MODULATIONAL STABILITY OF GROUND STATES TO NONLINEAR KIRCHHOFF EQUATIONS

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Abstract. We investigate the stability of ground states to a nonlinear focusing Schrödinger equation in presence of a Kirchhoff term. Through a spectral analysis of the linearized operator about ground states, we show a modulation stability estimate of ground states in the spirit of one due to Weinstein [*SIAM J. Math. Anal.*, 16(1985),472-491].

1. Introduction and main result

1.1. Overview. Let us consider the following nonlinear focusing Kirchhoff equation with a potential and an initial datum

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon = -\dfrac{1}{2}\left(\varepsilon^2 + \varepsilon\displaystyle\int_{\mathbb{R}^3}|\nabla u^\varepsilon|^2\right)\Delta u^\varepsilon + V(x)u^\varepsilon - |u^\varepsilon|^{2p}u^\varepsilon, & t>0,\ x\in\mathbb{R}^3, \\ u^\varepsilon(x,0) = u_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (0, 2/3)$ and $\varepsilon > 0$(referring to Plank's constant). Similar to [4, Theorem 6.1.1 and Corollary 6.1.2], problem (1.1) is globally well-posed, provided that $V \in L^m(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $m > 3/2$. Here we refer to [7] for the background of Kirchhoff equations. Of particular interest is *standing wave solutions* of (1.1), namely, special solutions of (1.1) have the form of

$$u^\varepsilon(x,t) = v_\varepsilon(x)e^{\frac{i}{\varepsilon}\theta t},\ x\in\mathbb{R}^3,\ t\in\mathbb{R}^+,\ \theta\in\mathbb{R}.$$

In this case, v_ε is a solution of the following singularly perturbed Kirchhoff equation

$$-\frac{1}{2}\left(\varepsilon^2 + \varepsilon\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\right)\Delta v_\varepsilon + \tilde{V}(x)v_\varepsilon = |v_\varepsilon|^{2p}v_\varepsilon,\quad x\in\mathbb{R}^3, \tag{1.2}$$

where $\tilde{V}(x) = V(x) + \theta$. An interesting class of solutions to (1.2) are families of solutions which develop a spike shape around some certain point (such as local minimal points, local maximal points, saddle points and degenerated or non-degenerated critical points of potential $\tilde{V}$) in $\mathbb{R}^3$ as $\varepsilon \to 0$. In view of physics, these standing wave solutions are referred to the semiclassical states for ε small. Initiated by Floer and Weinstein [6] for the Schrödinger equations

$$-\varepsilon^2\Delta v + W(x)v = f(v),$$

semiclasscial states have attracted a considerable attention in the last three decades. For the progress on this topic, we refer to Ambrosetti and Malchiodi [1] and the reference therein. In

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the study of the singularly perturbed problem (1.2), the following so called limit problem plays a crucial role

$$-\frac{1}{2}\left(1+\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=|u|^{2p}u,\quad u\in H^1(\mathbb{R}^3). \tag{1.3}$$

It is shown in [11] that the positive solution of (1.3) is, up to translation, unique. Denote by r the positive, radially symmetric solution of (1.3).

Another related topic is to associate to (1.1) a family of initial data u_0 which oscillate or concentrate with scale ε, and investigate the evolution of u^ε in time. Precisely, by choosing a suitable initial datum u_0 related to the ground state solution r, it can be expected that the evolution u^ε remains close to r locally uniformly in time in the semiclassical regime of ε going to zero. This kind of asymptotic behavior is called in the literature *soliton dynamics*. In this aspect, we refer the readers to a survey [16]. In [2], Bronski and Jerrard considered the following focusing Schrödinger equation with a potential

$$i\varepsilon\partial_t u^\varepsilon=-\frac{\varepsilon^2}{2}\Delta u^\varepsilon+V(x)u^\varepsilon-|u^\varepsilon|^{2p}u^\varepsilon,\quad t>0,\ x\in\mathbb{R}^N. \tag{1.4}$$

By using the conservation law (quantum and classical) and the stability of the ground state Q to the limit problem

$$-\frac{1}{2}\Delta u+u=|u|^{2p}u,\quad u\in H^1(\mathbb{R}^N),$$

they proved the solution of (1.4) exhibits the asymptotic soliton dynamics if the initial datum has the form of

$$Q\left(\frac{x-x_0}{\varepsilon}\right)e^{i\frac{x\cdot\xi_0}{\varepsilon}},\ x_0,\xi_0\in\mathbb{R}^N.$$

Subsequently, in [10], Keraani refined the method introduced by Bronski and Jerrard [2] and proved that, up to a time-dependent phase shift, the initial shape is conserved with parameters which are transported by the flow $(x(t),\xi(t))$:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\xi(t),\ \frac{\mathrm{d}\xi}{\mathrm{d}t}=-\nabla V(x(t)),\ x(0)=x_0,\ \xi(0)=\xi_0.$$

Later, in [15], Selvitella turned to study the Schrödinger equations

$$i\partial_t u^\varepsilon=-\frac{1}{2}\left(\frac{\varepsilon}{i}\nabla-A(x)\right)^2u^\varepsilon+V(x)u^\varepsilon-|u^\varepsilon|^{2p}u^\varepsilon,\quad t>0,\ x\in\mathbb{R}^N$$

with electric and magnetic field $B=\nabla\times A$. Combining the linearization agrument, the author adopted the idea due to Bronski and Jerrard [2] to show the asymptotic evolution of the semiclassical limit as $\varepsilon\to 0$. In [14], Squassina extended the result in [2,10] to the Schrödinger equations with an external magnetic potential A. Precisely, the author used the similar idea above to consider the semiclassical regime of the following problem

$$i\varepsilon\partial_t u^\varepsilon=-\frac{1}{2}\left(\frac{\varepsilon}{i}\nabla-A(x)\right)^2u^\varepsilon+V(x)u^\varepsilon-|u^\varepsilon|^{2p}u^\varepsilon,\quad t>0,\ x\in\mathbb{R}^N,$$

and explored the influence of the magnetic field B on asymptotic soliton dynamics. For more progress in this direction, we also would like to cite [5] for nonlocal Choquard equations and [13] for systems of weakly coupled Schrödinger equations.

