QUASI STATIC GROWTH OF BRITTLE CRACKS IN A LINEARLY ELASTIC FLEXURAL PLATE

FAUSTO ACANFORA AND MARCELLO PONSIGLIONE

Abstract. In this paper we propose a variational model for the irreversible quasi static growth in brittle fractures for a linearly elastic homogeneous isotropic plate, subject to a time dependent vertical displacement on a part of its lateral surface. The model is based on the Griffith's criterion for crack growth and is inspired by the model proposed in [11] by G.A. Francfort and J.-J. Marigo in the case of 3-D elasticity. We give a precise mathematical formulation of the model and in this framework we prove an existence result.

Keywords : Plates, variational methods for higher-order elliptic equations, energy minimization, free discontinuity problems, brittle fracture, crack propagation, quasi static growth.

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Contents

1. Introduction

Mathematical problems arising from the variational model of cracks propagation have been proposed by Francfort and Marigo in [11] . This theory is inspired to the classical *Griffith's criterion* for cracks growth, and its main characteristic is that it doesn't prescribe the path of the cracks but determines it through a competition between *bulk* and *surface energies.* In this model, the continuum growth of the cracks during the loading process is obtained as a limit of a discrete in time growth, determined by a step by step unilateral minimization problem.

The precise mathematical formulation of this model has been studied by G. Dal Maso and R. Toader [5] in the special case of linearized elasticity for anti-plane shear and with an a priori bound on the number of connected components of the cracks. This analysis has been extended to the case of plane elasticity by Chambolle in [3].

A weak formulation for the variational model of fracture growth in the framework of SBV spaces (that is the space of special functions with bounded variation, see [1]), has been proposed by G.A. Francfort and C.J. Larsen in [10] for anti-plane shear in higher dimensions. This approach is more natural since it is performed in any dimension and with no restrictions on the admissible cracks. However, the strong formulation in [5] based on the Hausdorff convergence of compact sets, is more elementary in dimension two and leads to the convergence in the Hausdorff metric of the cracks obtained in the discrete growth.

Recently, G. Dal Maso, G.A. Francfort and R. Toader in [4] have proved the existence of a quasi static growth in the framework of generalized SBV spaces for n-dimensional finite elasticity, with a quasiconvex bulk energy and with prescribed boundary deformations and applied loads.

In this paper we consider the case of a linearly elastic homogeneous isotropic plate, subject to a time dependent *vertical displacement* on a part of its lateral boundary. We provide a model of crack propagation which, according to Griffith's criterion, takes into account the competition between bulk and surface energies in the process of cracking, while it does not allow the appearing of kinks. We stress that (as far as we know) it is not completely decided in the literature whether the investigated model adequately portrays brittle fracture evolution in a plate submitted to bending.

The reference configuration is a bounded open set Ω of $\mathbb{R}^2$, which represents the middle surface of the plate, with Lipschitz continuous boundary $\partial\Omega$. Let $m > 0$ be a fixed integer. The set of admissible cracks is (as in [5]) the set $\mathcal{K}_m(\overline{\Omega})$ of all closed subsets K of $\overline{\Omega}$ whose elements have at most m connected components. Let $\partial_D\Omega$ be open and with a finite number of connected components. Given a crack $K \in \mathcal{K}_m(\overline{\Omega})$, the boundary datum is prescribed in the set $\partial_D\Omega \setminus K$, and is given by (the trace of) a function $g \in W^{2,2}(\Omega)$; we can not prescribe a boundary condition on $\partial_D\Omega \cap K$ because it is not transmitted through the crack. The remaining part of the boundary $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ and the cracks K are *traction free.* The displacement u relative to the crack K and subject to the boundary condition g is a function which may jump across K (we will introduce rigorously the space of admissible displacements in Section 3), which verifies the boundary condition and minimize the quadratic form

$$B(v,v) := \frac{2E}{3(1-k^2)} \int_\Omega |v_{xx}|^2 + |v_{yy}|^2 + (2-2k)|v_{xy}|^2 \, dxdy, \tag{1.1}$$

(see [6]), where the Poisson's coefficient $0 < k \le 1/2$ and the Young's modulus E measure the rigidity of the constituting material. We will consider for simplicity of notations $E = \frac{3(1-k^2)}{2}$, so that the leading coefficient in (1.1) is equal to 1. Finally, for every admissible crack $K \in \mathcal{K}_m(\overline{\Omega})$ and for every boundary datum g, let us introduce the *bulk energy* $\mathcal{E}_b$ and the *total energy* E defined by

$$\mathcal{E}_b(g,K) := B(u,u), \qquad E(g,K) := B(u,u) + \mathcal{H}^1(K), \tag{1.2}$$

where u is the displacement relative to K and g, and $\mathcal{H}^1(K)$ is the one dimensional Hausdorff measure of K.

We now describe our model of irreversible quasi static crack growth under the action of a time dependent boundary datum. Let $g(t) \in AC\big([0,1]; W^{2,2}(\Omega)\big)$ (i.e. the function $t \to g(t)$ is absolutely continuous) and let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ be a preexisting crack. In our model, the *irreversible quasi static crack growth* relative to the boundary datum g and to the preexisting crack K_0, is a function $\Gamma : (0,1) \to \mathcal{K}_m(\overline{\Omega})$ which verifies the following three properties:

(1) *Irreversibility of the process:*

$$K_0 \subseteq \Gamma(0) \subseteq \Gamma(t_1) \subseteq \Gamma(t_2) \quad \forall 0 \le t_1 \le t_2 \le 1;$$

(2) *Static equilibrium:*

$$E(g(0),\Gamma(0)) \le E(g(0),H) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H \text{ and}$$

$$E(g(t),\Gamma(t)) \le E(g(t),H) \quad \forall t \in (0,1], \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H;$$

(3) *Nondissipativity:*

the function $t \to E(g(t),\Gamma(t))$ is absolutely continuous and

$\frac{d}{dt}E(g(t),\Gamma(t)) = 2B(u(t),\dot{g}(t)),$

where $u(t)$ is the displacement relative to $\Gamma(t)$ and to $g(t)$.

The main result of this paper is Theorem 6.1, which establishes the existence of a quasi static evolution that verifies properties (1), (2) and (3) above. This quasi static growth is obtained as limit of a discrete in time growth $\Gamma_\delta(t)$. The construction of the step function $\Gamma_\delta(t)$ is inspired by the Griffith's criterion; more precisely, supposing to have constructed Γ_δ in the interval $[(i-1)\delta, i\delta)$, we define Γ_δ in $[i\delta,(i+1)\delta)$ as a solution of the minimum problem

$$\min_K \big\{E(g(i\delta),K),\ K \in \mathcal{K}_m(\overline{\Omega}) : \Gamma^\delta_{i-1} \subseteq K\big\}. \tag{1.3}$$

The main tool of this paper is the stability of these kind of *unilateral free-discontinuity problems* as $\delta \to 0$, that leads to the static equilibrium property of Γ; this is the subject of Section 5. The *stability result* is obtained through Theorem 5.1, that we will prove in Section 8. It is a new version in the framework of Sobolev spaces of the transfer of jumps Theorem given in [10], which enables to treat energies with derivatives of order greater than one. In fact the proof of the transfer of jumps Theorem given in [10] is based on a geometrical construction which uses the coarea formula, and therefore it needs an a priori bound on $\|\nabla u\|_{L^p(\Omega)}$, given by the fact that u minimizes the bulk energy. In our case the bulk energy involves only second derivatives and the domain (by the presence of cracks) is not regular, and hence Poincaré type inequalities does not hold in general. Therefore it is not clear how to provide a weak formulation which guarantees such a priori bound on the gradient of u in order to perform the same construction of [10]. These considerations motivated us to choose a strong formulation in the setting of Deny-Lions spaces

$$L^{2,2}(U) := \{u \in L^2_{\mathrm{loc}}(U) :\ D^2u \in L^2(u;\mathcal{M}^{2\times 2})\}.$$

We remark that also in the case of a uniform bound in $L^\infty(\Omega)$ for the boundary datum, by the presence of cracks the displacement is in general not well defined in the usual Sobolev space $W^{2,2}(\Omega)$. In our proof of the transfer of jumps we need also the technical assumption that the number of connected components of Γ is uniformly bounded, in order to perform a geometrical construction which does not use coarea formula. Finally we remark that stability results for energies involving derivatives of order one are been obtained in [7] still under the assumptions of a uniform bound on the connected components of the crack, but without using the tool of the transfer of jumps Theorem.

In Section 7 we consider the particular case where Γ is rectilinear. In this case in [6] is given a formula for the derivative of bulk energy with respect to the growth of the crack through a $3D-2D$ dimension reduction, under very strong regularity assumptions. Moreover in [16] is proved that this asymptotic quantity coincides with the derivative of the bulk energy $B(u_K, u_K)$ with respect to the growth of the crack (here u_K is the displacement relative to the crack K).

We prove that this quantity depends only on the singular part of the displacement u, and its explicit computation leads to

$$9\pi(1+k)^2\left(\frac{b_1^2}{(7+k)^2}+\frac{b_2^2}{(5+3k)^2}\right),$$

where b_1 and b_2 are coefficients which appear in the singular part of u around the tip (see [6]), and play a role analogous of the so called *mode III stress intensity factor* in elasticity. Moreover, we prove that during the load process

$$9\pi(1+k)^2\left(\frac{b_1(t)^2}{(7+k)^2}+\frac{b_2(t)^2}{(5+3k)^2}\right)\leq 1, \tag{1.4}$$

and that the tip moves if and only if (1.4) is satisfied with the equality. This is the Griffith's criterion for crack propagation in our model.

2. Notation and Preliminaries

In this section we introduce the main notations and the preliminary results employed in the rest of the paper. From now on Ω is an open bounded subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. For every $x \in \Omega$, we denote the open ball of radius r an centered at x by $B_r(x)$. Let $\mathcal{K}_m(\overline{\Omega})$ be the class of all closed subsets K of $\overline{\Omega}$ whose elements have at most m connected components.

