

THE SURFACE DIFFUSION FLOW WITH ELASTICITY IN THREE DIMENSIONS

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ABSTRACT. We establish short-time existence of a smooth solution to the surface diffusion equation with an elastic term and without an additional curvature regularization in three space dimensions. We also prove the asymptotic stability of strictly stable stationary sets.

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1. INTRODUCTION

Morphological evolution of strained elastic solids, driven by stress and surface mass transport occurs in many physical systems. One instance is the hetero-epitaxial growth of elastic films when a lattice mismatch between film and substrate is present. Another example is given by the phase separation in several small connected phases within a common elastic body, which takes place in certain alloys under specific temperature conditions. A third situation is represented by the nucleation and evolution of material voids inside a stressed elastic solid. From the mathematical point of view, such phenomena are related to a free energy functional, which is typically given by the sum of the stored elastic energy and the surface energy accounting for the surface tension along the interface between the phases. In this context the equilibria are identified with the local or global minimizers under a volume constraint of the aforementioned energy.

All these variational problems can be regarded as non-local *isoperimetric problems*, where the non-locality is given by the elastic term. They are very well studied in the physical and numerical literature, see for instance [26, 29, 40, 41, 42]. Concerning rigorous mathematical analysis, we refer to [6, 8, 10, 17, 21, 25, 28] for some existence, regularity and stability results

related to a variational model describing the equilibrium configurations of two-dimensional epitaxially strained elastic films, and to [9, 16] for results in three-dimensions. A hierarchy of variational principles to describe equilibrium shapes in the aforementioned contexts has been introduced in [30].

In what follows we consider the following prototypical energy

$$(1.1) \quad \mathcal{J}(F) := \frac{1}{2} \int_{\Omega \setminus F} \mathbb{C}E(u_F) : E(u_F) dx + \mathcal{H}^2(\partial F).$$

The associated minimum problem under a volume constraint can be used to describe the equilibrium shapes of voids in elastically stressed solids (see for instance [41]). Here, the set $F \subset\subset \Omega$ represents the shape of the void that has formed within the elastic body Ω (an open subset of \mathbb{R}^3), u_F stands for the equilibrium elastic displacement in $\Omega \setminus F$ subject to a prescribed boundary conditions $u_F = w_0$ on $\partial\Omega$ (see (2.12) below), \mathbb{C} is the elasticity tensor of the (linearly) elastic material, $E(u_F) := (Du_F + D^T u_F)/2$ denotes the elastic strain of u_F , and \mathcal{H}^2 stands for the surface measure. The presence of a nontrivial Dirichlet boundary condition $u_F = w_0$ on $\partial\Omega$ is what causes the solid $\Omega \setminus F$ to be elastically stressed. We refer to [15, 20] for related existence, regularity and stability results in two dimensions. See also [11] for a relaxation result valid in all dimensions for a variant of (1.1).

In this paper we study the morphological evolution of shapes towards equilibria of the functional (1.1), driven by stress and surface diffusion. Assuming that relaxation to equilibrium in the bulk occurs at a much faster time scale, see [38], we have, according to the Einstein-Nernst equation, that the evolution is governed by the following *volume preserving* law

$$(1.2) \quad V_t = \Delta_{\partial F_t} \mu_t \quad \text{on } \partial F_t$$

where V_t denotes the outer normal velocity of the evolving surface ∂F_t at time t and $\Delta_{\partial F_t} \mu_t$ stands for the Laplace-Beltrami operator acting on the chemical potential μ_t along ∂F_t . In turn, since μ_t is given by the *first variation* of the free-energy functional \mathcal{J} evaluated at F_t and taking into account (2.14) below, (1.2) reads as

$$(1.3) \quad V_t = \Delta_{\partial F_t} (H_{F_t} - Q(E(u_{F_t}))),$$

where H_{F_t} is the sum of the principal curvatures of ∂F_t , with the orientation given by the outer normal, u_{F_t} is the elastic equilibrium in $\Omega \setminus F_t$ subject to $u_{F_t} = w_0$ on $\partial\Omega$ and $Q(E(u_{F_t})) := \frac{1}{2} \mathbb{C}E(u_{F_t}) : E(u_{F_t})$. Note that the last quantity involves the traces of the gradient of the elastic equilibrium on the evolving boundary.

From the mathematical point of view, (1.3) is a fourth order geometric parabolic equation coupled with the elliptic Lamé system, which is solved time by time in the (evolving) bulk. Note also that when $w_0 = 0$ the elastic term vanishes and thus (1.3) reduces to the pure *surface diffusion flow*

$$(1.4) \quad V_t = \Delta_{\partial F_t} H_{F_t}$$

for evolving surfaces, studied in [19] (in the general n -dimensional case). Thus, we may also regard (1.3) as a sort of canonical nonlocal perturbation of (1.4) by an additive elastic contribution.

As observed already by Cahn and Taylor [14] for (1.4), the equation (1.3) can be seen formally as the gradient flow of the energy functional \mathcal{J} with respect to a suitable Riemannian metric of H^{-1} -type, see for instance [24, Remark 3.1].

Let us mention that in the physical literature a variant of the energy (1.1) with a *curvature regularization* term has also been considered, see [3, 12, 18, 31, 40, 41]. This in turn leads to

a variant of (1.3) with a sixth order regularization term. In particular, in [23] the following regularized energy

$$\mathcal{J}_\varepsilon(F) := \frac{1}{2} \int_{\Omega \setminus F} \mathbb{C}E(u_F) : E(u_F) dx + \int_{\partial F} \left(1 + \frac{\varepsilon}{p} |H_F|^p\right) d\mathcal{H}^2$$

and the associated evolution equation

$$(1.5) \quad V_t = \Delta_{\partial F_t} \left[H_{F_t} - Q(E(u_{F_t})) - \varepsilon \left(\Delta_{\partial F_t} (|H_{F_t}|^{p-2} H_{F_t}) - |H_{F_t}|^{p-2} H_{F_t} \left(\frac{p-1}{p} H_{F_t}^2 - 2K_{\partial F_t} \right) \right) \right]$$

are considered in the context of periodic graphs modeling the evolutions of epitaxially strained elastic films (see also [22] for the two-dimensional version of the same equation). Here $K_{\partial F_t}$ stands for the Gaussian curvature of ∂F_t , $\varepsilon > 0$ is a small parameter, and $p > 2$. The local-in-time existence and the asymptotic stability results proven in [23] (see also [22, 39]) rely heavily on the presence of the curvature regularization, which makes the elastic contribution a lower order term easily controlled by the sixth order leading terms of the equation. In fact, all the estimates provided there are ε -dependent and degenerate as $\varepsilon \rightarrow 0^+$. This is not surprising as the nonlocal elastic term in (1.1) cannot be treated simply as a lower order perturbation of the perimeter, as shown by the fact that its presence may lead to formation of singularities in the static case (see [25] and references therein). Thus the case $\varepsilon = 0$ requires completely different methods.

A first breakthrough in this direction has been obtained in [24], where short time existence result for (1.3) was proved in the two-dimensional case. In [24] we also proved the asymptotic stability of strictly stable stationary sets. However, the techniques developed there cannot be applied to higher dimensions, as some of the crucial estimates rely on the fact that an L^2 -bound of the curvature of the evolving curves provides uniform $C^{1,\alpha}$ -bounds. This is of course no longer true in higher dimensions. Moreover, the higher dimensional case is of course much more involved from the geometric point of view.

In this paper we are able to address equation (1.3) in the physical three-dimensional case and we establish short time existence and uniqueness of a solution starting from sufficiently regular initial sets, see Theorem 4.4. We highlight that Theorem 4.4 provides also quantitative estimates of the k -th order derivatives of the solution depending only on the H^3 -norm of the initial datum, somewhat in the spirit of those proved in [32]. We also remark that in general one cannot expect global-in-time existence. Indeed, even when no elasticity is present, singularities such as pinching may develop in finite time, see for instance [27].

In the second main result of the paper we establish global-in-time existence and study the long-time behavior for a class of initial data: we show that *strictly stable stationary sets*, that is, sets G that are stationary for the energy functional \mathcal{J} and with positive second variation $\partial^2 \mathcal{J}(G)$ are *exponentially stable* for the flow (1.3). More precisely, if the initial set F_0 is sufficiently close in H^3 to the strictly stable set G and has the same volume, then the flow (1.3) starting from F_0 exists for all times and converges to G exponentially fast in C^k for every k as $t \rightarrow +\infty$, see Theorem 5.1 for the precise statement.

A few comments on the proofs are in order. Concerning short-time existence, as in [24] our strategy is based on the natural idea of thinking of the elastic contribution Q as a forcing term. More precisely, we set up a fixed point argument on the map $f \mapsto Q(E(u_{F_t^f}))$, where F_t^f is the solution to the forced flow

$$(1.6) \quad V_t = \Delta_{\partial F_t} (H_{F_t} - f).$$

Major technical difficulties originate from the already mentioned fact that the nonlocal elastic term is not in general lower order with respect to the perimeter. One of the main technical breakthroughs obtained in the present paper is a new delicate elliptic estimate on the higher order derivatives of $Q(E(u_{F_t}))$ in terms of the higher order norms of the evolving boundaries ∂F_t , see Theorem 4.1. The crucial and somewhat surprising point of this result is the linear structure of the estimate, which allows us to show that the map $f \mapsto Q(E(u_{F_t^f}))$ is a contraction.

Concerning the asymptotic stability analysis, we adapt to the present situation the methods developed in [1] for the surface diffusion flow without elasticity (see also [24]). The rough idea is to look at the asymptotic behavior of the map

$$t \mapsto \int_{\partial F_t} |\nabla_{\partial F_t} (H_{F_t} - Q(E(u_{F_t})))|^2 d\mathcal{H}^2,$$

where $\nabla_{\partial F_t}$ stands for the tangential gradient on ∂F_t , and to show that it is decreasing and that in fact it vanishes with exponential rate as $t \rightarrow +\infty$. A crucial role in this analysis is played by the energy identity proven in Proposition 5.3 and by the estimates on the flow provided by Theorem 4.4. Let us remark that such estimates allow us also to considerably simplify the arguments of [1] and to obtain stronger asymptotic convergence results.

This paper is organized as follows. In Section 2 we set up the problem, introduce the main notation and present some differential geometry preliminaries that will be useful in the subsequent analysis. We also collect several auxiliary results concerning the energy functional \mathcal{J} in (1.1). In particular, we describe some properties of strictly stable stationary sets that are crucial for the asymptotic stability analysis carried out in Section 5. Section 3 is devoted to the study of (1.6), while the short-time existence theory for the flow (1.3) is addressed in Section 4. In Section 6 we briefly illustrate how to apply our main existence and asymptotic stability results in the case of evolving periodic graphs, that is in the geometric setting considered in [23]. In particular, in Theorem 6.1 we address the exponential asymptotic stability of *flat configurations*, thus extending to the evolutionary setting the results of [9]. In the final Appendix we collect the proofs of two technical lemmas and provide the derivation of the energy identity stated in Proposition 5.3.

We conclude this introduction by mentioning that it would be interesting to investigate whether the flow (1.5) studied in [23] converge to (1.3) as $\varepsilon \rightarrow 0^+$. This issue could be probably addressed by adapting the methods developed in [7].

2. PRELIMINARIES

2.1. Geometric preliminaries. In this section we introduce notation related to Riemannian geometry. As an introduction to the topic we refer to [4, 34]. Let $\Sigma \subset \mathbb{R}^n$ be a smooth $(n-1)$ -dimensional compact hypersurface without boundary. Since Σ is embedded in \mathbb{R}^n it has a natural metric, denoted by g , induced by the Euclidean metric. We thus have a Riemannian manifold (Σ, g) and we denote the inner product for vector fields X, Y as $\langle X, Y \rangle$,

$$\langle X, Y \rangle = g(X, Y) = g_{ij} X^i Y^j,$$

where the last expression is in local coordinates. Throughout the paper we adopt the Einstein summation convention. Similarly we define the inner product of covector fields ω, η , which in local coordinates can be written as

$$\langle \omega, \eta \rangle = g^{ij} \omega_i \eta_j,$$

where g^{ij} is the inverse matrix of g_{ij} . The inner product extends to $\binom{k}{0}$ -tensor fields $T = T_{i_1 \dots i_k}$ and $S = S_{j_1 \dots j_k}$ as

$$\langle T, S \rangle = g^{i_1 j_1} \dots g^{i_k j_k} T_{i_1 \dots i_k} S_{j_1 \dots j_k}.$$

The norm of a tensor T is then $|T| = \sqrt{\langle T, T \rangle}$ and we have the inequality $\langle T, S \rangle \leq |T||S|$. Given a $\binom{k}{0}$ -tensor field T we raise the first index by $T_{i_2 \dots i_k}^{i_1} = g^{i_1 l} T_{l i_2 \dots i_k}$ and thus we obtain a $\binom{k-1}{1}$ -tensor field. We may thus write the above inner product as

$$\langle T, S \rangle = T^{j_1 \dots j_k} S_{j_1 \dots j_k}.$$

The trace of a $\binom{k}{0}$ -tensor field T , with $k \geq 2$, on the first two indices is $\text{tr } T = g^{jl} T_{jl i_3 \dots i_k}$.

We denote the Riemannian connection on (Σ, g) by ∇ and $\nabla^k T = \nabla_{i_1} \dots \nabla_{i_k} T$ means the k -th covariant derivative of a tensor field T . There is a slight danger of confusion, since $\nabla^k f$ also denotes the k -th component of the gradient of a function f defined by raising the index of ∇f as $\nabla^k f = g^{ki} \nabla_i f$. However, the meaning of $\nabla^k f$ will be clear from the context. We also recall that ∇ is compatible with the metric g which means that $\nabla g = 0$.

In local coordinates the components of the covariant derivative of a vector field $X = X^i$ and of a covector field $\omega = \omega_k$ are

$$\nabla_j X^i = \frac{\partial X^i}{\partial x^j} + \Gamma_{jk}^i X^k \quad \text{and} \quad \nabla_j \omega_k = \frac{\partial \omega_k}{\partial x^j} - \Gamma_{jk}^l \omega_l,$$

where Γ_{ij}^k are the Christoffel symbols given in local coordinates by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

The covariant derivative of a $\binom{k}{l}$ -tensor field $T = T_{i_1 \dots i_k}^{j_1 \dots j_l}$ is thus a $\binom{k+1}{l}$ -tensor field which in local coordinates can be written as

$$\nabla_m T_{i_1 \dots i_k}^{j_1 \dots j_l} = \frac{\partial T_{i_1 \dots i_k}^{j_1 \dots j_l}}{\partial x^m} + \sum_{s=1}^l T_{i_1 \dots i_k}^{j_1 \dots p \dots j_l} \Gamma_{mp}^{j_s} - \sum_{s=1}^k T_{i_1 \dots p \dots i_k}^{j_1 \dots j_l} \Gamma_{m i_s}^p.$$

The divergence of a vector field X^i is $\text{div } X = \nabla_i X^i = \frac{\partial X^i}{\partial x^i} + \Gamma_{ik}^i X^k$ and the Laplace-Beltrami of a function f is

$$\Delta f = \text{div } \nabla f = \nabla_i \nabla^i f.$$

This can be written as the trace of the covariant Hessian $\nabla^2 f$ as

$$\Delta f = \text{tr } \nabla^2 f = g^{ij} \nabla_i \nabla_j f.$$

We recall the divergence theorem for compact manifolds (without boundary), which states that for a vector field X on Σ it holds

$$\int_{\Sigma} \text{div } X \, d\mathcal{H}^{n-1} = 0.$$

This yields the integration by parts formula for a function f and a vector field X

$$\int_{\Sigma} X^i \nabla_i f \, d\mathcal{H}^{n-1} = - \int_{\Sigma} f \text{div } X \, d\mathcal{H}^{n-1}.$$

The integration by parts formula generalizes to any $\binom{k}{0}$ -tensor field T and $\binom{k+1}{0}$ -tensor field S as

$$(2.1) \quad \int_{\Sigma} \langle \nabla T, S \rangle \, d\mathcal{H}^{n-1} = - \int_{\Sigma} \langle T, \text{tr } \nabla S \rangle \, d\mathcal{H}^{n-1},$$

where the trace is on the first two indices of ∇S .

The Riemann curvature endomorphism is a $\binom{3}{1}$ -tensor field R_{ijk}^l defined such that for vector fields X, Y, Z we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where ∇_X is the covariant derivative in direction of X . We adopt the convention to define the Riemann curvature tensor by lowering the index to the end, i.e., $R_{ijkl} = g_{lm} R_{ijk}^m$. The commutation formula of the covariant derivatives for a vector field X^k thus becomes

$$(2.2) \quad \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = g^{km} R_{ijlm} X^l$$

and for a covector field ω_k

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = -g^{ml} R_{ijkm} \omega_l.$$

Similar formulas hold for the commutation of higher order covariant derivatives. In particular, throughout the paper we will make repeated use of the fact that for any integer $k \geq 3$ there exists a constant $C > 0$ such that

$$(2.3) \quad |\nabla_{i_1} \dots \nabla_{i_k} f - \nabla_{i_{\sigma(1)}} \dots \nabla_{i_{\sigma(k)}} f| \leq C \sum_{l=1}^{k-2} |\nabla^l f|$$

for any choice of the indices i_1, \dots, i_k and for any permutation σ of $\{1, \dots, k\}$. We recall also that $\nabla_i \nabla_j f = \nabla_j \nabla_i f$ for any i, j .

Given a positive integer k and $p \in [1, \infty]$ we denote by $W^{k,p}(\Sigma)$ the Sobolev space endowed with the norm

$$\|f\|_{W^{k,p}(\Sigma)} := \sum_{m=0}^k \left(\int_{\Sigma} |\nabla^m f|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}},$$

when $p \in [1, \infty)$ and the obvious one when $p = \infty$. Here $\nabla^m f$ stands for the m -th covariant derivative of f . As customary, when $p = 2$ we shall always write H^k instead of $W^{k,2}$. We further define the norms $\|f\|_{C^{k,\alpha}(\Sigma)}$, $\|f\|_{H^{k+1/2}(\Sigma)}$ and $\|f\|_{H^{-1/2}(\Sigma)}$ with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, in a standard way using the partition of unity. Then the standard embedding theorems for smooth domains hold also in these spaces. Moreover, we recall the following well known interpolation inequalities, see [35, Proposition 6.5] and [5, Theorem 3.70].

Lemma 2.1. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth $(n-1)$ -dimensional compact manifold without boundary. Let l, m, k be integers such that $0 \leq l < m$, $k \geq 0$, $1 \leq q, r \leq \infty$. There exists a constant C with the following property: for every smooth covariant tensor T of order k , one has*

$$(2.4) \quad \|\nabla^l T\|_{L^p(\Sigma)} \leq C \|T\|_{W^{m,r}(\Sigma)}^\vartheta \|T\|_{L^q(\Sigma)}^{1-\vartheta},$$

where

$$\frac{1}{p} = \frac{l}{n-1} + \vartheta \left(\frac{1}{r} - \frac{m}{n-1} \right) + (1-\vartheta) \frac{1}{q}$$

for all $\vartheta \in [l/m, 1)$ for which p is nonnegative. Moreover, if f is a smooth function then

$$\|\nabla^l f\|_{L^p(\Sigma)} \leq C \|\nabla^m f\|_{L^r(\Sigma)}^\vartheta \|f\|_{L^q(\Sigma)}^{1-\vartheta},$$

for all $\vartheta \in [l/m, 1)$ for which p is nonnegative, provided $l \geq 1$.

Remark 2.2. Note that (2.4) implies also that

$$\|\nabla^l T\|_{L^p(\Sigma)} \leq C \|\nabla^m T\|_{L^r(\Sigma)}^\vartheta \|T\|_{L^q(\Sigma)}^{1-\vartheta} + C \|T\|_{L^{\max\{q,r\}}(\Sigma)}.$$

To see this it is enough to observe that $\|T\|_{W^{m,r}(\Sigma)} = \|T\|_{W^{m-1,r}(\Sigma)} + \|\nabla^m T\|_{L^r(\Sigma)}$ and that, in turn, for every $l = 1, \dots, m-1$ using (2.4) and Young's Inequality one gets

$$\|\nabla^l T\|_{L^r(\Sigma)} \leq \varepsilon \|T\|_{W^{m,r}(\Sigma)} + C_\varepsilon \|T\|_{L^r(\Sigma)}.$$

We also recall that the Morrey's inequality implies

$$\|f\|_{C^{1,\alpha}(\Sigma)} \leq C \|f\|_{W^{2,p}(\Sigma)}$$

for $p > n-1$ and $\alpha = 1 - (n-1)/p$.

We will also need the following result, (see the proof of [5, Theorem 4.19]).

