Equivalence of the ellipticity conditions for geometric variational problems.

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Abstract

We exploit the so called *atomic condition*, recently defined by De Philippis, De Rosa, and Ghiraldin in [DPDRG18, Comm. Pure Appl. Math.] and proved to be necessary and sufficient for the validity of the anisotropic counterpart of the Allard rectifiability theorem. In particular, we address an open question of this seminal work, showing that the atomic condition implies the strict Almgren geometric ellipticity condition.

1 Introduction

Since the pioneering works of Almgren [Alm68, Alm76], a deep effort has been devoted to the understanding of ellptic integrands in geometric variational problems. In particular, Almgren introduced the class of elliptic geometric integrands ([Alm76, IV.1(7)] or [Alm68, 1.6(2)]), further denoted AUE, which allowed him to prove regularity for minimisers in [Alm68].

Very recently, an ongoing interest on the anisotropic Plateau problem has lead to a series of reformulations and results in this direction, see [HP17, DPDRG16, DLDRG17, DPDRG17, DR18, FK18]. In particular, in [DPDRG18] (see also Definition 4.7) a new ellipticity condition, called the *atomic condition*, further denoted AC, has been introduced and proved to be necessary and sufficient to get an Allard type rectifiability result for varifolds whose anisotropic first variation is a Radon measure. The authors can prove that, in co-dimension one and in dimension one, AC is equivalent to the strict convexity of the integrand.

For general co-dimension there is no understanding of AC in the literature and this is stated as an open problem in [DPDRG18, Page 2]:

> "Since the atomic condition AC is essentially necessary to the validity of the rectifiability theorem, it is relevant to relate it to the previous known notions of *ellipticity* (or *convexity*) of F with respect to the "plane" variable. This task seems to be quite hard in the general case."

The aim of this paper is to address this open question, comparing condition AC with the classical notion of geometric ellipticity introduced by Almgren. We present for the moment an informal version of our main result, see 8.8:

Theorem A. *If a $\mathscr{C}^1$ integrand satisfies the atomic condition at some point $x \in \mathbf{R}^n$, then it also satisfies the strict Almgren ellipticity condition at x; see 8.8.*

In particular, if the co-dimension equals one, then strict convexity of the integrand implies the strict Almgren ellipticity.

It is worth to remark that there is no hope of improving Theorem A showing that the atomic condition implies the uniform Almgren ellipticity condition, see Remark 9.25. Indeed, if this was the case, in co-dimension one the strict convexity of the integrand (which is equivalent to the atomic condition) would imply the uniform Almgren ellipticity, which in turn implies the uniform convexity, leading to a contradiction.

In order to prove Theorem A, we need to get several auxiliary results of independent interest. In particular, in Section 4 we introduce another ellipticity condition for integrands, named BC, and in Section 7 we prove that it is equivalent to AC; see Definition 4.8 and Lemma 7.1. BC has the advantage of being more geometric than the algebraic condition AC, thus providing a useful tool not only for the proof of Theorem A, but also for future further understanding of the atomic condition. In Section 5 we show that the original Almgren ellipticity condition [Alm76, IV.1(7)] is the same as the condition used in [FK18, 3.16] which involves unrectifiable surfaces; see Corollary 5.13. To this end we provide a deformation theorem 5.8 which preserves unrectifiability of the unrectifiable part of a given set; see Theorem 5.8. Moreover, in Section 6, Theorem 6.7, we provide an independent proof of the existence of minimisers of anisotropic energies which satisfies a weaker version of BC, improving the recent solutions to the set theoretical approach to the anisotropic Plateau problem [DPDRG17, FK18]. Gathering these results, we provide in Section 8 the proof of Theorem A, see Theorem 8.8.

The last crucial point is that the proof of Theorem A in Section 8 requires the validity of a seemingly harmless property: the class of compact sets X used by Almgren to test the strict ellipticity considition (see [Alm76, IV.1(7)] or [Alm68, 1.6(2)]) is closed under gluing together many rescaled copies of X; see 8.5. In 9.23 we show indeed that this property is true, but our proof is quite complicated and employs some sophisticated tools of algebraic topology; see also the introduction to Section 9. Giving it some thought, Almgren's condition that X cannot be retracted onto its boundary sphere is topological in nature, so it is reasonable that topological arguments are indispensable. Moreover, the existence of the Adams' surface, which is retractible onto its boundary and is obtained by gluing together two surfaces that cannot be retracted onto their respective boundaries, supports the claim that the proof of Almgren's class being closed under the gluing operation is highly non-trivial; see 8.6. This question is fully addressed in Section 9.

2 Notation

For the whole article we fix two integers d and n satisfying $2 \le d \le n$.

In principle we shall follow the notation of Federer; see [Fed69, pp. 669–671]. In particular, given two sets A, B, we denote with $A \sim B$ their *set-theoretic difference* and, for every $a \in \mathbf{R}^n$ and $s \in \mathbf{R}$ we define the functions $\boldsymbol{\tau}_a(x) = a + x$ and $\boldsymbol{\mu}_s(x) = sx$; see [Fed69, 2.7.16, 4.2.8]. Concerning varifolds, we shall follow Allard [All72].

Following [Alm68] and [Alm00], if $S \in \mathbf{G}(n, d)$ is a d dimensional linear subspace of $\mathbf{R}^n$, then $S_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ shall denote the *orthogonal projection* onto S. In particular, if $p \in \mathbf{O}^*(n, d)$ is such that $\operatorname{im} p^* = S$, then $S_\natural = p^* \circ p$.

We divert in notation from [Fed69] in the following ways. To denote the *image of a set* $A \subseteq X$ under some map $f : X \to Y$ (more generally, under a relation $f \subseteq X \times Y$) we always use square brackets: $f[A]$. We employ the symbol id_X to denote the *identity map*

$X \to X$ and $\mathbb{1}_A$ to denote the *characteristic function* $X \to \{0,1\}$ of $A \subseteq X$. We also use abbreviations for *intervals*, e.g., $(a,b] = \{t : a < t \le b\}$. Moreover, we denote with $\mathbb{N}$ the set of *non-negative integers*, i.e., $\mathbb{N} = \mathscr{P} \cup \{0\}$. If (X,ρ) is a metric space, $A \subseteq X$, and $x \in X$, then we define $\operatorname{dist}(x,A) = \inf \rho[A \times \{x\}]$. We sometimes write $X \hookrightarrow Y$, $X \twoheadrightarrow Y$, or $X \xrightarrow{\simeq} Y$ to emphasis that a map is *injective*, *surjective*, or *bijective* respectively. We denote with ∂A the *topological boundary* of a set A. Whenever A, B are subsets of a vector space we write $A+B$ to denote the *algebraic sum* of A and B, i.e., $A+B = \{a+b : a \in A\,,\, b \in B\}$; in particular, if $\varepsilon \in (0,\infty)$, then $A + \mathbf{B}(0,\varepsilon)$ is the *ε-thickening* of A. If R is a ring and A, B are R-modules, then $A \oplus B$ denotes their *direct sum*; cf. [ES52, Chap. V, Def. 5.6]. For $a, b \in \mathscr{P}$ the symbol $\gcd(a,b)$ denotes the *greatest common divisor* of a and b and $a \bmod b$ means the *remainder* of the division of a by b.

In Sections 8 and 9 we shall need to use tools of algebraic topology. We shall work in the category of all pairs of topological spaces $\mathfrak{a}_1$ as defined in [ES52, Chap. I, §1, p. 5]. We write $\mathbf{H}_k(X,A;G)$ and $\mathbf{H}^k(X,A;G)$ for the k^{th} *singular homology* and *cohomology groups* of the pair (X,A) with coefficients in G; see [ES52, Chap. VII, Definition 2.9]. If $G = \mathbf{Z}$, then we omit G in the notation. Similarly, if $A = \varnothing$, we omit A. Given two maps $f, g : X \to Y$ between topological spaces we write $f \approx g$ to express that f and g are *homotopic*, i.e., there exists a continuous map $h : [0,1] \times X \to Y$ such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$. If X and Y are topological spaces which are *homotopy equivalent* we write $X \approx Y$ and if they are *homeomorphic* we write $X \simeq Y$.

2.1 Definition (cf. [ES52, Chap. XI, Def. 4.1])**.** Let $B \subseteq \mathbf{R}^n$ be homeomorphic to the standard k-dimensional sphere and $f : B \to B$ be continuous. Suppose σ is the generator of the k^{th} homology group $\mathbf{H}_k(B)$ of B and $f_* : \mathbf{H}_k(B) \to \mathbf{H}_k(B)$ is the map induced by f. The *topological degree* $\deg(f) \in \mathbf{Z}$ *of* f is the unique integer such that $f_*(\sigma) = \deg(f) \cdot \sigma$.

3 Basic definitions

3.1 Definition (cf. [Alm68, 1.2])**.** A function $F : \mathbf{R}^n \times \mathbf{G}(n,d) \to (0,\infty)$ of class $\mathscr{C}^k$ for some non-negative integer k is called a *$\mathscr{C}^k$ integrand.*

If $\inf \operatorname{im} F / \sup \operatorname{im} F \in (0,\infty)$, then we say that F is *bounded.*

3.2 Definition (cf. [Alm68, 3.1])**.** If $\varphi \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and F is an integrand, then the *pull-back* integrand $\varphi^\# F$ is given by

$$\varphi^\# F(x,T) = \begin{cases} F\big(\varphi(x), \mathrm{D}\varphi(x)[T]\big) \|{\textstyle\bigwedge_d} \mathrm{D}\varphi(x) \circ T_\natural\| & \text{if } \dim \mathrm{D}\varphi(x)[T] = d \\ 0 & \text{if } \dim \mathrm{D}\varphi(x)[T] < d\,. \end{cases}$$

If φ is a diffeomorphism, then the *push-forward* integrand is given by $\varphi_\# F = (\varphi^{-1})^\# F$.

3.3 Definition (cf. [Alm68, 1.2])**.** If F is a $\mathscr{C}^k$ integrand and $x \in \mathbf{R}^n$, then we define the *frozen $\mathscr{C}^k$* integrand F^x by the formula

$$F^x(y,S) = F(x,S) \quad \text{for every } y \in \mathbf{R}^n \text{ and } S \in \mathbf{G}(n,d)\,.$$

3.4 Remark. Since $F : \mathbf{R}^n \times \mathbf{G}(n,d) \to (0,\infty)$ and $\mathbf{G}(n,d)$ is compact, it follows that for any $x \in \mathbf{R}^n$ the frozen integrand F^x is bounded.

3.5 Definition. We say that $S \subseteq \mathbf{R}^n$ is a *d-set* if S is $\mathscr{H}^d$ measurable and $\mathscr{H}^d(S \cap K) < \infty$ for any compact set $K \subseteq \mathbf{R}^n$.

3.6 Definition. Assume $S \subseteq \mathbf{R}^n$ is a d-set. We define

$$\mathcal{R}(S) = \{x \in S : \boldsymbol{\Theta}^d(\mathscr{H}^d \llcorner S, x) = 1\} \quad \text{and} \quad \mathcal{U}(S) = S \sim \mathcal{R}(S)\,.$$

3.7 Remark. Observe that $\boldsymbol{\Theta}^d(\mathscr{H}^d \llcorner S, \cdot)$ is a Borel function, so $\mathcal{R}(S)$ is $\mathscr{H}^d$ measurable. Employing [Mat75] and [Fed69, 2.9.11], we observe that $\mathcal{R}(S)$ is countably $(\mathscr{H}^d, d)$ rectifiable and $\mathcal{U}(S)$ is purely $(\mathscr{H}^d, d)$ unrectifiable.

3.8 Remark. Recall that $\boldsymbol{\gamma}_{n,d}$ denotes the canonical probability measure on $\mathbf{G}(n,d)$ invariant under the action of the orthogonal group $\mathbf{O}(n)$, also called Haar measure; see [Fed69, 2.7.16(6)].

3.9 Definition (cf. [All72, 3.5])**.** Assume $S \subseteq \mathbf{R}^n$ is a d-set. We define $\mathbf{v}_d(S) \in \mathbf{V}_d(\mathbf{R}^n)$ by setting for every $\alpha \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,d))$

$$\mathbf{v}_d(S)(\alpha) = \int_{\mathcal{R}(S)} \alpha(x, \operatorname{Tan}^d(\mathscr{H}^d \llcorner \mathcal{R}(S), x)) \,\mathrm{d}\mathscr{H}^d(x) + \int_{\mathcal{U}(S)} \int \alpha(x,T) \,\mathrm{d}\boldsymbol{\gamma}_{n,d}(T) \,\mathrm{d}\mathscr{H}^d(x)\,.$$

3.10 Definition. If F is a $\mathscr{C}^k$ integrand, we define the functional $\Phi_F : \mathbf{V}_d(\mathbf{R}^n) \to [0,\infty]$ by the formula

$$\Phi_F(V) = \int F(x,S) \,\mathrm{d}V(x,S)\,.$$

3.11 Remark. If $\operatorname{spt}\|V\|$ is compact we have $\Phi_F(V) = V(\gamma F)$, whenever $\gamma \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ is such that $\operatorname{spt}\|V\| \subseteq \gamma^{-1}\{1\}$.

3.12 Definition. If $S \subseteq \mathbf{R}^n$ is a d-set, then we define

$$\Phi_F(S) = \Phi_F(\mathbf{v}_d(S))\,,$$

$$\Psi_F(S) = \Phi_F(S) + \int_{\mathcal{U}(S)} \big(\sup \operatorname{im} F^x - \textstyle\int F(x,T) \,\mathrm{d}\boldsymbol{\gamma}_{n,d}(T)\big) \,\mathrm{d}\mathscr{H}^d(x)\,.$$

For any other subset S of $\mathbf{R}^n$, we define $\Psi_F(S) = \Phi_F(S) = \infty$.

3.13 Remark. Assume $V \in \mathbf{V}_d(\mathbf{R}^n)$, $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is of class $\mathscr{C}^1$, and F is a $\mathscr{C}^0$ integrand. Then

$$\Phi_{\varphi^\# F}(V) = \Phi_F(\varphi_\# V)\,.$$

If $S \subseteq \mathbf{R}^n$ is a d-set, then

$$\varphi_\# \mathbf{v}_d(S) = \mathbf{v}_d(\varphi[S])$$

in the case φ is injective and S is countably $(\mathscr{H}^d, d)$ rectifiable, or in the case $\varphi = \boldsymbol{\mu}_r$ for some $r \in (0,\infty)$, or in the case $\varphi = \boldsymbol{\tau}_a$ for some $a \in \mathbf{R}^n$.

3.14 Remark. If S is a d-set, F is a $\mathscr{C}^0$ integrand and $x \in \mathbf{R}^n$, then

$$\Psi_{F^x}(S) = \Phi_{F^x}(\mathcal{R}(S)) + \mathscr{H}^d(\mathcal{U}(S)) \sup \operatorname{im} F^x\,.$$

3.15 Definition. For any set X and an element $x \in X$ we denote by $\operatorname{Dirac}(x)$ the measure over X with a single atom at x, i.e.,

$$\operatorname{Dirac}(x)(A) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{if } x \notin A\,, \end{cases} \quad \text{for } A \subseteq X\,.$$

The choice of X shall always be clear from the context.

3.16 Definition (cf. [All72, 4.9])**.** Assume $U \subseteq \mathbf{R}^n$ is open, $V \in \mathbf{V}_d(U)$, F is a $\mathscr{C}^1$ integrand. We define the *first variation of* V *with respect to* F to be the linear map $\delta_F V : \mathscr{X}(U) \to \mathbf{R}$ given by the formula

$$\delta_F V(g) = \frac{d}{dt}\Big|_{t=0} \Phi_F\big((\varphi_t)_\# V\big) \,,$$

where $g \in \mathscr{X}(U)$ is a smooth compactly supported vectorfield in U and $\varphi_t(x) = x + tg(x)$ for $x \in U$ and t in some neighbourhood of 0 in $\mathbf{R}$.

3.17 Remark. Note that if $T \in \mathbf{G}(n,d)$ and

$$\mathcal{G}_{n,d} = \big\{ P_\natural : P \in \mathbf{G}(n,d) \big\} \subseteq \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \,,$$

then

$$A \in \operatorname{Tan}(\mathcal{G}_{n,d}, T_\natural) \quad \iff \quad A^* = A \,, \quad T_\natural \circ A \circ T_\natural = 0 \,, \quad \text{and} \quad T_\natural^\perp \circ A \circ T_\natural^\perp = 0 \,.$$

For $x \in \mathbf{R}^n$ and $T \in \mathbf{G}(n,d)$ define

$$F_T : \mathbf{R}^n \to \mathbf{R} \quad \text{and} \quad F_x : \mathcal{G}_{n,d} \to \mathbf{R} \quad \text{by setting} \quad F_T(x) = F(x,T) = F_x(T_\natural) \,.$$

In [DPDRG18] the authors computed

$$\delta_F V(g) = \int \big\langle g(x), \mathrm{D}F_T(x) \big\rangle + B_F(x,T) \bullet \mathrm{D}g(x) \, \mathrm{d}V(x,T) \,,$$

where $B_F(x,T) \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is characterised by

$$B_F(x,T) \bullet L = F(x,T) T_\natural \bullet L + \big\langle T_\natural^\perp \circ L \circ T_\natural + (T_\natural^\perp \circ L \circ T_\natural)^*, \mathrm{D}F_x(T_\natural) \big\rangle \,,$$

whenever $L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$.

4 Notions of ellipticity

In this section we recall the notions of ellipticity we will work with.

4.1 Definition. We say that (S, D) is a *test pair* if there exists $T \in \mathbf{G}(n,d)$ such that

$$D = T \cap \mathbf{B}(0,1) \,, \quad B = T \cap \partial \mathbf{B}(0,1) \,, \quad S \subseteq \mathbf{R}^n \text{ is compact} \,, \quad \mathscr{H}^d(S) < \infty \,,$$
$$f[S] \neq B \quad \text{for all } f : \mathbf{R}^n \to \mathbf{R}^n \text{ satisfying } \operatorname{Lip} f < \infty \text{ and } f(x) = x \text{ for every } x \in B \,.$$

We say that (S, D) is a *rectifiable test pair* if, in addition, S is $(\mathscr{H}^d, d)$ rectifiable.

4.2 Remark. Using a standard extension procedure for Lipschitz functions (e.g. [EG92, 3.1.1, Theorem 1]), one sees that the last condition in Definition 4.1 means exactly that B is not a Lipschitz retract of S.

4.3 Example. Let $n = 3$, $d = 2$, $T = \mathbf{R}^2 \times \{0\}$, $D = T \cap \mathbf{B}(0,1)$, and S be a smoothly embedded Möbius strip with boundary $B = T \cap \partial \mathbf{B}(0,1)$. Observe, that S itself has the homotopy type of a 1-dimensional circle because a Möbius strip can easily be retracted onto the "middle circle". However, the inclusion $j : B \hookrightarrow S$ has topological degree 2, so given any continuous map $f : S \to B$ we have $j \circ f = f|_B : B \to B$ and we see that $\deg(f|_B) = \deg(j) \deg(f)$ is an *even* integer which means that $f|_B$ cannot equal the identity on B. Therefore, (S, D) is a rectifiable test pair.

4.4 Lemma. *Let (S, D) be a pair of compact sets in $\mathbf{R}^n$ with $\mathscr{H}^d(S) < \infty$ and $\{(S_i, D_i) : i \in \mathbb{N}\}$ be a sequence of test pairs such that*

$$\lim_{i\to\infty} d_{\mathscr{H}}(S_i, S) = 0 \quad \textit{and} \quad \lim_{i\to\infty} d_{\mathscr{H}}(D_i, D) = 0\,.$$

Then (S, D) is a test pair.

