

# STARSHAPED AND CONVEX SETS IN CARNOT GROUPS AND IN THE GEOMETRIES OF VECTOR FIELDS.

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**ABSTRACT.** The paper gives an overview on convex sets and starshaped sets in Carnot groups and in the more general case of geometries of vector fields, including in particular the Hörmander case. We develop some new notions and investigate their mutual relations and properties.

**Keywords:** Starshaped, starlike, convex, Heisenberg group, Carnot group, sub-Riemannian, Hörmander condition, vector fields.

## 1. INTRODUCTION.

The aim of this paper is to study some geometrical properties for sets in the context of degenerate geometries as Carnot groups or more general sub-Riemannian geometries. In particular we focus on starshapedness and convexity. In this setting convexity for functions has been extensively studied. In contrast with the Riemannian case, Monti and Rickly in [24] have proved that geodesic convexity is a bad notion for these more degenerate geometries: in fact in the Heisenberg group (that is the easiest but also the most significant example) the only geodesically convex functions are the constants. Danielli, Garofalo and Nhieu [10] and, independently, Lu, Manfredi and Stroffolini [21] (see also [20]) introduced the notion of horizontal convexity, named also H-convexity. This notion considers the scaling (or dilations) associated to the group along horizontal increments only (for a precise definition we refer to [10, 21]) and it turns out to be a suitable generalisation of the standard Euclidean convexity in the case of Carnot groups. Indeed, using horizontal convexity, many typical properties of standard convex functions has been retrieved, see e.g. [7, 10, 21, 20, 1, 11, 15, 18, 22, 25, 30]. On the other hand, this notion needs the Lie-group structure associated to a Carnot group and so it cannot be extended to more general sub-Riemannian geometries. A more general definition of convexity for functions in this setting has been developed by Bardi and Dragoni in [3, 4], by using the idea of convexity along vector fields. This second notion is equivalent to horizontal convexity in Carnot groups but applies to any given family of vector fields. Different from the case of convex functions, the literature about convex sets in the context of Carnot groups and sub-Riemannian geometries is very tiny. Some properties for horizontally convex sets in Carnot groups have been investigated by Danielli, Garofalo and Nhieu in [10]. In this paper we generalise the results therein to the geometry of vector fields by using the notion of  $\mathcal{X}$ -convexity developed in [3].

A notion which is deeply connected to convexity is *starshapedness*: in the Euclidean setting, a set  $\Omega \subset \mathbb{R}^N$  is said starshaped (or starlike) with respect to a point  $x_0 \in \Omega$  if for every  $x \in \Omega$  the whole segment  $(1 - t)x_0 + tx$ ,  $t \in (0, 1)$ , connecting  $x$  to  $x_0$ ,

is contained in  $\Omega$ . It is easily seen that a set is convex if and only if it is starshaped with respect to any one of its points. Then starshaped sets are surely interesting for their connections with convex sets, but also independently, since they provide a more general framework where some properties typically linked to convexity can be retrieved. For this reason they have been studied and used in many contexts, including PDEs, see for instance [14, 16, 17, 19, 27, 28, 26, 29]. In this paper we are interested in starshaped sets in Carnot groups, and also in the more general case of geometries of vector fields, including the Hörmander case as subcase.

Indeed, in this framework the notion of starshapedness is not very well-established. Some definitions and results on starshapedness for Carnot groups can be found in [8, 9, 12].

Here we consider two different notions of starshapedness in Carnot groups, namely respectively  $\mathbb{G}$ -starshapedness and weak starshapedness; the latter working also in the general case of geometries of vector fields. While the first definition has been introduced and then studied in the above papers, the second one is newly introduced in this work. We investigate their mutual relations and their relations with standard starshapedness, and some basic properties, which are useful when studying the geometry of solutions of PDEs, including their connections with horizontal convexity and  $\mathcal{X}$ -convexity.

The paper is organised as follow.

In Section 2 we recall some known properties of Euclidean starshaped sets.

In Section 3 we introduce Carnot groups and general sub-Riemannian geometries.

In Section 4 we study a first notion of starshapedness in Carnot groups, namely  $\mathbb{G}$ -starshapedness, by using the natural anisotropic scaling of the group (dilations).

We also look at the connections with horizontal convex sets.

In Section 5 we introduce a new notion of starshapedness in Carnot groups, namely weak  $\mathbb{G}$ -starshapedness, that applies also to more general sub-Riemannian geometries. This second notion is indeed equivalent to  $\mathcal{X}$ -convexity (and so to horizontal convexity in the particular case of Carnot group) whenever it holds true w.r.t. every internal point.

In Section 6 we investigate the relation between  $\mathbb{G}$ -starshaped sets, weakly starshaped sets and standard Euclidean starshaped sets.

In Section 7 we study some relations between  $\mathcal{X}$ -convex sets and  $\mathcal{X}$ -convex functions, so extending some results previously known in Carnot groups.

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## 2. STARSHAPED SETS IN THE EUCLIDEAN SPACE.

We recall some definitions and properties of starshaped sets in the Euclidean space  $\mathbb{R}^N$ . For more details and properties, we refer for instance to [29, 17, 26, 28].

**Definition 2.1.** A set  $\Omega \subseteq \mathbb{R}^N$  is said to be starshaped w.r.t. the point  $x_0 \in \Omega$  if for every  $x \in \Omega$  the line segment  $(1-t)x_0 + tx$ ,  $t \in [0, 1]$ , joining  $x_0$  to  $x$  is all contained in the set  $\Omega$ . We denote by  $\mathcal{S}_{x_0}$  the class of sets that are starshaped w.r.t.  $x_0$ . If  $\Omega \in \mathcal{S}_0$ , i.e. if  $x_0 = 0$ , we simply say that  $\Omega$  is starshaped.

Clearly, a starshaped set is connected; indeed it is simply connected. Moreover,  $\Omega$  is convex if and only if it is starshaped w.r.t. any one of its points. The class of starshaped sets is much larger than that one of convex sets; see Figure 1 for some examples. In general a union of convex sets is not convex but is starshaped w.r.t. all the points of the intersection. Thus the easiest way to construct sets not starshaped w.r.t. any internal point is to consider the union of 3 convex sets with empty intersection, as e.g. in Figure 2.

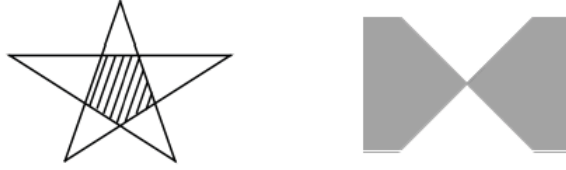


FIGURE 1. A star is starshaped w.r.t. every point in the marked region; a butterfly-like set is starshaped w.r.t. a single point.

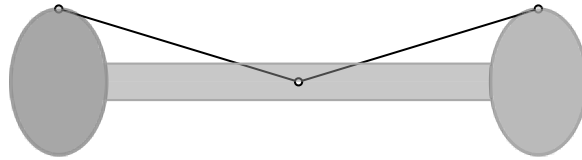


FIGURE 2. The dumbbell is not starshaped w.r.t. any point; e.g. the picture shows that it is not starshaped w.r.t. the centre.

Some other trivial properties are listed below: we denote by  $\overline{A}$  and  $A^\circ$  the closure and the interior of the set  $A$ , respectively and omit the proofs.

**Lemma 2.1.** Let  $A, B \subset \mathbb{R}^N$  and  $x_0 \in A \cap B$ . If  $A, B \in \mathcal{S}_{x_0}$  then:

- i)  $A \cap B \in \mathcal{S}_{x_0}$ ;
- ii)  $A \cup B \in \mathcal{S}_{x_0}$ ;
- iii)  $\overline{A} \in \mathcal{S}_{x_0}$ ;
- iv)  $A^\circ \in \mathcal{S}_{x_0}$  if and only if  $x_0 \in A^\circ$ .

To avoid problems possibly linked to the previous point (iv), we will usually consider *regular open* sets, i.e. sets that coincide with the interior of their closure.

Note that Definition 2.1 can be rewritten as

$$(1 - t)x_0 + t\Omega \subseteq \Omega, \quad \forall 0 \leq t \leq 1. \quad (1)$$

Whenever  $x_0 = 0$ , (1) can be clearly interpreted as a scaling property.

Another well-known characterisation is related to the direction of the normal at the boundary.

**Theorem 2.1.** Let  $\Omega \subseteq \mathbb{R}^N$  be a regular bounded open set with  $C^1$  boundary.

(i) If  $\Omega$  is starshaped w.r.t.  $x_0 \in \Omega$ , then

$$\langle n(x), x - x_0 \rangle \geq 0, \quad \forall x \in \partial\Omega, \quad (2)$$

where  $n(x)$  is the outer unit normal of  $\Omega$  at the point  $x$ .

(ii) Assume that the strict inequality in (2) holds true, then  $\Omega$  is starshaped w.r.t.  $x_0$ .

*Proof.* The result is well-known but we include a proof for completeness and for later generalisations. Fix a point  $x \in \partial\Omega$  and indicate by  $\gamma_{x_0,x}$  the segment line joining  $x_0$  to  $x$  in time 1, i.e.  $\gamma_{x_0,x} : [0, 1] \rightarrow \mathbb{R}^N$  defined as  $\gamma_{x_0,x}(t) = (x - x_0)t + x_0$ .