Lemma 2.3. *Let f be a smooth function on Σ and let k be a positive integer. There is a constant C , which depends on k and Σ , such that*

$$(2.5) \quad \|\nabla^{2k} f\|_{L^2(\Sigma)}^2 \leq \int_{\Sigma} (\Delta^k f)^2 d\mathcal{H}^{n-1} + C \|f\|_{H^{2k-1}(\Sigma)}^2$$

and

$$(2.6) \quad \|\nabla^{2k+1} f\|_{L^2(\Sigma)}^2 \leq \int_{\Sigma} |\nabla(\Delta^k f)|^2 d\mathcal{H}^{n-1} + C \|f\|_{H^{2k}(\Sigma)}^2.$$

Proof. We only proof (2.5) in the cases $k = 1, 2$, since the higher order cases and (2.6) are analogous. Recall that Ricci tensor is given by $R_{jm} = g^{ik} R_{ijmk}$. Thus from (2.2), with X equal to the covariant gradient of f and taking $k = i$, we get

$$\nabla_i \nabla_j \nabla^i f - \nabla_j \Delta f = R_{jl} \nabla^l f.$$

We multiply the above equality by $\nabla^j f$ and use the integration by parts formula (2.1) to obtain

$$- \int_{\Sigma} \nabla_i \nabla^j f \nabla_j \nabla^i f d\mathcal{H}^{n-1} + \int_{\Sigma} (\Delta f)^2 d\mathcal{H}^{n-1} = \int_{\Sigma} R_{ij} \nabla^i f \nabla^j f d\mathcal{H}^{n-1}.$$

This yields the claim since (recall that for a function $\nabla_i \nabla_j f = \nabla_j \nabla_i f$)

$$\nabla_i \nabla^j f \nabla_j \nabla^i f = \nabla^i \nabla^j f \nabla_i \nabla_j f = |\nabla^2 f|^2.$$

The argument in the case $k = 2$ is similar but more technical. We have by the previous statement

$$\int_{\Sigma} |\Delta^2 f|^2 d\mathcal{H}^{n-1} \geq \int_{\Sigma} |\nabla^2 \Delta f|^2 d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2.$$

Hence, we need to prove that

$$(2.7) \quad \int_{\Sigma} |\nabla^2 \Delta f|^2 d\mathcal{H}^{n-1} \geq \int_{\Sigma} |\nabla^4 f|^2 d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2.$$

First, by the integration by parts formula (2.1) we have

$$\begin{aligned} \int_{\Sigma} |\nabla^2 \Delta f|^2 d\mathcal{H}^{n-1} &= \int_{\Sigma} (\nabla^i \nabla^j \nabla_k \nabla^k f) (\nabla_i \nabla_j \nabla^l \nabla_l f) d\mathcal{H}^{n-1} \\ &= - \int_{\Sigma} (\nabla_i \nabla^i \nabla^j \nabla_k \nabla^k f) (\nabla_j \nabla^l \nabla_l f) d\mathcal{H}^{n-1}. \end{aligned}$$

Then, using (2.3), we obtain

$$\begin{aligned} \int_{\Sigma} |\nabla^2 \Delta f|^2 d\mathcal{H}^{n-1} &\geq - \int_{\Sigma} (\nabla_k \nabla_i \nabla^i \nabla^j \nabla^k f) (\nabla_j \nabla^l \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2 \\ &= - \int_{\Sigma} (\nabla^i \nabla^j \nabla^k f) (\nabla_i \nabla_k \nabla_j \nabla^l \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2, \end{aligned}$$

where the last equality follows by integration by parts. We proceed using formula (2.3) again and integration by parts to deduce

$$\begin{aligned} \int_{\Sigma} |\nabla^2 \Delta f|^2 d\mathcal{H}^{n-1} &\geq - \int_{\Sigma} (\nabla^i \nabla^j \nabla^k f) (\nabla^l \nabla_i \nabla_j \nabla_k \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2 \\ &= - \int_{\Sigma} (\nabla_i \nabla^l \nabla^i \nabla^j \nabla^k f) (\nabla_j \nabla_k \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2 \\ &\geq - \int_{\Sigma} (\nabla_i \nabla^i \nabla^j \nabla^k \nabla^l f) (\nabla_j \nabla_k \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2 \\ &= \int_{\Sigma} (\nabla^i \nabla^j \nabla^k \nabla^l f) (\nabla_i \nabla_j \nabla_k \nabla_l f) d\mathcal{H}^{n-1} - C \|f\|_{H^3(\Sigma)}^2. \end{aligned}$$

Thus we have (2.7), since $(\nabla^i \nabla^j \nabla^k \nabla^l f) (\nabla_i \nabla_j \nabla_k \nabla_l f) = |\nabla^4 f|^2$. \square

Remark 2.4. In the case $k = 1$ we have a more precise version of Lemma 2.3 for hypersurfaces. It is clear that the proof of Lemma 2.3 implies that

$$\int_{\Sigma} |\nabla^2 f|^2 d\mathcal{H}^{n-1} \leq \int_{\Sigma} (\Delta f)^2 d\mathcal{H}^{n-1} + (\sqrt{n-1} + 1) \int_{\Sigma} |B|^2 |\nabla f|^2 d\mathcal{H}^{n-1},$$

where B denotes the (scalar) second fundamental form (see [34] for definition). This follows from the fact that we may estimate the Ricci curvature by $|\text{Ric}| \leq (\sqrt{n-1} + 1)|B|^2$.

Remark 2.5. Using Lemma 2.1 we may write the statement of Lemma 2.3 in the following way. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|f\|_{H^{2k}(\Sigma)}^2 \leq (1 + \varepsilon) \int_{\Sigma} (\Delta^k f)^2 d\mathcal{H}^{n-1} + C_\varepsilon \|f\|_{L^2(\Sigma)}^2$$

and

$$\|f\|_{H^{2k+1}(\Sigma)}^2 \leq (1 + \varepsilon) \int_{\Sigma} |\nabla(\Delta^k f)|^2 d\mathcal{H}^{n-1} + C_\varepsilon \|f\|_{L^2(\Sigma)}^2.$$

Indeed, this follows by the interpolation inequality together with standard Young's inequality

$$\begin{aligned} \|\nabla^l f\|_{L^2(\Sigma)} &\leq C \|\nabla^h f\|_{L^2(\Sigma)}^\theta \|f\|_{L^2(\Sigma)}^{1-\theta} \\ &\leq \varepsilon \|\nabla^h f\|_{L^2(\Sigma)} + C(\varepsilon) \|f\|_{L^2(\Sigma)} \end{aligned}$$

for every $1 \leq l \leq h - 1$ and $\theta = \theta(h, l)$ is given by Lemma 2.1.

For clarity we denote the standard inner product between two vectors x, y in \mathbb{R}^n as $x \cdot y$ and the differential of the map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by DF to distinguish them from the inner product on manifold and from the covariant derivative. There is, however, a possibility of confusion when we denote the divergence of a vector field $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\text{div} X$, since “div” also denotes the divergence of a vector field on manifold. We will denote the divergence of a vector field on the manifold (Σ, g) by div_g and in \mathbb{R}^n by $\text{div}_{\mathbb{R}^n}$ if this is not clear from the context.

When the manifold Σ is given by a boundary of a smooth bounded set $F \subset \mathbb{R}^n$ it has a natural orientation and we denote by ν_F the unit outer normal. In this case we may extend the definition of divergence on Σ to vector fields which have values in \mathbb{R}^n . Let $X : U \rightarrow \mathbb{R}^n$ be a smooth vector field, where U is an open neighborhood of Σ . We define the tangential divergence of X on ∂F by

$$\operatorname{div}_\tau X := \operatorname{div} X - \langle DX\nu_F, \nu_F \rangle.$$

The divergence theorem states

$$\int_{\partial F} \operatorname{div}_\tau X \, d\mathcal{H}^{n-1} = \int_{\partial F} H_F (X \cdot \nu_F) \, d\mathcal{H}^{n-1},$$

where H_F denotes the sum of the principal curvatures of ∂F . We denote the second fundamental form of ∂F by B_F , which in our case is a symmetric $\binom{2}{0}$ -tensor (or equivalently a symmetric matrix). Finally we may project a vector field $X : U \rightarrow \mathbb{R}^n$ to the tangent space of ∂F by

$$(2.8) \quad X_\tau := X - (X \cdot \nu_F)\nu_F.$$

Then X_τ canonically defines a vector field on $(\partial F, g)$ and we denote by $\operatorname{div}_g X_\tau$ its divergence. For a given function $u : U \rightarrow \mathbb{R}$ we define the tangential gradient on $\Sigma = \partial F$ as the projection of its gradient Du

$$(2.9) \quad D_\tau u := (Du)_\tau.$$

The tangential gradient and the covariant gradient are canonically isomorphic. In particular, it holds

$$(2.10) \quad |\nabla u(x)|_g = |D_\tau u(x)| \quad \text{for } x \in \Sigma,$$

where $|\cdot|_g$ denotes the norm given by the metric tensor g , and $|\cdot|$ is the length of a vector in \mathbb{R}^n .

2.2. The energy functional. In this section we introduce the energy functional that underlies the flow. We also introduce the proper notions of stationary points and stability that will be needed in the study of the long-time behavior of the flow. As explained in the introduction, the free energy functional is the sum of the perimeter and of a bulk elastic term. Throughout the paper Ω will denote a fixed bounded open set of \mathbb{R}^3 with Lipschitz boundary.

Concerning the elastic part, for $F \subset\subset \Omega$ and for an elastic displacement $u : \Omega \setminus F \rightarrow \mathbb{R}^2$ we denote by $E(u)$ the symmetric part of Du , that is, $E(u) := \frac{Du + (Du)^T}{2}$. In what follows, \mathbb{C} stands for the *elasticity tensor* acting on 3×3 -matrices, such that $\mathbb{C}A = \frac{1}{2}\mathbb{C}(A + A^T)$ and $\mathbb{C}A$ is symmetric for all 3×3 -matrices A . Moreover, $\mathbb{C}A : A > 0$ if A is symmetric and $A \neq 0$. Finally we shall denote by $Q(A) := \frac{1}{2}\mathbb{C}A : A$ the *elastic energy density*.

We are now ready to write the energy functional. For a fixed *boundary displacement* $w_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we set

$$(2.11) \quad \mathcal{J}(F) := \int_{\Omega \setminus F} Q(E(u_F)) \, dx + \mathcal{H}^2(\partial F),$$

where u_F is the elastic equilibrium satisfying the Dirichlet boundary condition w_0 on a fixed relatively open subset $\partial_D \Omega \subseteq \partial\Omega$. More precisely, u_F is the unique solution in $H^1(\Omega \setminus F; \mathbb{R}^3)$

of the following elliptic system

$$(2.12) \quad \begin{cases} \operatorname{div} \mathbb{C}E(u_F) = 0 & \text{in } \Omega \setminus F, \\ \mathbb{C}E(u_F)[\nu_F] = 0 & \text{on } \partial F \cup (\partial\Omega \setminus \partial_D\Omega), \\ u_F = w_0 & \text{on } \partial_D\Omega. \end{cases}$$

Note that by the second condition for every $x \in \partial F$ the vector $\mathbb{C}E(u_F)(x)[e]$ belongs to the tangent space of ∂F at x for every vector e .

Next, we provide the first and the second variation formulas for (2.11). To this aim, for any vector field $X \in C_c^1(\mathbb{R}^3; \mathbb{R}^3)$, let $(\Phi_t)_{t \in (-1,1)}$ be the associated flow, that is the solution of

$$(2.13) \quad \begin{cases} \frac{\partial \Phi_t}{\partial t} = X(\Phi_t), \\ \Phi_0 = Id. \end{cases}$$

The first and the second variation of the functional (2.11) are stated in the following theorem. Recall that H_F denotes the sum of the principal curvatures and B_F the second fundamental form of ∂F . Sometimes, with a slight abuse of terminology, we will refer to H_F as the mean curvature of ∂F .

Theorem 2.6. *Let $F \subset\subset \Omega$ be a smooth set, $X \in C_c^1(\Omega; \mathbb{R}^2)$ and let $(\Phi_t)_{t \in (-1,1)}$ be the associated flow as in (2.13). Set $\psi := X \cdot \nu_F$ on ∂F and let X_τ be as in (2.8). Then,*

$$(2.14) \quad \frac{d}{dt} \mathcal{J}(\Phi_t(F)) \Big|_{t=0} = \int_{\partial F} (H_F - Q(E(u_F))) \psi \, d\mathcal{H}^2.$$

If in addition $\operatorname{div}_{\mathbb{R}^n} X = 0$ in a neighborhood of ∂F we have

$$(2.15) \quad \begin{aligned} \frac{d^2}{dt^2} \mathcal{J}(\Phi_t(F)) \Big|_{t=0} &= \int_{\partial F} |\nabla \psi|^2 - |B_F|^2 \psi^2 \, d\mathcal{H}^2 - 2 \int_{\Omega \setminus F} Q(E(u_\psi)) \, dx \\ &- \int_{\partial F} \partial_{\nu_F} (Q(E(u_F))) \psi^2 \, d\mathcal{H}^2 - \int_{\partial F} (H_F - Q(E(u_F))) \operatorname{div}_g(\psi X_\tau) \, d\mathcal{H}^2, \end{aligned}$$

where the function u_ψ is the unique solution in $H^1(\Omega \setminus F; \mathbb{R}^3)$, with $u_\psi = 0$ on $\partial_D\Omega$, of

$$(2.16) \quad \int_{\Omega \setminus F} \mathbb{C}E(u_\psi) : E(\varphi) \, dx = - \int_{\partial F} \operatorname{div}_g(\psi \mathbb{C}E(u_F)) \cdot \varphi \, d\mathcal{H}^2$$

for all $\varphi \in H^1(\Omega \setminus F; \mathbb{R}^2)$ such that $\varphi = 0$ on $\partial_D\Omega$.

Formulas (2.14) and (2.15) have been derived in [9] when F is the subgraph of a periodic function. The very same calculations apply to the more general situation considered here.

Throughout the paper we fix a smooth reference set $G \subset\subset \Omega$ and define the reference manifold as (Σ, g) , where $\Sigma = \partial G$ and g is the metric induced by the Euclidean metric in \mathbb{R}^3 . We denote the outer normal of G simply by ν . For every $\eta > 0$ we denote

$$\mathcal{N}_\eta(\Sigma) := \{x \in \mathbb{R}^3 : |d_G(x)| < \eta\},$$

where d_G denotes the signed distance function of G . Denote also π the orthogonal projection on the boundary of G . Since G is smooth,

$$(2.17) \quad \text{there exists } \eta_0 > 0 \text{ such that } d_G \text{ and } \pi \text{ are smooth in } \mathcal{N}_{2\eta_0}(\Sigma).$$

We denote by $\mathfrak{h}_M^k(\Sigma)$ the following class of sets, whose boundary is a suitable normal graph over Σ . Precisely, for any integer $k \geq 1$ and $M > 0$ we say

$$(2.18) \quad F \in \mathfrak{h}_M^k(\Sigma) \quad \text{if } \partial F = \{x + h_F(x)\nu(x) : x \in \Sigma\} \subset \mathcal{N}_{\eta_0}(\Sigma) \quad \text{with } \|h_F\|_{H^k(\Sigma)} \leq M.$$

In particular, by Morrey embedding any set in $\mathfrak{h}_M^3(\Sigma)$ is $C^{1,\alpha}$ -diffeomorphic to the reference set G for every $\alpha \in (0, 1)$. The space $\mathfrak{h}_M^{k,\alpha}(\Sigma)$, $\alpha \in (0, 1)$, is defined similarly in terms of the $C^{k,\alpha}$ -norm of the function h_F .

Let G_1, \dots, G_m be the bounded open sets enclosed by the connected components $\Gamma_{G,1}, \dots, \Gamma_{G,m}$ of the boundary ∂G . Note that the G_i 's are not in general the connected components of G and it may happen that $G_i \subset G_j$ for some $i \neq j$. If $F \in \mathfrak{h}_M^3(\Sigma)$, then F is C^1 -diffeomorphic to G and thus ∂F has the same number m of connected components $\Gamma_{F,1}, \dots, \Gamma_{F,m}$, which can be numbered in such a way that

$$(2.19) \quad \Gamma_{F,i} = \{x + h_F(x)\nu(x) : x \in \Gamma_{G,i}\},$$

for a suitable $h_F \in H^3(\Sigma)$. The boundaries $\Gamma_{F,i}$ then enclose the sets F_i , which in turn are diffeomorphic to G_i .

We are interested in area preserving variations, in the following sense.

Definition 2.7. Let $F \subset\subset \Omega$ be a smooth set. Given a vector field $X \in C_c^\infty(\Omega; \mathbb{R}^3)$, we say that the associated flow $(\Phi_t)_{t \in (-1,1)}$ is *admissible for F* if there exists $\varepsilon_0 \in (0, 1)$ such that

$$|\Phi_t(F_i)| = |F_i| \quad \text{for } t \in (-\varepsilon_0, \varepsilon_0) \text{ and } i = 1, \dots, m.$$

Remark 2.8. Note that if the flow associated with X is admissible in the sense of the previous definition, then for $i = 1, \dots, m$ we have

$$\int_{\Gamma_{F,i}} X \cdot \nu_F d\mathcal{H}^1 = 0.$$

In view of this remark it is convenient to introduce the space $\tilde{H}^1(\partial F)$ consisting of all functions $\psi \in H^1(\partial F)$ with zero average on each component of ∂F , i.e.,

$$\int_{\Gamma_{F,i}} \psi d\mathcal{H}^1 = 0 \quad \text{for every } i = 1, \dots, m.$$

Any admissible vector field X thus defines a function $\psi \in \tilde{H}^1(\partial F)$. Conversely, given $\psi \in \tilde{H}^1(\partial F) \cap C^\infty(\partial F)$ it is possible to construct a sequence of vector fields $X_n \in C_c^\infty(\Omega; \mathbb{R}^2)$, with $\operatorname{div}_{\mathbb{R}^n} X_n = 0$ in a neighborhood of \bar{F} , such that $X_n \cdot \nu_F \rightarrow \psi$ in $C^1(\partial F)$, see [2, Proof of Corollary 3.4] for the details. Note that in particular the flows associated with X_n are admissible.

Definition 2.9. Let $F \subset\subset \Omega$ be a set of class C^2 . We say that F is *stationary* if

$$\frac{d}{dt} \mathcal{J}(\Phi_t(F)) \Big|_{t=0} = 0$$

for all admissible flows in the sense of Definition 2.7.

Remark 2.10. By Remark 2.8 and in view of (2.14) it follows that a set $F \subset\subset \Omega$ of class C^2 is stationary if and only if there exist constants $\lambda_1, \dots, \lambda_m$ such that

$$H_F - Q(E(u_F)) = \lambda_i \quad \text{on } \Gamma_{F,i}$$

for every $i = 1, \dots, m$. Note that if F is a sufficiently regular (local) minimizer of (2.11) under the constraint $|F| = \text{const.}$, then there exists a constant λ such that

$$H_F - Q(E(u_F)) = \lambda \quad \text{on } \partial F.$$

Thus, our notion of stationarity differs from the usual notion of criticality just recalled. Note that by a bootstrap argument it can be proved that a stationary set is smooth. In fact, it can be shown that it is even analytic, see [33]. Note that if F is stationary, then the second variation formula (2.15) reduces to

$$(2.20) \quad \begin{aligned} \frac{d^2}{dt^2} \mathcal{J}(\Phi_t(F)) \Big|_{t=0} &= \int_{\partial F} |\nabla \psi|^2 - |B_F|^2 \psi^2 d\mathcal{H}^2 \\ &\quad - 2 \int_{\Omega \setminus F} Q(E(u_\psi)) dx - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F))) \psi^2 d\mathcal{H}^2, \end{aligned}$$

where we recall that $\psi = X \cdot \nu_F$ and u_ψ is the function satisfying (2.16).

In view of (2.20), for any set $F \subset\subset \Omega$ of class C^2 it is convenient to introduce the quadratic form $\partial^2 \mathcal{J}(F)$ defined on $\tilde{H}^1(\partial F)$ as

$$(2.21) \quad \begin{aligned} \partial^2 \mathcal{J}(F)[\psi] &:= \int_{\partial F} |\nabla \psi|^2 - |B_F|^2 \psi^2 d\mathcal{H}^2 \\ &\quad - 2 \int_{\Omega \setminus F} Q(E(u_\psi)) dx - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F))) \psi^2 d\mathcal{H}^2, \end{aligned}$$

where u_ψ is the unique solution of (2.16) under the Dirichlet condition $u_\psi = 0$ on $\partial_D \Omega$. We may finally give the definition of stability for a stationary point.

Definition 2.11. Let $F \subset\subset \Omega$ be a stationary set in the sense of Definition 2.9. We say that F is *strictly stable* if

$$(2.22) \quad \partial^2 \mathcal{J}(F)[\psi] > 0 \quad \text{for all } \psi \in \tilde{H}^1(\partial F) \setminus \{0\}.$$

It is not difficult to see that (2.22) is equivalent to the coercivity of $\partial^2 \mathcal{J}(F)$ on $\tilde{H}^1(\partial F)$. More precisely, (2.22) holds if and only if there exists $c_0 > 0$ such that

$$(2.23) \quad \partial^2 \mathcal{J}(F)[\psi] \geq c_0 \|\psi\|_{\tilde{H}^1(\partial F)}^2 \quad \text{for all } \psi \in \tilde{H}^1(\partial F),$$

see [9]. In turn the latter coercivity property is stable with respect to small H^3 -perturbations. More precisely, we have:

Lemma 2.12. *Assume that the reference set $G \subset\subset \Omega$ is a (smooth) strictly stable stationary set in the sense of Definition 2.11. Then, there exists $\sigma_0 > 0$ such that for all $F \in \mathfrak{h}_{\sigma_0}^3(\Sigma)$, defined in (2.18), we have*

$$\partial^2 \mathcal{J}(F)[\psi] \geq \frac{c_0}{2} \|\psi\|_{\tilde{H}^1(\partial F)}^2 \quad \text{for all } \psi \in \tilde{H}^1(\partial F),$$

where c_0 is the constant in (2.23).

Proof. The proof follows the argument in [9, Proof of Theorem 5.2 and Lemma 5.3], where the case of F being the subgraph of a periodic function is considered. Although the geometric framework here is more general, we may follow exactly the same line of argument up to the obvious changes due to the different setting. We note that in our case we may even simplify the aforementioned proof by taking advantage of the fact that $F \in \mathfrak{h}_{\sigma_0}^3(\Sigma)$ (while in [9] only $W^{2,p}$ -bounds were assumed). Indeed, under this assumption we have that u_F is of class H^3 in

a neighborhood of Σ , with the norm estimated by a constant depending on σ_0 (see the proof of Theorem 4.1). In turn, $\partial_{\nu_F}(Q(E(u_F))) \in H^{\frac{1}{2}}(\partial F)$ with a bound depending on σ_0 , which is a much stronger information than the boundedness in $H^{-\frac{1}{2}}(\partial F)$ proven in [9]. \square

We conclude this section by showing that in a sufficiently small H^3 -neighborhood of G the stationary sets are isolated, once we fix the areas enclosed by the connected components of the boundary.

Proposition 2.13. *Assume that the reference set $G \subset \subset \Omega$ is a smooth strictly stable stationary set in the sense of Definition 2.11 and let σ_0 be the constant provided by Lemma 2.12. There exists $\sigma_1 \in (0, \sigma_0)$ with the following property: Let $F_1, F_2 \in \mathfrak{h}_{\sigma_1}^3(\Sigma)$, defined in (2.18), be stationary sets in the sense of Definition 2.9 and (with the same notation as in (2.19)) assume that $|F_{1,i}| = |F_{2,i}|$ for $i = 1, \dots, m$. Then $F_1 = F_2$.*

Proof. Let F_1 and F_2 be in $\mathfrak{h}_{\sigma_1}^3(\Sigma)$, with $\sigma_1 \in (0, \sigma_0)$ to be chosen, and denote the components defined in (2.19) by $F_{i,1}, \dots, F_{i,m}$ for $i = 1, 2$. We begin by constructing a vector field $X : \mathcal{N}_{\eta_0}(\Sigma) \rightarrow \mathbb{R}^3$ such that the associated flow $(\Phi_t)_{t \in [0,1]}$ is admissible in the sense of Definition 2.8 and takes the set F_1 to F_2 . More precisely, it holds $\Phi_0(F_1) = F_1$, $\Phi_1(F_1) = F_2$ and $|\Phi_t(F_{1,i})| = |F_{1,i}|$ for every $t \in [0, 1]$ and $i = 1, \dots, m$. The construction can be done as in [37, Proposition 3.4] (see also [24, Lemma 2.8]) in such a way that $|X(x)| \leq 2|X(x) \cdot \nu_{F_t}(x)|$ for $x \in \partial F_t$ and for all $t \in [0, 1]$, and that

$$\partial F_t = \{x + h_{F_t}(x)\nu(x) : x \in \Sigma\} \quad \text{with} \quad \|h_{F_t}\|_{H^3(\Sigma)} \leq C\sigma_1 < \sigma_0,$$

where the last inequality holds provided that σ_1 is small enough. Recalling (2.15), (2.21), using the Lemma 2.12 and by integrating by parts we get

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{J}(\Phi_t(F_1)) &= \partial^2 \mathcal{J}(F_t)[X \cdot \nu_{F_t}] - \int_{\partial F_t} (H_{F_t} - Q(E(u_{F_t}))) \operatorname{div}_g((X \cdot \nu_{F_t})X_\tau) d\mathcal{H}^2 \\ &\geq \frac{c_0}{2} \|X \cdot \nu_{F_t}\|_{H^1(\partial F_t)}^2 + \int_{\partial F_t} \langle \nabla(H_{F_t} - Q(E(u_{F_t}))), (X \cdot \nu_{F_t})X_\tau \rangle d\mathcal{H}^2. \end{aligned}$$

We denote $R_t := H_{F_t} - Q(E(u_{F_t}))$ and estimate the last term by (5.3), which we will show later in the proof of Theorem 5.1, to get

$$\begin{aligned} \int_{\partial F_t} \langle \nabla R_t, (X \cdot \nu_{F_t})X_\tau \rangle d\mathcal{H}^2 &\leq \left(\int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \right)^{1/2} \left(\int_{\partial F_t} |(X \cdot \nu_{F_t})X_\tau|^2 d\mathcal{H}^2 \right)^{1/2} \\ &\leq C \|h_{F_t}\|_{H^3(\Sigma)}^{\theta/2} \left(\int_{\partial F_t} |X \cdot \nu_{F_t}|^4 d\mathcal{H}^2 \right)^{1/2} \\ &\leq C\sigma_1^{\theta/2} \|X \cdot \nu_{F_t}\|_{L^4(\partial F_t)}^2. \end{aligned}$$

Therefore we have by the Sobolev embedding

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{J}(\Phi_t(F_1)) &\geq \frac{c_0}{2} \|X \cdot \nu_{F_t}\|_{H^1(\partial F_t)}^2 - C\sigma_1^{\theta/2} \|X \cdot \nu_{F_t}\|_{L^4(\partial F_t)}^2 \\ &\geq \frac{c_0}{2} \|X \cdot \nu_{F_t}\|_{H^1(\partial F_t)}^2 - C\sigma_1^{\theta/2} \|X \cdot \nu_{F_t}\|_{H^1(\partial F_t)}^2 \geq \frac{c_0}{4} \|X \cdot \nu_{F_t}\|_{H^1(\partial F_t)}^2, \end{aligned}$$

provided that σ_1 is small enough.