Proof. For every $i \in \mathbb{N}$, let $T_i \in \mathbf{G}(n, d)$ be such that $D_i = T_i \cap \mathbf{B}(0, 1)$ and set $B_i = T_i \cap \partial\mathbf{B}(0, 1)$. First note that since $\{D_i : i \in \mathbb{N}\}$ is a Cauchy sequence with respect to the Hausdorff metric on compact sets, we obtain that $\{T_i : i \in \mathbb{N}\}$ is a Cauchy sequence in $\mathbf{G}(n, d)$ and there exists $T \in \mathbf{G}(n, d)$ such that $D = T \cap \mathbf{B}(0, 1)$. Set $B = T \cap \partial\mathbf{B}(0, 1)$.

Assume, by contradiction, that there exists $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $\operatorname{Lip} f < \infty$, $f(x) = x$ for every $x \in B$, and $f[S] = B$. Set $\delta = (\operatorname{Lip} f)^{-1} \in (0, 1]$. Then

$$f[S + \mathbf{B}(0, r)] \subseteq B + \mathbf{B}(0, r/\delta) \quad \text{for } r \in (0, \infty)\,.$$

Choose $i \in \mathbb{N}$ such that

$$S_i \subseteq S + \mathbf{B}(0, 2^{-5}\delta^2) \quad \text{and} \quad B \subseteq B_i + \mathbf{B}(0, 2^{-5}\delta)\,.$$

Then,

$$f[S_i] \subseteq B + \mathbf{B}(0, 2^{-5}\delta) \subseteq B_i + \mathbf{B}(0, 2^{-4}\delta)\,.$$

Define $g : S_i \to B_i$ by

$$\begin{aligned} g(y) &= f(y) \quad \text{for } y \in S_i \sim \big(B_i + \mathbf{B}(0, 2^{-4}\delta)\big)\,, \\ g(y) &= 2^4\delta^{-1} \operatorname{dist}(y, B_i)(f(y) - y) + y \quad \text{for } y \in S_i \cap \big(B_i + \mathbf{B}(0, 2^{-4}\delta)\big)\,. \end{aligned}$$

For any $y \in S_i$ with $\operatorname{dist}(y, B_i) \le 2^{-4}\delta$ we can find $x \in B_i$ and $z \in B$ such that $|x - y| \le 2^{-4}\delta$ and $|x - z| \le 2^{-5}\delta$; hence, $|y - z| \le 2^{-3}\delta$ and

$$\begin{aligned} \operatorname{dist}(g(y), B_i) &\le |g(y) - x| \le 2^4\delta^{-1} \operatorname{dist}(y, B_i)|f(y) - y| + |y - x| \\ &= |f(y) - f(z) + z - y| + |y - x| \le \delta^{-1}|y - z| + |z - y| + |y - x| \le 2^{-1}\,. \end{aligned}$$

This shows that $g[S_i] \subseteq B_i + \mathbf{B}(0, 2^{-1})$. Composing g with a Lipschitz map retracting $B_i + \mathbf{B}(0, 2^{-1})$ onto B_i yields a Lipschitz retraction of S_i onto B_i and a contradiction. □

4.5 Definition. Let $x \in \mathbf{R}^n$ and $\mathcal{P}$ be a set of pairs of compact d-sets in $\mathbf{R}^n$.

(a) *Almgren uniform ellipticity with respect to $\mathcal{P}$*: The class $\mathrm{AUE}_x(\mathcal{P})$ is defined to contain all $\mathscr{C}^0$ integrands F for which there exists $c > 0$ such that for all $(S, D) \in \mathcal{P}$ there holds

$$\Psi_{F^x}(S) - \Psi_{F^x}(D) \ge c\big(\mathscr{H}^d(S) - \mathscr{H}^d(D)\big)\,.$$

(b) *Almgren strict ellipticity with respect to $\mathcal{P}$*: The class $\mathrm{AE}_x(\mathcal{P})$ is defined to contain all $\mathscr{C}^0$ integrands F such that for all $(S, D) \in \mathcal{P}$ satisfying $\mathscr{H}^d(S) > \mathscr{H}^d(D)$ there holds

$$\Psi_{F^x}(S) - \Psi_{F^x}(D) > 0\,.$$

4.6 Remark. (a) If all elements of $\mathcal{P}$ are pairs of $(\mathscr{H}^d, d)$ rectifiable sets, then one can replace all occurrences of Ψ_{F^x} with Φ_{F^x}.

(b) If $\mathcal{P} = \varnothing$, then $\mathrm{AE}_x(\mathcal{P}) = \mathrm{AUE}_x(\mathcal{P})$ is the set of all $\mathscr{C}^0$ integrands.

(c) If $\mathcal{P}$ is the set of *rectifiable* test pairs, then $F \in \mathrm{AUE}_x(\mathcal{P})$ if and only if F is elliptic at x in the sense of [Alm76, IV.1(7)].

(d) If $\mathcal{P}$ is the set of *all* test pairs, then $F \in \mathrm{AUE}_x(\mathcal{P})$ if and only if F is elliptic at x in the sense of [FK18, 3.16].

4.7 Definition (cf. [DPDRG18, Definition 1.1])**.** Let $x \in \mathbf{R}^n$. The class AC_x is defined to contain all $\mathscr{C}^1$ integrands F satisfying the *atomic condition* at x, i.e., for any Radon probability measure μ over $\mathbf{G}(n,d)$, setting

$$A_x(\mu) = \int B_F(x,T)\,\mathrm{d}\mu(T) \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n),$$

there holds

(a) $\dim \ker A_x(\mu) \le n - d$;

(b) if $\dim \ker A_x(\mu) = n - d$, then $\mu = \mathrm{Dirac}(T_0)$ for some $T_0 \in \mathbf{G}(n,d)$.

To conclude, we introduce the following new notion of ellipticity, named BC. This will turn out to be equivalent to AC, see Lemma 7.1. Rephrasing AC as BC will be very useful for the proof of Theorem A and for a further understanding of AC. Indeed, Definition 4.8 is more geometric than the algebraic Definition 4.7, providing a better tool to relate AC with the other notions of ellipticity.

4.8 Definition. Let $x \in \mathbf{R}^n$. We define BC_x to be the class of all $\mathscr{C}^1$ integrands F such that for any $W \in \mathbf{V}_d(\mathbf{R}^n)$ of the form

$$W = (\mathscr{H}^k \llcorner T) \times \mu\,,$$

where μ is a Radon probability measure over $\mathbf{G}(n,d)$, $k \in \mathbb{N}$, and $T \in \mathbf{G}(n,k)$, there holds

(a) if $\delta_{F^x} W = 0$, then $k \ge d$,

(b) if $k = d$ and $\delta_{F^x} W = 0$, then $\mu = \mathrm{Dirac}(T)$.

The class wBC_x is defined by omitting condition (a).

5 Rectifiability of test pairs

Let $x \in \mathbf{R}^n$, $\mathcal{P}_1$ be the set of *all* test pairs, and $\mathcal{P}_2$ be the set of *rectifiable* test pairs. Here we prove (see Corollary 5.13) that $\mathrm{AE}_x(\mathcal{P}_1) = \mathrm{AE}_x(\mathcal{P}_2)$ and $\mathrm{AUE}_x(\mathcal{P}_1) = \mathrm{AUE}_x(\mathcal{P}_2)$, i.e., that the original Almgren's definition of ellipticity [Alm76, IV.1(7)] coincides with the definition used in [FK18, 3.16]. To this end we need to show an improved version of the deformation theorem, see 5.8. In contrast to similar theorems of Federer and Fleming [Fed69, 4.2.6-9], David and Semmes [DS00, Theorem 3.1], or Fang and Kolasiński [FK18, 7.13], this one has the special feature of preserving the unrectifiability of the purely unrectifiable part of the deformed set.

First, we introduce some notation (modelled on [Alm86]) needed to deal with cubes and cubical complexes.

5.1 Definition. Let $k \in \{0, 1, \ldots, n\}$ and $Q = [0,1]^k \subseteq \mathbf{R}^k$. We say that $R \subseteq \mathbf{R}^n$ is a *cube* if there exist $p \in \mathbf{O}^*(n,k)$, $o \in \mathbf{R}^n$ and $l \in (0,\infty)$ such that $R = \boldsymbol{\tau}_o \circ p^* \circ \boldsymbol{\mu}_l[Q]$. We call $\mathbf{o}(R) = o$ the *corner* of R and $\mathbf{l}(R) = l$ the *side-length* of R. We also set

- $\dim(R) = k$ – the *dimension* of R,
- $\mathbf{c}(R) = \mathbf{o}(R) + \frac{1}{2}\mathbf{l}(R)(1, 1, \ldots, 1)$ – the *centre* of R,
- $\partial_{\mathrm{c}} R = \boldsymbol{\tau}_{\mathbf{o}(R)} \circ p^* \circ \boldsymbol{\mu}_{\mathbf{l}(R)}[\partial Q]$ – the *boundary* of R,
- $\mathrm{Int}_{\mathrm{c}}(R) = R \sim \partial_{\mathrm{c}} R$ – the *interior* of R.

5.2 Definition. Let $k \in \{0, 1, \ldots, n\}$, $N \in \mathbf{Z}$, $Q = [0,1]^k \subseteq \mathbf{R}^k$, $e_1, \ldots, e_n$ be the standard basis of $\mathbf{R}^n$, and $f_1, \ldots, f_k$ be the standard basis of $\mathbf{R}^k$.

We define $\mathbf{K}^n_k(N)$ to be the set of all cubes $R \subseteq \mathbf{R}^n$ of the form $R = \boldsymbol{\tau}_v \circ p^* \circ \boldsymbol{\mu}_{2^{-N}}[Q]$, where $v \in \boldsymbol{\mu}_{2^{-N}}[\mathbf{Z}^n]$ and $p \in \mathbf{O}^*(n,k)$ is such that $p^*(f_i) \in \{e_1, \ldots, e_n\}$ for $i = 1, 2, \ldots, k$.

We also set

$$\mathbf{K}^n_k = \bigcup \big\{ \mathbf{K}^n_k(N) : N \in \mathbf{Z} \big\}, \quad \mathbf{K}^n = \mathbf{K}^n_n, \quad \mathbf{K}^n_* = \bigcup \big\{ \mathbf{K}^n_k : k \in \{0, 1, \ldots, n\} \big\}.$$

5.3 Definition. Let $k \in \{0, 1, \ldots, n\}$, $N \in \mathbf{Z}$, and $K \in \mathbf{K}^n_k(N)$. We say that $L \in \mathbf{K}^n_*$ is a *face* of K if and only if $L \subseteq K$ and $L \in \mathbf{K}^n_j(N)$ for some $j \in \{0, 1, \ldots, k\}$.

5.4 Definition (cf. [Alm86, 1.5])**.** A family of top-dimensional cubes $\mathcal{F} \subseteq \mathbf{K}^n$ is said to be *admissible* if

(a) $K, L \in \mathcal{F}$ and $K \neq L$ implies $\mathrm{Int}_{\mathrm{c}}(K) \cap \mathrm{Int}_{\mathrm{c}}(L) = \varnothing$,

(b) $K, L \in \mathcal{F}$ and $K \cap L \neq \varnothing$ implies $\frac{1}{2} \le \mathbf{l}(L)/\mathbf{l}(K) \le 2$,

(c) $K \in \mathcal{F}$ implies $\partial_{\mathrm{c}} K \subseteq \bigcup \{ L \in \mathcal{F} : L \neq K \}$.

5.5 Definition (cf. [Alm86, 1.8])**.** Let $\mathcal{F} \subseteq \mathbf{K}^n$ be admissible. We define the *cubical complex* $\mathbf{CX}(\mathcal{F})$ *of* $\mathcal{F}$ to consist of all those cubes $K \in \mathbf{K}^n_*$ for which

- K is a face of some cube in $\mathcal{F}$,
- if $\dim(K) > 0$, then $\mathbf{l}(K) \le \mathbf{l}(L)$ whenever L is a face of some cube in $\mathcal{F}$ with $\dim(K) = \dim(L)$ and $\mathrm{Int}_{\mathrm{c}}(K) \cap \mathrm{Int}_{\mathrm{c}}(L) \neq \varnothing$.

5.6 Definition. Let $k \in \mathbb{N}$, $Q \subseteq \mathbf{R}^k$ be closed convex with non-empty interior, and $a \in \mathrm{Int}\, Q$. We define the *central projection from a onto ∂Q* to be the locally Lipschitz map $\pi_{Q,a} : \mathbf{R}^k \sim \{a\} \to \mathbf{R}^k$ characterised by

$$\pi_{Q,a}(x) \in \partial Q \quad \text{and} \quad \frac{\pi_{Q,a}(x) - a}{|\pi_{Q,a}(x) - a|} = \frac{x - a}{|x - a|} \quad \text{for } x \in \mathrm{Int}\, Q \sim \{a\},$$
$$\pi_{Q,a}(x) = x \quad \text{for } x \in \mathbf{R}^k \sim \mathrm{Int}\, Q.$$

The following lemma is a counterpart of [Fed69, 4.2.7].

5.7 Lemma. *Assume*

$$k, N \in \mathbb{N}\,, \quad d < k \le n\,, \quad Q \subseteq \mathbf{R}^n \text{ is a cube}\,, \quad p \in \mathbf{O}^*(n,k)\,, \quad \operatorname{im} p^* = \operatorname{Tan}(Q, \mathbf{c}(Q))\,,$$
$$\mu_1, \ldots, \mu_N \text{ are Radon measures over } \mathbf{R}^n\,, \quad \Sigma = Q \cap \textstyle\bigcup_{i=1}^N \operatorname{spt} \mu_i\,, \quad \mathscr{H}^d(\Sigma) < \infty\,.$$

There exist $\Gamma = \Gamma(d,k,N)$ and $a \in Q$ such that

$$\operatorname{dist}(a, \Sigma) > 0\,, \quad \operatorname{dist}(a, \partial_{\mathrm{c}} Q) > \tfrac{1}{4}\mathbf{l}(Q)\,, \quad \int_Q \|\mathrm{D}(\pi_{Q,a} \circ p)\|^d \,\mathrm{d}\mu_i \le \Gamma \mu_i(Q) \quad \forall i \in \{1, \ldots, N\}\,.$$

Moreover, if $A \subseteq \Sigma$ is purely $(\mathscr{H}^d, d)$ unrectifiable, then $p^* \circ \pi_{Q,a} \circ p[A]$ is purely $(\mathscr{H}^d, d)$ unrectifiable.

Proof. Without loss of generality we shall assume $n = k$. Recall Definition 3.6 and Remark 3.7 and let $E = \mathcal{U}(\Sigma)$. Employing [Feu09, Lemma 6] with δ, E, d, k replaced by Q, E, d, k, we see that $\mathscr{H}^k(B) = 0$, where

$$B = \big\{ a \in Q : \pi_{Q,a}[E] \text{ is not purely } (\mathscr{H}^d, d) \text{ unrectifiable} \big\}\,.$$

Set $Q_0 = \{x \in Q : \operatorname{dist}(x, \partial_{\mathrm{c}} Q) > \frac{1}{4}\mathbf{l}(Q)\}$. From [FK18, 6.4] we deduce that there exists $\Gamma_0 = \Gamma_0(k) > 1$ such that

$$\|\mathrm{D}\pi_{Q,a}(x)\| \le \Gamma_0 |x - a|^{-1} \quad \text{for all } a \in Q_0 \text{ and all } x \in \mathbf{R}^k \sim \{a\}\,.$$

Since $d < k$, there exists $\Delta = \Delta(d,k) \in (0, \infty)$ such that for all $a \in \operatorname{Int} Q$ there holds $\int_Q |x - a|^{-d} \,\mathrm{d}\mathscr{H}^k(a) < \Delta$. Using the Fubini theorem [Fed69, 2.6.2] and arguing as in [FK18, 7.10] or in [Fed69, 4.2.7], we find out that there exists $\Gamma_1 = \Gamma(d,k,N)$ such that $\mathscr{H}^k(A) > 0$, where

$$A = \left\{ a \in Q_0 : \int_Q |x - a|^{-d} \,\mathrm{d}\mu_i(x) \le \Gamma_1 \mu_i(Q) \quad \text{for } i \in \{1, 2, \ldots, N\} \right\}\,.$$

We have $\mathscr{H}^k(\Sigma) = 0$ so $\mathscr{H}^k(A \sim \Sigma) > 0$. Hence, there exists $a \in A \sim (B \cup \Sigma)$ with all the desired properties. □

5.8 Theorem. *Assume*

$$\mathcal{F} \subseteq \mathbf{K}^n \text{ is admissible}\,, \quad \mathcal{A} \subseteq \mathcal{F} \text{ is finite}\,, \quad S \subseteq \mathbf{R}^n \text{ is a } d\text{-set}\,, \quad I = [0,1]\,,$$
$$J = [0,2]\,, \quad G = \operatorname{Int} \textstyle\bigcup \mathcal{A}\,, \quad \mathscr{H}^d(\bigcup \mathcal{A} \cap \operatorname{Clos} S) < \infty\,, \quad R = \mathcal{R}(S)\,, \quad U = \mathcal{U}(S)\,.$$

There exist $\Gamma = \Gamma(n,d) \in (1, \infty)$, a Lipschitz map $f : J \times \mathbf{R}^n \to \mathbf{R}^n$, a finite set $\mathcal{B} \subseteq \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}^n_d$, and an open set $V \subseteq \mathbf{R}^n$ such that

$$f(t,x) = x \quad \text{for } (t,x) \in \big(\{0\} \times \mathbf{R}^n\big) \cup \big(J \times (\mathbf{R}^n \sim G) \cup \textstyle\bigcup \mathcal{B}\big) \cup \big(I \times \bigcup(\mathbf{CX}(\mathcal{F}) \cap \mathbf{K}^n_d)\big)\,,$$
$$S \subseteq V\,, \quad f[J \times Q] \subseteq Q \quad \text{for } Q \in \mathcal{A}\,, \quad f[\{1\} \times V] \cap G \subseteq \textstyle\bigcup\big(\mathbf{CX}(\mathcal{F}) \cap \mathbf{K}^n_d\big)\,,$$
$$f[\{2\} \times V] \cap G = \textstyle\bigcup \mathcal{B} \cap G\,, \quad f[I \times (V \cap G)] \subseteq \bigcup \mathcal{A}\,,$$
$$\mathscr{H}^d(f(1,\cdot)[R \cap G]) \le \Gamma \mathscr{H}^d(R \cap G)\,, \quad \mathscr{H}^d(f(1,\cdot)[U \cap G]) \le \Gamma \mathscr{H}^d(U \cap G)\,,$$
$$\mathscr{H}^d(f(1,\cdot)[U] \cap G) = 0\,, \quad f(1,\cdot)[U] \text{ is purely } (\mathscr{H}^d, d) \text{ unrectifiable}\,,$$
$$f(2,\cdot)[f[J \times V]] = f[\{2\} \times V] \text{ and } f[\{2\} \times V] \text{ is a strong deformation retract of } f[J \times V]\,.$$

Proof. For each $Q \in \mathbf{CX}(\mathcal{F})$ we find $p_Q \in \mathbf{O}^*(n, \dim Q)$ such that $Q \subseteq \mathbf{c}(Q) + \operatorname{im} p_Q^*$. For $k \in \{0, 1, 2, \ldots, n\}$ set

$$\mathcal{A}_k = \left\{ Q \in \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}^n_k : Q \cap G \neq \varnothing \right\} .$$

We shall perform a central projection inside the cubes of $\mathcal{A}_k$ for $k = n, n-1, \ldots, d+1$. Note that $\partial G \cap \bigcup \mathcal{A}_k \neq \partial G$ for $k < n$. In fact, all the projections shall equal identity on ∂G.