1.2. **Main result.** In the works above, the nonlinear term is subcritical, namely, $p < 2/N$, where N is the dimension. It is well known that the ground states of the associated limit problems above are orbitally stable when $p < 2/N$. For more details, we refer the readers to [3, 4]. In the present paper, we also consider the subcritical case: $0 < p < 2/3$. Moreover, we should point out that in the works above, to establish the soliton dynamics of semiclassical states on finite time intervals, some kind of energy convexity plays an important role. More precisely, via a delicate spectral analysis of the linearized operator at the ground state of the limit problem (1.3), we establish a modulational stability result in term of Kirchhoff problems (1.1).

For any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, let

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{2p+2}.$$

Our main result can read as follows.

Theorem 1.1. *There exists $C > 0$ such that for any $\phi \in H^1(\mathbb{R}^3, \mathbb{C})$, there holds that*

$$\mathcal{E}(\phi) - \mathcal{E}(r) \geq C \inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \|\phi - e^{i\theta}r(\cdot - x)\|^2 + o\left(\inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \|\phi - e^{i\theta}r(\cdot - x)\|^2\right),$$

provided that $\|\phi\|_2 = \|r\|_2$ and

$$\inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \|\phi - e^{i\theta}r(\cdot - x)\| \leq \|r\|.$$

Remark 1.2. *With the help of Theorem 1.1, the evolution u^ε of (1.1) should remain close to r locally uniformly in time, provided a suitable initial datum u_0 related to the ground state solution r. We will subsequently deal with this topic for the Kirchhoff problem (1.1) elsewhere.*

Notations.

- For any $z \in \mathbb{C}$, $\bar{z}, \mathfrak{Re}(z), \mathfrak{Im}(z)$ denote the complex conjugate, real part and imaginary part of z, respectively.
- For any $z, w \in \mathbb{C}$, it holds that $\mathfrak{Re}(\bar{z}w) = \mathfrak{Re}(z\bar{w})$ and $\mathfrak{Im}(\bar{z}w) = -\mathfrak{Im}(z\bar{w})$.
- For any $z, w \in \mathbb{C}$, we define $z \cdot w = \mathfrak{Re}(z\bar{w}) = \frac{1}{2}(z\bar{w} + \bar{z}w)$.
- For any $x, y \in \mathbb{R}^3$, we denote by $x \cdot y$ the inner product between x and y.
- c, C denote (possibly different) positive constants which may change from line to line.
- $H^1(\mathbb{R}^3) = H^1(\mathbb{R}^3, \mathbb{R})$ and $H^1(\mathbb{R}^3, \mathbb{C})$ are real and complex Hilbert space respectively, endowed with the norm

$$\|u\| = \left(\frac{1}{2}\|\nabla u\|_2^2 + \|u\|_2^2\right)^{\frac{1}{2}}, \ u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

- Denote by (u, v) the scalar product in $L^2(\mathbb{R}^3, \mathbb{C})$ and

$$(u, v)_{H^1} = (u, v) + \frac{1}{2}(\nabla u, \nabla v), \text{ for } u, v \in H^1(\mathbb{R}^3, \mathbb{C}).$$

2. Preliminary results

In this section, we give a few basic properties about the ground state solutions to problem (1.3).

2.1. **The limit problem.** It is shown in [11, Theorem 1.2] that r is the unique radially symmetric solution of (1.3). Moreover, it is non-degenerate in the sense that

$$\mathrm{Ker} L_+ = \mathrm{span}\left\{\partial_{x_1} r, \partial_{x_2} r, \partial_{x_3} r\right\},$$

where L_+ is given as follows

$$L_+\varphi = -\frac{1}{2}\left(1+\int_{\mathbb{R}^3}|\nabla r|^2\right)\Delta\varphi - \left(\int_{\mathbb{R}^3}\nabla r\nabla\varphi\right)\Delta r + \varphi - (2p+1)r^{2p}\varphi,\ \ \varphi\in L^2(\mathbb{R}^3).$$

Moreover, $r\in C^\infty(\mathbb{R}^3)$, $r(0)=\max_{x\in\mathbb{R}^3} r(x)$ and $r, |\nabla r|$ exponentially decay at infinity.

Now, we consider the following minimization problem with a constraint. Let

$$m := \inf_{u\in\mathcal{M}} \mathcal{E}(u),\ \ \mathcal{M} := \left\{u\in H^1(\mathbb{R}^3,\mathbb{C}) : \|u\|_2 = \|r\|_2\right\}, \tag{2.1}$$

we have the following proposition:

Proposition 2.1. *If $p\in(0,2/3)$, then the following hold true*

(i) *$m\in(-\infty,0)$.*
(ii) *m can be achieved by r.*
(iii) *Any minimizer of m has the form as follows*

$$\left\{e^{i\theta} r(\cdot+y) : \theta\in\mathbb{R}, y\in\mathbb{R}^3\right\}.$$

Proof. (i) Noting that $2p+2\in(2,6)$, we have

$$\frac{1}{2p+2} = \frac{s}{2} + \frac{1-s}{6},\ \ s = \frac{3}{2p+2} - \frac{1}{2}.$$

