0$ we get

$$\varphi(z_s + v) - \varphi(z) \le |ZF(z)|\Big(-\frac{1}{2}(u_1 + s) + |u_2|\Big) + |z|^{2m}\omega(v_s,z) + C|z|^{2m}|v_s|^2.$$

Since $|u_2| \le \sigma_\lambda s$ and $|u_1| \le \sigma_\lambda s$, we have $|v_s| \le Cs$ and for small $\lambda > 0$, we get

$$\varphi(z_s + v) - \varphi(z) \le -\frac{s}{4}|ZF(z)| + |z|^{2m}(\omega(v,z) + sy + C_0 s^2). \tag{5.7}$$

Now we estimate the quantity H in the right-hand side of (5.6). Since $\varrho \le s \le \bar{s} \le \frac{1}{2}|z|$, and $u_3 \ge -\sigma_\lambda s^2$ we have the elementary inequalities

$$\begin{aligned} &\big||z_\varrho|^{2m} - |z|^{2m}\big| \le C|z|^{2m-1}\varrho, \\ &|z_s|^{2m}u_3 \ge -C|z|^{2m}\sigma_\lambda s^2, \quad \text{and} \\ &|z_s|^{2m}\omega(v, z_s) \ge |z|^{2m}\omega(v, z_s) - C|z|^{2m}s^2. \end{aligned} \tag{5.8}$$

Thus, we obtain

$$H \ge |z|^{2m}(\omega(v, z) + ys - C_1 s^2) \tag{5.9}$$

for a suitable absolute constant C_1.

By (5.7) and (5.9), the claim (5.6) is then implied by the inequality

$$(C_0 + C_1)|z|^{2m}s < \frac{1}{4}|ZF(z)|,$$

that holds for all $s \le \bar{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}}$ as soon as $\varepsilon_0 \le \frac{1}{4(C_0+C_1)}$.

Step 2. For $s \ge \bar{s}$, the curve γ is defined by the formula

$$\gamma(s) = \Big(z_{\bar{s}}, \varphi(z) + y\int_0^{\bar{s}} |z_\varrho|^{2m} d\varrho + s - \bar{s}\Big).$$

In the following we let $s - \bar{s} = \tau^{2m+2}$. As a function of τ, the diameter of γ restricted to $[0, s]$ satisfies

$$\Delta_\tau = \operatorname{diam}(\gamma|_{[0,\bar{s}+\tau^{2m+2}]}) \le C\Big(\bar{s} + \min\Big\{\tau, \frac{\tau^{m+1}}{|z|^m}\Big\}\Big) =: \bar{s} + m_\tau.$$

The proof of Theorem 1.4 shows in fact that the inequality above is a global equivalence for $\bar{s}, \tau \in [0, +\infty[$.

We claim that for any sufficiently small $\lambda > 0$ we have $B(\gamma(s), \lambda\Delta_\tau) \subset \operatorname{epi}(\varphi)$ for all $\tau \ge 0$. By Corollary 2.2, this claim is equivalent to

$$\operatorname{Box}_I(\gamma(s), \sigma_\lambda\Delta_\tau) \subset \operatorname{epi}(\varphi) \quad \text{and} \quad \operatorname{Box}_J(\gamma(s), \sigma_\lambda\Delta_\tau) \subset \operatorname{epi}(\varphi). \tag{5.10}$$

We prove the inclusion in the left-hand side of (5.10). We have $p \in \operatorname{Box}_I(\gamma(s), \sigma_\lambda\Delta_\tau)$ if and only if

$$p = \Big(z_{\bar{s}} + v, \varphi(z) + y\int_0^{\bar{s}} |z_\varrho|^{2m} d\varrho + |z_{\bar{s}}|^{2m}\big(u_3 + \omega(v, z_{\bar{s}})\big) + \tau^{2m+2}\Big), \tag{5.11}$$

with $\|u\|_{1,1,2} \le \sigma_\lambda\Delta_\tau$ and $u = (u_1, u_2, u_3) = (v, u_3)$. The point p belongs to $\operatorname{epi}(\varphi)$ if

$$L := \varphi(z_{\bar{s}} + v) - \varphi(z) < y\int_0^{\bar{s}} |z_\varrho|^{2m} d\varrho + |z_{\bar{s}}|^{2m}\big(u_3 + \omega(v, z_{\bar{s}})\big) + \tau^{2m+2} =: R.$$