Hausdorff metric. The *Hausdorff distance* between two closed subsets K_1 and K_2 of $\overline{\Omega}$ is defined by

$$d_H(K_1,K_2):=\max\left\{\sup_{x\in K_1}\operatorname{dist}(x,K_2),\ \sup_{x\in K_2}\operatorname{dist}(x,K_1)\right\},$$

with the conventions $\operatorname{dist}(x,\emptyset)=\operatorname{diam}(\Omega)$ and $\sup\emptyset=0$, so that

$$d_H(\emptyset\,,K)=\begin{cases}0 & \text{if } K=\emptyset,\\ \operatorname{diam}(\Omega) & \text{if } K\neq\emptyset.\end{cases}$$

Let (K_h) be a sequence of compact subsets of $\overline{\Omega}$. We say that K_h converges to K in the *Hausdorff metric* if $d_H(K_h\,,K)$ converges to 0. It is well-known (see e.g., [8, Blaschke's Selection Theorem]) that $\mathcal{K}_m(\overline{\Omega})$ is compact with respect to the Hausdorff convergence. Moreover the following semicontinuity result holds (for the proof see [5]).

Lemma 2.1. *Let $(K_n)\subset\mathcal{K}_m(\overline{\Omega})$ be such that $\mathcal{H}^1(K_n)$ is uniformly bounded. Then there exists $K\subset\mathcal{K}_m(\overline{\Omega})$ such that K_n converges to K in the Hausdorff metric, and*

$$\mathcal{H}^1(K)\leq\liminf_n\mathcal{H}^1(K_n).$$

Let S be a subset of $\mathbb{R}^2$ and let $x \in S$. For every positive $\lambda \in \mathbb{R}^+$ we set

$$D_\lambda(x)(S) := \Big\{x + \lambda(\xi - x), \quad \xi \in S\Big\}.$$

It is known that if K is connected and $\mathcal{H}^1(K)$ is finite, then $\mathcal{H}^1$-a.e. $x \in K$ admits an approximate normal vector in the sense of measure (see for instance [8]). Moreover the following lemma, proved in [12], holds.

Lemma 2.2. *Let $K \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K) < \infty$. Then for $\mathcal{H}^1$-a.e. $x \in K$ there exists a vector $\nu(x)$ with $|\nu(x)| = 1$ such that*

$$D_\lambda(x)(K \cap \overline{B_{1/\lambda}(x)}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)} \tag{2.1}$$

in the Hausdorff metric as $\lambda \to \infty$, where $\nu(x)^\perp$ is the space spanned by a vector orthogonal to $\nu(x)$.

The vector $\nu(x)$ is the so called *approximate normal to K at x*. We will need the following Lemma which easily follows by Lemma 2.2.

Lemma 2.3. *Let K, $H \in \mathcal{K}_m(\overline{\Omega})$ be such that $K \subset H$ and $\mathcal{H}^1(H) \leq \infty$. Then, for $\mathcal{H}^1$-a.e. $x \in K$ there exists a vector $\nu(x)$ with $|\nu(x)| = 1$ such that*

$$D_\lambda(x)(K \cap \overline{B_{1/\lambda}(x)}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)}, \tag{2.2}$$

and

$$D_\lambda(x)(H \cap \overline{B_{1/\lambda(x)}}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)}, \tag{2.3}$$

in the Hausdorff metric as $\lambda \to \infty$.

Deny-Lions spaces. Given an open subset U of $\mathbb{R}^2$ the Deny-Lions space $L^{2,2}(U)$ is defined by

$$L^{2,2}(U) := \{u \in L^2_{\text{loc}}(U) : \ D^2 u \in L^2(u; \mathcal{M}^{2\times 2})\},$$

where $\mathcal{M}^{2\times 2}$ are the 2×2 real matrices. The spaces $L^{2,2}(u)$ are endowed with the seminorm

$$\|u\|_{L^{2,2}(U)} := \|D^2 u\|_{L^2(U; \mathcal{M}^{2\times 2})} \qquad \forall u \in L^{2,2}(U).$$

Deny-Lions spaces are usually involved in minimization problems in non smooth domains where Poincaré inequalities do not hold in general. It is well known that $L^{2,2}(U)$ coincides with the Sobolev space $W^{2,2}(U)$ whenever U is bounded and has a Lipschitz continuous boundary, and that the set $\{D^2 u : \, u \in L^{2,2}(U)\}$ is a closed subspace of $L^2(U; \mathcal{M}^{2\times 2})$.

It is also known (see [14]) that if A is an open subset of $\mathbb{R}^2$ with Lipschitz boundary, there exists a continuous extension operator $E : L^{2,2}(A) \to L^{2,2}(\mathbb{R}^2)$. For every open set $A \subset \mathbb{R}^2$ and every $\varepsilon > 0$, let us now set

$$A_\varepsilon := \{\varepsilon\xi, \, \xi \in A\}.$$

From the existence of a continuous extension operator for a fixed domain, we deduce the following Lemma.

Lemma 2.4. *Let A be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. Then for every $\varepsilon > 0$ there exists a continuous extension operator $E_\varepsilon : L^{2,2}(A_\varepsilon) \to L^{2,2}(\mathbb{R}^2)$, such that*

$$\|E_\varepsilon(u)\|^2_{L^{2,2}(\mathbb{R}^2)} \leq C \, \|u\|^2_{L^{2,2}(A_\varepsilon)} \tag{2.4}$$

where C is a constant independent on ε.

Proof. Let E_1 be the extension operator relative to the set A. For every function $u \in L^{2,2}(A_\varepsilon)$, we define the function $\tilde{u} \in L^{2,2}(A)$ as follows: $\tilde{u}(x) := u(\varepsilon x)$. We consider the extension operator E_ε given by

$$E_\varepsilon(u)(x) := E_1(\tilde{u})(\frac{x}{\varepsilon}).$$

Then, by change of variable we get

$$\int_{\mathbb{R}^2} |D^2 E_\varepsilon(u)(x)|^2\, dx = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} |D^2 E_1(\tilde{u})(y)|^2\, dy \leq$$
$$\leq \frac{1}{\varepsilon^2} C \int_A |D^2 \tilde{u}(y)|^2\, dy = \frac{1}{\varepsilon^2}\, \varepsilon^2 C \int_{A_\varepsilon} |D^2 u(x)|^2\, dx,$$

where C is the constant in (2.4) relative to the extension operator E_1, and this concludes the proof. ■

For further properties of the spaces $L^{2,2}$ we refer the reader to [14].

3. Formulation of the problem

Static equilibrium for a clamped plate with cracks. We recall here the variational formulation for the static equilibrium of a homogeneous isotropic plate with crack $K \in \mathcal{K}_m(\overline{\Omega})$, subject to vertical displacement on a part of its boundary.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. We fix a subset $\partial_D\Omega$ of $\partial\Omega$ on which we prescribe a boundary condition; we assume that $\partial_D\Omega$ is non-empty, relatively open in $\partial\Omega$ and composed of a finite number of connected components, and we set $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. For every function $g \in W^{2,2}(\Omega)$ and for every crack $K \in \mathcal{K}_m(\overline{\Omega})$, we set

$$L^{2,2}_{g,\partial_D\Omega}(\Omega \setminus K) := \{u \in L^{2,2}(\Omega \setminus K) : u - g = 0,\ \frac{\partial}{\partial\nu}(u-g) = 0 \text{ a.e. on } \partial_D\Omega \setminus K\}.$$

Here the equality $u - g = 0$ and $\frac{\partial}{\partial\nu}(u-g) = 0$ a.e. on $\partial_D\Omega \setminus K$ are intended in the sense of traces as in [5].

Let us fix the so called Poisson coefficient $0 < k \leq 1/2$. However for most of materials (see [6]) k is strictly less than $1/2$, the case $k = 1/2$ corresponding to incompressible materials. Let us consider the bilinear form $B : L^{2,2}(\Omega \setminus K) \times L^{2,2}(\Omega \setminus K) \to \mathbb{R}$ defined by

$$B(u,v) := \int_{\Omega\setminus K} u_{xx}v_{xx} + u_{yy}v_{yy} + (2-2k)u_{xy}v_{xy}\, dxdy \quad \text{for every } u,\, v \in L^{2,2}(\Omega \setminus K).$$

Note that by definition

$$\|D^2 v\|^2 \leq B(v,v) \leq 2\|D^2 v\|^2, \tag{3.1}$$

where $\|\cdot\|$ denotes the L^2 norm. The displacement u corresponding to the boundary condition g is a solution of the following minimization problem:

$$\min_{v \in L^{2,2}_{g,\partial_D\Omega}(\Omega\setminus K)} B(v,v). \tag{3.2}$$

Using that the set $\{D^2 u : u \in L^{2,2}_{g,\partial_D\Omega}(\Omega)\}$ is closed (see [15]) and (3.1), from the direct method of calculus of variations it follows that the minimum problem (3.2) admits a solution $u \in L^{2,2}_{g,\partial_N\Omega}(\Omega \setminus K)$; moreover the functional $D^2 u \to B(u,u)$ is strictly convex, so that $D^2 u$ is uniquely determined. We remark also that u and ∇u are uniquely determined in every connected component A of Ω such that $\mathcal{H}^1(\partial A \cap \partial_D\Omega) > 0$, and that problem (3.2) is equivalent to finding $u \in L^{2,2}_{g,\partial_D\Omega}(\Omega)$ such that

$$B(u,v) = 0 \qquad \forall v \in L^{2,2}_{0,\partial_D\Omega}(\Omega). \tag{3.3}$$

For more details on the subject see for instance [6], [9],[16].

Finally let us introduce the *bulk energy* $\mathcal{E}_b : W^{2,2}(\Omega) \times \mathcal{K}_m(\overline{\Omega}) \to \mathbb{R}$ and the *total energy* $E : W^{2,2}(\Omega) \times \mathcal{K}_m(\overline{\Omega}) \to \mathbb{R}$ defined by

$$\mathcal{E}_b(g,K) := B(u,u), \qquad E(g,K) := B(u,u) + \mathcal{H}^1(K), \tag{3.4}$$

where u is a solution of problem (3.2).