It follows from the interpolation inequality that there exists $C>0$ such that, for any $u\in\mathcal{M}$, $\|u\|_{2p+2}^{2p+2} \le C\|\nabla u\|_2^{3p}$. Then

$$\inf_{u\in\mathcal{M}}\mathcal{E}(u) \ge \inf_{u\in\mathcal{M}}\left(\frac{1}{2}\|\nabla u\|_2^2 - C\|\nabla u\|_2^{3p}\right) \ge \min_{t\in[0,\infty)}\left(\frac{1}{2}t^2 - Ct^{3p}\right) > -\infty.$$

On the other hand, since r is a solution of (1.3), one can get that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla r|^2 + \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 + \int_{\mathbb{R}^3} r^2 = \int_{\mathbb{R}^3} r^{2p+2}.$$

By the Pohozaev identity

$$\frac{1}{4}\int_{\mathbb{R}^3}|\nabla r|^2 + \frac{1}{4}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 + \frac{3}{2}\int_{\mathbb{R}^3} r^2 = \frac{3}{2p+2}\int_{\mathbb{R}^3} r^{2p+2},$$

one can get that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla r|^2 + \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 = \left(\frac{3}{2} - \frac{3}{2p+2}\right)\int_{\mathbb{R}^3} r^{2p+2}.$$

It follows that

$$\begin{aligned} m \le \mathcal{E}(r) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla r|^2 + \frac{1}{4}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} r^{2p+2} \\ &= \left[\frac{1}{2}\int_{\mathbb{R}^3}|\nabla r|^2 + \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 - \frac{2}{2p+2}\int_{\mathbb{R}^3} r^{2p+2}\right] - \frac{1}{4}\left(\int_{\mathbb{R}^3}|\nabla r|^2\right)^2 \\ &= \left(\frac{3}{2} - \frac{5}{2p+2}\right)\|r\|_{2p+2}^{2p+2} - \frac{1}{4}\|\nabla r\|_2^4 < -\frac{1}{4}\|\nabla r\|_2^4 < 0. \end{aligned}$$

Here we used the fact that $p \in (0, 2/3)$.

(ii) Firstly, taking any minimization sequence $\{u_n\}$ of m, let u_n^* be its Symmetrization, since

$$\|u_n^*\|_2^2 = \|u_n\|_2^2,\ \|u_n^*\|_{2p+2}^{2p+2} = \|u_n\|_{2p+2}^{2p+2},\ \|\nabla u_n^*\|_2^2 \leq \|\nabla u_n\|_2^2,$$

one can get

$$m \leq \mathcal{E}(u_n^*) \leq \mathcal{E}(u_n)$$

and $\mathcal{E}(u_n^*) \to m$ as $n \to \infty$. So without loss of generality, (u_n) can be chosen to be nonnegative and radially symmetric. Since $\mathcal{E}(u_n) \to m$ as $n \to \infty$ and $\|u_n\|_2 = \|r\|_2$, thanks to $p \in (0, 2/3)$, one can show that $\{u_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^3)$. Up to a subsequence, for some $u_0 \in H^1_{rad}(\mathbb{R}^3)$, $u_n \to u_0$ weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^{2p+2}(\mathbb{R}^3)$ as $n \to \infty$. If $u_0 \equiv 0$, then by $\mathcal{E}(u_n) \to e$, we have

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 \to m < 0,\ n \to \infty,$$

which is a contradiction. Now, we claim that $\|u_0\|_2 = \|r\|_2$. Obviously, $\|u_0\|_2 \leq \|r\|_2$ and $\mathcal{E}(u_0) \leq m$. Then to show m can be achieved by u_0, it suffices to rule out the case: $\|u_0\|_2 < \|r\|_2$. If such case occurs, let

$$w(\cdot) = \frac{1}{s} u_0\left(\frac{\cdot}{t}\right),\ s, t \geq 0,\ s^{2p+2} = t^3,$$

then choosing $t > 0$ such that $w \in \mathcal{M}$, i. e.,

$$\|w\|_2^2 = \frac{t^3}{s^2}\|u_0\|_2^2 = \|r\|_2^2.$$

And we have $s^{2p+2} = t^3 > s^2$, which implies that $s > 1$ and $s^2 > t$ since $p \in (0, 2/3)$. Thus, $u_0(\cdot) = sw(t\cdot)$ and

$$\mathcal{E}(u_0) = \frac{s^2}{2t}\int_{\mathbb{R}^3} |\nabla w|^2 + \frac{s^4}{4t^2}\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |w|^{2p+2} > \mathcal{E}(w),$$

which contradicts the fact that $\mathcal{E}(w) \geq m$.

Secondly, we show that $u_0 = r$. Similar to [12], there exists $\lambda_0 > 0$ such that

$$-\frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla u_0|^2\right)\Delta u_0 + \lambda_0 u_0 = u_0^{2p+1},\ x \in \mathbb{R}^3.$$

By [9], let Q be the unique radially symmetric solution of

$$-\Delta Q + \lambda_0 Q = Q^{2p+1},\ Q > 0,\ Q \in H^1(\mathbb{R}^3),$$

then it follows from [11] that

$$u_0(x) = Q\left(\frac{x}{\sqrt{c}}\right),\ \sqrt{c} = \frac{1}{2}\left(\frac{1}{2}\|\nabla Q\|_2^2 + \sqrt{\frac{1}{4}\|\nabla Q\|_2^4 + 2}\right).$$

Similarly,

$$r(x) = \tilde{Q}\left(\frac{x}{\sqrt{d}}\right),\ \sqrt{d} = \frac{1}{2}\left(\frac{1}{2}\|\nabla \tilde{Q}\|_2^2 + \sqrt{\frac{1}{4}\|\nabla \tilde{Q}\|_2^4 + 2}\right),$$

where $\tilde{Q}$ is the unique radially symmetric solution of

$$-\Delta \tilde{Q} + \tilde{Q} = \tilde{Q}^{2p+1},\ \tilde{Q} > 0,\ \tilde{Q} \in H^1(\mathbb{R}^3).$$