Let us set

$$\mu_{1,n} = \mathscr{H}^d \llcorner (R \cap G) , \quad \mu_{2,n} = \mathscr{H}^d \llcorner (U \cap G) , \quad \mu_{3,n} = \mathscr{H}^d \llcorner (S \cap G) ,$$
$$E = \mathbf{R}^n \sim G , \quad \varphi_n(x) = \psi_n(t,x) = x \quad \text{for } (t,x) \in I \times \mathbf{R}^n , \quad \delta_{n+1} = 1 , \quad Z_{n+1} = \mathbf{R}^n .$$

For $k \in \{n-1, n-2, \ldots, d\}$ and $i \in \{1,2,3\}$ we shall define Lipschitz maps $\psi_k : I \times \mathbf{R}^n \to \mathbf{R}^n$ and $\varphi_k : \mathbf{R}^n \to \mathbf{R}^n$, Radon measures $\mu_{i,k}$ over $\mathbf{R}^n$, sets $Z_{k+1} \subseteq \bigcup \mathcal{A}_{k+1} \cup E$, and numbers $\delta_{k+1} \in (0,1)$ satisfying

$$(1) \qquad \begin{cases} \operatorname{spt} \mu_{i,k} = \varphi_k[\operatorname{spt} \mu_{i,k+1}] \subseteq E \cup \bigcup \mathcal{A}_k , \quad \psi_k[I \times Z_{k+1}] = Z_{k+1} , \\ \left(\operatorname{spt} \mu_{i,k+1} + \mathbf{U}(0, \delta_{k+1})\right) \cap \bigcup \mathcal{A}_{k+1} \subseteq Z_{k+1} , \quad \psi_k[\{1\} \times Z_{k+1}] \subseteq E \cup \bigcup \mathcal{A}_k , \\ \psi_k(t,x) = x \quad \text{for } (t,x) \in I \times (E \cup \bigcup \mathcal{A}_k) , \quad \varphi_k = \psi_k(1, \cdot) \circ \varphi_{k+1} . \end{cases}$$

We proceed inductively. Assume that for some $l \in \{n-1, \ldots, d+1\}$ we have defined ψ_k, φ_k, δ_{k+1}, Z_{k+1} and $\mu_{i,k}$ for $k \in \{n, n-1, \ldots, l+1\}$ and $i \in \{1,2,3\}$. For each $Q \in \mathcal{A}_{l+1}$ apply Lemma 5.7 to find $a_Q \in Q$ satisfying

$$(2) \qquad \begin{gathered} \operatorname{dist}(a_Q, \operatorname{spt} \mu_{3,l+1}) > 0 , \quad \operatorname{dist}(a_Q, \partial_{\mathrm{c}} Q) > \tfrac{1}{4} \mathbf{l}(Q) , \\ \int_Q \|\mathrm{D}(\pi_{Q,a_Q} \circ p_Q)\|^d \,\mathrm{d}\mu_{i,l+1} \le \Gamma_{5.7} \mu_{i,l+1}(Q) \quad \text{for } i \in \{1,2,3\} , \\ \text{if } A \subseteq \operatorname{spt} \mu_{3,l+1} \text{ is purely } (\mathscr{H}^d, d) \text{ unrectifiable,} \\ \text{then } p_Q^* \circ \pi_{Q,a_Q} \circ p_Q[A] \text{ is also purely } (\mathscr{H}^d, d) \text{ unrectifiable} . \end{gathered}$$

Let $\delta_{l+1} \in (0,1)$ be such that

$$(3) \qquad \operatorname{dist}(a_Q, \operatorname{spt} \mu_{3,l+1}) > 2\delta_{l+1} \quad \text{and} \quad \operatorname{dist}(a_Q, \partial_{\mathrm{c}} Q) > 2\delta_{l+1} \quad \text{for all } Q \in \mathcal{A}_{l+1} .$$

Set

$$Z_{l+1} = E \cup \left(\bigcup \mathcal{A}_{l+1} \sim \bigcup \left\{ \mathbf{B}(a_Q, \delta_{l+1}) : Q \in \mathcal{A}_{l+1} \right\} \right) .$$

Define $\tilde{\psi}_l : I \times Z_{l+1} \to Z_{l+1}$ by setting for $(t,x) \in I \times Z_{l+1}$

$$\tilde{\psi}_l(t,x) = \begin{cases} (1-t)x + t p_Q^* \circ \pi_{Q,a_Q} \circ p_Q(x) & \text{if } x \in \operatorname{Int}_{\mathrm{c}}(Q) \text{ for some } Q \in \mathcal{A}_{l+1} , \\ \tilde{\psi}_l(t,x) = x & \text{if } x \in E \cup \bigcup \mathcal{A}_l . \end{cases}$$

Since for $Q \in \mathcal{A}_{l+1}$ the map $p_Q^* \circ \pi_{Q,a_Q} \circ p_Q$ is Lipschitz continuous on $\mathbf{R}^n \sim \mathbf{U}(a_Q, \delta_l)$, equals the identity on $\partial_{\mathrm{c}} Q$, and Q is convex, we see that $\tilde{\psi}_l$ is well defined and Lipschitz continuous. Extend $\tilde{\psi}_l$ to a Lipschitz map $\psi_l : I \times \mathbf{R}^n \to \mathbf{R}^n$ using [Fed69, 2.10.43]. Next, for $i \in \{1,2,3\}$ set

$$\varphi_l = \psi_l(1, \cdot) \circ \varphi_{l+1} \quad \text{and} \quad \mu_{i,l} = (\varphi_l)_\#(\|\mathrm{D}\varphi_l\|^d \mu_{i,n}) .$$

Note that $\|\mathrm{D}\varphi_l\|^d$ is bounded and φ_l is proper, so $\mu_{i,l}$ is a Radon measure. Also, because we assumed $\operatorname{spt}\mu_{3,l+1} \subseteq E \cup \bigcup \mathcal{A}_{l+1}$, we readily verify that

$$\operatorname{spt}\mu_{3,l} \subseteq \varphi_l[\operatorname{Clos} S] \subseteq E \cup \bigcup \mathcal{A}_l \,.$$

Hence, ψ_l, φ_l, $\mu_{i,l}$ for $i \in \{1,2,3\}$, δ_{l+1}, and Z_{l+1} verify (1). This concludes the inductive construction.

Define

$$\mathcal{B} = \big\{Q \in \mathcal{A}_d : Q \subseteq \varphi_d[S]\big\}\,.$$

For $Q \in \mathcal{A}_d \sim \mathcal{B}$ we choose $a_Q \in \operatorname{Int}_{\mathrm{c}}(Q)$ so that (2) holds and we define $\delta_d \in (0,1)$ so that (3) is satisfied with $l+1 = d$. Set

$$Z_d = E \cup \big(\bigcup \mathcal{A}_d \sim \bigcup\big\{\mathbf{B}(a_Q,\delta_d) : Q \in \mathcal{B}\big\}\big)\,, \quad \tilde{\psi}_{d-1} : Z_d \to Z_d\,,$$

$$\tilde{\psi}_{d-1}(t,x) = \begin{cases} (1-t)x + tp_Q^* \circ \pi_{Q,a_Q} \circ p_Q(x) & \text{if } x \in \operatorname{Int}_{\mathrm{c}}(Q) \text{ for some } Q \in \mathcal{A}_d \sim \mathcal{B}\,, \\ \tilde{\psi}_l(t,x) = x & \text{if } x \in E \cup \bigcup \mathcal{B} \cup \bigcup \mathcal{A}_{d-1}\,. \end{cases}$$

Extend $\tilde{\psi}_{d-1}$ to a Lipschitz map $\psi_{d-1} : I \times \mathbf{R}^n \to \mathbf{R}^n$. Set $\varphi_{d-1} = \psi_{d-1}(1,\cdot) \circ \varphi_d$,

$$V_{d-1} = E \cup \big(\bigcup \mathcal{B} + \mathbf{U}(0,\delta_d)\big) \cap Z_d\,, \quad \text{and} \quad V_l = \tilde{\psi}_{l-1}(1,\cdot)^{-1}[V_{l-1}] \subseteq Z_l \quad \forall l \in \{d, d+1, \dots, n\}\,.$$

Note that V_l is relatively open in Z_l for $l \in \{n, n-1, \dots, d\}$; in particular, V_n is open in $\mathbf{R}^n$ and, setting $V = V_n$, we get

$$S \subseteq V\,, \quad \varphi_{d-1}[V] \cap G = \bigcup \mathcal{B} \cap G\,.$$

We set for $l \in \{1, 2, \dots, n-d\}$ and $(t,x) \in I \times \mathbf{R}^n$ satisfying $l - 1 \le (n-d)t < l$

$$f(t,x) = \psi_{n-l}\big((n-d)t - (l-1), \varphi_{n-l+1}(x)\big)$$

and for $(t,x) \in [1,2] \times \mathbf{R}^n$

$$f(t,x) = \psi_{d-1}\big(t-1, \varphi_d(x)\big)\,.$$

This defines a Lipschitz map $f : J \times \mathbf{R}^n \to \mathbf{R}^n$. From the construction it follows that $f[\{1\} \times U]$ is purely $(\mathscr{H}^d, d)$ unrectifiable and $f(1,\cdot)[U] \cap G \subseteq \bigcup(\mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_d^n)$, so

$$\mathscr{H}^d(f(1,\cdot)[U] \cap G) = 0\,.$$

Now, we need to verify the required estimates. For brevity of the notation let us set

$$g = f(1,\cdot) \quad \text{and} \quad \eta_k = \psi_k(1,\cdot) \quad \text{for } k \in \{d, d+1 \dots, n\}\,.$$

Observe that if $Q \in \mathcal{F}$, then $\mathscr{H}^0(\{R \in \mathcal{F} : R \cap Q \neq \varnothing\}) \le 4^n$. Note also that for $k \in \{d, d+1, \dots, n-1\}$ and $i \in \{1,2,3\}$ we have

$$\begin{aligned}(\varphi_{k+1})_\#\big(\|\mathrm{D}\varphi_{k+1}\|^d \mu_{i,n} \llcorner \varphi_k^{-1}[\textstyle\bigcup \mathcal{A}_k]\big) &= (\varphi_{k+1})_\#\big(\|\mathrm{D}\varphi_{k+1}\|^d \mu_{i,n}\big) \llcorner \varphi_{k+1}[\varphi_k^{-1}[\textstyle\bigcup \mathcal{A}_k]] \\ &= \mu_{i,k+1} \llcorner \eta_k^{-1}[\textstyle\bigcup \mathcal{A}_k] \le \mu_{i,k+1} \llcorner \textstyle\bigcup \mathcal{A}_{k+1}\,,\end{aligned}$$

so we obtain

$$\mu_{i,k}(\textstyle\bigcup \mathcal{A}_k) = \displaystyle\int_{\varphi_k^{-1}[\bigcup \mathcal{A}_k]} \|\mathrm{D}\varphi_k\|^d \,\mathrm{d}\mu_{i,n} \le \int_{\varphi_k^{-1}[\bigcup \mathcal{A}_k]} \|\mathrm{D}\eta_k \circ \varphi_{k+1}\|^d \|\mathrm{D}\varphi_{k+1}\|^d \,\mathrm{d}\mu_{i,n}$$
$$= \int_{\eta_k^{-1}[\bigcup \mathcal{A}_k]} \|\mathrm{D}\eta_k\|^d \,\mathrm{d}\mu_{i,k+1} \le \int_{\bigcup \mathcal{A}_{k+1}} \|\mathrm{D}\eta_k\|^d \,\mathrm{d}\mu_{i,k+1} \le \sum_{Q \in \mathcal{A}_{k+1}} \int_Q \|\mathrm{D}\eta_k\|^d \,\mathrm{d}\mu_{i,k+1}$$
$$= \sum_{Q \in \mathcal{A}_{k+1}} \int_Q \|\mathrm{D}(\pi_{Q,a_Q} \circ p_Q)\|^d \,\mathrm{d}\mu_{i,k+1} \le \Gamma_{5.7} \sum_{Q \in \mathcal{A}_{k+1}} \mu_{i,k+1}(Q) \le 4^n \Gamma_{5.7} \mu_{i,k+1}(\textstyle\bigcup \mathcal{A}_{k+1}) \,. \tag{4}$$

In particular, setting $\Sigma_1 = R \cap G$, $\Sigma_2 = U \cap G$ and employing [FK18, 7.12] we obtain for $i \in \{1, 2\}$

$$\mathscr{H}^d(g[\Sigma_i] \cap \textstyle\bigcup \mathcal{A}_d) = \mathscr{H}^d(\varphi_d[\Sigma_i] \cap \bigcup \mathcal{A}_d) \le \displaystyle\int_{\varphi_d^{-1}[\bigcup \mathcal{A}_d]} \|\mathrm{D}\varphi_d\|^d \,\mathrm{d}\mu_{i,n} = \mu_{i,d}(\textstyle\bigcup \mathcal{A}_d)$$
$$\le \big(4^n \Gamma_{5.7}\big)^{n-d} \mu_{i,n}(\textstyle\bigcup \mathcal{A}_n) = \big(4^n \Gamma_{5.7}\big)^{n-d} \mathscr{H}^d(\Sigma_i) \,.$$

Estimating as in (4), we also get

$$\mathscr{H}^d(g[\Sigma_i] \sim \textstyle\bigcup \mathcal{A}_d) = \mathscr{H}^d(\varphi_d[\Sigma_i] \sim \bigcup \mathcal{A}_d) \le \displaystyle\int_{G \cap \varphi_d^{-1}[\partial G]} \|\mathrm{D}\varphi_d\|^d \,\mathrm{d}\mu_{i,n}$$
$$\le \int_{\varphi_{d+1}[G] \cap \eta_d^{-1}[\partial G]} \|\mathrm{D}\eta_d\|^d \,\mathrm{d}\mu_{i,d+1} \le \int_{\bigcup \mathcal{A}_{d+1}} \|\mathrm{D}\eta_d\|^d \,\mathrm{d}\mu_{i,d+1}$$
$$\le 4^n \Gamma_{5.7} \mu_{i,d+1}(\textstyle\bigcup \mathcal{A}_{d+1}) \le (4^n \Gamma_{5.7})^{n-d} \mathscr{H}^d(\Sigma_i) \,.$$

This gives the desired estimates. □

5.9 Remark. Observe that

$$f(1,\cdot)[S] \cap G \subseteq \textstyle\bigcup \big(\mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_d^n\big) \quad \text{but} \quad f(1,\cdot)[S \cap G] \subseteq \bigcup \big(\mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_d^n\big) \cup \partial G \,.$$

5.10 Remark. Define

$$\tilde{Q} = \textstyle\bigcup \{R \in \mathcal{F} : R \cap Q \ne \varnothing\} \quad \forall Q \in \mathcal{F} \,, \quad H = \bigcup \{Q \in \mathcal{A} : \tilde{Q} \subseteq \bigcup \mathcal{A}\} \,, \quad \text{and} \quad W = V \cap G \,.$$

Assume that S is separated from $E = \mathbf{R}^n \sim G$ in the sense that $S \subseteq H$. Then W is an open neighborhood of S in $\mathbf{R}^n$ with

$$f[J \times S] \subseteq f[J \times W] \subseteq W$$

and $f(2,\cdot)[W] = \bigcup B$ is a strong deformation retract of S.

5.11 Lemma. *Assume*

$$(S, D) \text{ is a test pair} \,, \quad T = \operatorname{Tan}(D, 0) \,, \quad B = T \cap \partial \mathbf{B}(0,1) \,, \quad R = \mathcal{R}(S) \,, \quad I = \mathcal{U}(S) \,.$$

For each $\varepsilon \in (0,1)$ there exists a map $f : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\operatorname{Lip} f < \infty \,, \quad f(x) = x \quad \text{for } x \in B \,, \quad \mathscr{H}^d(f[I]) = 0 \,, \quad \mathscr{H}^d\big((R \sim f[R]) \cup (f[R] \sim R)\big) \le \varepsilon \,.$$

In particular, $f[S]$ is $(\mathscr{H}^d, d)$ rectifiable and $(f[S], D)$ is a rectifiable test pair.

Proof. Define
$$\iota = \big(2((31)^d + 1)\big)^{-1}\varepsilon\,.$$
Since $\mathscr{H}^d(B) = 0$ we can find $\delta_0 \in (0, 2^{-144})$ such that
$$\mathscr{H}^d\big((B + \mathbf{B}(0,\delta_0)) \cap S\big) \le \iota\,.$$
Set $T_0 = T$, $F_0 = T \cap (B + \mathbf{B}(0,\delta_0))$ and define $\psi_0 : T_0 \to T_0^\perp$ by $\psi_0(x) = 0$ for $x \in T_0$. Employing [Fed69, 3.2.29, 3.1.19(5), 2.8.18, 2.2.5] we find $Z \subseteq \mathbf{R}^n$ and for each $i \in \mathbb{N}$ a vectorspace $T_i \in \mathbf{G}(n,d)$, a compact set $K_i \subseteq T_i$, and a $\mathscr{C}^1$ map $\psi_i : T_i \to T_i^\perp$ such that
$$F_i = \{x + \psi_i(x) : x \in K_i\}\,, \quad F_i \cap F_j = \varnothing \quad \text{whenever } i \neq j\,,$$
$$R \sim F_0 = Z \cup \textstyle\bigcup_{i=1}^\infty F_i\,, \quad \mathscr{H}^d(Z) = 0\,, \quad \operatorname{Lip}\psi_i \le 2^{-44}\,.$$
Since $\mathscr{H}^d(R) < \infty$ we can find $N \in \mathbb{N}$ such that
$$F = \textstyle\bigcup_{i=0}^N F_i \quad \text{and} \quad \mathscr{H}^d(R \sim F) \le 2\iota\,.$$
Set
$$\delta = 2^{-144}\inf\{\delta_0\} \cup \big\{|x-y| : i,j \in \{0,1,\dots,N\},\, x \in F_i,\, y \in F_j,\, i \neq j\big\}\,.$$
Note that $\delta > 0$ because each F_i is compact. Let $L \in \mathbb{N}$ be such that $2^{-L} < \delta n^{-1/2} \le 2^{-L+1}$ so that $\operatorname{diam} Q < \delta$ whenever $Q \in \mathbf{K}_n^n(L)$. Define
$$\mathcal{F} = \mathbf{K}_n^n(L)\,, \quad \widetilde{Q} = \textstyle\bigcup\{R \in \mathcal{F} : R \cap Q \neq \varnothing\} \quad \text{for } Q \in \mathcal{F}\,,$$
$$\mathcal{A} = \big\{Q \in \mathcal{F} : \widetilde{Q} \cap I \neq \varnothing,\, Q \cap (F + \mathbf{B}(0,2\delta)) = \varnothing\big\}\,, \quad G = \operatorname{Int}\textstyle\bigcup \mathcal{A}\,.$$
Observe that
$$\big\{x \in I : \operatorname{dist}(x,F) \ge 4\delta\big\} \subseteq G\,.$$
Apply Theorem 5.8 to obtain a Lipschitz continuous map $f : \mathbf{R}^n \to \mathbf{R}^n$ such that
$$f(x) = x \quad \text{for } x \in \mathbf{R}^n \sim G\,, \quad f[I] \text{ is purely } (\mathscr{H}^d, d) \text{ unrectifiable}\,,$$
$$\mathscr{H}^d(f[R \cap G]) \le \Gamma_{5.8}\mathscr{H}^d(R \cap G) \le \Gamma_{5.8}\mathscr{H}^d(R \sim F) \le 2\iota\Gamma_{5.8}\,,$$
$$\mathscr{H}^d(f[I] \cap G) = 0\,, \quad \mathscr{H}^d(f[I \cap G]) \le \Gamma_{5.8}\mathscr{H}^d(I \cap G) < \infty\,.$$
For each $i \in \{0,1,\dots,N\}$ employ the Besicovitch-Federer projection theorem [Fed69, 3.3.15] to choose $P_i \in \mathbf{G}(n,d)$ such that
$$\|P_{i\natural} - T_{i\natural}\| \le 2^{-144} \quad \text{and} \quad \mathscr{H}^d(P_{i\natural} \circ f[I]) = 0\,.$$
Using the inverse function theorem [Fed69, 3.1.18] or [KSv15, Lamma 3.2] we argue that for $i \in \{0,1,\dots,N\}$ we can find a $\mathscr{C}^1$ function $\varphi_i : P_i \to P_i^\perp$ such that $\{x + \psi_i(x) : x \in T_i\} = \{x + \varphi_i(x) : x \in P_i\}$ and $\operatorname{Lip}\varphi_i \le 2^{-12}$. Next, for $i \in \{0,1,\dots,N\}$ we define the projection onto the graph of φ_i by the formula
$$\pi_i : \mathbf{R}^n \to \mathbf{R}^n\,, \quad \pi_i(x) = P_{i\natural}x + \varphi_i(P_{i\natural}x) \quad \text{for } x \in \mathbf{R}^n\,.$$
Note that $\operatorname{Lip}\pi_i \le 1 + \operatorname{Lip}\varphi_i \le 1 + 2^{-12}$. Choose a smooth map $\gamma : \mathbf{R} \to \mathbf{R}$ such that
$$\gamma(t) = 0 \quad \text{for } t > 10\delta\,, \quad \gamma(t) = 1 \quad \text{for } t < 5\delta\,, \quad -\frac{1}{\delta} \le \gamma'(t) \le 0$$

and define maps $f, \lambda_0, \lambda_1, \dots, \lambda_N : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\lambda_i(x) = \gamma(\operatorname{dist}(x, F_i))\pi_i(x) + (1 - \gamma(\operatorname{dist}(x, F_i)))x \quad \text{for } i \in \{0, 1, 2, \dots, N\}\,,$$
$$f = \lambda_1 \circ \dots \circ \lambda_N\,.$$