On the other hand by the stationarity of F_1 and F_2 we have

$$\frac{d}{dt} \mathcal{J}(\Phi_t(F_1)) \Big|_{t=0} = \frac{d}{dt} \mathcal{J}(\Phi_t(F_1)) \Big|_{t=1} = 0.$$

This means that $\frac{d^2}{dt^2} \mathcal{J}(\Phi_t(F_1)) = 0$ and therefore $X \cdot \nu_{F_t} = 0$ on ∂F_t for all $t \in (0, 1)$. Therefore $t \mapsto \Phi_t(F_1)$ is constant and $F_1 = F_2$. \square

3. SHORT TIME EXISTENCE FOR THE SURFACE DIFFUSION WITH A FORCING TERM

In the following we shall assume $n = 3$. Given a smooth function $f : \Sigma \times [0, +\infty) \rightarrow \mathbb{R}$ we shall consider the following forced surface diffusion equation

$$(3.1) \quad V_t = \Delta_{\partial F_t}(H_{F_t} + f(\cdot, t) \circ \pi)$$

where V_t denotes the outer normal velocity of ∂F_t and $\Delta_{\partial F_t}$ is the Laplace-Beltrami operator on ∂F_t endowed with the metric induced by the Euclidean metric. The goal in this section is to prove short time existence of a unique smooth solution of (3.1) starting from F_0 which is close to the reference set G . This will be done in Theorem 3.1.

3.1. The flow in coordinates. Given a sufficiently smooth function $h : \Sigma \rightarrow (-\eta_0, \eta_0)$, where η_0 is introduced in (2.17), we denote by F_h the bounded open set whose boundary is given by

$$\partial F_h = \{x + h(x)\nu(x) : x \in \Sigma\},$$

where ν is the outer unit normal to ∂G . Note that the projection $\pi|_{\partial F_h} : \partial F_h \rightarrow \Sigma$ is invertible and we denote by $\pi_{F_h}^{-1}$ its inverse. In this case we have $\pi_{F_h}^{-1}(x) = x + h(x)\nu(x)$.

We denote by ν the normal and by k_1, k_2 the principle curvatures of Σ , while τ_1, τ_2 denote the corresponding eigenvectors on the tangent plane. The exterior normal to F_h is

$$(3.2) \quad \nu_{F_h} \circ \pi_{F_h}^{-1} = \frac{1}{J}((1 + hk_1)(1 + hk_2)\nu - (1 + hk_2)\partial_{\tau_1} h \tau_1 - (1 + hk_1)\partial_{\tau_2} h \tau_2),$$

where $J^2 = (1 + hk_1)^2(1 + hk_2)^2 + (1 + hk_1)^2(\partial_{\tau_1} h)^2 + (1 + hk_2)^2(\partial_{\tau_2} h)^2$. We recall (see [36, p. 21]) that the mean curvature H_{F_h} of ∂F_h can be written as

$$H_{F_h} \circ \pi_{F_h}^{-1} = -(\nu_{F_h} \circ \pi_{F_h}^{-1} \cdot \nu)\Delta h + P(x, h, \nabla h),$$

where P is a smooth function such that $P(\cdot, 0, 0) = H_G$, the mean curvature of the boundary of G . We rewrite the above formula as

$$(3.3) \quad H_{F_h} \circ \pi_{F_h}^{-1} = -\Delta h + \langle A(x, h, \nabla h), \nabla^2 h \rangle + H_G + a(x, h, \nabla h),$$

where the tensor A and the function a are smooth and vanish when both h and ∇h are 0.

Let us denote by g_h the pull-back metric on Σ induced by the diffeomorphism $\pi_{F_h}^{-1} : \Sigma \rightarrow \partial F_h$. Since the manifold $(\partial F_h, g)$ endowed with the Euclidean metric g is isometric to (Σ, g_h) then for every smooth function f defined on Σ we have

$$(\Delta_{\partial F_h}(f \circ \pi)) \circ \pi_{F_h}^{-1} = \Delta_{g_h} f$$

where Δ_{g_h} is the Laplace-Beltrami operator on Σ with respect to the metric g_h . One can also check that (see [36, p. 21])

$$(g_h)_{ij} = g_{ij} + a_{ij}(\cdot, h, \nabla h),$$

where the functions a_{ij} are smooth and vanish when both h and ∇h vanish, and that we have the following expansion of the Christoffel symbols

$$(\Gamma_{g_h})_{jk}^i = (\Gamma_g)_{jk}^i + a_{jk}^i(x, h, \nabla h) + b_{jk}^{ilm}(x, h, \nabla h) \frac{\partial^2 h}{\partial x_l \partial x_m}.$$

Above b_{jk}^{ilm} is a smooth function and a_{jk}^i is a smooth function which vanish when h and ∇h vanish. We recall that the we may write the Laplace-Beltrami operator Δ_{g_h} as

$$\Delta_{g_h} f := (g_h)^{ij} \tilde{\nabla}_i \tilde{\nabla}_j f,$$

where $\tilde{\nabla}_i \tilde{\nabla}_j$ stands for the second order covariant derivatives with respect to g_h . Hence we get by the above formulas and after some straightforward calculations that

$$(3.4) \quad \Delta_{g_h} f = \Delta f + \langle A_1(x, h, \nabla h), \nabla^2 f \rangle + \langle A_2(x, h, \nabla h), \nabla f \rangle + \langle B(x, h, \nabla h), (\nabla^2 h \otimes \nabla f) \rangle.$$

Concerning the equation of interest, assume that a smooth flow $(F_t)_{t \in (0, T)}$ is a solution of (3.1) and that ∂F_t can be written as

$$\partial F_t = \{x + h(x, t)\nu(x) : x \in \Sigma\}.$$

Then the normal velocity is given by $V_t = \partial_t h(\nu_{F_t} \cdot \nu)$. Therefore, combining (3.3) and (3.4) and after long but straightforward calculations, we may rewrite the equation (3.1) as

$$(3.5) \quad \begin{aligned} \frac{\partial h}{\partial t} &= -\Delta^2 h + \langle A(x, h, \nabla h), \nabla^4 h \rangle \\ &\quad + J_1(x, h, \nabla h, \nabla^2 h, \nabla^3 h) + J_2(x, h, \nabla h, \nabla^2 h, \nabla f, \nabla^2 f), \end{aligned}$$

where as usual A is a smooth 4th-order tensor depending on $(x, h, \nabla h)$ vanishing when both h and ∇h vanish, J_1 is given by

$$(3.6) \quad \begin{aligned} J_1 &= \langle B_1, (\nabla^3 h \otimes \nabla^2 h) \rangle + \langle B_2, \nabla^3 h \rangle + \langle B_3, (\nabla^2 h \otimes \nabla^2 h \otimes \nabla^2 h) \rangle \\ &\quad + \langle B_4, (\nabla^2 h \otimes \nabla^2 h) \rangle + \langle B_5, \nabla^2 h \rangle + b_6 \end{aligned}$$

and J_2 is of the form

$$(3.7) \quad J_2 = \Delta f + \langle A_1, \nabla^2 f \rangle + \langle A_2, \nabla f \rangle + \langle B, (\nabla^2 h \otimes \nabla f) \rangle.$$

Here and throughout the paper we denote by A (possibly with a subscript) a smooth tensor-valued function depending on $(x, h, \nabla h)$ and vanishing at $(x, 0, 0)$, while B (possibly with a subscript) stands for a smooth tensor-valued function depending on $(x, h, \nabla h)$. We replace capital letters A and B with a and b , respectively, in case of scalar valued functions.

3.2. Short time existence and uniqueness. Let us fix an initial set $F_0 \in \mathfrak{h}_{K_0}^3(\Sigma)$ which is close to G . Finding a solution of (3.1) for a short time with initial set F_0 is equivalent to finding a solution h of (3.5) with initial datum $h(\cdot, 0) = h_{F_0} =: h_0$. This is the goal of this section and the result is stated in the following theorem.

Theorem 3.1. *Let $f : \Sigma \times [0, +\infty) \rightarrow \mathbb{R}$ be a smooth function. Given $\delta_0 > 0$ and $K_0 > 1$, there exist $\varepsilon_0, T_0 \in (0, 1)$ with the following property: if $F_0 \in \mathfrak{h}_{K_0}^3(\Sigma)$, defined in (2.18),*

$$(3.8) \quad \sup_{0 \leq t \leq T_0} \|f(\cdot, t)\|_{L^\infty(\Sigma)} + \int_0^{T_0} \|f(\cdot, t)\|_{H^3(\Sigma)}^2 dt \leq K_0,$$

and $\|h_0\|_{L^2(\Sigma)} < \varepsilon_0$, where $h_0 := h_{F_0}$, then the equation (3.5) has a unique smooth solution (F_t) in $C^\infty(0, T_0; C^\infty(\Sigma)) \cap H^1(0, T_0; H^1(\Sigma))$ and

$$(3.9) \quad \sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{L^2(\Sigma)} \leq \delta_0.$$

Moreover, for every integer $k \geq 0$ there exist constants $C_k, q_k > 0$, independent of δ_0 and K_0 , such that

$$(3.10) \quad \begin{aligned} & \sup_{0 \leq t \leq T} t^k \|h(\cdot, t)\|_{H^{2k+3}(\Sigma)}^2 + \int_0^T t^k \|h(\cdot, t)\|_{H^{2k+5}(\Sigma)}^2 dt \\ & \leq C_k \left(\|h_0\|_{H^3(\Sigma)}^2 + \int_0^T \left(1 + \|f\|_{L^\infty(\Sigma)}^{q_k} + \sum_{i=0}^k t^i \|f(\cdot, t)\|_{H^{2i+3}(\Sigma)}^2 \right) dt \right), \end{aligned}$$

for every $T \leq T_0$.

The proof of Theorem 3.1 is based on a fixed point argument in a carefully chosen function space and to this aim we need two lemmas. In the first one we estimate the derivatives of the nonlinear terms in (3.5).

Proposition 3.2. *Let h and f be of class $C^\infty(\Sigma)$. For every integer $k \geq 1$ there exist $\tilde{C}_k > 0$ and $p_k \geq 2$ such that given $M_0 > 0$ there is $\sigma_0 > 0$ with the property that if*

$$\|h\|_{H^3(\Sigma)}^2 \leq M_0 \quad \text{and} \quad \|h\|_{L^2(\Sigma)} \leq \sigma_0$$

then

$$\begin{aligned} \int_{\Sigma} |\nabla^k(\langle A, \nabla^4 h \rangle)|^2 + |\nabla^k J_1|^2 + |\nabla^k J_2|^2 d\mathcal{H}^2 & \leq \frac{1}{4} \int_{\Sigma} |\nabla^{k+4} h|^2 d\mathcal{H}^2 \\ & + \tilde{C}_k \left(1 + \|f\|_{L^\infty(\Sigma)}^{p_k} + \int_{\Sigma} |\nabla^{k+2} f|^2 d\mathcal{H}^2 \right), \end{aligned}$$

where A , J_1 , and J_2 are as in (3.5), (3.6), and (3.7).

Proof. Recall that $A(x, h, \nabla h)$ vanishes at $(x, 0, 0)$ and thus given $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that if $\|h\|_{C^1(\Sigma)} \leq \delta$, then by Leibniz formula

$$|\nabla^k(\langle A, \nabla^4 h \rangle)|^2 \leq \varepsilon |\nabla^{k+4} h|^2 + C \sum_{i=1}^k |\nabla^i(A(x, h, \nabla h))|^2 |\nabla^{k+4-i} h|^2.$$

On the other hand, the assumptions on h together with standard interpolation imply that $\|h\|_{C^1} \leq \delta$ and $\|h\|_{W^{2,4}} \leq 1$ when σ_0 is chosen small (depending on M_0). It turns out to be

convenient to set $w := \nabla h$. Since $\|w\|_\infty \leq \delta < 1$, one may check that

$$\begin{aligned} \sum_{i=1}^k |\nabla^i (A(x, h, \nabla h))|^2 |\nabla^{k+4-i} h|^2 &\leq C \sum_{i=1}^k |\nabla^{k+3-i} w|^2 \\ &\quad + C \sum_{i=1}^k \sum_{\substack{1 \leq j_1 \leq \dots \leq j_{m-1} \leq i \\ j_1 + \dots + j_{m-1} \leq i \\ m \geq 2}} |\nabla^{j_1} w|^2 \dots |\nabla^{j_{m-1}} w|^2 |\nabla^{k+3-i} w|^2 \\ &\leq C \sum_{i=1}^k |\nabla^{k+3-i} w|^2 + C \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq k+2 \\ j_1 + \dots + j_m \leq k+3 \\ m \geq 2}} |\nabla^{j_1} w|^2 \dots |\nabla^{j_m} w|^2. \end{aligned}$$

Then by Hölder's inequality we obtain

$$\begin{aligned} \int_{\Sigma} |\nabla^k (\langle A, \nabla^4 h \rangle)|^2 d\mathcal{H}^2 &\leq \int_{\Sigma} (\varepsilon |\nabla^{k+3} w|^2 + C \sum_{i=1}^k |\nabla^{k+3-i} w|^2) d\mathcal{H}^2 \\ &\quad + C \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq k+2 \\ j_1 + \dots + j_m \leq k+3 \\ m \geq 2}} \|\nabla^{j_1} w\|_{\frac{2(k+3)}{j_1}}^2 \dots \|\nabla^{j_m} w\|_{\frac{2(k+3)}{j_m}}^2. \end{aligned}$$

Observe that for every $l = 1, \dots, m-1$, it holds by the interpolation Lemma 2.1

$$\|\nabla^{j_l} w\|_{\frac{2(k+3)}{j_l}} \leq C \|w\|_{H^{k+3}}^{\theta_l} \|w\|_\infty^{1-\theta_l},$$

where $\theta_l = \frac{j_l}{k+3}$. To treat the last derivative we use a different interpolation:

$$\|\nabla^{j_m} w\|_{\frac{2(k+3)}{j_m}} \leq C \|w\|_{H^{k+3}}^{\theta_m} \|\nabla w\|_4^{1-\theta_m},$$

where $\theta_m = \frac{2j_m(k+2)}{(2k+3)(k+3)} - \frac{1}{2k+3} < \frac{j_m}{k+3}$ (recall that $3 \leq j_m < k+3$). Therefore, recalling that $\|w\|_\infty, \|\nabla w\|_4 \leq 1$, we get

$$\begin{aligned} \int_{\Sigma} |\nabla^k (\langle A, \nabla^4 h \rangle)|^2 d\mathcal{H}^2 &\leq \int_{\Sigma} (\varepsilon |\nabla^{k+4} h|^2 + C \sum_{i=1}^k |\nabla^{k+4-i} h|^2) d\mathcal{H}^2 \\ &\quad + C \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq k+2 \\ j_1 + \dots + j_m \leq k+3 \\ m \geq 2}} \prod_{l=1}^m \|w\|_{H^{k+3}}^{2\theta_l}. \end{aligned}$$

Observe that for every choice of j_1, \dots, j_m the sum of the corresponding θ_l satisfies

$$\sum_{l=1}^m \theta_l < \sum_{l=1}^m \frac{j_l}{k+3} \leq 1.$$

Therefore by Young's inequality, by Remark 2.2, and recalling that $\|w\|_\infty \leq 1$, we conclude from the above inequality that

$$(3.11) \quad \int_{\Sigma} |\nabla^k (\langle A, \nabla^4 h \rangle)|^2 d\mathcal{H}^2 \leq \frac{1}{20} \int_{\Sigma} |\nabla^{k+4} h|^2 d\mathcal{H}^2 + \tilde{C}_k.$$

Using again $\|w\|_\infty \leq 1$, we have that

$$|\nabla^k J_1| \leq C \sum_{i=1}^k |\nabla^{k+3-i} w| + C \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq 2+k \\ j_1 + \dots + j_m \leq 3+k \\ m \geq 2}} |\nabla^{j_1} w| \dots |\nabla^{j_m} w|.$$

Therefore, arguing exactly as above, we have

$$(3.12) \quad \int_{\Sigma} |\nabla^k J_1|^2 d\mathcal{H}^2 \leq \frac{1}{20} \int_{\Sigma} |\nabla^{k+4} h|^2 d\mathcal{H}^2 + \tilde{C}_k.$$

In order to control the derivatives of J_2 we need a slightly different argument, because we need to separate the terms involving f and h from each other. We recall (3.7) and begin by estimating

$$|\nabla^k (\Delta f + \langle A_1, \nabla^2 f \rangle)| \leq C \sum_{l=0}^k |\nabla^{l+2} f| + C \sum_{i=1}^k \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq i \\ j_1 + \dots + j_m \leq i \\ m \geq 1}} |\nabla^{j_1} w| \dots |\nabla^{j_m} w| |\nabla^{k+2-i} f|.$$

Therefore, using interpolation as above

$$\begin{aligned} \int_{\Sigma} |\nabla^k (\Delta f + \langle A_1, \nabla^2 f \rangle)|^2 d\mathcal{H}^2 &\leq C \left(\|f\|_\infty^2 + \|\nabla^{k+2} f\|_2^2 \right) \\ &\quad + C \sum_{i=1}^k \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq i \\ j_1 + \dots + j_m \leq i \\ m \geq 1}} \prod_{l=1}^m \|\nabla^{j_l} w\|_{\frac{2(2+k)}{j_l}}^2 \|\nabla^{k+2-i} f\|_{\frac{2(2+k)}{2+k-i}}^2 \\ &\leq C \left(\|f\|_\infty^2 + \|\nabla^{k+2} f\|_2^2 \right) \\ &\quad + C \sum_{i=1}^k \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq i \\ j_1 + \dots + j_m \leq i \\ m \geq 1}} \prod_{l=1}^m \|w\|_{H^{k+3}}^{2\theta(j_l)} \|w\|_\infty^{2(1-\theta(j_l))} \|\nabla^{2+k} f\|_2^{\frac{2(k+2-i)}{k+2}} \|f\|_\infty^{\frac{2i}{k+2}}, \end{aligned}$$

where $\theta(j_l) := \frac{j_l(k+1)}{(k+2)^2}$. Observe that since $j_1 + \dots + j_m \leq i$

$$\sum_{l=1}^m \left(2\theta(j_l) + 2 \frac{(2+k-i)}{k+2} \right) \leq \frac{2[(2+k)^2 - i]}{(2+k)^2} < 2.$$

Therefore, using Young's inequality, we may conclude that

$$(3.13) \quad \int_{\Sigma} |\nabla^k (\Delta f + \langle A_1, \nabla^2 f \rangle)|^2 d\mathcal{H}^2 \leq \frac{1}{20} \|\nabla^{k+4} h\|_2^2 + \tilde{C}_k \left(1 + \|f\|_\infty^{p_k} + \|\nabla^{k+2} f\|_2^2 \right).$$

A similar argument, whose details are left to the reader, shows that

$$\int_{\Sigma} |\nabla^k \langle A_2, \nabla f \rangle + \langle B, (\nabla^2 h \otimes \nabla f) \rangle|^2 d\mathcal{H}^2 \leq \frac{1}{20} \|\nabla^{k+4} h\|_2^2 + \tilde{C}_k \left(1 + \|f\|_\infty^{p_k} + \|\nabla^{k+2} f\|_2^2 \right).$$

The conclusion then follows by combining this inequality with (3.11), (3.12), and (3.13). \square

In the second lemma we “linearize” the terms J_1 and J_2 in the equation (3.5). The argument is similar to the previous one and therefore we postpone its proof until the Appendix.

Lemma 3.3. *Let $T \in (0, 1)$ and let $h_1, h_2, f : \Sigma \times (0, T) \rightarrow \mathbb{R}$ be smooth functions such that*

$$\sup_{0 \leq t \leq T} \|h_i(\cdot, t)\|_{H^3(\Sigma)}^2 + \int_0^T \int_{\Sigma} |\nabla^5 h_i|^2 d\mathcal{H}^2 dt \leq M_0,$$

and

$$\sup_{0 \leq t \leq T} \|f(\cdot, t)\|_{L^\infty(\Sigma)} + \int_0^T \int_{\Sigma} |\nabla^3 f|^2 d\mathcal{H}^2 dt \leq K_0.$$

Then, there exists $\theta \in (0, 1)$ with the following property: for any $\varepsilon > 0$ there exist $C = C(\varepsilon, K_0, M_0) > 0$ and $\delta = \delta(\varepsilon, M_0) > 0$ such that if $\sup_{0 \leq t \leq T} \|h_i(\cdot, t)\|_{L^2(\Sigma)} \leq \delta$, $i = 1, 2$, then

$$\int_0^T \int_{\Sigma} |J_{h_2} - J_{h_1}|^2 d\mathcal{H}^2 dt \leq \varepsilon \int_0^T \int_{\Sigma} |\nabla^4 h_2 - \nabla^4 h_1|^2 d\mathcal{H}^2 dt + CT^\theta \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2(\Sigma)}^2,$$

where J_h is defined as in (3.16).