Note that, without loss of generality, we can assume that there exists some  $C^1$  function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\partial\Omega = \{x \in \mathbb{R}^N \mid u(x) < 0\}$  and  $\partial\Omega = \{x \in \mathbb{R}^N \mid u(x) = 0\}$  (otherwise we can argue locally). In this case:

$$n(x) = \frac{\nabla u}{|\nabla u|} \quad \text{and} \quad \dot{\gamma}_{x_0,x}(1) = x - x_0. \quad (3)$$

Now assume  $\Omega$  starshaped w.r.t.  $x_0$ , this means that  $\gamma_{x_0,x}(t) \in \Omega \quad \forall t \in [0, 1]$ , i.e.

$$\begin{cases} u(\gamma_{x_0,x}(t)) < 0, & t \in [0, 1], \\ u(\gamma_{x_0,x}(1)) = u(x) = 0. \end{cases} \quad (4)$$

Then

$$\left. \frac{d}{dt} u(\gamma_{x_0,x}(t)) \right|_{t=1} = \lim_{t \rightarrow 1^-} \frac{u(\gamma_{x_0,x}(t)) - u(\gamma_{x_0,x}(1))}{t - 1} \geq 0. \quad (5)$$

On the other hand, using Equation (3) we find

$$\left. \frac{d}{dt} u(\gamma_{x_0,x}(t)) \right|_{t=1} = \langle \nabla u(x), \dot{\gamma}_{x_0,x}(1) \rangle = |\nabla u(x)| \langle n(x), x - x_0 \rangle. \quad (6)$$

Combining (5) and (6) we get (2).

To prove part (ii) we assume the strict inequality in (2). Then by (6) we deduce that there exists some  $\eta \in (0, 1]$  such that

$$u(\gamma_{x_0,x}(t)) < 0, \quad \forall t \in (1 - \eta, 1),$$

which means  $\gamma_{x_0,x}(t) \in \Omega, \quad \forall t \in (1 - \eta, 1)$ . Define

$$\bar{t} := \min \{t \in [0, 1] \mid \gamma_{x_0,x}(t) \in \Omega \quad \forall t \in (t, 1)\}.$$

Note that such a minimum exists by continuity and also  $\bar{t} \neq 1$  since  $\bar{t} \leq 1 - \eta < 1$ . If  $\bar{t} = 0$  we can conclude that  $\Omega$  is starshaped w.r.t.  $x_0$ . So let's assume that  $\bar{t} \in (0, 1)$ . In this case obviously

$$\bar{x} := \gamma_{x_0,x}(\bar{t}) \in \partial\Omega,$$

then we can apply the strict inequality in (2) at  $\bar{x}$ . Arguing as above we can find  $\tilde{\eta} \in (0, 1]$  such that  $\gamma_{x_0, \bar{x}}(t) \in \Omega$ ,  $\forall t \in (1 - \tilde{\eta}, 1)$ . On the other hand by using the structure of (Euclidean) lines and the fact that  $\Omega$  is a regular open set, this means that  $\gamma_{x_0, x}(t) \in \Omega$  for  $t \in (\bar{t} - \tilde{\eta}, 1)$ , which contradicts the minimality of  $\bar{t}$ .  $\square$

The following remark is essential to understand the large generality of the proof above.

**Remark 2.1.** In the above proof we can actually replace the segment line  $\gamma_{x_0, x}$  with any other family of  $C^1$  curves joining  $x_0$  to  $x$  in time 1 as soon as the following rescaling property is satisfied:

$$\dot{\gamma}_{x_0, x}(\bar{t}) = C \dot{\gamma}_{x_0, \bar{x}}(1), \quad \text{with } \bar{x} = \gamma_{x_0, x}(\bar{t}) \quad \text{and for some } C = C(\bar{t}) > 0. \quad (7)$$

In fact assumption (7) ensures that

$$\langle n(\bar{x}), \dot{\gamma}_{x_0, \bar{x}}(1) \rangle > 0 \implies \left. \frac{d}{dt} u(\gamma_{x_0, \bar{x}}(t)) \right|_{t=\bar{t}} < 0.$$

Obviously (7) is trivially true for (Euclidean) lines since  $\dot{\gamma}_{x_0, \bar{x}}(1) = (\bar{x} - x_0) = \bar{t}(x - x_0) = \bar{t}\dot{\gamma}_{x_0, x}(\bar{t})$  but this rescaling property can be easily checked in other families of curves which will be crucial in our later generalisations.

### 3. CARNOT GROUPS AND SUB-RIEMANNIAN GEOMETRIES.

Carnot groups are stratified non-commutative Lie groups and the Heisenberg group is the most famous example of a Carnot group. We briefly recall the main definitions and some properties about Carnot groups. For more details we refer to [6].

**Definition 3.1** (Carnot group). *A Carnot group is a simply connected nilpotent Lie group  $(\mathbb{G}, \circ)$  whose algebra  $\mathfrak{g}$  is stratified, i.e.*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k,$$

for some natural number  $k \geq 1$  and  $\mathfrak{g}_r = \{0\}$  for all  $r > k$ . Moreover  $\mathfrak{g}_i = [\mathfrak{g}_1, \mathfrak{g}_{i-1}]$  for all  $i = 2, \dots, k$ . The natural number  $k$  is called the step of the Carnot group.  $(\mathbb{G}, \circ)$  is non-commutative whenever  $k \geq 2$ .

Recall that  $\mathfrak{g}_1$  is the generator of the Lie algebra  $\mathfrak{g}$ . In general we indicate by  $N$  the dimension of  $\mathfrak{g}$  (which is equal to the dimension of  $\mathbb{G}$ ) while we indicate by  $m_i$  the dimension of each layer  $\mathfrak{g}_i$  for  $i = 1, \dots, k$ ; hence  $m_1 + \cdots + m_k = N$ . We usually write  $m = m_1$ .

Carnot groups are equipped with a family of automorphisms of the group (namely *dilations*), defined by

$$\delta_t \left( \exp \left( \sum_{i=1}^k \sum_{j=1}^{m_i} g_{j,i} X_{j,i} \right) \right) = \exp \left( \sum_{i=1}^k \sum_{j=1}^{m_i} t^i g_{j,i} X_{j,i} \right), \quad (8)$$

where  $\exp$  is the exponential map (see [6] for definition and properties) and  $g_{i,j}$  are the exponential coordinates of the point  $g \in \mathbb{G}$ , that is:

$g = \exp \left( \sum_{i=1}^k \sum_{j=1}^{m_i} g_{j,i} X_{j,i} \right)$  and  $X_{j,i}$  for  $j = 1, \dots, m_i$  are a basis for  $\mathfrak{g}_i$  with  $i = 1, \dots, k$ .

We also like to recall the following lemma.

**Lemma 3.1** ([6], Proposition 2.2.22). If  $\mathbb{G}$  is a simply connected nilpotent Lie group, then it is isomorphic to  $\mathbb{R}^N$  (so we indicate every element  $g$  by the corresponding point  $x$  in  $\mathbb{R}^N$ ) with a polynomial multiplication law  $(x^0, x) \rightarrow x^0 \circ x$  whose identity is 0 and inverse is  $x \rightarrow -x$ .

Then we identify  $\mathbb{G}$  with the triplet  $(\mathbb{R}^N, \circ, \delta_t)$ .

By the definition of dilations one can deduce the following properties (see e.g. [6]).

**Lemma 3.2** (Properties of dilations). Given a Carnot group  $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_t)$ , for all  $x, y$  and for all  $t, t_1, t_2$  the following properties hold:

- (1)  $\delta_1(x) = x$ ,
- (2)  $\delta_t(x) = (\delta_{1/t})^{-1}(x)$ ,
- (3)  $\delta_{t_1}(\delta_{t_2}(x)) = \delta_{t_1 t_2}(x)$ ,
- (4)  $\delta_t(x) \circ \delta_t(y) = \delta_t(x \circ y)$ ,
- (5) whenever  $k \geq 2$ , then  $\delta_{t_1}(x) \circ \delta_{t_2}(x) \neq \delta_{t_1+t_2}(x)$ .

We are interested in the non-commutative case (i.e.  $k \geq 2$ ), then in general the left-translations and the right-translations are different. We consider only *left-translations*, defined,  $\forall x \in \mathbb{R}^N$ , by the function  $L_x : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with

$$L_x(y) = x \circ y.$$

Carnot groups can be endowed by a distance and a norm defined in line with the stratification of the Lie algebra, as follows.

**Definition 3.2** (Homogeneous norm and homogeneous distance). A *homogeneous norm*  $\|\cdot\|_h$  is a continuous function from  $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$  to  $[0, +\infty)$  such that

$$\begin{aligned} \|x\|_h &= 0 \iff x = 0, \\ \|x^{-1}\|_h &= \|x\|_h, \forall x \in \mathbb{R}^N \\ \|\delta_\lambda(x)\|_h &= \lambda \|x\|_h, \forall x \in \mathbb{R}^N, \lambda > 0, \\ \|x \circ y\|_h &\leq \|x\|_h + \|y\|_h, \forall x, y \in \mathbb{R}^N. \end{aligned}$$

The induced homogenous distance between two given points  $x, y \in \mathbb{R}^N$  is defined by

$$d_h(x, y) = \|y^{-1} \circ x\|_h = \|(-y) \circ x\|_h.$$

Explicitly the homogenous norm can be written as:

$$\|x\|_h = \sum_{i=1}^k \|\underline{x}^i\|_{m_i}^{\frac{1}{2i}}, \quad (9)$$

where  $x = (\underline{x}^1, \cdot, \underline{x}^k)$ ,  $\exp^{-1}(\underline{x}^i) \in g_i$  and  $\|\underline{x}^i\|_{m_i}$  is the standard Euclidean norm in  $\mathbb{R}^{m_i}$ .

**Example 3.1** (The Heisenberg group.). The  $n$ -dimensional Heisenberg group  $\mathbb{H}^n$  is a Carnot group of step  $k = 2$  defined on  $\mathbb{R}^{2n+1}$ . Here  $N = 2n + 1$  and  $m = 2n$ . The group operation is  $x \circ y = \left(x_1 + y_1, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{x^2 \cdot y^1 - x^1 \cdot y^2}{2}\right)$ , where  $x = (\underline{x}^1, \underline{x}^2, x_{2n+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  and  $\underline{x}^1 = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\underline{x}^2 = (x_{n+1}, \dots, x_{2n}) \in \mathbb{R}^n$ . The dilations are given by  $\delta_t(x) = (tx_1, \dots, tx_{2n}, t^2 x_{2n+1})$ , and the homogenous norm (9) can be written as  $\|x\|_h = \left(\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right)^{1/4}$ .