Note that if $i \in \{0, 1, \dots, N\}$, $x \in \mathbf{R}^n$, $y \in F_i$ satisfy $|x-y| = \operatorname{dist}(x, F_i) \le 10\delta$, then $\pi_i(y) = y$ and

$$|x - \pi_i(x)| \le |x - y| + |\pi_i(y) - \pi_i(x)| + |y - \pi_i(y)| \le 10\delta + (1 + 2^{-12})10\delta \le 30\delta\,.$$

Therefore,

$$\operatorname{Lip} f \le \operatorname{Lip} \gamma \cdot 30\delta + 1 \le 31\,.$$

Observe also that

$$f(x) = x \quad \text{for } x \in F\,, \quad f[B + \mathbf{B}(0, \delta)] \subseteq T\,, \quad \mathscr{H}^d(f[I]) = 0\,;$$

hence, $(R \sim f[R]) \cup (f[R] \sim R) = f[R \sim F] \cup (R \sim F)$ and we get

$$\mathscr{H}^d\big((R \sim f[R]) \cup (f[R] \sim R)\big) \le ((31)^d + 1)\mathscr{H}^d(R \sim F) \le 2((31)^d + 1)\iota \le \varepsilon\,. \qquad \square$$

5.12 Remark. The difficulty in proving Lemma 5.11 stems from the situation when $\mathscr{H}^d(R \cap \operatorname{Clos} I) > 0$; cf. [Fed69, 4.2.25]. In this case one cannot argue that $\lim_{r \downarrow 0} \mathscr{H}^d((I + \mathbf{U}(0, r)) \cap R) = 0$ so it is not possible to separate the unrectifiable part of S from the rectifiable part. However, since R has a nice (rectifiable) structure and I can be easily squashed to a set of $\mathscr{H}^d$ measure zero by means of Besicovitch-Federer projection theorem [Fed69, 3.3.15], we can find nice Lipschitz deformations which produce "holes" in I and do not move most of R.

5.13 Corollary. *Let $x \in \mathbf{R}^n$, $\mathcal{P}_1$ be the set of all test pairs, and $\mathcal{P}_2$ be the set of rectifiable test pairs. Then*

$$\mathrm{AE}_x(\mathcal{P}_1) = \mathrm{AE}_x(\mathcal{P}_2) \quad \text{and} \quad \mathrm{AUE}_x(\mathcal{P}_1) = \mathrm{AUE}_x(\mathcal{P}_2)\,.$$

Proof. Since $\mathcal{P}_2 \subseteq \mathcal{P}_1$ we clearly have $\mathrm{AE}_x(\mathcal{P}_1) \subseteq \mathrm{AE}_x(\mathcal{P}_2)$ and $\mathrm{AUE}_x(\mathcal{P}_1) \subseteq \mathrm{AUE}_x(\mathcal{P}_2)$. Hence, it suffices to prove the reverse inclusions. Take any test pair $(S, D) \in \mathcal{P}_1$ and set

$$T = \operatorname{Tan}(D, 0)\,, \quad B = T \cap \mathbf{B}(0, 1)\,, \quad R = \mathcal{R}(S)\,, \quad \text{and} \quad I = \mathcal{U}(S)\,.$$

For each $k \in \mathbb{N}$ apply Lemma 5.11 with $\varepsilon = 1/k$ to obtain a map $f_k : \mathbf{R}^n \to \mathbf{R}^n$ satisfying

$$\operatorname{Lip} f_k < \infty\,, \quad f_k(x) = x \quad \text{for } x \in B\,,$$
$$\mathscr{H}^d(f[I]) = 0\,, \quad \mathscr{H}^d\big((R \sim f_k[R]) \cup (f_k[R] \sim R)\big) \le \tfrac{1}{k}\,.$$

Then $(S_k, D) = (f_k[S], D)$ is a rectifiable test pair for each $k \in \mathbb{N}$, hence for any integrand F we have

$$\Psi_{F^x}(S_k) - \Psi_{F^x}(D) = \Phi_{F^x}(S_k) - \Phi_{F^x}(D)\,.$$

Observe that

$$\big|\lim_{k \to \infty} \mathscr{H}^d(S_k) - \mathscr{H}^d(R)\big| = 0\,; \quad \text{hence, also} \quad \big|\lim_{k \to \infty} \Phi^d_{F^x}(S_k) - \Phi^d_{F^x}(R)\big| = 0\,.$$

Thus, if $F \in \mathrm{AUE}_x(\mathcal{P}_2)$, then

$$\begin{aligned}\Psi_{F^x}(S) - \Psi_{F^x}(D) = \Psi_{F^x}(I) + \lim_{k\to\infty} \Phi_{F^x}(S_k) - \Phi_{F^x}(D) &\geq \Psi_{F^x}(I) + c\big(\mathscr{H}^d(R) - \mathscr{H}^d(D)\big) \\ &\geq \inf\big(\{c\} \cup \operatorname{im} F^x\big)\big(\mathscr{H}^d(S) - \mathscr{H}^d(D)\big)\,.\end{aligned}$$

Similarly, if $F \in \mathrm{AE}_x(\mathcal{P}_2)$, then

$$\Psi_{F^x}(S) - \Psi_{F^x}(D) = \Psi_{F^x}(I) + \lim_{k\to\infty} \Phi_{F^x}(S_k) - \Phi_{F^x}(D) > \Psi_{F^x}(I) \geq 0\,. \qquad \square$$

5.14 Remark. Recalling Remark 4.6, from Corollary 5.13 we deduce that definitions [Alm76, IV.1(7)] and [FK18, 3.16] are equivalent.

6 Existence of a minimiser for an integrand in wBC

In this section we provide a solution to the set theoretical formulation of the anisotropic Plateau problem under the weak assumption wBC on the integrand. Since wBC is a weak version of BC and in turn BC will be proven to be equivalent to AC, see Lemma 7.1, this section improves [DPDRG17, Theorem 1.8], where the entire condition AC is required.

6.1 Definition. Let $U \subseteq \mathbf{R}^n$ be open. We say that $f : \mathbf{R}^n \to \mathbf{R}^n$ is a *basic deformation in* U if f is of class $\mathscr{C}^1$ and there exists a bounded convex open set $V \subseteq U$ such that

$$f(x) = x \quad \text{for every } x \in \mathbf{R}^n \sim V \quad \text{and} \quad f[V] \subseteq V\,.$$

If $f \in \mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ is a composition of a finite number of basic deformations, then we say that f is an *admissible deformation in* U. The set of all such deformations shall be denoted $\mathfrak{D}(U)$.

6.2 Definition (cf. [Fed69, 2.10.21])**.** Whenever $K \subseteq \mathbf{R}^n$ is compact and $A, B \subseteq \mathbf{R}^n$, we define $d_{\mathscr{H},K}(A, B)$ by

$$\begin{aligned}d_{\mathscr{H},K}(A,B) &= \sup\big\{|\operatorname{dist}(x,A) - \operatorname{dist}(x,B)| : x \in K\big\} \\ &= \max\big\{\sup\{\operatorname{dist}(x,A) : x \in K \cap B\}\,, \sup\{\operatorname{dist}(x,B) : x \in K \cap A\}\big\}\,.\end{aligned}$$

6.3 Definition. Let $U \subseteq \mathbf{R}^n$ be an open set. We say that $\mathcal{C}$ is a *good class in* U if

(a) $\mathcal{C} \neq \varnothing$;

(b) each $S \in \mathcal{C}$ is a closed subset of $\mathbf{R}^n$;

(c) if $S \in \mathcal{C}$ and $f \in \mathfrak{D}(U)$, then $f[S] \in \mathcal{C}$;

6.4 Remark. Definition 6.3 differs from [FK18, 3.4] by not assuming that the class is closed under Hausdorff convergence.

Combining [FK18, 11.2, 11.3, 11.7, 11.8(a)] we obtain the following.

6.5 Theorem. *Let $U \subset \mathbf{R}^n$ be an open set, $\mathcal{C}$ be a good class in U, and F be a bounded $\mathscr{C}^0$ integrand. Set $\mu = \inf\big\{\Phi_F(T \cap U) : T \in \mathcal{C}\big\}$.*

If $\mu \in (0,\infty)$, then there exist $V \in \mathbf{V}_d(U)$, $S \subseteq \mathbf{R}^n$ closed, and $\{S_i \in \mathcal{C} : i \in \mathbb{N}\}$ such that

(a) $S \cap U$ is $(\mathscr{H}^d, d)$ rectifiable. In particular $\mathscr{H}^d(S \cap U) < \infty$.

(b) $\lim_{i\to\infty} \mathbf{v}_d(S_i \cap U) = V$ *in* $\mathbf{V}_m(U)$.

(c) $\lim_{i\to\infty} \Phi_F(S_i \cap U) = \Phi_F(V) = \mu$.

(d) $\operatorname{spt}\|V\| \subseteq S \cap U$ *and* $\mathscr{H}^d(S \cap U \sim \operatorname{spt}\|V\|) = 0$.

(e) *The measures* $\|V\|$ *and* $\mathscr{H}^d \llcorner S$ *are mutually absolutely continuous.*

(f) $\lim_{i\to\infty} d_{\mathscr{H},K}(S_i \cap U, S \cap U) = 0$ *for any compact set* $K \subseteq U$.

(g) *For any compact set* $K \subseteq U$ *we have*

$$\lim_{i\to\infty} \sup\bigl\{r \in \mathbf{R} : \mathscr{H}^m(\{x \in S_i \cap K : \operatorname{dist}(x, \operatorname{spt}\|V\| \cup \mathbf{R}^n \sim U) \geq r\}) > 0\bigr\} = 0\,.$$

(h) *If* $\bar{S}_i = \mathcal{U}(S_i \cap U)$, *then*

$$\lim_{r\downarrow 0} \lim_{i\to\infty} r^{-d} \mathscr{H}^d(\bar{S}_i \cap \mathbf{B}(x,r)) = 0 \quad \text{for } \|V\|\text{-a.e. } x \quad \text{and} \quad \lim_{i\to\infty} \mathscr{H}^d(\bar{S}_i) = 0\,.$$

(i) $\mathbf{\Theta}^d(\|V\|, x) \geq 1$ *for* $\|V\|$ *almost all* x.

(j) *For* $\mathscr{H}^d$ *almost all* $x \in \operatorname{spt}\|V\|$ *we have*

$$\operatorname{Tan}^d(\|V\|, x) = \operatorname{Tan}(\operatorname{spt}\|V\|, x) \in \mathbf{G}(n,d)\,.$$

(k) *If* $\mathbf{R}^n \sim U$ *is compact and there exists a* Φ_F*-minimising sequence in* $\mathcal{C}$ *consisting only of compact sets (but not necessarily uniformly bounded), then*

$$\operatorname{diam}(\operatorname{spt}\|V\|) < \infty \quad \text{and} \quad \sup\bigl\{\operatorname{diam}(S_i \cap U) : i \in \mathbb{N}\bigr\} < \infty\,.$$

6.6 Lemma. *Assume* $U \subseteq \mathbf{R}^n$ *is open,* $V \in \mathbf{V}_d(U)$, $\mathcal{C}$ *is a good class,* F *is a bounded* $\mathscr{C}^0$ *integrand,* $\mu = \inf\{\Phi_F(P) : P \in \mathcal{C}\}$, $\Phi_F(V) = \mu$, *and either* $V = \mathbf{v}_d(S \cap U)$ *for some* $(\mathscr{H}^d, d)$ *rectifiable set* $S \in \mathcal{C}$, *or there exists a sequence* $\{S_i \in \mathcal{C} : i \in \mathbb{N}\}$ *such that*

$$\lim_{i\to\infty} \mathbf{v}_d(S_i \cap U) = V \quad \text{and} \quad \lim_{j\to\infty} \mathscr{H}^d(\mathcal{U}(S_j \cap U)) = 0\,.$$

Then

$$\delta_F V = 0\,.$$

Proof. The proof can be found, with a slightly different notation, in [DR18, Section 5.1]. For the sake of the exposition we report it below.

Assume there exists $g \in \mathscr{X}(U)$ such that $\delta_F V(g) \neq 0$. Since $\operatorname{spt} g$ is compact, using a partition of unity [Fed69, 3.1.13] one can decompose g into a finite sum $g = \sum_{i=1}^N g_i$, where $g_i \in \mathscr{X}(U)$ is supported in some ball contained in U for each $i \in \{1, 2, \ldots, N\}$. Recalling that $\delta_F V$ is linear we see that there exists an $i \in \{1, 2, \ldots, N\}$ such that $\delta_F V(g_i) \neq 0$. Set $h = g_i$ and $\varphi_t(x) = x + th(x)$ for $x \in U$ and t in some neighbourhood of 0 in $\mathbf{R}$. Clearly $\varphi_t \in \mathfrak{D}(U)$ is an injective admissible map whenever $|t|$ is small enough. Replacing possibly h with $-h$ we shall assume that $\delta_F V(h) < 0$. Then there exists $t_0 > 0$ such that $\Phi_F((\varphi_t)_\# V) < \Phi_F(V) = \mu$ for $t \in (0, t_0]$. Set $\psi = \varphi_{t_0}$.

In case $V = \mathbf{v}_d(S)$ for some $(\mathscr{H}^d, d)$ rectifiable set $S \in \mathcal{C}$, we have

$$\mu = \Phi_F(V) > \Phi_F(\psi_\# V) = \Phi_F(\psi[S]),$$

which contradicts the definition of μ.

In the other case, since $\psi_\# : \mathbf{V}_d(U) \to \mathbf{V}_d(U)$ is continuous and $V = \lim_{j\to\infty} \mathbf{v}_d(S_j \cap U)$, we have also $\psi_\# V = \lim_{j\to\infty} \psi_\# \mathbf{v}_d(S_j \cap U)$. For $j \in \mathbb{N}$ we set $\bar{S}_j = \mathcal{U}(S_j \cap U)$ and $\hat{S}_j = \mathcal{R}(S_j \cap U)$ to obtain

$$\begin{aligned}\mu > \lim_{j\to\infty} \Phi_F(\psi_\# \mathbf{v}_d(S_j \cap U)) \geq \lim_{j\to\infty} \Phi_F(\psi_\# \mathbf{v}_d(\hat{S}_j)) &= \lim_{j\to\infty} \Phi_F(\mathbf{v}_d(\psi[\hat{S}_j])) \\ &= \lim_{j\to\infty} \Phi_F(\psi[S_j \cap U]) - \Phi_F(\psi[\bar{S}_j])\,.\end{aligned}$$

Since $\lim_{j\to\infty} \mathscr{H}^d(\bar{S}_j) = 0$, we see that $\mu > \lim_{j\to\infty} \Phi_F(\psi[S_j \cap U])$ which contradicts the definition of μ. $\square$

6.7 Theorem. *Assume U, $\mathcal{C}$, F, μ, V, S, and $\{S_i : i \in \mathbb{N}\}$ are as in Theorem 6.5. Suppose that $F \in \mathrm{wBC}_x$ for all $x \in U$. Then*

(a) $T = \mathrm{Tan}^d(\|V\|, x)$ for V almost all (x, T).

(b) $\mathbf{\Theta}^d(\|V\|, x) = 1$ for $\|V\|$ almost all x.

In particular, $V = \mathbf{v}_d(S)$.

Proof. Proof of (a). Employing Lemma 6.6 together with [DPDRG18, 2.3, 2.4] and Theorem 6.5(a)(b)(c)(e)(h) we see that for $\|V\|$ almost all x and all $W \in \mathrm{VarTan}(V, x)$ there exists a Radon probability measure σ over $\mathbf{G}(n, d)$ such that

$$\mathrm{Tan}^d(\|V\|, x) = T \in \mathbf{G}(n,d)\,, \quad \mathbf{\Theta}^d(\|V\|, x) = \vartheta \in [1, \infty)\,, \tag{5}$$

$$W = \vartheta(\mathscr{H}^d \llcorner T) \times \sigma\,, \quad \text{and} \quad \delta_{F^x} W = 0\,. \tag{6}$$

Since $F \in \mathrm{wBC}_x$ it follows that $\mathrm{VarTan}(V, x) = \{\mathbf{\Theta}^d(\|V\|, x)\mathbf{v}_d(\mathrm{Tan}^d(\|V\|, x))\}$ for $\|V\|$ almost all x which proves (a).

