# An existence result for optimal obstacles

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**Abstract**

We consider the optimization problem

$$\min\{F(g)\ : g \in X(\Omega)\}$$

where $F(g)$ is a variational energy associated to the obstacle $g$ and the class $X(\Omega)$ of admissible obstacles is given by

$$X(\Omega) = \{g : \Omega \to \overline{I\!R} : g \le \psi \quad \text{on} \quad \Omega, \int_\Omega g\,dx = c\}$$

with $\psi \in W_0^{1,p}(\Omega)$ and $c \in I\!R$ fixed. Generally, this problem does not have a solution and it may happen that the "optimal" obstacle is of relaxed form. Under a monotonicity assumption on $F$, we prove the existence of a non-relaxed optimal obstacle in the family $X(\Omega)$ through a new method based on the notions of $\gamma$ and $w\gamma$-convergences.

## 1 Introduction

Optimization problems with varying obstacles in general do not posses a solution, even if the minimizing cost functional has good properties. What happens in general, is that a minimizing sequence of obstacles "*converges*" to the solution of a new problem which is of relaxed form. This question is mainly approached through the $\Gamma$- convergence theory and was widely studied in [2], [12], [15]. In order to avoid the relaxation phenomenon, a usual method used by several authors is to restrict the class of admissible obstacles, imposing some additional constraints (see for instance [1], [3], [4], [5], [10]).

On the other hand, in [9] it was observed that in some optimal design problems the existence of a classical non-relaxed solution is provided by a monotonicity assumption (with respect to the inclusion of domains) on the cost functional, without imposing any other constraints on the class of admissible domains. This is for example the case of functionals depending on the eigenvalues of the Laplace operator with Dirichlet boundary conditions.

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The main ideas of the proof come from the Γ-convergence theory and are strongly based on the exact description of the family of relaxed domains.

In this paper we consider a similar question for functionals depending on obstacles. More precisely, we consider optimization problems for obstacles, of the form

$$\min\{F(g)\ : g \in X(\Omega)\} \tag{1}$$

where $F$ is a given cost functional and $X(\Omega)$ indicates the class of admissible obstacles. A typical constraint on the elements of $X(\Omega)$ is of integral type:

$$\int_\Omega g\,dx = c$$

where $c$ is a given constant.

A possible method to prove that problem (1) admits a solution is the direct method of the calculus of variations. Therefore, one should find a compact convergence on $X(\Omega)$ which makes the cost $F$ lower semicontinuous. This is the greatest difficulty, since a " natural" convergence on $X(\Omega)$ can be identified, which makes $F$ lower semicontinuous, but this convergence is not compact, unless more restrictions on the regularity of admissible obstacles are imposed. Here we look for the minimum in (1) without further a priori assumptions on the competing obstacles, and actually it may happen that in general, the existence of an optimal obstacle may fail. This is why we shall concentrate our attention on a special class of cost functionals, verifying a monotonicity assumption. Therefore, the class $X(\Omega)$ will be considered as an ordered set, where the order between two admissible obstacles will be the usual order between functions.

The "natural" convergence on $X(\Omega)$ mentioned above will be the one associated to the Γ-convergence of the energy functionals

$$\int_\Omega \frac{1}{p}|Du|^p dx + \chi_{\{u\geq g\}}(u) \tag{2}$$

where $u \in W_0^{1,p}(\Omega)$ and, by analogy with some shape optimization problems already considered in the literature (see [6], [7], [8], [9]), we shall denote it by $\gamma$. As we already said, this convergence is not compact on $X(\Omega)$ and so, in general, we should expect only "relaxed" solutions for our problem (1). The "relaxed" obstacles, i.e. the $\gamma$-limits of sequences of obstacles, have been studied in detail in [2], [11], [15] and they are characterized as a subclass of triplets $(f, \mu, \nu)$ where $f$ is a real positive function on $\Omega \times \mathbb{R}$, and $\mu$ and $\nu$ two measures.

However, here we use a different approach and we introduce a general abstract framework which does not refer to the relaxed obstacles. Making use of monotonicity assumption on $F$ we introduce a second convergence, which we call weak-$\gamma$, which turns out to be compact, and which still makes $F$ lower semicontinuous.

## 2 General setting and main result

In this section we present a general framework in which the minimization problem of a monotone functional can be set. Consider an ordered space $(X, \leq)$ and a functional $F$ :

$X \to \overline{I\!R}$. Suppose that $X$ is endowed with two convergences denoted by $\gamma$ and $w\gamma$, the last convergence being weaker than the first and sequentially compact. Moreover suppose that the functional $F$ is $\gamma$ lower semicontinuous. The relation we assume between $\gamma$ and $w\gamma$ is the following one:

**Assumption (A)** *For every $x_n \xrightarrow{w\gamma} x$ there exist a sequence of integers $\{n_k\}$ and a sequence $\{y_{n_k}\}$ in $X$ such that $y_{n_k} \leq x_{n_k}$ and $y_{n_k} \xrightarrow{\gamma} x$.*

The monotonicity of $F$ becomes an important assumption because of the following result.

**Proposition 2.1** *If $F : X \to \overline{I\!R}$ is monotone increasing and $\gamma$ lower semicontinuous, then $F$ is $w\gamma$ lower semicontinuous.*

**Proof** Let us consider $x_n \xrightarrow{w\gamma} x$, and let $\{x_{n_k}\}$ be a subsequence such that

$$\lim_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n).$$

Using assumption (A) above there exists a subsequence (which we still denote by $\{x_{n_k}\}$) and $y_{n_k} \leq x_{n_k}$ such that $y_{n_k} \xrightarrow{\gamma} x$. The $\gamma$ lower semicontinuity of $F$ gives

$$F(x) \leq \liminf_{k\to\infty} F(y_{n_k})$$

and the monotonicity of $F$ gives $F(y_{n_k}) \leq F(x_{n_k})$. Therefore

$$F(x) \leq \liminf_{k\to\infty} F(y_{n_k}) \leq \liminf_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n)$$

which concludes the proof. $\square$

Consider now another functional $\Phi : X \to \overline{I\!R}$ and the following problem

$$\min\{F(x) : x \in X, \Phi(x) \leq 0\}; \tag{3}$$

then we have

**Theorem 2.2** *Let $F$ be an increasing $\gamma$ lower semicontinuous functional and let $\Phi$ be $w\gamma$ lower semicontinuous. Under the assumption (A) above, problem (3) admits at least one solution.*

**Proof** The proof follows straightforward by the direct method of the calculus of variations and proposition 2.1. $\square$

The general framework introduced above, even if quite trivial, applies very well in the case of shape optimization problems, and we shall apply it also in the case of obstacles. The main difficulty is to *"identify" the $w\gamma$ convergence, and to prove that assumption (A) is fulfilled.*

In the case of obstacles we shall consider a bounded open set $\Omega$ of $I\!R^N$ (for $N \geq 2$), a function $\psi \in W_0^{1,p}(\Omega)$, the family of admissible obstacles

$$X_\psi(\Omega) = \{g : \Omega \to \overline{I\!R} : g \leq \psi, \;\; g \text{ quasi upper semicontinuous}\},$$

and a cost functional $F : X_\psi(\Omega) \to \overline{I\!R}$ which is monotone increasing, i.e. for all $g_1, g_2 \in X_\psi(\Omega)$ with $g_1 \leq g_2$ we have $F(g_1) \leq F(g_2)$. Moreover suppose that $F$ is lower semicontinuous for the $\gamma$-convergence which is defined through the $\Gamma$-convergence of obstacle functionals (see definition 3.1).

