MULTIPHASE FREE DISCONTINUITY PROBLEMS: MONOTONICITY FORMULA AND REGULARITY RESULTS

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Abstract. The purpose of this paper is to analyze regularity properties of local solutions to free discontinuity problems characterized by the presence of multiple phases. Local solutions are meant according to an ad hoc, nonstandard notion of *multiphase local almost-quasi minimizers*. In particular this notion penalizes, among contacts between two different phases, only those which occur at jump points, leaving for free no-jump interfaces which may occur at the zero level of the corresponding state functions. In this setting, our main result states that the phases are open and the jump set (globally considered for all the phases) is essentially closed and Ahlfors regular. This is the same kind of regularity holding for one phase local almost-quasi minimizers of general free discontinuity problems. To achieve the same target in presence of multiple phases demands to set up new refined tools. They are a multiphase monotonicity formula and a multiphase decay lemma, which extend respectively the corresponding one phase results by Bucur-Luckhaus [4] and De Giorgi-Carriero-Leaci [16]. The proof of the former relies on a sharp collective Sobolev extension result for functions with disjoint supports on a sphere, which may be of independent interest.

1. Introduction and statement of the results

Free discontinuity problems characterized by the presence of multiple phases arise naturally in different contexts, such as image reconstruction or models in thermo-elasticity. The aim of this paper is to give a precise formulation of those problems and analyze regularity properties of their local solutions. Before entering into the description of the results, let us briefly address two model problems which fit into our framework. It is not our purpose to solve or describe these problems in full details, but just to serve them as meaningful motivations to attack the study of local minimizers to multiphase free discontinuity problems.

As a first prototype example, one can consider the celebrated segmentation problem by Mumford and Shah (see [20, 17] for a review). Actually, the original formulation of the problem presented in [24] was of multiphase type, the primary underlying idea being that of decomposing the domain of an input image into different regions, and then reconstructing a new image allowed to have jumps across the boundaries of such regions. The ensuing developments via different approaches (by De Giorgi-Carriero-Leaci [16] or Dal Maso-Morel-Solimini [14]) led genuinely to a single phase problem, in which the boundaries of the different regions are seen as the jump set of a single state function. By this way the notion of phase is fatally lost, because the jump set of the state function has no a priory reason to decompose the original image into disconnected regions. Nevertheless,

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the necessity to extract distinct objects (possibly exhibiting themselves inner jumps) is highly present in image reconstruction, as testified by the abundant literature devoted to the attempt of forcing phase separation in the original spirit by Mumford and Shah. With no attempt of completeness, see [10, 25, 11, 21, 23, 26, 27] and references therein. It is however an evidence that these models refer essentially to the simplistic case of piecewise constant functions, and are mainly focused on the numerical side. Thus, a satisfactory mathematical study of the Mumford-Shah functional in a multiphase context seems to be missing.

A rough formulation, suited for images with a uniform background (e.g. medical images with a reference black background) can be proposed as follows. Given a bounded open subset $\Omega \subseteq \mathbb{R}^d$, positive coefficients α_i, β_i, and a function f on Ω with values into $[0,1]$, consider the minimization problem

$$\min_{\substack{u_i \in SBV(\Omega) \\ u_i \cdot u_j = 0 \text{ for } i \neq j}} \sum_{i=1}^k \left[\alpha_i \int_\Omega |\nabla u_i|^2 dx + \beta_i \mathcal{H}^{d-1}(J_{u_i}) \right] + \int_\Omega \Big(\sum_{i=1}^k u_i - f \Big)^2 dx, \tag{1}$$

where $SBV(\Omega)$ is the space of functions with special bounded variation in Ω, and J_{u_i} is the jump set of u_i.

As a second example of a multiphase free discontinuity problem, let us address the thermal insulation of multiple obstacles, in the spirit of Caffarelli-Kriventsov [8] (see also [18] for a related problem). As above, we consider a bounded open set $\Omega \subseteq \mathbb{R}^d$, in which we now place a family of compact, pairwise disjoint subsets $G_1, \dots, G_k$ representing the obstacles. We assume that the background temperature is 0, while the temperature of the obstacles is 1. Then, if each obstacle is isolated by a conducting material with unitary cost $m > 0$ which is enveloped with an insulating thin layer of characteristic $\beta > 0$ and of unitary cost l, the multiphase insulation problem reads

$$\min \sum_{i=1}^k \left[\int_\Omega |\nabla u_i|^2 dx + \beta \int_{J_{u_i}} (u_i^+)^2 + (u_i^-)^2 d\mathcal{H}^{d-1} + l\mathcal{H}^{d-1}(J_{u_i}) + m|\{u_i > 0\}| \right], \tag{2}$$

the minimization being carried in the class

$$\Big\{ u_i \in SBV(\Omega) : u_i = 1 \text{ on } G_i,\ u_i \cdot u_j = 0 \text{ for } i \neq j \Big\}.$$

The analysis of different phenomena in applied sciences, such as for instance quasistatic crack evolution (Francfort-Marigo [19]) or columnar jointing of cooling lava (Jungen [22]), clusters of Cheeger sets (Caroccia [9]) or spectral partition problems related to the Robin Laplacian [3], provide other interesting examples of multiphase free discontinuity problems, that we do not detail here.

Let us now get to the heart of the matter, and drive our attention to the generic, local formulation of the problem. Since we deal with multiple phases, each one with possible free discontinuities, the natural functional framework is given by the class

$$\mathcal{U}(\Omega) := \Big\{ u = (u_1, \dots, u_k) \in (SBV(\Omega))^k \ :\ u_i \cdot u_j = 0 \quad \text{in } \Omega \text{ for } i \neq j \Big\}.$$

Here, as in the models described above, Ω is a bounded open subset of $\mathbb{R}^d$, and $SBV(\Omega)$ denotes the space of special functions of bounded variation in Ω introduced by De Giorgi-Ambrosio [15]; we refer to [1] for a detailed account of the mathematical background.

We say that $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ is a *multiphase local almost-quasi minimizer at the point* $x \in \Omega$ (with parameters $(\Lambda, \alpha, c_\alpha)$), if there exist constants $\Lambda \geq 1$, $\alpha > 0$, and $c_\alpha \geq 0$ such that, for every ball $B_\rho(x) \subset \Omega$ and every $(v_1, \dots, v_k) \in \mathcal{U}(\Omega)$ such that $\bigcup_i \{v_i \neq u_i\} \subseteq B_\rho(x)$, it holds

$$\sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \leq \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla v_i|^2 \, dx + \Lambda \mathcal{H}^{d-1}(J_{v_i} \cap \overline{B}_\rho(x)) \right) + c_\alpha \rho^{d-1+\alpha} .$$

The words "almost" and "quasi" in the terminology refer respectively to the presence of the coefficient $\Lambda \geq 1$ in front of the jump terms and to power decay of order higher than $(d-1)$ of the deviation from minimality. The definition above is a natural multiphase analogue of the one introduced in [4] for a *single* phase problem.

The leading idea to obtain primary regularity results is to search for a monotonicity formula, which now has to hold in the multiphase context. Clearly, the interplay between the phases makes this issue very delicate. The analysis of the cyclic inductive proof given in [4] reveals that the possibility of obtaining the monotonicity in the multiphase case rests upon an independent and quite intriguing question concerning sharp extensions of functions with disjoint support defined on the boundary of a ball, say B_ρ. The question is whether, given an arbitrary family of functions $(w_1, \dots, w_k) \in (H^1(\partial B_\rho))^k$, with $w_i \cdot w_j = 0$ on ∂B_ρ, they can be extended to functions $(\tilde{w}_1, \dots, \tilde{w}_k) \in (H^1(B_\rho))^k$, with $\tilde{w}_i \cdot \tilde{w}_j = 0$ inside B_ρ, in such a way that

$$\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 dx \leq \frac{\rho}{d-1} \sum_{i=1}^k \int_{\partial B_\rho} |\nabla_\tau w_i|^2 d\mathcal{H}^{d-1},$$

where ∇_τ denotes the tangential gradient.