Since Carnot groups are Lie groups, we can associate to them also a manifold structure: in fact  $\mathfrak{g}$  can be identified with the tangent space at the identity element  $e$ , thus  $\mathfrak{g}_1$  is an  $m$ -dimensional subspace at that point. By left translations we define an associated distribution by  $\exp^{-1}(x \circ \mathcal{G}_1)$  where  $\mathcal{G}_1 = \exp(\mathfrak{g}_1)$ , at any other point of the group (see [6] for more details). Choose  $X_1, \dots, X_m$  as basis of left-invariant vector fields for  $\mathfrak{g}_1$ , then the distribution spanned at any point by  $X_1, \dots, X_m$  is usually called *horizontal tangent space*. For more definitions and properties on left-invariant vector fields in Carnot groups we refer to [6], Section 2.17, Definition 2.1.41. Note that those vector fields satisfy the Hörmander condition with constant step  $k$  (namely the distribution is bracket generating), that means that

$$\text{Span}(\mathcal{L}(X_1, \dots, X_m)(x)) = T_x \mathbb{G}, \quad \forall x \in \mathbb{G}, \quad (10)$$

where  $\mathcal{L}(X_1, \dots, X_m)(x)$  is the Lie algebra (the set of all commutators) associated to the given family of vector fields at the point  $x$ .

**Example 3.2.** In the case of the Heisenberg group (Example 3.1) the left-invariant vector fields spanning  $\mathfrak{g}_1$  are

$$\begin{aligned} X_i(x) &= e_i - \frac{x_{n+i}}{2} e_{2n+1}, \\ X_{n+i}(x) &= e_{n+i} + \frac{x_i}{2} e_{2n+1}, \end{aligned} \quad \text{for } i = 1, \dots, n, \quad (11)$$

while  $X_{2n+1} = e_{2n+1}$  spans  $\mathfrak{g}_2$  with  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , and thus it is trivial to check the Hörmander condition with step 2; in fact  $[X_1(x), X_{2n+1}(x)] = e_{2n+1}$ , at any  $x$ .

Note that condition (10) is independent on the group structure and can be introduced in every manifold. This is the key idea behind sub-Riemannian manifolds.

**Definition 3.3** (Sub-Riemannian manifolds). *Given an  $N$ -dimensional manifold  $M$  and a family of vector fields  $\mathcal{X} = \{X_1, \dots, X_m\}$  defined on  $M$  satisfying the Hörmander condition (10), one can induce a Riemannian metric  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  on the associated distribution  $\mathcal{H}_x = \text{Span}(X_1(x), \dots, X_m(x))$ , for more details we refer to [23]. The triple  $(M, \mathcal{H}_x, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  is called sub-Riemannian manifold or sub-Riemannian geometry.*

On sub-Riemannian geometries which are not Carnot groups, it is in general not possible to define dilations, translations and the homogenous norm and distance. Still these geometries are metric spaces with the associated Carnot-Carathéodory distance (see e.g. [5, 23] for definitions and properties). A key notion in this setting is the one of admissible, namely horizontal, curves. Any absolutely continuous  $x : [0, 1] \rightarrow \mathbb{R}^N$  is horizontal whenever  $\exists \alpha_1, \dots, \alpha_m$  measurable such that

$$\dot{x}(t) = \sum_{i=1}^m \alpha_i(t) X_i(x(t)), \quad \text{a.e. } t \in (0, 1). \quad (12)$$

The  $m$ -valued measurable function  $\alpha = (\alpha_1, \dots, \alpha_m)$  is called *horizontal velocity*.

**Example 3.3.** The easiest example of a sub-Riemannian manifold which is not (globally) associated to any Carnot group is  $\mathbb{R}^2$  (with the usual topology of open sets) with the vector fields  $X_1(x_1, x_2) = (1, 0)^t$  and  $X_2(x_1, x_2) = (0, x_1)^t$  for all  $(x_1, x_2) \in \mathbb{R}^2$ , with the Riemannian metric induced by the Euclidean metric on the space generated by  $X_1$  and  $X_2$ . Such a geometry is known as the Grushin plane.

For more properties and examples on sub-Riemannian manifolds we refer to [23].

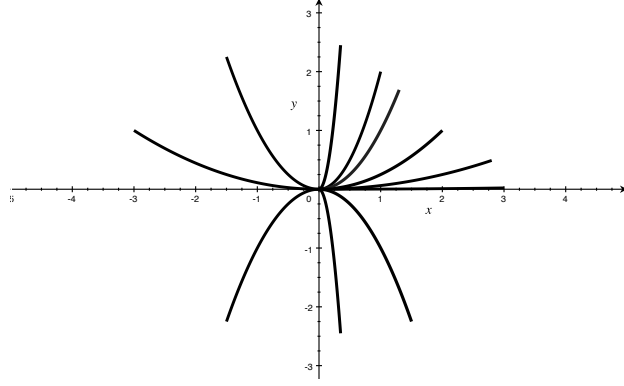


FIGURE 3. The dilation curves  $\gamma(t; x) = (tx_1, tx_2, t^2x_3)$  in  $\mathbb{H}^1$ , represented on the projection plane  $x_2 = 0$ .

#### 4. $\mathbb{G}$ -STARSHAPED SETS.

In every Carnot group  $\mathbb{G}$  we can define a notion of starshapedness in line with the Euclidean definition given by (1).

**Definition 4.1.** Consider a Carnot group  $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_t)$  and  $\Omega \subseteq \mathbb{R}^N$ , we say that  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. a point  $x_0 \in \Omega$  if and only if

$$x_0 \circ \delta_t(-x_0 \circ \Omega) \subset \Omega, \quad \forall t \in [0, 1]. \quad (13)$$

Note that by the Lemma 3.1 we know that  $x_0^{-1} = -x_0$ . The above definition gives back the standard Euclidean definition whenever  $\delta_t(x) = tx$  and  $x \circ y = x + y$ .

If  $x_0 = 0$  then (13) becomes simply  $\delta_t(\Omega) \subset \Omega$ ; thus it is evident as Definition 4.1 is actually a scaling property w.r.t. the natural scaling induced by the dilations on a Carnot group.

For a generic  $x_0$ , we can consider the left-translated set  $\Omega' = L_{-x_0}(\Omega) = -x_0 \circ \Omega$ , then (13) is equivalent to requiring  $\Omega'$   $\mathbb{G}$ -starshaped w.r.t. 0 (note that  $x_0 \in \Omega \Rightarrow 0 \in \Omega'$ ). Then Definition 4.1 is coherent with the Lie group structure of  $\mathbb{G}$ .

**Remark 4.1.** Unlike the standard Euclidean case, in Carnot groups in general  $x_0 \circ \delta_t(-x_0) \neq \delta_{1-t}(x_0)$ .

Next we give some easy examples.

**Example 4.1.** Consider the gauge ball  $B_R^h(0) := \{x \in \mathbb{R}^n \mid \|x\|_h < R\} \subseteq \mathbb{R}^N$ , where  $\|x\|_h$  is given by (9). By using  $\|\delta_t(x)\|_h = t\|x\|_h$  an easy computation shows that  $\delta_t(B_R^h(0)) = B_{tR}^h(0)$ , which trivially implies  $B_R^h(0)$   $\mathbb{G}$ -starshaped w.r.t. 0. In the same way  $B_R^h(x_0)$  is  $\mathbb{G}$ -starshaped w.r.t. a generic centre  $x_0$ .

**Remark 4.2** ( $\mathbb{G}$ -starshaped in the Heisenberg group). In the particular case of the 1-ional Heisenberg group  $\mathbb{H}^1$ , Definition 4.1 with  $x_0 = 0$  means that, for all  $x = (x_1, x_2, x_3) \in \Omega$ , the curve  $\gamma(t; x) = (tx_1, tx_2, t^2x_3)$  is contained in  $\Omega$  for all  $t \in [0, 1]$ . For any fixed  $x$ ,  $\gamma(t; x)$  are parabolas around the  $x_3$ -axis starting at the origin, passing trough the point  $x$  (see Figure 3). Note that the parabolas degenerate into Euclidean straight lines on the plane  $x_3 = 0$ . Moreover they become straight segment also on the  $x_3$ -axis but with velocity  $2x_3t$  (instead of constant



velocity as in the Euclidean case). Using these parabolic curves, it is easy to build plenty of examples of  $\mathbb{G}$ -starshaped sets.

In Section 6 we will give additional examples of  $\mathbb{G}$ -starshaped sets in the Heisenberg group; in particular we will give examples of  $\mathbb{G}$ -starshaped sets which are not starshaped in the standard Euclidean sense.

In the classic Euclidean setting, one of the characterisations for starshaped sets is known to be related to the normal at the boundary (see Theorem 2.1). An analogous characterisation can be proved in Carnot groups by using the infinitesimal generator induced by the anisotropic dilations (8), i.e.

$$\Gamma(x) := \left. \frac{d}{dt}(\delta_t(x)) \right|_{t=1}. \quad (14)$$

For example in the Euclidean case  $\Gamma(x) = x$  while in the  $n$ -dimensional Heisenberg group:

$$\Gamma(x) = \Gamma(x_1, \dots, x_{2n}, x_{2n+1}) = (x_1, \dots, x_{2n}, 2x_{2n+1}).$$

In general, by using the structures of dilations in Carnot groups, we have:

$$\delta_t(x) = (tx_1, \dots, tx_m, t^{\alpha_{m+1}}x_{m+1}, \dots, t^{\alpha_N}x_N),$$

with  $\alpha_{m+1}, \dots, \alpha_N \geq 1$  natural numbers, that implies

$$\Gamma(x) = (x_1, \dots, x_m, \alpha_{m+1}x_{m+1}, \dots, \alpha_N x_N).$$

More in general, for any given  $x_0 \in \mathbb{R}^N$ , we define

$$\Gamma(x; x_0) := \left. \frac{d}{dt} \left( x_0 \circ \delta_t(-x_0 \circ x) \right) \right|_{t=1}. \quad (15)$$

**Theorem 4.1.** Let us consider a Carnot group  $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_t)$  and  $\Omega \subseteq \mathbb{R}^N$  open, regular and bounded with  $C^1$ -boundary and a point  $x_0 \in \Omega$ .