Proof of (b). Let $T \in \mathbf{G}(n, d)$ and $\vartheta \in [1, \infty)$ satisfy (5)(6), and $x \in U$ be such that Theorem 6.5(h)(j) hold. Without loss of generality we shall assume $x = 0$. Assume, by contradiction, that $\vartheta > 1$. Define

$$\delta_r = \sup\left\{\frac{\mathrm{dist}(x, T)}{|x|} : x \in \mathrm{spt}\,\|V\| \cap \mathbf{U}(x, 2r) \sim \{0\}\right\} \quad \text{for } r \in (0, \infty)\,.$$

From Theorem 6.5(j), we see that $\delta_r \downarrow 0$ as $r \downarrow 0$. Set $\varepsilon_r = 12\delta_r^{1/2}$. For $r \in (0, 1)$ let $f_r, h_r \in \mathscr{C}^\infty(\mathbf{R}, [0,1])$ be such that

$$\begin{gathered} f_r(t) = 1 \quad \forall t \leq 1 - \varepsilon_r\,, \quad f_r(t) = 0 \quad \forall t \geq 1 - \tfrac{1}{2}\varepsilon_r\,, \quad \text{and} \quad |f_r'(t)| \leq 4/\varepsilon_r \quad \forall t \in \mathbf{R}\,, \\ h_r(t) = 1 \quad \forall t \leq 2\delta_r\,, \quad h_r(t) = 0 \quad \forall t \geq 3\delta_r\,, \quad \text{and} \quad |h_r'(t)| \leq 2/\delta_r \quad \forall t \in \mathbf{R}\,. \end{gathered}$$

For $r \in (0, 1)$ we define $p_r \in \mathscr{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$ by the formula

$$p_r(x) = T_\natural(x) + \big(1 - f_r(|T_\natural(x)|)h_r(|T_\natural^\perp(x)|)\big)T_\natural^\perp(x) \quad \text{for } x \in \mathbf{R}^n\,.$$

Clearly $p_r \in \mathfrak{D}(U)$ for $r \in (0,1)$ small enough. Note also that

$$
\begin{aligned}
&p_r(x) = x && \text{for } x \in \mathbf{R}^d \sim ((T \cap \mathbf{B}(0, 1-\varepsilon_r/2)) + \mathbf{B}(0, 3\delta_r)) \subseteq \mathbf{R}^d \sim \mathbf{U}(0,1)\,,\\
&p_r(x) = T_\natural x && \text{for } x \in (T \cap \mathbf{B}(0, 1-\varepsilon_r)) + \mathbf{B}(0, 2\delta_r)\,,\\
&&& \operatorname{Lip} p_r \le 8 + 12\frac{\delta_r}{\varepsilon_r} \le 8 + \delta_r^{1/2} \le 9 \quad \text{for } r \in (0,1)\,.
\end{aligned} \tag{7}
$$

Set $A_r = \mathbf{B}(0,1) \sim \mathbf{U}(0, 1-\varepsilon_r)$ and $\tilde{p}_r = \boldsymbol{\mu}_r \circ p_r \circ \boldsymbol{\mu}_{1/r}$. Let $C \in \operatorname{VarTan}(V, 0)$. By [All72, 3.4(2)] and (a) we get

$$
C = \lim_{r \downarrow 0} (\boldsymbol{\mu}_{1/r})_\# V = \lim_{r \downarrow 0} \lim_{i \to \infty} \mathbf{v}_d(\boldsymbol{\mu}_{1/r}[S_i]) = \vartheta \mathbf{v}_d(T)\,; \tag{8}
$$

Hence, we have $\|C\|(\partial \mathbf{B}(0,1)) = 0$, which implies that

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \mathscr{H}^d(\boldsymbol{\mu}_r[A_r] \cap S_i) = 0\,.
$$

In particular, employing (7),

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \Phi_F(\boldsymbol{\mu}_r[A_r] \cap S_i) = 0 \quad \text{and} \quad \lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \Phi_F(\tilde{p}_r[\boldsymbol{\mu}_r[A_r] \cap S_i]) = 0\,. \tag{9}
$$

For $r \in (0,1)$ and $i \in \mathbb{N}$ we have

$$
\begin{aligned}
\Phi_F(\tilde{p}_r[S_i \cap U]) = {} & \Phi_F(S_i \cap U) - \Phi_F(S_i \cap \mathbf{B}(0, (1-\varepsilon_r)r)) \\
& + \Phi_F(\tilde{p}_r[S_i \cap \mathbf{B}(0, (1-\varepsilon_r)r)]) - \Phi_F(S_i \cap \boldsymbol{\mu}_r[A_r]) + \Phi_F(\tilde{p}_r[S_i \cap \boldsymbol{\mu}_r[A_r]])\,.
\end{aligned} \tag{10}
$$

Since $\lim_{i \to \infty} \Phi_F(S_i \cap U) = \mu$, taking into account (9), to reach a contradiction it suffices to show that

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \Phi_F(\tilde{p}_r[S_i \cap \mathbf{B}(0, (1-\varepsilon_r)r)]) - r^{-d} \Phi_F(S_i \cap \mathbf{B}(0, (1-\varepsilon_r)r)) < 0\,. \tag{11}
$$

For $i \in \mathbb{N}$ and $r \in (0,1)$ we define

$$
S_{r,i} = \boldsymbol{\mu}_{1/r}[S_i] \cap \mathbf{B}(0,1)\,, \quad F_r = \boldsymbol{\mu}_r^\# F\,, \quad \text{and} \quad \hat{S}_{r,i} = \mathcal{R}(S_{r,i})\,.
$$

Observe that, using (9) and Theorem 6.5(h), claim (11) will follow from

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} \Phi_{F_r}(T_\natural[\hat{S}_{r,i}]) - \Phi_{F_r}(\hat{S}_{r,i}) < 0\,. \tag{12}
$$

In order to prove (12), we observe that (8) implies

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} \int_{\mathbf{B}(0,1)} \|P_\natural - T_\natural\| \, \mathrm{d}\mathbf{v}_d(\hat{S}_{r,i})(x, P) = 0\,.
$$

Since F is continuous, we obtain also

$$
\lim_{r \downarrow 0} \lim_{i \to \infty} \int_{\mathbf{B}(0,1)} |F(z, P) - F(z, T)| \, \mathrm{d}\mathbf{v}_d(\hat{S}_{r,i})(x, P) = 0 \quad \text{for any } z \in \mathbf{R}^n\,. \tag{13}
$$

We then estimate

$$
\begin{aligned}
\Phi_{F_r}(T_\natural[\hat{S}_{r,i}]) - \Phi_{F_r}(\hat{S}_{r,i}) &= \int_{T_\natural[\hat{S}_{r,i}]} F_r(y,T)\,\mathrm{d}\mathscr{H}^d(y) - \int F_r(x,P)\,\mathrm{d}\mathbf{v}_d(\hat{S}_{r,i})(x,P) \\
&\le \int_{T_\natural[\hat{S}_{r,i}]} F_r(0,T)\,\mathrm{d}\mathscr{H}^d(y) - \int F_r(0,T)\,\mathrm{d}\mathbf{v}_d(\hat{S}_{r,i}) \\
&\quad + \int_{T_\natural[\hat{S}_{r,i}]} |F_r(y,T) - F_r(0,T)|\,\mathrm{d}\mathscr{H}^d(y) \\
&\quad + \int |F_r(0,T) - F_r(0,P)| + |F_r(0,P) - F_r(x,P)|\,\mathrm{d}\mathbf{v}_d(\hat{S}_{r,i})(x,P)\,.
\end{aligned}
$$

Using continuity of F and (13), we see that the last two terms converge to zero when we first take the limit with $i \to \infty$ and then with $r \downarrow 0$. Therefore,

$$
\begin{aligned}
&\lim_{r\downarrow 0}\lim_{i\to\infty} \Phi_{F_r}(T_\natural[\hat{S}_{r,i}]) - \Phi_{F_r}(\hat{S}_{r,i}) \\
&\qquad = \lim_{r\downarrow 0}\lim_{i\to\infty} \int_{T_\natural[\hat{S}_{r,i}]} F_r(0,T)\,\mathrm{d}\mathscr{H}^d(y) - \int F_r(0,T)\,\mathrm{d}\mathbf{v}_d(\hat{S}_{r,i})(x,P) \\
&\quad = \lim_{r\downarrow 0}\lim_{i\to\infty} F_r(0,T)\big(\mathscr{H}^d(T_\natural[\hat{S}_{r,i}]) - \mathscr{H}^d(S_{r,i})\big) \le \boldsymbol{\alpha}(d)F_r(0,T)(1-\vartheta) = -\kappa < 0\,.
\end{aligned}
$$

Thus, we have proved (12), which in turn implies (11). Hence, recalling (10), we can choose $r \in (0,1)$ so that for all big enough $i \in \mathbb{N}$

$$
\Phi_F(\tilde{p}_r[S_i \cap U]) - \Phi_F(S_i \cap U) < -\tfrac{1}{2}\kappa r^d\,.
$$

Up to choosing a bigger $i \in \mathbb{N}$, we get $\Phi_F(\tilde{p}_r[S_i \cap U]) < \mu$, which contradicts the definition of μ. □

7 Equivalence of BC and AC

In this section we prove that the new condition BC can be used in place of AC.

7.1 Lemma. *Let $x \in \mathbf{R}^n$. We have $\mathrm{AC}_x = \mathrm{BC}_x$.*

Proof. Step 1 We first prove that $\mathrm{AC}_x \subseteq \mathrm{BC}_x$. Let $F \in \mathrm{AC}_x$, $k \in \{1,2,\dots,n\}$, μ be a Radon probability measure over $\mathbf{G}(n,d)$, and $T \in \mathbf{G}(n,k)$. We define the varifold

$$
W = (\mathscr{H}^k \llcorner T) \times \mu \in \mathbf{V}_d(\mathbf{R}^n)\,.
$$

Assume that $\delta_{F^x} W = 0$. We will show that $k \ge d$ and if $k = d$, then $\mu = \mathrm{Dirac}(T)$, i.e., that $F \in \mathrm{BC}_x$. By the very definition of anisotropic first variation, we deduce that for every test vector field $g \in \mathscr{X}(\mathbf{R}^n)$

$$
\begin{aligned}
(14)\quad 0 = \delta_{F^x} W(g) &= \int B_F(x,S) \bullet \mathrm{D}g(y)\,\mathrm{d}W(y,S) \\
&= \int\!\!\int B_F(x,S) \bullet \mathrm{D}g(y)\,\mathrm{d}(\mathscr{H}^k \llcorner T)(y)\,\mathrm{d}\mu(S) = \int A_x(\mu) \bullet \mathrm{D}g(y)\,\mathrm{d}(\mathscr{H}^k \llcorner T)(y)\,.
\end{aligned}
$$

Let $e_1, \dots, e_{n-k}$ be an orthonormal basis of $T^\perp$. For any $\varphi \in \mathscr{D}(T, \mathbf{R})$, $i, j \in \{1, 2, \dots, n-k\}$, we can find $g \in \mathscr{X}(\mathbf{R}^n)$ such that

$$g(y) = \varphi(T_\natural y)(y \bullet e_i)e_j \quad \text{whenever } y \in (T + \mathbf{B}(0,1))\,;$$

hence, equation (14) yields

$$\int \varphi(y) A_x(\mu) e_i \bullet e_j \, \mathrm{d}(\mathscr{H}^k \llcorner T)(y) = 0 \quad \text{for all } \varphi \in \mathscr{D}(T, \mathbf{R}) \text{ and } i, j \in \{1, 2, \dots, n-k\}\,,$$

which shows that

$$T^\perp \subseteq \ker A_x(\mu)\,.$$

Since $\dim T^\perp = n - k$, we deduce that $\dim \ker A_x(\mu) \geq n - k$. By Definition 4.7(a) we obtain $n - k \leq \dim \ker A_x(\mu) \leq n - d$, so $k \geq d$ and we get Definition 4.8(a).

If $k = d$, then it follows from Definition 4.7(b) that $\mu = \mathrm{Dirac}(S)$ for some $S \in \mathbf{G}(n, d)$. Then

$$A_x(\mu) = B_F(x, S)\,.$$

Directly from the definition of $B_F(x, S)$ it follows that $S^\perp \subseteq \ker B_F(x, S)$. Therefore, since $\dim \ker B_F(x, S) = n - d$ and $T^\perp \subseteq \ker B_F(x, S) = \ker A_x(\mu)$, we see that $S = T$, which settles Definition 4.8(b).

Step 2 We prove now that $\mathrm{BC}_x \subseteq \mathrm{AC}_x$. Assume $F \in \mathrm{BC}_x$. Given a Radon probability measure μ over $\mathbf{G}(n, d)$, we define

$$T = \mathrm{im}(A_x(\mu)^*)\,, \quad k = \dim T\,, \quad V = (\mathscr{H}^k \llcorner T) \times \mu \in \mathbf{V}_d(\mathbf{R}^n)\,.$$

Note that $T^\perp = [\mathrm{im}(A_x(\mu)^*)]^\perp = \ker A_x(\mu)$. Thus, by equation (14), we get that for every $g \in \mathscr{X}(\mathbf{R}^n)$

$$\delta_{F^x} V(g) = A_x(\mu) \bullet \int \mathrm{D}(g \circ T_\natural)(y) \, \mathrm{d}(\mathscr{H}^k \llcorner T)(y) + \int A_x(\mu) \bullet \left(\mathrm{D}g(y) \circ T_\natural^\perp\right) \mathrm{d}(\mathscr{H}^k \llcorner T)(y) = 0.$$

By Definition 4.8, we obtain $k \geq d$ and conclude that

$$\dim \ker A_x(\mu) = n - \dim T \leq n - d\,,$$

which is Definition 4.7(a). Moreover, if $\dim \ker A_x(\mu) = n - d$, then $\dim T = d$ and we can apply Definition 4.8 to the varifold V and deduce that $\mu = \mathrm{Dirac}(T)$, which is precisely Definition 4.7(b). □

8 The inclusion $\mathrm{wBC} \subseteq \mathrm{AE}(\mathcal{P})$

In this section we work with cubical test pairs (S, Q), where Q is now a d-dimensional cube; see Definition 8.1. Cubical test pairs give rise to the same classes of Almgren elliptic integrands as the test pairs defined in Definition 4.1; see Remark 8.2.

The main result is Theorem 8.8, which shows that $\mathrm{wBC}_x \subseteq \mathrm{AE}_x(\mathcal{P})$ given $\mathcal{P}$ is closed under Lipschitz deformations leaving the boundary fixed and under gluing together several rescaled copies of an element of $\mathcal{P}$; see Definition 8.5.

The second closedness property for $\mathcal{P}$ is needed to be able to perform a "blow-down - homogenization" argument. More precisely, given a minimiser P of Φ_{F^x} in $\{R : (R, Q) \in \mathcal{P}\}$

we construct the varifold W, occurring in Definition 4.8, so that $W \llcorner Q \times \mathbf{G}(n,d)$ is a limit of a sequence of varifolds $\tilde{W}_N = \mathbf{v}_d(P_N)$, where P_N is constructed, for $N \in \mathbb{N}$, by gluing together 2^{Nd} rescaled copies of P. A crucial observation is that P_N has the same Φ_{F^x} energy as P which, in turn, is a minimiser of Φ_{F^x} in $\mathcal{P}$. This allows us to deduce that $\delta_{F^x} W_N = 0$ using Lemma 6.6, provided P_N is a competitor (or a limit of competitors), i.e., if $(P_N, Q) \in \mathcal{P}$ for an appropriate choice of the cube Q.

It is not at all obvious that 8.8 is valid with $\mathcal{P}$ being the set of all cubical test pairs; see Remark 8.6. The proof that such family $\mathcal{P}$ has the necessary closedness property requires some subtle topological arguments and is postponed to Section 9; see 9.23.

8.1 Definition. Let $Q_0 = [-1,1]^d \subseteq \mathbf{R}^d$. We say that (S,Q) is a *cubical test pair* if there exists $p \in \mathbf{O}^*(n,d)$ such that

$$Q = p^*[Q_0]\,, \quad B = p^*[\partial Q_0]\,, \quad S \subseteq \mathbf{R}^n \text{ is compact and } (\mathscr{H}^d, d) \text{ rectifiable}\,,$$
$$f[S] \neq B \text{ for all } f : \mathbf{R}^n \to \mathbf{R}^n \text{ satisfying } \operatorname{Lip} f < \infty \text{ and } f(x) = x \text{ for } x \in B\,.$$

8.2 Remark. In the rest of the paper we will work for simplicity on cubical test pairs, but it's worth to remark that the two notions are perfectly equivalent for our purposes. Indeed, if we denote with $\mathcal{P}_1$ the set of rectifiable test pairs and with $\mathcal{P}_2$ the set of cubical test pairs, then we easily verify that for every F being a $\mathscr{C}^0$ integrand and $x \in \mathbf{R}^n$, it holds $\mathrm{AE}_x(\mathcal{P}_1) = \mathrm{AE}_x(\mathcal{P}_2)$ and $\mathrm{AUE}_x(\mathcal{P}_1) = \mathrm{AUE}_x(\mathcal{P}_2)$. To show this, we denote $\rho = \sqrt{d}$ and $Q_0 = [-1,1]^d$.

Given $(S,Q) \in \mathcal{P}_2$, we find $p \in \mathbf{O}^*(n,d)$ such that $Q = p^*[Q_0]$ and construct $(R,D) \in \mathcal{P}_1$ by setting

$$T = \operatorname{im} p^*\,, \quad D = T \cap \mathbf{B}(0,1)\,, \quad \bar{D} = \boldsymbol{\mu}_\rho[D]\,, \quad \bar{R} = S \cup (\bar{D} \sim Q)\,, \quad R = \boldsymbol{\mu}_{1/\rho}[\bar{R}]\,.$$

Then

$$\rho^d\big(\Phi_{F^x}(R) - \Phi_{F^x}(D)\big) = \Phi_{F^x}(\bar{R}) - \Phi_{F^x}(\bar{D}) = \Phi_{F^x}(S) - \Phi_{F^x}(Q)\,.$$

Given $(R,D) \in \mathcal{P}_1$ we choose $p \in \mathbf{O}^*(n,d)$ such that $D \subseteq \operatorname{im} p^*$ and construct $(S,Q) \in \mathcal{P}_2$ by setting

$$Q = p^*[Q_0]\,, \quad \bar{Q} = \boldsymbol{\mu}_\rho[\bar{Q}]\,, \quad \bar{S} = R \cup (\bar{Q} \sim D)\,, \quad S = \boldsymbol{\mu}_{1/\rho}[\bar{S}]\,.$$

Then

$$\rho^d\big(\Phi_{F^x}(S) - \Phi_{F^x}(Q)\big) = \Phi_{F^x}(\bar{S}) - \Phi_{F^x}(\bar{Q}) = \Phi_{F^x}(R) - \Phi_{F^x}(D)\,.$$

Therefore, $\mathrm{AE}_x(\mathcal{P}_1) = \mathrm{AE}_x(\mathcal{P}_2)$ and $\mathrm{AUE}_x(\mathcal{P}_1) = \mathrm{AUE}_x(\mathcal{P}_2)$.

8.3 Definition. Let Q be a d-dimensional cube in $\mathbf{R}^n$ (see Definition 5.1), and $X \subseteq \mathbf{R}^n$. We say that (Y,Q) is a *multiplication* of (X,Q) if there exist $k \in \mathscr{P}$ and a finite set $\mathcal{A}$ of d-dimensional cubes in $\mathbf{R}^n$ of side-length $\mathbf{l}(Q)/k$ such that

$$Q = \bigcup \mathcal{A}\,, \quad \mathrm{Int_c}(K) \cap \mathrm{Int_c}(L) = \varnothing \ \forall K \neq L \in \mathcal{A}\,, \quad Y = \bigcup \big\{ \boldsymbol{\tau}_{\mathbf{c}(K)} \circ \boldsymbol{\mu}_{1/k} \circ \boldsymbol{\tau}_{-\mathbf{c}(Q)}[X] : K \in \mathcal{A} \big\}\,.$$

8.4 Remark. Observe that a multiplication (Y,Q) of (X,Q) is uniquely determined by the parameter k occurring in Definition 8.3. Thus, we may define the *k-multiplication of* (X,Q) to be exactly (Y,Q).

8.5 Definition. We say that a set $\mathcal{Q}$ of pairs of subsets of $\mathbf{R}^n$ is a *good family* if

(a) all elements of $\mathcal{Q}$ are cubical test pairs;

(b) if $(X, Q) \in \mathcal{Q}$, $N \in \mathbb{N}$, and (Y, Q) is the 2^N-multiplication of (X, Q), then $(Y, Q) \in \mathcal{Q}$;

(c) if $(X, Q) \in \mathcal{Q}$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is Lipschitz, and $f(x) = x$ for $x \in \partial_c Q$, then $(f[X], Q) \in \mathcal{Q}$.

8.6 Remark. It is plausible that the set of all cubical test pairs is a good family and, indeed, in Section 9 we prove it is. However, this is not at all obvious.