The answer turns out to be affirmative. This is our first result, which is of crucial importance for the sequel and may have an autonomous interest:

Theorem 1 (Multiphase extension result). *For every $w = (w_1, \dots w_k) \in (H^1(\partial B_\rho))^k$ with $w_i \cdot w_j = 0$ on ∂B_ρ, it holds*

$$\min \left\{ \sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx \, : \, \tilde{w}_i|_{\partial B_\rho} = w_i \, , \, \tilde{w}_i \cdot \tilde{w}_j = 0 \text{ in } B_\rho \right\} \leq \frac{\rho}{d-1} \sum_{i=1}^k \int_{\partial B_\rho} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} .$$

The difficult part in the proof of the previous result is to show that the inequality holds true with the sharp constant $\rho/(d-1)$ for every given setting of data on ∂B_ρ. In this respect, let us stress that the radial extensions are not enough accurate to obtain the inequality, as there are equality cases in which the optimal functions $\tilde{w}_i$'s are not radial (equality occurs for $k = 1, 2$, corresponding to positive and negative parts of an affine function). Actually, the optimal extension of the boundary functions is based on the solution of a multiphase free boundary problem with homogeneous Dirichlet conditions, which has been firstly considered in [13]. We get the result by manipulating some integral identities for the solution to this problem, which take into account in a subtle way their fine regularity properties established by Conti-Terracini-Verzini [13] and Caffarelli-Lin [7, 6].

A consequence of Theorem 1, is:

Theorem 2 (**Multiphase monotonicity formula).** *There exists a dimensional constant c_d such that, if $(u_1, \dots, u_k) \in \mathcal{U}(\Omega)$ is a multiphase local almost-quasi minimizer at $x \in \Omega$, with parameters $(\Lambda, \alpha, c_\alpha)$, then the mapping*

$$\rho \mapsto F_u(\rho) := \left[\frac{1}{\rho^{d-1}} \sum_{i=1}^{k} \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \right] \wedge \frac{c_d \Lambda^{2-d}}{d-1} + (d-1) \frac{c_\alpha}{\alpha} \rho^\alpha$$

is non decreasing on $(0, dist(x, \partial\Omega))$.

This monotonicity formula leads quite directly to the essential closedness of the jump sets for multiphase local almost-quasi minimizers, but taken alone it does not suffice to obtain its Ahlfors-regularity. To that aim, we need to set up another fine tool, which is a generalized version of the decay lemma due to De Giorgi-Carriero-Leaci [16] holding in the multiphase context. The interaction of the different phases at non-jump points of their common boundaries plays a crucial role in the control of the decay. Let us set

$$\Phi(u, A) := \sum_{i=1}^{k} \left[\int_A |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1} \left(J_{u_i} \cap A \right) \right] ,$$

where $u := (u_1, \dots, u_k) \in \mathcal{U}(\Omega)$ and $A \subseteq \Omega$. Setting $Dev(u, A)$ as the minimum $\lambda \geq 0$ such that

$$(3) \qquad \Phi(u, A) \leq \Phi(v, A) + \lambda \qquad \text{for every } v \in \mathcal{U}(\Omega) \text{ with } \{u \neq v\} \subset\subset A \, ,$$

we prove:

Theorem 3 (**Multiphase decay estimate).** *There exists $C_d > 0$ such that for every $\tau \in]0, 1[$ there exist $\varepsilon(\tau), \vartheta(\tau)$ such that if $u = (u_1, \dots, u_k) \in \mathcal{U}(\Omega)$, $B_\rho(x) \subset \Omega$ and*

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) < \varepsilon(\tau) \rho^{d-1}, \qquad Dev(u, B_\rho(x)) \leq \vartheta(\tau) \Phi(u, B_\rho(x)),$$

then

$$\Phi(u, B_{\tau\rho}(x)) \leq C_d \tau^d \Phi(u, B_\rho(x)).$$

Relying on Theorems 2 and 3, we obtain our main result:

Theorem 4 (**Regularity of multiphase local almost-quasi minimizers).** *Let $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ be a local almost-quasi minimizer of a multiphase free discontinuity problem at every point $x \in \Omega$.*

- *The function u is Hölder continuous on $\Omega \setminus \overline{\bigcup_i J_{u_i}}$, so that each phase $\Omega_i := \{u_i \neq 0\}$ is open in $\mathbb{R}^d$.*
- *The union of the jump sets $\bigcup_i J_{u_i}$ is essentially closed and Ahlfors regular, meaning respectively that*

$$\mathcal{H}^{d-1} \left(\overline{\bigcup_i J_{u_i}} \setminus \bigcup_i J_{u_i} \right) = 0 \, ,$$

and that there exist $c > 0$ and $\rho_0 > 0$ such that, for every $x \in \bigcup_i J_{u_i}$ and every $B_\rho(x) \subset \Omega$ with $\rho < \rho_0$, it holds

$$c\rho^{d-1} \leq \mathcal{H}^{d-1}\left(\bigcup_i J_{u_i} \cap B_\rho(x)\right) \leq \frac{1}{c}\rho^{d-1}\,. \tag{4}$$

Once the above regularity for the union of the jump sets is settled, some words of comments are in order as regards "no-jump interfaces", *i.e.* interfaces occurring at the zero level of two adjacent phases. To clarify what we mean, if we associate with an element $u = (u_1, \dots, u_k)$ in $\mathcal{U}(\Omega)$ the k phases $\Omega_i = \{u_i \neq 0\}$, the interaction between two distinct phases Ω_i, Ω_j may be of different nature, according to whether it occurs or not at jump points of u_i, u_j. Since a state function vanishes on the support of the other ones, a no-jump passage may actually occur at the zero level only.

A first comment in this respect is that a complete analysis of local minimizers to multiphase free discontinuity problems should not overlook also the study of the regularity of no-jump interfaces. Since the length of the boundaries of the phases are not counted in our notion of multiphase local almost-quasi minimizer, at this level the problem becomes of free boundary type, in the spirit of [13]. However an important different feature must be underlined: one deals with quasi-minimality instead of minimality, while the class of admissible test functions is larger. If we go back to the model problems, where global minimizers are investigated (the multiphase Mumford-Shah problem (1), and the multiphase thermal insulation problem 2), the regularity of the no-jump interfaces follows from the analysis of multiphase free boundary problems (see for instance [6], [7], [13]).

Another remark, arising especially from the specific point of view of image segmentation, is that penalizing also the boundary of the phases in the length term would be a reasonable choice as well. This is indeed the case especially when the reference background is not uniform, but the outcoming model would be different, and it is not our purpose to discuss it here. We postpone to a forthcoming paper a detailed study of the multiphase Mumford-Shah functional in a general form, in which the energy includes also the length of the boundary of the phases (plus some statistical term which allows the research for a given pattern).

The remaining of the paper is organized into four sections, where we give in the order the proofs of Theorem 1, 2, 3, 4, along with the required auxiliary results.

