

Isoperimetric inequality under Measure-Contraction property

Fabio Cavalletti* and Flavia Santarcangelo†

Abstract

We prove that if (X, d, \mathfrak{m}) is an essentially non-branching metric measure space with $\mathfrak{m}(X) = 1$, having Ricci curvature bounded from below by K and dimension bounded from above by $N \in (1, \infty)$, understood as a synthetic condition called Measure-Contraction property, then a sharp isoperimetric inequality à la Lévy-Gromov holds true. Measure theoretic rigidity is also obtained.

1 Introduction

The isoperimetric problem is one of the most classical problems in mathematics; it addresses the following natural problem: given a space X what is the minimal amount of area needed to enclose a fixed volume v . If the space X has a simple structure or has many symmetries the problem can be completely solved and the “optimal shapes” can be explicitly described (e.g. Euclidean space and the sphere). In the general case however one cannot hope to obtain a complete solution to the problem and a comparison result is already completely satisfactory. Probably the most popular result in this direction is the Lévy-Gromov isoperimetric inequality [30, Appendix C] stating that if E is a (sufficiently regular) subset of a Riemannian manifold (M, g) of dimension N and Ricci bounded below by $K > 0$, then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|}, \quad (1.1)$$

where B is a spherical cap in the model sphere S , i.e. the N -dimensional round sphere with constant Ricci curvature equal to K , and $|M|, |S|, |\partial E|, |\partial B|$ denote the appropriate N or $N - 1$ dimensional volume, and where B is chosen so that $|E|/|M| = |B|/|S|$.

Lévy-Gromov isoperimetric inequality has been then extended to more general settings; for the scope of this note, the most relevant progress was the one obtained by E. Milman [37] for smooth manifolds with densities, i.e. smooth Riemannian manifold whose volume measure has been multiplied by a smooth non-negative integrable density function, having Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by N in a generalized sense, i.e. verifying the so called Curvature-Dimension condition $CD(K, N)$ introduced in the 1980's by Bakry and Émery [6, 7]. E. Milman detected a

*SISSA, Trieste 34136, Italy.

†SISSA, Trieste 34136, Italy.

model isoperimetric profile $\mathcal{I}_{K,N,D}^{\text{CD}}$ such that if a Riemannian manifold with density verifying $\text{CD}(K, N)$ has diameter at most $D > 0$, then the isoperimetric profile function of the weighted manifold is bounded from below by $\mathcal{I}_{K,N,D}^{\text{CD}}$.

After the works of Cordero-Erausquin–McCann–Schmuckenschläger [25], Otto–Villani [42] and von Renesse–Sturm [47], it was realized that the $\text{CD}(K, \infty)$ condition in the smooth setting may be equivalently formulated synthetically as a certain convexity property of an entropy functional along W_2 Wasserstein geodesics (associated to L^2 -Optimal-Transport). This idea led Lott–Villani [36] and Sturm [51, 52], to propose a successful (and compatible with the classical one) synthetic definition of $\text{CD}(K, N)$ for a general (complete, separable) metric space (X, \mathbf{d}) endowed with a (locally-finite Borel) reference measure \mathbf{m} (“metric-measure space”, or m.m.s.); the theory of m.m.s.’s verifying $\text{CD}(K, N)$ has then extensively developed leading to a rich and fruitful approach to the geometry of m.m.s.’s by means of Optimal-Transport [2, 3, 4, 27, 26, 5, 38, 29, 33, 13]. See also [1] for a recent account on the topic.

Building on the work by Klartag [34] and the localization paradigm developed by Payne–Weinberger [44], Gromov–Milman [31] and Kannan–Lovász–Simonovits [32], the first author with Mondino [19] managed to extend Lévy–Gromov–Milman isoperimetric inequality to the class of *essentially non-branching* (see Section 2 for the definition) m.m.s.’s verifying $\text{CD}(K, N)$ with $\mathbf{m}(X) = 1$; in particular [19] proves that

$$\mathbf{m}^+(A) \geq \mathcal{I}_{K,N,D}^{\text{CD}}(\mathbf{m}(A)), \quad (1.2)$$

whenever $A \subset X$ and

$$\mathbf{m}^+(A) = \liminf_{\varepsilon \rightarrow 0} \frac{\mathbf{m}(A^\varepsilon) - \mathbf{m}(A)}{\varepsilon},$$

is the Minkowski content of A and A^ε is the ε -enlargement of A given by $A^\varepsilon = \{x \in X : \mathbf{d}(x, A) < \varepsilon\}$. The isoperimetric inequality (1.2) is equivalent to the following inequality

$$\mathcal{I}_{(X,\mathbf{d},\mathbf{m})}(v) \geq \mathcal{I}_{K,N,D}^{\text{CD}}(v),$$

for all $v \in (0, 1)$. Here $\mathcal{I}_{(X,\mathbf{d},\mathbf{m})}$ denotes the isoperimetric profile function of the m.m.s. $(X, \mathbf{d}, \mathbf{m})$ defined as follows

$$\mathcal{I}_{(X,\mathbf{d},\mathbf{m})}(v) := \inf\{\mathbf{m}^+(A) : A \subset X \text{ Borel, } \mathbf{m}(A) = v\}.$$

In some cases, given a one-dimensional density h defined on the real interval (a, b) integrating to 1, we will adopt the shorter notation \mathcal{I}_h to denote the isoperimetric profile function $\mathcal{I}_{((a,b),|\cdot|,h\mathcal{L}^1)}$.

1.1 Isoperimetric inequality under Measure-Contraction property

The Measure Contraction Property $\text{MCP}(K, N)$ was introduced independently by Ohta in [40] and Sturm in [52] as a weaker variant of $\text{CD}(K, N)$. Roughly, the idea is to only require the $\text{CD}(K, N)$ condition to hold not for any couple of probability measures μ_0, μ_1 absolutely continuous with respect to the reference measure \mathbf{m} , but when μ_1 degenerates to a delta-measure at $o \in \text{supp}(\mathbf{m})$.

Still retaining a weaker synthetic lower bound on the Ricci curvature, an upper bound on the dimension and stability in the measured Gromov-Hausdorff sense (see also [41] for

further properties), $\text{MCP}(K, N)$ includes a larger family of spaces than $\text{CD}(K, N)$. It is now well known for instance that the Heisenberg group equipped with a left-invariant measure, which is the simplest sub-Riemannian structure, does not satisfy any form of $\text{CD}(K, N)$ and do satisfy $\text{MCP}(0, N)$ for a suitable choice of N , see [35]. It is worth mentioning that MCP was first investigated in Carnot groups in [35, 48], see also [9].

Recently, interpolation inequalities à la Cordero-Erausquin–McCann–Schmuckenshläger [25] have been obtained, under suitable modifications, by Barilari and Rizzi [10] in the ideal sub-Riemannian setting and by Balogh, Kristly and Sipos [8] for the Heisenberg group. As a consequence, an increasing number of examples of spaces verifying MCP and not CD is at disposal, e.g. the Heisenberg group, generalized H-type groups, the Grushin plane and Sasakian structures (for more details, see [10]). In all the previous examples a sharp isoperimetric inequality is not at disposal yet; due to lack of regularity of minimizers, sharp isoperimetric inequality has been proved for subclasses of competitors having extra regularity or additional symmetries; in particular, Pansu Conjecture [43] is still unsolved. For more details we refer to [39, 49, 50, 12] and references therein.

In this paper we address the isoperimetric inequality à la Lévy-Gromov within the class of spaces verifying MCP . In particular, we identify a family of one-dimensional $\text{MCP}(K, N)$ -densities, each for every choice of K, N , volume v and diameter D , not verifying $\text{CD}(K, N)$, and having optimal perimeter; we thus call the optimal perimeter $\mathcal{I}_{K,N,D}(v)$ and obtain the main result of this note.

Theorem 1.1. *[Theorem 4.1] Let $K, N \in \mathbb{R}$ with $N > 1$ and let $(X, \mathbf{d}, \mathbf{m})$ be an essentially non-branching m.m.s. verifying $\text{MCP}(K, N)$ with $\mathbf{m}(X) = 1$ and having diameter less than D .*

For any $A \subset X$,

$$\mathbf{m}^+(A) \geq \mathcal{I}_{K,N,D}(\mathbf{m}(A)). \quad (1.3)$$

Moreover (1.3) is sharp, i.e. for each $v \in [0, 1]$, K, N, D there exists a m.m.s. $(X, \mathbf{d}, \mathbf{m})$ with $\mathbf{m}(X) = 1$ and $A \subset X$ with $\mathbf{m}(A) = v$ such that (1.3) is an equality.

Finally for each K, N, D and $v \in (0, 1)$, $\mathcal{I}_{K,N,D}(v) < \mathcal{I}_{K,N,D}^{\text{CD}}(v)$.