Consider the Adams' surface; see [Rei60, Example 8 on p. 81]. The Möbius strip M and the triple Möbius strip T are both homotopy equivalent to the 1-dimensional sphere and both can be continuously embedded in some $\mathbf{R}^n$ so that (M, Q) and (T, Q) become cubical test pairs, where $Q = [0, 1]^2 \times \{0\}^{n-2}$. However, if one puts M and T side by side touching only along one 1-dimensional face of Q, then one obtains the Adams' surface A, which retracts onto its boundary. This, as explained in [Rei60, Example 8 on p. 81], is a consequence of the fact that the inclusion of the boundary of M into M has degree 2, the inclusion of the boundary of T into T has degree 3, these numbers are relatively prime, and A is homotopy equivalent to the wedge sum (a.k.a. "bouquet"; see 9.6) of two circles so, defining $f : A \to \mathbb{S}^1$ to be of degree -1 on M and of degree 1 on T, we get a map such that $f \circ j$ is of degree one, where $j : \mathbb{S}^1 \to A$ is a parameterization of the boundary of A. One can then construct a Lipschitz retraction of A onto its boundary; see 9.5. Luckily for us, the situation is different if one puts together many copies of *the same* set X. We prove in 9.16 that if (X, Q) is a cubical test pair, then one cannot have two maps $f, g : X \to \partial_c Q$ such that $\deg(f|_{\partial_c Q})$ and $\deg(g|_{\partial_c Q})$ are relatively prime.

Before stating and proving the main theorem of this section, we need the following lemma, which, roughly speaking, will be used as an *almost* uniqueness result for minimizers of the area functional in the class of cubical test pairs:

8.7 Lemma. *Given a cubical test pair (R, Q) as in Definition 8.1 and $x \in \mathbf{R}^n$. If*

$$\Phi_{F^x}(R) < \Phi_{F^x}(Q)\,, \tag{15}$$

then

$$\mathscr{H}^d(R) > \mathscr{H}^d(Q)\,. \tag{16}$$

Proof. Assume by contradiction that (16) does not hold. Thus in particular

$$\mathscr{H}^d(R \cap (Q \times \mathbf{R}^{n-d})) \le \mathscr{H}^d(R) \le \mathscr{H}^d(Q)\,. \tag{17}$$

Denoting with T the d-plane containing Q, we observe that

$$\mathscr{H}^d(R \cap (Q \times \mathbf{R}^{n-d})) \ge \mathscr{H}^d(T_\natural(R \cap (Q \times \mathbf{R}^{n-d}))) \ge \mathscr{H}^d(Q)\,, \tag{18}$$

otherwise there would exist a d-dimensional open ball $B \subset Q$ such that

$$(B \times \mathbf{R}^{n-d}) \cap R = \emptyset\,. \tag{19}$$

Since R is compact, then (19) would imply the existence of $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfying $\operatorname{Lip} f < \infty$ and $f(x) = x$ for $x \in \partial_c Q$, such that $f[R] = \partial_c Q$, which would contradict the property of

(R, Q) being a cubical test pair. By (18) and the area formula (a.f.) [Fed69, 3.2.20], we compute

$$\begin{aligned} \mathscr{H}^d(Q) &\overset{(18)}{\leq} \mathscr{H}^d(T_\natural(R \cap (Q \times \mathbf{R}^{n-d}))) \leq \int_Q \mathscr{H}^0(T_\natural^{-1}(y) \cap R) \, \mathrm{d}\mathscr{H}^d(y) \\ &\overset{(a.f.)}{=} \int_{R \cap (Q \times \mathbf{R}^{n-d})} \operatorname{ap} J_d T_\natural(y) \, \mathrm{d}\mathscr{H}^d(y) \leq \mathscr{H}^d(R \cap (Q \times \mathbf{R}^{n-d})) \overset{(17)}{\leq} \mathscr{H}^d(Q) \,. \end{aligned} \tag{20}$$

Then the inequalities in (20) are all equality, which implies that $\operatorname{ap} J_d T_\natural(y) = 1$ for $\mathscr{H}^d$-a.e. $y \in R \cap (Q \times \mathbf{R}^{n-d})$. Hence,

$$\operatorname{Tan}^d(\mathscr{H}^d \llcorner R, y) = T, \qquad \text{for } \mathscr{H}^d\text{-a.e. } y \in R \cap (Q \times \mathbf{R}^{n-d}) \,. \tag{21}$$

We can then compute the following chain of inequalities, which provides a contradiction

$$\begin{aligned} \Phi_{F^x}(Q) = \int_Q F^x(T) \, \mathrm{d}\mathscr{H}^d(y) &\overset{(18)}{\leq} \int_{R \cap (Q \times \mathbf{R}^{n-d})} F^x(T) \, \mathrm{d}\mathscr{H}^d(y) \\ &\overset{(21)}{\leq} \Phi_{F^x}(R \cap (Q \times \mathbf{R}^{n-d})) \leq \Phi_{F^x}(R) \overset{(15)}{<} \Phi_{F^x}(Q) \,. \end{aligned}$$

□

We can finally prove the following:

8.8 Theorem. *Assume $x \in \mathbf{R}^n$ and $\mathcal{P}$ is a good family (cf. Definition 8.5). Then* $\mathrm{wBC}_x \subseteq \mathrm{AE}_x(\mathcal{P})$.

Proof. We proceed by contradiction. Assume $F \in \mathrm{wBC}_x \sim \mathrm{AE}_x(\mathcal{P})$. Then there exists $(S, Q) \in \mathcal{P}$ such that

$$\mathscr{H}^d(S) > \mathscr{H}^d(Q) \quad \text{and} \quad \Phi_{F^x}(S) \leq \Phi_{F^x}(Q) \,.$$

Define

$$B = \partial_c Q \quad \text{and} \quad \mathcal{C} = \{S : (S, Q) \in \mathcal{P}\} \,.$$

Note that $\mathcal{C}$ is a good class in $\mathbf{R}^n \sim B$ in the sense of Definition 6.3.

Next, we employ Theorem 6.7 with F^x in place of F together with Theorem 6.5(c)(a)(k) to find a compact $(\mathscr{H}^d, d)$ rectifiable set $R \subseteq \mathbf{R}^n$ such that

$$\Phi_{F^x}(R) = \inf\{\Phi_{F^x}(P) : P \in \mathcal{C}\} \leq \Phi_{F^x}(S) \leq \Phi_{F^x}(Q) \,.$$

Proceeding as in Lemma 4.4 we see that (R, Q) is a cubical test pair (may be not in $\mathcal{P}$). In case $\Phi_{F^x}(R) < \Phi_{F^x}(Q)$, by Lemma 8.7 we get $\mathscr{H}^d(R) > \mathscr{H}^d(Q)$, and we set $P = R$. Otherwise, we have $\Phi_{F^x}(R) = \Phi_{F^x}(Q) = \Phi_{F^x}(S)$ and we set $P = S$. In any case, setting $V = \mathbf{v}_d(P) \in \mathbf{V}_d(\mathbf{R}^n)$ and using Lemma 6.6, we obtain

$$\infty > \mathscr{H}^d(P) > \mathscr{H}^d(Q) \quad \text{and} \quad \delta_{F^x} V(g) = 0 \quad \text{for } g \in \mathscr{X}(\mathbf{R}^n \sim B) \,.$$

Let $p \in \mathbf{O}^*(n, d)$ and $T \in \mathbf{G}(n, d)$ be such that $p^*[Q_0] = Q \subseteq T$, where $Q_0 = [-1, 1]^d$. For each $N \in \mathbb{N}$ we define P_N and $\mathcal{A}_N$ so that (P_N, Q) is the 2^N-multiplication of (P, Q) and $\mathcal{A}_N$ is the corresponding set of d-dimensional cubes covering Q as in Definition 8.3. We also set

$$W_N = \sum_{v \in \mathbf{Z}^d} \mathbf{v}_d(\boldsymbol{\tau}_{p^*(2v)}[P_N]) \in \mathbf{V}_d(\mathbf{R}^n) \quad \text{and} \quad R_K = \boldsymbol{\tau}_{\mathbf{c}(K)} \circ \boldsymbol{\mu}_{2^{-N+1}}[P] \quad \text{for } K \in \mathcal{A}_N \,.$$

Observe that for $N \in \mathbb{N}$ and $\rho \in (0, \infty)$ there are at most $\boldsymbol{\alpha}(d)\big(\rho + \operatorname{diam} P\big)^d$ translated copies of P_N in $\operatorname{spt} \|W_N\| \cap \mathbf{B}(0, \rho)$; therefore,

$$\|W_N\| \, \mathbf{B}(0, \rho) \leq \boldsymbol{\alpha}(d)\big(\rho + \operatorname{diam} P\big)^d \mathscr{H}^d(P_N) = \boldsymbol{\alpha}(d)\big(\rho + \operatorname{diam} P\big)^d \mathscr{H}^d(P) \quad \text{for } \rho \in (0, \infty).$$

So W_N is a Radon measure and there exists a subsequence $\{W_{N_i} : i \in \mathbb{N}\}$ which converges to some varifold W in $\mathbf{V}_d(\mathbf{R}^n)$. Moreover, we have

$$R_K \subseteq T + \mathbf{B}(0, 2^{-N} \operatorname{diam} P) \quad \text{for } K \in \mathcal{A}_N \quad \text{so } \operatorname{spt} \|W\| \subseteq T \,.$$

Directly from the construction and by density of base 2 rational numbers in $\mathbf{R}$, it follows also that W is translation invariant in T, i.e., $(\boldsymbol{\tau}_v)_\# W = W$ for all $v \in T$. Hence, there exists $\vartheta \in (0, \infty)$ and a Radon probability measure μ over $\mathbf{G}(n, d)$ such that

$$W = \vartheta(\mathscr{H}^d \llcorner T) \times \mu \quad \text{and} \quad \vartheta = \frac{\mathscr{H}^d(P)}{\mathscr{H}^d(Q)} > 1 \,.$$

We define

$$\tilde{W}_N = \mathbf{v}_d(P_N) \in \mathbf{V}_d(\mathbf{R}^n) \quad \text{for } N \in \mathbb{N} \quad \text{and} \quad \tilde{W} = \lim_{i \to \infty} \tilde{W}_{N_i} = \vartheta(\mathscr{H}^d \llcorner Q) \times \mu \,.$$

We also record that

$$\mathscr{H}^d(P_N) = \mathscr{H}^d(P) \quad \text{and} \quad \Phi_{F^x}(P_N) = \Phi_{F^x}(P) \quad \text{for } N \in \mathbb{N} \,,$$

and since the supports of $\|\tilde{W}_N\|$ for $N \in \mathbb{N}$ all lie in a fixed compact set (cf. Remark 3.11) we also have

$$\Phi_{F^x}(\tilde{W}) = \lim_{i \to \infty} \Phi_{F^x}(\tilde{W}_{N_i}) = \lim_{i \to \infty} \Phi_{F^x}(P_{N_i}) = \Phi_{F^x}(P) \,. \tag{22}$$

We claim that

$$\delta_{F^x} W = 0. \tag{23}$$

First we observe that this would immediately give a contradiction and conclude the proof. Indeed, since $F \in \mathrm{wBC}_x$, we deduce from (23) and Definition 4.8 that $\mu = \mathrm{Dirac}(T)$. This, in turn, yields the following contradiction

$$\Phi_{F^x}(Q) < \vartheta \Phi_{F^x}(Q) = \Phi_{F^x}(\tilde{W}) \overset{(22)}{=} \Phi_{F^x}(P) \leq \Phi_{F^x}(Q) \,.$$

We are just left to prove the claim (23). To this end, since W is invariant under translations in T, it suffices to show that

$$\delta_{F^x} \tilde{W}_N(g) = 0 \quad \text{for } N \in \mathbb{N} \text{ and } g \in \mathscr{X}(\mathbf{R}^n \sim B) \,.$$

If $P = S \in \mathcal{C}$, since $\mathcal{C}$ is a good family, then $P_N \in \mathcal{C}$ and $\tilde{W}_N = \mathbf{v}_d(P_N)$ and $\|\tilde{W}_N\|(\mathbf{R}^n) = \mathscr{H}^d(P) = \inf\{\Phi_{F^x}(K) : K \in \mathcal{C}\}$ for $N \in \mathbb{N}$; hence, applying Lemma 6.6, we see that $\delta_{F^x} \tilde{W}_N(g) = 0$ for $g \in \mathscr{X}(\mathbf{R}^n \sim B)$ and $N \in \mathbb{N}$.

In case $P = R$, we use Theorem 6.5 to find a minimising sequence $\{S_i \in \mathcal{C} : i \in \mathbb{N}\}$ such that $\mathbf{v}_d(P) = V = \lim_{i \to \infty} \mathbf{v}_d(S_i \cap \mathbf{R}^n \sim B)$. Defining $S_{i,N} \in \mathcal{C}$ so that $(S_{i,N}, Q)$ is the 2^N-multiplication of (S_i, Q) we get $\tilde{W}_N = \lim_{i \to \infty} \mathbf{v}_d(S_{i,N})$. Recalling Theorem 6.5(b)(c)(h) we may once again apply Lemma 6.6 to see that also in this case $\delta_{F^x} \tilde{W}_N(g) = 0$ for $g \in \mathscr{X}(\mathbf{R}^n \sim B)$ and $N \in \mathbb{N}$ so the proof is done. □

9 Cubical test pairs form a good family

Here we prove that the family of all cubical test pairs is good in the sense of 8.5. To our surprise the proof had to employ a few sophisticated (yet classical) tools of algebraic topology. Given a cubical test pair (X, Q) and its 2^N-multiplication (Y, Q) we need to show that $S = \partial_{\mathrm{c}} Q$ is not a Lipschitz retract of Y, which is the same as showing that there is no continuous map $f : Y \to S$ with $\deg(f|_S) = 1$; cf. 9.5. This becomes a topological problem of independent interest. We first sketch the idea of the proof, highlighting the main points of the argument.

Let (X, Q) be a cubical test pair. To be able to use tools of algebraic topology we need to pass from an arbitrary compact set X satisfying $0 < \mathscr{H}^d(X) < \infty$ to an open set U containing X and having homotopy type of a d-dimensional CW-complex. We achieve this by applying the deformation theorem 5.8 to X, obtaining an open set $U \subseteq \mathbf{R}^n$ with $X \subseteq U$ and a d-dimensional cubical complex $E \subseteq U$ such that $\partial_{\mathrm{c}} Q \subseteq E \subseteq U$ and E is a strong deformation retract of U; see 9.18. Moreover, we get that (U, E) is a *Borsuk pair*, i.e., has the homotopy extension property HEP; see 9.2 and 9.3, which will be a useful tool to get suitable homotopy equivalences.

The topological part of the argument works as follows. Consider a 2-multiplication $(\tilde{Y}, Q)$ of (U, Q) and assume there exists a retraction $\tilde{r} : \tilde{Y} \to \partial_{\mathrm{c}} Q$. Note that $\partial_{\mathrm{c}} Q$ is a topological $(d-1)$-dimensional sphere and set $S = \partial_{\mathrm{c}} Q$. Different copies of $\boldsymbol{\mu}_{1/2}[U \sim S]$ may, in general, intersect inside $\tilde{Y}$. Thus, we define the *lifted* 2-*multiplication* (Y, Q) of (U, Q) in order to prevent this intersection and we observe that $\tilde{r}$ gives rise to a retraction $r : Y \to S$; cf. 9.20. Next, we consider the pairwise orthogonal affine $(d-1)$-planes, lying in the affine d-plane spanned by Q, parallel to the sides of Q, and passing through the center of Q. We denote with R the union of these planes intersected with Q. Since R is contractible, by the aforementioned HEP, we deduce that Y is homotopy equivalent to Y/R which, in turn, is homotopy equivalent to the wedge sum Z of 2^d copies of U; see 9.6. Let Σ be the wedge sum of 2^d copies of S, $\pi_i : \Sigma \to S$ be projections onto particular components of Σ, $\tau_i : S \hookrightarrow \Sigma$ be inclusions of components, and $j : \Sigma \hookrightarrow Z$ be the inclusion map; cf. 9.7. The inclusion $S \hookrightarrow Y$ composed with the homotopy equivalences yields a map $\alpha : S \to \Sigma \subset Z$ such that $\deg(\pi_i \circ \alpha) = 1$ for all $i \in \{1, 2, \ldots, 2^d\}$. In particular, since $\mathbf{H}_{d-1}(\Sigma) \simeq \bigoplus_{i=1}^{2^d} \mathbf{H}_{d-1}(S) = \mathbf{Z}^{2^d}$ by [Hat02, Corollary 2.25], we get

$$\alpha_* = \textstyle\sum_{i=1}^{2^d} \tau_{i*} : \mathbf{H}_{d-1}(S) \to \mathbf{H}_{d-1}(\Sigma)\,. \tag{24}$$

If $\rho : Z \to S$ is obtained by composing the retraction r with the homotopy equivalences, then $\deg(\rho \circ j \circ \alpha) = 1$. The following homotopy commutative diagram presents the situation.

$$\begin{tikzcd}
 & & S \arrow[r, "\tau_i"] & \Sigma = \bigvee_{i=1}^{2^d} S \arrow[r, "\pi_i"] \arrow[d, "j"] & S \\
S \arrow[r] \arrow[urrr, bend left=10, "\alpha"] & Y \arrow[r, shift left, "\approx"] \arrow[rrr, bend right, "r"'] & Y/R \arrow[l, shift left] \arrow[r, shift left, "\approx"] & Z = \bigvee_{i=1}^{2^d} U \arrow[l, shift left] \arrow[r, "\rho"] & S
\end{tikzcd}$$

Recalling (24) we see that the degree of $\rho \circ j \circ \alpha$ is a linear combination with integer coefficients of the numbers $m_i = \deg(\rho \circ j \circ \tau_i)$. Hence, the Euclidean algorithm shows that the greatest common divisor of $m_1, \ldots, m_{2^d}$ equals one. Since Z is a wedge sum of copies of *the same* space U, we get 2^d maps $f_i : U \to S$ and integers $a_i \in \mathbf{Z}$ such that $\deg(f_i|_S) = m_i$ and $\sum_{i=1}^{2^d} a_i m_i = 1$. The question now is whether there exists $g : U \to S$ which induces the map

$$\sum_{i=1}^{2^d} a_i f_{i*} : \mathbf{H}_{d-1}(U) \to \mathbf{H}_{d-1}(S) = \mathbf{Z}.$$

If so, then $\deg(g|_S) = 1$ and g yields a retraction $U \to S$ by 9.5.

This is the point where we need to employ algebra and algebraic topology. We prove in 9.13 that if E is a d-dimensional CW-complex, then any homomorphism $\zeta : \mathbf{H}_{d-1}(E) \to \mathbf{Z}$ is induced by some map $g : E \to S$. The cellular homology of E (which coincides with the singular homology) is computed from the chain complex $(C_k, \delta_k)_{k=0}^d$, where the group of k-dimensional *chains* C_k is the free abelian group generated by the k-dimensional cells (or cubes) of E. Observe that if G is a torsion group (i.e. every element has finite order), then there exists only one homomorphism $G \to \mathbf{Z}$, namely, the one sending all elements of G to zero. Therefore, we do not lose any information by composing the homomorphism ζ with the projection $p : \ker \delta_{d-1} \twoheadrightarrow \ker \delta_{d-1} / \operatorname{im} \delta_d = \mathbf{H}_{d-1}(E)$, which yields a homomorphism $\xi = \zeta \circ p$ defined on *cycles*. Since C_{d-1} and C_{d-2} are *free groups* (in particular, projective $\mathbf{Z}$-modules), the group C_{d-1} splits into a direct sum $C_{d-1} = \ker(\delta_{d-1}) \oplus H$ and we can extend ξ to all *chains* by setting $\xi|_H = 0$; cf. 9.12. Hence we can define g on any $(d-1)$-dimensional cell σ of E as $g|_\sigma = h_\sigma \circ \pi$, where $\pi : \sigma \twoheadrightarrow \sigma/\partial_c \sigma \simeq S$ and $h_\sigma : S \to S$ is a map of degree $\xi(\sigma)$. The next step is to extend g to all the d-dimensional cells of E. To this end we employ the *obstruction theory*, which is a sophisticated version of the Brouwer fixed-point theorem and its consequence: the fact that a map $S \to S$ extends to a map $Q \to S$ if and only if its topological degree is zero. Given a d-dimensional cell ω of E, we need to ensure that $g|_{\partial_c \omega}$ has topological degree zero. Recalling that $\xi(\delta_d \omega) = \zeta \circ p(\delta_d \omega) = 0$ whenever $\omega \in C_d$, the required condition on g follows.