Irreversible quasi static growth. We consider now the case of time-dependent boundary conditions and we introduce the notion of *irreversible quasi static growth.* Let $g \in AC([0,1]; W^{2,2}(\Omega))$, where $AC([0,1]; W^{2,2}(\Omega))$ is the space of all absolutely continuous functions defined in $[0,1]$ with values in $W^{2,2}(\Omega)$ (for details on the spaces of absolutely continuous functions see [2]). It is well-known that for a.e. $x \in [0,1]$ there exists the time derivative of g, denoted by $\dot{g}$, and that $\dot{g}$ is a Bochner integrable function with values in $W^{2,2}(\Omega)$.

Let us fix a positive integer $m > 0$. Given a pre-existing crack $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with finite length, an *irreversible quasi static growth* relative to the initial crack K_0 and to the boundary datum $g(t)$, is a function

$$\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$$

such that the following three properties hold.

(1) *Irreversibility of the process:*

$$K_0 \subseteq \Gamma(0) \subseteq \Gamma(t_1) \subseteq \Gamma(t_2) \quad \forall\, 0 \le t_1 \le t_2 \le 1;$$

(2) *Static equilibrium:*

$$E(g(0),\Gamma(0)) \le E(g(0),H) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H \text{ and}$$

$$E(g(t),\Gamma(t)) \le E(g(t),H) \quad \forall t \in (0,1], \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H;$$

(3) *Nondissipativity:*

the function $t \to E(g(t),\Gamma(t))$ is absolutely continuous and

$$\tfrac{d}{dt}E(g(t),\Gamma(t)) = 2B\big(u(t),\dot{g}(t)\big),$$

where $u(t)$ is the solution of the minimum problem in (3.2) with K replaced by $\Gamma(t)$ and g replaced by $g(t)$.

4. Discrete growth of the cracks

In this section we construct a discrete in time approximation of the quasi static growth described previously.

Let Ω, $\partial_D\Omega$ and $\partial_N\Omega$ be as defined in the previous section. Let m be a fixed positive integer, let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K_0) \le \infty$, and let $g \in AC([0,1]; W^{2,2}(\Omega))$. For any $\delta > 0$, let N_δ be the largest integer such that $\delta(N_\delta - 1) \le 1$; for $0 \le i \le N_\delta - 1$ we set $t_i^\delta := i\delta$, and $t^\delta_{N_\delta} = 1$. We discretize the boundary data setting $g_i^\delta := g(t_i^\delta)$, and we construct the discrete growth as follows: we set $\Gamma^\delta_{-1} = K_0$ and, supposing to have constructed Γ^δ_{i-1}, we proceed recursively setting $\Gamma_i^\delta \in \mathcal{K}_m(\overline{\Omega})$ as a solution of

$$\min_K \big\{E(g_i^\delta, K),\ K \in \mathcal{K}_m(\overline{\Omega}) : \Gamma_{i-1}^\delta \subseteq K\big\}, \tag{4.1}$$

and setting u_i^δ as a solution of the minimum problem 3.2 in $\Omega \setminus \Gamma_i^\delta$ with boundary datum g_i^δ.

Lemma 4.1. *Problem* (4.1) *admits a solution.*

Proof. Let (K_n) be a minimizing sequence for problem (4.1) and let u_n be a solution of problem (3.2) in $\Omega \setminus K_n$. By the fact that g_i^δ is an admissible function in (3.2) and by (3.1), we can assume that there exists a positive constant C such that

$$\int_{\Omega\setminus K_n} |D^2 u_n|^2 \le C, \qquad \mathcal{H}^1(K_n) \le C. \tag{4.2}$$

By Lemma 2.1 there exists $K \in \mathcal{K}_m(\overline{\Omega})$ such that, up to a subsequence, $K_n \to K$ in the Hausdorff metric and

$$\mathcal{H}^1(K) \le \liminf \mathcal{H}^1(K_n). \tag{4.3}$$

Moreover, by the fact that $\Gamma_{i-1}^\delta \subset K_n$ for every n and $K_n \to K$, we have that $\Gamma_{i-1}^\delta \subset K$. Now, let $A \subset \overline{A} \subset \Omega \setminus K$ be open; by the Hausdorff convergence of K_n to K and since $K \cap A = \emptyset$, it follows that for n big enough $K_n \cap A = \emptyset$. By (4.2) we have that

$$\int_A |D^2 u_n|^2 \, dx \le C,$$

so that there exists $u \in L^{2,2}(A)$ such that, up to a subsequence, $D^2 u_n$ converges to $D^2 u$ weakly in $L^{2,2}(A, \mathcal{M}^{2\times 2})$. Since A is arbitrary, u can actually be defined in $L^{2,2}(\Omega \setminus K)$. Moreover it is easy to see that $u \in L^{2,2}_{g_i^\delta, \partial_D\Omega}(\Omega \setminus K)$. By lower semicontinuity

$$B(u,u) \le \liminf B(u_n, u_n). \tag{4.4}$$

From (4.3) and (4.4) it follows that the pair (u, K) minimizes $B(v,v) + \mathcal{H}^1(H)$ among all $H \in \mathcal{K}_m(\overline{\Omega})$ with $\Gamma_{i-1}^\delta \subset H$, and all $v \in L^{2,2}_{g_i^\delta, \partial_D\Omega}(\Omega \setminus H)$, so that the proof is concluded. ■

Note that by construction we have that

$$K_0 \subseteq \Gamma_i^\delta \subseteq \Gamma_j^\delta \qquad \forall\, 0 \le i \le j \le N_\delta.$$

Moreover, the minimality property (4.1) is equivalent to

$$B(u_i^\delta, u_i^\delta) + \mathcal{H}^1(\Gamma_i^\delta) \le B(v,v) + \mathcal{H}^1(H), \tag{4.5}$$

for every $H \in \mathcal{K}_m(\overline{\Omega})$ which contains Γ_i^δ and for every $v \in L^{2,2}_{g_i^\delta, \partial_D\Omega}(\Omega \setminus H)$. From (4.5), comparing u_i^δ with g_i^δ and by (3.1), we have that for every i

$$\int_\Omega |D^2 u_i^\delta|^2 \, dx \le C \qquad \forall\, 0 \le i \le N_\delta, \tag{4.6}$$

where C is a constant independent on i, δ.

Now we define the step functions

$$g^\delta := g_i^\delta, \quad u^\delta := u_i^\delta, \quad \Gamma^\delta := \Gamma_i^\delta, \tag{4.7}$$

for $t_i^\delta \leq t < t_{i+1}^\delta$. By construction and by (4.6), we have that

$$K_0 \subseteq \Gamma^\delta(t_1) \subseteq \Gamma^\delta(t_2) \qquad \forall 0 \leq t_1 \leq t_2 \leq 1,$$

and

$$\int_\Omega |D^2 u^\delta(t)|^2\, dx \leq C \qquad \forall\, 0 \leq t \leq 1.$$

Next lemma gives an estimate from above for the discrete energy $E(g_i^\delta, \Gamma_i^\delta)$.

Lemma 4.2. *For every $1 \leq i \leq N_\delta$ we have*

$$E(g_i^\delta, \Gamma_i^\delta) \leq E(g_0^\delta, \Gamma_0^\delta) + 2\int_0^{t_i^\delta} B(u^\delta(t), \dot g(t))\, dt + o(\delta), \tag{4.8}$$

where $o(\delta) \to 0$ as $\delta \to 0$.

Proof. By (4.5), comparing u_{j+1}^δ with $u_j^\delta - g_j^\delta + g_{j+1}^\delta$, we have that for every $0 \leq j \leq N_\delta - 1$

$$\begin{aligned} B\Big(u_{j+1}^\delta, u_{j+1}^\delta\Big) + \mathcal{H}^1\Big(\Gamma_{j+1}^\delta\Big) &\leq B\Big(u_j^\delta + g_{j+1}^\delta - g_j^\delta, u_j^\delta + g_{j+1}^\delta - g_j^\delta\Big) + \mathcal{H}^1\Big(\Gamma_i^\delta\Big) \leq \\ &\leq B\Big(u_j^\delta, u_j^\delta\Big) + \mathcal{H}^1\Big(\Gamma_j^\delta\Big) + 2B\Big(u_j^\delta, \int_{t_j^\delta}^{t_{j+1}^\delta} \dot g(t)\, dt\Big) + 2\|D^2 g_{j+1}^\delta - D^2 g_j^\delta\|^2 \leq \\ &\leq B\Big(u_j^\delta, u_j^\delta\Big) + \mathcal{H}^1\Big(\Gamma_j^\delta\Big) + 2\int_{t_j^\delta}^{t_{j+1}^\delta} B\Big(u_j^\delta, \dot g(t)\Big)\, dt + S(\delta)\int_{t_j^\delta}^{t_{j+1}^\delta} \|D^2 \dot g(t)\|\, dt, \end{aligned} \tag{4.9}$$

where

$$S(\delta) := 2 \max_{0 \leq l \leq N_\delta - 1} \int_{t_l^\delta}^{t_{l+1}^\delta} \|D^2 \dot g(t)\|\, dt.$$

Considering the sum for $j = 0$ to $i-1$ in (4.9), we obtain

$$\begin{aligned} B\Big(u_i^\delta, u_i^\delta\Big) + \mathcal{H}^1(\Gamma_i^\delta) &\leq B\Big(u_0^\delta, u_0^\delta\Big) + \mathcal{H}^1(\Gamma_0^\delta) + \\ &+ 2\int_0^{t_i^\delta} B\Big(u^\delta(t), \dot g(t)\Big)\, dt + S(\delta)\int_0^{t_i^\delta} \|D^2 \dot g\|\, dt, \end{aligned}$$

that implies (4.8) by choosing $o(\delta) := S(\delta)\int_0^1 \|D^2 \dot g(t)\|\, dt$. ■

5. Stability of the unilateral free-discontinuity problem

In the minimum problem (4.1), the unknown set Γ_i^δ minimizes the energy $E(g, H)$ among all $H \in \mathcal{K}_m(\overline{\Omega})$ such that $\Gamma_{i-1}^\delta \subseteq H$. In particular Γ_i^δ minimizes the energy among all H larger then Γ_i^δ. This is a so called *unilateral free-discontinuity problem*.