The result we are going to prove is the following.

**Theorem 2.3** *Under the assumptions above, for any constant $c \in I\!R$, the problem*

$$\min\{F(g) : g \in X_\psi(\Omega), \int_\Omega g\,dx = c\}$$

*admits a solution.*

# 3 Preliminaries

In this section we recall some basic concepts about the obstacle problem: position of the problem, capacity, $\Gamma$-convergence. Let us begin with the variational definition of the harmonic and metaharmonic capacity relatively to an open set $\Omega$ of $I\!R^N$, $N \geq 2$. We also fix a real number $p \in ]1, +\infty[$.

If $\Omega$ is bounded the harmonic capacity of an arbitrary set $E \subseteq \Omega$ is (see [16])

$$C'_p(E) := \inf\{\int_\Omega |\nabla u|^p dx, \quad u \in W_0^{1,p}(\Omega), u \geq 1 \text{ on a neighborhood of } E\};$$

if $\Omega$ is unbounded the metaharmonic capacity of an arbitrary set $E \subseteq \Omega$ is

$$C_p(E) := \inf\{\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx, \quad u \in W_0^{1,p}(\Omega), u \geq 1 \text{ on a neighborhood of } E\}.$$

We observe that if $\Omega$ is bounded the capacities $C'_p$ and $C_p$ are equivalent, in the sense that for each $\varepsilon > 0$ there exists a constant $k = k(\varepsilon, p, \Omega)$ such that

$$C'_p(E) \leq C_p(E) \leq kC'_p(E), \text{ for every } E \subseteq \Omega, d(E, \partial\Omega) \geq \varepsilon.$$

It is said that a property holds *$p$-quasi everywhere* (shortly $p$-q.e.) if it holds outside a set of $p$-capacity zero. A set $A \subseteq I\!R^N$ is called *$p$-quasi open*, if for any $\epsilon > 0$ there exists an open set $A_\epsilon$ such that $C_p(A_\epsilon) < \epsilon$ and $A \cup A_\varepsilon$ is open. Sobolev spaces are naturally defined on quasi open sets (see [19]).

A function $u$ is said *$p$-quasi continuous* if for any $\epsilon > 0$ there exists an open set $A_\epsilon$ such that $C_p(A_\epsilon) < \epsilon$ and $u|_{\Omega \setminus A_\epsilon}$ is continuous in $\Omega \setminus A_\epsilon$. Any function $u \in W^{1,p}(\Omega)$ has a unique (up to a set of $p$-capacity zero) $p$-quasi continuous representative. A function $u$ is said *$p$-quasi upper semicontinuous* if for any $\epsilon > 0$ there exists an open set $A_\epsilon$ such that $C_p(A_\epsilon) < \epsilon$ and $u$ is upper semicontinuous in $\Omega \setminus A_\epsilon$. We recall that for a $p$-quasi upper semicontinuous function $f$ its level sets $\{f < t\}$ are quasi open for every $t \in I\!R$. For more properties concerning quasi continuous and quasi upper semicontinuous functions we refer to [12], [18] and [20].

The choice of obstacles as quasi upper semicontinuous functions is natural, since one can replace an arbitrary obstacle by a suitable upper quasi semicontinuous one (see [2], [12]).

For a quasi upper semicontinuous function $g : \Omega \to \overline{I\!R}$ we define the set

$$K_g = \{u \in W_0^{1,p}(\Omega) : u \geq g\}$$

so that, for every $f \in L^p(\Omega)$, the solution of the obstacle problem associated to $f$ and $g$ is given by

$$\min\{\int_\Omega \frac{1}{p}|\nabla u|^p dx - \int_\Omega f u\, dx : u \in K_g\}. \tag{4}$$

For some fixed $f$ the study of the solution of problem (4) when the obstacle $g$ varies is done by the classical tool of the $\Gamma$-convergence related to the energy functional (2) (see [2], [15]). For this purpose, we also recall the notion of Mosco convergence: for a sequence of arbitrary convex sets $G_n \subseteq W_0^{1,p}(\Omega)$ it is said that $G_n$ converges in the sense of Mosco to $G$ if

1. $\forall u \in G$, $\exists u_n \in G_n$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$;
2. if $u_{n_k} \in G_{n_k}$ and $u_{n_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, then $u \in G$.

If $\{g_n\}$ is a sequence of admissible obstacles and if $K_{g_n}$ converges in the sense of Mosco to $K_g$ then it easy to see that the solutions $u_n$ of problem (4) associated to $g_n$ converge weakly in $W_0^{1,p}(\Omega)$ to the solution of (4) corresponding to $g$. It is also well known that the Mosco convergence of the convex sets $K_{g_n}$ to $K_g$ is equivalent to the $\Gamma$-convergence of the energy functionals (2) associated to $g_n$ and $g$.

**Definition 3.1** *If one of the two previous equivalent situations occurs we say that $g_n$ $\gamma$-converges to $g$.*

We recall the following result of [12] which gives a characterization of the $\gamma$-convergence in terms of the behaviour of the level sets.

**Lemma 3.2** *Let $g_n, g$ be a family of quasi upper semicontinuous functions $g_n : \Omega \to \overline{I\!R}$. Then $g_n \xrightarrow{\gamma} g$ and $K_g \neq \emptyset$ if and only if*

1. *there exists $T \subseteq ]0, +\infty[$ with $0 \in \overline{T}$ such that for every $t \in T$*
$$\lim_{n\to\infty} C_p(\{g_n > t\}) = C_p(\{g > t\});$$
2. *there exist a dense set $D \subseteq I\!R$ and a dense set $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ such that for every $t \in D$ and every $B \in \mathcal{B}$*
$$\lim_{n\to\infty} C_p(\{g_n > t\} \cap B) = C_p(\{g > t\} \cap B);$$
3. $\displaystyle\lim_{t\to\infty} \limsup_{n\to\infty} \int_t^{+\infty} C_p(\{g_n > s\})(s-t)^{p-1} ds = 0;$
4. $\displaystyle\lim_{t\to 0} \limsup_{n\to\infty} \int_0^t C_p(\{g_n > s\}) s^{p-1} ds = 0.$

Note that when replacing the metaharmonic capacity $C_p$ with the harmonic one $C_p'$ in the previous type of results, the analogous of lemma 3.2 may fail to be true (see [12], theorems 5.6, 5.9).

Giving an obstacle $g$ we shall consider the family of its level sets $\{g < t\}$ for $t \in I\!R$; these level sets are quasi open and therefore we recall some facts about the behaviour of quasi open sets.