Notation. Throughout the paper, for every pair of closed sets $A, B \subseteq \mathbb{R}^d$ we will denote with $dist(A, B)$ the distance between A and B. We will also write $A \subset\subset B$ if the closure $\bar{A}$ is compact and contained in B. If $E \subseteq \mathbb{R}^d$, $|E|$ will denote its Lebesgue measure, while $\mathcal{H}^\alpha(E)$ will stand for its α-dimensional Hausdorff measure. $B_\rho(x)$ will stand for the open ball of center $x \in \mathbb{R}^d$ and radius $\rho > 0$. When $x = 0$, we will write simply B_ρ: we set $\omega_d := |B_1|$.

Concerning functional spaces, if $A \subseteq \mathbb{R}^d$ is open, $H^1(A)$ will stand for the usual space Sobolev functions which are square integrable together with their weak partial derivatives, while $SBV(A)$ will denote the space of special functions of bounded variation on A. We will consider also functions in H^1 or SBV on open subsets of spheres, which are defined locally through coordinate systems.

Finally for $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$.

2. Proof of Theorem 1

Up to rescaling, it is not restrictive to assume $\rho = 1$. In addition, by considering positive and negative parts, we can prove the result for nonnegative functions.

Definition 5. We say that w is an admissible datum if $w = (w_1, \dots w_k) \in (H^1(\partial B_1))^k$, with $w_i \geq 0$ and $w_i \cdot w_j = 0$ on ∂B_1.

For any admissible datum, we consider the multiphase extension problem

$$\mathcal{P}(w) \qquad \min\Big\{ \sum_{i=1}^k \int_{B_1} |\nabla \tilde{w}_i|^2 \, dx \; : \; \tilde{w}_i \in H^1(B_1)\,,\; \tilde{w}_i|_{\partial B_1} = w_i\,,\; \tilde{w}_i \cdot \tilde{w}_j = 0 \text{ in } B_1 \Big\}\,.$$

We proceed as follows:

- in Section 2.1 (Proposition 6), we recall some known facts about the above multiphase extension problem;
- in Section 2.2 (Proposition 7), we prove an integral identity holding for solutions to the multiphase extension problem under suitable regularity assumptions;
- in Section 2.3 (Proposition 10), we prove an approximation result which allows us to reduce ourselves to the smooth setting considered in the previous subsection;
- finally in Section 2.4 we give the proof of Theorem 1.

2.1. Some known facts on the multiphase extension problem.

Proposition 6. *Let w be an admissible datum. Then*

(i) *Problem $\mathcal{P}(w)$ admits a unique solution.*

(ii) *The solution $\tilde{w}$ to problem $\mathcal{P}(w)$ is Lipschitz in B_1, and it is Lipschitz up to the boundary in case w is smooth.*

(iii) *For every $i = 1, \dots, k$, the set $\Omega_i := \{\tilde{w}_i > 0\}$ is open and*

$$\Delta \tilde{w}_i = 0 \quad \textit{in} \quad \Omega_i.$$

(iv) *If $\{i \, : \, |\Omega_i \cap B_r(x)| > 0$ for r small enough$\} = \{i_1, i_2\}$, then*

$$\lim_{\Omega_{i_1} \ni y \to x} \nabla \tilde{w}_{i_1}(y) = - \lim_{\Omega_{i_2} \ni y \to x} \nabla \tilde{w}_{i_2}(y)\,.$$

(v) *The family of walls $(\cup_{i=1}^k \partial\Omega_i) \cap B_1$ is the disjoint union of an analytic hypersurface, and a relatively closed set having Hausdorff dimension at most $d-2$ and zero capacity. We shall refer to the latter as to the* singular set *of $\tilde{w}$.*

(vi) *If a sequence of admissible boundary data w^n converges strongly to some w in $H^{1/2}(\partial B_1)$, the sequence of solutions $\tilde{w}^n$ to problems $\mathcal{P}(w^n)$ converges strongly in $H^1(B_1)$ to the solution $\tilde{w}$ to problem $\mathcal{P}(w)$.*

Proof. For items (i)-(ii)-(iii)-(iv), we refer to [13], see respectively Theorem 3.1, Theorem 4.2, Theorem 5.1, Theorems 8.3 and 8.4, Remark 6.4. For item (v), we refer to [6, Section 1.6]. For item (vi), see again [13], Theorem 3.2. □

2.2. **The regular case.** In this section we are going to focus attention on a particular class of admissible data. Namely we will consider

$$v \in C^\infty(\partial B_1)^k, \qquad v_i \geq 0 \text{ on } B_1 \tag{5}$$

such that, setting

$$S_i := \{x \in \partial B_1 \,:\, v_i(x) > 0\}, \tag{6}$$

we have

$$\partial B_1 = E \cup \bigcup_{i=1}^k S_i, \qquad S_i \cap S_j = \emptyset, \quad E \text{ is a smooth hypersurface of dimension } d-2. \tag{7}$$

We prove:

Proposition 7. *Let v be an admissible datum satisfying* (5) *and* (7)*, and let $\tilde v = (\tilde v_1, \dots, \tilde v_k)$ be the solution to $\mathcal{P}(v)$. Then, denoting by ∇_τ the tangential gradient, we have*

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau v_i|^2 \, d\mathcal{H}^{d-1} = (d-1) \sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx + \frac{1}{2} \sum_{i=1}^k \int_{\Omega_i} (1 - |x|^2) |D^2 \tilde v_i|^2 \, dx \,, \tag{8}$$

where $\Omega_i := \{\tilde v_i > 0\}$. In particular, we have $\int_{\Omega_i} (1-|x|^2)|D^2 \tilde v_i|^2 \, dx < +\infty$ for every $i = 1, \dots, k$.

The proof of Proposition 7 is based on the following two lemmas.

Lemma 8. *Under the same assumptions of Proposition 7, it holds*

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} = d \sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx + \sum_{i=1}^k \int_{\Omega_i} (1 - |x|^2) |D^2 \tilde v_i|^2 \, dx \,.$$

Lemma 9. *Under the same assumptions of Proposition 7, it holds*

$$(d-2) \sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx = \sum_{i=1}^k \int_{\partial B_1} |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} - 2 \sum_{i=1}^k \int_{\partial B_1} \left| \frac{\partial \tilde v_i}{\partial n} \right|^2 \, d\mathcal{H}^{d-1}.$$

Before giving the proofs of Lemmas 8 and 9, let us set up a geometric construction which will be exploited in both of them. Let v and $\tilde v$ be as in Proposition 7, and let Σ be the singular set of $\tilde v$ according to Proposition 6 (v). From Proposition 6 (ii) and (v), we infer that $\overline{\Sigma} \cap \partial B_1 \subset E$, so that $\overline{\Sigma}$ has zero capacity. Consequently, we can find a sequence $(\mathcal{U}_\varepsilon, \varphi_\varepsilon)_{\varepsilon > 0}$ satisfying the following conditions:

- $\mathcal{U}_\varepsilon$ is a sequence of open neighborhoods of Σ (in $\mathbb{R}^d$), monotone decreasing with respect to inclusions, such that $\partial \mathcal{U}_\varepsilon$ is a smooth hypersurface and $\partial \mathcal{U}_\varepsilon \cap \partial \Omega_i$ is $\mathcal{H}^{d-1}$-negligible;
- φ_ε is a sequence of smooth functions, pointwise monotone increasing, such that, for every ε:

  $$1 - \varphi_\varepsilon \in \mathcal{C}_0^\infty(B_2, [0,1]) \,, \qquad \varphi_\varepsilon \equiv 0 \text{ in } \mathcal{U}_\varepsilon \,, \qquad \Delta \varphi_\varepsilon = 0 \text{ in } B_2 \setminus \overline{\mathcal{U}_\varepsilon} \,,$$

  and, in the limit as $\varepsilon \to 0^+$,

  $$\varphi_\varepsilon \to 1 \text{ in } H^1(B_2) \,.$$

In particular, thanks to the regularity of $\partial\Omega_i \setminus \overline{\Sigma}$ and since $\tilde{v}_i$ is harmonic on Ω_i, for every $\varepsilon > 0$ we have (by elliptic regularity)

$$\tilde{v}_i \in C^\infty(\overline{\Omega_i \setminus \overline{\mathcal{U}}_\varepsilon}). \tag{9}$$