Proof of Theorem 3.1. Given K_0 , let us define the set \mathcal{S} of functions in $C^\infty(0, T_0; C^\infty(\Sigma)) \cap H^1(0, T_0; H^1(\Sigma))$, which satisfy

$$(3.14) \quad \sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{L^2(\Sigma)} \leq \sigma_0 \quad \text{and} \quad \sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{H^3(\Sigma)}^2 + \int_0^{T_0} \|h(\cdot, t)\|_{H^5(\Sigma)}^2 dt \leq M_0,$$

where the constants M_0 and σ_0 will be chosen later. We also define a subclass $\mathcal{S}' \subset \mathcal{S}$ of functions which satisfy the additional requirement (3.10), where the constants C_k and q_k will again be chosen later. The goal is to obtain a solution of (3.5) in \mathcal{S}' which is unique in \mathcal{S} .

We begin by assuming that h_0 is smooth with $\|h_0\|_{H^3(\Sigma)} < K_0$ and $2\|h_0\|_{L^2(\Sigma)} \leq \sigma_0$. We now define a map $\mathcal{L} : \mathcal{S} \rightarrow C^\infty(0, T_0; C^\infty(\Sigma))$ by setting $\mathcal{L}(h) := \tilde{h}$, where $\tilde{h} : \Sigma \times [0, \infty) \rightarrow \mathbb{R}$ is the solution of

$$(3.15) \quad \begin{cases} \frac{\partial \tilde{h}}{\partial t} = -\Delta^2 \tilde{h} + J_h(x, t) \\ \tilde{h}(\cdot, 0) = h_0 \end{cases}$$

and where we have set

$$(3.16) \quad J_h(x, t) := \langle A(x, h, \nabla h), \nabla^4 h \rangle + J_1(x, h, \nabla h, \nabla^2 h, \nabla^3 h) + J_2(x, h, \nabla h, \nabla^2 h, \nabla f, \nabla^2 f)$$

with A, J_1, J_2 as in (3.5).

We note that the set \mathcal{S}' is nonempty when the constants C_k are chosen properly. To see this consider the solution \bar{h} of

$$(3.17) \quad \begin{cases} \frac{\partial \bar{h}}{\partial t} = -\Delta^2 \bar{h} \\ \bar{h}(\cdot, 0) = h_0. \end{cases}$$

By classical regularity estimates \bar{h} is smooth and satisfies $\sup_{0 \leq t \leq 1} \|\bar{h}(\cdot, t)\|_{L^2(\Sigma)} \leq \|h_0\|_{L^2(\Sigma)}$ and

$$\sup_{0 \leq t \leq 1} t^k \|h(\cdot, t)\|_{H^{2k+3}(\Sigma)}^2 + \int_0^1 t^k \|h(\cdot, t)\|_{H^{2k+5}(\Sigma)}^2 dt \leq C'_k \|h_0\|_{H^3(\Sigma)}^2$$

for all integers $k \geq 0$, and therefore $\bar{h} \in \mathcal{S}'$ provided that we choose M_0 sufficiently large. We remark that in Steps 1 and 2 below we give an argument which can be applied to prove the above estimate.

Step 1: In this step we prove that if $h \in \mathcal{S}$ then $\tilde{h} = \mathcal{L}(h) \in \mathcal{S}$ for a suitable choice of M_0 , σ_0 and T_0 .

To prove this we multiply (3.15) by $\Delta^3 \tilde{h}$. Integrating by parts both sides we get

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_{\Sigma} |\nabla(\Delta \tilde{h})|^2 d\mathcal{H}^2 &= - \int_{\Sigma} \frac{\partial \tilde{h}}{\partial t} \Delta^3 \tilde{h} d\mathcal{H}^2 = \int_{\Sigma} (\Delta^2 \tilde{h} - J_h) \Delta^3 \tilde{h} d\mathcal{H}^2 \\ &= \int_{\Sigma} (-|\nabla(\Delta^2 \tilde{h})|^2 + \langle \nabla J_h, \nabla(\Delta^2 \tilde{h}) \rangle) d\mathcal{H}^2. \end{aligned}$$

By Proposition 3.2 it follows that if σ_0 is sufficiently small then

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_{\Sigma} |\nabla(\Delta \tilde{h})|^2 d\mathcal{H}^2 &\leq -\frac{1}{2} \int_{\Sigma} |\nabla(\Delta^2 \tilde{h})|^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma} |\nabla J_h|^2 d\mathcal{H}^2 \\ &\leq -\frac{1}{2} \int_{\Sigma} |\nabla(\Delta^2 \tilde{h})|^2 d\mathcal{H}^2 + \frac{3}{8} \int_{\Sigma} |\nabla^5 h|^2 d\mathcal{H}^2 \\ &\quad + \frac{3}{2} \tilde{C}_1 \left(1 + \|f\|_{L^\infty(\Sigma)}^{q_0} + \int_{\Sigma} |\nabla^3 f|^2 d\mathcal{H}^2 \right), \end{aligned}$$

where $q_0 = p_1$ and \tilde{C}_1 are from the Proposition 3.2. Integrate this over $(0, t)$ with $t \leq T_0$, where T_0 will be chosen later, and get

(3.18)

$$\begin{aligned} &\int_{\Sigma} |\nabla(\Delta \tilde{h}(\cdot, t))|^2 d\mathcal{H}^2 - \int_{\Sigma} |\nabla(\Delta h_0)|^2 d\mathcal{H}^2 + \int_0^t \int_{\Sigma} |\nabla(\Delta^2 \tilde{h})|^2 d\mathcal{H}^2 ds \\ &\leq \frac{3}{4} \int_0^{T_0} \int_{\Sigma} |\nabla^5 h(\cdot, t)|^2 d\mathcal{H}^2 dt + 3\tilde{C}_1 \int_0^{T_0} \left(1 + \|f(\cdot, t)\|_{L^\infty(\Sigma)}^{q_0} + \int_{\Sigma} |\nabla^3 f(\cdot, t)|^2 d\mathcal{H}^2 \right) dt. \end{aligned}$$

From this estimate, from the fact that h satisfies (3.14), f satisfies (3.8), $\|h_0\|_{H^3(\Sigma)} < K_0$ and using Remark 2.5 (with a sufficiently small ε) we obtain

$$\begin{aligned} (3.19) \quad &\sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{H^3} + \int_0^{T_0} \|\tilde{h}(\cdot, t)\|_{H^5}^2 dt \\ &\leq C \sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{L^2} + K_0^2 + \frac{4}{5} M_0 + 4\tilde{C}_1 ((T_0 + T_0 K_0^{q_0}) + K_0). \end{aligned}$$

In order to estimate the L^2 -norm of \tilde{h} , we multiply the equation (3.15) by \tilde{h} . Recalling (3.16) and using the H^3 -bound on h and the interpolation Lemma 2.1 we get

$$\begin{aligned}
& \int_{\Sigma} \frac{\partial \tilde{h}}{\partial t} \tilde{h} d\mathcal{H}^2 = - \int_{\Sigma} \Delta^2 \tilde{h} \tilde{h} d\mathcal{H}^2 + \int_{\Sigma} J_h \tilde{h} d\mathcal{H}^2 \\
& \leq \int_{\Sigma} (-|\Delta \tilde{h}|^2 + C_{\eta} \tilde{h}^2) d\mathcal{H}^2 + \eta \int_{\Sigma} J_h^2 d\mathcal{H}^2 \\
(3.20) \quad & \leq C_{\eta} \int_{\Sigma} \tilde{h}^2 d\mathcal{H}^2 \\
& \quad + C_{\eta} \int_{\Sigma} (1 + (1 + |\nabla^2 h|^2) |\nabla^3 h|^2 + |\nabla^2 h|^6 + (1 + |\nabla^2 h|^2) (|\nabla f|^2 + |\nabla^2 f|^2)) d\mathcal{H}^2 \\
& \leq C_{\eta} \int_{\Sigma} \tilde{h}^2 d\mathcal{H}^2 + C_{\eta} \left(1 + \|f\|_{L^{\infty}}^2 + \int_{\Sigma} (|\nabla^5 h|^2 + |\nabla^3 f|^2) d\mathcal{H}^2 \right),
\end{aligned}$$

for some $C > 0$ depending on M_0 and K_0 . Integrating this over $(0, t)$ and using the fact that h satisfies (3.14) and f satisfies (3.8) yield

$$\begin{aligned}
\frac{1}{2} \int_{\Sigma} \tilde{h}(\cdot, t)^2 d\mathcal{H}^2 - \frac{1}{2} \int_{\Sigma} h_0^2 d\mathcal{H}^2 & \leq C_{\eta} T_0 \sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{L^2(\Sigma)}^2 \\
& \quad + \tilde{C}_{\eta} \left(T_0 + T_0 K_0^2 + M_0 + K_0 \right).
\end{aligned}$$

Hence, recalling that $\|h_0\|_{L^2(\Sigma)} \leq \frac{\sigma_0}{2}$ we have

$$\sup_{0 \leq t \leq T} \|\tilde{h}(\cdot, t)\|_{L^2}^2 \leq \frac{\sigma_0^2}{4} + 2C_{\eta} T_0 \sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{L^2}^2 + 2\tilde{C}_{\eta} \left(T_0 + T_0 K_0^2 + M_0 + K_0 \right).$$

From this inequality, choosing η and T_0 sufficiently small (depending on M_0 and K_0) we conclude that

$$\sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{L^2(\Sigma)} \leq \sigma_0.$$

In turn, since $\sigma_0 \leq 1$, we may choose M_0 sufficiently large (depending on K_0) and T_0 smaller if needed to deduce that from (3.19) that

$$\sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{H^3(\Sigma)} + \int_0^{T_0} \|\tilde{h}(\cdot, t)\|_{H^5(\Sigma)}^2 dt \leq M_0.$$

This concludes the proof of the fact that $\tilde{h} = \mathcal{L}(h)$ satisfies (3.14) and thus belongs to \mathcal{S} .

Step 2: Let us now prove that if $h \in \mathcal{S}'$ then $\tilde{h} = \mathcal{L}(h) \in \mathcal{S}'$, i.e., it satisfies (3.10) with h replaced by \tilde{h} . We begin by observing that the case $k = 0$ can be proven by a similar argument as the one used in Step 1, by combining (3.18), (3.20) and replacing T_0 by $T \leq T_0$. We proceed by induction and assume that (3.10) holds for $k - 1$ and prove it for k . We argue similarly as in the previous step and multiply the equation (3.15) by $\Delta^{2k+3} \tilde{h}$, and after integrating by parts the left-hand side $(2k + 3)$ -times and the right-hand side $(2k + 1)$ -times

and using Proposition 3.2 with k replaced by $2k + 1$ we get

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_{\Sigma} |\nabla(\Delta^{k+1}\tilde{h})|^2 d\mathcal{H}^2 &\leq -\frac{1}{2} \int_{\Sigma} |\nabla(\Delta^{k+2}\tilde{h})|^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma} |\nabla^{2k+1} J_h|^2 d\mathcal{H}^2 \\ &\leq -\frac{1}{2} \int_{\Sigma} |\nabla(\Delta^{k+2}\tilde{h})|^2 d\mathcal{H}^2 + \frac{3}{8} \int_{\Sigma} |\nabla^{2k+5} h|^2 d\mathcal{H}^2 \\ &\quad + \frac{3}{2} \tilde{C}_{2k+1} \left(1 + \|f(\cdot, t)\|_{L^\infty}^{p_{2k+1}} + \int_{\Sigma} |\nabla^{2k+3} f|^2 d\mathcal{H}^2 \right). \end{aligned}$$

From this estimate we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(t^k \int_{\Sigma} |\nabla(\Delta^{k+1}\tilde{h})|^2 d\mathcal{H}^2 \right) &\leq k t^{k-1} \int_{\Sigma} |\nabla(\Delta^{k+1}\tilde{h})|^2 d\mathcal{H}^2 - t^k \int_{\Sigma} |\nabla(\Delta^{k+2}\tilde{h})|^2 d\mathcal{H}^2 \\ &\quad + \frac{3}{4} t^k \int_{\Sigma} |\nabla^{2k+5} h|^2 d\mathcal{H}^2 + 3\tilde{C}_{2k+1} t^k \left(1 + \|f(\cdot, t)\|_{L^\infty}^{p_{2k+1}} + \int_{\Sigma} |\nabla^{2k+3} f|^2 d\mathcal{H}^2 \right). \end{aligned}$$

Integrating this inequality over $(0, t)$ for $t \leq T$ yields

$$\begin{aligned} &\sup_{0 \leq t \leq T} t^k \int_{\Sigma} |\nabla(\Delta^{k+1}\tilde{h})|^2 d\mathcal{H}^2 + \int_0^T t^k \int_{\Sigma} |\nabla(\Delta^{k+2}\tilde{h})|^2 d\mathcal{H}^2 dt \\ &\leq k \int_0^T t^{k-1} \int_{\Sigma} |\nabla^{2k+3}\tilde{h}|^2 d\mathcal{H}^2 dt \\ &\quad + \int_0^T t^k \left(\frac{3}{4} \int_{\Sigma} |\nabla^{2k+5} h|^2 d\mathcal{H}^2 + 3\tilde{C}_{2k+1} (1 + \|f(\cdot, t)\|_{L^\infty}^{p_{2k+1}} + \int_{\Sigma} |\nabla^{2k+3} f|^2 d\mathcal{H}^2) \right) dt. \end{aligned}$$

By using the fact that \tilde{h} satisfies (3.10) with $k - 1$, and h satisfies (3.10) we deduce

$$\begin{aligned} &\sup_{0 \leq t \leq T} t^k \|\nabla(\Delta^{k+1}\tilde{h})\|_{L^2(\Sigma)}^2 + \int_0^T t^k \int_{\Sigma} |\nabla(\Delta^{k+2}\tilde{h})|^2 d\mathcal{H}^2 dt \\ &\leq (kC_{k-1} + \frac{3}{4}C_k + 3\tilde{C}_{2k+1}) \left(\int_0^{T_0} (\|h_0\|_{H^3(\Sigma)}^2 + 3 + 3\|f(\cdot, t)\|_{L^\infty(\Sigma)}^{q_k} + \sum_{i=0}^k t^i \|f(\cdot, t)\|_{H^{2i+3}}^2) dt \right) \end{aligned}$$

when we choose $q_k \geq \max\{q_{k-1}, p_{2k+1}\}$. Using the fact that $\sup_{0 \leq t \leq T_0} \|\tilde{h}(\cdot, t)\|_{L^2} \leq \sigma_0$ and by Remark 2.5, we obtain the estimate (3.10) for \tilde{h} by choosing C_k large enough.

Step 3: In this step we prove that the map \mathcal{L} introduced in the previous step is a contraction with respect to a suitable norm, provided that σ_0 and T_0 are chosen sufficiently small.

To this aim, let $h_1, h_2 \in \mathcal{S}$ and let $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}$ be the corresponding solutions of (3.15). Multiplying the equation satisfied by \tilde{h}_i by $\Delta^2(\tilde{h}_2 - \tilde{h}_1)$, subtracting and integrating by parts we get

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{1}{2} \int_{\Sigma} |\Delta(\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t))|^2 d\mathcal{H}^2 \\ &= - \int_{\Sigma} |\Delta^2(\tilde{h}_2 - \tilde{h}_1)(\cdot, t)|^2 d\mathcal{H}^2 + \int_{\Sigma} \Delta^2(\tilde{h}_2 - \tilde{h}_1)(\cdot, t) (J_{h_2}(\cdot, t) - J_{h_1}(\cdot, t)) d\mathcal{H}^2 \\ &\leq -\frac{1}{2} \int_{\Sigma} |\Delta^2(\tilde{h}_2 - \tilde{h}_1)(\cdot, t)|^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma} |J_{h_2}(\cdot, t) - J_{h_1}(\cdot, t)|^2 d\mathcal{H}^2. \end{aligned}$$

Fix $\varepsilon > 0$ small. By choosing σ_0 smaller in (3.14) if needed, we may integrate the above inequality over $(0, t)$, with $t < T_0$, and use Remark 2.5 and Lemma 3.3 to obtain

$$\begin{aligned}
& \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{H^2(\Sigma)}^2 + \int_0^{T_0} \int_{\Sigma} |\nabla^4(\tilde{h}_2 - \tilde{h}_1)|^2 d\mathcal{H}^2 dt \\
& \leq C \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{L^2(\Sigma)}^2 + C \int_0^{T_0} \int_{\Sigma} |\tilde{h}_2 - \tilde{h}_1|^2 d\mathcal{H}^2 dt \\
(3.21) \quad & + \varepsilon \int_0^{T_0} \int_{\Sigma} |\nabla^4(h_2 - h_1)|^2 d\mathcal{H}^2 dt + CT_0^\theta \sup_{0 \leq t \leq T_0} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2(\Sigma)}^2 \\
& \leq C \sup_{0 \leq t \leq T_0} \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{L^2(\Sigma)}^2 \\
& + \varepsilon \int_0^{T_0} \int_{\Sigma} |\nabla^4(h_2 - h_1)|^2 d\mathcal{H}^2 dt + CT_0^\theta \sup_{0 \leq t \leq T_0} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2(\Sigma)}^2.
\end{aligned}$$

Next we have to estimate the first term on the right-hand side. To this aim we multiply the equations satisfied by \tilde{h}_1 and \tilde{h}_2 by $\tilde{h}_2 - \tilde{h}_1$, subtract and get

$$\begin{aligned}
& \frac{\partial}{\partial t} \frac{1}{2} \int_{\Sigma} |\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)|^2 d\mathcal{H}^2 = \int_{\Sigma} (\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)) \frac{\partial}{\partial t} (\tilde{h}_2 - \tilde{h}_1)(\cdot, t) d\mathcal{H}^2 \\
& = - \int_{\Sigma} (\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)) \Delta^2 (\tilde{h}_2 - \tilde{h}_1)(\cdot, t) d\mathcal{H}^2 \\
& + \int_{\Sigma} (\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)) (J_{h_2}(\cdot, t) - J_{h_1}(\cdot, t)) d\mathcal{H}^2 \\
& \leq - \int_{\Sigma} |\Delta(\tilde{h}_2 - \tilde{h}_1)(\cdot, t)|^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma} |\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)|^2 d\mathcal{H}^2 \\
& + \frac{1}{2} \int_{\Sigma} |J_{h_2}(\cdot, t) - J_{h_1}(\cdot, t)|^2 d\mathcal{H}^2.
\end{aligned}$$

Integrating over $(0, t)$, with $t < T_0$, and using again Lemma 3.3 we get

$$\begin{aligned}
& \int_{\Sigma} |\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)|^2 d\mathcal{H}^2 \leq T_0 \sup_{0 \leq t \leq T_0} \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{L^2(\Sigma)}^2 \\
& + \varepsilon \int_0^{T_0} \int_{\Sigma} |\nabla^4 h_1 - \nabla^4 h_2|^2 d\mathcal{H}^2 dt + CT_0^\theta \sup_{0 \leq t \leq T_0} \|h_1(\cdot, t) - h_2(\cdot, t)\|_{H^2(\Sigma)}^2,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
(3.22) \quad & \sup_{0 \leq t \leq T_0} \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{L^2(\Sigma)}^2 \leq 2\varepsilon \int_0^{T_0} \int_{\Sigma} |\nabla^4 h_1 - \nabla^4 h_2|^2 d\mathcal{H}^2 dt \\
& + 2CT_0^\theta \sup_{0 \leq t \leq T_0} \|h_1(\cdot, t) - h_2(\cdot, t)\|_{H^2(\Sigma)}^2,
\end{aligned}$$

provided that $T_0 \leq \frac{1}{2}$. Combining (3.21) and (3.22), and taking ε small and T_0 smaller if needed, we deduce that

$$(3.23) \quad \sup_{0 \leq t \leq T_0} \|\tilde{h}_2(\cdot, t) - \tilde{h}_1(\cdot, t)\|_{H^2(\Sigma)}^2 + \int_0^{T_0} \int_{\Sigma} |\nabla^4(\tilde{h}_2 - \tilde{h}_1)|^2 d\mathcal{H}^2 dt \\ \leq \frac{1}{2} \left(\sup_{0 \leq t \leq T_0} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2(\Sigma)}^2 + \int_0^{T_0} \int_{\Sigma} |\nabla^4(h_2 - h_1)|^2 d\mathcal{H}^2 dt \right).$$

Step 4. (Conclusion) We may proceed with a standard argument, by recursively setting $h_1 = \bar{h}$, with \bar{h} defined as in (3.17), and $h_n := \mathcal{L}(h_{n-1})$ and for every $n \geq 2$. From (3.23) we have that there exists h such that $h_n \rightarrow h$ in $L^\infty(0, T_0; H^2(\Sigma)) \cap L^2(0, T_0; H^4(\Sigma))$. Moreover, from Step 1 and Step 2 we have also that $h_n \rightharpoonup h$ weakly in $H_{loc}^1(0, T; H^k(\Sigma))$ and that h satisfies (3.9) and (3.10). Using these convergences one can easily pass to the limit in the equations satisfied by the h_n 's to conclude that h is a solution of (3.1). We remark that the smoothness of h in time follows from the equation and from the regularity in space of h . Note that the smoothness assumption on h_0 can be removed by a standard approximation argument. Finally, the uniqueness follows from the same argument used to prove (3.23). \square

4. SHORT TIME EXISTENCE FOR THE SURFACE DIFFUSION FLOW WITH ELASTICITY

Here we will prove the existence of the flow

$$(4.1) \quad V_t = \Delta_{\partial F_t}(H_{F_t} - Q(E(u_{F_t})))$$

where u_{F_t} is the minimizer of the elastic energy, that is the solution to (2.12), with F replaced by F_t .

The most crucial point for the proof of the short time existence of (4.1), is to prove sharp regularity estimates for u_F up to the boundary ∂F in terms of regularity of ∂F . We prove this in the theorem below.

Theorem 4.1. *Let $K > 0$, $\alpha \in (0, 1)$, and let $k \geq 3$ be an integer. There exists $C_k = C_k(K) > 0$ such that if $h \in H^k(\Sigma)$ and $F_h \in \mathfrak{H}_K^{1, \alpha}(\Sigma)$, defined as in (2.18), then*

$$(4.2) \quad \|Q(E(u_{F_h})) \circ \pi_{F_h}^{-1}\|_{H^{k-\frac{3}{2}}(\Sigma)} \leq C_k(\|h\|_{H^k(\Sigma)} + 1).$$

Moreover if $h_1, h_2 \in H^3(\Sigma)$ and $F_{h_i} \in \mathfrak{H}_K^3(\Sigma)$ for $i = 1, 2$, then there exists $C = C(K) > 0$ such that

$$(4.3) \quad \|u_{F_{h_2}} \circ \pi_{F_{h_2}}^{-1} - u_{F_{h_1}} \circ \pi_{F_{h_1}}^{-1}\|_{H^{3/2}(\Sigma)} \leq C\|h_2 - h_1\|_{H^2(\Sigma)}.$$

Proof. We begin by proving (4.2). By standard approximation argument we may assume that h is smooth, which implies that u_{F_h} is smooth up to the boundary ∂F_h .