It is said that a sequence of quasi open sets $\{A_n\}$ $\gamma_p$-converges to a quasi open set $A$, if the Sobolev spaces $W_0^{1,p}(A_n)$ converge in the sense of Mosco to $W_0^{1,p}(A)$.

We also define a weak $\gamma_p$-convergence (simply denoted by $w\gamma_p$) for domains as in [7] in the following way: $\{A_n\}$ weakly $\gamma_p$-converges to a quasi open set $A$ if $w_{A_n} \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$ and $A = \{w > 0\}$. Here $w_A$ denotes the solution of

$$\begin{cases} -\Delta_p w_A = 1 & \text{in } A \\ w_A \in W_0^{1,p}(A). \end{cases}$$

The $\gamma_p$-convergence of a sequence of domains is equivalent (see [8]) to the strong convergence of $w_{A_n}$ to $w_A$ in $W_0^{1,p}(\Omega)$. Moreover, in [13] and [16] a characterization of the $\gamma_p$-convergence in terms of the behaviour of the local capacity is given: if $\Omega$ is a bounded open set in $I\!R^n$ and $A_n, A$ are quasi open subsets of $\Omega$, then $A_n \overset{\gamma_p}{\to} A$ if and only if there exists a dense set $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ such that for every $B \in \mathcal{B}$

$$\lim_{n\to\infty} C_p'(B \setminus A_n) = C_p'(B \setminus A). \tag{5}$$

If $\Omega$ is unbounded, the same result is true if in relation (5) one use the metaharmonic capacity $C_p$ (see [16]). In order to apply lemma 3.2 for a bounded set $\Omega$ with the metaharmonic capacity, we recall the following.

**Lemma 3.3** *Let $\Omega$ be a bounded open set in $I\!R^n$ and let $A_n, A$ be quasi open sets of $\Omega$. Then $A_n \overset{\gamma_p}{\to} A$ if and only if there exists a dense set $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ such that for every $B \in \mathcal{B}$*

$$\lim_{n\to\infty} C_p(B \setminus A_n) = C_p(B \setminus A).$$

The result of this lemma is proved in [12] for unbounded sets, the passage to bounded sets $\Omega$ being immediate. We are then allowed to reformulate lemma 3.2 only in terms of the $\gamma_p$-convergence of level sets in the family of obstacles which are uniformly bounded from above (see theorem 4.2). In order to solve the minimization problem (3), we prove a result similar to assumption (A) for quasi open sets, which is a generalization of the result proved in [9] for $p = 2$ (see also [6], [7]). The following is proved in [6] and [9].

**Lemma 3.4** *Let $\{A_n\}$ be a sequence of quasi open subsets of $\Omega$ and let $w \in W_0^{1,p}(\Omega)$ be a function such that $w_{A_n} \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$. Let $u_n \in W_0^{1,p}(\Omega)$ be such that $u_n = 0$ $p$-q.e. on $\Omega \setminus A_n$ and suppose that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then $u = 0$ $p$-q.e. on $\{w = 0\}$.*

In the case of domains, assumption (A) is a consequence of the following.

**Lemma 3.5** *Let $\{A_n\}$ be a sequence of quasi open sets which weak $\gamma_p$ -converges to a quasi open set $A$. Then there exist a subsequence $\{A_{n_k}\}$ and a sequence of quasi open sets $G_k \subseteq \Omega$ such that $A_{n_k} \subseteq G_k$ and $G_k$ $\gamma_p$-converges to $A$.*

**Proof** From the definition of weak $\gamma_p$-convergence we have $w_{A_n} \to w$ weakly in $W_0^{1,p}(\Omega)$ and $A = \{w > 0\}$; moreover, following [17], we have $w \le w_A$. For every $\varepsilon > 0$ we define the $p$-quasi open set $A^\varepsilon = \{w_A > \varepsilon\}$ (see [9] for $p = 2$). For a subsequence, still denoted by the same indices, we can suppose that

$$w_{A_n \cup A^\varepsilon} \rightharpoonup w^\varepsilon \text{ weakly in } W_0^{1,p}(\Omega),$$

and by the comparison principle we have that $w^\varepsilon \ge u_{A^\varepsilon}$. But $w^\varepsilon \in W_0^{1,p}(A)$ from the Mosco convergence applied to $\{A_n \cup A^\varepsilon\}$. Indeed, defining $v^\varepsilon = 1 - \frac{1}{\varepsilon}\min\{w_A, \varepsilon\}$ we get $0 \le v^\varepsilon \le 1$, $v^\varepsilon = 0$ on $A^\varepsilon$, $v^\varepsilon = 1$ on $\Omega \setminus A$. Taking $w_n = \min\{v^\varepsilon, w_{A_n \cup A^\varepsilon}\}$ we get $w_n = 0$ on $A^\varepsilon$ and $w_n = 0$ on $\Omega \setminus (A_n \cup A^\varepsilon)$, and so $w_n = 0$ on $\Omega \setminus A_n$. Moreover $w_n \rightharpoonup \min\{v^\varepsilon, w^\varepsilon\}$ weakly in $W_0^{1,p}(\Omega)$ and hence from lemma 3.4 we have that $\min\{v^\varepsilon, w^\varepsilon\} = 0$ $p$-q.e. on $\{w = 0\}$. Since $v^\varepsilon = 1$ on $\Omega \setminus A$ we get that $w^\varepsilon = 0$ $p$-q.e. on $\Omega \setminus A$. ¿From theorem 5.1 of [17] (see also [9]) and from the fact that $-\Delta_p w_{A_n \cup A^\varepsilon} \le 1$ in $\Omega$ we get $-\Delta_p w^\varepsilon \le 1$ and hence $w^\varepsilon \le w_A$. Therefore $w_{A^\varepsilon} \le w^\varepsilon \le w_A$, and by a diagonal procedure we get that for a suitable $\epsilon_n \to 0$

$$w_{A_n \cup A^{\varepsilon_n}} \rightharpoonup w_A \quad \text{weakly in } W_0^{1,p}(\Omega).$$

To conclude, is sufficient to choose $G_n = A_n \cup A^{\varepsilon_n}$ for the diagonal sequence above. □

**Remark 3.6** In the previous lemma, the quasi open sets $G_n$ can be replaced by open sets. This follows from the fact that any quasi open set can be approached in capacity with an exterior sequence of open sets.

# 4 The obstacle problem

In this section we define the $w\gamma$-convergence for the obstacle problem, and we prove that assumption (A) is fulfilled for this convergence. Let us begin with a technical lemma.