Let

$$Q(\cdot) = \lambda_0^{\frac{1}{2p}} \bar{Q}(\lambda_0^{\frac{1}{2}}\cdot),$$

then

$$-\Delta\bar{Q}+\bar{Q}=\bar{Q}^{2p+1},\ \bar{Q}>0,\ \bar{Q}\in H^1(\mathbb{R}^3).$$

Then we know $\bar{Q}\equiv\tilde{Q}$ and

$$\begin{aligned}\|u_0\|_2^2&=c^{\frac{3}{2}}\|Q\|_2^2=c^{\frac{3}{2}}\lambda_0^{\frac{1}{p}-\frac{3}{2}}\|\bar{Q}\|_2^2\\&=\frac{1}{8}\left(\frac{1}{2}\lambda_0^{\frac{1}{p}-\frac{1}{2}}\|\nabla\bar{Q}\|_2^2+\sqrt{\frac{1}{4}\lambda_0^{\frac{2}{p}-1}\|\nabla\bar{Q}\|_2^4+2}\right)^3\lambda_0^{\frac{1}{p}-\frac{3}{2}}\|\bar{Q}\|_2^2.\end{aligned}$$

Since

$$\|r\|_2^2=\frac{1}{8}\left(\frac{1}{2}\|\nabla\tilde{Q}\|_2^2+\sqrt{\frac{1}{4}\|\nabla\tilde{Q}\|_2^4+2}\right)^3\|\tilde{Q}\|_2^2$$

and $\|u_0\|_2=\|r\|_2$, we get that $\lambda_0=1$, where we used the fact that $p\in(0,2/3)$. Therefore, u_0 is a radially symmetric positive solution of problem (1.3) and we get the claim as desired.

(iii) The proof is similar to [3, Theorem II.1]. So we omit the details. □

2.2. **The linearized problem.** Let L be the linearization of (1.3) at r acting on $L^2(\mathbb{R}^3,\mathbb{C})$ with domain in $H^2(\mathbb{R}^3,\mathbb{C})$. Precisely, for any $\xi\in H^2(\mathbb{R}^3,\mathbb{C})$,

$$L\xi=-\frac{1}{2}\left(1+\int_{\mathbb{R}^3}|\nabla r|^2\right)\Delta\xi-\frac{1}{2}\left(\int_{\mathbb{R}^3}\nabla r\nabla(\xi+\bar{\xi})\right)\Delta r+\xi-r^{2p}\left[p(\xi+\bar{\xi})+\xi\right],$$

and

$$L\xi=L_+\mathfrak{Re}(\xi)+iL_-\mathfrak{Im}(\xi).$$

If $\eta\in H^2(\mathbb{R}^3,\mathbb{R})$, then

$$L_-\eta=-\frac{1}{2}\left(1+\int_{\mathbb{R}^3}|\nabla r|^2\right)\Delta\eta+\eta-r^{2p}\eta.$$

It is easy to check that L_+,L_- are self-adjoint. Recalling that $L_-r=0$, we know r is an eigenfunction of the operator

$$\tilde{L}:=-\frac{1}{2}\left(1+\int_{\mathbb{R}^3}|\nabla r|^2\right)\Delta+1$$

in $H^2(\mathbb{R}^3,\mathbb{R})\cap L^2(\mathbb{R}^3,r^{2p}\mathrm{d}x)$. Since $r(x)>0,\ x\in\mathbb{R}^3$, we know 1 is the first eigenvalue of $\tilde{L}$ which is simple and the associated eigenfunction is r. Then

$$\mathrm{Ker}L_-=\mathrm{span}\{r\},$$

and $\langle L_-\eta,\eta\rangle\geq 0$ for any $\eta\in H^1(\mathbb{R}^3)$.

3. Proof of Theorem 1.1

In this section, we are in position to investigate the modulational stability of ground states to problem (1.3).

3.1. **Spectral estimates of $L_\pm$.** To start the proof, we give some crucial lemmas as follows.

Lemma 3.1. [5, Lemma 2.5] *For any $\phi \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|\phi\|_2 = \|r\|_2$ and*

$$\inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \|\phi - e^{i\theta} r(\cdot - x)\| \le \|r\|,$$

then the minimization problem

$$\inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \|\phi - e^{i\theta} r(\cdot - x)\|$$

is achieved at some $(x_0, \gamma) \in \mathbb{R}^3 \times [0, 2\pi)$.

Lemma 3.2. *For $\phi, (x_0, \gamma)$ given above, let*

$$w := u + iv = e^{-i\gamma}\phi(\cdot + x_0) - r(\cdot),$$

then $\|w\| \le \|r\|$, $\|w + r\|_2 = \|r\|_2$. Moreover,

$$(v, r)_{H^1} = (u, \partial_{x_j} r)_{H^1} = 0,\ j = 1, 2, 3. \tag{3.1}$$

Proof. In fact, for any $(x, \theta) \in \mathbb{R}^3 \times [0, 2\pi)$, consider the function

$$\begin{aligned}\Upsilon(x,\theta) &= \|\phi - e^{i\theta} r(\cdot - x)\|^2 \\ &= \|\phi\|^2 + \|r\|^2 - 2\mathfrak{Re}\int_{\mathbb{R}^3} e^{i\theta}\bar{\phi}(y)\left(-\frac{1}{2}\Delta r + r\right)(y - x)\,\mathrm{d}y.\end{aligned}$$