Proof of Lemma 8. Let i be a fixed index in $\{1, \dots, k\}$. We are going to compute, for every $\varepsilon > 0$,

$$\int_{\Omega_i} \varphi_\varepsilon(1 - |x|^2)\, |D^2\tilde{v}_i|^2\, dx = \int_{\Omega_i \setminus \overline{\mathcal{U}}_\varepsilon} \varphi_\varepsilon(1 - |x|^2)\, |D^2\tilde{v}_i|^2\, dx, \tag{10}$$

where $\Omega_i := \{\tilde{v}_i > 0\}$, while $\mathcal{U}_\varepsilon$ and φ_ε are defined as above.

We point out that, by construction (due to Proposition 6 (v) and the choice of $\mathcal{U}_\varepsilon$), we have

$$\partial\Omega_i \setminus \overline{\mathcal{U}_\varepsilon} \subset S_i \cup \Gamma_i,$$

where $S_i \subseteq \partial B_1$ is given in (6) while $\Gamma_i \subset B_1$ is a smooth hypersurface.

In the following computations, we adopt Einstein convention on repeated indices. Moreover, for simplicity and with an abuse of notation, until otherwise stated we set $v := \tilde{v}_i$, $S = S_i$, $\Gamma := \Gamma_i$, $\mathcal{U} := \mathcal{U}_\varepsilon$, and $\varphi := \varphi_\varepsilon$; further, we denote by n the unit outer normal vector defined (everywhere) on $\partial\Omega \setminus \overline{\mathcal{U}}$.

Notice that we can perform integration by parts on the right hand side of (10) thanks to (9) and since $\Omega \setminus \overline{\mathcal{U}}$ is piecewise regular with a singular part which is $\mathcal{H}^{d-1}$-negligible (in particular it has finite perimeter): since boundary integrals take place only on $S \cup \Gamma$ (as $\varphi = 0$ in $\overline{\mathcal{U}}$), we obtain

$$\begin{aligned}
&\int_\Omega \varphi(1 - |x|^2)\, |D^2 v|^2\, dx = \int_{\Omega \setminus \overline{\mathcal{U}}} \varphi(1 - |x|^2)\, |D^2 v|^2\, dx \\
= \;& \int_S \varphi(1 - x_k x_k)\, \partial_{ij} v\, \partial_j v\, n_i\, d\mathcal{H}^{d-1} + \int_\Gamma \varphi(1 - x_k x_k)\, \partial_{ij} v\, \partial_j v\, n_i\, d\mathcal{H}^{d-1} \\
&- \int_\Omega \partial_i\varphi(1 - x_k x_k)\, \partial_{ij} v\, \partial_j v\, dx + 2\int_\Omega \varphi x_i\, \partial_{ij} v\, \partial_j v\, dx - \int_\Omega \varphi(1 - x_k x_k)\, \partial_{iij} v\, \partial_j v\, dx\,.
\end{aligned}$$

The first and last integrals in the latter sum vanish, respectively because $|x| = 1$ on $S \subseteq \partial B_1$, and because v is harmonic in Ω. Thus we have

$$\int_\Omega \varphi(1 - |x|^2)\, |D^2 v|^2\, dx = I_A + I_B + I_C\,, \tag{11}$$

being

$$\begin{aligned}
I_A &:= \int_\Gamma \varphi(1 - |x|^2)\, \partial_{ij} v\, \partial_j v\, n_i\, d\mathcal{H}^{d-1} \\
I_B &:= -\int_\Omega \partial_i\varphi(1 - |x|^2)\, \partial_{ij} v\, \partial_j v\, dx \\
I_C &:= 2\int_\Omega \varphi x_i\, \partial_{ij} v\, \partial_j v\, dx\,.
\end{aligned}$$

We now analyze separately the three integrals above.

(a) *Computation of I_A.* Recall the general formula

$$\Delta v = \Delta_\Gamma v + (d-1)\frac{\partial v}{\partial n} H_\Gamma + \frac{\partial^2 v}{\partial n^2} \qquad \text{on } \Gamma,$$

being H_Γ the scalar mean curvature. Since v is harmonic in Ω and vanishes on Γ, we get

$$\nabla v = -|\nabla v| n \qquad \text{and} \qquad (d-1)\frac{\partial v}{\partial n} H_\Gamma + \frac{\partial^2 v}{\partial n^2} = 0 \quad \text{on } \Gamma.$$

Moreover we may write

$$\partial_{ij} v \partial_j v n_i = -|\nabla v| (\partial_{ij} v) n_i n_j = -|\nabla v| \frac{\partial^2 v}{\partial n^2},$$

so that

$$I_A = -(d-1)\int_\Gamma \varphi(1-|x|^2) H_\Gamma \, |\nabla v|^2 \; d\mathcal{H}^{d-1} \,. \tag{12}$$

(b) *Computation of I_B.* We perform a further integration by parts. Exploiting the equality $|x| = 1$ on ∂B_1, and the fact that φ is harmonic in $\Omega \setminus \mathcal{U}$, we get

$$I_B = -\int_\Gamma \partial_i \varphi \, n_i (1-|x|^2) \partial_j v \partial_j v \, d\mathcal{H}^{d-1} - 2\int_\Omega \partial_i \varphi \, x_i \partial_j v \partial_j v \, dx - I_B \,,$$

so that

$$I_B = -\frac{1}{2}\int_\Gamma \frac{\partial \varphi}{\partial n}(1-|x|^2)|\nabla v|^2 \, d\mathcal{H}^{d-1} - \int_\Omega (\nabla\varphi \cdot x)|\nabla v|^2 \, dx \,. \tag{13}$$

(c) *Computation of I_C.* Also in this case, we perform a further integration by parts. Exploiting the equality $x = n$ on ∂B_1, and the identity $\operatorname{div}(x) = d$, we get

$$\begin{aligned} I_C = 2\int_S \varphi \partial_j v \partial_j v n_i n_i \, d\mathcal{H}^{d-1} + 2\int_\Gamma \varphi \partial_j v \partial_j v x_i n_i \, d\mathcal{H}^{d-1} - 2\int_\Omega \partial_i \varphi \, x_i |\nabla v|^2 \, dx \\ - 2d\int_\Omega \varphi |\partial_j v|^2 \, dx - I_C \end{aligned}$$

so that

$$I_C = \int_S \varphi |\nabla v|^2 \, d\mathcal{H}^{d-1} + \int_\Gamma \varphi |\nabla v|^2 (x \cdot n) \, d\mathcal{H}^{d-1} - \int_\Omega (\nabla\varphi \cdot x)|\nabla v|^2 \, dx - d\int_\Omega \varphi |\nabla v|^2 \, dx \,. \tag{14}$$

Now, we sum up the equalities (11) over all the phases, that we resume to denote by $\tilde{v}_i$, for $i = 1, \dots, k$. Notice carefully that, in doing so, we have cancellations of all the integrals over the hypersurfaces Γ_i coming from the expressions of I_A, I_B and I_C as computed respectively in (12), (13), and (14). Namely

$$\begin{aligned} (d-1)\sum_{i=1}^k \int_{\Gamma_i} \varphi(1-|x|^2) H_{\Gamma_i} \, |\nabla \tilde{v}_i|^2 \; d\mathcal{H}^{d-1} &= \sum_{i=1}^k \int_{\Gamma_i} \frac{\partial \varphi}{\partial n}(1-|x|^2)|\nabla \tilde{v}_i|^2 \, d\mathcal{H}^{d-1} \\ &= \sum_{i=1}^k \int_{\Gamma_i} \varphi |\nabla \tilde{v}_i|^2 (x \cdot n) \, d\mathcal{H}^{d-1} = 0 \,. \end{aligned}$$

Indeed, for two adjacent phases, all the integrands above have the same modulus (thanks to Proposition 6 (iv)), and opposite sign (due to the sign change respectively of the terms H_{Γ_i}, $\frac{\partial \varphi}{\partial n}$, and $(x \cdot n)$ when passing from a phase to an adjacent one).