More precisely let $g \in W^{2,2}(\Omega)$, let $K \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K) \leq \infty$ and let u be a solution of the minimum problem (3.2). We say that the pair (u, K) is an unilateral minimum relative to the boundary condition g if

$$E(g, K) \leq E(g, H) \text{ for all } H \in \mathcal{K}_m(\overline{\Omega}) \text{ such that } K \subset H. \tag{5.1}$$

The aim of this section is to study the stability of the unilateral minimality property (5.1) among a sequence of closed sets (K_h) (Theorem 5.2), and this result will be a key point for the proof of the equilibrium condition for the crack $\Gamma(t)$. We need the following version of the transfer of jumps Theorem, proved in the setting of BV functions in [10]. The proof is postponed to Section 8.

Theorem 5.1 (transfer of jumps Theorem)**.** *Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $\mathcal{H}^1(K_h) \leq C$ for some fixed positive constant C. Let (g_h) be a sequence in $W^{2,2}(\Omega)$ which converges to g strongly in $W^{2,2}(\Omega)$ and let H in $\mathcal{K}_m(\overline{\Omega})$ with $K \subseteq H$ and $\mathcal{H}^1(H) \leq C$. Then there exists a sequence $(H_h) \subseteq \mathcal{K}_m(\overline{\Omega})$ converging to H in the Hausdorff metric, with $K_h \subseteq H_h$ for every h, and such that the following properties hold.*

i) $\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K)$.

ii) *For every $v \in L^{2,2}_{g,\partial_D\Omega}(\Omega \setminus H)$ there exists $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$ such that*
$$D^2 v_h \to D^2 v \qquad \text{strongly in } L^2(\Omega, \mathcal{M}^{2\times 2}).$$

We are now in position to prove the main result of this section.

Theorem 5.2. *Let (g_h) be a sequence in $W^{2,2}(\Omega)$ which converges to some g strongly in $W^{2,2}(\Omega)$. Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K_h) \leq C$, and let $u_h \in L^{2,2}(\Omega \setminus K_h)$ be such that the pair (u_h, K_h) is an unilateral minimum relative to the boundary condition g_h. Finally let us assume that*
$$D^2 u_h \rightharpoonup D^2 u \quad \text{weakly in } L^2(\Omega, \mathcal{M}^{2\times 2}), \qquad K_h \to K \text{ in the Hausdorff metric.}$$
Then the pair (u, K) is an unilateral minimum relative to the boundary condition g. Moreover $D^2 u_h$ converges to $D^2 u$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$.

Proof. Let us prove that the pair (u, K) is an unilateral minimum relative to the boundary condition g. To this aim, let $H \in \mathcal{K}_m(\overline{\Omega})$ with $K \subset H$ and let $v \in L^{2,2}(\Omega \setminus K)$. Let us consider the sequences (H_h) and (v_h) given by Theorem 5.1. By the fact that $K_h \subset H_h$, we have that $\mathcal{H}^1(H_h \setminus K_h) = \mathcal{H}^1(H_h) - \mathcal{H}^1(K_h)$. Hence by the unilateral minimality of the pair (u_h, K_h), we get

$$B(u_h, u_h) \leq B(v_h, v_h) + \mathcal{H}^1(H_h \setminus K_h). \tag{5.2}$$

Passing to the limit for $h \to \infty$ and using Theorem 5.1, we get

$$B(u, u) \leq \liminf_h B(u_h, u_h) \leq \limsup_h B(v_h, v_h) + \limsup \mathcal{H}^1(H_h \setminus K_h) = B(v, v) + \mathcal{H}^1(H \setminus K), \tag{5.3}$$

which, by the fact that $K \subset H$, is equivalent to the unilateral minimality condition. Choosing now $H = K$ and $v = u$ in (5.3), we obtain

$$B(u, u) \leq \liminf_h B(u_h, u_h) \leq \limsup B(u_h, u_h) \leq \limsup_h B(v_h, v_h) + \limsup \mathcal{H}^1(H_h \setminus K_h) = B(v, v) + \mathcal{H}^1(H \setminus K) = B(u, u).$$

We deduce that $B(u_h, u_h) \to B(u, u)$, which (together with $D^2 u_h \rightharpoonup D^2 u$) implies that $D^2 u_h$ converges to $D^2 u$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$, and this concludes the proof. ∎

Remark. In Theorem 5.2 the assumption that the minima u_h are unilateral can not be removed in order to get the stability. In fact let us consider $\Omega := (-1, 1)^2$, and
$$K_h := [-1, -1/h] \cup [1/h, 1] \times 0,$$

which converges to $K := [-1,1] \times 0$ in the Hausdorff metric. Let moreover

$$\partial_D\Omega := [-1,1] \times \{-1\} \cup [-1,1] \times \{1\},$$

and let $g_h \equiv g$ be a fixed function with normal derivative equal to 0 on $\partial_D\Omega$ and with

$$g = -1 \text{ on } [-1,1] \times \{-1\} \quad \text{and} \quad g = 1 \text{ on } [-1,1] \times \{1\}.$$

Let $Q^- := \Omega \cap \{x_2 < 0\}$ and $Q^+ := \Omega \cap \{x_2 > 0\}$. The solution u of (3.2) in $\Omega \setminus K$ is clearly the function with zero energy defined by

$$u(x) := \begin{cases} 1 & \text{if } x \in Q^+; \\ -1 & \text{if } x \in Q^-. \end{cases}$$

If u_h are the solutions of (3.2) in $\Omega \setminus K_h$, it is easy to see that D^2u_h does not converge to D^2u weakly in $L^2(\Omega, \mathcal{M}^{2\times 2})$; otherwise we will have that u_h converges to -1 uniformly in Q^- and to $+1$ uniformly in Q^+. On the other hand by symmetry we have that $u_h(0,0) \equiv 0$, and this gives a contradiction.

6. Irreversible quasi static growth of the cracks

In this Section we prove the main result of the paper, that is the existence of an irreversible quasi static growth of brittle cracks as formulated in Section 2.

Theorem 6.1. *Let m be a positive integer, let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with finite length and let $g \in AC([0,1]; W^{2,2}(\Omega))$. Then there exists an irreversible quasi static growth $\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$ relative to the initial crack K_0 and to the boundary datum g.*

Proof. Let Γ_δ be the step function defined in (4.7). As proved in [5, Theorem 6.3.], there exists a sequence $\delta_n \to 0$ and an increasing function $\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$, such that for every $t \in [0,1]$

$$\Gamma_{\delta_n}(t) \to \Gamma(t) \quad \text{in the Hausdorff metric.} \tag{6.1}$$

We claim that Γ is a quasi static growth. For every $t \in [0,1]$, we set $u(t)$ as a solution of (3.2) in $\Omega \setminus \Gamma(t)$ relative to the boundary condition $g(t)$. We have that

$$E\Big(g(0), \Gamma(0)\Big) \le E\Big(g(t), H\Big) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H. \tag{6.2}$$

In fact $\Gamma_\delta(0)$ does not depend on δ, that is $\Gamma_\delta(0) \equiv \Gamma(0)$ for every δ. Then (6.2) follows directly by (4.5) with $i = 0$. Now we prove that

$$E(g(t), \Gamma(t)) \le E(g(t), H) \quad \forall t \in (0,1], \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s \le t}\Gamma(s) \subseteq H. \tag{6.3}$$

To this aim, note that for every fixed $t \in [0,1]$ the pair $(u_{\delta_n}(t), \Gamma_{\delta_n}(t))$ is an unilateral minimum relative to the boundary condition $g_{\delta_n}(t)$. For every t we have by construction that $\Gamma^{\delta_n}(t) \to \Gamma(t)$ in the Hausdorff metric. Moreover, up to a subsequence $D^2u^{\delta_n}(t) \rightharpoonup D^2\tilde{u}$ for some $\tilde{u} \in L^{2,2}(\Omega \setminus \Gamma(t))$. Recalling that $g_{\delta_n}(t)$ converges to $g(t)$ strongly in $W^{2,2}(\Omega)$, we are in position to apply Theorem 5.2, so that the pair $(\tilde{u}, \Gamma(t))$ is an unilateral minimum relative to the boundary condition $g(t)$. By the fact that both $\tilde{u}$ and u are minimizers of (3.2), we deduce that $D^2\tilde{u} = D^2u(t)$, and hence for every t the pair $(u(t), \Gamma(t))$ is an unilateral minimum relative to the boundary condition $g(t)$, that is (6.3) holds. Moreover, as a consequence of Theorem 5.2 we also get

$$D^2u^{\delta_n}(t) \to D^2u(t) \quad \text{strongly in } L^2(\Omega, \mathcal{M}^{2\times 2}) \text{ for every } t \in [0,1]. \tag{6.4}$$

Now we prove that all properties defining the quasi static growth are satisfied.

1) Irreversibility of the process. This property holds by construction.