So $\inf_{(x,\theta)\in\mathbb{R}^3\times[0,2\pi)} \Upsilon(x,\theta) = \min_{(x,\theta)\in\mathbb{R}^4} \Upsilon(x,\theta) = \Upsilon(x_0,\gamma)$. Recalling that $r \in C^\infty(\mathbb{R}^3)$, we know $\Upsilon \in C^1(\mathbb{R}^4)$ and then

$$\partial_\theta \Upsilon(x_0,\gamma) = \partial_{x_j}\Upsilon(x_0,\gamma) = 0,\ j = 1, 2, 3.$$

Since

$$\begin{aligned}\partial_\theta \Upsilon(x_0,\gamma) &= -2\mathfrak{Re}\int_{\mathbb{R}^3} ie^{i\gamma}\bar{\phi}(y)\left(-\frac{1}{2}\Delta r + r\right)(y - x_0)\,\mathrm{d}y \\ &= 2\mathfrak{Im}\int_{\mathbb{R}^3} \overline{e^{-i\gamma}\phi(y + x_0)}\left(-\frac{1}{2}\Delta r + r\right)(y)\,\mathrm{d}y \\ &= 2\mathfrak{Im}\int_{\mathbb{R}^3} \overline{u + r + iv}\left(-\frac{1}{2}\Delta r + r\right)(y)\,\mathrm{d}y \\ &= 2\mathfrak{Im}\int_{\mathbb{R}^3} (u + r - iv)\left(-\frac{1}{2}\Delta r + r\right)(y)\,\mathrm{d}y \\ &= -2\int_{\mathbb{R}^3} v\left(-\frac{1}{2}\Delta r + r\right)\,\mathrm{d}y\end{aligned}$$

and

$$(v, r)_{H^1} = \int_{\mathbb{R}^3}\left(\frac{1}{2}\nabla r\nabla v + rv\right) = \int_{\mathbb{R}^3} v\left(-\frac{1}{2}\Delta r + r\right),$$

we get that

$$\partial_\theta \Upsilon(x_0,\gamma) = -2(v, r)_{H^1}.$$

For $j=1,2,3$,

$$\begin{aligned}\partial_{x_j}\Upsilon(x_0,\gamma) &= 2\mathfrak{Re}\int_{\mathbb{R}^3} e^{i\gamma}\bar{\phi}(y)\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y-x_0)\,\mathrm{d}y \\ &= 2\mathfrak{Re}\int_{\mathbb{R}^3} e^{i\gamma}\bar{\phi}(y+x_0)\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y \\ &= 2\mathfrak{Re}\int_{\mathbb{R}^3} \overline{e^{-i\gamma}\phi(y+x_0)}\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y \\ &= 2\mathfrak{Re}\int_{\mathbb{R}^3} \overline{u+r+iv}\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y \\ &= 2\mathfrak{Re}\int_{\mathbb{R}^3} (u+r-iv)\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y \\ &= 2\int_{\mathbb{R}^3} (u+r)\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y.\end{aligned}$$

Noting that r is even and $\Delta(\partial_{x_j}r),\partial_{x_j}r$ are odd, we know

$$\int_{\mathbb{R}^3} r\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y=0.$$

So

$$\partial_{x_j}\Upsilon(x_0,\gamma)=2\int_{\mathbb{R}^3} u\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right].$$

On the other hand, since

$$(u,\partial_{x_j}r)_{H^1}=\int_{\mathbb{R}^3}\left[\frac{1}{2}\nabla u\nabla(\partial_{x_j}r)+u\partial_{x_j}r\right](y)\,\mathrm{d}y=\int_{\mathbb{R}^3} u\left[-\frac{1}{2}\Delta(\partial_{x_j}r)+\partial_{x_j}r\right](y)\,\mathrm{d}y,$$

we know

$$\partial_{x_j}\Upsilon(x_0,\gamma)=2(u,\partial_{x_j}r)_{H^1}.$$

Thus, we get (3.1). □

Similar to [5, Lemma 2.4], let

$$\mathcal{V}:=\left\{u\in H^1(\mathbb{R}^3):(u,r)=0\right\},$$

then we have

Lemma 3.3. $\inf_{u\in\mathcal{V}}\langle L_+u,u\rangle=0$.

Set

$$\mathcal{V}_0:=\left\{u\in H^1(\mathbb{R}^3):(u,r)=(u,\partial_{x_j}r)_{H^1}=0,\ j=1,2,3\right\},$$

then $\mathcal{V}_0$ is regular. In fact, for any $u\in\mathcal{V}_0$, let

$$F_0(u)=(u,r),\ \ F_j(u)=(u,\partial_{x_j}r)_{H^1},\ j=1,2,3.$$

For any $v\in H^1(\mathbb{R}^3)$, for any $t\in\mathbb{R}$, one can get that

$$F_0(u+tv)=F_0(u)+t(v,r),\ \ F_j(u+tv)=F_j(u)+t(v,\partial_{x_j}r)_{H^1}.$$

These yield that $F_j's$ are Fréchet-differentiable and their derivatives at any $u\in\mathcal{V}_0$ are given as follows

$$F_0'(u)v=(v,r),\ \ F_j'(u)v=(v,\partial_{x_j}r)_{H^1},\ j=1,2,3,\ \forall v\in H^1(\mathbb{R}^3).$$

Obviously, $F_0'(u)\neq 0$ and $F_j'(u)\neq 0$, $j=1,2,3$. Moreover, $F_i'(u)$, $i=0,1,2,3$, are linearly independent. Otherwise, if for some $a_i\in\mathbb{R}$, $i=0,1,2,3$ with at lease one of $a_i's$ is non zero, there holds that $\sum_{i=0}^3 F_i'(u) = 0$, then $\sum_{i=0}^3 F_i'(u)r = \sum_{i=0}^3 F_i'(u)\partial_{x_j} r = 0$, $j = 1, 2, 3$. Noting that $(r, \partial_{x_j} r)_{H^1} = 0$, $(\partial_{x_j} r, r) = 0$, $j = 1, 2, 3$, one can know that $a_i's$ are zero, which is a contradiction.