So far, we have obtained

$$\sum_{i=1}^{k}\int_{\Omega_i}\varphi(1-|x|^2)|D^2\tilde v_i|^2\,dx = \sum_{i=1}^{k}\int_{\partial B_1}\varphi|\nabla\tilde v_i|^2\,d\mathcal{H}^{d-1} - d\sum_{i=1}^{k}\int_{B_1}\varphi|\nabla\tilde v_i|^2\,dx -2\sum_{i=1}^{k}\int_{\Omega_i}(\nabla\varphi\cdot x)|\nabla\tilde v_i|^2\,dx\,.$$

Finally, we recall that $\varphi=\varphi_\varepsilon$, and obtain the lemma by passing to the limit as $\varepsilon\to0$ in the identity above. Indeed, since the sequence $\{\varphi_\varepsilon\}$ converges increasingly to 1 as $\varepsilon\to0^+$, by monotone convergence we can pass to the limit in the first three integrals above. On the other hand, since $\nabla\varphi_\varepsilon\to0$ strongly in $L^2(B_1)$ and $|\nabla\tilde v_i|\le M$ (thanks to the smoothness of the boundary data v and to Proposition 6 (iii)), we can pass to the limit also in the fourth integral and see that it is infinitesimal as $\varepsilon\to0$.

□

Proof of Lemma 9. In the case of a single harmonic function in the ball, the proof of the lemma is standard and can be found, for instance, in [12, Appendix A]. In the multiphase context, the proof is still based on the same key, namely the minimality of the total Dirichlet energy, but contains some additional technical difficulties, due to the presence of the interfaces and their singularities.

For $t\in(-\tau,\tau)$, consider a one-parameter family of bi-Lipschitz homeomorphisms of B_1 into itself of the form

$$\Phi_t(x)=x+t\Psi_{\varepsilon,\delta}(x)\,,\qquad x\in B_1,$$

where

$$\Psi_{\varepsilon,\delta}(x)=\varphi_\varepsilon(x)\gamma_\delta(|x|)\,x\,,\qquad x\in B_1.$$

Above φ_ε is defined as done just above Lemma 8, while $\gamma_\delta:\mathbb{R}\to\mathbb{R}$ is an even function defined, for $\delta\in(0,1/2)$, by

$$\gamma_\delta(t):=\begin{cases}1 & \text{if } 0\le t\le 1-\delta\\ \frac{1}{\delta}(1-t) & \text{if } 1-\delta\le t\le 1\\ 0 & \text{if } t\ge1\,.\end{cases}$$

Throughout the proof, for simplicity and by abuse of notation we simply write v_i in place of $\tilde v_i$. Set

$$v_i^t(x):=v_i(\Phi_t(x))\,,\qquad x\in B_1\,.$$

By the minimality of $v=(v_1,\dots,v_k)$, the map $t\mapsto E(t):=\sum_{i=1}^k\int_{B_1}|\nabla v_i^t|^2\,dx$ has a critical point at $t=0$. In order to compute $E'(0)$, we can argue in the same way as in the classical case of harmonic functions (see [12]). Indeed, the functions v_i are smooth on the support of $\Psi_{\varepsilon,\delta}$ (by the presence of the cut-off function φ_ε in the definition of $\Psi_{\varepsilon,\delta}$ itself). Thus, starting from the identity

$$\nabla v_i^t(x)=\nabla v_i(\Phi_t(x))\big(I+tD\Psi_{\varepsilon,\delta}(x)\big)\,,\quad x\in B_1\,,$$

and, letting $\Omega_i := \{v_i > 0\}$, we get

$$
\begin{aligned}
E'(0) = \sum_{i=1}^{k} \int_{B_1} \frac{d}{dt} |\nabla v_i^t|^2 \Big|_{t=0} \, dx \\
= 2\sum_{i=1}^{k} \int_{\Omega_i} (D^2 v_i \cdot \nabla v_i) \cdot \Psi_{\varepsilon,\delta} \, dx + 2\sum_{i=1}^{k} \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta} \cdot \nabla v_i) \, dx \\
= -\sum_{i=1}^{k} \int_{B_1} |\nabla v_i|^2 \operatorname{div} \Psi_{\varepsilon,\delta} \, dx + 2\sum_{i=1}^{k} \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta} \cdot \nabla v_i) \, dx \,.
\end{aligned}
$$

(The last equality follows by noticing that $\sum_{i=1}^{k} 2\int_{B_1} (D^2 u_i \cdot \nabla v_i \cdot \Psi_{\varepsilon,\delta}) \, dx$ is the derivative at $t = 0$ of the map $t \mapsto \sum_{i=1}^{k} \int_{B_1} |\nabla v_i(\Phi_t(x))|^2 \, dx = \sum_{i=1}^{k} \int_{B_1} |\nabla v_i|^2 \det(D\Phi_t)^{-1}(x) \, dx$).

Thus the stationarity condition $E'(0) = 0$ gives us the identity

$$
\sum_{i=1}^{k} \int_{B_1} |\nabla v_i|^2 \operatorname{div} \Psi_{\varepsilon,\delta} \, dx = 2\sum_{i=1}^{k} \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta} \cdot \nabla v_i) \, dx \,. \tag{15}
$$

It is straightforward to compute $\operatorname{div} \Psi_{\varepsilon,\delta}$ and $D\Psi_{\varepsilon,\delta}$ as

$$
\begin{aligned}
\operatorname{div} \Psi_{\varepsilon,\delta} &= d\, \varphi_\varepsilon(x)\gamma_\delta(|x|) + \varphi_\varepsilon(x)\gamma_\delta'(|x|)|x| + \gamma_\delta(|x|) \big(x \cdot \nabla\varphi_\varepsilon(x)\big) \\
D\Psi_{\varepsilon,\delta} &= \varphi_\varepsilon(x)\gamma_\delta(|x|)I + \gamma_\delta(|x|)(x \otimes \nabla\varphi_\varepsilon(x)) + \gamma_\delta'(|x|)\varphi_\varepsilon(x) \left(x \otimes \frac{x}{|x|}\right).
\end{aligned}
$$