Via localization paradigm for MCP -spaces (see Section 4 for details), following [34, 19], the proof of Theorem 1.1 is reduced to the proof of the corresponding statement in the one-dimensional setting. However, contrary to the CD framework, due to lack of any form of concavity, the isoperimetric problem for a general one-dimensional $\text{MCP}(K, N)$ -density seems to be out of reach. We instead directly exhibit, for each K, N, D and v , an optimal one-dimensional $\text{MCP}(K, N)$ -density, denoted by $h_{K,N,D,v}$ that will be optimal only for that choice of K, N, D and v . In particular,

$$\mathcal{I}_{K,N,D}(v) = \begin{cases} h_{K,N,D,v}(a_{K,N,D}(v)), & K \leq 0, \\ \min_{D' \leq D} h_{K,N,D',v}(a_{K,N,D'}(v)), & K > 0. \end{cases} \quad (1.4)$$

where $a_{K,N,D}(v)$ is the unique point of $[0, D]$ such that $\int_{[0, a_{K,N,D}(v)]} h_{K,N,D,v}(x) dx = v$; in particular

$$\mathcal{I}_{h_{K,N,D,v}}(v) = h_{K,N,D,v}(a_{K,N,D}(v)),$$

for all K, N, D and v . To explain (1.4), we underline that for each K, N, D and v , $h_{K,N,D,v}(a_{K,N,D}(v))$ is the optimal perimeter when minimization is constrained to all one-dimensional $\text{MCP}(K, N)$ -densities (integrating to 1) having support of *exactly* length D , see Theorem 3.7. Denoting the optimal value of the latter minimization problem by $\tilde{\mathcal{I}}_{K,N,D}(v)$, the previous sentence reads as

$$\tilde{\mathcal{I}}_{K,N,D}(v) = \mathcal{I}_{h_{K,N,D,v}}(v). \quad (1.5)$$

Hence (1.4) is a direct consequence of the following fact: $\tilde{\mathcal{I}}_{K,N,D}(v)$ is strictly decreasing as a function of D only if $K \leq 0$, showing a remarkable difference with the CD-framework (see [37]).

The rigidity property of Lévy-Gromov isoperimetric inequality is a well-known fact: if a Riemannian manifold verifies the equality case in (1.1) then it is isometric to the round sphere of the correct dimension [30]; if equality is attained in (1.2) and the metric measure space verifies the stronger $\text{RCD}(K, N)$ condition (see [2, 3, 27, 26, 5] and references therein), then it is isomorphic in the metric-measure sense to a spherical suspension (see [19] for details). At the present generality, i.e. the class of m.m.s.'s verifying $\text{MCP}(K, N)$, competitors are less regular and a weaker rigidity is valid.

In particular, the proof of Theorem 3.7 is sufficiently stable to imply one-dimensional rigidity (Theorem 3.11), valid for each choice of K, N, D and v . Building on this and on the monotonicity in D of $\tilde{\mathcal{I}}_{K,N,D}(v)$, we show that whenever $K \leq 0$ the optimal metric measure space has a product structure in a measure theoretic sense (see Theorem 4.2 for the precise result).

We conclude the Introduction presenting the structure of the paper. Section 2 contains some basics on the theory of m.m.s.'s verifying synthetic lower bounds on Ricci curvature. Section 3 proves the new main one-dimensional facts on $\text{MCP}(K, N)$ -densities; Section 4 contains the main results of the paper and a general overview on localization technique.

2 Backgrounds

A triple $(X, \mathbf{d}, \mathbf{m})$ is called metric measure space (or m.m.s.) if (X, \mathbf{d}) a Polish space (i.e. a complete and separable metric space) and \mathbf{m} is a positive Radon measure over X ; in this work however we will always assume $\mathbf{m}(X) = 1$.

We denote by

$$\text{Geo}(X) := \{\gamma \in C([0, 1], X) : \mathbf{d}(\gamma_s, \gamma_t) = |s - t| \mathbf{d}(\gamma_0, \gamma_1), \text{ for every } s, t \in [0, 1]\}.$$

the space of constant speed geodesics. The metric space (X, \mathbf{d}) is a *geodesic space* if and only if for each $x, y \in X$ there exists $\gamma \in \text{Geo}(X)$ so that $\gamma_0 = x, \gamma_1 = y$. For complete geodesic spaces, local compactness is equivalent to properness (a metric space is proper if every closed ball is compact).

$\mathcal{P}(X)$ denotes the space of all Borel probability measures over X and with $\mathcal{P}_2(X)$ the space of probability measures with finite second moment. $\mathcal{P}_2(X)$ can be endowed with the L^2 -Kantorovich-Wasserstein distance W_2 defined as follows: for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, set

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi} \int_{X \times X} \mathbf{d}^2(x, y) \pi(dx dy), \quad (2.1)$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal. The space (X, d) is geodesic if and only if the space $(\mathcal{P}_2(X), W_2)$ is geodesic.

For any $t \in [0, 1]$, let e_t denote the evaluation map:

$$e_t : \text{Geo}(X) \rightarrow X, \quad e_t(\gamma) := \gamma_t.$$

Any geodesic $(\mu_t)_{t \in [0, 1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\text{Geo}(X))$, so that $(e_t)_\# \nu = \mu_t$ for all $t \in [0, 1]$.

Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\text{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\text{Geo}(X))$ for which $(e_0, e_1)_\# \nu$ realizes the minimum in (2.1). Such a ν will be called *dynamical optimal plan*. If (X, d) is geodesic, then the set $\text{OptGeo}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$.

We will also consider the subspace $\mathcal{P}_2(X, d, \mathfrak{m}) \subset \mathcal{P}_2(X)$ formed by all those measures absolutely continuous with respect with \mathfrak{m} .

In the paper we will only consider essentially non-branching spaces, let us recall their definition (introduced in [46]).

A set $G \subset \text{Geo}(X)$ is a set of non-branching geodesics if and only if for any $\gamma^1, \gamma^2 \in G$, it holds:

$$\exists \bar{t} \in (0, 1) \text{ such that } \forall t \in [0, \bar{t}] \quad \gamma_t^1 = \gamma_t^2 \implies \gamma_s^1 = \gamma_s^2, \quad \forall s \in [0, 1].$$

Definition 2.1. A metric measure space (X, d, \mathfrak{m}) is *essentially non-branching* (e.n.b. for short) if and only if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, with μ_0, μ_1 absolutely continuous with respect to \mathfrak{m} , any element of $\text{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

It is clear that if (X, d) is a smooth Riemannian manifold then any subset $G \subset \text{Geo}(X)$ is a set of non-branching geodesics, in particular any smooth Riemannian manifold is essentially non-branching.

It is worth stressing that the restriction to essentially non-branching m.m.s.'s is done to avoid pathological cases: as an example of possible pathological behaviour we mention the failure of the local-to-global property of $\text{CD}(K, N)$ within this class of spaces; in particular, a heavily-branching m.m.s. verifying $\text{CD}_{loc}(0, 4)$ which does not verify $\text{CD}(K, N)$ for any fixed $K \in \mathbb{R}$ and $N \in [1, \infty]$ was constructed by Rajala in [45], while the local-to-global property of $\text{CD}(K, N)$ has been recently proved to hold [18] for essentially non-branching m.m.s.'s.

2.1 Measure-Contraction Property

We briefly describe the MCP condition encapsulating generalized Ricci curvature lower bounds coupled with generalized dimension upper bounds.

Definition 2.2 ($\sigma_{K, \mathcal{N}}$ -coefficients). Given $K \in \mathbb{R}$ and $\mathcal{N} \in (0, \infty]$, define:

$$D_{K, \mathcal{N}} := \begin{cases} \frac{\pi}{\sqrt{K/\mathcal{N}}} & K > 0, \mathcal{N} < \infty \\ +\infty & \text{otherwise} \end{cases}.$$

In addition, given $t \in [0, 1]$ and $0 < \theta < D_{K, \mathcal{N}}$, define:

$$\sigma_{K, \mathcal{N}}^{(t)}(\theta) := \begin{cases} \frac{\sin(t\theta\sqrt{\frac{K}{\mathcal{N}}})}{\sin(\theta\sqrt{\frac{K}{\mathcal{N}}})} & K > 0, \mathcal{N} < \infty \\ t & K = 0 \text{ or } \mathcal{N} = \infty, \\ \frac{\sinh(t\theta\sqrt{\frac{-K}{\mathcal{N}}})}{\sinh(\theta\sqrt{\frac{-K}{\mathcal{N}}})} & K < 0, \mathcal{N} < \infty \end{cases}$$

and set $\sigma_{K, \mathcal{N}}^{(t)}(0) = t$ and $\sigma_{K, \mathcal{N}}^{(t)}(\theta) = +\infty$ for $\theta \geq D_{K, \mathcal{N}}$. Finally given $K \in \mathbb{R}$ and $N \in (1, \infty]$, define:

$$\tau_{K, N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K, N-1}^{(t)}(\theta)^{1 - \frac{1}{N}}.$$

When $N = 1$, set $\tau_{K, 1}^{(t)}(\theta) = t$ if $K \leq 0$ and $\tau_{K, 1}^{(t)}(\theta) = +\infty$ if $K > 0$.