Lemma 3.4. $\inf_{u\in\mathcal{V}_0} \frac{\langle L_+u,u\rangle}{\|u\|^2} > 0$.

Proof. It suffices to show that

$$\inf_{u\in\mathcal{V}_0} \frac{\langle L_+u,u\rangle}{\|u\|_2^2} > 0. \tag{3.2}$$

Indeed, if (3.2) holds true, then we have

$$\inf_{u\in\mathcal{V}_0} \frac{\langle L_+u,u\rangle}{\|u\|^2} > 0.$$

If not, there exists $\{u_n\} \subset \mathcal{V}_0$ satisfying $\|u_n\| = 1$ and $\langle L_+u_n, u_n\rangle \to 0$ as $n \to \infty$. By (3.2), $u_n \to 0$ strongly in $L^2(\mathbb{R}^3)$ and then weakly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. So

$$\begin{aligned}\langle L_+u_n, u_n\rangle &= \frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla r|^2\right)\|\nabla u_n\|_2^2 + \left(\int_{\mathbb{R}^3} \nabla r\nabla u_n\right)^2 + \int_{\mathbb{R}^3}\left[1-(2p+1)r^{2p}\right]u_n^2\\ &= \frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla r|^2\right)\|\nabla u_n\|_2^2 + o_n(1), \text{ as } n\to\infty.\end{aligned}$$

It yields that $u_n \to 0$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$, which contradicts the fact that $\|u_n\| = 1$ for any n.

In the following, we only need to show (3.2) is true. If not, there exists $\{u_n\} \subset \mathcal{V}_0$ with $\|u_n\|_2 = 1$ such that $\langle L_+u_n, u_n\rangle \to 0$ as $n \to \infty$. Noting that

$$\langle L_+u_n, u_n\rangle = \frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla r|^2\right)\|\nabla u_n\|_2^2 + \left(\int_{\mathbb{R}^3} \nabla r\nabla u_n\right)^2 + \|u_n\|_2^2 - (2p+1)\int_{\mathbb{R}^3} r^{2p}u_n^2,$$

we get

$$\begin{aligned}&\limsup_{n\to\infty}\left[\frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla r|^2\right)\|\nabla u_n\|_2^2 + \|u_n\|_2^2\right]\\ &= \limsup_{n\to\infty}\left[(2p+1)\int_{\mathbb{R}^3} r^{2p}u_n^2 - \left(\int_{\mathbb{R}^3} \nabla r\nabla u_n\right)^2\right],\end{aligned} \tag{3.3}$$

and $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \le 2(2p+1)\|r\|_\infty^{2p}$. So, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \to u$ weakly in $H^1(\mathbb{R}^3)$ and a. e. in $\mathbb{R}^3$ as $n \to \infty$. Obviously, $u \in \mathcal{V}_0$ and $\langle L_+u, u\rangle \ge 0$. On the other hand, since $r(x) \to 0$ as $|x| \to \infty$, up to a subsequence, $\int_{\mathbb{R}^3} r^{2p}u_n^2 \to \int_{\mathbb{R}^3} r^{2p}u^2$ as $n \to \infty$. Thanks to the weak lower semi-continuity of the norm, we have

$$\langle L_+u, u\rangle \le \liminf_{n\to\infty}\langle L_+u_n, u_n\rangle = 0,$$

which implies that $\langle L_+u, u\rangle = 0$. That is,

$$\frac{1}{2}\left(1 + \int_{\mathbb{R}^3} |\nabla r|^2\right)\|\nabla u\|_2^2 + \|u\|_2^2 = (2p+1)\int_{\mathbb{R}^3} r^{2p}u^2 - \left(\int_{\mathbb{R}^3} \nabla r\nabla u\right)^2. \tag{3.4}$$

Noting that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \nabla r\nabla u_n = \int_{\mathbb{R}^3} \nabla r\nabla u,$$

it follows from (3.3) and (3.4) that $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$ and $\|u\|_2 = 1$. Then by the Lagrange Multiplier Theorem [8], there exist Lagrange multipliers $\lambda, \mu, \lambda_1, \lambda_2, \lambda_3$ such that for any $\eta \in H^1(\mathbb{R}^3)$,

$$\langle L_+ u, \eta\rangle = \lambda(u,\eta) + \mu(r,\eta) + \sum_{i=1}^{3} \lambda_i(\partial_{x_i} r, \eta)_{H^1}. \tag{3.5}$$

Thanks to $u \in \mathcal{V}_0$, $\lambda = 0$. For any j,

$$\langle L_+ u, \partial_{x_j} r\rangle = \langle L_+ \partial_{x_j} r, u\rangle = 0,$$

where we used the fact that $\mathrm{Ker} L_+ = \mathrm{span}\{\partial_{x_1} r, \partial_{x_2} r, \partial_{x_3} r\}$. Then taking $\eta = \partial_{x_j} r$, for $j = 1, 2, 3$, we have

$$\lambda_j(\partial_{x_j} r, \partial_{x_j} r)_{H^1} = 0,$$

and $\lambda_j = 0$. Here we used the fact that $(r, \partial_{x_j} r) = 0,\ j = 1, 2, 3$, and $(\partial_{x_i} r, \partial_{x_j} r)_{H^1} = 0,\ i \neq j$. In turn, for any $\eta \in H^1(\mathbb{R}^3)$, it holds true that $\langle L_+ u, \eta\rangle = \mu(r, \eta)$. If $\mu = 0$, then $u \in \mathrm{Ker} L_+$, which contradicts the fact that $\|u\|_2 = 1$ and $(u, \partial_{x_j} r)_{H^1} = 0,\ j = 1, 2, 3$. So $\mu \neq 0$.