First we get an upper bound for L. Observe first that assumption (5.4) and the inequalities $\bar{s} \le \frac{1}{2}|z|$ and $|v| \le \sigma_\lambda(\bar{s} + m_\tau)$ give

$$\langle ZF(z), v_{\bar{s}}\rangle \le |ZF(z)|\Big(-\frac{1}{4}\bar{s} + \sigma_\lambda m_\tau\Big).$$

Thus, using formula (3.5) we obtain

$$\begin{aligned} L &= \varphi(z+v_{\bar s})-\varphi(z) = \langle ZF(z), v_{\bar s}\rangle + |z|^{2m}\omega(v_{\bar s},z) + \max\big\{|z|^{2m},|v_{\bar s}|^{2m}\big\}O\big(|v_{\bar s}|^2\big)\\ &\le |ZF(z)|\Big(-\frac14\bar s+\sigma_\lambda m_\tau\Big)+|z|^{2m}\omega(v_{\bar s},z)+C\big(|z|^{2m}+\sigma_\lambda m_\tau^{2m}\big)(\bar s^2+\sigma_\lambda^2 m_\tau^2)\\ &\le |ZF(z)|\Big(-\frac14\bar s+\sigma_\lambda m_\tau\Big)+|z|^{2m}\Big(\omega(v,z)+\bar s y+C_0\bar s^2+\sigma_\lambda m_\tau^2\Big)+\sigma_\lambda m_\tau^{2m+2}. \end{aligned} \tag{5.12}$$

We compute a lower bound for the right-hand side R. Using (5.8) we get

$$\begin{aligned} R &= y\int_0^{\bar s}|z_\varrho|^{2m}d\varrho+|z_{\bar s}|^{2m}\big(u_3+\omega(v,z_{\bar s})\big)+\tau^{2m+2}\\ &\ge y|z|^{2m}\bar s - C|z|^{2m}\bar s^2+\tau^{2m+2}-\sigma_\lambda|z|^{2m}m_\tau^2+|z|^{2m}(\omega(v,z)-u_2\bar s)\\ &\ge \tau^{2m+2}+|z|^{2m}\Big(\omega(v,z)+\bar s y-C_1\bar s^2-\sigma_\lambda\bar m_\tau^2\Big). \end{aligned} \tag{5.13}$$

Then, the inequality $L<R$ follows from

$$\sigma_\lambda m_\tau|ZF(z)|+|z|^{2m}\Big(C_2\bar s^2+\sigma_\lambda m_\tau^2\Big)+\sigma_\lambda m_\tau^{2m+2}<\frac14\bar s|ZF(z)|+\tau^{2m+2}. \tag{5.14}$$

To prove (5.14), we start from the second term. By the definition of $\bar s$, we have

$$C_2\bar s^2|z|^{2m}=\varepsilon_0C_2|ZF(z)|\bar s\le\frac14|ZF(z)|\bar s,$$

as soon as ε_0 satisfies $\varepsilon_0C_2<1/4$. This is the last time we modify the choice of ε_0.

Next we look at the first term. Observe that

$$\sigma_\lambda|ZF(z)|m_\tau=\sigma_\lambda|ZF(z)|\min\Big\{\tau,\frac{\tau^{m+1}}{|z|^m}\Big\}\le\sigma_\lambda|ZF(z)|\frac{\tau^{m+1}}{|z|^m}\le\sigma_\lambda\Big(\frac{|ZF(z)|^2}{|z|^{2m}}+\tau^{2m+2}\Big).$$

Then, since $\frac14|ZF(z)|\bar s+\tau^{2m+2}=\frac{\varepsilon_0}{4}\frac{|ZF(z)|^2}{|z|^{2m}}+\tau^{2m+2}$, we can finish the estimate as soon as σ_λ is small with respect to absolute constants (which include ε_0, now).