Definition 2.3 (MCP(K, N)). A m.m.s. $(X, \mathbf{d}, \mathbf{m})$ is said to satisfy MCP(K, N) if for any $o \in \text{supp}(\mathbf{m})$ and $\mu_0 \in \mathcal{P}_2(X, \mathbf{d}, \mathbf{m})$ of the form $\mu_0 = \frac{1}{\mathbf{m}(A)} \mathbf{m} \llcorner_A$ for some Borel set $A \subset X$ with $0 < \mathbf{m}(A) < \infty$ (and with $A \subset B(o, \pi\sqrt{(N-1)/K})$ if $K > 0$), there exists $\nu \in \text{OptGeo}(\mu_0, \delta_o)$ such that:

$$\frac{1}{\mathbf{m}(A)} \mathbf{m} \geq (e_t)_\# (\tau_{K, N}^{(1-t)}(\mathbf{d}(\gamma_0, \gamma_1))^N \nu(d\gamma)) \quad \forall t \in [0, 1]. \quad (2.2)$$

If $(X, \mathbf{d}, \mathbf{m})$ is a m.m.s. verifying MCP(K, N), then $(\text{supp}(\mathbf{m}), \mathbf{d})$ is Polish, proper and it is a geodesic space. With no loss in generality for our purposes we will assume that $X = \text{supp}(\mathbf{m})$. Many additional results on the structure of W_2 -geodesics can be obtained just from the MCP condition together with the essentially non-branching assumption (see [21]).

To conclude, referring to [40, 52] for more general results, we report the following important fact [40, Theorem 3.2]: if (M, g) is n -dimensional Riemannian manifold with $n \geq 2$, the m.m.s. (M, d_g, vol_g) verifies MCP(K, n) if and only if $\text{Ric}_g \geq Kg$, where d_g is the geodesic distance induced by g and vol_g the volume measure.

A relevant case for our purposes (due to the crucial use of the localization technique) is the one of one-dimensional spaces $(X, \mathbf{d}, \mathbf{m}) = (I, |\cdot|, h\mathcal{L}^1)$. It is a standard fact that the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$ verifies MCP(K, N) if and only if the non-negative Borel function h satisfies the following inequality:

$$h(tx_1 + (1-t)x_0) \geq \sigma_{K, N-1}^{(1-t)}(|x_1 - x_0|)^{N-1} h(x_0). \quad (2.3)$$

for all $x_0, x_1 \in I$ and $t \in [0, 1]$. We will call h an MCP(K, N)-density.

Inequality (2.3) implies several known properties that we recall for readers convenience. To write them in a unified way, we define for $\kappa \in \mathbb{R}$ the function $s_\kappa : [0, +\infty) \rightarrow \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa < 0. \end{cases} \quad (2.4)$$

For the moment we confine ourselves to the case $I = (a, b)$ with $a, b \in \mathbb{R}$; hence (2.3) implies (actually is equivalent to)

$$\left(\frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)} \right)^{N-1}, \quad (2.5)$$

for $x_0 \leq x_1$. In particular, h is locally Lipschitz in the interior of I and continuous up to the boundary. The next lemma was stated and proved in [18, Lemma A.8] under the CD condition; as the proof only uses MCP(K, N) we report it in this more general version.

Lemma 2.4. *Let h denote a MCP(K, N) density on a finite interval (a, b) , $N \in (1, \infty)$, which integrates to 1. Then:*

$$\sup_{x \in (a, b)} h(x) \leq \frac{1}{b-a} \begin{cases} N & K \geq 0 \\ (\int_0^1 (\sigma_{K, N-1}^{(t)}(b-a))^{N-1} dt)^{-1} & K < 0 \end{cases}. \quad (2.6)$$

In particular, for fixed K and N , h is uniformly bounded from above as long as $b-a$ is uniformly bounded away from 0 (and from above if $K < 0$).

3 One-dimensional analysis

The isoperimetric problem for a one-dimensional density h verifying MCP(K, N) for some $K, N \in \mathbb{R}$ and $N > 1$ will be addressed in this section.

Without loss of generality we can assume h to be defined over $[0, D]$ (recall that $D \leq \pi\sqrt{(N-1)/K}$, whenever $K > 0$). Recall that the case $K > 0$ and $D = \pi\sqrt{(N-1)/K}$ is trivial as (2.5) forces the density to coincide with the model density $\sin^{N-1}(t)$ (that in particular is also a CD(K, N)-density).

Proposition 3.1 (Lower Bound). *Define the following strictly positive function*

$$f_{K, N, D}(x) := \left(\int_{(0, x)} \left(\frac{s_{K/(N-1)}(D-y)}{s_{K/(N-1)}(D-x)} \right)^{N-1} dy + \int_{(x, D)} \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(x)} \right)^{N-1} dy \right)^{-1}$$

for $x \in (0, D)$ and equal 0 for $x = 0, D$. Then

- i) $f_{K, N, D}$ is strictly increasing over $(0, D/2)$;
- ii) $f_{K, N, D}(x) = f_{K, N, D}(D-x)$;
- iii) if $h : [0, D] \rightarrow \mathbb{R}$ is an MCP(K, N)-density integrating to 1, then $h(x) \geq f_{K, N, D}(x)$.

Proof. The second claim is straightforward to check. For the first one, being $f_{K, N, D}$ a smooth function and strictly positive in $(0, D)$, it will be enough to show that $f'_{K, N, D}(x) = 0$ has no solution for $x \in (0, D/2)$. Imposing $f'_{K, N, D}(x) = 0$ is equivalent to

$$\frac{s'_{K/(N-1)}(D-x)}{s_{K/(N-1)}^N(D-x)} \int_{(0, x)} s_{K/(N-1)}^{N-1}(D-y) dy = \frac{s'_{K/(N-1)}(x)}{s_{K/(N-1)}^N(x)} \int_{(x, D)} s_{K/(N-1)}^{N-1}(y) dy,$$

that can be rewritten as

$$\frac{s'_{K/(N-1)}(D-x)}{s_{K/(N-1)}^N(D-x)} \int_{(D-x,D)} s_{K/(N-1)}^{N-1}(y) dy = \frac{s'_{K/(N-1)}(x)}{s_{K/(N-1)}^N(x)} \int_{(x,D)} s_{K/(N-1)}^{N-1}(y) dy.$$

Since $D-x \geq x$, the previous identity implies

$$\frac{|s'_{K/(N-1)}(D-x)|}{s_{K/(N-1)}^N(D-x)} > \frac{|s'_{K/(N-1)}(x)|}{s_{K/(N-1)}^N(x)}. \quad (3.1)$$

For $K=0$, (3.1) becomes

$$\frac{1}{(D-x)^N} > \frac{1}{x^N},$$

giving a contradiction. For negative $K = -(N-1)$ (the other negative cases follow similarly) (3.1) implies

$$\frac{\cosh(D-x)}{\sinh(D-x)^N} > \frac{\cosh(x)}{\sinh(x)^N}$$

forcing

$$\frac{\cosh(D-x)}{\sinh(D-x)} > \left(\frac{\sinh(D-x)}{\sinh(x)} \right)^{N-1} \frac{\cosh(x)}{\sinh(x)} > \frac{\cosh(x)}{\sinh(x)},$$

giving a contradiction with monotonicity of \tanh . Finally, for $K = N-1$, (3.1) becomes

$$\frac{\cos(D-x)}{\sin(D-x)^N} > \frac{\cos(x)}{\sin(x)^N}, \quad \text{sgn}(\cos(D-x)) = \text{sgn}(\cos(x));$$

the second identity implies that $x < D-x < \pi/2$ or $\pi/2 < x < D-x$. The second case would imply that $D > 2x > \pi$ giving a contradiction. Hence we are left with $x < D-x < \pi/2$:

$$1 > \frac{\cos(D-x)}{\cos x} > \left(\frac{\sin(D-x)}{\sin x} \right)^N,$$

giving a contradiction.

The third claim follows simply observing that (2.5) gives

$$\begin{aligned} 1 &= \int_{(0,x)} h(y) dy + \int_{(x,D)} h(y) dy \\ &\leq \frac{h(x)}{s_{K/(N-1)}^{N-1}(D-x)} \int_{(0,x)} s_{K/(N-1)}^{N-1}(D-y) dy + \frac{h(x)}{s_{K/(N-1)}^{N-1}(x)} \int_{(x,D)} s_{K/(N-1)}^{N-1}(y) dy, \end{aligned}$$

and the claim is proved. \square

Starting from the lower bound of Proposition 3.1, we define a distinguished family of $\text{MCP}(K, N)$ densities, depending on four parameters, that will be the model one-dimensional isoperimetric density:

$$h_{K,N,D}^a(x) := f_{K,N,D}(a) \begin{cases} \left(\frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(D-a)} \right)^{N-1}, & x \leq a, \\ \left(\frac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(a)} \right)^{N-1}, & x \geq a. \end{cases} \quad (3.2)$$

Notice that $h_{K,N,D}^{D-a}(D-x) = h_{K,N,D}^a(x)$ and

$$h_{K,N,D}^a(zD/D') = h_{(D/D')^2K,N,D'}^{aD'/D}(z),$$

showing that it will no be restrictive to assume for some of the next proofs $K = N - 1$ or $K = -(N - 1)$, letting D vary.