In the following, we show that we can reach a contradiction: $\mu = 0$. In fact, by computation, one can get that

$$\Delta(x \cdot \nabla r) = 2\Delta r + \sum_{j=1}^{3} x_j \partial_{x_j} \Delta r,\ \int_{\mathbb{R}^3} \nabla r \nabla(x \cdot \nabla r) = -\frac{1}{2}\|\nabla r\|_2^2.$$

Then

$$\begin{aligned} L_+(x \cdot \nabla r) &= -\left(1 + \frac{1}{2}\|\nabla r\|_2^2\right)\Delta r - \sum_{j=1}^{3} x_j \left[-\frac{1}{2}(1 + \|\nabla r\|_2^2)\Delta r + r - r^{2p+1}\right] \\ &= -\left(1 + \frac{1}{2}\|\nabla r\|_2^2\right)\Delta r. \end{aligned} \tag{3.6}$$

Meanwhile, since r is a solution of (1.3), we have

$$\begin{aligned} L_+ r &= -\frac{1}{2}(1 + 3\|\nabla r\|_2^2)\Delta r + r - (2p+1)r^{2p+1} \\ &= \left[p + (p-1)\|\nabla r\|_2^2\right]\Delta r - 2pr. \end{aligned} \tag{3.7}$$

So by (3.6)-(3.7), we get that

$$L_+\left(\frac{r}{2p} + \frac{p + (p-1)\|\nabla r\|_2^2}{p(2 + \|\nabla r\|_2^2)}(x \cdot \nabla r)\right) = -r.$$

Recalling that $L_+ u = \mu r$, for some $\vartheta \in \mathbb{R}^3$, we have

$$u = \vartheta \cdot \nabla r - \mu\left[\frac{r}{2p} + \frac{p + (p-1)\|\nabla r\|_2^2}{p(2 + \|\nabla r\|_2^2)}(x \cdot \nabla r)\right].$$

Thanks to the fact that

$$\int_{\mathbb{R}^3} ru = \int_{\mathbb{R}^3} r\partial_{x_j} r = 0,\ j = 1, 2, 3,$$

we reach that

$$\mu \int_{\mathbb{R}^3}\left[\frac{r^2}{2p} + \frac{p + (p-1)\|\nabla r\|_2^2}{p(2 + \|\nabla r\|_2^2)}(x \cdot \nabla r) r\right] = 0.$$

Since

$$\int_{\mathbb{R}^3}(x\cdot\nabla r)r=-\frac{3}{2}\int_{\mathbb{R}^3}r^2,$$

then by $p\in(0,2/3)$,

$$\int_{\mathbb{R}^3}\left[\frac{r^2}{2p}+\frac{p+(p-1)\|\nabla r\|_2^2}{p(2+\|\nabla r\|_2^2)}(x\cdot\nabla r)r\right]=\frac{(2-3p)+(4-3p))\|\nabla r\|_2^2}{2p(2+\|\nabla r\|_2^2)}\int_{\mathbb{R}^3}r^2>0.$$

It follows that $\mu=0$. The proof is complete. □

Lemma 3.5. *Let $w=u+iv\in H^1(\mathbb{R}^3,\mathbb{C})$ with $u,v\in H^1(\mathbb{R}^3)$. If $\|w+r\|_2=\|r\|_2$ and*

$$(u,\partial_{x_j}r)_{H^1}=0,\ j=1,2,3,$$

then there exist D,D_1,D_2 such that

$$\langle L_+u,u\rangle\geq D\|u\|^2-D_1\|w\|^4-D_2\|w\|^3.$$

Proof. By $\|w+r\|_2=\|r\|_2$, we get $(u,r)=-\frac{1}{2}\|w\|_2^2$. Without loss of generality, we assume that $\|r\|_2=1$. Let

$$u=u_\|+u_\perp,\ \ u_\|=(u,r)r,$$

then $(u_\perp,r)=0$. Noting that $(r,\partial_{x_j}r)_{H^1}=0$, $j=1,2,3$, we have $(u_\perp,\partial_{x_j}r)_{H^1}=0$, $j=1,2,3$ and $u_\perp\in\mathcal{V}_0$. It follows that $\langle L_+u_\perp,u_\perp\rangle\geq C\|u_\perp\|_2^2$ for some $C>0$. Similar to [5, Proposition 2.2],

$$\langle L_+u_\perp,u_\perp\rangle\geq C(\|u\|^2-\|w\|_2^4). \tag{3.8}$$

On the other hand, since r is a solution of (1.3), we have

$$\frac{1}{2}(1+\|\nabla r\|_2^2)\|\nabla r\|_2^2+\|r\|_2^2=\|r\|_{2p+2}^{2p+2}.$$

It follows that

$$\langle L_+r,r\rangle=-p\|\nabla r\|_2^2-2p\|r\|_2^2+(1-p)\|\nabla r\|_2^4\geq-2p\|r\|^2,$$

and then

$$\langle L_+u_\|,u_\|\rangle=\frac{1}{4}\|w\|_2^4\langle L_+r,r\rangle\geq-\frac{p}{2}\|w\|_2^4\|r\|^2. \tag{3.9}$$