**Lemma 4.1** *For every $B \subseteq \Omega$ and for every function $g$ quasi upper semicontinuous there exists an at most countable set $D(B)$ in $I\!R$ such that*

$$C_p(\{g > t\} \cap B) = C_p(\{g \ge t\} \cap B) \quad \forall t \in I\!R \setminus D(B).$$

**Proof** Fix $B \subseteq \Omega$ and set for all $t \in I\!R$

$$U_t = \{g > t\} \cap B, \quad \tilde{U}_t = \{g \ge t\} \cap B.$$

We have $U_t \subseteq \tilde{U}_t$, from which it follows

$$C_p(U_t) \le C_p(\tilde{U}_t) \quad \forall t \in I\!R. \tag{6}$$

The function $t \mapsto C_p(\tilde{U}_t)$ is decreasing in $t$, so it is continuous on $N = I\!R \setminus D(B)$, with $D(B) \subseteq I\!R$ at most countable. Let us fix $\tau \in N$. For each $t \in I\!R$ such that $\tau < t$, we have $C_p(U_\tau) \geq C_p(U_t)$. Making $t \to \tau$ we have

$$C_p(U_\tau) \geq C_p(\tilde{U}_\tau) \quad \forall \tau \in N. \tag{7}$$

Now from (6) and (7) we deduce $C_p(U_\tau) = C_p(\tilde{U}_\tau)$ which concludes the proof. □

The following result is a very useful characterization of the $\gamma$-convergence of a sequence of obstacles in terms of the behaviour of the level sets. It mainly states that a sequence of obstacles is $\gamma$-convergent if and only if their level sets are $\gamma_p$-convergent as quasi open sets.

**Theorem 4.2** *Let $g_n, g \in X_\psi(\Omega)$. Then $g_n \xrightarrow{\gamma} g$ if and only if there exists a family $T \subseteq I\!R$ such that $I\!R \setminus T$ is at most countable and*

$$\{g_n < t\} \xrightarrow{\gamma_p} \{g < t\} \quad \forall t \in T.$$

**Proof** The proof is based on lemma 3.2 and lemma 3.3. Let us suppose that $g_n \xrightarrow{\gamma} g$. Following lemma 3.2 there exists a dense set $T \subseteq \overline{I\!R}$ with $\overline{I\!R} \setminus T$ at most countable, and a countable dense family $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ such that

$$\lim_{n\to\infty} C_p(\{g_n > t\} \cap B) = C_p(\{g > t\} \cap B)$$

for every $t \in T$ and every $B \in \mathcal{B}$. Using lemma 4.1 for any $B_k \in \mathcal{B}$ and for any $n \in I\!N$ the family of $t \in I\!R$ such that

$$C_p(\{g_n > t\} \cap B_k) \neq C_p(\{g_n \geq t\} \cap B_k)$$

is at most countable. Therefore eliminating all $t \in T$ for all $k, n \in I\!N$ such that the previous relation holds, one can find a set $T'$ such that $I\!R \setminus T'$ is at most countable and such that

$$\lim_{n\to\infty} C_p(\{g_n \geq t\} \cap B) = C_p(\{g \geq t\} \cap B)$$

for every $t \in T'$ and every $B \in \mathcal{B}$. Using lemma 3.3 we have that for all $t \in T'$ $\{g_n < t\} \xrightarrow{\gamma_p} \{g < t\}$. Since $T'$ is dense in $\overline{I\!R}$ we conclude the necessity.

Suppose now that for a dense family $T \subseteq \overline{I\!R}$ we have $\{g_n < t\} \xrightarrow{\gamma_p} \{g < t\}$. We prove that conditions $1), 2), 3), 4)$ of lemma 3.2 are satisfied. From the fact that $g_n, g \leq \psi$ conditions 3) and 4) are satisfied. Following lemma 3.3 condition 2) is also satisfied, by eliminating again an at most countable family of $t \in I\!R$ such that $C_p(\{g_n > t\} \cap B_k) \neq C_p(\{g_n \geq t\} \cap B_k)$. It remains to prove 1).

Let us fix some $t > 0$ and set $K_t = \{\psi \geq t\}$ which is a quasi closed subset of $\Omega$. Since $g_n \leq \psi$ we get

$$C_p(\{g_n \geq t\} \cap K_t) = C_p(\{g_n \geq t\}).$$

The idea is to find a set $B \in \mathcal{B}$ "between" $K_t$ and $\Omega$; this is not immediately possible since $K_t$ is not closed but only quasi closed, nevertheless, for any $\varepsilon > 0$ there exists a closed set $K_\varepsilon \subseteq K_t$ such that $C_p(K_t \setminus K_\varepsilon) < \varepsilon$. Then

$$|C_p(\{g_n \geq t\} \cap K_t) - C_p(\{g_n \geq t\} \cap K_\varepsilon)| \leq C_p(K_t \setminus K_\varepsilon) < \varepsilon.$$

Choosing a set $B \in \mathcal{B}$ such that $K_\varepsilon \subseteq B \subseteq \Omega$ and for which

$$C_p(\{g_n > t\} \cap B) \to C_p(\{g > t\} \cap B)$$

we have

$$|C_p(\{g > t\}) - C_p(\{g_n > t\})| = |C_p(\{g > t\} \cap K_t) - C_p(\{g_n > t\} \cap K_t)| \leq$$

$$\leq |C_p(\{g > t\} \cap K_t) - C_p(\{g > t\} \cap B)| +$$

$$+|C_p(\{g > t\} \cap B) - C_p(\{g_n > t\} \cap B)| + |C_p(\{g_n > t\} \cap K_t) - C_p(\{g_n > t\} \cap B)|.$$

The first and the last term of the right hand side are less than $\varepsilon$ by the choice of $B$, and the middle term vanishes for $n \to \infty$. Hence we get

$$\lim_{n \to +\infty} C_p(\{g_n > t\} = C_p(\{g > t\}).$$

Therefore, the point 1) of lemma 3.2 is also proved and so the proof is concluded. $\square$

**Remark 4.3** *The family $\mathbb{R} \setminus T$ in the previous theorem can be replaced by a smaller one, as for example a dense set in $\mathbb{R}$.*

**Corollary 4.4** *Let $g_n, g \in X_\psi(\Omega)$. Then $g_n \xrightarrow{\gamma} g$ if and only if there exists a countable dense family $D \subseteq \mathbb{R}$ such that*

$$\{g_n < t\} \xrightarrow{\gamma_p} \{g < t\} \quad \forall t \in D. \tag{8}$$

**Proof** For the necessity see the first step of theorem 4.2. Conversely, suppose that (8) holds for a set $D \subseteq \mathbb{R}$ countable and dense. We prove that (8) holds for $t \in T$ where $\mathbb{R} \setminus T$ is at most countable (which implies, by theorem 4.2, that $g_n \xrightarrow{\gamma} g$).

For every $t \in \mathbb{R}$, possibly passing to subsequences, we have $w_{\{g_{n_k} < t\}} \rightharpoonup u_t$ weakly in $W_0^{1,p}(\Omega)$ for a suitable function $u_t$. It will be enough to prove that $u_t = w_{\{g<t\}}$ up to an at most countable set.