Corollary 3.2 (Rigidity of lower bound). *Let $h : [0, D] \rightarrow \mathbb{R}$ be a MCP(K, N)-density integrating to 1. Assume $h(y) = f_{K,N,D}(y)$ for some $y \in (0, D)$; then $h = h_{K,N,D}^y$.*

Proof. From the proof Proposition 3.1, point *iii*), and (2.5) one deduces that

$$h(x) = h(y) \begin{cases} \left(\frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(D-a)} \right)^{N-1}, & x \leq a, \\ \left(\frac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(a)} \right)^{N-1}, & x \geq a. \end{cases}$$

The claim then follows. □

To avoid cumbersome notation, the dependence of $h_{K,N,D}^a$ on K, N, D will be omitted and we will use h_a .

Lemma 3.3. *For every $a \in (0, D)$, the function h_a integrates to 1 and it is an MCP(K, N)-density.*

Proof. Each h_a has by definition integral 1. To check MCP(K, N) it will be enough to verify that the inequality (2.5) is satisfied.

We start observing that the function

$$\frac{s_{K/(N-1)}(D-\cdot)}{s_{K/(N-1)}(\cdot)} \tag{3.3}$$

is decreasing in $[0, D]$; this will be proved showing its first derivative to be negative:

$$\frac{s'_{K/(N-1)}(D-a)}{s_{K/(N-1)}(a)} + \frac{s_{K/(N-1)}(D-a)s'_{K/(N-1)}(a)}{s_{K/(N-1)}^2(a)} \geq 0.$$

The previous inequality is straightforward for $K \leq 0$; for $K > 0$, assuming without loss of generality $K = N - 1$, it reduces to $\sin(a) \cos(D-a) + \sin(D-a) \cos(a) = \sin(D) \geq 0$, that is always verified with the strict inequality except for the trivial case $D = \pi$ (where the function (3.3) is identically equal to one).

Using the result just obtained, we are able to check (2.5) distinguishing three cases.

If $x_0 \leq x_1 \leq a$:

$$\left(\frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)} \right)^{N-1} = \frac{h_a(x_1)}{h_a(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)} \right)^{N-1}.$$

If $a \leq x_0 \leq x_1$:

$$\left(\frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)} \right)^{N-1} \leq \frac{h_a(x_1)}{h_a(x_0)} = \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)} \right)^{N-1}.$$

If $x_0 \leq a \leq x_1$:

$$\frac{h_a(x_1)}{h_a(x_0)} = \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)} \right)^{N-1} \cdot \left(\frac{s_{K/(N-1)}(D-a)}{s_{K/(N-1)}(D-x_0)} \right)^{N-1};$$

using again the fact that (3.3) is decreasing, we get the claim. \square

Lemma 3.4. *For every choice of K , N and D , except the case in which $K > 0$ and $D = \pi\sqrt{(N-1)/K}$, the density h_a defined in (3.2) does not verify $\text{CD}(K, N)$.*

Proof. Recall that a non-negative Borel function h defined on an interval $I \subset \mathbb{R}$ is called a $\text{CD}(K, N)$ density if for every $t \in [0, 1]$ and for all $x_0, x_1 \in I$ such that $x_0 < x_1$, it holds:

$$h((1-t)x_0 + tx_1)^{\frac{1}{N-1}} \geq \sigma_{K, N-1}^{(1-t)}(x_1 - x_0)h(x_0)^{\frac{1}{N-1}} + \sigma_{K, N-1}^{(t)}(x_1 - x_0)h(x_1)^{\frac{1}{N-1}}. \quad (3.4)$$

In order to prove our claim we will discuss several cases.

If $K = 0$, the inequality (3.4) simply reduces to the concavity of $h^{\frac{1}{N-1}}$. We will prove now that (3.4) fails for the density $h_a(\cdot)$ exactly for convex combinations that give out the point a . Pick $x_0 < a < x_1$ and let $t \in (0, 1)$ be such that $a = (1-t)x_0 + tx_1$. It follows that

$$\begin{aligned} (1-t)h_a(x_0)^{\frac{1}{N-1}} + th_a(x_1)^{\frac{1}{N-1}} &= f_{0, N, D}(a)^{\frac{1}{N-1}} \left[(1-t) \left(\frac{D-x_0}{D-a} \right) + t \left(\frac{x_1}{a} \right) \right] \\ &> f_{0, N, D}(a)^{\frac{1}{N-1}} = h_a(a)^{\frac{1}{N-1}}, \end{aligned}$$

hence(3.4) is not satisfied.

If $K \neq 0$, we argue as follows. Since $a = (1-t)x_0 + tx_1$, it should be $t = \frac{a-x_0}{x_1-x_0}$ and $1-t = \frac{x_1-a}{x_1-x_0}$. Hence, we can rewrite the second member of the inequality (3.4) in this form

$$f_{K, N, D}(a)^{\frac{1}{N-1}} \left[\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_1-x_0)} \cdot \frac{s_{K/(N-1)}(D-x_0)}{s_{K/(N-1)}(D-a)} + \frac{s_{K/(N-1)}(a-x_0)}{s_{K/(N-1)}(x_1-x_0)} \cdot \frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)} \right]; \quad (3.5)$$

using now that (3.3) is a strictly decreasing function, we get that the quantity above is strictly greater than

$$f_{K, N, D}(a)^{\frac{1}{N-1}} \left[\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_1-x_0)} \cdot \frac{s_{K/(N-1)}(x_0)}{s_{K/(N-1)}(a)} + \frac{s_{K/(N-1)}(a-x_0)}{s_{K/(N-1)}(x_1-x_0)} \cdot \frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)} \right]. \quad (3.6)$$

If $K < 0$, assuming without loss of generality that $K = -(N-1)$, we get that (3.6) can be rewritten in the following way

$$f_{-(N-1), N, D}(a)^{\frac{1}{N-1}} \left[\frac{\sinh(x_1-a)\sinh(x_0) + \sinh(a-x_0)\sinh(x_1)}{\sinh(a)\sinh(x_1-x_0)} \right] = f_{-(N-1), N, D}(a)^{\frac{1}{N-1}},$$

by straightforward computations. Arguing in the same way in the case $K > 0$ (assuming as usual that $K = N - 1$), we get that (3.6) can be rewritten in this form

$$f_{N-1,N,D}(a)^{\frac{1}{N-1}} \left[\frac{\sin(x_1 - a) \sin(x_0) + \sin(a - x_0) \sin(x_1)}{\sin(a) \sin(x_1 - x_0)} \right] = f_{N-1,N,D}(a)^{\frac{1}{N-1}}.$$

Hence the claim follows also in this case. \square

3.1 One dimensional isoperimetric inequality

To properly formulate the one-dimensional minimization problem, let us consider the following set of probabilities

$$\tilde{\mathcal{F}}_{K,N,D} = \{ \mu \in \mathcal{P}(\mathbb{R}) : \mu = h_\mu \mathcal{L}^1, h_\mu : [0, D] \rightarrow \mathbb{R}, \text{ MCP}(K, N) \text{ density} \},$$

and consider the following “restricted” minimization: for each $v \in (0, 1)$

$$\tilde{\mathcal{I}}_{K,N,D}(v) := \inf \{ \mu^+(A) : A \subset [0, D], \mu(A) = v, \mu \in \tilde{\mathcal{F}}_{K,N,D} \}.$$

The term “restricted” is motivated by the choice of fixing the domain of the $\text{MCP}(K, N)$ densities. For the “unrestricted” one-dimensional minimization we will adopt the classical notation

$$\mathcal{I}_{K,N,D}(v) := \inf \{ \mu^+(A) : A \subset [0, D], \mu(A) = v, \mu \in \mathcal{F}_{K,N,D} \}, \quad (3.7)$$

where $\mathcal{F}_{K,N,D} = \cup_{D' \leq D} \tilde{\mathcal{F}}_{K,N,D'}$.

The final claim will be to prove that each h_a is a minimum of the isoperimetric problem for the volume equal to $\int_{(0,a)} h_a(x) dx$. We will therefore show that each volume $v \in (0, 1)$ is reached in this manner.

Lemma 3.5. *The map*

$$(0, D) \ni a \mapsto v(a) := \int_{(0,a)} h_a(x) dx \in (0, 1),$$

is invertible.