3) Nondissipativity. Using (6.4) and (4.8), we easily get

$$B(u(t),u(t))+\mathcal{H}^1(\Gamma(t))\leq \liminf \big(B(u^{\delta_n}(t),u^{\delta_n}(t))+\mathcal{H}^1(\Gamma^{\delta_n}(t)\big)\leq$$
$$\leq E(g_0^{\delta_n},\Gamma_0^{\delta_n})+\liminf 2\int_0^t B(u^{\delta_n}(\tau),\dot g(\tau))\,d\tau = E(g(0),\Gamma(0))+2\int_0^t B(u(\tau),\dot g(\tau))d\tau.$$

To prove the inverse inequality, given $t\in[0,1]$ and given a positive integer k, let us set $s_i^k:=\frac{i}{k}t$ for all $i=0,\dots,k$. By (6.3), comparing $u(s_i^k)$ with $u(s_{i+1}^k)-g(s_{i+1}^k)+g(s_i^k)$, we get

$$\begin{aligned}
B(u(s_i^k),u(s_i^k))+\mathcal{H}^1(\Gamma((s_i^k))\leq\\
B\Big(\big(u(s_{i+1}^k)-g(s_{i+1}^k)+g(s_i^k)\big),\big(u(s_{i+1}^k)-g(s_{i+1}^k)+g(s_i^k)\big)\Big)+\mathcal{H}^1(\Gamma(s_{i+1}^k))=\\
B\big(u(s_{i+1}^k),u(s_{i+1}^k)\big)+\mathcal{H}^1(\Gamma(s_{i+1}^k))+B\Big(\big(g(s_{i+1}^k)-g(s_i^k)\big),\big(g(s_{i+1}^k)-g(s_i^k)\big)\Big)\\
-2\int_{s_i^k}^{s_{i+1}^k} B\big(u(s_{i+1}^k),\dot g(\tau)\big)\,d\tau.
\end{aligned}\tag{6.5}$$

Summing for $i=0$ to k in (6.5), and setting $u^k(t)=u(s_{i+1}^k)$ for $s_i^k\leq t<s_{i+1}^k$, we get

$$B(u(0),u(0))+\mathcal{H}^1(\Gamma(0))+2\int_0^t B(u^k(\tau),\dot g(\tau))\,d\tau\leq B(u(t),u(t))+\mathcal{H}^1(\Gamma(t))+o_k,\tag{6.6}$$

where $o_k\to 0$ as $k\to\infty$. Let us set now $\Gamma^k(t)=\Gamma(s_{i+1}^k)$ for $s_i^k\leq t<s_{i+1}^k$. By construction we have that $(u^k(t),\Gamma^k(t))$ is an unilateral minimum for every t. If t is a continuity point for the function $l\to\mathcal{H}^1(\Gamma(l))$ it is easy to check that $\Gamma^k(t)$ converges to $\Gamma(t)$ in the Hausdorff metric. Arguing as in the proof of (6.4) we have that $D^2u^k(t)$ converges to $D^2u(t)$ strongly in $L^2(\Omega,\mathcal{M}^{2\times2})$ so that $B(u^k(\tau,\dot g(\tau))$ converges to $B(u^k(\tau,\dot g(\tau))$ for a.e. τ. Therefore passing to the limit for $k\to\infty$ in (6.6) we obtain

$$E(g(t),\Gamma(t))\geq E(g(0),\Gamma(0))+2\int_0^t B(u(\tau),\dot g(\tau))d\tau.\tag{6.7}$$

2) Static equilibrium. Let us fix $t\in(0,1)$ and let (s_h) be an increasing sequence converging to t. By (6.3) we get

$$B(u(s_h),u(s_h))+\mathcal{H}^1(\Gamma(s_h))\leq B(v-g(t)+g(s_h),v-g(t)+g(s_h))+\mathcal{H}^1(H)$$

for every $H\in\mathcal{K}_m(\overline{\Omega}):\cup_{s<t}\Gamma(s)\subseteq H$ and for every $v\in L^{2,2}_{g(t),\partial_D\Omega}(\Omega\setminus H)$. Letting $h\to\infty$ and using that the function $t\to E(g(t),\Gamma(t))$ is continuous by the nondissipativity condition, we deduce

$$B(u(t),u(t))+\mathcal{H}^1(\Gamma(t))\leq B(v,v)+\mathcal{H}^1(H),$$

for every $H\in\mathcal{K}_m(\overline{\Omega}):\cup_{s<t}\Gamma(s)\subseteq H$ and for every $v\in L^{2,2}_{g(t),\partial_D\Omega}(\Omega\setminus H)$, so that also static equilibrium property holds. ■

7. Griffith's criterion for crack growth

In this section we shall see that, in the model case where the crack $\Gamma(t)$ is rectilinear, it satisfies Griffith's criterion for crack growth. More precisely let Ω be open and connected, and let $\partial_D\Omega$ be a (non empty) open subset of $\partial\Omega$ composed of a finite number of connected components. We consider a quasi static growth $\Gamma(t)$, relative to a boundary datum $g \in AC([0,1]; W^{2,2}(\Omega))$, of the following type (see Fig. 1):

$$\Gamma(t) := [0, x_1(t)] \times \{x_2\}, \tag{7.1}$$

where $x_1 : [0,1] \to [l_1, l_2]$ is an increasing function and $[0, l_1] \times \{x_2\}$ is a preexisting crack which touches the boundary of Ω at the point $(0, x_2)$. For every $x_1 \in [l_1, l_2]$ we set

$$K(x_1) := [0, x_1] \times x_2.$$

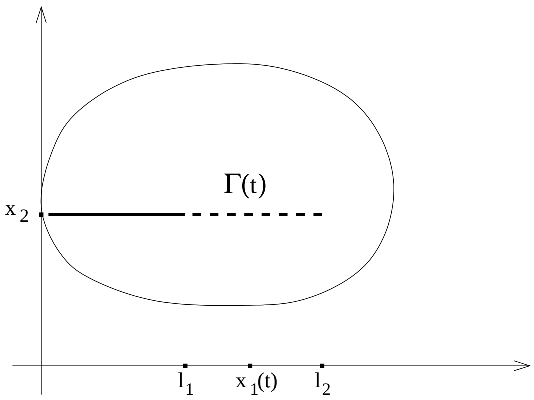

Fig. 1

We want to compute the derivative of the bulk energy $\mathcal{E}_b(g(t), K(x_1))$ defined in (3.4) with respect to the growth of the crack (that is with respect to x_1) at the point $x_1(t)$. For every function $v \in L^{2,2}(\Omega \setminus \Gamma(t))$, we set

$$\begin{aligned} M_{11}[v] &:= v_{x_1x_1} + k\, v_{x_2x_2}; \\ M_{22}[v] &:= v_{x_2x_2} + k\, v_{x_1x_1}; \\ M_{12}[v] &= M_{21}[v] := (1-k)\, v_{x_1x_2}. \end{aligned}$$

Let now C be a smooth closed path around the point $(x_1(t), x_2)$ and let $u(t)$ be a solution of (3.2) in $\Omega \setminus \Gamma(t)$. In [16], [6], is proved that the functional $x_1 \to \mathcal{E}_b(g(t)), K(x_1))$ is C^1, and that the following formula holds.

$$\begin{aligned} \frac{d}{dx_1}\mathcal{E}_b(g(t)), K(x_1))\,|_{x_1 = x_1(t)} = &\sum_{i,j\in\{1,2\}} \Big(-\frac{1}{2}\int_C M_{ij}[u(t)]\, u_{x_ix_j}(t)\nu_1 \\ &+ \int_C M_{ij}[u(t)]\, u_{x_1x_j}(t)\nu_i - \int_C \frac{\partial}{\partial x_j} M_{ij}[u(t)] \frac{\partial}{\partial x_1} u(t)\nu_i(t)\Big) \\ &+ \big(M_{12}[u(t)]\frac{\partial}{\partial x_1} u(x^+)(t)\big) - \big(M_{12}[u(t)]\frac{\partial}{\partial x_1} u(x^-)(t)\big), \end{aligned} \tag{7.2}$$

where $\nu = (\nu_1, \nu_2)$ is the inner normal to C and x^+ and x^- are as in Fig. 2.

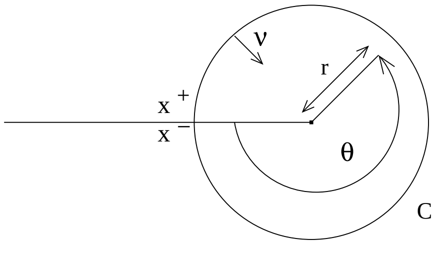

Fig. 2

It is well known that the solution $u(t)$ has the following behavior in a neighborhood of the tip $(x_1(t), x_2)$ (see [6], [13]).

$$u(t)(r,\theta) = r^{3/2}\Big(b_1(t)\big(\sin(\frac{3\theta}{2}) + \frac{3(1-k)}{7+k}\sin(\frac{\theta}{2})\big) + b_2(t)\big(\cos(\frac{3\theta}{2}) + \frac{3(1-k)}{5+3k}\cos(\frac{\theta}{2})\big)\Big) + u^R(r,\theta), \tag{7.3}$$

where (r,θ) are the polar coordinates as in Figure 2, and $u^R \in W^{3,2}(\Omega)$.

For every b_1, $b_2 \in \mathbb{R}$ we set

$$u^S(b_1,b_2)(r,\theta) := r^{3/2}\Big(b_1\big(\sin(\frac{3\theta}{2})+\frac{3(1-k)}{7+k}\sin(\frac{\theta}{2})\big)+b_2\big(\cos(\frac{3\theta}{2})+\frac{3(1-k)}{5+3k}\cos(\frac{\theta}{2})\big)\Big), \tag{7.4}$$

so that $u(t) = u^S(b_1(t), b_2(t)) + u^R(t)$. Now let us fix a radius $\varepsilon > 0$. For every v, $w \in L^{2,2}(\Omega \setminus \Gamma(t))$, we consider the bilinear form $b^\varepsilon : L^{2,2}(\Omega\setminus\Gamma(t)) \times L^{2,2}(\Omega\setminus\Gamma(t)) \to \mathbb{R}$ defined by

$$b^\varepsilon(v,w) := \sum_{i,j\in\{1,2\}} \Big(-\frac{1}{2}\int_{B_\varepsilon((x_1(t),x_2))} M_{ij}[v] w_{x_i x_j}\nu_1 + \int_{B_\varepsilon((x_1(t),x_2))} M_{ij}[v] w_{x_1 x_j}\nu_i - \int_{B_\varepsilon((x_1(t),x_2))} \frac{\partial}{\partial x_j} M_{ij}[v]\frac{\partial}{\partial x_1} w\nu_i(t)\Big) + \big(M_{12}[u(t)]\frac{\partial}{\partial x_1}w(x^+)(t)\big) - \big(M_{12}[u(t)]\frac{\partial}{\partial x_1}w(x^-)(t)\big), \tag{7.5}$$

Finally, for every b_1, $b_2 \in \mathbb{R}$ we define the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$q(b_1,b_2) := -b^\varepsilon(u^S(b_1,b_2), u^S(b_1,b_2)). \tag{7.6}$$

From (7.4) and (7.5) it easily follows that q does not depend on ε. The explicit computation of the right hand-side of (7.6), leads to the following expression

$$q(b_1,b_2) = 9\pi(1+k)^2\left(\frac{b_1^2}{(7+k)^2} + \frac{b_2^2}{(5+3k)^2}\right).$$

In order to prove that $q(b_1(t), b_2(t))$ is the only contribution that does not vanish in (7.2) as ε tends to zero, we will need the following Lemma.