By assumption, $D$ is dense in $\mathbb{R}$ and so we can find $t_1, t_2 \in D$ with $t_1 \leq t \leq t_2$. Then

$$w_{\{g<t_1\}} \leq u_t \leq w_{\{g<t_2\}}. \tag{9}$$

Let us define

$$A_\tau = \{g < \tau\} \text{ for any } \tau \in D, \quad \tau \leq t,$$
$$B_\tau = \{g < \tau\} \text{ for any } \tau \in D, \quad \tau \geq t.$$

The sets $A_\tau$ and $B_\tau$ are quasi open and we have that $\{A_\tau\}_{\tau \in D, \tau \leq t}$ is increasing and $\{B_\tau\}_{\tau \in D, \tau \geq t}$ is decreasing with respect to the set inclusions. Now, from the theory of the $\gamma_p$-convergence (see [13], [19]), we have

$$A_\tau \xrightarrow{\gamma_p} \bigcup_{\tau \in D, \tau \leq t} A_\tau = A_t \quad \text{as } \tau \to t \tag{10}$$

and

$$w_{\{g<t\}} = \sup \{w_{\{g<\tau\}} : \tau \in D, \tau \leq t\}. \tag{11}$$

One has to pay attention to the fact that an arbitrary union of quasi open sets is not generally quasi open. In this case, the monotonicity plays an essential role. In fact any increasing sequence of quasi open sets is $\gamma_p$-convergent to their union. Moreover

$$\{g < t\} \subseteq B_\tau \quad \forall \tau \in D,\ \tau \geq t, \tag{12}$$

then

$$w_{\{g<t\}} \leq w_{\{g<\tau\}} \quad \forall \tau \in D,\ \tau \geq t, \tag{13}$$

from which it follows

$$w_{\{g<t\}} \leq \inf \{w_{\{g<\tau\}} : \quad \tau \in D,\ \tau \geq t\}. \tag{14}$$

Now from (9),...,(14) we have

$$\begin{aligned} w_{\{g<t\}} &= \sup \{w_{\{g<\tau\}} : \quad \tau \in D,\ \tau \leq t\} \\ &\leq u_t \leq \inf \{w_{\{g<\tau\}} : \quad \tau \in D,\ \tau \geq t\}. \end{aligned} \tag{15}$$

Let us consider the mapping $t \mapsto \|w_{\{g<t\}}\|_{L^p(\Omega)}$ which is increasing, and hence it has an at most countable points of discontinuity. If $t$ is a point where this mapping is continuous we have

$$\sup \{w_{\{g<\tau\}} : \quad \tau \in D,\ \tau \leq t\} = \inf \{w_{\{g<\tau\}} : \quad \tau \in D,\ \tau \geq t\}$$

and so from (15) we have $u_t = w_{\{g<t\}}$, which concludes the proof. $\square$

Let us introduce now the weak $\gamma$-convergence for sequences of obstacles.

**Definition 4.5** *We say that a sequence $\{g_n\}_{n\in I\!N}$ weak $\gamma$-converges to $g$ (and we write $g_n \stackrel{w\gamma}{\rightharpoonup} g$) if there exists a dense set $D \subseteq I\!R$ such that*

$$\{g_n < t\} \stackrel{w\gamma_p}{\rightharpoonup} \{g < t\} \quad \forall t \in D.$$

Remark that this is a correct definition in the sense that if $g_n \stackrel{w\gamma}{\rightharpoonup} g$ and $g_n \stackrel{w\gamma}{\rightharpoonup} g'$ then $g = g'$. Thanks to this, there is no ambiguity in the choice of the dense set $D$.

**Proposition 4.6** *Let $g_n, g \in X_\psi(\Omega)$. The following conditions are equivalent:*

*(i) there exists a dense family $D \in I\!R$ such that $\{g_n < t\} \stackrel{w\gamma_p}{\rightharpoonup} \{g < t\}$ for all $t \in D$;*

*(ii) there exists an at most countable family $N \in I\!R$ such that $\{g_n < t\} \stackrel{w\gamma_p}{\rightharpoonup} \{g < t\}$ for all $t \in I\!R \setminus N$.*

**Proof** If (ii) is true, obviously (i) follows. Conversely, let us suppose that (i) is true. For all $s \in D$ we have $w_{\{g_n<t\}} \rightharpoonup w_s$ in $W_0^{1,p}(\Omega)$, $\{w_s > 0\} = \{g < s\}$ and, from the upper semicontinuity of $g$, for all $t \in I\!R$,

$$\{g < t\} = \bigcup_{s\in D, s\leq t} \{g < s\}.$$

We define for any $t \in I\!R$ $w_t := \sup_{s\in D, s\leq t} w_s$. ¿From the monotonicity of the mapping $s \mapsto w_s$ the definition above is consistent and we have

$$\{w_t > 0\} = \bigcup_{s\in D, s\leq t} \{w_s > 0\} = \bigcup_{s\in D, s\leq t} \{g < s\} = \{g < t\}.$$

¿From the boundedness of $w_{\{g_n<t\}}$, possibly passing to a subsequence still denoted by the same indices, we have

$$w_{\{g_n<t\}} \rightharpoonup u_t.$$

So, to conclude, it is sufficient to prove that there exists an at most countable set $N$ such that $u_t(x) = w_t(x)$ $p$-q.e. for $t \in I\!R \setminus N$. We observe that, by monotonicity, for every $s \in D$ with $s \leq t$ we have

$$w_{\{g_n<s\}} \leq w_{\{g_n<t\}}, \tag{16}$$

so passing to the limit in (16) as $i \to +\infty$ we obtain

$$w_s \leq u_t \quad \forall s \in D, s \leq t,$$

from which

$$u_t \geq \sup_{s\in D, s\leq t} w_s = w_t. \tag{17}$$

On the other hand, again by the monotonicity, we also have

$$u_t \leq w_s \quad s \in D, s \geq t,$$

from which

$$u_t \leq \inf_{s\in D, s\geq t} w_s. \tag{18}$$

Now, denoting by $\tilde{w}_t$ the right hand side of (18), for $t \in I\!R \setminus N$, with $N$ at most countable, we have $w_t = \tilde{w}_t$ since the mapping $t \mapsto \|w_t\|_{L^p(\Omega)}$ is monotone increasing and it has only at most countable points of discontinuity. So we obtain from (17) and (18) that $u_t = w_t$ for every $t \in I\!R \setminus N$ which concludes the proof. □

**Proposition 4.7** *Suppose that $g_n \xrightarrow{\gamma} g$. Then $g_n \xrightarrow{w\gamma} g$.*

**Proof** It is an immediate consequence of theorem 4.2 and definition 4.5. □

The compactness is one of the key properties of the $w\gamma$-convergence. This is proved in the following.