Proof. It will be convenient to rewrite the function in the following way

$$v(a) = \frac{f_{K,N,D}(a)}{s_{K/(N-1)}^{N-1}(D-a)} \int_{(0,a)} s_{K/(N-1)}^{N-1}(D-x) dx \quad (3.8)$$

implying differentiability. Given the strict monotonicity of the integral with respect to the variable a , it is sufficient to prove that also the other factor is an increasing function. Since

$$\left(\frac{s_{K/(N-1)}^{N-1}(D-a)}{f_{K,N,D}(a)} \right)' = \left[\left(\frac{s_{K/(N-1)}(D-a)}{s_{K/(N-1)}(a)} \right)^{N-1} \right]' \int_{(a,D)} s_{K/(N-1)}^{N-1}(x) dx,$$

it follows that the previous derivative has the same sign of the derivative of (3.3), thus it is non positive and the claim follows. \square

Hence for each K, N, D it is possible to define the inverse map of $v(a)$ from Lemma 3.5:

$$(0, 1) \ni v \longmapsto a_{K,N,D}(v) \in (0, D),$$

with $a_{K,N,D}(v)$ the unique element such that

$$\int_{(0, a_{K,N,D}(v))} h_{a_{K,N,D}(v)}(x) dx = v. \quad (3.9)$$

For ease of notation we will prefer in few places the shorter notation a_v to denote $a_{K,N,D}(v)$.

Remark 3.6. The function $v \mapsto a_v$ enjoys a simple symmetric property: by definition we have that

$$\begin{aligned} 1 - v &= \frac{f_{K,N,D}(a_v)}{s_{K/(N-1)}^{N-1}(a_v)} \int_{(a_v, D)} s_{K/(N-1)}^{N-1}(x) dx \\ &= \frac{f_{K,N,D}(D - a_v)}{s_{K/(N-1)}^{N-1}(a_v)} \int_{(0, D - a_v)} s_{K/(N-1)}^{N-1}(D - x) dx \\ &= v(D - a_v), \end{aligned}$$

where the last identity follows from (3.8). Since there exists a unique value $a_{1-v} \in (0, D)$ such that $v(a_{1-v}) = 1 - v$, it turns out that $a_{1-v} = D - a_v$.

The first main result of this note is the following explicit formula for $\tilde{\mathcal{I}}_{K,N,D}$.

Theorem 3.7. *For each volume $v \in (0, 1)$, it holds*

$$\tilde{\mathcal{I}}_{K,N,D}(v) = f_{K,N,D}(a_{K,N,D}(v)).$$

In particular, since $f_{K,N,D}(a_{K,N,D}(v)) = h_{a_{K,N,D}(v)}(a_{K,N,D}(v))$, the lower bound is attained.

For the proof of Theorem 3.7 will be useful to consider the function $A_{K,N,D} : [0, D] \rightarrow [0, \infty)$ defined as follows:

$$A_{K,N,D}(a) := \frac{v(a)}{f_{K,N,D}(a)} = \int_{(0, a)} \left(\frac{s_{K/(N-1)}(D - x)}{s_{K/(N-1)}(D - a)} \right)^{N-1} dx. \quad (3.10)$$

We will use that $[0, D] \ni a \mapsto A_{K,N,D}(a)$ is increasing; we postpone the proof of this fact at the end of the section. From the symmetric property of a_v observed few lines above, we obtain the analogous one for $A_{K,N,D}$:

$$\frac{1 - v}{A_{K,N,D}(D - a_v)} = \frac{v(D - a_v) f_{K,N,D}(D - a_v)}{v(D - a_v)} = f_{K,N,D}(a_v). \quad (3.11)$$

Proof of Theorem 3.7. Fix $K, N, D \in \mathbb{R}$ with $N > 1$ and any $v \in (0, 1)$. Consider h_{a_v} and $h_{a_{1-v}}$ and notice that

$$\int_{(0, a_v)} h_{a_v}(x) dx = \int_{(a_{1-v}, D)} h_{a_{1-v}}(x) dx = v$$

and

$$h_{a_v}(a_v) = f_{K,N,D}(a_v) = f_{K,N,D}(a_{1-v}) = h_{a_{1-v}}(a_{1-v}),$$

where the second equality follows from $a_{1-v} = D - a_v$ and the symmetric property of $f_{K,N,D}$. Hence it is enough to show that for any MCP(K, N) density $h : [0, D] \rightarrow [0, \infty)$, the following inequality is valid

$$\mathcal{I}_h(v) \geq f_{K,N,D}(a_{K,N,D}(v)).$$

In the one-dimensional setting, taking the lowest possible Minkowski content or the lowest possible perimeter with respect to h makes no difference (see [22, Corollary 3.2]). Hence fix any h as above and a set E of finite perimeter with respect to $h\mathcal{L}^1$. It follows that, up to a Lebesgue negligible set, $E = \cup_{i \in \mathcal{I}} [a_i, b_i] \subseteq [0, D]$, where $\mathcal{I} \subseteq \mathbb{N}$ is a set of indices, so that (see [22, Proposition 3.1])

$$P_h(E) = \sum_i h(a_i) + h(b_i),$$

where P_h denotes the perimeter with respect to h . First notice that if any a_i, b_i is in the interval having as boundary points a_v and $D - a_v$, the claim is proved

$$h(x) \geq f_{K,N,D}(x) \geq \inf_{y \in [a_v, D-a_v]} f_{K,N,D}(y) = f_{K,N,D}(a_v);$$

the same chain of inequalities is valid if $2a_v \geq D$. So for each $i \in \mathcal{I}$, points $a_i, b_i \notin (a_v, D - a_v)$ if $a_v \leq D/2$, or $a_i, b_i \notin (D - a_v, a_v)$ if $a_v \geq D/2$.

It is convenient to assume with no loss in generality that $a_v \leq D - a_v$ and consider the following subsets of indices

$$\mathcal{I}_1 := \{i \in \mathcal{I} : a_i \geq D - a_v\}, \quad \mathcal{I}_2 := \{i \in \mathcal{I} : b_i \leq a_v\};$$

notice that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$.

Case 1. $\mathcal{I} = \mathcal{I}_1$.

Then

$$\begin{aligned} v &= \sum_{i \in \mathcal{I}} \int_{a_i}^{b_i} h(y) dy \leq \sum_{i \in \mathcal{I}} h(a_i) \int_{a_i}^D \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(a_i)} \right)^{N-1} dy \\ &= \sum_{i \in \mathcal{I}} h(a_i) A(D - a_i) \leq A(a_v) \sum_{i \in \mathcal{I}} h(a_i). \end{aligned}$$

Hence, we get

$$\sum_{i \in \mathcal{I}} (h(a_i) + h(b_i)) \geq \sum_{i \in \mathcal{I}} h(a_i) \geq \frac{v}{A(a_v)} = f_{K,N,D}(a_v).$$

Case 2. $\mathcal{I} = \mathcal{I}_2$.

It holds true

$$\begin{aligned} v &= \sum_{i \in \mathcal{I}} \int_{a_i}^{b_i} h(y) dy \leq \sum_{i \in \mathcal{I}} h(b_i) \int_{a_i}^{b_i} \left(\frac{s_{K/(N-1)}(D - y)}{s_{K/(N-1)}(D - b_i)} \right)^{N-1} dy \\ &\leq \sum_{i \in \mathcal{I}} h(b_i) A(b_i) \\ &\leq A(a_v) \sum_{i \in \mathcal{I}} h(b_i), \end{aligned}$$

for the increasing monotonicity of the function $A(\cdot)$.

Case 3. $\mathcal{I} \neq \mathcal{I}_1 \cup \mathcal{I}_2$.

There exists $i \in \mathcal{I}$ such that $a_i \leq a_v, D - a_v \leq b_i$. Then

$$\begin{aligned} 1 - v &\leq \int_0^{a_i} h(y) dy + \int_{b_i}^D h(y) dy \\ &\leq h(a_i) \int_0^{a_i} \left(\frac{s_{K/(N-1)}(D-y)}{s_{K/(N-1)}(D-a_i)} \right)^{N-1} dy + h(b_i) \int_{b_i}^D \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(b_i)} \right)^{N-1} dy \\ &= h(a_i)A(a_i) + h(b_i)A(D-b_i) \\ &\leq A(D-a_v)[h(a_i) + h(b_i)], \end{aligned}$$

proving the claim.

Case 4. $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.