Lemma 7.1. *Let $B_h(z)$ be an open ball in $\mathbb{R}^2$ and let $f \in L^p(B_h(z))$. Then there exists a subset $I \subset [0,h]$ such that*

$$\lim_{l\to 0}\frac{|I\cap[0,l]|}{l} = 1,$$

and such that for every sequence $\{\varepsilon_n\} \subset I$ *with* $\{\varepsilon_n\} \to 0$, *we have that* f *is defined for* $\mathcal{H}^1$*-a.e.* $x \in \partial B_{\varepsilon_n}(z)$, *and*

$$\lim_{n\to\infty} \quad \varepsilon_n^{\frac{2-p}{p}} \int_{\partial B_{\varepsilon_n}(z)} |f(x)|\, dx \ \to 0.$$

Proof. We have

$$\int_{B_h(z)} |f|^p = \int_0^h \int_{\partial B_r(z)} |f|^p = \int_0^h 2\pi r \Big(\frac{1}{2\pi r} \int_{\partial B_r(z)} |f|^p \Big).$$

By the fact that $f \in L^p(B_h(z))$, using Jensen inequality, there exists a positive constant C such that

$$\int_0^h r^{1-p} \left(\int_{\partial B_r(z)} |f| \right)^p \leq C.$$

Now suppose by contradiction that there exist $\delta_1, \delta_2 > 0$ such that, setting

$$U := \{ r : r^{(2-p)/p} \int_{\partial B_r(z)} |f| \geq \delta_1 \},$$

there exist arbitrary small intervals $[0, J]$ with

$$\frac{|U \cap [0, J]|}{J} \geq \delta_2.$$

We deduce that

$$\int_{[0,J]} r^{1-p} \left(\int_{\partial B_r(z)} |f| \right)^p = \int_{[0,J]} \frac{1}{r} \left(r^{(2-p)/p\xi} \int_{\partial B_r(z)} |f| \right)^p \geq$$
$$\int_{[0,J]\cap U} \frac{1}{r}\, \delta_1^p \geq \frac{1}{J}\, \delta_1^p \, |[0, J] \cap U| \geq \delta_1^p \, \delta_2,$$

and this, by the arbitrariness of J, is in contradiction with the equi-integrability of the L^1 function $r^{1-p}\big(\int_{\partial B_r(z)} |f| \big)^p$. ■

Next theorem gives a more explicit formula then (7.2) for the derivative of the bulk energy with respect to the growth of the crack. We will see that this derivative actually depends only on the coefficients in (7.3) of the singular part of u.

Theorem 7.2. *Let* $\Gamma(\cdot)$ *be a quasi static growth of the type* (7.1), *and let* $b_1(\cdot)$, $b_2(\cdot)$ *be the coefficients in* (7.3). *Then for every* $t \in (0, 1)$

$$\frac{d}{dx_1} \mathcal{E}_b(g(t)), K(x_1)) \,|_{x_1 = x_1(t)} = -q(b_1(t), b_2(t)).$$

Proof. By (7.2), we have that there exists $h > 0$ such that for every $\varepsilon \leq h$

$$\begin{aligned} &\frac{d}{dx_1} \mathcal{E}_b(g(t)), K(x_1)) \,|_{x_1 = x_1(t)} = b^\varepsilon \big(u^S(b_1(t), b_2(t)) + u^R(t), u^S(b_1(t), b_2(t)) + u^R(t) \big) = \\ &\quad -q\big(b_1(t), b_2(t)\big) + b^\varepsilon \big(u^S(b_1(t), b_2(t)), u^R(t) \big) + b^\varepsilon \big(u^R(t), u^S(b_1(t), b_2(t)) \big) + \\ &\quad b^\varepsilon \big(u^R(t), u^R(t) \big). \end{aligned} \tag{7.7}$$

We claim that there exists subsets $I, L, M \subset (0, h)$ with the property

$$\lim_{l\to 0} \frac{I \cap [0, l]}{l} = 1, \qquad \lim_{l\to 0} \frac{L \cap [0, l]}{l} = 1, \qquad \lim_{l\to 0} \frac{M \cap [0, l]}{l} = 1, \tag{7.8}$$

such that the following hold:

1) $\lim_{n\to\infty} b^{\varepsilon_n}(u^S(b_1(t),b_2(t)),u^R(t))=0 \qquad \forall\{\varepsilon_n\}\subset I,\ \varepsilon_n\to 0;$

2) $\lim_{n\to\infty} b^{\varepsilon_n}(u^R(t),u^S(b_1(t),b_2(t)))=0 \qquad \forall\{\varepsilon_n\}\subset L,\ \varepsilon_n\to 0;$

3) $\lim_{n\to\infty} b^{\varepsilon_n}(u^R(t),u^R(t))=0 \qquad \forall\{\varepsilon_n\}\subset M,\ \varepsilon_n\to 0.$

Therefore, it is sufficient to consider a sequence (ε_n) in $I\cap L\cap M$ (which exists in view of (7.8)): along this sequence, the right hand side of (7.7) tends to $-q(b_1(t),b_2(t))$, and this concludes the proof. So let us prove 1), the proof of 2) and 3) being similar.

To this aim, note that we can always assume that $\nabla u^R(t)(x_1(t),x_2)=0$. In fact, for every fixed $\xi\in\mathbb{R}^n$ and for every $s\in(0,1)$, we have that the function

$$u^S(b_1(s),b_2(s))+(u^R(s)+\xi\cdot x)$$

is a solution of (3.2) relative to the boundary condition $g(s)+\xi\cdot x$, and

$$\mathcal{E}_b\big((g(s)),K(x_1(s))\big)=\mathcal{E}_b\big((g(s)+\xi\cdot x(s)),K(x_1(s))\big).$$

We have

$$\begin{aligned} b^\varepsilon\big(u^S(b_1(t),b_2(t)),u^R(t)\big) &\le C\int_{\partial B_\varepsilon((x_1(t),x_2))}\varepsilon^{-1/2}|f(x)|+ \\ &+\sum_{i,j\in\{1,2\}}\int_{\partial B_\varepsilon((x_1(t),x_2))}\varepsilon^{-3/2}|\frac{\partial}{\partial x_1}u^R(t)|+ \end{aligned} \tag{7.9}$$

$$\big(M_{12}[u^S(b_1(t),b_2(t))]\frac{\partial}{\partial x_1}u^R(x^+)(t)\big)-\big(M_{12}[u^S(b_1(t),b_2(t))]\frac{\partial}{\partial x_1}u^R(x^-)(t)\big),$$

where C is a positive constant and $f\in H^1(\Omega)$. By Lemma 7.1, noting that by the Sobolev embedding Theorem $f\in L^p\big(B_\varepsilon(x_1(t))\big)$ for every $p\ge 1$, there exists a subset $I\subset(0,h)$ that verifies (7.8) and such that

$$\lim_{n\to\infty} C\int_{\partial B_{\varepsilon_n}(x_1(t)}\varepsilon_n^{-1/2}f(x)=0 \qquad \forall\{\varepsilon_n\}\subset I,\ \varepsilon_n\to 0.$$

Concerning the second term, note that the function $\frac{\partial}{\partial x_1}u^R(t)$ is in $H^2(\Omega)$, so that it is holder continuous with coefficient greater than 1/2 (it is for instance in $C^{0,2/3}(\Omega)$), and hence, recalling that $\nabla u^R(t)(x_1(t),x_2)=0$, we have that for every i,j and for ε small enough

$$\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-3/2}|\frac{\partial}{\partial x_1}u^R(t)|\le C\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-3/2}\varepsilon^{2/3}=C\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-5/6}$$

which tends to zero as $\varepsilon\to 0$. Finally last term is equal to zero for every ε because

$$\frac{\partial}{\partial x_1}u^R(x^+)(t)=\frac{\partial}{\partial x_1}u^R(x^-)(t), \qquad M_{12}[u^S(b_1(t),b_2(t))](x^+)=M_{12}[u^S(b_1(t),b_2(t))](x^-).$$

■

By Theorem 7.2, we deduce the following formula for the derivative of the total energy with respect to the growth of the crack.

$$\frac{d}{dx_1}E(g(t)),K(x_1))\,|_{x_1=x_1(t)}=1-q(b_1(t),b_2(t)). \tag{7.10}$$

Moreover, arguing as in [5], it is possible to prove that for every $t \in (0,1)$

$$\frac{d}{ds}E(g(t),\Gamma(s))\,|_{s=t} = 0.$$

We are now in position to state the main result of this section.

Theorem 7.3. *Let $\Gamma(t)$ be a quasi static growth of the type* (7.1)*. Then*

$$\dot{x}_1(t) \geq 0 \qquad \text{for a.e. } t \in (0,1), \tag{7.11}$$

$$1 - q(b_1(t), b_2(t)) \geq 0 \qquad \text{for every } t \in (0,1), \tag{7.12}$$

$$\big(1 - q(b_1(t), b_2(t))\big)\dot{x_1}(t) = 0 \qquad \text{for a.e. } t \in (0,1). \tag{7.13}$$

Proof. The first condition comes directly from the irreversibility of the process. The second condition comes directly from (7.10) and from the static equilibrium condition.