(i) If  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t.  $x_0$ , then

$$\langle \Gamma(x, x_0), n(x) \rangle \geq 0, \quad \forall x \in \partial\Omega, \quad (16)$$

where  $\Gamma(x)$  is the infinitesimal generator of the dilations defined in (15) and  $n(x)$  is the outer unit normal of  $\Omega$  at the point  $x$ .

(ii) Assume that the strict inequality in (16) holds true, then  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. the  $x_0$ .

*Proof.* A proof of this result can be found in [9] (see also [12]). Here we prove the result simply by checking the rescaling property (7) for the smooth curves

$$\gamma_{x_0, x}(t) := x_0 \circ \delta_t(-x_0 \circ x) = x_0 \circ \delta_t(-x_0) \circ \delta_t(x);$$

then the proof of the Euclidean result applies (see Theorem 2.1 plus Remark 2.1). To this purpose, by using  $\delta_t(x \circ y) = \delta_t(x) \circ \delta_t(y)$ , we first compute:

$$\gamma_{x_0, \bar{x}}(t) = x_0 \circ \delta_t(-x_0) \circ \delta_t \left( x_0 \circ \delta_{\bar{t}}(-x_0) \circ \delta_{\bar{t}}(x) \right) = x_0 \circ \delta_{\bar{t}t}(-x_0) \circ \delta_{\bar{t}t}(x).$$

Thus setting  $s = \bar{t}t$ , we get:

$$\dot{\gamma}_{x_0, \bar{x}}(1) = \left. \frac{d}{dt} \left( x_0 \circ \delta_t(-x_0) \circ \delta_t(\bar{x}) \right) \right|_{t=1} = \left. \frac{d}{ds} \left( x_0 \circ \delta_s(-x_0) \circ \delta_s(x) \right) \right|_{s=\bar{t}} \frac{ds}{dt} = \bar{t} \dot{\gamma}_{x_0, x}(\bar{t}).$$

□

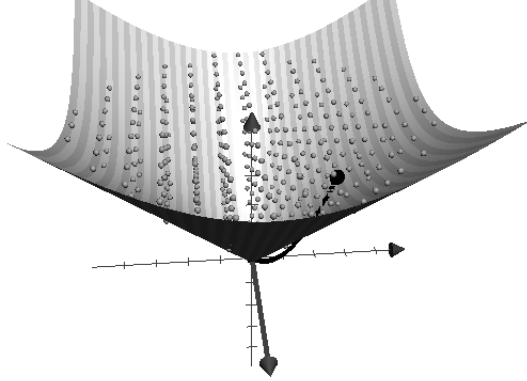


FIGURE 4. A Euclidean convex set (hence also horizontally convex in the Heisenberg group) which is not  $\mathbb{G}$ -starshaped in  $\mathbb{H}^1$  w.r.t the origin. As one can see the dilation curve joining the origin with the internal point  $(\frac{13}{10}, 1, 2)$  goes outside the set (see Example 4.2).

As in the standard Euclidean case, one can approximate sets by  $\mathbb{G}$ -starshaped sets.

**Definition 4.2** ( $\mathbb{G}$ -starshaped hull). *Given an open subset  $\Omega \subseteq \mathbb{G}$ , we define the  $\mathbb{G}$ -starshaped hull of the set  $\Omega$  w.r.t. a given point  $x_0$ , and we simply indicate by  $\Omega_{x_0}^{*\mathbb{G}}$ , the set given by the intersection of all the sets which are  $\mathbb{G}$ -starshaped w.r.t.  $x_0$  and contain  $\Omega$ . This can be written as*

$$\Omega_{x_0}^{*\mathbb{G}} = \bigcup_{t \in [0,1]} (x_0 \circ \delta_t(-x_0) \circ \delta_t(\Omega)), \quad (17)$$

that simply writes  $\Omega^{*\mathbb{G}} = \bigcup_{t \in [0,1]} \delta_t(\Omega)$  in the case  $x_0 = 0$ .

**Remark 4.3.** Trivially, given an open subset  $\Omega \subseteq \mathbb{R}^N$ ,  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t.  $x_0$  if and only if  $\Omega = \Omega_{x_0}^{*\mathbb{G}}$ .

In the Euclidean setting, it is known that starshapedness w.r.t. each internal point is equivalent to convexity. This is not anymore true in the case of  $\mathbb{G}$ -starshapedness and horizontal convexity. Horizontally convex sets (namely also  $H$ -convex sets) have been introduced and studied in [10]. In Section 6 we will show that  $\mathbb{G}$ -starshapedness w.r.t. each internal point implies horizontal convexity (see Proposition 6.2) while the opposite implication is in general false. It is actually quite easy to construct counterexamples already in the Heisenberg group  $\mathbb{H}^1$ , as we see in the next two examples. First we need to recall that all the Euclidean convex sets are in particular also horizontally convex in  $\mathbb{H}^1$ : this can be easily shown by using the equivalence of horizontal convexity with  $v$ -convexity (see [21, 20]) or by using the equivalence of horizontal convexity with  $\mathcal{X}$ -convexity (see [3, 4]).

**Example 4.2.** Consider the cone  $\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > \sqrt{x_1^2 + x_2^2} - \frac{1}{10} \right\}$  (see Figure 4), then  $\Omega$  is convex in the standard Euclidean sense so it is also horizontally convex in  $\mathbb{H}^1$  but it is not  $\mathbb{G}$ -starshaped w.r.t.  $(0, 0, 0) \in \Omega$ . In fact given the point  $p = (\frac{13}{10}, 1, 2) \in \Omega$ , the corresponding dilation curve  $\delta_t(p) = (\frac{13}{10}t, t, 2t^2)$  does not all belong to  $\Omega$  for  $t \in (0, 1)$ , e.g. for  $t = \frac{1}{4}$ .

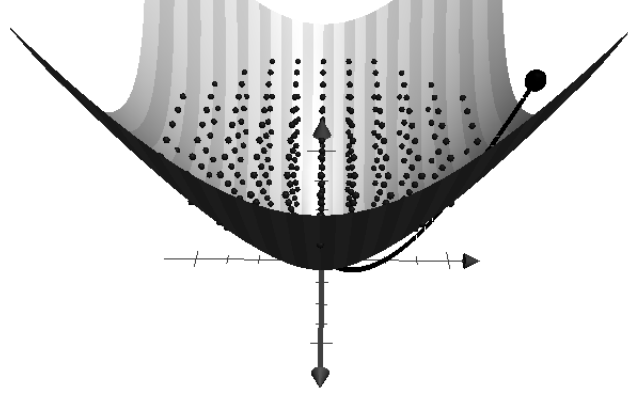


FIGURE 5. An smooth and strictly Euclidean convex set (hence also horizontally convex) which is not  $\mathbb{G}$ -starshaped in  $\mathbb{H}^1$  w.r.t. the origin (see Example 4.3).

**Example 4.3.** This example is very similar to the previous one but in this case the set does not have corners at the boundary and it is also strictly Euclidean convex. In fact, consider  $\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > (x_1^2 + x_2^2)^{\frac{3}{4}} - 10^{-4} \right\}$  (see Figure 5), then  $\Omega$  has a smooth boundary and it is strictly Euclidean convex and therefore (strictly) horizontally convex in  $\mathbb{H}^1$  but it is not  $\mathbb{G}$ -starshaped w.r.t. the origin. This can be proved using the same point given in the previous example.

Very interesting examples of horizontally convex sets, which are not  $\mathbb{G}$ -starshaped w.r.t. to an open set of internal points, have been given by Danielli and Garofalo in [8]). In that work the authors proved that the homogeneous ball in the Heisenberg group is not  $\mathbb{G}$ -starshaped w.r.t. a open subset of internal points, located near the characteristic points.

From what we have remarked above, we can also deduce another difference between this notion and the standard Euclidean notion. In fact, while exactly as in the standard Euclidean case, the union of  $\mathbb{G}$ -starshaped sets w.r.t. the same point  $x_0$  is still trivially  $\mathbb{G}$ -starshaped, the union instead of two horizontal convex sets could be not anymore  $\mathbb{G}$ -starshaped w.r.t. all the points of the intersection, since horizontally convex sets are not anymore necessarily  $\mathbb{G}$ -starshaped w.r.t. all internal points. In the next section we introduce a different definition of starshapedness in Carnot groups, that can be generalised also to more general sub-Riemannian geometries and characterises horizontal convexity.

## 5. WEAKLY STARSHAPED SETS.

In the previous section we have studied a notion of starshapedness related to the natural scaling defined on Carnot groups. In this section we introduce a new notion of starshapedness related to more geometrical curves. We show that this newly introduced notion is weaker than the previous one but can be applied to the very general setting of geometries of vector fields and so in particular the Hörmander case. Moreover this notion trivially characterises  $\mathcal{X}$ -convexity (see [3]), thus it characterises also horizontal convexity in the case of Carnot groups. We start

recalling the notion of  $\mathcal{X}$ -lines, which are horizontal curves with constant horizontal velocity (recall the curves defined in (12)). More precisely.