We use the estimates of **Case 2.** for \mathcal{I}_1 and the ones in **Step 1.** for \mathcal{I}_2 , so:

$$v = \sum_{i \in \mathcal{I}} \int_{a_i}^{b_i} h(y) dy = \sum_{i \in \mathcal{I}_1} \int_{a_i}^{b_i} h(y) dy + \sum_{j \in \mathcal{I}_2} \int_{a_j}^{b_j} h(y) dy \leq A(a_v) \left(\sum_{i \in \mathcal{I}_1} h(a_i) + \sum_{j \in \mathcal{I}_2} h(b_j) \right).$$

Hence, the claim is proved also in this class. \square

Lemma 3.8. *The function $A_{K,N,D}(\cdot)$ is strictly increasing on $[0, D)$.*

Proof. If we are in the case $K = 0$, we get that

$$A_{0,N,D}(a) = \int_{(0,a)} \left(\frac{D-x}{D-a} \right)^{N-1} dx$$

and so $A_{0,N,D}(\cdot)$ is trivially increasing. If $K < 0$, without loss of generality we can assume $K = -(N-1)$. In this case we have

$$A_{-(N-1),N,D}(a) = \int_{(0,a)} \left(\frac{\sinh(D-x)dx}{\sinh(D-a)} \right)^{N-1} dx$$

and so again we get the claim by the monotonicity of the hyperbolic sine. If $K > 0$, we can directly deal with the case $D < \pi\sqrt{(N-1)/K}$. Assuming $K = N-1$, we can rewrite (3.10) in the following way:

$$A_{N-1,N,D}(a) = \int_{(0,a)} \left(\frac{\sin(D-x)}{\sin(D-a)} \right)^{N-1} dx.$$

For sure this function is increasing for $a \in [D - \pi/2, D)$ by the monotonicity of $\sin(D - \cdot)$; so, if $D \leq \pi/2$, we are done. If this is not the case, i.e. $D > \pi/2$, we have to prove that the same result holds in $[0, D - \pi/2)$. Computing the first derivative we obtain that

$$\begin{aligned} A'_{N-1,N,D}(a) &= 1 + (N-1) \frac{\cos(D-a)}{\sin^N(D-a)} \int_{(0,a)} \sin^{N-1}(D-x) dx \\ &= 1 + \frac{N-1}{\tan(D-a)} A_{N-1,N,D}(a); \end{aligned} \tag{3.12}$$

so $A(\cdot)$ is solution of a differential equation. In order to prove that $A(\cdot)$ is an increasing function, we will check that its first derivative is positive, i.e.

$$A_{N-1,N,D}(a) \leq -\frac{\tan(D-a)}{N-1} := g(a), \quad \forall a \in [0, D - \pi/2).$$

For $a = 0$ we have $A_{N-1,N,D}(a) = 0$ and $g(a) = -\frac{\tan D}{N-1} > 0$, hence the inequality at the initial point holds true. In order to prove that it holds for every $a \in [0, D - \pi/2)$, we will check that g verifies the following differential inequality:

$$g'(a) > 1 + \frac{N-1}{\tan(D-a)} \cdot g(a).$$

Since the choice of g makes the second member identically equals to zero, it is sufficient to prove that $g'(a) > 0$ for every $a \in [0, D - \pi/2)$. This trivially holds true since

$$g'(a) = \frac{1}{(N-1)\cos^2(D-a)} > 0.$$

Hence, the claim follows also in this case. \square

We now analyse the dependence of $\tilde{\mathcal{I}}_{K,N,D}(v)$ on the diameter.

Lemma 3.9. *Fix $N, D > 0$ and $v \in (0, 1)$.*

- if $K \leq 0$, the map $D \mapsto \tilde{\mathcal{I}}_{K,N,D}(v)$ is strictly decreasing;
- if $K > 0$, the map $D \mapsto D\tilde{\mathcal{I}}_{K,N,D}(v)$ is non-decreasing;

Proof. Given any MCP(K, N) density h with domain $[0, D]$, and any other D' defining $g(x) := \frac{D}{D'}h(\frac{Dx}{D'})$, for each $x \in [0, D']$, one easily gets that g is an MCP(K', N) with domain $[0, D']$ and $K' = K(D/D')^2$. Moreover for any $A \subset [0, D]$,

$$\mathbf{P}_g\left(A\frac{D'}{D}\right) = \frac{D}{D'}\mathbf{P}_h(A),$$

where \mathbf{P}_g is the perimeter with respect to g and \mathbf{P}_h the one with respect to h . Assume h is the optimal density and A the optimal set, one gets

$$\tilde{\mathcal{I}}_{K',N,D'} \leq \frac{D}{D'}\tilde{\mathcal{I}}_{K,N,D}.$$

Hence if $K \leq 0$ and $D' \geq D$: $\tilde{\mathcal{I}}_{K,N,D} \geq \frac{D'}{D}\tilde{\mathcal{I}}_{K,N,D'} \geq \tilde{\mathcal{I}}_{K,N,D'}$; if $K > 0$ and $D \geq D'$: $D\tilde{\mathcal{I}}_{K,N,D} \geq D'\tilde{\mathcal{I}}_{K,N,D'}$. The claim follows. \square

We then obtain straightforwardly the next fact.

Corollary 3.10. *The one-dimensional isoperimetric profile function (3.7) has the following representation:*

$$\mathcal{I}_{K,N,D}(v) = \begin{cases} f_{K,N,D}(a_{K,N,D}(v)) & \text{if } K \leq 0, \\ \inf_{D' \leq D} f_{K,N,D'}(a_{K,N,D'}(v)) & \text{if } K > 0. \end{cases} \quad (3.13)$$

In the case $K > 0$ we expect the map $D \mapsto f_{K,N,D'}(a_{K,N,D'}(v))$ to be strictly convex as some explicit calculations for particular choices of v would suggest. However at the moment we cannot conclude the existence of a unique minimizer $\bar{D} = \bar{D}(K, N, D, v) < D$ representing $\mathcal{I}_{K,N,D}(v)$ in the case $K > 0$. This in turn affects rigidity of the equality case of the isoperimetric inequality in the regime $K > 0$.

3.2 One-dimensional rigidity

Building on Corollary 3.2, we prove that the one-dimensional isoperimetric inequality obtained in Theorem 3.7 is rigid.

Theorem 3.11. *Let $h : [0, D] \rightarrow \mathbb{R}$ be a MCP(K, N) density which integrates to 1. Assume there exists $v \in (0, 1)$ such that $\mathcal{I}_h(v) = \tilde{\mathcal{I}}_{K,N,D}(v)$. Then either $h = h_{a_v}$ or $h = h_{a_{1-v}}$.*

Proof. Assume the existence of a sequence of sets $E_i \subset [0, D]$ so that

$$\int_{E_i} h(x) dx = v, \quad \lim_{i \rightarrow \infty} (h\mathcal{L}^1 \llcorner_{[0,D]})(E_i) = \tilde{\mathcal{I}}_{K,N,D}(v).$$

Then one can find a sequence of sets having perimeter with respect to h converging to $\tilde{\mathcal{I}}_{K,N,D}(v)$ still with volume v . By lower-semicontinuity we deduce the existence of a set $\cup_{i \in \mathcal{J}} [a_i, b_i]$ of volume v such that

$$\sum_i h(a_i) + h(b_i) = f_{K,N,D}(a_{K,N,D}(v)).$$

We then proceed as in the proof of Theorem 3.7.

In the **Case 1.**, $\mathcal{J} = \mathcal{J}_1$, the first chain of inequalities yields that $\cup_{i \in \mathcal{J}} [a_i, b_i] = [a_1, D]$ and strict monotonicity of $A_{K,N,D}$ implies that $D - a_1 = a_v$. The second chain of inequalities then implies

$$h(D - a_v) = f_{K,N,D}(a_{K,N,D}(v)) = f_{K,N,D}(D - a_{K,N,D}(v)).$$

Corollary 3.2 yields $h = h_{D-a_v}$ and the set $\cup_{i \in \mathcal{J}} [a_i, b_i] = [D - a_v, D]$. Equality in **Case 2.**, $\mathcal{J} = \mathcal{J}_2$, implies, repeating the same argument, that $h = h_{a_v}$ and the set $\cup_{i \in \mathcal{J}} [a_i, b_i] = [0, a_v]$. Equality in **Case 3.** cannot be achieved: the chain of inequality implies that $\cup_{i \in \mathcal{J}} [a_i, b_i] = [a_1, b_1]$ and $a_1 = a_v$ and $b_1 = D - a_v$; coupled with the chain of inequality implies

$$f_{K,N,D}(a_v) = h(a_v) + h(D - a_v) \geq 2f_{K,N,D}(a_v),$$

giving a contradiction. The same argument implies that also equality in **Case 4.** cannot be achieved. \square

Exploiting Lemma 3.9, in the case $K \leq 0$ one can obtain the following stronger rigidity

Corollary 3.12. *Let $h : [0, D'] \rightarrow \mathbb{R}$ be a MCP(K, N) density which integrates to 1 with $K \leq 0$. Assume there exists $v \in (0, 1)$ such that $\mathcal{I}_h(v) = \mathcal{I}_{K,N,D}(v)$ with $D' \leq D$. Then $D = D'$ and either $h = h_{a_v}$ or $h = h_{a_{1-v}}$.*

Proof. Lemma 3.9 forces $D' = D$ and then Theorem 3.11 applies. \square

To conclude we present another application of one-dimensional rigidity. Since $\text{CD}(K, N) \subset \text{MCP}(K, N)$, we already know that $\tilde{\mathcal{I}}_{K,N,D}(v) \leq \tilde{\mathcal{I}}_{K,N,D}^{\text{CD}}(v)$. We can now prove that the inequality is always strict, made exception of a single case.

Corollary 3.13. *For every choice of K, N and D , except the case in which $K > 0$ and $D = \pi\sqrt{(N-1)/K}$, it holds*

$$\tilde{\mathcal{I}}_{K,N,D}(v) < \mathcal{I}_{K,N,D}^{\text{CD}}(v).$$

In particular, $\mathcal{I}_{K,N,D}(v) < \mathcal{I}_{K,N,D}^{\text{CD}}(v)$.