The estimate of the third term is easy:

$$\sigma_\lambda|z|^{2m}m_\tau^2=\sigma_\lambda|z|^{2m}\Big(\min\Big\{\tau,\frac{\tau^{m+1}}{|z|^m}\}\Big)^2\le\sigma_\lambda\tau^{2m+2},$$

which is correctly estimated, provided that σ_λ is small enough. Finally, we have

$$\sigma_\lambda m_\tau^{2m+2}=\sigma_\lambda\Big(\min\Big\{\tau,\frac{\tau^{m+1}}{|z|^m}\Big\}\Big)^{2m+2}\le\sigma_\lambda\tau^{2m+2},$$

which again satisfies the required estimate.

To conclude the proof, we have to check the inclusion in the right-hand side of (5.10). In this case the box $\mathrm{Box}_J(\gamma(s),\sigma_\lambda(\bar s+m_\tau))$ is made of points of the form

$$\Big(z_{\bar s}+v,\varphi(z)+y\int_0^{\bar s}|z_\varrho|^{2m}d\varrho+\tau^{2m+2}+u_3+|z_{\bar s}|^{2m}\omega(v,z_{\bar s})\Big).$$

The unique difference with (5.11) is that the term u_3 replaces the term $|z_{\bar{s}}|^{2m}u_3$, and now $|u_3| \leq \sigma_\lambda(\bar{s}+m_\tau)^{2m+2}$.

The estimate from above for L remains unchanged, because it does not involve u_3. In the estimate from below for R, we need the following evaluation for the term u_3:

$$u_3 \geq -\sigma_\lambda(\bar{s}+m_\tau)^{2m+2} \geq -\sigma_\lambda|z|^{2m}\bar{s}^2 - \sigma_\lambda m_\tau^{2m+2}.$$

Therefore, the inequality (5.14) remains unchanged and the proof can be concluded arguing as in the previous case. □

6. Uniform property of entire admissible epigraphs

We recall the definition of a uniform domain, specialized to the metric space $(\mathbb{R}^3, d)$.

Definition 6.1. An open set $\Omega \subset \mathbb{R}^3$ is a uniform domain if there exist $\varepsilon > 0$ and $\delta > 0$ with the following property. For any pair of points $x, y \in \Omega$ there is a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,

$$\operatorname{diam}(\gamma) \leq \delta^{-1} d(x,y), \tag{6.1}$$

and, letting $\Delta_t = \min\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\}$, for any $t \in [0,1]$ we have

$$B(\gamma(t), \varepsilon\Delta_t) \subset \Omega. \tag{6.2}$$

Uniform domains are also known as (ε,δ)-domains. As for John domains, the curves in our definition are not required to be rectifiable. By the results of [MS79], this is equivalent to the more standard definition which requires rectifiability.

We consider an unbounded domain of the epigraph type $\Omega = \operatorname{epi}(\varphi) = \{(z,t) \in \mathbb{R}^3 : t > \varphi(z)\}$, where $\varphi \in C^\infty(\mathbb{R}^2)$ is an m-admissible function. For any pair of points $p, q \in \Omega$, we construct a curve connecting them and satisfying the conditions (6.1) and (6.2) with uniform constants δ and ε. The case of a bounded domain is discussed in Section 7.

Theorem 6.2. *Let $\varphi \in C^\infty(\mathbb{R}^2)$ be a function satisfying* (1.2)*. Then, the epigraph $\Omega = \operatorname{epi}(\varphi)$ is a uniform domain.*

Proof. Let $p = (z, \varphi(z)+b)$ and $q = (\zeta, \varphi(\zeta)+\beta)$, with $b, \beta > 0$, be points in the epigraph of φ. We can without loss of generality assume that

$$\frac{|ZF(z)|}{|z|^{2m}} = \max\left\{\frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}}\right\} \geq 0. \tag{6.3}$$

The maximum can be 0, even for arbitrarily close points. This happens for instance in Example 3.1. By assumption (1.2) we can define continuously $\frac{|ZF(z)|}{|z|^{2m}} = 0$ for $z = 0$.