**Definition 5.1.** *Given  $\mathcal{X} = \{X_1, \dots, X_m\}$  family of vector fields on  $\mathbb{R}^N$ , we call  $\mathcal{X}$ -line any absolutely continuous curve  $x : [0, 1] \rightarrow \mathbb{R}^N$  satisfying*

$$\dot{x}(t) = \sum_{i=1}^m \alpha_i X_i(x(t)), \quad \text{a.e. } t \in [0, 1], \quad (18)$$

for some  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ . The vector  $\alpha$  is the constant horizontal velocity.

We usually indicate a  $\mathcal{X}$ -line corresponding to the horizontal constant velocity  $\alpha$  by  $x^\alpha$ . We will always assume that the vector fields are at least  $C^1$ . Therefore, by standard results for ODEs, we know that, on bounded domains in  $\mathbb{R}^N$ , for all  $\alpha \in \mathbb{R}^m$  there exists a unique  $x^\alpha$  starting from a given point  $x_0$  at time  $t = 0$ ; we indicate such a curve by  $x_{x_0}^\alpha$ .

Note that  $\mathcal{X}$ -lines have at least the same regularity of the vector fields; thus in the case of Carnot groups they are smooth.

**Remark 5.1** (Matrix associated to the vector fields). Later we will also often use the matrix  $\sigma(x)$  associated to  $\mathcal{X}$ , that is defined as the  $N \times m$ -matrix whose columns are the vectors  $X_1(x), \dots, X_m(x)$ , at every point  $x \in \mathbb{R}^N$ . In this notation, the  $\mathcal{X}$ -line associated to the constant horizontal velocity  $\alpha$  can be written as

$$\dot{x}^\alpha(t) = \sigma(x^\alpha(t)) \alpha.$$

**Example 5.1.** In  $\mathbb{H}^1$  (see Example 3.1), an easy computation shows that the  $\mathcal{X}$ -line corresponding to constant horizontal velocity  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$  is the (Euclidean) straight line  $x_{x_0}^\alpha(t) = (x_1^0 + \alpha_1 t, x_2^0 + \alpha_2 t, x_3^0 + \frac{x_1^0 \alpha_2 - x_2^0 \alpha_1}{2} t)$ .

**Example 5.2.** Given the Grushin plane defined in Example 3.3, the  $\mathcal{X}$ -line corresponding to constant horizontal velocity  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$  is the parabolic curve  $x_{x_0}^\alpha(t) = (x_1^0 + \alpha_1 t, x_2^0 + \alpha_2 x_1^0 t + \frac{\alpha_1 \alpha_2}{2} t^2)$ .

Next we introduce the  $\mathcal{X}$ -plane w.r.t. a given point  $x_0$ , which is the set of all the points which can be connected to  $x_0$  through a  $\mathcal{X}$ -line, i.e.

$$\mathbb{V}_{x_0} := \{y \in \mathbb{R}^N \mid \exists \alpha \in \mathbb{R}^m \text{ and } x^\alpha \mathcal{X}\text{-line such that } x^\alpha(0) = x_0, x^\alpha(1) = y\}. \quad (19)$$

For more examples and properties for  $\mathcal{X}$ -lines and  $\mathcal{X}$ -planes, we refer to [3, 4].

**Definition 5.2** (Weakly starshaped set). *Given a family of  $C^1$  vector fields  $\mathcal{X} = \{X_1, \dots, X_m\}$  defined on  $\mathbb{R}^N$  (with in general  $m \leq N$ ), we say that  $\Omega \subseteq \mathbb{R}^N$  is weakly starshaped w.r.t. some point  $x_0 \in \Omega$  whenever for all  $y \in \mathbb{V}_{x_0} \cap \Omega$  the  $\mathcal{X}$ -segment joining  $x_0$  to  $y$  belongs to  $\Omega$ , i.e.  $x^\alpha(t) \in \Omega$ , for all  $t \in [0, 1]$  and for all  $\alpha \in \mathbb{R}^m$ , where  $x^\alpha$  is the  $\mathcal{X}$ -line defined in (18) with  $x^\alpha(0) = x_0$  and  $x^\alpha(1) = y$ .*

Next we show that the above definition characterises  $\mathcal{X}$ -convexity. A definition of  $\mathcal{X}$ -convex sets can be easily introduced following the ideas in [3].

**Definition 5.3** ( $\mathcal{X}$ -convex sets). *Given a family  $\mathcal{X} = \{X_1, \dots, X_m\}$  of  $C^1$  vector fields defined on  $\mathbb{R}^N$  (with in general  $m \leq N$ ) we say that  $\Omega \subset \mathbb{R}^N$  is  $\mathcal{X}$ -convex if*

$$x_{x_0}^\alpha(t) \in \Omega, \quad \forall t \in (0, 1), \forall x_0 \in \Omega \text{ and } \forall y \in \Omega \cap \mathbb{V}_{x_0},$$

where  $x_{x_0}^\alpha$  is the  $\mathcal{X}$ -segment joining  $x_0$  to  $y$  at time  $t = 1$ .

In Section 7 we will study some properties of  $\mathcal{X}$ -convex sets. In this section we only consider their relations with weakly starshaped sets.

**Lemma 5.1.** Given any family of vector fields  $\mathcal{X} = \{X_1, \dots, X_m\}$  defined on  $\mathbb{R}^N$  (with in general  $m \leq N$ ), a set  $\Omega \subseteq \mathbb{R}^N$  is  $\mathcal{X}$ -convex if and only if  $\Omega$  is weakly starshaped w.r.t. every point  $x_0 \in \Omega$ .

*Proof.* The proof is trivial. In fact a function is  $\mathcal{X}$ -convex if and only if it is convex in  $t$  along all the  $\mathcal{X}$ -lines starting from every  $x_0 \in \Omega$  (see [3]), then the result follows immediately.  $\square$

**Remark 5.2** (Weakly starshaped sets in Carnot groups). Whenever we have a Carnot group  $\mathbb{G}$ , we can consider the family of left-invariant vector fields  $\mathcal{X} = \{X_1, \dots, X_m\}$  associated to the first layer of the Lie algebra  $\mathfrak{g}$ ; thus the definition of weakly starshaped applies in particular also to Carnot groups.

By using the equivalence between  $\mathcal{X}$ -convex functions and horizontally convex functions in Carnot groups (see e.g. Lemma 4.1 in [3]) we have the following lemma.

**Lemma 5.2.** Given a Carnot group  $\mathbb{G}$ , a set  $\Omega \subseteq \mathbb{G} \equiv \mathbb{R}^N$  is horizontally convex (namely also H-convex) if and only if  $\Omega$  is weakly starshaped w.r.t. every point  $x_0 \in \Omega$ , where  $\mathcal{X} = \{X_1, \dots, X_m\}$  is the family of left-invariant vector fields associated to the first layer of the Lie algebra  $\mathfrak{g}$ .

We want to highlight that, unlike the associated  $\mathcal{X}$ -convexity where the possibility to vary the starting point  $x_0$  gives a good control of the set in all directions, the definition of weakly starshaped gives a sort of control of the behaviour of the set only at the points of the set which belong to  $\mathbb{V}_{x_0}$ . For example in  $\mathbb{H}^1$ , taking  $x_0 = 0$ , we have no control of the set in the vertical direction (i.e. in the direction of  $x_3$ ).

Also for weakly starshaped sets we can deduce a characterisation at the boundary.

**Theorem 5.1.** Given a family of  $C^1$  vector fields  $\mathcal{X} = \{X_1, \dots, X_m\}$  defined on  $\mathbb{R}^N$  (with  $m \leq N$ ), consider a regular open bounded set  $\Omega \subset \mathbb{R}^N$  with  $C^1$ -boundary and a point  $x_0 \in \Omega$ .

- (i) If  $\Omega$  is weakly starshaped w.r.t.  $x_0$ , then

$$\langle \sigma(x) \alpha_x, n(x) \rangle \geq 0, \quad \forall x \in \mathbb{V}_{x_0} \cap \partial\Omega, \quad (20)$$

where  $\sigma(x)$  is the matrix associated to the vector fields (see Remark 5.1),  $n(x)$  is the standard (Euclidean) outer unit normal of  $\Omega$  at the point  $x$ , and  $\alpha_x \in \mathbb{R}^m$  is the unique constant horizontal velocity such that the corresponding  $\mathcal{X}$ -line starting from  $x_0$  reaches  $x$  at time  $t = 1$ .

- (ii) Assume that the strict inequality in (20) holds true, then  $\Omega$  is weakly starshaped w.r.t.  $x_0$ .

*Proof.* To prove the result we first remark that  $\mathcal{X}$ -lines are  $C^1$  since the vector fields are assumed to be  $C^1$ . Thus if we prove that the rescaling property (7) is satisfied, then the proof of the Euclidean result applies (see Theorem 2.1 plus Remark 2.1). To check (7) we need only to show that, for all  $\bar{t} \in (0, 1)$  and  $\bar{x} = x_{x_0}^{\alpha_x}(\bar{t})$ , we have

$$\gamma_{x_0, \bar{x}}(t) := x_{x_0}^{\alpha_{\bar{x}}}(t) = x_{x_0}^{\bar{\alpha}}(t), \quad \text{with } \bar{\alpha} = \bar{t} \alpha_x. \quad (21)$$

Claim (21) trivially implies

$$\dot{\gamma}_{x_0, \bar{x}}(1) = \sigma(\gamma_{x_0, \bar{x}}(1))(\bar{\alpha}) = \bar{t} \sigma(\bar{x}) \alpha_x = \bar{t} \dot{\gamma}_{x_0, x}(\bar{t}).$$

To check (21) it is not hard: in fact, set for simplicity  $y(s) := x_{x_0}^{\alpha\bar{\tau}}(\bar{t}s)$ , then  $y(0) = x_0$  and

$$\dot{y}(s) = \bar{t} \dot{x}_{x_0}^{\alpha}(\bar{t}s) = \bar{t} \sigma(\dot{x}_{x_0}^{\alpha}(\bar{t}s)) = \sigma(y(s))\bar{\alpha}.$$

By uniqueness for ODEs with  $C^1$  coefficients we deduce (21).  $\square$

In the particular case of Carnot groups, using canonical coordinates, one can show that the first  $m$ -components of  $\mathcal{X}$ -lines are actually (Euclidean) straight lines (see e.g. Lemma 2.2, [4]), that means

$$\alpha_x = \pi_m(x - x_0), \quad \forall x \in \mathbb{V}_{x_0},$$

where by  $\pi_m : \mathbb{R}^N \rightarrow \mathbb{R}^m$  is the projection on the first  $m$ -components. Then we can rewrite the previous result in the following easier way.