To conclude the argument, we observe that the 2^N-multiplication of (X, Q) is the same as the 2-multiplication of (W, Q), where W is the 2^{N-1}-multiplication of (X, Q); thus, we get the result by induction.

9.1 Definition. For $k \in \mathbb{N}$ we set $\mathbb{S}^k = \mathbf{R}^{k+1} \cap \partial \mathbf{B}(0, 1)$.

9.2 Definition (cf. [Hat02, Chap. 0, p. 14])**.** Let X be a topological space and $A \subseteq X$ be a subspace. Set $I = [0, 1] \subseteq \mathbf{R}$. We say that the pair (X, A) has the *homotopy extension property HEP* if for every topological space Y every continuous function $h : (X \times \{0\}) \cup (A \times I) \to Y$ extends to a continuous homotopy $H : X \times I \to Y$.

9.3 Remark (cf. [Hat02, Chap. 0, Example 0.15, p. 15]). If $k \in \mathscr{P}$, $A \subseteq X \subseteq \mathbf{R}^n$, A is compact of dimension k, and there exists an open set $U \subseteq \mathbf{R}^n$ such that $A \subseteq U \subseteq X$ and U is homeomorphic to $A \times \mathbf{R}^{n-k}$ (i.e. U is a trivial vector bundle over A with fiber $\mathbf{R}^{n-k}$), then (X, A) has the HEP. In particular, if A is a sum of a finite set of k-dimensional cubes and $A \subseteq \operatorname{Int} X$, then (X, A) has the HEP.

Note also that if $X, Y \subseteq \mathbf{R}^n$, $A = X \cap Y$, and both (X, A) and (Y, A) have the HEP, than $(X \cup Y, A)$ has the HEP.

9.4 Remark (cf. [Hat02, Chap. 0, Prop. 0.17, p. 15]). If (X, A) has the HEP and A is contractible, then X and X/A are homotopy equivalent.

9.5 Lemma. *Assume $S, X \subseteq \mathbf{R}^n$ are compact, $S \subseteq X$, $\varepsilon \in (0,1)$, and there exists a Lipschitz retraction $\pi : S + \mathbf{B}(0,\varepsilon) \to S$. Let $j : S \to \mathbf{R}^n$ be the inclusion map.*

The following properties are equivalent:

(a) *S is a Lipschitz retract of X;*

(b) *S is a retract of X;*

(c) *there exists $\delta \in (0,\varepsilon)$ such that S is a retract of $X + \mathbf{B}(0,\delta)$;*

(d) *there exist a continuous map $f : X \to S$ such that* $\deg(f \circ j) = 1$.

Proof. Clearly the implications $(a) \Rightarrow (b)$, $(c) \Rightarrow (b)$, $(b) \Rightarrow (d)$ hold.

Proof of $(b) \Rightarrow (a)$*:* Assume $r : X \to S$ is a retraction. Using the Tietze extension theorem (see e.g. [Kel75, Chap. 7, Problem O, p. 242]), we extend r to a continuous function $\tilde{R} : \mathbf{R}^n \to \mathbf{R}^n$. We mollify $\tilde{R}$ to obtain a smooth function $R : \mathbf{R}^n \to \mathbf{R}^n$ such that $|R(x) - r(x)| \le 2^{-12}\varepsilon$ for $x \in X$; in particular, $\operatorname{dist}(R(x), S) \le 2^{-12}\varepsilon$ for $x \in X$ so $\pi \circ R : X \to S$ is well defined. Since $r(x) = \pi(x)$ for $x \in S$, there exists $\delta \in (0,\varepsilon)$ such that $|R(x) - \pi(x)| \le 2^{-8}\varepsilon$ for $x \in S + \mathbf{B}(0,\delta)$. Finally, we define a Lipschitz retraction $f : X \to S$ by

$$f(x) = \begin{cases} \pi(x) & \text{if } \operatorname{dist}(x,S) \le 2^{-8}\delta\,, \\ \pi(R(x)) & \text{if } \operatorname{dist}(x,S) \ge 2^{-7}\delta\,, \\ \pi\big((1-t)\pi(x) + t\pi(R(x))\big) & \text{if } t = 2^8 \operatorname{dist}(x,S)/\delta - 1 \in (0,1)\,. \end{cases}$$

Proof of $(b) \Rightarrow (c)$*:* Assume $r : X \to S$ is a retraction. Once again we extend r to a continuous function $R : \mathbf{R}^n \to \mathbf{R}^n$. Note that R is uniformly continuous on every compact subset of $\mathbf{R}^n$; hence, there exists $\delta \in (0,1)$ such that $R[X + \mathbf{B}(0,\delta)] \subseteq S + \mathbf{B}(0,\varepsilon)$. We get that $\pi \circ R|_{X+\mathbf{B}(0,\delta)}$ is the desired retraction.

Proof of $(d) \Rightarrow (b)$*:* Let $f : X \to S$ be continuous and such that $\deg(f \circ j) = 1$. Then there exists a continuous homotopy $h : S \times I \to S$ such that $h(x,0) = f(x)$ and $h(x,1) = x$ for $x \in S$. We extend f to a continuous function $F : \mathbf{R}^n \to \mathbf{R}^n$ using the Tietze extension theorem and we find $\delta \in (0,1)$ such that $F[X + \mathbf{B}(0,\delta)] \subseteq S + \mathbf{B}(0,\varepsilon)$. Set $Y = X + \mathbf{B}(0,\delta)$. Observe that $\pi \circ F|_Y : Y \to S$ is well defined. Recalling 9.3, we see that the pair (Y,S) has the HEP. Therefore, we may extend h to a homotopy $H : Y \times I \to S$ such that $H(x,0) = \pi(F(x))$ for every $x \in Y$. The desired retraction $r : X \to S$ is then given by $r(x) = H(x,1)$ for $x \in X$. □

9.6 Definition. Assume J is an index set and for each $\alpha \in J$ we are given a pointed topological space (X_α, x_α). We define the *wedge sum* to be the pointed topological space

$$\textstyle\bigvee_{\alpha\in J}(X_\alpha, x_\alpha) = \big(\bigcup\big\{X_\alpha \times \{\alpha\} : \alpha \in J\big\}\big)/\big\{(x_\alpha,\alpha) : \alpha \in J\big\}$$

endowed with the quotient topology.

If $J = \{1, 2, \dots N\}$ for some $N \in \mathscr{P}$, then we use the notation

$$\textstyle\bigvee_{\alpha\in J}(X_\alpha, x_\alpha) = \bigvee_{i=1}^N (X_i, x_i) = (X_1, x_1) \vee (X_2, x_2) \vee \cdots \vee (X_N, x_N)\,.$$

9.7 Remark. (a) Let $Z = \bigvee_{\alpha\in J}(X_\alpha, x_\alpha)$ and $\alpha \in J$. There exist continuous maps $\tau_\alpha : X_\alpha \hookrightarrow Z$ and $\pi_\alpha : Z \twoheadrightarrow X_\alpha$. The first one is simply the inclusion and the second comes from the projection $Z \twoheadrightarrow Z/\bigvee_{\beta\in J\sim\{\alpha\}}(X_\beta, x_\beta)$.

(b) For each $\alpha \in J$ assume (X_α, x_α) and (Y_α, y_α) are pointed topological spaces and there exist maps $f_\alpha : (X_\alpha, x_\alpha) \to (Y_\alpha, y_\alpha)$ and $g_\alpha : (Y_\alpha, y_\alpha) \to (X_\alpha, x_\alpha)$ such that $f_\alpha \circ g_\alpha \approx \mathrm{id}_{Y_\alpha}$ and $g_\alpha \circ f_\alpha \approx \mathrm{id}_{X_\alpha}$. Then $\bigvee_{\alpha\in J}(X_\alpha, x_\alpha)$ and $\bigvee_{\alpha\in J}(Y_\alpha, y_\alpha)$ are homotopy equivalent.

9.8 Definition (cf. [FFG86, §3])**.** A *CW-complex* is a topological space X such that for $l \in \mathbb{N}$ there exist: an index set J_l, a family of l-dimensional balls $\{\sigma_i^l : i \in J_l\}$, and for each $i \in J_l$ there is a continuous *characteristic map* $\varphi_i^l : \sigma_i^l \to X$ such that

(a) setting $X^{-1} = \varnothing$ and $X^k = \bigcup_{l=0}^k \bigcup_{i\in J_l} \operatorname{im} \varphi_i^l$ for $k \in \mathbb{N}$, we have $X = \bigcup_{k=0}^\infty X^k$;

(b) φ_i^l restricted to $\operatorname{Int} \sigma_i^l$ is a homeomorphic embedding;

(c) the image of $\partial\sigma_i^l$ under φ_i^l is contained in X^{l-1};

(d) the image of φ_i^l intersects only finitely many images of other characteristic maps;

(e) a set $F \subseteq X$ is closed in X if and only if $(\varphi_i^l)^{-1}[F]$ is closed in σ_i^l for all $l \in \mathbb{N}$ and $i \in J_l$.

The image of any φ_i^l shall be called an *l-dimensional cell* of X and the set X^l the *l-skeleton* of X. If $X = X^k$ for some $k \in \mathbb{N}$, then we say that X is *k-dimensional* and if, in addition, all the sets J_l for $l \in \{0, 1, \dots, k\}$ are finite, then we say that X is a *finite CW-complex.*

9.9 Remark. A CW-complex X can also be seen as constructed inductively by attaching cells σ_i^l to X^{l-1} via maps $\varphi_i^l|_{\partial\sigma_i^l}$; cf. [Hat02, Chap. 0, p. 5].

9.10 Remark. If $\mathcal{A} \subseteq \mathbf{K}_*^n$, then $X = \bigcup \mathcal{A}$ is a CW-complex with $X^k = \bigcup\{Q \in \mathbf{K}_k^n : Q \subseteq X\}$ for $k \in \{0, 1, \dots, n\}$. If $\mathcal{A}$ is finite, then X is a finite CW-complex.

9.11 Remark. Assume X is a CW-complex. We shall use cellular homology of X; see [FFG86, §12] or [Hat02, §2.2, p. 137]. Recall that for $l \in \mathbb{N}$ the chain group

$$C_l(X) = \mathbf{H}_l(X^l, X^{l-1})$$

is the free abelian group with basis $\{\sigma_i^l : i \in J_l\}$. Next, define the differentials

$$\begin{gathered} d_0 : C_0 \to \{0\} \quad \text{and} \quad d_l : C_l(X) \to C_{l-1}(X) \\ \text{by} \quad d_l(\sigma_i^l) = \sum_{j\in J_{l-1}} \deg(\psi_{i,j}^l)\sigma_j^{l-1} \quad \text{for } l \in \mathscr{P}\,, \end{gathered} \tag{25}$$

where $\psi_{i,j}^l$ is defined as the composition

$$\partial\sigma_i^l \xrightarrow{\varphi_i^l|_{\partial\sigma_i^l}} X^l \twoheadrightarrow X^l/(X^l \sim \sigma_j^{l-1}) \xrightarrow{\simeq} \mathbb{S}^{l-1}\,.$$

Clearly, by 9.8(d), the sum in (25) is finite. Moreover, $(C_l(X), d_l)_{l=0}^\infty$ defines a chain-complex whose homology groups coincide with singular homology groups of X; see [Hat02, Theorem 2.35] or [FFG86, §12, p. 94].

9.12 Remark. Let F be a free abelian group. The following observations shall become particularly useful:

(a) If G is a subgroup of F, then G is itself a free abelian group; cf. [Lan02, I,§7,Theorem 7.3].

(b) If G is another free abelian group and $d : F \to G$, then F splits into a direct sum $F = \ker d \oplus H$ for some subgroup H of F.

To prove the above claim (b), let $A = \operatorname{im} d \subseteq G$. Then A is a subgroup of G; hence, A is a free abelian group. Let $\{a_i : i \in J\}$ be a basis of A. In order to prove the existence of a splitting, it suffices to define a homomorphism $f : A \to F$ such that $d \circ f = \mathrm{id}_A$. For each $i \in J$ we choose arbitrarily $b_i \in F$ such that $d(b_i) = a_i$ and set $f(a_i) = b_i$. Then f extends to a homomorphism $A \to F$ simply because A is free.

Next, we prove that if X is a $(k+1)$-dimensional CW-complex, then any homomorphism from the k^{th} homology group $\mathbf{H}_k(X)$ to the group of integers $\mathbf{Z}$ is induced by some map $X \to \mathbb{S}^k$.

9.13 Lemma. *Assume $k \in \mathbb{N}$, X is a $(k+1)$-dimensional CW-complex, and there is given a homomorphism $\zeta : \mathbf{H}_k(X) \to \mathbf{Z}$. Then there exists $f : X \to \mathbb{S}^k$ such that $f_* = \zeta$.*

Proof. For $l \in \{0, 1, 2, \dots, k+1\}$ let J_l be the set indexing l-dimensional cells of X and for $i \in J_l$ let $\{\sigma_i^l : i \in J_l\}$, $\varphi_i^l : \sigma_i^l \to X$, d_l, $C_l(X)$, X^l be defined as in 9.8 and 9.11.

By definition $C_k(X)$ are free abelian groups. Set $K = \ker d_k \subseteq C_k(X)$ and employ 9.12(b) to find another subgroup $L \subseteq C_k(X)$ such that $C_k(X) = K \oplus L$. Let $p : K \twoheadrightarrow \mathbf{H}_k(X)$ and $q : K \oplus L \twoheadrightarrow K$ be canonical projections. Define $\xi : C_k(X) \to \mathbf{Z}$ as the composition

$$C_k(X) \xrightarrow{q} K \xrightarrow{p} \mathbf{H}_k(X) \xrightarrow{\zeta} \mathbf{Z}\,.$$

We record now some trivial observations

$$\zeta(x) = 0 \quad \text{whenever } x \in \mathbf{H}_k(X) \text{ has finite order}\,, \quad \zeta \circ p = \xi|_K\,, \quad \xi \circ d_{k+1} = 0\,. \tag{26}$$

We shall first construct $\gamma : X^k \to \mathbb{S}^k$ such that $\gamma_* : \mathbf{H}_k(X^k) \to \mathbf{Z}$ equals $\zeta \circ p$ and then extend γ to $f : X^{k+1} \to \mathbb{S}^k$ using a bit of obstruction theory.

For each $i \in J_k$ the space $\sigma_i^k / \partial\sigma_i^k$ is homeomorphic to $\mathbb{S}^k$ and we define

$$\gamma_i : \sigma_i^k / \partial\sigma_i^k \to \mathbb{S}^k \quad \text{so that} \quad \deg(\gamma_i) = \xi(\sigma_i^k)\,.$$

Note that the space X^k / X^{k-1} is homeomorphic to the wedge sum of topological spheres $\bigvee_{i \in J_k} (\sigma_i^k / \partial\sigma_i^k, [\partial\sigma_i^k])$. We construct the map

$$\tilde{\gamma} : X^k / X^{k-1} \to \mathbb{S}^k \quad \text{so that} \quad \tilde{\gamma}|_{\sigma_i^k / \partial\sigma_i^k} = \gamma_i \quad \text{for } i \in J_k\,.$$

Let $\pi : X^k \twoheadrightarrow X^k / X^{k-1}$ be the projection. Finally, set

$$\gamma = \tilde{\gamma} \circ \pi\,.$$

Note that $\mathbf{H}_k(X^k) = K$. One readily verifies that $\gamma_* = \xi|_K = \zeta \circ p$.

Now we need to extend γ to the $(k+1)$-dimensional cells in X. Employing the obstruction theory [FFG86, §17] this is possible if for each $j \in J_{k+1}$ the composition

$$\partial\sigma_j^{k+1} \xrightarrow{\varphi_j^{k+1}|_{\partial\sigma_j^{k+1}}} X^k \xrightarrow{\gamma} \mathbb{S}^k$$

has topological degree zero. However, this degree equals exactly $\xi(d_{k+1}(\sigma_j^{k+1}))$ which is zero by (26). Therefore, there exists $f : X \to \mathbb{S}^k$ such that $f|_{X^k} = \gamma$; in particular, $f_* : \mathbf{H}_k(X) \to \mathbf{Z}$ equals ζ. $\square$

9.14 Remark. Employing some more sophisticated tools of algebraic topology, a shorter proof of Lemma 9.13 can be given as follows. The universal coefficient theorem [Hat02, Theorem 3.2] provides an epimorphism

$$h : \mathbf{H}^k(X; \mathbf{Z}) \twoheadrightarrow \operatorname{Hom}(\mathbf{H}_k(X), \mathbf{Z}) .$$

On the other hand, there exists an isomorphism (see [Hat02, Theorem 4.57])

$$T : \big[X, K(\mathbf{Z}, k)\big]_{\mathrm{htp}} \xrightarrow{\simeq} \mathbf{H}^k(X; \mathbf{Z}) ,$$

where $[X, K(\mathbf{Z}, k)]_{\mathrm{htp}}$ denotes the set of homotopy classes of maps $X \to K(\mathbf{Z}, k)$ and $K(\mathbf{Z}, k)$ is the Eilenberg-MacLane space; cf. [Hat02, §4.2, p. 365]. Therefore, any homomorphism $\mathbf{H}_k(X) \to \mathbf{Z}$ is induced by some map $X \to K(\mathbf{Z}, k)$. Observing, that $K(\mathbf{Z}, k)$ is a CW-complex obtained from the sphere $\mathbb{S}^k$ by gluing in cells of dimension at least $k + 2$, we see, since X is $(k+1)$-dimensional and the homotopy groups $\pi_l(\mathbb{S}^{k+2}) = 0$ for $l \in \{1, 2, \dots, k+1\}$, that any map $X \to K(\mathbf{Z}, k)$ is homotopic to a map whose image lies in $\mathbb{S}^k$.

9.15 Remark. The bound on the dimension of X plays a crucial role in 9.13. Indeed, if the dimension of X is bigger than $k+1$, then an element of $\operatorname{Hom}(\mathbf{H}_k(X), \mathbf{Z})$ might not be induced by a map $X \to \mathbb{S}^k$ as the following example shows. Let $k = 2$ and X be the complex projective space of real dimension 4 (often denoted $\mathbf{CP}^2$). Then X is a CW-complex constructed by attaching a 4-dimensional cell to $\mathbb{S}^2$ via the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. We have

$$\mathbf{H}_2(X) = \mathbf{H}^2(X) = \mathbf{H}^4(X) = \mathbf{Z} .$$

Recall that $\mathbf{H}^*(X)$ is the graded ring $\mathbf{Z}[\sigma]/\sigma^3$, where σ is the generator of $\mathbf{H}^2(X)$; cf. [Hat02, Theorem 3.12]. Finally, since all the homology and cohomology groups of X are free, the universal coefficient theorem provides a natural isomorphism

$$j : \mathbf{H}^2(X) \xrightarrow{\simeq} \operatorname{Hom}(\mathbf{H}_2(X), \mathbf{Z}) .$$

Assume there exists a map $f : X \to \mathbb{S}^2$ such that $f_* : \mathbf{H}_2(X) \to \mathbf{H}_2(\mathbb{S}^2)$ is an isomorphism. In consequence, $f^* : \mathbf{H}^2(\mathbb{S}^2) \to \mathbf{H}^2(X)$ is also an isomorphism. However, the map f^* is a *homomorphism of graded rings* and this gives a contradiction because the square of the generator of $\mathbf{H}^2(\mathbb{S}^2)$ is zero while the square of the generator of $\mathbf{H}^2(X)$ is the generator of $\mathbf{H}^4(X)$.