We consider a diffeomorphism $\Phi_h : \Omega \setminus G \rightarrow \Omega \setminus F_h$ such that

$$\Phi_h(x) = x + h(\pi(x))\nu(\pi(x))$$

in $\mathcal{N}_{\eta_0}^+(G)$, where for any $\sigma > 0$ $\mathcal{N}_{\sigma}^+(G) = \{x \in \Omega \setminus G : d_G \leq \sigma\}$ is the one-sided neighborhood of Σ . Note that we may construct Φ_h such that $\|\Phi_h - I\|_{H^k(\Omega \setminus G)} + \|\Phi_h^{-1} - I\|_{H^k(\Omega \setminus G)} \leq C\|h\|_{H^k(\Sigma)}$.

Let us fix $x_0 \in \Sigma$. There exists a smooth diffeomorphism Φ from a neighborhood U of x_0 to a ball B_{2R} which straightens the boundary such that $\Phi(U \cap (\Omega \setminus G)) = B_{2R}^+ = B_{2R} \cap \{x_3 > 0\}$. Setting $v = u_{F_h} \circ \Phi_h \circ \Phi^{-1}$ and $\bar{h} := h \circ \pi \circ \Phi^{-1}$, v is a solution of a system of the form

$$(4.4) \quad \int_{B_{2R}^+} \mathbb{A}(x, \bar{h}, D\bar{h}) Dv : D\varphi \, dx = 0$$

for all $\varphi \in C^\infty(B_{2R}^+; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$, where the tensor \mathbb{A} is smooth. In particular, by using the explicit definition of \bar{h} and Lemma 7.1 it holds $\|\bar{h}\|_{H^k(B_{2R}^+)} \leq C(k)(1 + \|h\|_{H^k(\Sigma)})$ for every $k \in \mathbb{N}$. Moreover, by using Korn's inequality, one may check that \mathbb{A} is elliptic in the sense that

$$(4.5) \quad \int_{B_{2R}^+} \mathbb{A}(x, \bar{h}, D\bar{h}) D\varphi : D\varphi \, dx \geq c_0 \int_{B_{2R}^+} |D\varphi|^2 \, dx,$$

for all $\varphi \in C^\infty(B_{2R}^+; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$.

Let us fix $k \geq 3$ and a multi-index $\beta = (\beta_1, \beta_2, 0)$, with $\beta_1 + \beta_2 = k - 1$. By differentiating the equation (4.4) in the β -directions we have

$$\int_{B_{2R}^+} D^\beta(\mathbb{A}(x, \bar{h}, D\bar{h}) Dv) : D\varphi \, dx = 0.$$

Let $\eta \in C_0^\infty(B_{2R})$ be a standard cut-off function such that $\eta \equiv 1$ in B_R and $0 \leq \eta \leq 1$. By choosing $\varphi = D^\beta v \eta^2$ as a test function in (4.4) and by expanding the term $D^\beta((\mathbb{A}(x, \bar{h}, D\bar{h}) Dv))$ by Leibniz formula we deduce

$$\begin{aligned} \int_{B_{2R}^+} (\mathbb{A}(x, \bar{h}, D\bar{h}) D D^\beta v) : D D^\beta v \eta^2 \, dx &\leq 2 \int_{B_{2R}^+} |\mathbb{A}(x, \bar{h}, D\bar{h})| |D D^\beta v| |D\eta| \eta |D^\beta v| \, dx \\ &+ C \sum_{i=1}^{k-1} \int_{B_{2R}^+} |D^i \mathbb{A}(x, \bar{h}, D\bar{h})| |D^{k-i} v| (|D D^\beta v| \eta^2 + |D^\beta v| |D\eta| \eta) \, dx \end{aligned}$$

By the ellipticity condition (4.5) we have

$$\begin{aligned} \frac{c_0}{2} \int_{B_{2R}^+} |D(D^\beta v)|^2 \eta^2 \, dx &\leq c_0 \int_{B_{2R}^+} |D(D^\beta v \eta)|^2 \, dx + c_0 \int_{B_{2R}^+} |D^\beta v|^2 |D\eta|^2 \, dx \\ &\leq \int_{B_{2R}^+} (\mathbb{A}(x, \bar{h}, D\bar{h}) D(D^\beta v \eta)) : D(D^\beta v \eta) \, dx + c_0 \int_{B_R^+} |D^\beta v|^2 |D\eta|^2 \, dx \\ &\leq \int_{B_{2R}^+} (\mathbb{A}(x, \bar{h}, D\bar{h}) D D^\beta v) : D D^\beta v \eta^2 \, dx \\ &\quad + C \int_{B_{2R}^+} |D D^\beta v| |D\eta| \eta |D^\beta v| + |D^\beta v|^2 |D\eta|^2 \, dx, \end{aligned}$$

where in the last inequality we have used fact that $\|\bar{h}\|_{C^{1,\alpha}} \leq C$, which in turn implies that $\mathbb{A}(x, \bar{h}, D\bar{h})$ is bounded. Combining the previous estimates and using Young's inequality we obtain

$$(4.6) \quad \int_{B_R^+} |D(D^\beta v)|^2 \, dx \leq C \int_{B_{2R}^+} |D^{k-1} v|^2 \, dx + C \sum_{i=1}^{k-1} \int_{B_{2R}^+} |D^i \mathbb{A}(x, \bar{h}, D\bar{h})|^2 |D^{k-i} v|^2 \, dx.$$

We denote $w = D\bar{h}$ and estimate by Leibniz formula

$$\sum_{i=1}^{k-1} |D^i \mathbb{A}(x, \bar{h}, D\bar{h})|^2 |D^{k-i} v|^2 \leq C \sum_{i=1}^{k-1} |D^{k-i} v|^2 + C \sum_{i=1}^{k-1} \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq i \\ j_1 + \dots + j_m \leq i \\ m \geq 1}} |D^{j_1} w|^2 \dots |D^{j_m} w|^2 |D^{k-i} v|^2.$$

Then by Hölder's inequality we get

$$\begin{aligned} \sum_{i=1}^{k-1} \int_{B_{2R}^+} |D^i \mathbb{A}(x, \bar{h}, D\bar{h})|^2 |D^{k-i} v|^2 dx &\leq C \|v\|_{H^{k-1}(B_{2R}^+)}^2 \\ &+ C \sum_{i=1}^{k-1} \sum_{\substack{1 \leq j_1 \leq \dots \leq j_m \leq i \\ j_1 + \dots + j_m \leq i \\ m \geq 1}} \|D^{j_1} w\|_{\frac{2(k-1)}{j_1}}^2 \dots \|D^{j_m} w\|_{\frac{2(k-1)}{j_m}}^2 \|D^{k-i} v\|_{\frac{2(k-1)}{k-i-1}}^2, \end{aligned}$$

where all the norms in the last line are evaluated in B_{2R}^+ . Note that if $i = k - 1$ then in the last term it is understood that $\|D^{k-i} v\|_{\frac{k-1}{k-1-i}} = \|Dv\|_{L^\infty}$. Note that by standard Schauder estimates the assumption $\|h\|_{C^{1,\alpha}(\Sigma)} \leq K$ implies that $\|Dv\|_{L^\infty(B_{2R}^+)} \leq C$. We use Lemma 2.1 to estimate

$$\|D^{j_l} w\|_{\frac{2(k-1)}{j_l}} \leq C \|w\|_{H^{k-1}}^{\theta(j_l)} \|w\|_{L^\infty}^{1-\theta(j_l)} \leq C \|w\|_{H^{k-1}}^{\theta(j_l)}$$

for $\theta(j_l) := \frac{j_l}{k-1}$. By the same lemma we also have

$$\|D^{k-i} v\|_{\frac{2(k-1)}{k-i-1}} \leq C \|v\|_{H^k}^\theta \|Dv\|_{L^\infty}^{1-\theta} \leq C \|v\|_{H^k}^\theta$$

for $\theta = \frac{k-i-1}{k-1}$. Since $\theta(j_1) + \dots + \theta(j_m) \leq \frac{i}{k-1}$, from (4.6) and from the previous estimate we have by Young's inequality

$$\begin{aligned} \int_{B_R^+} |D(D^\beta v)|^2 dx &\leq C \|v\|_{H^{k-1}(B_{2R}^+)}^2 + C \sum_{i=1}^{k-1} (\|w\|_{H^{k-1}(B_{2R}^+)}^{\frac{2i}{k-1}} + 1) \|v\|_{H^k(B_{2R}^+)}^{\frac{2(k-i-1)}{k-1}} \\ &\leq \varepsilon \|D^k v\|_{L^2(B_{2R}^+)}^2 + C \|v\|_{H^{k-1}(B_{2R}^+)}^2 + C(1 + \|h\|_{H^k(\Sigma)}^2). \end{aligned}$$

In order to control the remaining derivatives we use the equation (4.4) in the strong form

$$\operatorname{div}(\mathbb{A}(x, \bar{h}, D\bar{h}) Dv) = 0.$$

Indeed, observe that we have estimated all the derivatives of the type $D^\beta(Dv)$, where $\beta = (\beta_1, \beta_2, 0)$, with $\beta_1 + \beta_2 = k - 1$. Using these estimates and differentiating the equation $k - 2$ times with respect to the horizontal directions and once in the vertical direction, we may estimate $D^\beta(D_{x_3 x_3} v)$ for all $\beta = (\beta_1, \beta_2, 0)$, with $\beta_1 + \beta_2 = k - 2$, by using an interpolation argument as before to control the lower order derivatives. Then we proceed by induction by differentiating the equation $k - 3$ times with respect to the horizontal directions and twice in the vertical direction, and so on, until we differentiate the equation $k - 1$ times only in the vertical direction. As a result we obtain

$$\int_{B_R^+} |D^k v|^2 dx \leq \varepsilon \|D^k v\|_{L^2(B_{2R}^+)}^2 + C \|v\|_{H^{k-1}(B_{2R}^+)}^2 + C(1 + \|h\|_{H^k(\Sigma)}^2).$$

The previous estimate holds at every point on ∂F_h . Thus we may cover $\mathcal{N}_{\sigma_1}^+(F_h)$, with $\sigma_1 < \frac{\eta_0}{2}$, by a finite union of balls and use the previous estimate in every ball of the covering. Precisely,

we go back to the original map, set $u = u_{F_h} \circ \Phi_h$ for simplicity, use Lemma 7.1 and conclude that there are $0 < \sigma_1 < \sigma_2$ such that

$$\begin{aligned} \int_{\mathcal{N}_{\sigma_1}^+} |D^k u|^2 dx &\leq C\varepsilon \int_{\mathcal{N}_{\sigma_2}^+} |D^k u|^2 dx + C\|u\|_{H^{k-1}(\mathcal{N}_{\sigma_2}^+)}^2 + C(1 + \|h\|_{H^k(\Sigma)}^2) \\ &\leq 2C\varepsilon \int_{\mathcal{N}_{\sigma_2}^+} |D^k u|^2 dx + C\|u\|_{L^2(\mathcal{N}_{\sigma_2}^+)}^2 + C(1 + \|h\|_{H^k(\Sigma)}^2), \end{aligned}$$

where the last inequality follows from standard interpolation inequality. Choosing ε small we obtain

$$\int_{\mathcal{N}_{\sigma_1}^+} |D^k u|^2 dx \leq 2 \int_{\mathcal{N}_{\sigma_2}^+ \setminus \mathcal{N}_{\sigma_1}^+} |D^k u|^2 dx + C\|u\|_{L^2(\mathcal{N}_{\sigma_2}^+)}^2 + C(1 + \|h\|_{H^k(\Sigma)}^2).$$

By standard interior regularity it holds

$$\int_{\mathcal{N}_{\sigma_2}^+ \setminus \mathcal{N}_{\sigma_1}^+} |D^k u|^2 dx \leq C\|u_{F_h}\|_{L^2(\Omega \setminus F_h)}^2.$$

Again by standard interpolation we have that

$$\|u\|_{H^k(\mathcal{N}_{\sigma_1}^+)} \leq C(1 + \|u\|_{L^2(\mathcal{N}_{\sigma_1}^+)} + \|h\|_{H^k(\Sigma)}).$$

By the minimality and by Poincaré inequality we have that $\|u_{F_h}\|_{L^2(\Omega \setminus F_h)}$ is bounded by the boundary value w_0 . Using again Lemma 7.1 and the C^1 estimates on u_{F_h} , we have from the above inequality that

$$\|Q(E(u_{F_h})) \circ \Phi_h\|_{H^{k-1}(\mathcal{N}_{\sigma_1}^+)} \leq C(1 + \|h\|_{H^k(\Sigma)}).$$

From this inequality the first claim follows by the trace theorem.

As for the second part of the lemma, let Φ_i be a diffeomorphism constructed as above from $\Omega \setminus G$ to $\Omega \setminus F_{h_i}$. Note that, since h_1 and h_2 are bounded in $C^{1,\alpha}$, we may construct the Φ_i 's in such a way that

$$\|\Phi_2 - \Phi_1\|_{H^1(\Omega \setminus G)} \leq C\|h_2 - h_1\|_{H^1(\Sigma)}.$$

As before we fix $x_0 \in \Sigma$ and denote as before by Φ the diffeomorphism that straightens Σ . Setting $v_i = u_{F_{h_i}} \circ \Phi_i \circ \Phi^{-1}$ and $\bar{h}_i = h_i \circ \pi \circ \Phi$, we have that

$$\int_{B_{2R}^+} \mathbb{A}(x, \bar{h}_i, D\bar{h}_i) Dv_i : D\varphi dx = 0$$

for all $\varphi \in C^\infty(B_{2R}^+; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$, where \mathbb{A} is the same tensor as before.

Differentiating the equations in the x_j -direction, $j = 1, 2$, and subtracting the two resulting equations we obtain

$$\begin{aligned} \int_{B_{2R}^+} \mathbb{A}(x, \bar{h}_2, D\bar{h}_2) D(D_j(v_2 - v_1)) : D\varphi dx &= - \int_{B_{2R}^+} D_j(\mathbb{A}(x, \bar{h}_2, D\bar{h}_2)) D(v_2 - v_1) : D\varphi dx \\ &\quad - \int_{B_{2R}^+} [\mathbb{A}(x, \bar{h}_2, D\bar{h}_2) - \mathbb{A}(x, \bar{h}_1, D\bar{h}_1)] DD_j v_1 : D\varphi dx \\ &\quad - \int_{B_{2R}^+} D_j[\mathbb{A}(x, \bar{h}_2, D\bar{h}_2) - \mathbb{A}(x, \bar{h}_1, D\bar{h}_1)] Dv_1 : D\varphi dx. \end{aligned}$$

We choose $\varphi = D_j(v_2 - v_1)\eta^2$ as a testfunction and get by arguing as before

$$\begin{aligned} \int_{B_R^+} |D(D_j(v_2 - v_1))|^2 dx &\leq C \int_{B_{2R}^+} (1 + |D^2\bar{h}_2|^2 + |D^2\bar{h}_1|^2) |Dv_2 - Dv_1|^2 dx \\ &\quad + C \int_{B_{2R}^+} (|\bar{h}_2 - \bar{h}_1|^2 + |D\bar{h}_2 - D\bar{h}_1|^2 + |D^2\bar{h}_2 - D^2\bar{h}_1|^2) |Dv_1|^2 dx \\ &\quad + C \int_{B_{2R}^+} (|\bar{h}_2 - \bar{h}_1|^2 + |D\bar{h}_2 - D\bar{h}_1|^2) |D^2v_1|^2 dx. \end{aligned}$$

Recall first that as before $\|Dv_1\|_{L^\infty} \leq C$. Moreover, we assume that $\|h_i\|_{H^3(\Sigma)} \leq K$ and therefore by the proof of the first stament we conclude that $\|v_i\|_{H^3(B_{2R}^+)} \leq C$. Using interpolation we get

$$\int_{B_{2R}^+} |D^2\bar{h}_1|^2 |Dv_2 - Dv_1|^2 dx \leq \|D^2\bar{h}_1\|_{L^4}^2 \|Dv_2 - Dv_1\|_{L^4}^2 \leq C \|\bar{h}_1\|_{H^3}^2 \|v_2 - v_1\|_{H^2}^{\frac{3}{2}} \|v_2 - v_1\|_{L^2}^{\frac{1}{2}}.$$

Estimating the other terms similarly and using the equation to estimate $D_{33}(v_2 - v_1)$, we get for any $\varepsilon \in (0, 1)$

$$\begin{aligned} \int_{B_R^+} |D^2(v_2 - v_1)|^2 &\leq C \|v_2 - v_1\|_{H^2(B_{2R}^+)}^{\frac{3}{2}} \|v_2 - v_1\|_{L^2(B_{2R}^+)}^{\frac{1}{2}} + C \|\bar{h}_2 - \bar{h}_1\|_{H^2(B_{2R}^+)}^2 \\ &\leq \varepsilon \int_{B_{2R}^+} |D^2(v_2 - v_1)|^2 + C \int_{B_{2R}^+} |v_2 - v_1|^2 + C \|h_2 - h_1\|_{H^2(\Sigma)}^2. \end{aligned}$$

Using a simple covering argument as before, going back to the original functions and arguing as above we get

$$\|D^2(u_{F_{h_2}} \circ \Phi_{h_2} - u_{F_{h_1}} \circ \Phi_{h_1})\|_{L^2(\mathcal{N}_{\sigma_1}^+)} \leq C \|u_{F_{h_2}} \circ \Phi_{h_2} - u_{F_{h_1}} \circ \Phi_{h_1}\|_{L^2(\mathcal{N}_{\sigma_2}^+)} + C \|h_2 - h_1\|_{H^2(\Sigma)}.$$

Observe now that writing down the equations satisfied by $u_{F_{h_i}} \circ \Phi_{h_i}$ in $\Omega \setminus G$ and using as an admissible test function $\varphi = u_{F_{h_1}} \circ \Phi_{h_1} - u_{F_{h_2}} \circ \Phi_{h_2}$, one may check that

$$\|D(u_{F_{h_1}} \circ \Phi_{h_1} - u_{F_{h_2}} \circ \Phi_{h_2})\|_{L^2(\Omega \setminus G)} \leq C \|\Phi_1 - \Phi_2\|_{H^1(\Omega \setminus G)} \leq C \|h_1 - h_2\|_{H^1(\Sigma)}.$$

The conclusion follows from this estimate and from the previous one by the Poincaré inequality. \square

Remark 4.2. Let h_{F_i} and u_{F_i} for $i = 1, 2$ be as in Theorem 4.1. The inequality at the end of the proof of the lemma implies that

$$\|u_{F_{h_2}} \circ \pi_{F_{h_2}}^{-1} - u_{F_{h_1}} \circ \pi_{F_{h_1}}^{-1}\|_{H^{1/2}(\Sigma)} \leq C \|h_2 - h_1\|_{H^1(\Sigma)}.$$

Moreover, if in addition to the assumptions of the second part of Theorem 4.1 we know also that $\|h_i\|_{C^1(\Sigma)}$ is sufficiently small for $i = 1, 2$, then the proof of the inequality (4.3) also gives the estimate

$$\|(Du_{F_{h_2}}) \circ \pi_{F_{h_2}}^{-1} - (Du_{F_{h_1}}) \circ \pi_{F_{h_1}}^{-1}\|_{L^2(\Sigma)} \leq C \|h_2 - h_1\|_{H^2(\Sigma)}.$$

Let us consider the smooth flow $(F_t)_{t \in (0, T_0)}$ with initial set F_0 , which is a solution of (3.1) with smooth forcing term $f : \Sigma \times [0, T_0) \rightarrow \mathbb{R}$. Here T_0 is the existence time provided by Theorem 3.1. For every given time $t \in (0, T_0)$ we consider the elastic equilibrium u_t in $\Omega \setminus F_t$ defined in (2.12) and we use the regularity estimates from Theorem 4.1 to establish the following lemma.

Lemma 4.3. *Let $K_0 > 1$ be such that $\|Q(E(u_G))\|_{L^\infty(\Sigma)} < K_0/4$. There exist $T > 0$ and $\tilde{\varepsilon} > 0$ with the following property: if $\|h_0\|_{H^3(\Sigma)} < K_0$, and $\|h_0\|_{L^2(\Sigma)} < \tilde{\varepsilon}$, and f is a smooth function satisfying (3.8) then the solution of (3.1), with initial datum h_0 , provided by Theorem 3.1 exists for the time interval $(0, T)$ and it holds*

$$(4.7) \quad \sup_{0 \leq t \leq T} \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty(\Sigma)} + \int_0^T \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^3(\Sigma)}^2 dt \leq K_0.$$

Moreover, for every $k \in \mathbb{N}$ there exists $C'_k(K_0) > 0$ such that

$$(4.8) \quad \sum_{i=0}^k \int_0^T t^i \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{2i+3}(\Sigma)}^2 dt \leq \frac{1}{2} \left(C'_k(K_0) + \sum_{i=0}^k \int_0^T t^i \|f(\cdot, t)\|_{H^{2i+3}(\Sigma)}^2 dt \right).$$

Proof. We begin by proving (4.7). Let us fix $\alpha \in (0, 1)$. Given $\delta_0 > 0$ to be chosen later and taking $\tilde{\varepsilon}$ equal to the corresponding ε_0 , let $h(\cdot, t)$ be the solution defined on $(0, T_0)$, provided by Theorem 3.1. Note that from (3.9) and (3.10) we have $\sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{H^3} \leq C(K_0)$ and $\sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{L^2} \leq \delta_0$. In turn, by interpolation $\sup_{0 \leq t \leq T_0} \|h(\cdot, t)\|_{C^{1,\alpha}} \leq C\delta_0^\theta < 1$ for some $\theta \in (0, 1)$. Recall also that by choosing $\tilde{\varepsilon}$ small we can make δ_0 as small as we wish. By standard elliptic estimates we have that

$$\sup_{0 \leq t \leq T_0} \|u_t \circ \pi_{F_t}^{-1} - u_G\|_{C^{1,\alpha}(\Sigma)} \leq \omega(\delta_0),$$

and $\omega(\delta_0) \rightarrow 0$ as $\delta_0 \rightarrow 0$. In turn, we conclude that for every $t \in (0, T_0)$ it holds

$$\|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty} \leq \|Q(E(u_t)) - Q(E(u_G)) \circ \pi_{F_t}^{-1}\|_{L^\infty} + \|Q(E(u_G)) \circ \pi_{F_t}^{-1}\|_{L^\infty} \leq \frac{K_0}{3}$$

provided $\tilde{\varepsilon}$ (and thus δ_0) is small enough.

Concerning the second term on the left-hand side of (4.7), we have by a well-known interpolation result and by (4.2) for $k = 5$ from Theorem 4.1

$$\begin{aligned} & \int_0^T \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^3(\Sigma)}^2 dt \\ & \leq C \int_0^T \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{\frac{7}{2}}(\Sigma)}^{2\theta} \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty(\Sigma)}^{2(1-\theta)} dt \\ & \leq C \int_0^T (1 + \|h(\cdot, t)\|_{H^5(\Sigma)}^{2\theta}) K_0^{2(1-\theta)} dt \\ & \leq \eta \int_0^T \|h(\cdot, t)\|_{H^5(\Sigma)}^2 dt + C_\eta K_0^2 T \\ & \leq \eta C \left(K_0^2 + \int_0^T (1 + \|f(\cdot, t)\|_{L^\infty(\Sigma)}^{q_0} + \|f(\cdot, t)\|_{H^3(\Sigma)}^2) dt \right) + C_\eta K_0^2 T \\ & \leq \eta C (K_0^2 + T + TK_0^{q_0} + K_0) + C_\eta K_0^2 T, \end{aligned}$$

where the second last inequality follows from (3.10). The inequality (4.7) follows by choosing η and $T \leq T_0$ sufficiently small.