So, let us pass to the prove of the third condition, that actually is the Griffith's criterion for crack growth in our model. let $t \in [0,1]$ be a point of differentiability for $x_1(t)$. We have

$$0 = \frac{d}{ds}E(g(t),\Gamma(s))\,|_{s=t} = \frac{d}{dx_1}E(g(t),K(x_1))\,|_{x_1=x_1(t)}\,\dot{x}_1(t) = \big(1 - q(b_1(t), b_2(t))\big)\,\dot{x}_1(t).$$

and this concludes the proof. ■

8. Proof of the transfer of jumps Theorem

In this section we prove the transfer of jumps Theorem (Theorem 5.1). In the proof we will need the following lemma, which is a particular case of [5][Lemma 3.6].

Lemma 8.1. *Let U be an open bounded subset of $\mathbb{R}^2$ with Lipschitz continuous boundary, let $p \geq m \geq 0$ and let (K_h) be a sequence in $\mathcal{K}_p(\overline{U})$ converging to some $K \in \mathcal{K}_m(\overline{U})$ and uniformly bounded in length. Then there exists $(J_h) \subset \mathcal{K}_m(\overline{U})$ converging to K, with $K_h \subset J_h$, and such that*

$$\lim_h \mathcal{H}^1(J_h \setminus K_h) \to 0.$$

We are now in position to prove the transfer of jumps Theorem.

Proof. [Proof of Theorem 5.1] For every $x \in K$ which satisfies (2.2), for every $0 < \delta < 1$ and for every $r > 0$ let us set

$$R_r(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : |(z-x)\cdot\nu(x)| < (\delta/2)r\};$$
$$B_r^+(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : (z-x)\cdot\nu(x) > \delta r\};$$
$$B_r^-(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : (z-x)\cdot\nu(x) < -\delta r\};$$
$$L_r(x) := \partial B_r(x) \cap \partial R_r(x).$$

The idea of the proof is the following. We would like to recover K with small balls $B_r(x)$ such that (up to small errors in length) K cuts every $B_r(x)$ into two connected components. Then, in order to have the same geometrical configuration for the sequence K_h, we have to enlarge a bit them, obtaining a new sequence of closed sets which still cut every $B_r(x)$ into two connected components which we denote now by $D_r^+(x)$ and $D_r^-(x)$, so that $B_r^+(x) \subset D_r^+(x)$ and $B_r^-(x) \subset D_r^-(x)$. We add to this sequence of enlarged K_h the set $\overline{H \setminus K}$, obtaining a sequence which looks like the H_h of Theorem 5.1. Now we have to approximate v with functions $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$. This procedure is called *transfer of the jump's set* . We set

$v_h = v$ far from K, while around K we consider the restriction of v on every $B_r^+(x)$ (respectively on $B_r^-(x)$) and we extend it on $D_r^+(x)$ (respectively on $D_r^-(x)$), obtaining in this way a function whose jumps are contained in H_h. With a further modification we also obtain the right boundary datum. However the rigorous proof presents some additional difficulties; for instance it will need some technical effort in order to ensure H_h to be in $\mathcal{K}_m(\overline{\Omega})$. In order to keep rigorous this rough idea, let us claim as follows.

Claim. For every $0 < \delta < 1$ and for every $\varepsilon > 0$ there exists a finite family of disjoint balls $\{B_{r_1}(x_1), \dots B_{r_N}(x_N)\}$ (where N depends on ε), and there exists a sequence $(H_h^{\delta,\varepsilon}) \subset \mathcal{K}_m(\overline{\Omega})$ of closed sets, such that for every i the following properties hold.

a) $H \cap B_{r_i}(x_i) \subset R_{r_i}(x_i)$;

b) Either $B_{r_i}^+(x_i) \subset \Omega$ or $B_{r_i}^+(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

c) Either $B_{r_i}^-(x_i) \subset \Omega$ or $B_{r_i}^-(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

d) For h large enough $H_h^{\delta,\varepsilon} \cap (B_{r_i}^+(x_i) \cup B_{r_i}^-(x_i)) = \emptyset$. Moreover $B_{r_i}^+(x_i)$ and $B_{r_i}^-(x_i)$ are in two different connected components of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$;

e) $\mathcal{H}^1(K \setminus \cup_{i=1}^N B_{r_i}(x_i)) \le \varepsilon$;

f) $r_i \le \varepsilon$;

g) $K_h \cup \overline{H \setminus K} \subseteq H_h^{\delta,\varepsilon}$. Moreover
$$\lim_h \mathcal{H}^1(H_h^{\delta,\varepsilon} \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta), \qquad \text{where } o(\delta) \to 0 \text{ as } \delta \to 0.$$

Using the claim, we construct a sequence $v_h^{\delta,\varepsilon} \in L^{2,2}(\Omega \setminus H_h^{\delta,\varepsilon})$ as follows. For every $1 \le i \le N$, by property d) we can define (for h large enough) $D_{i,h}^+$ as the connected component of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$ containing $B_{r_i}^+(x_i)$, and similarly $D_{i,h}^-$ as the connected component of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$ containing $B_{r_i}^-(x_i)$. Let us define the function $v_h^{\delta,\varepsilon}$ on every $D_{i,h}^+ \cap \Omega$ (the case $D_{i,h}^- \cap \Omega$ being similar). If $B_{i,h}^+$ is contained in $\mathbb{R}^2 \setminus \overline{\Omega}$, we define $v_h^{\delta,\varepsilon} = g$ on $D_{i,h}^+ \cap \Omega$. Otherwise, by property b) we have that $B_{i,h}^+ \subset \Omega$. Let v_i^+ be the restriction of v on $B_{r_i}^+(x_i)$. By property a) we have that $v_i^+ \in L^{2,2}(B_{r_i}^+(x_i))$, so that we can consider its extension $E(v_i^+)$ on $\mathbb{R}^2$ given by Lemma 2.4. We define $v_h^{\delta,\varepsilon} = E(v_i^+)$ on $D_{i,h}^+$. Finally we define $v_h^{\delta,\varepsilon} = v$ on
$$\Omega \setminus \bigcup_i \Big(D_{i,h}^+ \cup D_{i,h}^- \Big).$$

Note that by construction $v_h^{\delta,\varepsilon} \in L^{2,2}_{g,\partial_D \Omega}(\Omega \setminus H_h^{\delta,\varepsilon})$. Moreover by Lemma 2.4 there exists a positive constant C_δ (independent on ε) such that

$$\int_\Omega |D^2 v - D^2 v_h^{\delta,\varepsilon}|^2 \, dx \le C_\delta \sum_i \int_{(D_i^+ \cup D_i^-)\cap\Omega} \|D^2 v\|^2 \, dx. \tag{8.1}$$

Let us fix two sequences $(\delta_h) \to 0$ and $(\varepsilon_h) \to 0$, and let us repeat the construction of the sets $H_h^{\delta_h,\varepsilon_h}$ as described above. Using property f) and the equi-integrability of $\|D^2 v\|^2$, we can assume without loss of generality that (δ_h) and (ε_h) are chosen such that right hand side of (8.1) tends to zero as h tends to infinity. Moreover by a diagonal argument (i.e. by freezing δ_h and ε_h) we can also assume that property d) holds for every h with $H_h^{\delta_h,\varepsilon_h}$ in place of $H_h^{\delta,\varepsilon}$, so that for every h we can construct the functions $v_h^{\delta_h,\varepsilon_h}$ as described previously. We set

$$H_h := H_h^{\delta_h,\varepsilon_h} \qquad v_h := v_h^{\delta_h,\varepsilon_h} - g + g_h. \tag{8.2}$$

By property g) we have $K_h \subset H_h$, $H_h \to H$ in the Hausdorff metric, and

$$\lim_h \mathcal{H}^1(H_h \setminus K_h) = \mathcal{H}^1(H \setminus K).$$

On the other hand $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$, and by (8.1) and the choice of δ_h, ε_h we have that $D^2 v_h \to D^2 v$ strongly in $L^2(\Omega; \mathcal{M}^{2\times 2})$, so that using the claim the proof of the Theorem is completed.

Let us pass to the proof of the claim. From now on $0 < \delta < 1$ and $\varepsilon > 0$ are keep fixed. For almost every $x \in K$, if r is small enough (depending on x) the following properties hold.

i) $H \cap B_r(x) \subset R_r(x)$;
ii) Either $B_r^+(x) \subset \Omega$ or $B_r^+(x) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;
iii) Either $B_r^-(x) \subset \Omega$ or $B_r^-(x) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;
iv) There exists a closed segment $S_r(x) \subset R_r(x)$ with $\mathcal{H}^1(S_r(x)) \leq \delta r$ and such that $(K \cap B_r(x)) \cup L_r(x) \cup S_r(x)$ is connected.

In fact, by Lemma 2.2 we can assume that $x \in K$ is a point satisfying (2.2) and (2.3). Property i) follows by (2.3) for r small enough. Properties ii) and iii) are trivial if $x \in \Omega$ and $r < d(x, \partial\Omega)$, while if $x \in \partial\Omega$, it holds at every x which admits the approximate normal to $\partial\Omega$ with r small enough; the fact that $\partial\Omega$ is Lipschitz ensure that such x have full measure in $K \cap \partial\Omega$. Let us pass to the proof of iv). Let m be the minimum of the diameter of the connected components of K which are not single points (so that $m > 0$). We can always assume that there are not isolated points in $K \cap B_r(x)$ and that $2r < m$. We deduce that every connected component of $K \cap B_r(x)$ intersect $\partial B_r(x)$, otherwise there will be a connected component of K with diameter smaller than m. On the other hand by (2.2) for r small enough $K \cap B_r(x) \subset R_r(x)$, and hence every connected component of $K \cap B_r(x)$ intersects $L_r(x)$. Let us denote by $L_r^L(x)$ and $L_r^R(x)$ the two connected components of $L_r(x)$ and let $K_r^L(x)$ (respectively $K_r^R(x)$) be the union of all connected components of $K \cap B_r(x)$ which intersect $L_r^L(x)$ (respectively $L_r^R(x)$). By (2.2) we have that

$$\frac{d_H(K_r^L(x) \cup L_r^L(x), K_r^R(x) \cup L_r^R(x))}{r} \to 0 \quad \text{as } r \to 0.$$

We deduce that there are two points $a_r \in K_r^L(x) \cup L_r^L(x)$ and $b_r \in K_r^R(x) \cup L_r^R(x)$ with

$$\frac{|a_r - b_r|}{r} \to 0 \quad \text{as } r \to 0. \tag{8.3}$$

We set S_r as the closed segment with end points a_r and b_r. By construction we have that $(K \cap B_r(x)) \cup L_r(x) \cup S_r$ is connected, and by (8.3) for r small enough $\mathcal{H}^1(S_r) \leq \delta r$, and this concludes the proof of property iv).