**Corollary 5.1.** Given a Carnot group  $\mathbb{G}$  and the associated family  $\mathcal{X}$  of left-invariant vector fields associated to the first layer of the stratified Lie algebra  $\mathfrak{g}$  in canonical coordinates, consider a set  $\Omega \subset \mathbb{G}$  and  $x_0 \in \mathbb{G}$ .

(i) If  $\Omega$  is weakly starshaped w.r.t.  $x_0$ , then

$$\langle \sigma(x)\pi_m(x - x_0), n(x) \rangle \geq 0, \quad \forall x \in \mathbb{V}_{x_0} \cap \partial\Omega, \quad (22)$$

where  $n(x)$  is the standard (Euclidean) outer unit normal at the point  $x$  and  $\pi_m$  is the projection on the first  $m$ -components.

(ii) Assume that the strict inequality in (20) holds true, then  $\Omega$  is weakly star-shaped w.r.t.  $x_0$ .

Again we want to point out as Theorem 5.1 and Corollary 5.1 give a characterisation for the weakly starshaped sets only on a subset of the boundary since they consider only points in  $\partial\Omega \cap \mathbb{V}_{x_0}$ .

**Remark 5.3** (Characteristic points). Note that strict inequalities in (20) (and respectively (22)) are more difficult to get than in the standard Euclidean case. In fact the inequality vanishes at every characteristic point. Characteristic points of a set  $\Omega$  are points of the boundary where the Euclidean normal  $n(x)$  is perpendicular to the horizontal space  $\mathcal{H}_x$ . We recall that the horizontal space  $\mathcal{H}_x$  is defined as the vector space spanned by the vectors  $X_1(x), \dots, X_m(x)$ . Therefore the velocity of  $\mathcal{X}$ -lines at a point  $x$  always belongs by definition to  $\mathcal{H}_x$ , which trivially implies that (20) (and so (22)) vanishes at all characteristic points.

**Remark 5.4** (Non-characteristic points and horizontal normal). When  $x \in \partial\Omega \cap \mathbb{V}_{x_0}$  is a non-characteristic point, then (20) and (22) can be respectively rewritten as

$$\langle \alpha_x, n_0(x) \rangle \geq 0 \quad \text{and} \quad \langle \pi_m(x - x_0), n_0(x) \rangle \geq 0,$$

where the scalar product is now the standard scalar product on  $\mathbb{R}^m$  and by  $n_0(x)$  we indicate the outer unit horizontal normal at the point  $x$ , that in our notation is

$$n_0(x) = \frac{\sigma^t(x)n(x)}{|\sigma^t(x)n(x)|_{\mathcal{X}}},$$

where for all points  $v(x) \in \mathcal{H}_x$  the sub-Riemannian norm induced by  $\mathcal{X}$  is defined as  $|v(x)|_{\mathcal{X}} = \sqrt{\sum_{i=1}^m \beta_i(x)}$  for  $v(x) = \sum_{i=1}^m \beta_i(x) X_i(x)$ .

Note that at characteristic points instead  $\sigma^t(x)n(x) = 0$ , which means that  $n_0(x)$  is not defined.

6.  $\mathbb{G}$ -STARSHAPED VS WEAKLY STARSHAPED AND EUCLIDEAN STARSHAPED.

In this section we always refer to Carnot groups looking at the associated  $\mathbb{R}^N$  with a polynomial non-commutative group law (see Lemma 3.1); thus, fixed a point  $x_0 \in \mathbb{G}$ , we can introduce three different notions of starshapedness: the standard (Euclidean) notion in  $\mathbb{R}^N$ ,  $\mathbb{G}$ -starshapedness and weak starshapedness. In this section we want to fully understand the mutual relations between these three notions. Next we show that weak starshapedness is indeed a weaker notion than  $\mathbb{G}$ -starshapedness in the sense that the second one always implies the first one while the reverse is in general false.

**Proposition 6.1.** Given a Carnot group  $\mathbb{G}$  (identified with  $\mathbb{R}^N$  by Lemma 3.1) and consider the family  $\mathcal{X} = \{X_1, \dots, X_m\}$  of left-invariant vector fields associated to the first layer of the stratified Lie algebra. Let  $\Omega \subseteq \mathbb{R}^N$  and  $x_0 \in \Omega$ , if  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t.  $x_0$  (according to Definition 4.1), then  $\Omega$  is weakly starshaped w.r.t.  $x_0$  (according to Definition 5.2).

*Proof.* For sake of simplicity we consider first the case  $x_0 = 0$ . So assume that  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. 0, that means  $\delta_t(y) \in \Omega$ , for all  $t \in [0, 1]$  and for all  $y \in \Omega$ . To show that  $\Omega$  is weakly starshaped w.r.t. 0, we need to consider a generic point  $y \in \mathbb{V}_0 \cap \Omega$ , i.e.  $y = x^\alpha(1)$  for some  $\alpha \in \mathbb{R}^m$  with  $x^\alpha$  corresponding  $\mathcal{X}$ -line starting at the time  $t = 0$  at the origin; then we claim that:

$$x^\alpha(t) = \delta_t(y). \quad (23)$$

Assuming claim (23), it is immediate to conclude: in fact, weakly starshaped can be written again as  $\delta_t(y) \in \Omega$  for all  $t \in [0, 1]$  but only for all  $y \in \mathbb{V}_0 \cap \Omega \subseteq \Omega$ . Then the implication follows. We remain to show claim (23). At this purpose, we need to recall a non trivial result for the left-invariant vector fields of Carnot groups in exponential coordinates ([6] Proposition 1.3.5 and Corollary 1.3.19). Let  $\sigma(x)$  be the smooth  $N \times m$ -matrix associated to left-invariant vector fields  $X_1, \dots, X_m$  (see Remark 5.1), then in exponential coordinates the matrix  $\sigma(x)$  can be written as

$$\sigma(x) = \begin{pmatrix} Id_{m \times m} \\ A(x) \end{pmatrix},$$

where  $Id_{m \times m}$  is the  $m \times m$ -identity matrix while  $A(x) = (a_{ji}(x))_{i,j}$  for  $i = 1, \dots, N$  and  $j = 1, \dots, m$  and the coefficients  $a_{ji}(x)$  are polynomial functions in the variables  $x_1, \dots, x_m$  of degree exactly equal to  $k - 1$ , whenever the corresponding component  $m + i$  scales as  $t^k$  w.r.t. the associated family of dilations  $\delta_t$  defined in (8). Using this result, we can show that, for all constant horizontal velocity  $\alpha \in \mathbb{R}^m$ , the  $\mathcal{X}$ -line  $x^\alpha$  starting at  $t = 0$  at the origin can be written as

$$x_1^\alpha(t) = \alpha_1 t, \dots, x_m^\alpha(t) = \alpha_m t \quad \text{and} \quad x_i^\alpha(t) = P_{i,k-1}(\alpha_1, \dots, \alpha_m) t^k,$$

whenever  $x_i$  scales as  $t^k$  in the associated dilation  $\delta_t$  and where by  $P_{i,k-1}(x_1, \dots, x_m)$  we indicate a generic polynomial of order  $k - 1$ .

Taking  $t = 1$ , we find that for all  $y \in \mathbb{V}_0$  then

$$\begin{aligned} y &= (\alpha_1, \dots, \alpha_m, \dots, P_{i,k-1}(\alpha_1, \dots, \alpha_m), \dots), \quad \text{and} \\ \delta_t(y) &= (\alpha_1 t, \dots, \alpha_m t, \dots, P_{i,k-1}(\alpha_1, \dots, \alpha_m) t^k, \dots), \end{aligned}$$

which proves (23).

We now consider the general case  $x_0 \neq 0$ , then we recall that  $\mathbb{G}$ -starshaped means

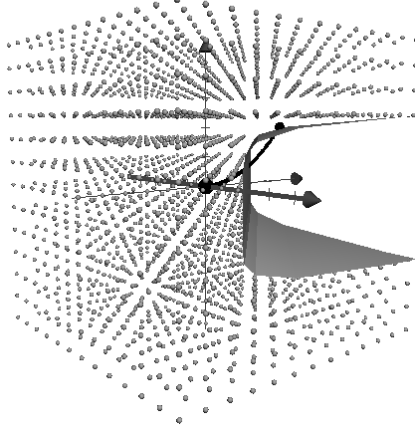


FIGURE 6. A weakly starshaped set w.r.t. the origin in  $\mathbb{H}^1$ , which is not  $\mathbb{G}$ -starshaped (Example 6.1). The set is indicated by the spheres; one can see that the dilation curve goes outside the set.

that  $x_0 \circ \delta_t(-x_0) \circ \delta_t(\Omega) \subset \Omega$ , for all  $t \in [0, 1]$ . Recall that property (4) in Lemma 3.2 tells  $\delta_t(-x_0) \circ \delta_t(y) = \delta_t(-x_0 \circ y)$ . Hence for all  $y \in \mathbb{V}_{x_0}$  we need to show that

$$x_0 \circ \delta_t(-x_0 \circ y) = x_{x_0}^\alpha(t), \quad (24)$$

where by  $x_{x_0}^\alpha$  we indicate the  $\mathcal{X}$ -line with constant horizontal velocity  $\alpha$  and such that  $x_{x_0}^\alpha(0) = x_0$  and  $x_{x_0}^\alpha(1) = y$ . Since  $y \in \mathbb{V}_{x_0}$  if and only if  $-x_0 \circ y \in \mathbb{V}_0$ , then there exists some constant  $\bar{\alpha}$  such that the corresponding  $\mathcal{X}$ -line  $x_0^{\bar{\alpha}}$  connects 0 to  $-x_0 \circ y$  at time 1. From (23) we know that  $\delta_t(-x_0 \circ y) = x_0^{\bar{\alpha}}(t)$ . Define the curve  $\tilde{x} = x_0 \circ x_0^{\bar{\alpha}}$ . By left-invariant property for the vector fields, we have that  $\tilde{x}$  is still a horizontal curve with constant horizontal velocity  $\bar{\alpha}$ . Moreover it is easy to check that  $\tilde{x}(0) = x_0$  and  $\tilde{x}(1) = y$ , so by uniqueness of  $\mathcal{X}$ -lines joining two given points in Carnot group, we find  $\tilde{x} = x_{x_0}^\alpha$ , that implies (24) and concludes the proof.  $\square$

The previous result implies trivially the following result.