The inequality (4.8) follows by a similar argument. For all $i = 1, \dots, k$ we have again by interpolation and by (4.2) that

$$\begin{aligned} & \int_0^T t^i \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{2i+3}(\Sigma)}^2 dt \\ & \leq C \int_0^T t^i \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{2i+\frac{7}{2}}(\Sigma)}^{2\theta} \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty(\Sigma)}^{2(1-\theta)} dt \\ & \leq C_k \int_0^T t^i (1 + \|h(\cdot, t)\|_{H^{2i+5}}^{2\theta}) K_0^{2(1-\theta)} dt \\ & \leq \eta \int_0^T t^i \|h(\cdot, t)\|_{H^{2i+5}}^2 dt + C_{k,\eta} K_0^2 T. \end{aligned}$$

The conclusion then follows by estimating the last integral by means of (3.10) and choosing η sufficiently small and $C'_k(K_0)$ sufficiently large. \square

Theorem 4.4. *Let $K_0 > 1$ be such that $\|Q(E(u_G))\|_{L^\infty(\Sigma)} < K_0/4$ and fix $\delta_0 > 0$. There exist $T \in (0, 1)$ and $\varepsilon_1 \in (0, 1)$ with the following property: if $F_0 \in \mathfrak{h}_{K_0}^3(\Sigma)$, defined in (2.18), with $\|h_0\|_{L^2(\Sigma)} < \varepsilon_1$ then there exists a unique solution h to (4.1) in $H^1(0, T; H^1(\Sigma)) \cap L^\infty(0, T; H^3(\Sigma))$. Moreover, the solution belongs to $H_{loc}^1(0, T; H^k(\Sigma))$ for every $k \geq 1$ and it holds*

$$(4.9) \quad \sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{L^2(\Sigma)} < \delta_0$$

and

$$(4.10) \quad \sup_{0 \leq t \leq T} t^k \|h(\cdot, t)\|_{H^{2k+3}(\Sigma)}^2 + \int_0^T t^k \|h(\cdot, t)\|_{H^{2k+5}(\Sigma)}^2 dt \leq C(k, K_0).$$

Proof. We divide the proof into three steps.

Step 1. Let K_0, T be as in Lemma 4.3. Let \mathcal{S} be the set of functions in $C^\infty(0, T; C^\infty(\Sigma))$ that satisfy

$$\sup_{0 \leq t \leq T} \|f(\cdot, t)\|_{L^\infty(\Sigma)} + \int_0^T \|f(\cdot, t)\|_{H^3(\Sigma)}^2 dt \leq K_0$$

and

$$\sum_{i=0}^k \int_0^T (t^i \|f(\cdot, t)\|_{H^{2i+3}(\Sigma)})^2 dt \leq C'_k(K_0)$$

for every $k \in \mathbb{N}$, where $C'_k(K_0)$ are the constants from (4.8). We define a map $\mathcal{L} : \mathcal{S} \rightarrow \mathcal{S}$ as $\mathcal{L}(f)(\cdot, t) := -Q(E(u_t)) \circ \pi_{F_t}^{-1}$ for all $t \in (0, T)$, where F_t is the solution of (3.1) with initial datum h_0 and forcing term f , and where u_t stands for u_{F_t} , that is for the elastic equilibrium in $\Omega \setminus F_t$. Lemma 4.3 implies that the map $\mathcal{L} : \mathcal{S} \rightarrow \mathcal{S}$ is well defined, provided that $\varepsilon_1 \leq \tilde{\varepsilon}$. Note also that \mathcal{S} is clearly nonempty as the zero function belongs to \mathcal{S} .

We will show that $\mathcal{L} : \mathcal{S} \rightarrow \mathcal{S}$ is a contraction with respect to a suitable norm.

Step 2. Fix $\mu \in (0, 1)$. Let f_1 and f_2 be two smooth functions in \mathcal{S} and let h_1 and h_2 be the corresponding solutions of (3.5) with initial datum h_0 . The goal in this step is to show that it holds

$$(4.11) \quad \int_0^T \int_\Sigma (h_2(\cdot, t) - h_1(\cdot, t))^2 d\mathcal{H}^2 dt \leq \mu \int_0^T \int_\Sigma (f_2(\cdot, t) - f_1(\cdot, t))^2 d\mathcal{H}^2 dt,$$

by possibly decreasing the time T . We recall that by Theorem 3.1 we have that

$$\sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{L^2(\Sigma)} \leq \delta_0 \quad \text{and} \quad \sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{H^3(\Sigma)} \leq C(K_0),$$

provided that $\varepsilon_1 < \varepsilon_0$. By interpolation these imply that $\sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{C^{1,\alpha}(\Sigma)} \leq C\delta_0^\theta < 1$ for some $\theta \in (0, 1)$. In turn, by standard Schauder estimates the corresponding elastic equilibria in $F_{h(\cdot, t)}$ are uniformly bounded in $C^{1,\alpha}$ up to the boundary, i.e., $\sup_{0 \leq t \leq T} \|u_t \circ \pi_{F_t}^{-1}\|_{C^{1,\alpha}(\Sigma)} \leq C$. We will use these facts repeatedly in the proof.

We denote by $F_{t,i}$ the set related to $h_i(\cdot, t)$ with $\partial F_{t,i} = \{x + h_i(x, t)\nu(x) : x \in \Sigma\}$. We multiply (3.1) for $i = 1, 2$ by $((h_2 - h_1) \circ \pi)((J_i \circ \pi)\nu_{F_{t,i}} \cdot (\nu \circ \pi))^{-1}$, where J_i stands for the tangential Jacobian on Σ of the map $x \mapsto x + h_i(x)\nu(x)$ and π for the projection on Σ . We then get

$$\begin{aligned} & \int_{\partial F_{t,i}} (\partial_t h_i(\cdot, t) \circ \pi) \frac{(h_1 - h_2) \circ \pi}{J_i \circ \pi} d\mathcal{H}^2 \\ &= \int_{\partial F_{t,i}} \Delta_{\partial F_{t,i}} [H_{\partial F_{t,i}} + f_i(\cdot, t) \circ \pi] ((h_2 - h_1) \circ \pi) ((J_i \circ \pi)\nu_{F_{t,i}} \cdot (\nu \circ \pi))^{-1} d\mathcal{H}^2. \end{aligned}$$

Recall that, denoting by $\partial_{\tau_1} h_i$ and $\partial_{\tau_2} h_i$ the tangential derivatives of h_i in the directions of the principal curvatures, we have

$$J_i = \sqrt{(1 + h_i k_1)^2 (1 + h_i k_2)^2 + (1 + h_i k_1)^2 (\partial_{\tau_1} h_i)^2 + (1 + h_i k_2)^2 (\partial_{\tau_2} h_i)^2},$$

where k_1, k_2 are the principal curvatures of Σ . Therefore we have by the formula for the outer normal (3.2) that

$$((J_i \circ \pi)\nu_{F_{t,i}} \cdot (\nu \circ \pi))^{-1} = \frac{1}{(1 + h_i k_1)(1 + h_i k_2)} \circ \pi =: R(\cdot, h_i) \circ \pi.$$

By integrating by parts we get

$$\begin{aligned} & \int_{\partial F_{t,i}} (\partial_t h_i(\cdot, t) \circ \pi) \frac{(h_1 - h_2) \circ \pi}{J_i \circ \pi} d\mathcal{H}^2 \\ &= \int_{\partial F_{t,i}} (H_{\partial F_{t,i}} + f_i(\cdot, t) \circ \pi) \Delta_{\partial F_{t,i}} [(h_1 - h_2) \circ \pi R(\cdot, h_i) \circ \pi] d\mathcal{H}^2 \end{aligned}$$

Rewriting the integrals above on Σ and subtracting, we have

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \int_{\Sigma} (h_2 - h_1)^2 d\mathcal{H}^2 \\ &= \int_{\Sigma} (J_2 H_{\partial F_{t,2}} \circ \pi_{F_{t,2}}^{-1} - J_1 H_{\partial F_{t,1}} \circ \pi_{F_{t,1}}^{-1} + J_2 f_2 - J_1 f_1) \Delta_{\partial F_{t,2}} [(h_2 - h_1) \circ \pi R(\cdot, h_2) \circ \pi] \circ \pi_{F_{t,2}}^{-1} d\mathcal{H}^2 \\ & \quad + \int_{\Sigma} J_1 (H_{\partial F_{t,1}} \circ \pi_{F_{t,1}}^{-1} + f_1) (\Delta_{\partial F_{t,2}} [(h_2 - h_1) \circ \pi R(\cdot, h_2) \circ \pi] \circ \pi_{F_{t,2}}^{-1} \\ & \quad \quad \quad - \Delta_{\partial F_{t,1}} [(h_2 - h_1) \circ \pi R(\cdot, h_1) \circ \pi] \circ \pi_{F_{t,1}}^{-1}) d\mathcal{H}^2. \end{aligned}$$

We recall (3.3) and (3.4), where the coefficients A, A_1 and A_2 vanish as $(h, \nabla h) = 0$. We recall also that $\|h_i(\cdot, t)\|_{C^{1,\alpha}}$ is small uniformly in time and that f_i are uniformly bounded

with respect to time. After straightforward calculations we have

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \int_{\Sigma} (h_2 - h_1)^2 d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma} |\Delta(h_2 - h_1)|^2 d\mathcal{H}^2 \leq \varepsilon \int_{\Sigma} |\nabla^2(h_2 - h_1)|^2 d\mathcal{H}^2 \\
& + C \int_{\Sigma} (1 + |\nabla^2 h_1| + |\nabla^2 h_2|)(|h_2 - h_1| + |\nabla(h_2 - h_1)|) \cdot \\
& \quad \cdot (|h_2 - h_1| + |\nabla(h_2 - h_1)| + |\nabla^2(h_2 - h_1)|) d\mathcal{H}^2 \\
& + C \int_{\Sigma} |f_2 - f_1| ((1 + |\nabla^2 h_1| + |\nabla^2 h_2|)(|h_2 - h_1| + |\nabla(h_2 - h_1)|) + |\nabla^2(h_2 - h_1)|) d\mathcal{H}^2 \\
& + C \int_{\Sigma} (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 + |\nabla(h_2 - h_1)|^2) d\mathcal{H}^2 =: RHS.
\end{aligned}$$

Using Young's Inequality we obtain

$$\begin{aligned}
RHS & \leq \varepsilon \int_{\Sigma} |\nabla^2(h_2 - h_1)|^2 d\mathcal{H}^2 \\
& + C \int_{\Sigma} (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 + |\nabla(h_2 - h_1)|^2) d\mathcal{H}^2 + C \int_{\Sigma} |f_2 - f_1|^2 d\mathcal{H}^2.
\end{aligned}$$

Observe now that by interpolation, by controlling the second derivatives of h_i with the H^3 -norms, and using the fact that $\|h(\cdot, t)\|_{H^3}$ is bounded uniformly with respect to time we have

$$\begin{aligned}
& \int_{\Sigma} (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 + |\nabla(h_2 - h_1)|^2) d\mathcal{H}^2 \\
& \leq C(1 + \|\nabla^2 h_1\|_{L^4}^2 + \|\nabla^2 h_2\|_{L^4}^2) \|h_2 - h_1\|_{W^{1,4}}^2 \leq C \|h_2 - h_1\|_{H^2}^{\frac{3}{2}} \|h_2 - h_1\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

From the previous inequalities we get

$$\begin{aligned}
\frac{1}{2} \frac{\partial}{\partial t} \int_{\Sigma} (h_2 - h_1)^2 d\mathcal{H}^2 & \leq -\frac{1}{2} \int_{\Sigma} |\Delta(h_2 - h_1)|^2 d\mathcal{H}^2 + \varepsilon \int_{\Sigma} |\nabla^2(h_2 - h_1)|^2 d\mathcal{H}^2 \\
& + C_{\varepsilon} \int_{\Sigma} (|\nabla(h_2 - h_1)|^2 + (h_2 - h_1)^2 + (f_2 - f_1)^2) d\mathcal{H}^2.
\end{aligned}$$

Using now Remark 2.5 we in turn obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Sigma} (h_2 - h_1)^2 d\mathcal{H}^2 + \frac{1}{4} \int_{\Sigma} |\nabla^2(h_2 - h_1)|^2 d\mathcal{H}^2 \leq C \int_{\Sigma} (|h_2 - h_1|^2 + (f_2 - f_1)^2) d\mathcal{H}^2.$$

Integrating this with respect to time over $(0, t)$, with $t \in (0, T)$, we have

$$\begin{aligned}
(4.12) \quad & \int_{\Sigma} (h_2(\cdot, t) - h_1(\cdot, t))^2 d\mathcal{H}^2 + \frac{1}{2} \int_0^t \int_{\Sigma} |\nabla^2(h_2(\cdot, s) - h_1(\cdot, s))|^2 d\mathcal{H}^2 ds \\
& \leq C \int_0^t \int_{\Sigma} (|h_2(\cdot, s) - h_1(\cdot, s)|^2 + (f_2(\cdot, s) - f_1(\cdot, s))^2) d\mathcal{H}^2 ds.
\end{aligned}$$

Integrating the above inequality with respect to time over $(0, T)$ we obtain (4.11) when T is sufficiently small.

Step 3. Here we finally prove that the map $\mathcal{L} : \mathcal{S} \rightarrow \mathcal{S}$ is a contraction with respect to the $L^2(0, T; L^2(\Sigma))$ -norm. To be more precise, let f_1 and f_2 be two functions in \mathcal{S} and h_1 and h_2

the corresponding solutions of (3.1). For simplicity we denote the elastic equilibrium for F_{h_i} as $u_i(\cdot, t) := u_{F_{t,i}}$, for $i = 1, 2$. Then $\mathcal{L}(f_i) = -Q(E(u_i)) \circ \pi_{F_{t,i}}^{-1}$ and our goal is to show

$$(4.13) \quad \int_0^T \|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)}^2 dt \leq \frac{1}{2} \int_0^T \|f_2(\cdot, t) - f_1(\cdot, t)\|_{L^2(\Sigma)}^2 dt.$$

Let us fix $t \in (0, T)$. We begin by proving

$$(4.14) \quad \begin{aligned} & \|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)} \\ & \leq C \|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{L^2(\Sigma)} \\ & \quad + \varepsilon \|\nabla^2(h_2(\cdot, t) - h_1(\cdot, t))\|_{L^2(\Sigma)}^2 + C \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^1(\Sigma)}. \end{aligned}$$

To shorten the notation we denote $U_i := Du_i \circ \pi_{F_{t,i}}^{-1}$, $\nu_i = \nu_{F_{t,i}} \circ \pi_{F_{t,i}}^{-1}$ and $h_i = h_i(\cdot, t)$ for $i = 1, 2$. Recall that $Q(E(u_i(\cdot, t))) \circ \pi_{F_{t,i}}^{-1} = \frac{1}{2} \mathbb{C}U_i : U_i$. We may thus write

$$\|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)} = \frac{1}{2} \|\mathbb{C}(U_2 + U_1) : (U_2 - U_1)\|_{L^2(\Sigma)}.$$

We estimate this simply as

$$(4.15) \quad \begin{aligned} \|\mathbb{C}(U_2 + U_1) : (U_2 - U_1)\|_{L^2(\Sigma)} & \leq \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(I - \nu \otimes \nu))\|_{L^2(\Sigma)} \\ & \quad + \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(\nu \otimes \nu))\|_{L^2(\Sigma)} \end{aligned}$$

Note that by the second condition in (2.12) it holds $\mathbb{C}U_i[\nu_i] = \mathbb{C}E(u_i) \circ \pi_{F_{t,i}}^{-1}[\nu_i] = 0$ on Σ . We use this equality to estimate the last term in (4.15) by

$$\begin{aligned} & \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(\nu \otimes \nu))\|_{L^2(\Sigma)} \\ & \leq \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(\nu \otimes (\nu - \nu_2)))\|_{L^2(\Sigma)} + \|\mathbb{C}U_1 : ((U_2 - U_1)(\nu \otimes \nu_2))\|_{L^2(\Sigma)} \\ & = \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(\nu \otimes (\nu - \nu_2)))\|_{L^2(\Sigma)} + \|\mathbb{C}U_1 : ((U_2 - U_1)(\nu \otimes (\nu_2 - \nu_1)))\|_{L^2(\Sigma)}. \end{aligned}$$

Using the expression (3.2) for the normal ν_2 and the uniform $C^{1,\alpha}$ -bound for h_i we deduce that $\|\nu - \nu_2\|_{L^\infty(\Sigma)} \leq C\delta_0^\theta$ and $\|\nu_2 - \nu_1\|_{L^2(\Sigma)} \leq C\|h_2 - h_1\|_{H^1(\Sigma)}$. Moreover, by the $C^{1,\alpha}$ -bound for u_i we have that $\|U_i\|_{L^\infty} \leq C$ and by the second inequality in Remark 4.2 it holds $\|U_2 - U_1\|_{L^2(\Sigma)} \leq C\|h_2 - h_1\|_{H^2(\Sigma)}$. Therefore we may estimate the above inequality as

$$\|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(\nu \otimes \nu))\|_{L^2(\Sigma)} \leq \varepsilon \|h_2 - h_1\|_{H^2(\Sigma)} + C \|h_2 - h_1\|_{H^1(\Sigma)}.$$

Thus we deduce by (4.15) that

$$\begin{aligned} \|\mathbb{C}(U_2 + U_1) : (U_2 - U_1)\|_{L^2(\Sigma)} & \leq \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(I - \nu \otimes \nu))\|_{L^2(\Sigma)} \\ & \quad + \varepsilon \|h_2 - h_1\|_{H^2(\Sigma)} + C \|h_2 - h_1\|_{H^1(\Sigma)}. \end{aligned}$$

The inequality (4.14) then follows from (2.10) as

$$\begin{aligned}
& \|\mathbb{C}(U_2 + U_1) : ((U_2 - U_1)(I - \nu \otimes \nu))\|_{L^2(\Sigma)} \\
&= \|\mathbb{C}(U_2 + U_1) : ((Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})(I - \nu \otimes \nu))\|_{L^2(\Sigma)} \\
&\leq C\|(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})_\tau\|_{L^2(\Sigma)} \\
&\leq C\|[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1})D\pi_{F_{t,2}}^{-1} - (Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})D\pi_{F_{t,1}}^{-1}]_\tau\|_{L^2(\Sigma)} \\
&\quad + C\|[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1})(D\pi_{F_{t,2}}^{-1} - D\pi_{F_{t,1}}^{-1})]_\tau\|_{L^2(\Sigma)} \\
&\quad + C\|[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})(I - D\pi_{F_{t,1}}^{-1})]_\tau\|_{L^2(\Sigma)} \\
&\leq C\|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{L^2(\Sigma)} + C\|h_2 - h_1\|_{H^1(\Sigma)} + \varepsilon\|h_2 - h_1\|_{H^2(\Sigma)},
\end{aligned}$$

where in the last inequality we used the second estimate in Remark 4.2 and the fact that the C^1 -norm of h_1 is small.

We proceed by using (4.14) and interpolation to deduce

$$\begin{aligned}
& \|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)} \\
&\leq C\|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{H^{\frac{1}{2}}(\Sigma)}^{\frac{1}{2}} \|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{H^{-\frac{1}{2}}(\Sigma)}^{\frac{1}{2}} \\
&\quad + \varepsilon\|h_2 - h_1\|_{H^2(\Sigma)} + C\|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^1(\Sigma)}.
\end{aligned}$$

By the estimate (4.3) in Theorem 4.1 we have

$$\begin{aligned}
& \|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{H^{\frac{1}{2}}(\Sigma)} \\
&\leq \|u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1}\|_{H^{3/2}(\Sigma)} \leq C\|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}.
\end{aligned}$$

Moreover by using the well-known inequality $\|\nabla g\|_{H^{-\frac{1}{2}}(\Sigma)} \leq C\|g\|_{H^{\frac{1}{2}}(\Sigma)}$ and Remark 4.2 we have

$$\begin{aligned}
& \|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{H^{-\frac{1}{2}}(\Sigma)} \\
&\leq C\|u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1}\|_{H^{\frac{1}{2}}(\Sigma)} \leq C\|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^1(\Sigma)}.
\end{aligned}$$

Collecting the previous three inequalities, using standard interpolation

$$\|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^1(\Sigma)} \leq C\|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2(\Sigma)}^{1/2} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{L^2(\Sigma)}^{1/2},$$

and by Young's inequality we obtain

$$\begin{aligned}
& \|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2}^2 \\
&\leq 2\varepsilon\|\nabla^2(h_2(\cdot, t) - h_1(\cdot, t))\|_{L^2}^2 + C_\varepsilon\|h_2(\cdot, t) - h_1(\cdot, t)\|_{L^2}^2.
\end{aligned}$$

Integrating the previous inequality over $(0, T)$ and using (4.11) and (4.12), we obtain

$$\begin{aligned}
& \int_0^T \|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2}^2 dt \\
&\leq ((C_\varepsilon + \varepsilon C)\mu + \varepsilon C) \int_0^T \|f_2(\cdot, s) - f_1(\cdot, s)\|_{L^2}^2 d\mathcal{H}^1 ds \\
&\leq \frac{1}{2} \int_0^T \|f_2(\cdot, s) - f_1(\cdot, s)\|_{L^2}^2 d\mathcal{H}^1 ds,
\end{aligned}$$

provided that ε and then μ are chosen sufficiently small. This proves (4.13) and we conclude that $\mathcal{L} : \mathcal{S} \rightarrow \mathcal{S}$ is a contraction with respect to the $L^2(0, T; L^2(\Sigma))$ -norm.

Step 4. (Conclusion) We may proceed with a standard argument, by recursively setting $f_1 = 0$, $f_n := \mathcal{L}(f_{n-1})$ and for every $n \geq 1$ letting h_n be the solution to (3.1) with f replaced by f_n . From Step 2 and Step 3 we have that there exist f and h such that $f_n \rightarrow f$ and $h_n \rightarrow h$ in $L^2(0, T; L^2(\Sigma))$. Moreover, using (4.8) and (3.10), we conclude easily that for every $n \geq 1$ the functions h_n satisfy (4.9) and (4.10) for every $k \in \mathbb{N}$, with constants depending only on k and K_0 . Thus, we have that $h_n \rightharpoonup h$ weakly in $H^1(0, T; H^1(\Sigma)) \cap L^\infty(0, T; H^3(\Sigma))$. Moreover using the equation satisfied by h_n and (3.10) we also have that $\partial_t h_n$ is bounded in $L^2_{loc}(0, T; H^k(\Sigma))$ for every $k \in \mathbb{N}$. Therefore we have that $h_n \rightharpoonup h$ weakly in $H^1_{loc}(0, T; H^k(\Sigma))$ and thus strongly in $L^2_{loc}(0, T; H^k(\Sigma))$ and that h satisfies (4.9) and (4.10). Using these convergences one can easily pass to the limit in the equations satisfied by the h_n 's to conclude that h is a solution of (4.1). The uniqueness follows from the same argument used in Step 2 and Step 3. \square

5. ASYMPTOTIC STABILITY

In this section we study the flow when the initial set is close to a smooth strictly stable stationary set G , which will be our reference set, i.e., we set $\Sigma = \partial G$. Throughout this section we denote

$$R_t := H_{F_t} - Q(E(u_{F_t})).$$

Moreover, in what follows we shall drop the subscript ∂F_t (and similar) in all the covariant differential operators, when no danger of confusion arises. Here is the main result.