Since $\gamma_\delta'(|x|) = -1/\delta$ on $B_1 \setminus B_{1-\delta}$ and $\gamma_\delta'(|x|) = 0$ on $B_{1-\delta}$, we infer that

$$
\begin{aligned}
\sum_{i=1}^{k} \int_{B_1} |\nabla v_i|^2 \operatorname{div} V \, dx \quad &= d\sum_{i=1}^{k} \int_{B_1} \varphi_\varepsilon(x)\gamma_\delta(|x|)|\nabla v_i(x)|^2 \, dx \\
&+ \sum_{i=1}^{k} \int_{B_1} \gamma_\delta(|x|) \big(x \cdot \nabla\varphi_\varepsilon(x)\big)|\nabla v_i(x)|^2 \, dx \\
&- \frac{1}{\delta} \sum_{i=1}^{k} \int_{B_1 \setminus B_{1-\delta}} \varphi_\varepsilon(x)|x||\nabla v_i(x)|^2 \, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
2\sum_{i=1}^{k} \int_{B_1} \nabla v_i D\Psi_{\varepsilon,\delta} \cdot \nabla v_i \, dx \quad &= 2\sum_{i=1}^{k} \int_{B_1} \varphi_\varepsilon(x)\gamma_\delta(|x|)|\nabla v_i(x)|^2 \, dx \\
&+ 2\sum_{i=1}^{k} \int_{B_1} \gamma_\delta(|x|) \big(x \cdot \nabla v_i(x)\big)\big(\nabla v_i(x) \cdot \nabla\varphi_\varepsilon(x)\big) \, dx \\
&- \frac{2}{\delta} \sum_{i=1}^{k} \int_{B_1 \setminus B_{1-\delta}} \varphi_\varepsilon(x) \frac{1}{|x|} \big(\nabla v_i(x) \cdot x\big)^2 \, dx \,.
\end{aligned}
$$

Passing to the limit as $\delta \to 0$ in (15), we obtain

$$\sum_{i=1}^{k} d \int_{B_1} \varphi_\varepsilon(x)|\nabla v_i(x)|^2\,dx + \sum_{i=1}^{k}\int_{B_1} \big(x\cdot\nabla\varphi_\varepsilon(x)\big)|\nabla v_i(x)|^2\,dx$$
$$-\sum_{i=1}^{k}\int_{\partial B_1}\varphi_\varepsilon(x)|\nabla v_i(x)|^2\,d\mathcal{H}^{d-1} = \sum_{i=1}^{k} 2\int_{B_1}\varphi_\varepsilon(x)|\nabla v_i(x)|^2\,dx$$
$$+\sum_{i=1}^{k}2\int_{B_1}\big(x\cdot\nabla v_i(x)\big)\big(\nabla v_i(x)\cdot\nabla\varphi_\varepsilon(x)\big) - \sum_{i=1}^{k}2\int_{\partial B_1}\varphi_\varepsilon(x)\Big(\frac{\partial v_i}{\partial n}\Big)^2 d\mathcal{H}^{d-1}\,.$$

Eventually, we obtain the lemma by passing to the limit as $\varepsilon \to 0$ in the above equality, since $\varphi_\varepsilon \to 1$ strongly in $H^1(B_1)$, and since $|\nabla v_i| \le M$ (thanks to the smoothness of the boundary data v and to Proposition 6 (ii)). □

Proof of Proposition 7. Setting

$$T := \sum_{i=1}^{k}\int_{\partial B_1}|\nabla_\tau \tilde v_i|^2\,dx\,, \qquad N := \sum_{i=1}^{k}\int_{\partial B_1}\Big|\frac{\partial \tilde v_i}{\partial n}\Big|^2$$

from Lemma 8 and Lemma 9 we have respectively

$$T + N = d\sum_{i=1}^{k}\int_{B_1}|\nabla\tilde v_i|^2\,dx + \sum_{i=1}^{k}\int_{\Omega_i}(1-|x|^2)|D^2\tilde v_i|^2\,dx$$
$$T - N = (d-2)\sum_{i=1}^{k}\int_{B_1}|\nabla\tilde v_i|^2\,dx\,.$$

Solving in (T,N), we find (8). □

2.3. **Approximation.**

Proposition 10. *Any admissible datum $w = (w_1,\dots,w_k)$ can be approximated, in the strong topology of $(H^1(\partial B_1))^k$, by a sequence $v^n = (v_1^n,\dots,v_k^n)$ of admissible data satisfying* (5) *and* (7)*.*

For the proof we shall need the following technical lemma. Recall that, by definition, a sequence of equi-bounded quasi-open sets Ω_n γ-converges to Ω if the corresponding torsion functions w_{Ω_n}, namely the unique minimizers in $H^1_0(\Omega_n)$ of the functional $J(u) = \int_{\mathbb{R}^d}\big(\frac12|\nabla u|^2 - u\big)$, converge in $L^1(\mathbb{R}^d)$ to w_Ω.

Lemma 11. *Let D be an open bounded subset of $\mathbb{R}^d$. Given two quasi-open subsets A_1, A_2 of D such that* $\mathrm{cap}(A_1\cap A_2) = 0$*, there exist two sequences (A_1^n, A_2^n) of open subsets of D such that $A_1^n \cap A_2^n = \emptyset$ for every n and, as $n \to +\infty$, $(A_1^n, A_2^n) \to (A_1, A_2)$ in the sense of γ-convergence.*

Proof. For $i = 1, 2$, let w_i denote the torsion function of the quasi-open set A_i, defined as above. By [5, Proposition 2.1], every point of $\mathbb{R}^d$ is a Lebesgue point of w_i and w_i is upper-semicontinuous on $\mathbb{R}^d$. Then, for a fixed $\varepsilon > 0$, we consider the set

$$F_1^\varepsilon := \{w_1 \ge \varepsilon\}\,.$$

By the upper semicontinuity of w_1, this set is closed. Moreover, since $F_1^\varepsilon \subset A_1$, $\mathrm{cap}(A_1 \cap A_2) = 0$, and $w_2 = 0$ q.e. on $D \setminus A_2$, we have that $w_2 = 0$ q.e. on F_1^ε, or equivalently $w_2 \in$

$H^1_0(D\setminus F_1^\varepsilon)$. Then, since the set $D\setminus F_1^\varepsilon$ is open, the quasi-open set A_2 can be approximated in the sense of γ-convergence by a sequence $A_2^{\varepsilon,n}$ of open sets contained into $D\setminus F_1^\varepsilon$. Further, possibly passing to smaller open sets $\widetilde{A}_2^{\varepsilon,n} \subset A_2^{\varepsilon,n}$ such that $\text{dist}(\widetilde{A}_2^{\varepsilon,n}, \partial A_2^{\varepsilon,n}) > 0$ and still $\widetilde{A}_2^{\varepsilon,n} \to A_2$ in the sense of γ-convergence, we can assume that

$$\{w_1 > \varepsilon\} \subset D \setminus \overline{A_2^{\varepsilon,n}}\,.$$

Then the quasi-open set $\{w_1 > \varepsilon\}$ can be approximated in the sense of γ-convergence by a sequence $A_1^{\varepsilon,n}$ of open sets contained into $D \setminus \overline{A_2^{\varepsilon,n}}$. Again, possibly replacing $A_1^{\varepsilon,n}$ by smaller open sets $\widetilde{A}_1^{\varepsilon,n}$, we can assume that

$$\text{dist}(\partial A_1^{\varepsilon,n}, \partial A_2^{\varepsilon,n}) > 0\,.$$

So far, for every $\varepsilon > 0$ we have found two sequences $(A_1^{\varepsilon,n}, A_2^{\varepsilon,n})$ of open subsets of D such that $A_1^{\varepsilon,n} \cap A_2^{\varepsilon,n} = \emptyset$ for every n and, as $n \to +\infty$, γ-converge to $(\{w_1 > \varepsilon\}, A_2)$.

The proof is achieved by letting $\varepsilon \to 0$ and taking diagonal sequences. □

Proof of Proposition 10. We prove the statement for $k = 2$ phases, being the proof the same if $k > 2$. So we consider the case of two functions $(w_1, w_2) \in (H^1(\partial B_1))^2$, with $w_i \geq 0$ and $w_1 \cdot w_2 = 0$ on ∂B_1.