Let $\mu > 0$ be a parameter that will be fixed along the proof. We distinguish two cases:

$$d(p,q) < \mu \max\left\{\frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}}\right\} \qquad \text{(Case A);}$$

$$d(p,q) \geq \mu \max\left\{\frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}}\right\} \qquad \text{(Case B).}$$

We can without loss of generality assume that (5.4) holds at the point z, i.e.: $-XF(z) = |XF(z)| > \frac{1}{2}ZF(z)|$. Then, if we denote by ε_0 and $\lambda > 0$ the parameters fixed in Section 5, we know that the curve

$$\gamma_z(s) = \begin{cases} \Big(z_s, \varphi(z) + b + y\int_0^s |z_\varrho|^{2m} d\varrho\Big), & \text{if } s \le \bar{s} = \varepsilon_0 \dfrac{|ZF(z)|}{|z|^{2m}} \\ \Big(z_{\bar{s}}, \varphi(z) + b + y\int_0^{\bar{s}} |z_\varrho|^{2m} d\varrho + s - \bar{s}\Big), & \text{if } s \ge \bar{s}, \end{cases}$$

is a John curve with parameter λ.

Analysis of Case A. We claim that there exists $\mu > 0$ such that the curve

$$\gamma_\zeta(s) = \begin{cases} \Big(\zeta_s, \varphi(\zeta) + \beta + \eta\int_0^s |\zeta_\varrho|^{2m} d\varrho\Big), & \text{if } s \le \bar{\bar{s}} = \varepsilon_0 \dfrac{|ZF(\zeta)|}{|\zeta|^{2m}} \\ \Big(\zeta_{\bar{\bar{s}}}, \varphi(\zeta) + \beta + \eta\int_0^{\bar{\bar{s}}} |\zeta_\varrho|^{2m} d\varrho + s - \bar{\bar{s}}\Big), & \text{if } s \ge \bar{\bar{s}} \end{cases}$$

is a John curve with parameter λ. To prove this claim, it suffices to show that $-XF(\zeta) > \frac{1}{4}|ZF(\zeta)|$ if Case A holds and μ is small enough.

From (1.2) it follows that $|\nabla ZF(z)| \le C|z|^{2m}$ for all $z \in \mathbb{R}^2$ and thus the function $z \mapsto |ZF(z)|/|z|^{2m}$ is globally Lipschitz continuous on $\mathbb{R}^2$. Let L be the Lipschitz constant. By (6.3) and by the Case A with sufficiently small μ, we have

$$\begin{aligned} -\frac{XF(\zeta)}{|\zeta|^{2m}} &\ge -\frac{XF(z)}{|z|^{2m}} - L|\zeta - z| \ge \frac{1}{2}\frac{|ZF(z)|}{|z|^{2m}} - Ld(p,q) \ge \frac{1}{2}\frac{|ZF(z)|}{|z|^{2m}} - L\mu\frac{|ZF(z)|}{|z|^{2m}} \\ &\ge \frac{1}{4}\frac{|ZF(z)|}{|z|^{2m}} \ge \frac{1}{4}\frac{|ZF(\zeta)|}{|\zeta|^{2m}}. \end{aligned}$$

Also the mapping $z \mapsto \bar{s}(z)$ in (5.3) is Lipschitz continuous. Then, for μ small enough, in the Case A the times $\bar{s} = \bar{s}(z)$ and $\bar{\bar{s}} = \bar{s}(\zeta)$ satisfy

$$\frac{1}{2}\bar{s} \le \bar{\bar{s}} \le \frac{3}{2}\bar{s}. \tag{6.4}$$

Finally, we also have $|\zeta - z| \le d(p,q) \le \mu\frac{|ZF(z)|}{|z|^{2m}} \le C\mu|z| \le \frac{1}{2}|z|$, for μ sufficiently small.

We are now ready to define the curve joining p and q and satisfying (6.1), (6.2). For a suitable $H > 0$, let

$$\widehat{s} = Hd(p,q). \tag{6.5}$$

Then, the curve γ is the concatenation of $\gamma_z\big|_{[0,\widehat{s}]}$, a length-minimizing path $\widehat{\gamma}$ joining $\widehat{\gamma}(0) = \gamma_z(\widehat{s})$ and $\widehat{\gamma}(1) = \gamma_\zeta(\widehat{s})$, and the opposite of $\gamma_\zeta\big|_{[0,\widehat{s}]}$.