Theorem 5.1. *Let $G \subset\subset \Omega$ be a regular strictly stable stationary set in the sense of Definition 2.11. There exists $\delta > 0$ such that if $F_0 \in \mathfrak{h}_\delta^3(\Sigma)$, then the unique solution $(F_t)_{t>0}$ of the flow (4.1) with initial datum F_0 is defined for all times $t > 0$.*

Moreover $F_t \rightarrow F_\infty$ exponentially fast, where F_∞ is the unique stationary set near G such that $|F_{\infty,i}| = |F_{0,i}|$ for $i = 1, \dots, m$. In particular, if $|F_{0,i}| = |G_i|$ for $i = 1, \dots, m$, then $F_t \rightarrow G$ exponentially fast. Here G_i denote the open bounded sets enclosed by the components $\Gamma_{G,1}, \dots, \Gamma_{G,m}$ of ∂G , $F_{\infty,i}$ and $F_{0,i}$ are diffeomorphic to G_i , and $\partial F_{0,i}$ and $\partial F_{\infty,i}$ are the components of ∂F_0 and ∂F_∞ respectively.

Remark 5.2. By exponential convergence of F_t to F_∞ we mean precisely the following: writing $\partial F_t := \{x + \tilde{h}(x, t)\nu_{F_\infty}(x) : x \in \partial F_\infty\}$, we have that for every $k \in \mathbb{N}$ there exists $c_k > 0$ and $C_k > 1$ such that

$$\|\tilde{h}(\cdot, t)\|_{C^k(\partial F_\infty)} \leq C_k e^{-c_k t}$$

for $t \geq 1$.

The proof of stability is based on the following energy identity.

Proposition 5.3. *Let $(F_t)_{t \in [0, T]}$ be the solution of (4.1) provided by Theorem 4.4. Then the function*

$$t \mapsto \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2$$

is absolutely continuous and for almost every $t \in (0, T)$ we have the following energy identity

$$(5.1) \quad \frac{d}{dt} \left(\int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \right) = -2\partial^2 \mathcal{J}(F_t)[\Delta R_t] \\ - 2 \int_{\partial F_t} B_{F_t}[\nabla R_t, \nabla R_t](\Delta R_t) d\mathcal{H}^2 + \int_{\partial F_t} H_{F_t} |\nabla R_t|^2 (\Delta R_t) d\mathcal{H}^2,$$

where $\partial^2 \mathcal{J}(F_t)$ is defined as in (2.21).

The proof of the proposition is similar to [24, Proposition 4.3] (see also [1, Lemma 4.4]) and therefore we shift it to the appendix.

In order to control the two last terms in (5.1) we need the following interpolation result on the evolving boundaries. The proof of the next lemma is precisely the same as [1, Lemma 4.7] and therefore we omit it.

Lemma 5.4. *If $F \subset U$ is such that $\partial F = \{x + h_F(x)\nu(x) : x \in \Sigma\}$ with $\|h_F\|_{C^{1,\alpha}(\Sigma)} \leq M$, then for every smooth function $f \in C^\infty(\partial F)$ it holds*

$$\int_{\partial F} |B_F \|\nabla f\|^2 |\Delta f| d\mathcal{H}^2 \leq C \left(1 + \|H_F\|_{L^6(\partial F)}^3 \right) \|\nabla \Delta f\|_{L^2(\partial F)}^2 \|\nabla f\|_{L^2(\partial F)}.$$

The constant C depends only on M and Σ .

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. For any set $F \in \mathfrak{h}_1^3(\Sigma)$ consider

$$D(F) := \int_{F \Delta G} \text{dist}(x, \Sigma) dx$$

and note that

$$(5.2) \quad \frac{1}{C} \|h_F\|_{L^2(\partial G)}^2 \leq D(F) \leq C \|h_F\|_{L^2(\partial G)}^2$$

for a constant depending only on G . Moreover, we define

$$R_F := H_F - Q(E(u_F))$$

which is defined on ∂F .

Step 1. (Preliminary estimates) In this step we show that if $F \in \mathfrak{h}_1^3(\Sigma)$ and $\|h_F\|_{C^1(\Sigma)} \leq \delta$ for δ sufficiently small, then it holds

$$(5.3) \quad \frac{1}{C} \|h_F\|_{H^3(\Sigma)}^{1/\theta} \leq D(F) + \int_{\partial F} |\nabla R_F|^2 d\mathcal{H}^2 \leq C \|h_F\|_{H^3(\Sigma)}^\theta.$$

for $\theta \in (0, 1)$ and for constant $C > 1$.

We begin by proving the first inequality. We use interpolation, (4.2) and the second inequality in Remark 4.2 to deduce that

$$(5.4) \quad \begin{aligned} & \|\nabla(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G)))\|_{L^2(\Sigma)} \\ & \leq C \|Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_F)) \circ \pi_F^{-1}\|_{H^{\frac{3}{2}}(\Sigma)}^{\theta'} \|Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\|_{L^2(\Sigma)}^{1-\theta'} \\ & \leq (C + \|Q(E(u_F)) \circ \pi_F^{-1}\|_{H^{3/2}(\Sigma)}^{\theta'}) \|(Du_F) \circ \pi_F^{-1} - (Du_G)\|_{L^2(\Sigma)}^{1-\theta'} \\ & \leq (C + \|h_F\|_{H^3(\Sigma)}^{\theta'}) \|h_F\|_{H^2(\Sigma)}^{1-\theta'} \\ & \leq C \|h_F\|_{H^2(\Sigma)}^{1-\theta'} \end{aligned}$$

for $\theta' \in (0, 1)$. Since G is a stationary set it holds $\nabla R_G = 0$ on Σ . Therefore we conclude by the above inequality that

$$\begin{aligned} & \|\nabla(H_F \circ \pi_F^{-1} - H_G)\|_{L^2(\partial F)}^2 \\ & \leq 2 \int_{\Sigma} |\nabla(R_F \circ \pi_F^{-1})|^2 d\mathcal{H}^2 + 2\|\nabla(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G)))\|_{L^2(\Sigma)}^2 \\ & \leq 2C \int_{\partial F} |\nabla R_F|^2 d\mathcal{H}^2 + C\|h_F\|_{H^2(\Sigma)}^{2(1-\theta')}. \end{aligned}$$

We use (2.6), (3.3) and the fact that $\|h_F\|_{C^1(\Sigma)} \leq \delta$ to deduce with straightforward calculations

$$\|h_F\|_{H^3(\Sigma)}^2 \leq C\|\nabla(H_F \circ \pi_F^{-1} - H_G)\|_{L^2(\Sigma)}^2 + C\|h_F\|_{H^2(\Sigma)}^2.$$

Therefore, from the two previous inequalities and by interpolation we obtain that

$$\begin{aligned} \|h_F\|_{H^3(\Sigma)}^2 & \leq C \int_{\partial F} |\nabla R_F|^2 d\mathcal{H}^2 + C\|h_F\|_{H^2}^{2(1-\theta')} + C\|h_F\|_{H^2}^2 \\ & \leq C \int_{\partial F} |\nabla R_F|^2 d\mathcal{H}^2 + \frac{1}{2}\|h_F\|_{H^3}^2 + C\|h_F\|_{L^2}^{\theta''}, \end{aligned}$$

for a suitable $\theta'' \in (0, 1)$. The first inequality in (5.3) then follows from the previous the previous estimate and from (5.2), recalling that since $\|h_F\|_{H^3(\Sigma)} \leq 1$ we have also $\|\nabla R_F\|_{L^2(\partial F)} \leq C$.

To prove the second inequality in (5.3) we argue similarly as above and use (3.3) to conclude that

$$\|\nabla(H_F \circ \pi_F^{-1} - H_G)\|_{L^2(\Sigma)}^2 \leq C\|h_F\|_{H^3(\Sigma)}^2$$

Moreover by (5.4) we have that

$$\|\nabla(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G)))\|_{L^2(\Sigma)} \leq C\|h_F\|_{H^2(\Sigma)}^{1-\theta'}$$

for $\theta' \in (0, 1)$. Therefore since G is a critical set we obtain

$$\begin{aligned} \int_{\partial F} |\nabla R_F|^2 d\mathcal{H}^2 & \leq C \int_{\Sigma} |\nabla(R_F \circ \pi_F^{-1} - R_G)|^2 d\mathcal{H}^2 \\ & \leq C \int_{\Sigma} |\nabla(H_F \circ \pi_F^{-1} - H_G)|^2 d\mathcal{H}^2 + C \int_{\Sigma} |\nabla(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G)))|^2 d\mathcal{H}^2 \\ & \leq C\|h_F\|_{H^3(\Sigma)}^2 + C\|h_F\|_{H^2(\Sigma)}^{2(1-\theta')} \leq C\|h_F\|_{H^3(\Sigma)}^{\theta}. \end{aligned}$$

Hence, we have (5.3).

Step 2. (*Global existence*) Let us assume that the initial set F_0 is in $\mathfrak{h}_\delta^3(\Sigma)$ with $\delta < \varepsilon_1$, where $\varepsilon_1 \in (0, 1)$ is the constant provided by Theorem 4.4 corresponding to the choice $\delta_0 = 1$, $K_0 = \max\{2, 5\|Q(E(u_G))\|_{L^\infty(\Sigma)}\}$. Then the flow $(F_t)_{t \in [0, T]}$ starting from F_0 which is a solution of (4.1) exists for a time interval $(0, T)$, with T bounded from below by a positive constant which depends only G . Let $\sigma > 0$ be a small number which will be chosen later. Note that by (5.3) and by continuity we have

$$(5.5) \quad D(F_t) + \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \leq C\|h(\cdot, t)\|_{H^3(\Sigma)}^{\theta} \leq C\delta^{\theta} < \sigma$$

for some time interval $(0, T')$, where the last inequality holds provided that δ is small enough. Note that by (5.3) it follows that

$$(5.6) \quad \|h(\cdot, t)\|_{H^3(\Sigma)} < C\sigma^\theta < \min\{\varepsilon_1, \sigma_1\} \quad \text{for every } t \in (0, T')$$

when σ is small enough, where σ_1 is the constant provided by Proposition 2.13. In particular, we conclude from Theorem 4.4 that as long as the flow $(F_t)_{t \in (0, T)}$ satisfies (5.5) it is well defined. In other words, if $(0, T^*)$ is the maximal time of existence and if it satisfies (5.5) for every $t \in (0, T^*)$, then $T^* = \infty$, i.e., the flow exists for all times.

Let us denote by $[0, T')$ the maximal time interval where the flow satisfies (5.5). We claim that if $\|h_0\|_{H^3(\Sigma)} < \delta$ for δ small enough, then the flow satisfies (5.5) for every $t \in (0, T^*)$ and thus $T^* = T' = +\infty$.

We start by recalling that by Lemma 2.12 and (5.6), since $\sigma_1 < \sigma_0$, we have

$$\partial^2 \mathcal{J}(F_t)[\Delta R_t] \geq \frac{c_0}{2} \|\Delta R_t\|_{H^1(\partial F_t)}^2 \quad \text{for every } t \in (0, T').$$

Thus, from the energy identity (5.1), using also Lemma 5.4 and again (5.5), we may estimate

$$(5.7) \quad \begin{aligned} \frac{d}{dt} \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 &\leq -2\partial^2 \mathcal{J}(F_t)[\Delta R_t] + C \int_{\partial F_t} |B_{F_t}| |\nabla R_t|^2 |\Delta R_t| d\mathcal{H}^2 \\ &\leq -c_0 \|\Delta R_t\|_{H^1(\partial F_t)}^2 + C(1 + \|H_{F_t}\|_{L^6(\partial F_t)}^3) \|\nabla \Delta R_t\|_{L^2(\partial F_t)}^2 \|\nabla R_t\|_{L^2(\partial F_t)} \\ &\leq -c_0 \|\Delta R_t\|_{H^1(\partial F_t)}^2 + C\sqrt{\sigma} \|\nabla \Delta R_t\|_{L^2(\partial F_t)}^2 \\ &\leq -\frac{c_0}{2} \|\Delta R_t\|_{H^1(\partial F_t)}^2, \end{aligned}$$

where the last inequality holds by taking σ smaller if needed.

Next we show that

$$(5.8) \quad \|\nabla R_t\|_{L^2(\partial F_t)} \leq C \|\Delta R_t\|_{L^2(\partial F_t)}$$

for some constant which depends on Σ . Let us fix a component of ∂F_t and denote it by Γ_t . Since F_t is diffeomorphic to G we denote the component of Σ diffeomorphic to Γ_t by Γ . Since Γ is smooth, compact and connected Riemannian manifold we conclude by [5, Theorem 3.67] that the Poincaré inequality holds on Γ , i.e., for every $\varphi \in C^\infty(\Gamma)$ with $\int_\Gamma \varphi d\mathcal{H}^2 = 0$ it holds

$$\|\varphi\|_{L^2(\Gamma)} \leq C \|\nabla \varphi\|_{L^2(\Gamma)}.$$

Therefore since $\Gamma_t = \Phi_t(\Gamma)$ with $\Phi_t(x) = x + h(x, t)\nu(x)$ and $\|h(\cdot, t)\|_{C^{1,\alpha}} \leq C$ the Poincaré inequality holds also on Γ_t . In particular, we have

$$\|R_t - \bar{R}_t\|_{L^2(\Gamma_t)} \leq C \|\nabla R_t\|_{L^2(\Gamma_t)},$$

where \bar{R}_t denotes the average of R_t on Γ_t and the constant depends on Σ . Then by integration by parts we get

$$\begin{aligned} \int_{\Gamma_t} |\nabla R_t|^2 d\mathcal{H}^2 &= - \int_{\Gamma_t} (R_t - \bar{R}_t) \Delta R_t d\mathcal{H}^2 \\ &\leq \|R_t - \bar{R}_t\|_{L^2(\Gamma_t)} \|\Delta R_t\|_{L^2(\Gamma_t)} \leq C \|\nabla R_t\|_{L^2(\Gamma_t)} \|\Delta R_t\|_{L^2(\Gamma_t)}. \end{aligned}$$

We obtain (5.8) by repeating the above argument for every component of ∂F_t .

By (5.7) and (5.8) we conclude that

$$\frac{d}{dt} \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \leq -c \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2$$

for every $t \in (0, T')$. Integrating this over $(0, t)$, using (5.3) and $\|h_0\|_{H^3(\Sigma)} \leq \delta$ yield

$$(5.9) \quad \int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \leq C e^{-ct} \delta^\theta.$$

On the other hand by differentiating $D(F_t)$ with respect to time and using the same calculations as in [24, Lemma 3.3] we get

$$\begin{aligned} \frac{d}{dt} D(F_t) &= \int_{\partial F_t} d_G \Delta R_t d\mathcal{H}^2 = - \int_{\partial F_t} \langle \nabla d_G, \nabla R_t \rangle d\mathcal{H}^2 \\ &\leq \mathcal{H}^2(\partial F_t)^{1/2} \left(\int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \right)^{1/2} \leq C e^{-\frac{c}{2}t} \delta^{\frac{\theta}{2}}. \end{aligned}$$

Integrating this over $(0, t)$, using (5.2) and $\|h_0\|_{H^3(\Sigma)} \leq \delta$ yield

$$(5.10) \quad D(F_t) \leq D(F_0) + C e^{-\frac{c}{2}t} \delta^{\frac{\theta}{2}} \leq C \delta^2 + C e^{-\frac{c}{2}t} \delta^{\frac{\theta}{2}} < \sigma$$

when δ is chosen small enough. Hence, we have that (5.5) holds for the whole life span of the flow $(0, T^*)$ and by the previous discussion this implies that $T^* = \infty$.

Step 3. (Convergence) Combining (5.3) and (5.5) we have that $\sup_{t>0} \|h(\cdot, t)\|_{H^3(\Sigma)} \leq C \sigma^\theta$. Therefore there exists a subsequence such that

$$h(\cdot, t_m) \rightarrow h_\infty(\cdot) \quad \text{in } H^2(\Sigma).$$

We denote the target set by F_∞ , i.e., $\partial F_\infty = \{x + h_\infty(x)\nu(x) : x \in \Sigma\}$. By (5.9) we deduce that $\nabla R_{F_\infty} = 0$, i.e., F_∞ is a stationary set. We will show that $F_t \rightarrow F_\infty$ exponentially fast.

To this aim we define

$$D_\infty(F) := \int_{F \Delta F_\infty} \text{dist}(x, F_\infty) dx.$$

Repeating the calculations leading to (5.10) we get

$$\left| \frac{d}{dt} D_\infty(F_t) \right| = \left| \int_{\partial F_t} d_{F_\infty} \Delta R_t d\mathcal{H}^2 \right| \leq \mathcal{H}^2(\partial F_t)^{1/2} \left(\int_{\partial F_t} |\nabla R_t|^2 d\mathcal{H}^2 \right)^{1/2} \leq C e^{-\frac{c}{2}t} \delta^{\frac{\theta}{2}},$$

where the last inequality follows from (5.9). This implies that $\lim_{t \rightarrow \infty} D_\infty(F_t)$ exists and the choice of F_∞ implies that $D_\infty(F_t) \rightarrow 0$. Therefore integrating the above inequality over (t, ∞) we get

$$D_\infty(F_t) \leq C e^{-\frac{c}{2}t} \delta^{\frac{\theta}{2}}$$

for every $t > 0$. We change the reference set from $\Sigma = \partial G$ to ∂F_∞ and write $\partial F_t = \{x + \tilde{h}(x, t)\nu_{F_\infty}(x) : x \in \partial F_\infty\}$. Then by inequality (5.2), with ∂G replaced by ∂F_∞ , and by the above inequality we have

$$\|\tilde{h}(\cdot, t)\|_{L^2(\partial F_\infty)} \leq C e^{-\frac{c}{4}t} \delta^{\frac{\theta}{4}}.$$

Moreover, since $\|h(\cdot, t)\|_{H^3(\Sigma)} \leq C \sigma^\theta$ for all $t > 0$ then also $\|\tilde{h}(\cdot, t)\|_{H^3(\partial F_\infty)} \leq C$ for all $t > 0$. By Theorem 4.4 we conclude that $\|\tilde{h}(\cdot, t)\|_{H^{2k+3}(\partial F_\infty)} \leq C(k, \sigma)$ for all $t \geq 1$ and for every $k \in \mathbb{N}$. Thus we deduce by interpolation that

$$\|\tilde{h}(\cdot, t)\|_{C^k(\partial F_\infty)} \leq C_k e^{-c_k t} \quad \text{for all } t \geq 1$$

for some constants $c_k > 0$ and $C_k > 1$ depending on k and K_0 .

To conclude the proof, for every $t \in [0, +\infty]$ denote by $(\Gamma_{F_t, i})_{i=1, \dots, m}$ the connected components of ∂F_t , numbered according to (2.19). Denote also by $F_{t, i}$ the bounded open set enclosed by $\Gamma_{F_t, i}$ and recall that the flow preserves the volume of each $F_{t, i}$. Indeed,

$$\frac{d}{ds} |F_{t+s, i}|_{s=0} = \int_{\Gamma_{F_t, i}} V_t d\mathcal{H}^2 = \int_{\Gamma_{F_t, i}} \Delta R_t d\mathcal{H}^2 = 0.$$

Thus, recalling (5.6) and Proposition 2.13, we may conclude that F_∞ is the unique stationary set in $\mathfrak{h}_{\sigma_1}^3(\partial G)$ such that $|F_{\infty, i}| = |F_{0, i}|$ for $i = 1, \dots, m$. \square

6. EVOLUTION OF EPITAXIALLY STRAINED ELASTIC FILMS

In this section we briefly describe how our main results read in the context of evolving periodic graphs.

In this framework, given a (sufficiently regular) non-negative function $h : \mathbb{R}^2 \rightarrow [0, +\infty)$, 1-periodic with respect to both variables x_1, x_2 , the free energy associated with it reads

$$(6.1) \quad \mathcal{J}(h) := \int_{\Omega_h} Q(E(u_h)) dx + \mathcal{H}^2(\Gamma_h),$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^2$, Γ_h , Ω_h denote the graph and the subgraph of h , respectively, over the periodic cell, i.e.,

$$\begin{aligned} \Omega_h &:= \{(x_1, x_2, x_3) \in (0, 1)^2 \times \mathbb{R} : 0 < x_3 < h(x_1, x_2)\}, \\ \Gamma_h &:= \{(x_1, x_2, x_3) \in (0, 1)^2 \times \mathbb{R} : x_3 = h(x_1, x_2)\}, \end{aligned}$$

and u_h is the elastic equilibrium in Ω_h , namely the solution of the elliptic system

$$(6.2) \quad \begin{cases} \operatorname{div} \mathbb{C}E(u_h) = 0 & \text{in } \Omega_h, \\ \mathbb{C}E(u_h)[\nu_{\Omega_h}] = 0 & \text{on } \Gamma_h, \\ Du_h(\cdot, x_3) & \text{is 1-periodic,} \\ u(x_1, x_2, 0) = e_0(x_1, x_2, 0), \end{cases}$$

for a suitable fixed constant $e_0 \neq 0$. The above energy relates to a variational model for epitaxial growth, see the introduction. Precisely, the graph Γ_h describes the (free) profile of the elastic film, which occupies the region Ω_h and is grown on a (rigid) and much thicker substrate, while the *mismatch strain* constant e_0 appearing in the Dirichlet condition for u_h at the interface $\{x_1 = 0\}$ between film and substrate measures the mismatch between the characteristic atomic distances in the lattices of the two materials. In this framework, the (local) minimizers of (6.1) under an area constraint on Ω_h describe the equilibrium configurations of epitaxially strained elastic films, see [21, 22, 23, 25] and the references therein.

In the context of periodic graphs, given an initial 1-periodic profile $h_0 \in H_{loc}^3(\mathbb{R}^2)$ (in short $h_0 \in H_{per}^3((0, 1)^2)$), we look for a local-in-time solution $h(\cdot, t)$ of the following problem:

$$(6.3) \quad \begin{cases} \frac{1}{J_t} \partial_t h = \Delta_{\Gamma_t} (H_t + Q(E(u_t))) & \text{on } \Gamma_t \text{ and for all } t \in (0, T), \\ h(\cdot, t) \text{ is 1-periodic} & \text{for all } t \in (0, T), \\ h(\cdot, 0) = h_0, \end{cases}$$

where $J_t := \sqrt{1 + |Dh(\cdot, t)|^2}$, u_t stands for the solution of (6.2), with Ω_{h_t} in place of Ω_h , we wrote Γ_t instead of Γ_{h_t} , and H_t denotes the mean curvature of Γ_t . Note that in the first

equation of (6.3) we have $+Q(E(u_t))$ instead of $-Q(E(u_t))$. This is due to the fact that in (6.1) the vector ν_{Ω_h} now points outwards with respect to the elastic body.