Step 1: *There exist two sequences $(w_1^n, w_2^n) \in (\mathcal{C}_0^\infty(\partial B_1))^2$ such that $w_i^n \geq 0$ on ∂B_1 ($i = 1, 2$), $\text{dist}\big(\{w_1^n > 0\}, \{w_2^n > 0\}\big) > 0$ for every n and, as $n \to +\infty$, $(w_1^n, w_2^n) \to (w_1, w_2)$ strongly in $(H^1(\partial B_1))^2$.*

This is obtained as an immediate consequence of Lemma 11 (applied replacing D with ∂B_1 by local coordinates). Indeed, the sets $A_i := \{w_i > 0\}$ are quasi-open and satisfy the condition $\text{cap}(A_1 \cap A_2) = 0$ (since $w_1 \cdot w_2 = 0$ on ∂B_1). Then Lemma 11 ensures the existence of two sequences of open sets (A_1^n, A_2^n) such that $A_1^n \cap A_2^n = \emptyset$ for every n and, as $n \to +\infty$, $(A_1^n, A_2^n) \to (A_1, A_2)$ in the sense of γ-convergence. In particular, since $w_i \in H_0^1(A_i)$, by the Mosco-convergence of the spaces $H_0^1(A_i^n)$ to $H_0^1(A_i)$ (*cf.* [2, Proposition 4.5.3]), we can find $(v_1^n, v_2^n) \in H_0^1(A_1^n) \times H_0^1(A_2^n)$ which converge to (w_1, w_2) strongly in $(H^1(\partial B_1))^2$. Possibly passing to $\max\{v_i^n, 0\}$ it is not restrictive to assume that $v_i^n \geq 0$ ($i = 1, 2$). In turn, since the sets A_i^n are open, for every fixed n we can find sequences $(\varphi_1^{n,k}, \varphi_2^{n,k}) \subset \mathcal{C}_0^\infty(A_1^n) \times \mathcal{C}_0^\infty(A_2^n)$ (still with non-negative values), which converge to (v_1^n, v_2^n) strongly in $H_0^1(A_1^n) \times H_0^1(A_2^n)$ as $k \to +\infty$. Passing to a diagonal sequence $(w_1^n, w_2^n) := (\varphi_1^{n,k(n)}, \varphi_2^{n,k(n)})$ we get the claim of Step 1.

Step 2: *Let $(w_1^n, w_2^n) \in (\mathcal{C}_0^\infty(\partial B_1))^2$ be as in Step 1. For every fixed n, we can approximate (w_1^n, w_2^n) strongly in $(H^1(\partial B_1))^2$ by a sequence $(\varphi_1^{n,k}, \varphi_2^{n,k}) \in (\mathcal{C}^\infty(\partial B_1))^2$ such that, for every k, setting $S_i^{n,k} := \{\varphi_i^{n,k} > 0\}$ it holds*

$$\partial B_1 = S_1^{n,k} \cup S_2^{n,k} \cup E^{n,k},$$

where $E^{n,k}$ is a smooth $(d-2)$-dimensional manifold, and the sets $S_1^{n,k}, S_2^{n,k}, E^{n,k}$ are mutually disjoint.

It is enough to enlarge the disjoint supports of the functions (w_1^n, w_2^n) constructed in Step 1, increasing them to sets whose disjoint union covers ∂B_1 and whose boundary is locally made by smooth $(d-2)$-dimensional manifolds; moreover, possibly adding a positive constant (infinitesimal as $n \to +\infty$), we make the functions strictly positive on their supports.

Step 3: Conclusion. Let $(\varphi_1^{n,k}, \varphi_2^{n,k}) \in (\mathcal{C}^\infty(\partial B_1))^2$ be as in Step 2. Passing to a diagonal sequence, namely setting $(v_1^n, v_2^n) := (\varphi_1^{n,k(n)}, \varphi_2^{n,k(n)})$ we obtain the proposition. Indeed, by construction, (v_1^n, v_2^n) are admissible data satisfying (5) and (7), and converging strongly to (w_1, w_2) in $H^1(\partial B_1)$. □

2.4. **Proof of Theorem 1.** Let $w = (w_1, \dots, w_k) \in (H^1(\partial B_1))^k$ be an admissible datum. Let $v^n = (v_1^n, \dots, v_k^n) \subset (\mathcal{C}^\infty(\partial B_1))^k$ be a sequence as given by Proposition 10. By Proposition 7, for every n a solution $\tilde v^n = (\tilde v_1^n, \dots, \tilde v_k^n)$ to problem $\mathcal{P}(v_n)$ satisfies

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau v_i^n|^2 \, d\mathcal{H}^{d-1} = (d-1)\sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i^n|^2 \, dx + \frac{1}{2}\sum_{i=1}^k \int_{\Omega_i^n} (1-|x|^2)|D^2 \tilde v_i^n|^2 \, dx \,, \tag{16}$$

where $\Omega_i^n := \{\tilde v_i^n > 0\}$. Moreover, by Proposition 6 (vi), the sequence $\{\tilde v^n = (\tilde v_1^n, \dots, \tilde v_k^n)\}$ converges strongly in $(H^1(B_1))^k$ to the solution $\tilde w = (\tilde w_1, \dots, \tilde w_k)$ to problem $\mathcal{P}(v)$.

Then we pass to the limit as $n \to +\infty$ in (16). Since the quantities

$$\int_{\partial B_1} |\nabla_\tau v_i|^2 \, dx \qquad \text{and} \qquad \int_{B_1} |\nabla \tilde v_i|^2 \, dx$$

are strongly continuous respectively in $H^1(\partial B_1)$ and in $H^1(B_1)$, we obtain

$$\begin{aligned}\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} &= (d-1)\sum_{i=1}^k \int_{B_1} |\nabla \tilde w_i|^2 \, dx + \frac{1}{2}\liminf_n \sum_{i=1}^k \int_{\Omega_i^n} (1-|x|^2)|D^2 \tilde v_i^n|^2 \, dx \\ &\geq (d-1)\min\Big\{ \sum_{i=1}^k \int_{B_1} |\nabla \tilde w_i|^2 \, dx \, : \, \tilde w_i|_{\partial B_1} = w_i \,, \, \tilde w_i \cdot \tilde w_j = 0 \text{ in } B_1 \Big\} .\end{aligned}$$

□

3. Proof of Theorem 2

Up to a translation, it is not restrictive to assume $x = 0$. For any $0 < \rho < dist(0, \partial\Omega)$, we set

$$\mathcal{E}_u(\rho) := \sum_{i=1}^k \left[\int_{\partial B_\rho} |\nabla_\tau u_i|^2 \, d\mathcal{H}^{d-1} + \mathcal{H}^{d-2}(J_{u_i}) \right], \qquad u \in (SBV(\partial B_\rho))^k$$

$$E_u(\rho) := \sum_{i=1}^k \left(\int_{B_\rho} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i}) \right), \qquad u \in (SBV(B_\rho))^k \,.$$

We proceed along the same line as in [4], setting up a cyclic inductive argument based on the following two lemmas.