We claim that there exist $H > 0$ and $\mu > 0$ such that the curve γ satisfies (6.1) and (6.2).

We preliminarily show that:

(i) $\widehat{s} \le \min\{\bar{s}, \bar{\bar{s}}\}$, i.e., the points $\gamma_z(\widehat{s})$ and $\gamma_\zeta(\widehat{s})$ belong to the first piece of the curves γ_z and γ_ζ, respectively;
(ii) $d(\gamma_\zeta(\widehat{s}), \gamma_z(\widehat{s})) \le \frac{\lambda}{2}\widehat{s}$, where λ is the John constant of γ_ζ and γ_z;
(iii) $\operatorname{diam}(\gamma) \le Cd(p,q)$.

Condition (6.1) is (iii). We show that (i)–(iii) imply (6.2). For $s \le \widehat{s}$, by $\Delta_s \le \operatorname{diam}(\gamma_z|_{[0,s]})$ and by the cone property (5.1) we have

$$B(\gamma(s), \lambda\Delta_s) \subset B(\gamma_z(s), \lambda \operatorname{diam}(\gamma_z|_{[0,s]})) \subset \operatorname{epi}(\varphi).$$

Then (6.2) holds with $\varepsilon = \lambda$. The same happens for points $\gamma_\zeta(s)$ with $s \le \widehat{s}$. Finally, for a point $\widehat{\gamma}(s^*)$ in the intermediate part, by (ii) we have

$$\begin{aligned} \operatorname{dist}(\widehat{\gamma}(s^*), \operatorname{gr}(\varphi)) &\ge \operatorname{dist}(\gamma_z(\widehat{s}), \operatorname{gr}(\varphi)) - \frac{\lambda}{2}\widehat{s} \\ &\ge \lambda \operatorname{diam}(\gamma_z|_{[0,\widehat{s}]}) - \frac{\lambda}{2} \operatorname{diam}(\gamma_z|_{[0,\widehat{s}]}) = \frac{\lambda}{2} \operatorname{diam}(\gamma_z|_{[0,\widehat{s}]}). \end{aligned} \tag{6.6}$$

In order to get a lower bound for the last diameter, we use the length-minimizing property of $\widehat{\gamma}$ and property (ii), which give

$$\operatorname{diam}(\widehat{\gamma}_{[0,s^*]}) \le d(\widehat{\gamma}(0), \widehat{\gamma}(1)) \le \frac{\lambda}{2}\widehat{s} \le \frac{\lambda}{2} \operatorname{diam}\left(\gamma_z|_{[0,\widehat{s}]}\right).$$

Therefore, we have $\operatorname{diam}\left(\gamma_z|_{[0,\widehat{s}]} + \widehat{\gamma}|_{[0,s^*]}\right) \le 2 \operatorname{diam}\left(\gamma_z|_{[0,\widehat{s}]}\right)$ and then it is easy to conclude that (6.2) holds with $\varepsilon = \frac{\lambda}{4}$.

Now we prove (i). By (6.4) this is implied by $\widehat{s} \le \frac{1}{2}\bar{s}$. By (6.5), Case A, (6.3), we have $\widehat{s} \le H\mu\frac{|ZF(z)|}{|z|^{2m}} = H\mu\frac{\bar{s}}{\varepsilon_0}$. Thus, we deduce that (i) holds provided that

$$H\mu \le \frac{1}{2}\varepsilon_0. \tag{6.7}$$

This is the first requirement on H and μ. This restriction is compatible with further conditions made below.

Next we prove (ii). Theorem 1.4 gives

$$d(\gamma_z(\widehat{s}), \gamma_\zeta(\widehat{s})) \le C_0|\zeta - z| + C_0 \min\left\{\frac{|\Theta|^{1/2}}{|z|^m}, |\Theta|^{\frac{1}{2m+2}}\right\} \tag{6.8}$$

where

$$\Theta = \varphi(\zeta) - \varphi(z) + \beta - b + \int_0^{\widehat{s}} (\eta|\zeta_\varrho|^{2m} - y|z_\varrho|^{2m})d\varrho + |z_{\widehat{s}}|^{2m}\omega(z_{\widehat{s}}, \zeta_{\widehat{s}}).$$