**Proposition 4.8** *For every sequence $\{g_n\}_{n\in I\!N}$ of elements of $X_\psi(\Omega)$ there exist a subsequence $\{g_{n_k}\}$ and an obstacle $g \in X_\psi(\Omega)$ such that $g_{n_k} \xrightarrow{w\gamma} g$.*

**Proof** Consider an enumeration $\{r_1, r_2, ...\}$ of the set $\mathbb{Q}$ of rational numbers. For the level $r_1$ there exists a subsequence of $\{g_n\}_{n\in I\!N}$ (still denoted with the same indices) such that $w_{\{g_n<r_1\}} \rightharpoonup w_{r_1}$ weakly in $W_0^{1,p}(\Omega)$. For the level $r_2$ there exists a subsequence of the previous one such that $w_{\{g_n<r_2\}} \rightharpoonup w_{r_2}$ weakly in $W_0^{1,p}(\Omega)$. In such a way for any $r_k \in \mathbb{Q}$ one can extract a subsequence (of the sequence established for $r_{k-1}$) such that $w_{\{g_n<r_k\}} \rightharpoonup w_{r_k}$ weakly in $W_0^{1,p}(\Omega)$. By a diagonal procedure we can choose an element of the first sequence such that

$$\|w_{\{g_{n_1}<r_1\}} - w_{r_1}\|_{L^p(\Omega)} < 1,$$

then a second element of the second sequence such that

$$\|w_{\{g_{n_2}<r_1\}} - w_{r_1}\|_{L^p(\Omega)} < \frac{1}{2} \text{ and } \|w_{\{g_{n_2}<r_2\}} - w_{r_2}\|_{L^p(\Omega)} < \frac{1}{2},$$

and continuing this procedure, one construct a subsequence $\{g_{n_i}\}_i$ such that

$$\forall k \in I\!N \quad w_{\{g_{n_i}<r_k\}} \overset{L^p(\Omega)}{\to} w_{r_k} \quad \text{for} \quad i \to \infty.$$

This means exactly that for all $t \in \mathbb{Q}$

$$\{g_{n_i} < t\} \overset{w\gamma_p}{\rightharpoonup} \{w_t > 0\} \quad \text{for} \quad i \to \infty.$$

One define then the limit obstacle through its level sets

$$\{g < t\} = \{w_t > 0\} \quad \text{for all} \quad t \in \mathbb{Q}. \tag{19}$$

If $t_1, t_2 \in \mathbb{Q}$ with $t_1 < t_2$ then obviously $\{g < t_1\} \subseteq \{g < t_2\}$. The function $g$ is defined by

$$g(x) = \inf\{t \in \mathbb{Q} : x \in \{g < t\}\}.$$

Hence for every $t \in I\!R \setminus \mathbb{Q}$ we have

$$\{g < t\} = \bigcup_{s\in\mathbb{Q}, s<t} \{g < s\}. \tag{20}$$

One can see that

(i) $g$ is correctly defined.

(ii) $g$ is quasi upper semicontinuous, because its level sets $\{g < t\}$ are quasi open, being by definition, countable unions of quasi open sets (see (19),(20)).

(iii) $g \in X_\psi(\Omega)$, because $\{\psi < t\} \supseteq \{g < t\}$ for every $t \in I\!R$. Indeed, let us consider first the case $t \in \mathbb{Q}$. From hypothesis we have $g_{n_i} \le \psi$, for all $i$, hence

$$\{\psi < t\} \supseteq \{g_{n_i} < t\}, \quad \forall i \in I\!N,$$

from which

$$w_{\{g_{n_i}<t\}} \le w_{\{\psi<t\}}.$$

Passing into the limit as $i \to +\infty$, we get

$$w_t \leq w_{\{\psi<t\}},$$

so that

$$\{w_{\{\psi<t\}} > 0\} = \{\psi < t\} \supseteq \{w_t > 0\} = \{g < t\}.$$

Fix now $t \in I\!\!R \setminus \mathbb{Q}$. We have

$$\{g < t\} = \bigcup_{s<t, s\in\mathbb{Q}} \{g < s\} \subseteq \bigcup_{s<t, s\in\mathbb{Q}} \{\psi < s\} \subseteq \{\psi < t\}$$

which shows that $g \in X_\psi(\Omega)$.

To conclude, it remains to prove that $g_{n_i} \xrightarrow{w\gamma} g$, or equivalently, that there exists a dense set $D \subseteq I\!\!R$ such that

$$w_{\{g_{n_i}<t\}} \rightharpoonup w_t, \quad \{w_t > 0\} = \{g < t\}, \quad \forall t \in D.$$

If $t \in \mathbb{Q}$ the above is true from the definition of $g$, which concludes the proof. $\square$

In order to prove assumption (A) for the $w\gamma$ and $\gamma$-convergences, we give a technical result for domains, necessary for the construction of the $\gamma$-convergent sequence deriving from the $w\gamma$-convergent one.

**Lemma 4.9** *Let us consider two sequences of quasi open sets $X_n \xrightarrow{\gamma_R} X$ and $Y_n \xrightarrow{\gamma_R} Y$. Then $X_n \cap Y_n \xrightarrow{\gamma_R} X \cap Y$.*

**Proof** The proof consists in verifying the two Mosco conditions. First, consider a weak convergent sequence $w_n \in W_0^{1,p}(X_n \cap Y_n)$, $w_n \rightharpoonup w$. By hypotheses we get $w \in W_0^{1,p}(X) \cap W_0^{1,p}(Y)$ which means $w \in W_0^{1,p}(X \cap Y)$. Consider now some $w \in W_0^{1,p}(X \cap Y)$, and $w \geq 0$ (if $w$ changes sign we apply the argument to its positive and negative parts $w^+$ and $w^-$). There exists a sequence $x_n \in W_0^{1,p}(X_n)$ and a sequence $y_n \in W_0^{1,p}(Y_n)$ strongly convergent to $w$. These sequences can be chosen positive. Then $\min\{x_n, y_n\} \in W_0^{1,p}(X_n \cap Y_n)$ and it converges strongly to $w$. $\square$

Assumption (A) is given by the following result.

**Theorem 4.10** *Consider a sequence of obstacles $\{g_n\}_{n\in I\!\!N} \in X_\psi(\Omega)$ such that $g_n \xrightarrow{w\gamma} g$. There exists a subsequence $\{g_{n_k}\}_k$ and a sequence $\{f_k\}_k$ with $f_k \leq g_{n_k}$ such that $f_{n_k} \xrightarrow{\gamma} g$.*

**Proof** We prove this assertion in two steps. We consider first the particular case of obstacles whose ranges are in finite set. Let us consider the numbers

$$-\infty = l_1 < l_2 < \ldots < l_q$$

and the special family of obstacles

$$\mathcal{A}[l_1, \ldots, l_q] = \{g \in X_\psi(\Omega) \; : \; g(x) \in \{l_1, \ldots, l_q\}\}.$$

We construct the sets

$$A_1(g) = \{g < l_2\},\ \ A_2(g) = \{g < l_3\}, \dots, A_{q-1}(g) = \{g < l_q\}$$

which are quasi open and such that $A_1 \subseteq A_2 \subseteq \dots \subseteq A_{q-1}$. Consider now a sequence $\{g_n\}_{n\in I\!N} \in \mathcal{A}[l_1, \dots, l_q]$ which weak $\gamma$-converges to a function $g$. To every function $g_n$ we associate as before the sets $g_n \to A_1(g_n), \dots, A_{q-1}(g_n)$ and, using the compactness of the weak $\gamma_p$-convergence for sets, we can write (for a subsequence still denoted with the same indices)

$$A_i(g_n) \stackrel{w\gamma_p}{\rightharpoonup} A_i \quad \forall i = 1, \dots, q-1.$$

By the definition of the weak $\gamma$-convergence we have

$$g = l_1 \text{ on } A_1, \quad g = l_2 \text{ on } A_2 \setminus A_1, \dots, g = l_q \text{ on } \Omega \setminus (A_1 \cup \dots \cup A_{q-1}).$$

Moreover, $g$ is quasi upper semicontinuous and $g \le \psi$. For this last inequality it is sufficient to prove for any $i = 2, \dots, q$ that $\{g < l_i\} \subseteq \{\psi < l_i\}$. This follows from the fact that $\{g_n < l_i\} \subseteq \{\psi < l_i\}$ and from the properties of the weak $\gamma_p$-convergence of sets.