By properties i)-iv) above, applying Vitali-Besicovitch covering Theorem (see for instance [1]), we can consider a finite family of disjoint balls, $\{B_{r_1}(x_1), \dots B_{r_N}(x_N)\}$ (where N depends on ε), such that for every $1 \leq i \leq N$ the following properties hold:

1) $H \cap B_{r_i}(x_i) \subset R_{r_i}(x_i)$;
2) Either $B_{r_i}^+(x_i) \subset \Omega$ or $B_{r_i}^+(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

3) Either $B^-_{r_i}(x_i) \subset \Omega$ or $B^-_{r_i}(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;
4) There exists a closed segment $S_{r_i}(x_i) \subset R_{r_i}(x_i)$ with $\mathcal{H}^1(S_{r_i}(x_i)) \leq \delta r_i$ and such that $(K \cap B_{r_i}(x_i)) \cup L_{r_i}(x_i) \cup S_{r_i}(x_i)$ is connected;
5) $\mathcal{H}^1(K \setminus \cup_{i=1}^N B_{r_i}(x_i)) \leq \varepsilon$;
6) $r_i \leq \varepsilon$.

Properties 1), 2), 3), 5) and 6) are exactly properties a), b), c) e) and f) of the Claim. In order to prove properties d) and g) let us fix $1 \leq i \leq N$ and let us set

$$\tilde{K} := K \cup S_{r_i}(x_i) \cup L_{r_i}(x_i), \qquad \tilde{K}_h := K_h \cup S_{r_i}(x_i) \cup L_{r_i}(x_i).$$

Note that $\tilde{K}$ and $\tilde{K}_h$ have at most $m+3$ connected components. By property 4) $\tilde{K} \cap \overline{B}_{r_i}(x_i)$ is connected, and hence there exists a connected component $\tilde{K}^i$ of $\tilde{K}$ which contains $\tilde{K} \cap \overline{B}_{r_i}(x_i)$. Let $C^1_h, \dots, C^l_h$ be the connected components of $\tilde{K}_h$ converging to the sets $C^1, \dots, C^l$ composing $\tilde{K}^i$, i.e. such that $\cup_{j=1}^l C^j = \tilde{K}^i$. We have $l \leq m+3$ and we can thus apply Lemma 8.1 to the sequence $\cup_{j=1}^l C^j_h$, obtaining that there exists a sequence of connected sets J^i_h in $\overline{\Omega}$ which still converges to $\tilde{K}^i$ in the Hausdorff metric and such that $\lim_h \mathcal{H}^1(J^i_h \setminus \tilde{K}_h) = 0$. Therefore we have

$$\limsup_h \mathcal{H}^1(J^i_h \setminus K_h) \leq \mathcal{H}^1(L_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{8.4}$$

Let us enlarge $L_{r_i}(x_i)$; more precisely let us set

$$\tilde{L}_{r_i}(x_i) := \{x \in \partial B_{r_i}(x_i) : d(x, L_{r_i}(x_i)) \leq a\},$$

where a is a positive constant chosen such that $\tilde{L}_{r_i}(x_i)$ does not intersect neither $B^+_{r_i}(x_i)$ nor $B^-_{r_i}(x_i)$. The sequence $J^i_h \cap \overline{B}_{r_i}(x_i)$ converges to $\tilde{K}^i \cap \overline{B}_{r_i}(x_i) = \tilde{K} \cap \overline{B}_{r_i}(x_i)$ in the Hausdorff metric, which is contained in $\overline{R}_{r_i}(x_i)$. We deduce that for h large enough every connected component of $J^i_h \cap \overline{B}_{r_i}(x_i)$ can intersect $B_{r_i}(x_i)$ only on $\tilde{L}_{r_i}(x_i)$. Therefore, recalling that J^i_h is connected, we have that $(J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ has at most three connected components and it converges to the connected set $(\tilde{K} \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ in the Hausdorff metric. Applying again Lemma 8.1 to the sequence $(J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ we deduce that there exists a sequence of connected sets I^i_h in $\overline{B}_{r_i}(x_i)$ converging to $(\tilde{K} \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ and such that

$$\lim_h \mathcal{H}^1\Big(I^i_h \setminus \big((J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)\big)\Big) = 0. \tag{8.5}$$

Note that by the fact that I^i_h is connected, contains $\tilde{L}_{r_i}(x_i)$, and for h large enough does not intersect neither $B^+_{r_i}(x_i)$ nor $B^-_{r_i}(x_i)$, it follows that

$$B^+_{r_i}(x_i) \text{ and } B^-_{r_i}(x_i) \text{ are in two different connected components of } B_{r_i}(x_i) \setminus I^i_h. \tag{8.6}$$

By (8.4) we have

$$\limsup_h \mathcal{H}^1\big((J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i) \setminus K_h\big) \leq \mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{8.7}$$

By (8.5) and (8.7) we obtain

$$\limsup_h \mathcal{H}^1(I^i_h \setminus K_h) \leq \mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{8.8}$$

Let us repeat the construction above for every $1 \le i \le N$, and let us set

$$I_h^{\delta,\varepsilon} := \bigcup_{i=1}^{N} I_h^i \cup K_h \cup \overline{H \setminus K}. \tag{8.9}$$

For every i we have

$$\mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)) \le C\delta r_i, \tag{8.10}$$

for a positive constant C (independent on δ and ε). Moreover by property 4) it follows that

$$\mathcal{H}^1(K \cap B_{r_i}(x_i)) \ge (1-\delta) r_i,$$

and hence

$$|\cup_{i=1}^{N} \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)| \le C \frac{\delta}{1-\delta} \mathcal{H}^1(K). \tag{8.11}$$

By (8.8), (8.9) and (8.11) we obtain

$$\lim_h \mathcal{H}^1(I_h^{\delta,\varepsilon} \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta), \tag{8.12}$$

where $o(\delta) \to 0$ as $\delta \to 0$.

By the fact that $K \subset H$, $K \in \mathcal{K}_m(\overline{\Omega})$, $H \in \mathcal{K}_m(\overline{\Omega})$, and that every I_h^i is connected, we deduce that the number of connected components of $I_h^{\delta,\varepsilon}$ is uniformly bounded with respect to h. Moreover $I_h^{\delta,\varepsilon}$ converges to $H \cup_{i=1}^{N} \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)$, which by construction and by property 4) has at most m connected components. By Lemma 8.1 there exists a sequence $\tilde{I}_h^{\delta,\varepsilon} \in \mathcal{K}_m(\overline{\Omega})$ which contain $I_h^{\delta,\varepsilon}$, which still converge to $H \cup_{i=1}^{N} \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)$ in the Hausdorff metric and with

$$\lim_h \mathcal{H}^1(\tilde{I}_h^{\delta,\varepsilon} \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta). \tag{8.13}$$

The construction of $\tilde{I}_h^{\delta,\varepsilon}$ does not ensure that $\tilde{I}_h^{\delta,\varepsilon}$ are contained in $\overline{\Omega}$. Therefore we have to project every $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ on $\partial\Omega$ as follows. For every connected component C of $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ we set $\partial_C \Omega$ as the connected subset of $\partial\Omega$ with minimal length which contains $C \cap \partial\Omega$. By the fact that $\partial\Omega$ is Lipschitz, we deduce that there exists a positive constant L such that

$$\mathcal{H}^1(\partial_C \Omega) \le L \mathcal{H}^1(C). \tag{8.14}$$

Therefore, substituting every connected component C of $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ with the corresponding $\partial_C \Omega$ we obtain a sequence $H_h^{\delta,\varepsilon}$ which by construction is in $\mathcal{K}_m(\overline{\Omega})$, by (8.9) contains $K_h \cup \overline{H \setminus K}$, so that by (8.13) and (8.14) satisfies property g) of the Claim. Moreover by construction and by (8.6) we deduce that also property d) holds. This concludes the proof of the Claim and therefore of the Theorem. ■

9. Conclusions and remarks

In Theorem 6.1 we proved the existence of an irreversible quasi static growth of cracks for a plate clamped on a part of its boundary. More general boundary conditions can be treated with these methods. We mention for instance the case of the so called *hinged plate*, where no conditions are imposed on the normal derivative of the displacement u on $\partial_D\Omega$. In this case, it is sufficient to set the minimum problem (3.2) in the space

$$\{u \in L^{2,2}(\Omega \setminus K) : u - g = 0 \text{ on } \partial_D\Omega \text{ in the sense of traces}\}.$$

The main tool used is Theorem 5.1, which leads to the stability of unilateral minimality problems like (4.1). Note that the proof of Theorem 5.1 is based on a geometrical construction, and can be extended in the framework of $L^{k,p}$ spaces (i.e. the space of functions in L^p_{loc} with derivatives of order k in L^p (see [14]). Therefore the stability of unilateral minimum problems like (4.1) holds for more general energies $E : L^{k,p}(\Omega) \to \mathbb{R}$ depending on the k-order derivatives of u and with standard p-growth hypothesis. It is also possible to treat energies depending on the point x of the reference configuration Ω, as in the case of shells.

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