Let $\Theta = \Theta_1 + \Theta_1 + \Theta_3$, with

$$\begin{aligned} \Theta_1 &= \varphi(\zeta) - \varphi(z) + \beta - b + |z|^{2m}\omega(z, \zeta), \\ \Theta_2 &= |z_{\widehat{s}}|^{2m}\omega(z_{\widehat{s}}, \zeta_{\widehat{s}}) - |z|^{2m}\omega(z, \zeta), \\ \Theta_3 &= \int_0^{\widehat{s}} (\eta|\zeta_\varrho|^{2m} - y|z_\varrho|^{2m})d\varrho. \end{aligned}$$

The first term in the right-hand side of (6.8) can be estimated as follows

$$C_0|\zeta - z| \le C_0 d(p, q) \le \frac{\lambda}{8} H d(p, q) = \frac{\lambda}{8}\widehat{s},$$

as soon as H is large enough to ensure that

$$C_0 \leq \frac{\lambda}{8} H, \tag{6.9}$$

where C_0 is the absolute constant in (6.8). We used definition (6.5) of $\widehat{s}$.

Concerning the second term in the right-hand side of (6.8), we claim that for all $j = 1, 2, 3$ we have

$$C_0 \min\Big\{ |\Theta_j|^{\frac{1}{2m+2}}, \frac{|\Theta_j|^{1/2}}{|z|^m} \Big\} \leq \frac{\lambda}{8} H d(p,q). \tag{6.10}$$

By Theorem 1.4, we have

$$C_0 \min\{ |\Theta_1|^{\frac{1}{2m+2}}, |\Theta_1|^{1/2}/|z|^m \} \leq C_0 \delta(p,q) \leq C C_0 d(p,q) \leq \frac{\lambda}{8} H d(p,q),$$

as soon as H is large enough so that

$$C C_0 \leq \frac{\lambda}{8} H. \tag{6.11}$$

To evaluate the term with Θ_2, we apply the inequalities

$$\begin{aligned} |\Theta_2| &= \Big| \Big(|z_{\widehat{s}}|^{2m} - |z|^{2m} \Big) \omega(z,\zeta) + |z_{\widehat{s}}|^{2m} \widehat{s} (\eta - y) \Big| \leq C |z|^{2m} |\zeta - z| \widehat{s} \\ &\leq C |z|^{2m} d(p,q) \widehat{s} = C |z|^{2m} H d(p,q)^2. \end{aligned}$$

Thus, we deduce that for some absolute constant $C_2 > 0$ we have

$$\frac{|\Theta_2|^{1/2}}{|z|^m} \leq C_2 d(p,q) \sqrt{H} \leq C_0^{-1} \frac{\lambda}{8} H d(p,q)$$

as soon as

$$C_2 \leq C_0^{-1} \frac{\lambda}{8} \sqrt{H}. \tag{6.12}$$

Finally, we estimate Θ_3:

$$\begin{aligned} |\Theta_3| &\leq |y - \eta| \int_0^{\widehat{s}} |z_\varrho|^{2m} d\varrho + |\eta| \Big| \int_0^{\widehat{s}} (|z_\varrho|^{2m} - |\zeta_\varrho|^{2m}) d\varrho \Big| \\ &\leq C \widehat{s} |z|^{2m} \delta(p,q) + C \widehat{s} |\eta| |z|^{2m-1} |z - \zeta| \\ &\leq C \widehat{s} |z|^{2m} d(p,q) = C |z|^{2m} d(p,q)^2 H, \end{aligned}$$

and we end up with again with the requirement (6.12).

To conclude the proof, we choose $H > 0$ large enough so that (6.9), (6.11) and (6.12) hold. This implies (ii). Then we choose $\mu > 0$ such that (6.7) holds. This implies (i). The diameter estimate in (iii) holds in terms of such constants and the proof of Case A is concluded.