**Proposition 6.2.** Given a Carnot group  $\mathbb{G}$ , and  $\Omega \subset \mathbb{G}$ , if  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. each point  $x_0 \in \Omega$ , then  $\Omega$  is horizontally convex.

*Proof.* Since  $\mathbb{G}$ -starshapedness implies weakly starshapedness (see Proposition 6.1) then Lemma 5.1 gives the implication we were looking for.  $\square$

The reverse implication of the previous result is in general false, unless we are in the commutative case (i.e.  $\text{step} = 1$ ), where  $\mathbb{V}_x = \mathbb{R}^N$  for all  $x$ . To prove this statement we give a counterexample in the case of the 1-dimensional Heisenberg group  $\mathbb{H}^1$ .

**Example 6.1** (A weakly starshaped but not  $\mathbb{G}$ -starshaped set in  $\mathbb{H}^1$ ). Let us consider the set  $\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3^8 < 1\}$ . Using that the  $\mathcal{X}$ -plane at the origin in  $\mathbb{H}^1$  is  $\mathbb{V}_0 = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$  (see e.g. [4]), then it is trivial to show that  $\Omega$  is weakly starshaped w.r.t. 0 since its projection on the first two components is the whole  $(x_1, x_2)$ -plane. Instead  $\Omega$  is not  $\mathbb{G}$ -starshaped w.r.t. 0 in  $\mathbb{H}^1$ . In fact, if we look at the point  $p = (0, 1.8, 1) \in \Omega$ , then the dilation curve  $\delta_t(p)$  for  $t \in (0, 1)$  is not all contained in  $\Omega$ , e.g.  $\delta_{3/4}(p) \notin \Omega$  (see Figure 6).



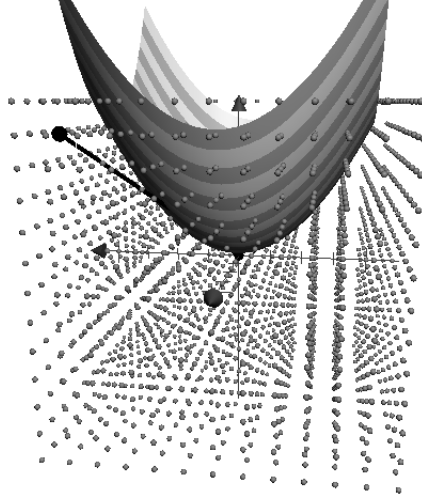


FIGURE 7. A  $\mathbb{G}$ -starshaped set in  $\mathbb{H}^1$  which is not Euclidean star-shaped (see Example 6.2); the set is indicated by the spheres.

As we have already noticed, roughly speaking, in the case  $x_0 = 0$  for a set  $\Omega$  to be  $\mathbb{G}$ -starshaped, the set needs to be “starshaped” along the dilation curves  $\gamma(t) = \delta_t(x)$  for all fixed  $x \in \Omega$ . The standard Euclidean starshapedness means instead to look at the behaviour along straight lines. With this difference in mind, it is not difficult to construct counterexamples to show that the two notions are completely non equivalent, in the sense that none of them implies the other. We again look for counterexamples in the 1-dimensional Heisenberg group  $\mathbb{H}^1$  defined in Example 4.2. To show that Euclidean starshapedness does not imply  $\mathbb{G}$ -starshapedness, the easiest way is to consider the Euclidean convex sets given in Examples 4.2 and 4.3, see Figures 4 and 5. To prove that  $\mathbb{G}$ -starshapedness in  $\mathbb{H}^1$  does not imply Euclidean starshapedness, we can consider any non-convex set (in the standard Euclidean sense) such that the boundary is foliated by the dilation curves in  $\mathbb{H}^1$ , e.g. see the next example.

**Example 6.2** ( $\mathbb{G}$ -starshaped set in  $\mathbb{H}^1$  but not Euclidean starshaped). Consider the set  $\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < x_1^2 + x_2^2 + \frac{1}{30}\}$ , then the origin is an internal point and so is  $(1, 1, 1)$ . The set  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. the origin: in fact, set  $\tilde{p} = \delta_t(p) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$  then, for all  $t \leq 1$ , we find

$$\tilde{x}_3 = t^2 x_3 < t^2 \left( x_1^2 + x_2^2 + \frac{1}{30} \right) = \tilde{x}_1^2 + \tilde{x}_2^2 + \frac{t^2}{30} \leq \tilde{x}_1^2 + \tilde{x}_2^2 + \frac{1}{30}, \quad \text{for } t \leq 1.$$

This shows that  $\delta_t(p) \in \Omega$ , for all  $t \in [0, 1]$  and for all  $p \in \Omega$ ; thus  $\Omega$  is  $\mathbb{G}$ -starshaped w.r.t. the origin. Look now at the (Euclidean) straight segment-line joining the origin to  $p = (1, 1, 1)$  at time  $t = 1$ , that is  $l(t) = p t$ ; then e.g. for  $\bar{t} = \frac{1}{3} \in (0, 1)$ ,  $l(\bar{t}) \notin \Omega$ , thus  $\Omega$  is not Euclidean starshaped w.r.t. the origin (see Figure 7).

We now look at the relations between weakly starshaped sets and Euclidean starshaped sets. In the  $n$ -dimensional Heisenberg group it is easy to show that Euclidean starshaped sets are always weakly starshaped since the  $\mathcal{X}$ -lines are Euclidean straight lines (see Example 5.1). Instead this same implication is trivially false whenever the  $\mathcal{X}$ -lines are not Euclidean straight lines, e.g. in the Grushin plane

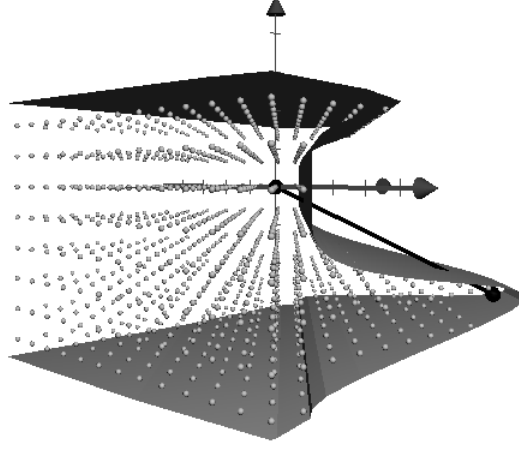


FIGURE 8. A weakly starshaped set in  $\mathbb{H}^1$  which is not Euclidean starshaped; the set is indicated by the spheres. The straight line joins to internal points but exits the set (see Example 6.3).

(see Example 5.2). Explicit counterexamples are easy to build in the Grushin plane but also in Carnot groups with step  $> 2$  (e.g. in the Engel group); we omit them. We now deal with the opposite implication: to construct weakly starshaped sets which are not Euclidean starshaped is possible in every non commutative Carnot group. We give as usual an explicit counterexample in  $\mathbb{H}^1$ .

**Example 6.3** (A weakly starshaped set in  $\mathbb{H}^1$  not Euclidean starshaped). Consider the set  $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3^6(x_3 + 2)^3(x_3 - 1) \leq \frac{1}{2}\}$ , then  $\Omega$  is weakly starshaped w.r.t. 0 in  $\mathbb{H}^1$ . In fact, take any point  $q \in \mathbb{V}_0 \cap \Omega$  then  $q = (\alpha_1, \alpha_2, 0)$ , for some  $\alpha_1, \alpha_2 \in \mathbb{R}$ . Then the  $\mathcal{X}$ -line joining the origin to  $q$  in time  $t = 1$  is  $x_0^\alpha(t) = (\alpha_1 t, \alpha_2 t, 0)$ ; thus for  $0 \leq t \leq 1$  and using that  $q \in \Omega$  that implies  $\alpha_1 + \alpha_2 \leq \frac{1}{2}$ , we easily find  $\alpha_1 t + \alpha_2 t = (\alpha_1 + \alpha_2)t \leq (\alpha_1 + \alpha_2) \leq \frac{1}{2}$ , that means  $x_0^\alpha(t) \in \Omega$ , i.e.  $\Omega$  is a weakly starshaped w.r.t. the origin. Instead  $\Omega$  is not Euclidean starshaped w.r.t 0: in fact if we consider the point  $(2, 2, -1.4) \in \Omega \setminus \mathbb{V}_0$ , then the Euclidean line joining the origin to  $(2, 2, -1.4)$  does not all belong to  $\Omega$ , e.g.  $t = \frac{1}{2} \in (0, 1)$  (see Figure 8).