We construct now the sequence $f_n$ of admissible obstacles such that $f_n \stackrel{\gamma}{\to} g$ with $f_n \le g_n$. Using lemma 3.5 for subsequences still denoted with the same indices there exist sets $G_i^n \supseteq A_i(g_n)$ such that $G_i^n \stackrel{\gamma_p}{\to} A_i$. For fixed $n$, the sets $G_1^n, \dots, G_{q-1}^n$ are not ordered. Therefore one apply lemma 4.9 and consider

$$\tilde{G}_1^n = G_1^n \cap G_2^n \cap \dots \cap G_{q-1}^n \tag{21}$$

$$\tilde{G}_2^n = G_2^n \cap G_3^n \cap \dots \cap G_{q-1}^n \tag{22}$$

$$\dots \tag{23}$$

$$\tilde{G}_{q-1}^n = G_{q-1}^n. \tag{24}$$

Then $\tilde{G}_1^n \subseteq \tilde{G}_2^n \subseteq \dots \subseteq \tilde{G}_{q-1}^n$ and $\tilde{G}_i^n \stackrel{\gamma_p}{\to} A_i$. Moreover, $\tilde{G}_i^n \supseteq A_i(g_n)$. Defining the obstacle $f_n$ with $\tilde{G}_1^n, \dots, \tilde{G}_{q-1}^n$ in the following way: $f_n = l_1$ on $\tilde{G}_1^n$, $f_n = l_2$ on $\tilde{G}_2^n \setminus \tilde{G}_1^n, \dots,$ $f_n = l_q$ on $\Omega \setminus \tilde{G}_{q-1}^n$, we get that $f_n$ is quasi upper semicontinuous, $f_n \le g_n$ and $f_n \stackrel{\gamma}{\to} g$.

In a second step of the proof, a general obstacle is approached by obstacles with a finite range. We define for any $k \in I\!N$ the family of levels

$$\mathcal{R}_k = \{l_1, \dots, l_k\} \cup -\infty$$

where $\{l_k\}_k$ is a dense set in $\overline{I\!R}$ with the property that the weak $\gamma_p$-convergence holds on levels $l_i$. We have $\mathcal{R}_{k_1} \subseteq \mathcal{R}_{k_2}$ if $k_1 \le k_2$. Denote by $\mathcal{R} = \bigcup_{k\in I\!N} \mathcal{R}_k$.

For some obstacle $g \in X_\psi(\Omega)$ we define the truncation on $\mathcal{R}_k$ of $g$ in the following way

$$T_k(g)(x) = \sup\{l \in \mathcal{R}_k : g(x) \ge l\}.$$

Obviously $T_k(g) \in \mathcal{A}[\mathcal{R}_k]$ and $T_k(g) \le g$. Moreover, if $l \in \mathcal{R}_k$ then for all $k' \ge k$ we have

$$\{T_{k'}(g) < l\} = \{g < l\}.$$

As in the first step, for $k = 1$ there exists a subsequence, still denoted with the same indices, such that

$$T_1(g_n) \overset{w\gamma}{\rightharpoonup} T_1(g);$$

we consider a subsequence and a sequence $f_n^1 \leq T_1(g_n)$ with $f_n^1 \overset{\gamma}{\to} T_1(g)$. For $k = 2$ there exists (as in step one) a subsequence such that

$$T_2(g_n) \overset{w\gamma}{\rightharpoonup} T_2(g);$$

again, we consider a subsequence and a sequence $f_n^2 \leq T_2(g_n)$ with $f_n^2 \overset{\gamma}{\to} T_2(g)$. We continue this procedure for any $k \in \mathbb{N}$ and we chose a diagonal sequence $\{f_{n_k}^k\}_k$ with the property that $d_\gamma(f_{n_k}^k, T_k(g)) \leq \frac{1}{k}$. Here by $d_\gamma$ one denotes the distance which generates the same topology of the $\gamma$-convergence (see [14] for the metrizability of the $\gamma$-convergence). On the other hand, using theorem 4.2 we have $T_k(g) \overset{\gamma}{\to} g$ and therefore we found a subsequence $\{f_{n_k}^k\}_k$ which satisfies the desired properties. $\square$

Since assumption (A) is fullfiled, the general framework presented in section 2 and in particular theorem 2.2 could be applied. Nevertheless, in the case of obstacles the integral constraint plays a very important role.

# 5 Lower semicontinuity of the constraint and proof of theorem 2.3

In order to prove that an integral constraint of the type

$$\int_\Omega g\, dx \geq c$$

is weak $\gamma$-compact, we give the following.

**Proposition 5.1** *Let $g_n, g \in X_\psi(\Omega)$, with $g_n \overset{w\gamma}{\rightharpoonup} g$. Then*

$$g(x) \geq \limsup_{n\to\infty} g_n(x) \quad \textit{for a.e. } x \in \Omega.$$

**Proof** Indeed, consider some $x \in \Omega$ and $l \in \mathbb{R}$ with $g(x) < l$. From the weak $\gamma$-convergence, there exists some $l'$ between $g(x)$ and $l$, such that

$$\{g_n < l'\} \overset{w\gamma_p}{\rightharpoonup} \{g < l'\};$$

then $1_{\{g<l'\}} \leq \liminf_{n\to\infty} 1_{\{g_n<l'\}}$, where we denote by $1_C$ the function such that $1_C(x) = 1$ if $x \in C$ and and $1_C(x) = 0$ otherwise. In particular, this means that $g_n(x) < l'$ for $n$ large enough, that is

$$\limsup_{n\to\infty} g_n(x) \leq l'.$$

Since $l$ was arbitrary, and $g(x) < l' < l$ the proof is concluded. $\square$

**Proposition 5.2** *Let $g_n, g \in X_\psi(\Omega)$, such that $g_n \overset{w\gamma}{\rightharpoonup} g$. Then* $\int_\Omega g\,dx \geq \limsup_{n\to\infty} \int_\Omega g_n dx.$

**Proof** ¿From proposition 5.1 we have $g \geq \limsup_{n\to\infty} g_n$. The conclusion now follows by Fatou's lemma. □

**Remark 5.3** A constraint of the type $\int_\Omega \Phi(x, g(x))dx \geq c$ can replace the previous one in the general framework of theorem 2.2 provided the function $\Phi(x, \cdot)$ is increasing and upper semicontinuous.

Let us conclude the paper by giving the proof of theorem 2.3, which is derived from theorem 2.2 and proposition 5.1. We point out the fact that in theorem 2.3 the constraint is given by an equality, while in theorem 2.2 the general form of the constraint is expressed by an inequality.