Analysis of Case B. Let us consider the second piece of the curve from $(z, \varphi(z) + b)$,

$$\gamma_z(s) = \Big(z_{\overline{s}}, t + b + y \int_0^{\overline{s}} |z_\varrho|^{2m} d\varrho + s - \overline{s} \Big) \quad \text{for } s \geq \overline{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}}.$$

Let also $(\zeta, \varphi(\zeta)+\beta)$ be such that Case B holds. Then, there is a unit vector $w=(u,v)\in \mathbb{R}^2$ such that the curve γ_ζ is a λ-John curve in $\mathrm{epi}(\varphi)$ starting from $(\zeta,\varphi(\zeta)+\beta)$. When $s\geq \bar{\bar{s}}=\varepsilon_0|ZF(\zeta)|/|\zeta|^{2m}$, the curve is

$$\gamma_\zeta(s)=\Big(\zeta+\bar{\bar{s}}w, \varphi(\zeta)+\beta+\omega(w,\zeta)\int_0^{\bar{\bar{s}}}|\zeta+\varrho w|^{2m}d\varrho+s-\bar{\bar{s}}\Big).$$

Note that the numbers $\bar{s}$ and $\bar{\bar{s}}$ could both vanish. Furthermore, we will assume without loss of generality that $\mathrm{diam}\left(\gamma_z\big|_{[0,\widehat{s}_z]}\right)\geq \mathrm{diam}\left(\gamma_\zeta\big|_{[0,\widehat{s}_\zeta]}\right)$.

For $\tau\geq 0$ consider the points $\gamma_z(\widehat{s}_z)$ and $\gamma_\zeta(\widehat{s}_\zeta)$, where

$$\widehat{s}_z=\bar{s}+\tau^{2m+2},\qquad \widehat{s}_\zeta=\bar{\bar{s}}+\tau^{2m+2}.$$

We claim that there exists $M>0$ such that for all $p=(z,\varphi(z)+b)$ and $q=(\zeta,\varphi(\zeta)+\beta)$ for which Case B holds, if $\tau\geq 0$ satisfies

$$\mathrm{diam}\left(\gamma_z\big|_{[0,\widehat{s}_z]}\right)=\max\Big\{\mathrm{diam}\left(\gamma_z\big|_{[0,\widehat{s}_z]}\right),\mathrm{diam}\left(\gamma_\zeta\big|_{[0,\widehat{s}_\zeta]}\right)\Big\}=Md(p,q), \tag{6.13}$$

then we have

$$d\Big(\gamma_z(\widehat{s}_z),\gamma_\zeta(\widehat{s}_\zeta)\Big)\leq\frac{\lambda}{2}\,\mathrm{diam}\left(\gamma_z\Big|_{[0,\widehat{s}_z]}\right), \tag{6.14}$$

where λ is the John constant of the curves.

Notice that for any M,p,q there is always a τ such that (6.13) holds because the left-hand side of (6.13) is increasing in τ and tends to $+\infty$, as $\tau\to+\infty$.

We prove the claim. By the invariance of the distance with respect to vertical translations we have

$$\begin{aligned}
d\Big(\gamma_z(\bar{s}+\tau^{2m+2}),\gamma_\zeta(\bar{\bar{s}}+\tau^{2m+2})\Big)&=d\big(\gamma_z(\bar{s}),\gamma_\zeta(\bar{s})\big)\\
&\leq d(\gamma_z(\bar{s}),\gamma_z(0))+d(\gamma_z(0),\gamma_\zeta(0))+d(\gamma_\zeta(\bar{\bar{s}}),\gamma_\zeta(0))\\
&\leq \varepsilon_0\frac{|ZF(z)|}{|z|^{2m}}+d(p,q)+\varepsilon_0\frac{|ZF(\zeta)|}{|\zeta|^{2m}}\\
&\leq \frac{2\varepsilon_0}{\mu}d(p,q)+d(p,q)\\
&=\frac{1}{M}\Big(\frac{2\varepsilon_0}{\mu}+1\Big)\max\{\mathrm{diam}(\gamma_z|_{[0,\widehat{s}_z]}),\mathrm{diam}(\gamma_\zeta|_{[0,\widehat{s}_\zeta]})\},
\end{aligned}$$

by (6.13). Thus (6.14) holds if M is large enough, and the claim is proved.