Proof. Suppose by contradiction the existence of K, N, D, v such that $\tilde{\mathcal{I}}_{K,N,D}(v) = \mathcal{I}_{K,N,D}^{\text{CD}}(v)$. As proved in [37](see Corollary 1.4)

$$\mathcal{I}_{K,N,D}^{\text{CD}}(v) = \tilde{\mathcal{I}}_{K,N,D}^{\text{CD}}(v),$$

and there exists (see [37, Corollary A.3]) a $\text{CD}(K, N)$ -density, and therefore an $\text{MCP}(K, N)$ -density g defined on $[0, D]$ and integrating to 1 such that $\mathcal{I}_{([0,D],g)}(v) = \mathcal{I}_{K,N,D}^{\text{CD}}(v)$. As observed in the Theorem 3.11, this would force the density g to be exactly h_{a_v} or $h_{a_{1-v}}$ contradicting Lemma 3.4. The final claim simply follows observing that $\inf_{D' \leq D} \tilde{\mathcal{I}}_{K,N,D'}(v) \leq \mathcal{I}_{K,N,D}(v)$. \square

4 Isoperimetric inequality

We now deduce Theorem 1.1 from the one-dimensional results of Theorem 3.7 and Lemma 3.9 via localization techniques; we now briefly recall few facts on localization.

The localization paradigm, developed by Payne–Weinberger [44], Gromov–Milman [31] and Kannan–Lovász–Simonovits [32], permits to reduce various analytic and geometric inequalities to appropriate one-dimensional counterparts. The original approach by these authors was based on a bisection method, and thus inherently confined to \mathbb{R}^n . In 2015 [34], Klartag extended the localization paradigm to the weighted Riemannian setting, by disintegrating the reference measure \mathbf{m} on L^1 -Optimal-Transport geodesics associated to the inequality under study, and proving that the resulting conditional one-dimensional measures inherit the Curvature-Dimension properties of the underlying manifold.

The first author and Mondino in [19] extended the localization paradigm to the framework of essentially non-branching geodesic m.m.s.'s $(X, \mathbf{d}, \mathbf{m})$ verifying $\text{CD}_{loc}(K, N)$, $N \in (1, \infty)$: the Curvature-Dimension information encoded in the W_2 -geodesics is transferred to the individual rays along which a given W_1 -geodesic evolves; this has permitted to obtain several new results in the field [20, 23, 17].

Localization for $\text{MCP}(K, N)$ was, partially and in a different form, already known in 2009, see [11, Theorem 9.5], for non-branching m.m.s.. The case of essentially non-branching m.m.s.'s and an effective reformulation (after the work of Klartag [34]) has been recently discussed in [24, Section 3] to which we refer for all the missing details (see in particular [24, Theorem 3.5]). Here we only report the next fact:

If $(X, \mathbf{d}, \mathbf{m})$ is an essentially non-branching m.m.s. with $\text{supp}(\mathbf{m}) = X$ and satisfying $\text{MCP}(K, N)$, for some $K \in \mathbb{R}, N \in (1, \infty)$, then, for any 1-Lipschitz function $u : X \rightarrow \mathbb{R}$,

the non-branching transport set \mathcal{T}_u^b associated with u (roughly coinciding, up to a set of \mathbf{m} -measure zero, with $\{|\nabla u| = 1\}$) admits a disjoint family of unparametrized geodesics $\{X_\alpha\}_{\alpha \in Q}$ such that $\mathbf{m}(\mathcal{T}_u^b \setminus \cup_\alpha X_\alpha) = 0$ and the corresponding disintegration of \mathbf{m} is as follows

$$\mathbf{m} \llcorner_{\mathcal{T}_u^b} = \int_Q \mathbf{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q}(Q) = 1, \quad \mathfrak{q}\text{-a.e. } \mathbf{m}_\alpha(X) = \mathbf{m}_\alpha(X_\alpha) = 1. \quad (4.1)$$

Moreover, \mathfrak{q} -a.e. \mathbf{m}_α is a Radon measure with $\mathbf{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \ll \mathcal{H}^1 \llcorner_{X_\alpha}$ and $(X_\alpha, d, \mathbf{m}_\alpha)$ verifies MCP(K, N).

This permits to obtain the next main result and to prove Theorem 1.1; notice that the second part of Theorem 1.1 will then follow by Theorem 3.7.

Theorem 4.1. *Let (X, d, \mathbf{m}) be an essentially non-branching metric measure space with $\mathbf{m}(X) = 1$ and $\text{diam}(X) \leq D$. If (X, d, \mathbf{m}) satisfies MCP(K, N) for some $K \in \mathbb{R}, N \in [1, \infty)$, then*

$$\mathcal{I}_{(X, d, \mathbf{m})}(v) \geq \mathcal{I}_{K, N, D}(v), \quad \forall v \in [0, 1]$$

where $\mathcal{I}_{K, N, D}$ is explicitely given in (3.13).

Even though the proof is a standard consequence of localization, we present it below for readers' convenience.

Proof. Fix $v \in (0, 1)$ and let $A \subset X$ be a Borel set with $\mathbf{m}(A) = v$. Define the \mathbf{m} -measurable function $f := \chi_A - v$ having zero integral with respect to \mathbf{m} , and study the L^1 -Optimal Transport problem from $\mu_0 := f^+ \mathbf{m}$ to $\mu_1 := f^- \mathbf{m}$, where f^\pm denotes the positive and the negative part of f respectively. The associated Kantorovich potential u has $|\nabla u| = 1$ \mathbf{m} -a.e. implying the existence of a family of unparametrized geodesics $\{X_\alpha\}_{\alpha \in Q}$ (of length at most D) such that $\mathbf{m}(X \setminus \cup_\alpha X_\alpha) = 0$ and

$$\mathbf{m} = \int_Q \mathbf{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q}\text{-a.e. } \mathbf{m}_\alpha(X) = \mathbf{m}_\alpha(X_\alpha) = 1;$$

moreover $\mathbf{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ and h_α is a MCP(K, N)-density. From the localization of the constraint, it follows that for \mathfrak{q} -a.e. $\mathbf{m}_\alpha(A) = \mathbf{m}(A) = v$. Hence

$$\begin{aligned} \mathbf{m}^+(A) &= \liminf_{\varepsilon \rightarrow 0} \frac{\mathbf{m}(A^\varepsilon) - \mathbf{m}(A)}{\varepsilon} \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_Q \frac{\mathbf{m}_\alpha((A \cap X_\alpha)^\varepsilon) - \mathbf{m}_\alpha(A)}{\varepsilon} \mathfrak{q}(d\alpha), \\ &\geq \int_Q \mathbf{m}_\alpha^+(A \cap X_\alpha) \mathfrak{q}(d\alpha) \\ &\geq \int_Q \mathcal{I}_{K, N, D}(v) \mathfrak{q}(d\alpha) \\ &= \mathcal{I}_{K, N, D}(v). \end{aligned}$$

□

In the case $K \leq 0$, one-dimensional rigidity (Theorem 3.11) implies the following measure rigidity.

Theorem 4.2. *Let $(X, \mathbf{d}, \mathbf{m})$ be an essentially non-branching metric measure space satisfying MCP(K, N) for $K \leq 0$, $N \in [1, \infty)$ with $\mathbf{m}(X) = 1$ and $\text{diam}(X) \leq D$.*

If there exists $v \in (0, 1)$ such that $\mathcal{I}_{(X, \mathbf{d}, \mathbf{m})}(v) = \mathcal{I}_{K, N, D}(v)$, then $\text{diam}(X) = D$, there exist a measure space (Q, \mathbf{q}) and a measurable isomorphism between $(0, D) \times Q$ and $X' \subset X$ with $\mathbf{m}(X') = 1$.

Moreover, the measure \mathbf{m} admits the following representation

$$\mathbf{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha} \mathbf{q}(d\alpha),$$

and \mathbf{q} -a.e., $h_\alpha = h_{a_{K, N, D}(v)}$ or $h_\alpha = h_{a_{K, N, D}(1-v)}$.