**Proof of theorem 2.3:** Consider a minimizing sequence $\{g_n\}_{n\in I\!N}$ of admissible obstacles. By theorem 2.2 one can extract a subsequence such that $g_n \overset{w\gamma}{\rightharpoonup} g$ where $w\gamma$ is defined by definition 4.5, and

$$F(g) \leq \liminf_{n\to\infty} F(g_n).$$

By proposition 5.2 we have $\int_\Omega g\,dx \geq c$. If $\int_\Omega g\,dx = c$ then the obstacle $g$ is admissible and gives the minimum. If $\int_\Omega g\,dx > c$ then the new obstacle $\tilde{g}$ defined by

$$\tilde{g}(x) = g(x) - \frac{1}{|\Omega|}\Big(\int_\Omega g(y)dy - c\Big)$$

is admissible (i.e. $\tilde{g} \in X_\psi(\Omega)$ and $\int_\Omega \tilde{g}\,dx = c$) and using the monotonicity of $F$ we get $F(\tilde{g}) \leq F(g)$. □

# 6 Further remarks and examples

The results of this paper can be extended by analogy to the bilateral obstacle problem. In order to consider bilateral problems, we define the family

$$\tilde{X}_\psi(\Omega) = \{g : \Omega \to \overline{I\!R} : g \text{ quasi l.s.c. } g \geq \psi\}$$

with $\psi$ fixed in $W_0^{1,p}(\Omega)$; the associated convex sets are

$$\tilde{K}_g = \{u \in W_0^{1,p}(\Omega) : u \leq g\}.$$

In this case, the $\gamma$-convergence, respectively $w\gamma$-convergence, are defined as in definitions 3.1 and 4.5 by using the upper level sets: $\{g > t\}$. We denote them by $\tilde{\gamma}$, respectively $w\tilde{\gamma}$.

The new assumptions for theorem 2.3 are:

1. $F$ is lower semicontinuous with respect to the $\tilde{\gamma}$-convergence;

2. $F$ is monotone decreasing with respect to the usual order of functions.

We can consider then the bilateral problem. Fix first a function $\psi \in W_0^{1,p}(\Omega)$; the admissible set is now

$$X(\Omega) = \{(g_1, g_2) : g_i : \Omega \to \overline{I\!R}, i = 1, 2, \;\; g_1 \text{ quasi u.s.c.}, \;\; g_2 \text{ quasi l.s.c.}, g_1 \leq \psi \leq g_2\}.$$

We define

$$K_{g_1,g_2} = \{u \in W_0^{1,p}(\Omega) : g_1 \leq u \leq g_2\}$$

and consider a functional $F : X(\Omega) \to \overline{I\!R}$. Theorem 2.3 still holds if

1. $F(.,.)$ is lower semicontinuous with respect to the $(\gamma, \tilde{\gamma})$-convergence;

2. $F(.,.)$ is monotonously increasing with respect to the first variable and decreasing with respect to the second one.

In this case, the constraint is of the form

$$\int_\Omega g_1 dx = c, \quad \int_\Omega g_2 dx = \tilde{c},$$

with, of course, $c \leq \tilde{c}$.

The shape optimization problem solved in [9] for $p = 2$ is a particular case of the previous result and follows directly considering the family of obstacles

$$\mathcal{O}(\Omega) = \{(g_1, g_2) : \;\; g_1 = -\infty \cdot 1_A, \;\; g_2 = +\infty \cdot 1_A, \quad A \subseteq \Omega, \;\; A \;\; \text{quasi open}\}.$$

It is sufficient to see that $\mathcal{O}(\Omega)$ is closed with respect to the $(w\gamma, \tilde{w\gamma})$-convergence. In fact, a bilateral obstacle $(-\infty \cdot 1_A, +\infty \cdot 1_A)$ is identified with the quasi open set $A$. In this case, the bilateral obstacle problem becomes a Dirichlet problem with homogeneous boundary condition associated to the $p$-Laplacian on the quasi open set $A$.

Let consider $F : X_\psi(\Omega) \to \overline{I\!R}$ be a functional of the type

$$F(g) = \int_\Omega f(x, u_g) dx$$

where $u_g$ is the minimum point of

$$E_g(u) = \begin{cases} \frac{1}{p} \int_\Omega |Du|^p \, dx & \text{if } u \geq g, \\ +\infty & \text{otherwise,} \end{cases}$$

and $f(x, u)$ is a fixed non negative Carathèodory function, nondecreasing in $u$. Obviously $F$ is $\gamma$-lower semicontinuous and, from classical comparison principles and from the monotonicity assumption on $f$, it satisfies the monotonicity property, hence we can apply theorem 2.3 for this class of functionals.

Let now $\tilde{F} : W_0^{1,p}(\Omega) \to \overline{I\!R}$ be a given functional. A variational functional $F$ of the type

$$F(g) = \min_{u \in K_g} \tilde{F}(u)$$

has the monotonicity property. Moreover, if $\tilde{F}$ is weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$, then $F$ is $\gamma$-lower semicontinuous and hence one can apply theorem 2.3.

For this very particular class of functionals $F$, and if moreover one drops the constraint on the admissible obstacle $g \leq \psi$, the minimization problem

$$\min\left\{\tilde{F}(u)\ :\ u \in K_g,\ g \text{ quasi u.s.c.},\ \int_\Omega g\,dx \geq c\right\} \tag{25}$$

is equivalent to the following one

$$\min\left\{\tilde{F}(u)\ :\ u \in W_0^{1,p}(\Omega),\ \int_\Omega u\,dx \geq c\right\}. \tag{26}$$

Indeed, if $(g, u)$ is a solution for (25), then $u$ is a minimizer for (26). Conversely if $u^*$ minimizes for (26), then the couple $(u^*, u^*)$ is a minimizer for (25). Hence the existence of a solution for problem (25) derives from a solution of problem (26), which in this case can also be obtained by the direct method of the calculus of variations.

However, this method does not apply to more general situations when the optimization problem is not written in the form (25). For instance, in shape optimization, this would be the case of the second eigenvalue of the Laplace operator with Dirichlet boundary conditions.

An example in which the previous method works is the minimal energy associated to an obstacle. For some fixed $f \in L^q(\Omega)$ (here $q$ is the conjugate exponent of $p$) and for any admissible obstacle $g \in X(\Omega)$ the energy is given by

$$\tilde{F}(u) = \frac{1}{p}\int_\Omega |Du|^p dx - \int_\Omega f u\,dx. \tag{27}$$

Using the equivalence between (25) and (26), one can write the necessary optimality conditions for the solution $u^*$ of problem (27) performing directional derivatives and using the Lagrange multiplier in (26). For some positive constant $\lambda$ we then have

$$-\Delta_p u^* = f + \lambda \ \text{ in }\ W_0^{1,p}(\Omega). \tag{28}$$

The regularity of $u^*$ depends then on the regularity of $f$, hence for quite smooth data $f$ and $\Omega$, the optimal solution of (27) belongs to the class of regular functions.

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