To conclude the proof, we show that the path γ, given by the concatenation of $\gamma_\zeta\big|_{[0,\widehat{s}_\zeta]}$, a length minimizing path $\widehat{\gamma}$ connecting $\widehat{\gamma}(0)=\gamma_z(\widehat{s}_z)$ and $\widehat{\gamma}(1)=\gamma_\zeta(\widehat{s}_\zeta)$ and the reverse of $\gamma_z\big|_{[0,\widehat{s}_z]}$ satisfies the (ε,δ)-condition. Since the diameter estimate (6.1) is contained in the claim above, we are left with the proof of (6.2).

Let q be a point of γ. If $q=\gamma_z(s)$ with $s\leq\widehat{s}_z$ or $q=\gamma_\zeta(s)$ with $s\leq\widehat{s}_\zeta$, then (6.2) follows with $\varepsilon=\lambda$ from the John property (5.1). If $q=\widehat{\gamma}(s^*)$, then we argue as in (6.6). precisely

$$\begin{aligned}
\mathrm{dist}(\widehat{\gamma}(s^*),\mathrm{gr}\,\varphi)&\geq\mathrm{dist}(\gamma_z(\widehat{s}_z),\mathrm{gr}\,\varphi)-d(\widehat{\gamma}(s^*),\gamma_z(\widehat{s}_z))\\
&\geq\lambda\,\mathrm{diam}(\gamma_z|_{[0,\widehat{s}_z]})-\mathrm{diam}(\widehat{\gamma})\geq\frac{\lambda}{2}\,\mathrm{diam}(\gamma_z|_{[0,\widehat{s}_z]}),
\end{aligned} \tag{6.15}$$

by (6.14). Finally, to get a lower estimate of the latter diameter with $\operatorname{diam}(\gamma_z|_{[0,\widehat{s}_z]} + \widehat{\gamma}|_{[0,s^*]})$, which will give the John property, it suffices to use the length minimizing property of $\widehat{\gamma}$

$$\operatorname{diam}(\widehat{\gamma}|_{[0,s^*]}) \leq d(\widehat{\gamma}(0), \widehat{\gamma}(1)) \leq \frac{\lambda}{2} \operatorname{diam}\left(\gamma_z\big|_{[0,\widehat{s}_z]}\right).$$

Thus, as in Case A, we get the correct lower bound for the last line of (6.15) and the proof is easily concluded. □

7. Bounded admissible domains are uniform

In this section we prove Theorems 1.2 and 1.3 in the case of a *bounded* m-admissible domain. Now we assume that $m \in \mathbb{N}$ is an integer.

Proof of Theorem 1.2. Let $\Omega \subset \mathbb{R}^3$ be an m-admissible domain. By a standard localization argument (see e.g. [MM05a, Proposition 2.5]), it suffices to show that for all $p_0 \in \partial\Omega$ there is a neighborhood A_{p_0} of p_0 in $\mathbb{R}^3$ such that for all $p, q \in A_{p_0}$ there is a continuous curve $\gamma : [0,1] \to \Omega \cap A_{p_0}$ satisfying $\gamma(0) = p$ and $\gamma(1) = q$ and such that (6.1) and (6.2) hold.

There are two cases:

1. p_0 is a noncharacteristic point, i.e., $\operatorname{span}\{X(p_0), Y(p_0)\}$ is not contained in $T_{p_0}\partial\Omega$.
2. p_0 is a characteristic point of $\partial\Omega$.

In Case 1, the claim is proved in [MM05a, Theorem 1.1]. To use this result, we need a C^∞ boundary and smooth vector fields. For this reason we require $m \in \mathbb{N}$.

In the Case 2, in a neighborhood of p_0 the boundary of Ω is a graph of the type $t = \varphi(z)$ for an m-admissible function $\varphi \in C^\infty(D)$ for some open set $D \subset \mathbb{R}^2$. The claim is proved in Sections 5 and 6. □

Proof of Theorem 1.3. By compactness, we can cover $\partial\Omega$ with a finite union of m-admissible graphs, together with a compact subset $K \subset \partial\Omega$ containing only noncharacteristic points.

At points $p \in K$, the Ahlfors estimates (1.4) is proved in [MM02, Corollary 1]. To use this result, we need a smooth boundary and smooth vector fields ($m \in \mathbb{N}$).

On m-admissible graphs, the Ahlfors estimate is proved in Section 4. □

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