9.16 Corollary. *Let $k \in \mathbb{N}$, X be a $(k+1)$-dimensional CW-complex, and $j : \mathbb{S}^k \to X$ be continuous. Define*

$$D = \big\{ |\deg(f \circ j)| : f : X \to \mathbb{S}^k \textit{ continuous} \big\} \sim \{0\} .$$

If $D \neq \varnothing$ and $A = \min D$, then

$$D = \{ mA : m \in \mathscr{P} \} .$$

Proof. If $D = \varnothing$ there is nothing to prove, so we assume $D \neq \varnothing$. Let $f_1, f_2 : X \to \mathbb{S}^k$ be two continuous maps such that $d_i = |\deg(f_i \circ j)| \in \mathscr{P}$ for $i \in \{1, 2\}$. Set $d = \gcd(d_1, d_2) \in \mathscr{P}$. By the Euclidean algorithm, there exist integers c_1, c_2 such that $d = c_1 d_1 + c_2 d_2$. We employ 9.13 to find a map $f : X \to \mathbb{S}^k$ such that $f_* = c_1 f_{1*} + c_2 f_{2*}$. Then $|\deg(f \circ j)| = d \in D$.

We have shown that whenever $d_1, d_2 \in D \subseteq \mathscr{P}$, then $\gcd(d_1, d_2) \in D$. Moreover, if $f : X \to \mathbb{S}^k$, $|\deg(f \circ j)| = A \in D$, and $m \in \mathscr{P}$, then $mA \in D$ because one can post-compose f with a map $\mathbb{S}^k \to \mathbb{S}^k$ of degree m. $\square$

9.17 Corollary. *Let $k, N \in \mathbb{N}$, X be $(k+1)$-dimensional CW-complex, $x_0 \in X$, $Z = \bigvee_{i=1}^{N}(X, x_0)$ and $j : \mathbb{S}^k \to Z$ be continuous. For $l \in \{1, 2, \ldots, N\}$ define $\pi_l : Z \to X$ as in 9.7. Assume there exists $\varphi : \mathbb{S}^k \to X$ such that for $l \in \{1, 2, \ldots, N\}$ the map $\pi_l \circ j : \mathbb{S}^k \to X$ is homotopic either to φ or to the constant map and $\pi_1 \circ j \approx \varphi$. Set*

$$
\begin{aligned}
D &= \left\{ |\deg(f \circ j)| : f : Z \to \mathbb{S}^k \text{ continuous} \right\}, \\
E &= \left\{ |\deg(g \circ \varphi)| : g : X \to \mathbb{S}^k \text{ continuous} \right\}.
\end{aligned}
$$

Then $D = E$.

Proof. For $l \in \{1, 2, \ldots, N\}$ let $\tau_l : X \to Z$ be the injection as in 9.7. If $g : X \to \mathbb{S}^k$ is continuous, then $f = g \circ \pi_1 : Z \to \mathbb{S}^k$ is homotopic to $g \circ \varphi$ so $\deg(g \circ \varphi) = \deg(f \circ j)$ and we get $E \subseteq D$. On the other hand if $f : Z \to \mathbb{S}^k$, then we consider the maps $f_l = f \circ \tau_l : X \to \mathbb{S}^k$ for $l \in \{1, 2, \ldots, N\}$ to see that

$$
D \ni |\deg(f \circ j)| = \left| \textstyle\sum_{l=1}^{N} \deg(f_l \circ \pi_l \circ j) \right| \in E \quad \text{by 9.16};
$$

thus, $D \subseteq E$. □

9.18 Lemma. *Let $J = [0, 2]$, $\varepsilon \in (0, \infty)$ and assume*

$$
Q \in \mathbf{K}_d^n, \quad S = \partial_{\mathrm{c}} Q, \quad X \subseteq \mathbf{R}^n \text{ is compact}, \quad S \subseteq X, \quad \mathscr{H}^d(X) < \infty.
$$

Then there exist: a Lipschitz map $f : I \times \mathbf{R}^n \to \mathbf{R}^n$, a compact set $E \subseteq \mathbf{R}^n$, an open set $U \subseteq \mathbf{R}^n$, and a finite set $\mathcal{B} \subseteq \mathbf{K}_d^n$ such that

$$
\begin{gathered}
S \subseteq E = \bigcup \mathcal{B} = f[\{2\} \times U], \quad X \subseteq U \subseteq X + \mathbf{B}(0, \varepsilon), \quad f[J \times U] \subseteq V, \\
f(t, x) = x \quad \text{for } (t, x) \in I \times E, \quad E \text{ is a strong deformation retract of } U.
\end{gathered}
$$

Proof. For $R \in \mathbf{K}^n$ denote by $\tilde{R}$ the n-dimensional cube with the same center as R and side-length three times bigger than R. Let $N \in \mathscr{P}$ be such that $2^{-N+4}\sqrt{n} < \min\{\varepsilon, \mathbf{l}(Q)\}$ and define

$$
\mathcal{A} = \left\{ R \in \mathbf{K}_n^n(N) : \tilde{R} \cap X \neq \varnothing \right\}.
$$

Apply 5.8 with $\mathbf{K}_n^n$, $\mathcal{A}$, X in place of $\mathcal{F}$, $\mathcal{A}$, S to obtain a Lipschitz map $f : J \times \mathbf{R}^n \to \mathbf{R}^n$, an open set $V \subseteq \mathbf{R}^n$, and a finite set $\mathcal{B} \subseteq \mathbf{K}_d^n(N)$. Set $E = \bigcup \mathcal{B}$ and $U = V \cap \operatorname{Int} \bigcup \mathcal{A}$ and recall 5.10. Since $S \subseteq \bigcup \mathbf{K}_{d-1}^n(N)$ we get $S \subseteq E$. □

For convenience and brevity of the notation we introduce the following definition.

9.19 Definition. We define $\mathbf{R}^\infty$ to be the direct sum of countably many copies of $\mathbf{R}$ and for $i \in \mathscr{P}$ we let $e_i \in \mathbf{R}^\infty$ be the standard basis vector of the i^{th} copy of $\mathbf{R}$. Thus, $\mathbf{R}^\infty$ is the set of all finite linear combinations of the vectors $\{e_i : i \in \mathscr{P}\}$.

We want to compare, up to homotopy, a multiplication (Y, Q) of some cubical test pair (X, Q) with the wedge sum of certain number of copies of X. However, it might happen that two copies of X placed side by side intersect outside $\partial_{\mathrm{c}} Q$. To prevent this, we define a lifted multiplication so that different copies of X intersect only along $\partial_{\mathrm{c}} Q$.

9.20 Definition. Let X, Q, k, $\mathcal{A} = \{K_1, \dots, K_{k^d}\}$ be as in 8.3. Let e_i for $i \in \mathscr{P}$ be as in 9.19. Define $j : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^\infty$, $p : \mathbf{R}^n \times \mathbf{R}^\infty \to \mathbf{R}^n$, and $\eta_i : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^\infty$ for $i \in \{1, 2, \dots, k^d\}$ by

$$j(x) = (x, 0)\,, \quad p(x, y) = x\,, \quad \eta_i(x) = j \circ \boldsymbol{\tau}_{\mathbf{c}(K_i)} \circ \boldsymbol{\mu}_{1/k} \circ \boldsymbol{\tau}_{-\mathbf{c}(Q)}(x) + \operatorname{dist}(x, \partial_{\mathrm{c}} Q) e_i\,.$$

We say that $(Y, j[Q])$ is the *lifted k-multiplication* of (X, Q) if

$$Y = \bigcup \bigl\{ \eta_i[X] : i \in \{1, 2, \dots, k^d\} \bigr\} \subseteq \mathbf{R}^n \times \mathbf{R}^\infty\,.$$

9.21 Lemma. *Assume*

$$\begin{gathered} U \subseteq \mathbf{R}^n \text{ is open}\,, \quad Q = [0,1]^d \times \{0\}^{n-d} \in \mathbf{K}^n_d(0)\,, \quad S = \partial_{\mathrm{c}} Q\,, \quad N \in \mathscr{P}\,, \\ \mathcal{B} \subseteq \mathbf{K}^n_d \text{ is finite}\,, \quad S \subseteq E = \textstyle\bigcup \mathcal{B} \subseteq U\,, \quad E \text{ is a strong deformation retract of } U\,, \\ j \text{ and } p \text{ are as in 9.20}\,, \quad (Y, j[Q]) \text{ is the lifted } 2^N\text{-multiplication of } (U, Q)\,, \\ (Z, j[Q]) \text{ is the lifted } 2^{N-1}\text{-multiplication of } (U, Q)\,. \end{gathered}$$

If $j[S]$ is a Lipschitz retract of Y, then $j[S]$ is a Lipschitz retract of Z.

Proof. Suppose there exists a Lipschitz retraction $r : Y \to j[S]$. Due to 9.5 it suffices to show that there exists a continuous map $h : Z \to S$ such that $\deg(h \circ j|_S) = 1$. Set $J = \{1, 2, \dots, 2^d\}$. Let $(X, j[Q])$ be the lifted 2^{N-1}-multiplication of (E, Q) and $(F, j[Q])$ be the lifted 2^N-multiplication of (E, Q). Observe that Y contains 2^d copies of $\boldsymbol{\mu}_{1/2}[Z]$; let us denote these copies Z_1, Z_2, $\dots$, Z_{2^d} and the corresponding cubes Q_1, Q_2, $\dots$, Q_{2^d} so that

$$Y = \bigcup \{ Z_i : i \in J \} \quad \text{and} \quad j[Q] = \bigcup \{ Q_i : i \in J \}\,.$$

We also define

$$S_i = \partial_{\mathrm{c}} Q_i \quad \text{and} \quad X_i = F \cap Z_i \quad \text{for } i \in J\,.$$

Let $T = \mathbf{R}^d \times \{0\}^{n-d} \in \mathbf{G}(n, d)$. Then $Q \subseteq \mathbf{o}(Q) + T$. Let $(v_1, v_2, \dots, v_n)$ be the standard basis of $\mathbf{R}^n$ and define

$$\begin{gathered} T_i = \operatorname{span}\{v_i\}^\perp \cap T \in \mathbf{G}(n, d-1) \quad \text{for } i \in \{1, 2, \dots, d\}\,, \\ R = j\bigl[\textstyle\bigcup \bigl\{ (\mathbf{c}(Q) + T_i) \cap Q : i \in \{1, 2, \dots, d\} \bigr\} \bigr] \subseteq Y\,. \end{gathered}$$

Note that R and $R \cap Z_i$ for $i \in J$ are contractible. Since U is open, we have $S \subseteq \operatorname{Int} U$ so the pairs (Y, R) and $(Z_i, R \cap Z_i)$ for $i \in \{1, 2, \dots, d\}$ all have the HEP by 9.3. Therefore, R and Y/R are homotopy equivalent by 9.4. Similarly, Z_i and $Z_i/(R \cap Z_i)$ are homotopy equivalent for $i \in J$. Let $q_0 = j(\mathbf{c}(Q))$. We shall write $[q_0]$ for the equivalence class of q_0 in a given quotient space. Denoting homotopy equivalence by "$\approx$" and homeomorphism by "$\simeq$" we obtain

$$Y \approx Y/R \simeq \textstyle\bigvee_{i=1}^{2^d} (Z_i/(Z_i \cap R), [q_0]) \approx \bigvee_{i=1}^{2^d} (Z_i, q_0)\,.$$

Set

$$W = \textstyle\bigvee_{i=1}^{2^d} (Z_i, q_0)\,, \quad M = \bigvee_{i=1}^{2^d} (X_i, q_0)\,, \quad \Sigma = \bigvee_{i=1}^{2^d} (S_i, q_0)\,,$$

and note that $\Sigma \subseteq M \subseteq W$. Let $\varphi : Y \to W$ and $\psi : W \to Y$ be such that $\varphi \circ \psi \approx \mathrm{id}_W$ and $\psi \circ \varphi \approx \mathrm{id}_Y$. For $i \in J$ let $\pi_i : \Sigma \to S_i$ be the projection defined in 9.7. Observe that

$$\varphi \circ j[S] = \Sigma \quad \text{and} \quad \deg(\pi_i \circ \varphi \circ j|_S) = 1 \quad \text{for } i \in J\,.$$

Recall that E is a strong deformation retract of U; hence, if $\xi : M \hookrightarrow W$ is the inclusion map, there exists a continuous maps $\zeta : W \to M$ such that $\xi \circ \zeta \approx \mathrm{id}_W$ and $\zeta \circ \xi \approx \mathrm{id}_M$. Moreover, $\xi|_\Sigma = \zeta|_\Sigma = \mathrm{id}_\Sigma$. Since $E = \bigcup \mathcal{B}$ we see that E and M are d-dimensional CW-complexes by 9.10. Hence, we may apply 9.17 to deduce that

$$\big\{ |\deg(f \circ \zeta \circ \varphi \circ j|_S)| : f : M \to S \text{ continuous} \big\} = \big\{ |\deg(g|_S)| : g : X \to S \text{ continuous} \big\} .$$

However, if we take $f = p \circ r \circ \psi \circ \xi : M \to S$, then

$$f \circ \zeta \circ \varphi \circ j|_S = p \circ r \circ \psi \circ \xi \circ \zeta \circ \varphi \circ j|_S \approx p \circ r \circ j|_S = \mathrm{id}_S .$$

Therefore, there exists $g : X \to S$ such that $\deg(g \circ j|_S) = 1$. Let $\alpha : X_1 \to X$ and $\beta : Z \to Z_1$ be homeomorphisms composed of homotheties and translations. Then, recalling $\zeta|_\Sigma = \mathrm{id}_\Sigma$, the composition

$$S \xrightarrow{j|_S} Z \xrightarrow{\beta} Z_1 \xrightarrow{\zeta|_{Z_1}} X_1 \xrightarrow{\alpha} X \xrightarrow{g} S$$

equals $g \circ j|_S$ and has degree one. Employing 9.5 we obtain a Lipschitz retraction $Z \to S$. $\square$

9.22 Corollary. *If S and U are as in 9.21, then S is a Lipschitz retract of U.*

Proof. We assume $j[S]$ is a Lipschitz retract of Y, where Y is the lifted 2^N-multiplication of (U, Q). We proceed by induction with respect to $N \in \mathbb{N}$. If $N = 0$, we have $j[U] = Y$ so S is a Lipschitz retract of U by assumption. The inductive step is now a direct application of 9.21. $\square$

9.23 Theorem. *Assume $N \in \mathscr{P}$, (X, Q) is a cubical test pair, and (Y, Q) is the 2^N-multiplication of (X, Q). Then (Y, Q) is a cubical test pair.*

Proof. Using homotheties and rotations we may and shall assume that $Q = [0, 1]^d \times \{0\}^{n-d} \in \mathbf{K}^n_d(0)$. We only need to show that $S = \partial_c Q$ is not a Lipschitz retract of Y. Let p and j be as in 9.20. Assume, by contradiction, that there is a Lipschitz retraction of Y onto S. Employing 9.5 we find $\delta \in (0, 1)$ such that S is a retract of $Y + \mathbf{B}(0, 2^{-N}\delta)$. Apply 9.18 with X, Q, δ in place of X, Q, ε to obtain a finite set $\mathcal{B} \subseteq \mathbf{K}^n_d$ and an open set $U \subseteq X + \mathbf{B}(0, \delta)$ such that $E = \bigcup \mathcal{B}$ is a strong deformation retract of U and $X \subseteq U$. Let $(Z, j[Q])$ be the lifted 2^N-multiplication of (U, Q). Clearly $p[Z] = Y$ and $p \circ j|_S = \mathrm{id}_S$, so $j[S]$ is a Lipschitz retract of Z. Applying 9.21 to U, Q, N, $\mathcal{B}$ and then 9.22, we conclude that S is a Lipschitz retract of U which contains X, so S is also a Lipschitz retract of X and this contradicts the assumption that (X, Q) is a cubical test pair. $\square$

9.24 Remark. To conclude we gather all our results in one place. Let $x \in \mathbf{R}^n$, $\mathcal{C}$ be the set of all cubical test pairs, $\mathcal{P}$ be the set of all test pairs, $\mathcal{R}$ be the set of all rectifiable test pairs. Then

(a) if $U \subseteq \mathbf{R}^n$ is open, $F \in \mathrm{wBC}_x$ for all $x \in U$, F is bounded, and $\mathcal{G}$ is a good class in the sense of [FK18, 3.4], then there exists $S \in \mathcal{G}$ such that $\Phi_F(S) = \inf\{\Phi_F(R) : R \in \mathcal{G}\}$;

(b) $\mathrm{AE}_x(\mathcal{P}) = \mathrm{AE}_x(\mathcal{C}) = \mathrm{AE}_x(\mathcal{R})$ and $\mathrm{AUE}_x(\mathcal{P}) = \mathrm{AUE}_x(\mathcal{C}) = \mathrm{AUE}_x(\mathcal{R})$;

(c) $\mathrm{AC}_x = \mathrm{BC}_x \subseteq \mathrm{wBC}_x \subseteq \mathrm{AE}_x(\mathcal{C})$.

Moreover, if $n = d+1$, then by [DPDRG18, Theorem 1.3] we know that $F \in \mathrm{AC}_x$ if and only if the function

$$G(x,\nu) = |\nu| F(x, \operatorname{span}\{\nu\}^{\perp}) \quad \text{for every } x, \nu \in \mathbf{R}^n \tag{27}$$

is strictly convex in all but the radial directions, namely

$$G(x,\nu) > \langle D_\nu G(x,\bar\nu), \nu\rangle \qquad \text{for every } x \in \mathbf{R}^n,\ \bar\nu, \nu \in \mathbb{S}^{n-1} \text{ and } \nu \neq \pm\bar\nu\,.$$

Hence, given $n = d+1$,

(d) if F is a $\mathscr{C}^1$ integrand such that the corresponding function G, as in (27), is strictly convex, then $F \in \mathrm{AE}_x(\mathcal{P})$.

9.25 Remark. In [Alm76, IV.1(7), p. 88] Almgren observes that uniformly convex functions give rise to anisotropic lagrangians satisfying $\mathrm{AUE}_x(\mathcal{P})$ in co-dimension 1 and vice-versa, where $\mathcal{P}$ is the class of test pairs. Our result shows that functions that are just strictly convex give rise to anisotropic lagrangians satisfying $\mathrm{AE}_x(\mathcal{P})$ in co-dimension 1, for every good family $\mathcal{P}$. In particular we deduce that there is no hope of improving Theorem 8.8 showing that $\mathrm{wBC}_x \subseteq \mathrm{AUE}_x(\mathcal{P})$ (and neither $\mathrm{BC}_x \subseteq \mathrm{AUE}_x(\mathcal{P})$). Indeed, if this was the case, in co-dimension one the strict convexity of the integrand would give rise to an anisotropic lagrangian satisfying BC_x and consequently also $\mathrm{AUE}_x(\mathcal{P})$, which in turn would imply the uniform convexity of the integrand.

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