Finally, since r satisfies (1.3), we get

$$L_+r=p(1+\|\nabla r\|_2^2)\Delta r-\|\nabla r\|_2^2\Delta r-2pr.$$

Then for some $C>0$,

$$\begin{aligned}\langle L_+u_\perp,r\rangle=\langle L_+r,u_\perp\rangle&=\Big[(1-p)\|\nabla r\|_2^2-p\Big]\int_{\mathbb{R}^3}\nabla r\nabla u_\perp\\&=\Big[(1-p)\|\nabla r\|_2^2-p\Big]\left(\int_{\mathbb{R}^3}\nabla r\nabla u-\int_{\mathbb{R}^3}\nabla r\nabla u_\|\right)\\&=\Big[(1-p)\|\nabla r\|_2^2-p\Big]\left(\int_{\mathbb{R}^3}\nabla r\nabla u+\frac{1}{2}\|w\|_2^2\int_{\mathbb{R}^3}|\nabla r|^2\right)\\&\leq C(\|\nabla u\|_2+\|w\|_2^2).\end{aligned}$$

So

$$\begin{aligned}\langle L_+u_\perp,u_\|\rangle=-\frac{1}{2}\|w\|_2^2\langle L_+u_\perp,r\rangle&\geq-\frac{C}{2}\|w\|_2^2(\|\nabla u\|_2+\|w\|_2^2)\\&\geq-C(\|w\|^3+\|w\|^4).\end{aligned} \tag{3.10}$$

Thus, the result as claimed is yielded by (3.8)-(3.10). □

Lemma 3.6.

$$\inf_{\substack{v\in H^1(\mathbb{R}^3)\setminus\{0\}\\ (v,r)_{H^1}=0}} \frac{\langle L_- v, v\rangle}{\|v\|^2} > 0.$$

Proof. It suffices to show that

$$\omega := \inf_{\substack{v\in H^1(\mathbb{R}^3)\setminus\{0\}\\ \|v\|_2=1,\,(v,r)_{H^1}=0}} \langle L_- v, v\rangle > 0.$$

Since $r(x)\to 0$ as $|x|\to\infty$, similar to [5, Proposition 2.3], we know $\omega\geq 0$. If $\omega=0$, taking any minimizing sequence $\{v_n\}$, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and for some $v\in H^1(\mathbb{R}^3)$, we have $v_n\to v$ weakly in $H^1(\mathbb{R}^3)$ and a. e. in $\mathbb{R}^3$ as $n\to\infty$. So $(v,r)_{H^1}=0$ and by the decay of r,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} r^{2p}v_n^2 = \int_{\mathbb{R}^3} r^{2p}v^2.$$

$$0\leq \langle L_- v, v\rangle \leq \liminf_{n\to\infty}\langle L_- v_n, v_n\rangle = 0$$

Then $\langle L_- v, v\rangle = 0$. Furthermore, we know $v_n\to v$ strongly in $H^1(\mathbb{R}^3)$ as $n\to\infty$ and $\|v\|_2=1$. In turn, there exist λ,μ such that

$$\langle L_- v,\eta\rangle = \lambda(v,\eta)+\mu(r,\eta)_{H^1},\ \ \eta\in H^1(\mathbb{R}^3).$$

By taking $\eta=v$, then $\lambda=0$. Finally, we take $\eta=r$ and get that

$$\mu\|r\|_{H^1}^2 = \langle L_- v, r\rangle = \langle L_- r, v\rangle = 0.$$

That is, $\mu=0$ and $L_- v=0$. Recalling that $\mathrm{Ker}L_- = \mathrm{Span}\{r\}$, we get that $v=\theta r$ for some $\theta\in\mathbb{R}$. Noting that $(v,r)_{H^1}=0$, $\theta=0$, which contradicts the fact that $\|v\|_2=1$. □

3.2. **Toward to Proof of Theorem 1.1.**

Proof. Take $\phi = r+w, (x_0,\gamma)$ given in Lemma 3.1 and w,u,v given in Lemma 3.2. Let $I(\phi)=\mathcal{E}(\phi)+\|\phi\|_2^2$, we get that $I'(r)=0$ in $H^{-1}(\mathbb{R}^3)$ and then by Proposition 2.1, $I(\phi)\geq I(r)$. By the Taylor expend, for some $\theta\in(0,1)$, we have

$$\begin{aligned} I(\phi)-I(r) = I(r+w)-I(r) &= \frac{1}{2}\langle I''(r+\theta w)w,w\rangle \\ &:= \langle L_+u,u\rangle + \langle L_- v, v\rangle + J + K, \end{aligned}$$

where

$$\begin{aligned} J =& \frac{1}{2}\left(\|\nabla(r+\theta u)\|_2^2 + \theta^2\|\nabla v\|_2^2 - \|\nabla r\|_2^2\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) \\ &+ \left[\theta\|\nabla u\|_2^2 + \theta\|\nabla v\|_2^2 + \int_{\mathbb{R}^3}\nabla r\nabla u\right]^2 - \left(\int_{\mathbb{R}^3}\nabla r\nabla u\right)^2, \end{aligned}$$

and

$$\begin{aligned} K =& \int_{\mathbb{R}^3}[(2p+1)r^{2p}u^2 + r^{2p}v^2 - |w|^2|r+\theta w|^{2p}] \\ &- 2p\int_{\mathbb{R}^3}|r+\theta w|^{2p-2}|ru+\theta u^2+\theta v^2|^2. \end{aligned}$$

It is easy to check that $J\geq -C(\|w\|^3+\|w\|^4)$ for some $C>0$. Similar to the inequality in (2.5) in [18], by an interpolation estimate of Nirenberg and Gagliardo, one can get that $K\geq -C(\|w\|^{2+\tau}+\|w\|^6)$, where $\tau>0$ and $C>0$. Finally, the claim is concluded by (3.1), Lemma 3.2, Lemma 3.5 and Lemma 3.6. □

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