Proof. We will prove that X has diameter D . Arguing by contradiction, let us suppose that there exists $\varepsilon > 0$ such that $\text{diam}(X) = D - \varepsilon$. From (3.13), $K \leq 0$ and Lemma 3.9, for any $v \in (0, 1)$ the function $\mathcal{I}_{K, N, D}(v)$ is strictly decreasing in D . Hence, there exists $\eta > 0$ such that

$$\mathcal{I}_{K, N, D'}(v) \geq \mathcal{I}_{K, N, D}(v) + \eta, \quad \forall D' \in (0, D - \varepsilon].$$

Let $A \subset X$ be such that $\mathbf{m}(A) = v$ and $\mathbf{m}^+(A) \leq \mathcal{I}_{K, N, D}(v) + \eta/2$. Arguing as in the proof of Theorem 4.1, we get that

$$\begin{aligned} \mathcal{I}_{K, N, D}(v) + \eta/2 \geq \mathbf{m}^+(A) &\geq \int_Q \mathbf{m}_\alpha^+(A \cap X_\alpha) \mathbf{q}(d\alpha) \\ &\geq \int_Q \tilde{\mathcal{I}}_{K, N, |\text{supp } h_\alpha|}(v) \mathbf{q}(d\alpha) \\ &\geq \mathcal{I}_{K, N, D}(v) + \eta \end{aligned}$$

where the last inequality is due to the fact that $\text{supp}(h_\alpha)$ is isometric to a geodesic X_α of (X, \mathbf{d}) and hence $|\text{supp } h_\alpha| \leq D - \varepsilon$ and from $K \leq 0$ together with Lemma 3.9. Thus the contradiction is obtained.

The same argument implies that $|\text{supp}(h_\alpha)| = D$ for \mathbf{q} -a.e. α and

$$\mathcal{I}_{K, N, D}(v) = \tilde{\mathcal{I}}_{K, N, |\text{supp } h_\alpha|}(v);$$

the claim follows from the one-dimensional rigidity obtained in Corollary 3.12. \square

References

- [1] L. Ambrosio. Calculus and curvature-dimension bounds in metric measure spaces. *Proceedings of the ICM 2018*, preprint available at <http://cvgmt.sns.it/paper/3779/>.
- [2] L. Ambrosio, N. Gigli and G. Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, **195**, no. 2, (2014), 289–391.
- [3] ———: *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. Journ., **163**, no. 7, (2014), 1405–1490.
- [4] ———: *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, Ann. Probab., **43**, no. 1, (2015), 339–404.

- [5] L. Ambrosio, A. Mondino and G. Savaré. Nonlinear diffusion equations and curvature conditions in metric measure spaces. *Mem. Amer. Math. Soc.*, in press.
- [6] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on probability theory (Saint-Flour, 1992)*, volume 1581 of *Lecture Notes in Math.*, pages 1–114. Springer, Berlin, 1994.
- [7] D. Bakry and M. Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.
- [8] Z. M. Balogh, A. Kristly, and K. Sipos. Geometric inequalities on Heisenberg groups. *Calc. Var. Partial Differential Equations*, 57 (2018), no. 2, Art. 61, 41.
- [9] D. Barilari and L. Rizzi. Sharp measure contraction property for generalized H -type Carnot groups. *Commun. Contemp. Math.* 20 (2018), no. 06, 1750081.
- [10] D. Barilari and L. Rizzi. Sub-Riemannian interpolation inequalities. Preprint arXiv:1705.05380.
- [11] S. Bianchini and F. Cavalletti. The Monge problem for distance cost in geodesic spaces. *Comm. Math. Phys*, 318:615 – 673, 2013.
- [12] L. Capogna, D. Danielli, S. D. Pauls and J. Tyson. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. *Progress in Mathematics*, 259. Birkhäuser Verlag, Basel, 2007. xvi+223 pp.
- [13] E. Bruè and D. Semola. Constancy of the dimension for $RCD(K, N)$ spaces via regularity of Lagrangian flows. *Comm. Pure Appl. Math.*, accepted for publication.
- [14] F. Cavalletti. Monge problem in metric measure spaces with Riemannian curvature-dimension condition. *Nonlinear Anal.*, **99**, (2014), 136–151.
- [15] F. Cavalletti. Decomposition of geodesics in the Wasserstein space and the globalization property. *Geom. Funct. Anal.*, 24:493 – 551, 2014.
- [16] F. Cavalletti. An Overview of L^1 optimal transportation on metric measure spaces. *Book Chapter*, to appear in “Measure Theory in Non-Smooth Spaces”, edited by N. Gigli, De Gruyter Open.
- [17] F. Cavalletti, F. Maggi and A. Mondino. Quantitative isoperimetry à la Levy-Gromov. *Comm. Pure Appl. Math.*, to appear.
- [18] F. Cavalletti and E. Milman. The Globalization Theorem for the Curvature-Dimension Condition. Preprint arXiv:1612.07623.
- [19] F. Cavalletti and A. Mondino. Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *Invent. Math.*, **208**, 3, (2017), 803–849.
- [20] ———, Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. *Geom. Topol.*, to appear, arXiv:1502.06465.
- [21] ———: Optimal maps in essentially non-branching spaces, *Commun. Contemp. Math.*, **19**, (6), (2017), 1750007 (27 pages).
- [22] ———: Isoperimetric inequalities for finite perimeter sets under lower Ricci curvature bounds. *Rend. Lincei Mat. Appl.*, 29 (2018), 413430 DOI 10.4171/RLM/814.
- [23] ———: Almost euclidean Isoperimetric Inequalities in spaces satisfying local Ricci curvature lower bounds. *Int. Math. Res. Not.*, <https://doi.org/10.1093/imrn/rny070>.

- [24] ———: New formulas for the Laplacian of distance functions and applications. *Preprint*, arXiv:1803.09687.
- [25] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math.*, 146:219–257, 2001.
- [26] M. Erbar, K. Kuwada and K.T. Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. *Invent. Math.*, **201**, no. 3, (2015), 993–1071.
- [27] N. Gigli. On the differential structure of metric measure spaces and applications. *Mem. Amer. Math. Soc.*, **236**, no. 1113, (2015).
- [28] ———: *The splitting theorem in non-smooth context*, preprint arXiv:1302.5555, (2013).
- [29] N. Gigli and E. Pasqualetto. Behaviour of the reference measure on RCD spaces under charts. *Comm. Anal. Geom.*, to appear, arXiv:1607.05188..
- [30] M. Gromov. Metric structures for Riemannian and non Riemannian spaces, *Modern Birkhäuser Classics*, (2007).
- [31] M. Gromov and V. D. Milman. Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. *Compositio Math.*, 62(3):263–282, 1987.
- [32] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995.
- [33] M. Kell and A. Mondino. On the volume measure of non-smooth spaces with Ricci curvature bounded below. *Annali SNS-Classe di Scienze*, DOI Number: 10.2422/2036 – 2145.201608_007.
- [34] B. Klartag. Needle decomposition in Riemannian geometry. *Mem. Amer. Math. Soc.*, **249**, no.1180, (2017).
- [35] N. Juillet. Geometric Inequalities and Generalized Ricci Bounds in the Heisenberg Group. *Int. Math. Res. Not.*, **13**, (2009), 2347–2373.
- [36] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Annals of Math.*, **169**, (2009), 903–991.
- [37] E. Milman. Sharp Isoperimetric Inequalities and Model Spaces for Curvature-Dimension-Diameter Condition. *J. Europ. Math. Soc.*, **17**, (5), (2015), 1041–1078.
- [38] A.Mondino and A.Naber. Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds. *J. Europ. Math. Soc.*, to appear, arXiv:1405.2222..
- [39] R. Monti. Isoperimetric problem and minimal surfaces in the Heisenberg group. In: Ambrosio L. (eds) Geometric Measure Theory and Real Analysis. *Publications of the Scuola Normale Superiore*, vol 17. Edizioni della Normale, Pisa
- [40] S.I. Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, **82** (2007), 805–828.
- [41] ———: Products, cones, and suspensions of spaces with the measure contraction property. *J. London Math. Soc. (2)*, **76**, (2007), 225–236.
- [42] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.*, 173(2):361–400, 2000.

- [43] P. Pansu. Une inégalité isopérimétrique sur le groupe de Heisenberg. *C. R. Acad. Sci. Paris Sér. I Math.* 295 (1982), 127–130.
- [44] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292, 1960.
- [45] T. Rajala. Failure of the local-to-global property for $CD(K, N)$ spaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 16:45–68, 2016.
- [46] T. Rajala and K.T. Sturm. Non-branching geodesics and optimal maps in strong $CD(K, \infty)$ -spaces. *Calc. Var. Partial Differential Equations*, **50**, (2014), 831–846.
- [47] M.-K. von Renesse and K.-T. Sturm. Transport inequalities, gradient estimates, entropy and Ricci curvature, *Comm. Pure Appl. Math.*, 58:923–940, 2005.
- [48] L. Rifford. Ricci curvatures in Carnot groups. *Math. Control Relat. Fields*, **3**, (4), (2013), 467–487.
- [49] M. Ritoré. Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group H^1 with low regularity. *Calc. Var. Partial Differential Equations*, 34 (2009), no. 2, 179–192.
- [50] M. Ritoré. A proof by calibration of an isoperimetric inequality in the Heisenberg group H^n . *Calc. Var. Partial Differential Equations* 44 (2012), no. 1-2, 47–60.
- [51] K.T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, **196** (2006), 65–131.
- [52] ———: On the geometry of metric measure spaces. II. *Acta Math.*, **196** (2006), 133–177.
- [53] C. Villani. Optimal transport. Old and new. *Grundlehren der Mathematischen Wissenschaften*, **338**, Springer-Verlag, Berlin, (2009).