Summing up we have showed that  $\mathbb{G}$ -starshapedness is a stronger notion than weakly starshapedness in the sense that the first one implies the second one and that in general both of them are non-equivalent in both directions to Euclidean starshapedness (but the particular case of the Heisenberg group where the weaker notion is always implied by the Euclidean notion).

7.  $\mathcal{X}$ -CONVEX FUNCTIONS VS  $\mathcal{X}$ -CONVEX SETS.

In this last section we generalise some relations between the  $\mathcal{X}$ -convex sets (see Definition 5.3) and the  $\mathcal{X}$ -convex functions introduced in [3, 4]. Similar properties have been already proved in the case of Carnot groups by Danielli, Garofalo and Nhieu in [10] by using the equivalent concept of  $H$ -convexity.

First we look at the property of sub-level sets.

**Proposition 7.1.** Let  $\mathcal{X} = \{X_1, \dots, X_m\}$  be a family of  $C^1$  vector fields on  $\mathbb{R}^N$  and  $\Omega \subseteq \mathbb{R}^N$  be an open and bounded set, and consider a function  $u : \Omega \rightarrow \mathbb{R}$ . If  $u$  is a  $\mathcal{X}$ -convex function, then for any  $a \in \mathbb{R}$  the sub-level set

$$\Omega_a := \{x \in \Omega \mid u(x) \leq a\}$$

is  $\mathcal{X}$ -convex.

*Proof.* Let us fix  $a \in \mathbb{R}$  and  $x \in \Omega_a$  and let us consider the  $\mathcal{X}$ -line  $x_x^\alpha$ , starting at  $x$  with constant horizontal velocity  $\alpha$ , for all  $\alpha \in \mathbb{R}^m$ , and such that  $x_x^\alpha(1) \in \Omega$ . Then  $\Omega_a$  is  $\mathcal{X}$ -convex if and only if  $x_x^\alpha(t) \in \Omega_a$  for all  $t \in [0, 1]$ . Recall that  $x_x^\alpha(t) \in \Omega_a$  if and only if  $u(x_x^\alpha(t)) \leq a$ . By using that  $u$  is  $\mathcal{X}$ -convex, we find that

$$\begin{aligned} u(x_x^\alpha(t)) &= u(x_x^\alpha(1-t)0 + t) \leq (1-t)u(x_x^\alpha(0)) + tu(x_x^\alpha(1)) = (1-t)u(x) + tu(y) \\ &\leq (1-t)a + ta = a, \end{aligned}$$

thus all the level sets  $\Omega_a$  are  $\mathcal{X}$ -convex in  $\Omega$ .  $\square$

Next we look at the characterisation by the epigraph. Recall that, given a function  $u : \Omega \rightarrow \mathbb{R}$  with  $\Omega \subset \mathbb{R}^N$ , the *epigraph* of  $u$  is the subset of  $\Omega \times \mathbb{R}$  defined as

$$\text{epi}(u) := \{(x, a) \in \Omega \times \mathbb{R} \mid u(x) \leq a\}.$$

**Theorem 7.1.** Let  $\mathcal{X} = \{X_1, \dots, X_m\}$  be a family of  $C^1$  vector fields on  $\mathbb{R}^N$  and  $\Omega \subseteq \mathbb{R}^N$  be an open and bounded set, and consider a function  $u : \Omega \rightarrow \mathbb{R}$ . Then  $u$  is  $\mathcal{X}$ -convex if and only if  $\text{epi}(u)$  is  $\tilde{\mathcal{X}}$ -convex, where by  $\tilde{\mathcal{X}}$  we denote the family of vector fields on  $\mathbb{R}^N \times \mathbb{R}$ , defined by  $\tilde{\mathcal{X}} = \{\tilde{X}_1, \dots, \tilde{X}_m, \tilde{X}_{m+1}\}$  where

$$\tilde{X}_i(x, a) := \begin{pmatrix} X_i(x) \\ 0 \end{pmatrix} \quad \text{for } i = 1, \dots, m \quad \text{and} \quad \tilde{X}_{m+1}(x, a) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

*Proof.* Let us consider a point  $(x, a) \in \text{epi}(u) \subseteq \Omega \times \mathbb{R}$  and fix  $(y, b) \in \mathbb{V}_{(x,a)}^{\tilde{\mathcal{X}}} \cap \text{epi}(u)$ . Note that

$$(y, b) \in \mathbb{V}_{(x,a)}^{\tilde{\mathcal{X}}} \iff x \in \mathbb{V}_x = \mathbb{V}_x^{\mathcal{X}} \quad \text{and} \quad b \in \mathbb{R}. \quad (25)$$

Then let us consider any  $\tilde{\mathcal{X}}$ -line segment  $\xi : [0, 1] \rightarrow \mathbb{R}^{N+1}$  joining  $(x, a) \in \text{epi}(u)$  to  $(y, b) \in \mathbb{V}_{(x,a)}^{\tilde{\mathcal{X}}} \cap \text{epi}(u)$ . To prove that  $\text{epi}(u)$  is  $\tilde{\mathcal{X}}$ -convex, we need to show that

$$\xi(t) \in \text{epi}(u), \quad \forall t \in [0, 1], \quad \text{where } \xi(t) = (x_x^\alpha(t), (b-a)t + a), \quad (26)$$

with  $x_x^\alpha$  any  $\mathcal{X}$ -line joining  $x$  to  $y \in \mathbb{V}_x \cap \Omega$ . Recall that (26) is equivalent to requiring

$$u(x_x^\alpha(t)) \leq (b-a)t + a, \quad \forall t \in [0, 1]. \quad (27)$$

To prove (27) we use the fact that the function  $u$  is a  $\mathcal{X}$ -convex function, which means

$$\begin{aligned} u(x_x^\alpha(t)) &= u(x_x^\alpha((1-t)0 + t)) \leq (1-t)u(x_x^\alpha(0)) + (1-t)u(x_x^\alpha(1)) \\ &= (1-t)u(x) + tu(y) \leq (1-t)a + tb, \end{aligned}$$

where we have used that  $(x, a), (y, b) \in \text{epi}(u)$ . Thus (27) is proved, i.e.  $\text{epi}(u)$  is a  $\tilde{\mathcal{X}}$ -convex set.

To prove the opposite implication let us assume that  $\text{epi}(u)$  is  $\tilde{\mathcal{X}}$ -convex, i.e. (27) holds with  $x_x^\alpha(0) = x$  and  $x_x^\alpha(1) = y$  and  $a, b \in \mathbb{R}$  such that  $(x, a) \in \text{epi}(u)$  and  $(y, b) \in \text{epi}(u)$ . We want to prove that  $u$  is a  $\mathcal{X}$ -convex function, i.e. that

$$u(x_x^\alpha((1-t)t_1 + tt_2)) \leq (1-t)u(x_x^\alpha(t_1)) + tu(x_x^\alpha(t_2)), \quad (28)$$

holds for all  $t, t_1, t_2 \in [0, 1]$  and for any  $\mathcal{X}$ -line  $x_x^\alpha$ .

We define  $f_x^\alpha(t) := u(x_x^\alpha(t))$ , for all fixed  $x_x^\alpha$ . Now (28) can be rewritten as

$$f_x^\alpha((1-t)t_1 + tt_2) \leq (1-t)f_x^\alpha(t_1) + tf_x^\alpha(t_2). \quad (29)$$

Hence (28) is equivalent to proving:  $f_x^\alpha : [0, 1] \rightarrow \mathbb{R}$  is a Euclidean convex 1-dimensional function for all  $\alpha \in \mathbb{R}^m$ . If we can prove that  $\text{epi}(f_x^\alpha)$  is a convex set in  $\mathbb{R} \times \mathbb{R}$  then we can use the known Euclidean 1-dimensional case to conclude: in fact  $\text{epi}(f_x^\alpha)$  convex implies the function  $f_x^\alpha$  is convex.

Recall that for  $\text{epi}(f_x^\alpha)$  to be convex in the standard Euclidean sense means that for all  $t_1, t_2, \in [0, 1]$  and for all  $a, b \in \mathbb{R}$  such that  $(t_1, a), (t_2, b) \in \text{epi}(f_x^\alpha)$  then

$$((1-t)t_1 + tt_2, (1-t)a + tb) \in \text{epi}(f_x^\alpha), \quad t \in [0, 1],$$

that is equivalent to requiring

$$f_x^\alpha((1-t)t_1 + tt_2) \leq (1-t)a + tb. \quad (30)$$

Note that  $(t_1, a), (t_2, b) \in \text{epi}(f_x^\alpha)$  is equivalent to  $(x_x^\alpha(t_1), a) \in \text{epi}(u)$  and  $(x_x^\alpha(t_2), b) \in \text{epi}(u)$ . Then since  $\text{epi}(u)$  is  $\tilde{\mathcal{X}}$ -convex then

$$((1-t)x_x^\alpha(t_1) + tx_x^\alpha(t_2), (1-t)a + tb) \in \text{epi}(u), \quad \forall t \in [0, 1],$$

which gives (30). Thus  $\text{epi}(f_x^\alpha)$  is (Euclidean) convex  $\Rightarrow f_x^\alpha$  is a convex function, which gives (29) and then (28). This concludes the proof.  $\square$

Similarly one can find the characterisation by the indicator function. We recall that the indicator function of a set  $\Omega$  is the function  $1_\Omega : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  defined as

$$1_\Omega(x) := \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{if } x \notin \Omega. \end{cases}$$

**Proposition 7.2.** Let  $\mathcal{X} = \{X_1, \dots, X_m\}$  be a family of vector fields on  $\mathbb{R}^N$  and  $\Omega \subseteq \mathbb{R}^N$  be an open and bounded set, then  $\Omega$  is  $\mathcal{X}$ -convex if and only if the indicator function  $1_\Omega$  is a  $\mathcal{X}$ -convex function.

*Proof.* The proof is very similar to the previous one so we omit it. For details see [13].  $\square$

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