Although the setting is a bit different from that of the previous sections, the short-time existence theory of Section 4 clearly extends also to the present situation, with the same arguments. In this way we improve upon the results of [23] at least in the case of isotropic surface energy density.

Also the stability analysis of Section 5 applies without any essential changes, thus showing that strictly stable stationary 1-periodic configurations are exponentially stable in the sense of Theorem 5.1.

A particular class of critical configurations to which our stability theorem applies are the flat configurations, that is, in the case of constant profiles $h \equiv d$, provided that $d > 0$ is sufficiently small. Indeed in [9, Proposition 7.3] it is shown that if d is sufficiently small then the flat configuration $h \equiv d$ is strictly stable for the functional \mathcal{J} . Therefore, we may state the following theorem.

Theorem 6.1. *There exists $d_0 > 0$ with the following property: Let $d \in (0, d_0)$. Then, there exists $\delta > 0$ such that if*

$$\|h_0 - d\|_{H^3((0,1)^2)} \leq \delta \quad \text{and} \quad \int_{(0,1)^2} h_0 \, dx = d,$$

then the unique solution $h(\cdot, t)$ of (6.3) exists for all $t > 0$ and for every integer $k \geq 1$ we have

$$\|h(\cdot, t) - d\|_{C^k([0,1]^2)} \leq C_k e^{-c_k t} \quad \text{for all } t > 1$$

and for suitable positive constants C_k, c_k .

7. APPENDIX: TECHNICAL LEMMAS

In this appendix we collect a few technical results and we give the proof of Lemma 3.3 and of Proposition 5.3.

Lemma 7.1. *Let Σ be an m -dimensional smooth compact manifold in \mathbb{R}^n and let $k \geq 1$. If $f, g \in H^k(\Sigma) \cap L^\infty(\Sigma)$, then $fg \in H^k(\Sigma)$ and $\|fg\|_{H^k(\Sigma)} \leq C(\|f\|_{H^k(\Sigma)}\|g\|_{L^\infty(\Sigma)} + \|g\|_{H^k(\Sigma)}\|f\|_{L^\infty(\Sigma)})$. Moreover, if $A \in C^\infty(\mathbb{R})$ then $A(f) \in H^k(\Sigma)$ and $\|A(f)\|_{H^k(\Sigma)} \leq C(1 + \|f\|_{H^k(\Sigma)})$ where the constant depends on A and on $\|f\|_{L^\infty(\Sigma)}$.*

Finally, if $U \subset \mathbb{R}^m$ is an open set $\Phi : \bar{U} \rightarrow \Phi(\bar{U}) \subset \Sigma$ is a diffeomorphism of class $H^k \cap C^1$, $k \geq 1$, and $f \in H^k(\Phi(U)) \cap C^1(\Phi(\bar{U}))$, then $\|f \circ \Phi\|_{H^k(U)} \leq C(\|Df\|_\infty, \|D\Phi\|_\infty)(\|f\|_{H^k} + \|\Phi\|_{H^k})$.

Proof. The first two statements of the lemma are classical, see for instance [43, Propositions 3.7 and 3.9]. The third one can be proven by an induction argument from the first one. \square

We now prove Lemma 3.3.

Proof of Lemma 3.3. First, recall (3.16) and observe that from the assumption on h_i we have $\sup_{0 \leq t \leq T} \|h_i(\cdot, t)\|_{C^{1,\alpha}(\Sigma)} \leq C\delta^{\theta'}$ for a suitable $C > 0$ and $\theta' \in (0, 1)$. We begin by estimating

for $\varepsilon > 0$

$$\begin{aligned}
(7.1) \quad & \int_0^T \int_{\Sigma} |\langle A(x, h_2, \nabla h_2), \nabla^4 h_2 \rangle - \langle A(x, h_1, \nabla h_1), \nabla^4 h_1 \rangle|^2 d\mathcal{H}^2 dt \\
& \leq 2 \int_0^T \int_{\Sigma} |A(x, h_2, \nabla h_2)|^2 |\nabla^4 h_2 - \nabla^4 h_1|^2 d\mathcal{H}^2 dt \\
& \quad + 2 \int_0^T \int_{\Sigma} |\nabla^4 h_1|^2 |A(x, h_2, \nabla h_2) - A(x, h_1, \nabla h_1)|^2 d\mathcal{H}^2 dt \\
& \leq \varepsilon \int_0^T \int_{\Sigma} |\nabla^4 h_2 - \nabla^4 h_1|^2 d\mathcal{H}^2 dt \\
& \quad + C \int_0^T \int_{\Sigma} |\nabla^4 h_1|^2 (|h_2 - h_1|^2 + |\nabla h_2 - \nabla h_1|^2) d\mathcal{H}^2 dt.
\end{aligned}$$

To estimate the last term, we use the Sobolev inequality and the interpolation Lemma 2.1, and have

$$\begin{aligned}
(7.2) \quad & \int_0^T \int_{\Sigma} |\nabla^4 h_2|^2 (|h_2 - h_1|^2 + |\nabla h_2 - \nabla h_1|^2) d\mathcal{H}^2 dt \\
& \leq C \int_0^T \|h_2(\cdot, t) - h_1(\cdot, t)\|_{W^{1,4}}^2 \|\nabla^4 h_2(\cdot, t)\|_{L^4}^2 dt \\
& \leq C \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|h_2(\cdot, t)\|_{H^5}^{\frac{5}{3}} \|\nabla h_2(\cdot, t)\|_{L^\infty}^{\frac{1}{3}} dt \\
& \leq C \delta^{\frac{\theta'}{3}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 T^{\frac{1}{6}} \left(\int_0^T \|h_2(\cdot, t)\|_{H^5}^2 dt \right)^{\frac{5}{6}} \\
& \leq C(M_0) T^{\frac{1}{6}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2.
\end{aligned}$$

Concerning the estimate of

$$\int_0^T \int_{\Sigma} |J_1(x, h_2, \nabla h_2, \nabla^2 h_2, \nabla^3 h_2) - J_1(x, h_1, \nabla h_1, \nabla^2 h_1, \nabla^3 h_1)|^2 d\mathcal{H}^2 dt$$

we observe that

$$\begin{aligned}
& \int_0^T \int_{\Sigma} |\langle B_1(x, h_2, \nabla h_2), \nabla^3 h_2 \otimes \nabla^2 h_2 \rangle - \langle B_1(x, h_1, \nabla h_1), \nabla^3 h_1 \otimes \nabla^2 h_1 \rangle|^2 d\mathcal{H}^2 dt \\
& \leq C \int_0^T \int_{\Sigma} |B_1(x, h_2, \nabla h_2) - B_1(x, h_1, \nabla h_1)|^2 |\nabla^3 h_2 \otimes \nabla^2 h_2|^2 d\mathcal{H}^2 dt \\
& \quad + C \int_0^T \int_{\Sigma} |B_1(x, h_1, \nabla h_1)|^2 |\nabla^3 h_2 - \nabla^3 h_1|^2 |\nabla^2 h_2|^2 d\mathcal{H}^2 dt \\
& \quad + C \int_0^T \int_{\Sigma} |B_1(x, h_1, \nabla h_1)|^2 |\nabla^2 h_2 - \nabla^2 h_1|^2 |\nabla^3 h_1|^2 d\mathcal{H}^2 dt \\
& \leq C \int_0^T \int_{\Sigma} (|h_2 - h_1|^2 + |\nabla h_2 - \nabla h_1|^2) |\nabla^3 h_2|^2 |\nabla^2 h_2|^2 d\mathcal{H}^2 dt \\
& \quad + C \int_0^T \int_{\Sigma} |\nabla^2 h_2 - \nabla^2 h_1|^2 |\nabla^3 h_1|^2 d\mathcal{H}^2 dt \\
& \quad + C \int_0^T \int_{\Sigma} |\nabla^3 h_2 - \nabla^3 h_1|^2 |\nabla^2 h_2|^2 d\mathcal{H}^2 dt =: I_1 + I_2 + I_3.
\end{aligned}$$

By a simple interpolation argument, we have

$$\begin{aligned}
I_3 & \leq \int_0^T \|\nabla^3 h_2 - \nabla^3 h_1\|_{L^4}^2 \|\nabla^2 h_2\|_{L^4}^2 dt \leq CM_0 \int_0^T \|h_1 - h_2\|_{H^4}^{\frac{3}{2}} \|\nabla^2 h_2 - \nabla^2 h_1\|_{L^2}^{\frac{1}{2}} \\
& \leq \varepsilon \int_0^T \|\nabla^4 h_2 - \nabla^4 h_1\|_{L^2}^2 dt + C_{\varepsilon}(M_0)T \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
I_2 & \leq \int_0^T \|\nabla^2 h_2 - \nabla^2 h_1\|_{L^4}^2 \|\nabla^3 h_1\|_{L^4}^2 dt \\
& \leq C \int_0^T \|h_2 - h_1\|_{H^4}^{\frac{1}{2}} \|h_2 - h_1\|_{H^2}^{\frac{3}{2}} \|h_1\|_{H^5}^{\frac{1}{2}} \|\nabla^3 h_1\|_{L^2}^{\frac{3}{2}} dt \\
& \leq \varepsilon \int_0^T \|\nabla^4 h_2 - \nabla^4 h_1\|_{L^2}^2 dt + C_{\varepsilon}(M_0) \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T 1 + \|h_1\|_{H^5}^{\frac{2}{3}} dt \\
& \leq \varepsilon \int_0^T \|\nabla^4 h_2 - \nabla^4 h_1\|_{L^2}^2 dt + C_{\varepsilon}(M_0)T^{\frac{2}{3}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2.
\end{aligned}$$

Finally, arguing similarly as above,

$$\begin{aligned}
I_1 & \leq \int_0^T \|h_1 - h_2\|_{W^{1,6}}^2 \|\nabla^3 h_2\|_{L^6}^2 \|\nabla^2 h_2\|_{L^6}^2 dt \\
& \leq CM_0 \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|h_2\|_{H^5}^{\frac{2}{3}} \|\nabla^3 h_2\|_{L^2}^{\frac{4}{3}} dt \\
& \leq C(M_0)T^{\frac{2}{3}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2.
\end{aligned}$$

Since the difference of the remaining terms in J_1 can be treated in a similar (in fact easier) way, we conclude that

$$(7.3) \quad \begin{aligned} & \int_0^T \int_{\Sigma} |J_1(x, h_2, \nabla h_2, \nabla^2 h_2, \nabla^3 h_2 - J_1(x, h_1, \nabla h_1, \nabla^2 h_1, \nabla^3 h_1))|^2 d\mathcal{H}^2 dt \\ & \leq \varepsilon \int_0^T \|\nabla^4 h_2(\cdot, t) - \nabla^4 h_1(\cdot, t)\|_2^2 dt + C_\varepsilon(M_0)T^\theta \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2. \end{aligned}$$

We are left to show that

$$(7.4) \quad \begin{aligned} & \int_0^T \int_{\Sigma} |J_2(x, h_2, \nabla h_2, \nabla^2 h_2, \nabla f, \nabla^2 f - J_2(x, h_1, \nabla h_1, \nabla^2 h_1, \nabla f, \nabla^2 f))|^2 d\mathcal{H}^2 dt \\ & \leq \varepsilon \int_0^T \|\nabla^4 h_1(\cdot, t) - \nabla^4 h_2(\cdot, t)\|_2^2 dt + C_\varepsilon(M_0, K_0)T^\theta \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2. \end{aligned}$$

As before we only prove the estimate for

$$I_4 := \int_0^T \int_{\Sigma} |\langle A_1(x, h_2, \nabla h_2) - A_1(x, h_1, \nabla h_1), \nabla^2 f \rangle|^2 d\mathcal{H}^2 dt,$$

the other terms being similar (or easier). Using once again Lemma 2.1 we have

$$\begin{aligned} I_4 & \leq \int_0^T \|h_2 - h_1\|_{W^{1,4}}^2 \|\nabla^2 f\|_{L^4}^2 dt \\ & \leq C \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|\nabla^3 f\|_2^{\frac{3}{2}} \|f\|_{L^\infty}^{\frac{1}{2}} dt \\ & \leq CK_0^{\frac{1}{2}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|\nabla^3 f\|_{L^2}^{\frac{3}{2}} dt \\ & \leq CK_0^{\frac{5}{4}} T^{\frac{1}{4}} \sup_{0 \leq t \leq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2. \end{aligned}$$

The conclusion then follows by collecting (7.1)-(7.4). \square

Finally we give the proof of Proposition 5.3.

Proof of Proposition 5.3. The proof is similar to the proof of [24, Lemma 3.3]. For this reason we adopt the same notation as there and extend every function on ∂F_t using the signed distance function d_{F_t} . In particular, the normal $\nu_t = \nu_{F_t}$, the second fundamental form $B_t = B_{F_t}$ and the mean curvature $H_t = H_{F_t}$ are extended to a tubular neighborhood of ∂F_t . Recall that D_τ denotes the tangential gradient defined in (2.9) and div_τ denotes the tangential divergence, which is defined as $\operatorname{div}_\tau X = \operatorname{div} X - (DX\nu_t) \cdot \nu_t$. The Laplace-Beltrami operator on F_t can be written as $\Delta v = \operatorname{div}_\tau(D_\tau v)$, the second fundamental form as $B_t = D_\tau \nu_t$ and the mean curvature as $H_t = \operatorname{div}_\tau \nu_t$.

The regularity properties of h stated in Theorem 4.4 imply that for every integer $k \geq 1$ $\nabla^k h \in H_{loc}^1(0, T; L^2(\Sigma))$. Therefore, in what follows all the time derivatives are well defined almost everywhere. In turn, this allows us to differentiate $u_t := u_{F_t}$ with respect to time. More precisely, setting $\dot{u}_t := \frac{\partial u_{t+s}}{\partial s} \Big|_{s=0}$, we can argue as in [9, Theorem 4.1] to conclude that \dot{u} solves

$$(7.5) \quad \int_{\Omega \setminus F_t} \mathbb{C}E(\dot{u}_t) : E(\varphi) dx = - \int_{\partial F_t} \operatorname{div}_\tau(\Delta R_t \mathbb{C}E(u_t)) \cdot \varphi d\mathcal{H}^2$$

for all $\varphi \in H^1(\Omega \setminus F_t; \mathbb{R}^3)$ such that $\varphi = 0$ on $\partial_D \Omega$. Note also that $\dot{u}_t = 0$ on $\partial_D \Omega$.

Let us fix time $t > 0$. To continue we observe that, by redefining the velocity field X associated with the flow (4.1) if needed (in a time interval centered at t), we may assume that X_t has only a normal component on ∂F_t ; that is,

$$X_t = (X_t \cdot \nu_t) \nu_t = (\Delta R_t) \nu_t \quad \text{on } \partial F_t.$$

Since we extended all the geometric quantities by means of the gradient of the signed distance from F_t we have the following equality (see [13])

$$\dot{\nu}_t = -D_\tau(X_t \cdot \nu_t) = -D_\tau(\Delta R_t) \quad \text{on } \partial F_t.$$

This implies (see the proof of [1, eq. (5.15)])

$$(7.6) \quad \dot{H}_t := \frac{\partial}{\partial s} H_{t+s} \Big|_{s=0} = -\Delta^2 R_t \quad \text{on } \partial F_t.$$

Moreover we have (see [13])

$$(7.7) \quad \partial_{\nu_t} H_t = -|B_t|^2 \quad \text{on } \partial F_t.$$

Denoting by $D_{\tau_{t+s}}$ the tangential gradient on ∂F_{t+s} and by $J_\tau \Phi_s$ the tangential Jacobian of Φ_s , we have

$$(7.8) \quad \begin{aligned} & \frac{d}{ds} \left(\frac{1}{2} \int_{\partial F_{t+s}} |D_\tau R_{t+s}|^2 d\mathcal{H}^2 \right) \Big|_{s=0} \\ &= \frac{d}{ds} \left(\frac{1}{2} \int_{\partial F_t} (|D_{\tau_{t+s}} R_{t+s}|^2 \circ \Phi_s) J_\tau \Phi_s d\mathcal{H}^2 \right) \Big|_{s=0} \\ &= \frac{1}{2} \int_{\partial F_t} |D_\tau R_t|^2 \operatorname{div}_\tau(\Delta R_t \nu_t) d\mathcal{H}^2 + \int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau_{t+s}} R_{t+s} \circ \Phi_s) \Big|_{s=0} d\mathcal{H}^2 \\ &= \frac{1}{2} \int_{\partial F_t} H_t |D_\tau R_t|^2 \Delta R_t d\mathcal{H}^2 + \int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau_{t+s}} R_{t+s} \circ \Phi_s) \Big|_{s=0} d\mathcal{H}^2 \end{aligned}$$

We write the last term as

$$D_{\tau_{t+s}} R_{t+s} \circ \Phi_s = [I - \nu_{t+s} \circ \Phi_s \otimes \nu_{t+s} \circ \Phi_s] D R_{t+s} \circ \Phi_s$$

and get (recall $\dot{\Phi} = X_t = (\Delta R_t) \nu_t$)

$$\begin{aligned} \frac{\partial}{\partial s} (D_{\tau_{t+s}} R_{t+s} \circ \Phi_s) \Big|_{s=0} &= [I - \nu_t \otimes \nu_t] (D \dot{R}_t + D^2 R_t X_t) + (-\dot{\nu}_t \otimes \nu_t - \nu_t \otimes \dot{\nu}_t) D R_t \\ &= D_\tau \dot{R}_t + \Delta R_t ((I - \nu_t \otimes \nu_t) D^2 R_t) [\nu_t] + (D R_t \cdot \nu_t) D_\tau \Delta R_t - (D R_t \cdot \dot{\nu}_t) \nu_t. \end{aligned}$$

Note that $D_\tau(D R_t \cdot \nu_t) = B_t D_\tau R_t + ((I - \nu_t \otimes \nu_t) D^2 R_t) [\nu_t]$. Thus we have

$$\begin{aligned} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau_{t+s}} R_{t+s} \circ \Phi_s) \Big|_{s=0} &= (D_\tau R_t \cdot D_\tau \dot{R}_t) - \Delta R_t (B_t [D_\tau R_t, D_\tau R_t]) \\ &\quad + \Delta R_t (D_\tau R_t \cdot D_\tau (D R_t \cdot \nu_t)) + (D_\tau R_t \cdot D_\tau \Delta R_t) (D R_t \cdot \nu_t). \end{aligned}$$

Therefore by integrating by parts the first and the third terms we obtain

$$\begin{aligned}
& \int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau+s} R_{t+s} \circ \Phi_s) \Big|_{s=0} d\mathcal{H}^2 \\
&= \int_{\partial F_t} (D_\tau R_t \cdot D_\tau \dot{R}_t) - \Delta R_t (B_t [D_\tau R, D_\tau R_t]) d\mathcal{H}^2 \\
&\quad + \int_{\partial F_t} \Delta R_t (D_\tau R \cdot D_\tau (DR_t \cdot \nu_t)) + (D_\tau R_t \cdot D_\tau \Delta R_t) (DR_t \cdot \nu_t) d\mathcal{H}^2 \\
&= \int_{\partial F_t} -\Delta R_t \dot{R}_t - \Delta R_t (B_t [D_\tau R, D_\tau R_t]) d\mathcal{H}^2 \\
&\quad + \int_{\partial F_t} -(DR_t \cdot \nu_t) \operatorname{div}_\tau (\Delta R_t D_\tau R_t) + (D_\tau R_t \cdot D_\tau \Delta R_t) (DR_t \cdot \nu_t) d\mathcal{H}^2 \\
&= \int_{\partial F_t} -\Delta R_t \dot{R}_t - (DR_t \cdot \nu_t) (\Delta R_t)^2 - \Delta R_t (B_t [D_\tau R, D_\tau R_t]) d\mathcal{H}^2.
\end{aligned}$$

Let us denote $u_t = u_{F_t}$ and $\dot{u}_t = \frac{\partial}{\partial t} u_t$. By (7.6) it holds

$$\dot{R}_t = \dot{H}_t + \frac{\partial}{\partial t} Q(E(u_t)) = -\Delta^2 R_t + \mathbb{C}E(\dot{u}_t) : E(u_t)$$

and by (7.7) we have

$$(DR_t, \nu_t) = \partial_{\nu_t} H_t + \partial_{\nu_t} Q(E(u_t)) = -|B_t|^2 + \partial_{\nu_t} Q(E(u_t)).$$

Therefore we get

$$\begin{aligned}
& \int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau+s} R_{t+s} \circ \Phi_s) \Big|_{s=0} d\mathcal{H}^2 = \int_{\partial F_t} \Delta R_t \Delta^2 R_t - \mathbb{C}E(\dot{u}_t) : E(u_t) \Delta R_t d\mathcal{H}^2 \\
&\quad + \int_{\partial F_t} |B_t|^2 (\Delta R_t)^2 - \partial_{\nu_t} Q(E(u_t)) (\Delta R_t)^2 - \Delta R_t (B_t [D_\tau R, D_\tau R_t]) d\mathcal{H}^2.
\end{aligned}$$

Observe now that using the second equation in (2.12) and (7.5) we have

$$\begin{aligned}
& \int_{\partial F_t} \mathbb{C}E(\dot{u}_t) : E(u_t) \Delta R_t d\mathcal{H}^2 = \int_{\partial F_t} \mathbb{C}E(u_t) : D(\dot{u}_t) \Delta R_t d\mathcal{H}^2 \\
&= \int_{\partial F_t} \mathbb{C}E(u_t) : D_\tau(\dot{u}_t) \Delta R_t d\mathcal{H}^2 = - \int_{\partial F_t} \operatorname{div}_\tau (\Delta R_t \mathbb{C}E(u_t)) \cdot \dot{u}_t = 2 \int_{\Omega \setminus F_t} Q(E(\dot{u}_t)) dx.
\end{aligned}$$

Collecting the previous three identities we then get

$$\begin{aligned}
& \int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s} (D_{\tau+s} R_{t+s} \circ \Phi_s) \Big|_{s=0} d\mathcal{H}^2 = - \int_{\partial F_t} |\nabla \Delta_\tau R_t|^2 + 2Q(E(\dot{u}_t)) \Delta R_t d\mathcal{H}^2 \\
&\quad + \int_{\partial F_t} |B|^2 (\Delta R_t)^2 - \partial_{\nu_t} Q(E(u_t)) (\Delta R_t)^2 - B_t [\nabla R_t, \nabla R_t] \Delta R_t d\mathcal{H}^2.
\end{aligned}$$

We notice that the first four terms coincide with $-\partial^2 J(F_t)[\Delta R_t]$ (see (2.21)). Thus, combining the last identity with (7.8), we obtain (5.1). \square

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