Lemma 12. *Let $d \geq 2$. There exists a dimensional constants c_d such that, for every $u = (u_1, \dots, u_k) \in (SBV(\partial B_\rho))^k$, with $u_i \cdot u_j = 0$ on ∂B_ρ, which satisfies*

$$\mathcal{E}_u(\rho) \leq c_d \Lambda^{2-d} \rho^{d-2} \,,$$

there exists $w = (w_1, \dots, w_k) \in (H^1(\partial B_\rho))^k$, with $w_i \cdot w_j = 0$ on ∂B_ρ, such that

$$\sum_{i=1}^k \left[\int_{\partial B_\rho} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} + \frac{\Lambda(d-1)}{\rho} \mathcal{H}^{d-1}\big(\{u_i \neq w_i\}\big) \right] \leq \mathcal{E}_u(\rho) \,.$$

Lemma 13. *Let $d \geq 2$. For every u as in Lemma 12, and any w associated with u according to such lemma, there exists $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_k) \in (H^1(B_\rho))^k$, with $\tilde{w}_i = w_i$ on ∂B_ρ, such that*

$$\sum_{i=1}^{k} \left[\int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx + \Lambda \mathcal{H}^{d-1}\big(\{u_i \neq \tilde{w}_i\} \cap \partial B_\rho\big) \right] \leq \frac{\rho}{d-1} \mathcal{E}_u(\rho) \, .$$

Proof of Lemma 12 in $\mathbb{R}^2$. It is enough to take $c_2 < 1$. Indeed, if $u = (u_1, \ldots, u_k) \in (SBV(\partial B_\rho))^k$, with $u_i \cdot u_j = 0$ on ∂B_ρ, satisfies

$$\mathcal{E}_u(\rho) = \sum_{i=1}^{k} \left[\int_{B_\rho} |\nabla_\tau u_i|^2 \, dx + \mathcal{H}^0(J_{u_i}) \right] < 1 \, ,$$

necessarily we have $\mathcal{H}^0(J_{u_i}) = 0$ for every $i = 1, \ldots, k$. Hence $u \in (H^1(\partial B_\rho))^k$, and it is enough to take $w = u$. □

Proof of Lemma 12 in $\mathbb{R}^d$ $\Rightarrow$ Lemma 13 in $\mathbb{R}^d$. Let u be as in Lemma 12, and let w be associated with u according to such lemma. Let $\tilde{w}$ be the solution of the extension problem appearing in Theorem 1 (with boundary data w). We have

$$\begin{aligned}
&\sum_{i=1}^{k} \Big[\int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx + \Lambda \mathcal{H}^{d-1}\big(\{u_i \neq \tilde{w}_i\} \cap \partial B_\rho\big) \Big] \\
&\leq \sum_{i=1}^{k} \Big[\frac{\rho}{d-1} \int_{\partial B_\rho} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} + \Lambda \mathcal{H}^{d-1}\big(\{u_i \neq w_i\}\big) \Big] \\
&= \frac{\rho}{d-1} \sum_{i=1}^{k} \Big[\int_{\partial B_\rho} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} + \frac{\Lambda(d-1)}{\rho} \mathcal{H}^{d-1}\big(\{u_i \neq w_i\}\big) \Big] \leq \frac{\rho}{d-1} \mathcal{E}_u(\rho) \, ,
\end{aligned}$$

where the first inequality follows from Theorem 1, and the second one from Lemma 12. □

Proof of Lemma 13 in $\mathbb{R}^d$ $\Rightarrow$ Theorem 2 in $\mathbb{R}^d$. Let us show that Theorem 2 holds by taking the same dimensional constant which appears in Lemma 12.

Let $(u_1, \ldots, u_k) \in \mathcal{U}(\Omega)$ be a local $(\Lambda, \alpha, c_\alpha)$-almost-quasi minimizer of a multiphase free boundary problem at 0 in $\mathbb{R}^d$. For $\rho \in (0, dist(0, \partial\Omega))$, we can write $F_u(\rho)$ as

$$F_u(\rho) = \frac{E_u(\rho)}{\rho^{d-1}} \wedge \frac{c_d \Lambda^{2-d}}{d-1} + (d-1) \frac{c_\alpha}{\alpha} \rho^\alpha \, .$$

In the remaining of the proof, we write for brevity F, E, and $\mathcal{E}$ in place of F_u, E_u, and $\mathcal{E}_u$. Since the map $\rho \mapsto E(\rho)$ is non decreasing on $I := (0, dist(0, \partial\Omega))$, we have that E (and consequently also F) belong to $BV_{\rm loc}(I)$. The distributional derivative of E is given by

$$DE = D^a E + D^s E = E'(\rho) \, d\rho + \mu \, ,$$

where $E'(\rho)$ denotes the density of the absolutely continuous part $D^a E$, and μ is a singular positive measure. Hence, in order to show that F is non decreasing, it is enough to prove that the density $F'(\rho)$ of the absolutely continuous part of the measure DF is non negative.

By using Leibniz formula for BV function and the locality of the distributional derivative, we get

$$F'(\rho) = \begin{cases} (d-1)c_\alpha \rho^{\alpha-1} & \mathcal{L}^1\text{-a.e. on } I_+ := \left\{ \frac{E(\rho)}{\rho^{d-1}} \geq \frac{c_d \Lambda^{2-d}}{d-1} \right\} \\ \dfrac{E'(\rho)}{\rho^{d-1}} - (d-1)\dfrac{E(\rho)}{\rho^d} + (d-1)c_\alpha \rho^{\alpha-1} & \mathcal{L}^1\text{-a.e. on } I_- := \left\{ \frac{E(\rho)}{\rho^{d-1}} < \frac{c_d \Lambda^{2-d}}{d-1} \right\}. \end{cases}$$

It follows immediately that $F'(\rho) \geq 0$ $\mathcal{L}^1$-a.e. on I_+.

Let us show that the same holds true also on I_-. Assume by contradiction that $F'(\rho) < 0$ $\mathcal{L}^1$-a.e. on a subset J of I_- with positive measure. In this case, for $\mathcal{L}^1$-a.e. $\rho \in J$, we have

$$\mathcal{E}(\rho) \leq E'(\rho) < (d-1)E(\rho)\rho^{-1} - (d-1)c_\alpha \rho^{\alpha+d-2} < c_d \rho^{d-2}\Lambda^{2-d},$$

where the first inequality follows from the coarea formula, the second one from the above expression of $F'(\rho)$, and the third one from the definition of I_-. Then, applying Lemma 13, we get the existence of a function $\tilde{w} \in (H^1(B_\rho))^k$, such that

$$\sum_{i=1}^k \Big[\int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx + \Lambda \mathcal{H}^{d-1}\big(\{u_i \neq \tilde{w}_i\} \cap \partial B_\rho\big)\Big] \leq \frac{\rho}{d-1}\mathcal{E}(\rho) < E(\rho) - c_\alpha \rho^{d-1+\alpha}.$$

This contradicts the fact the u is a local $(\Lambda, \alpha, c_\alpha)$-almost-quasi minimizer (by taking the trial function $z = (z_1, \ldots, z_k)$ defined by $z_i = \tilde{w}_i \chi_{B_\rho} + u_i \chi_{\Omega \setminus B_\rho}$ and noticing that $\mathcal{H}^{d-1}(J_{z_i} \cap \overline{B}_\rho) \leq \mathcal{H}^{d-1}(\{u_i \neq \tilde{w}_i\} \cap \partial B_\rho)$).

Proof of Theorem 2 in $\mathbb{R}^d \Rightarrow$ Lemma 12 in $\mathbb{R}^{d+1}$. Up to rescaling, it is not restrictive to prove the lemma for $\rho = 1$. Let B^{d+1} denote the unit ball in $\mathbb{R}^{d+1}$, and let $u = (u_1, \ldots, u_k) \in (SBV(\partial B^{d+1}))^k$, with $u_i \cdot u_j = 0$ on ∂B^{d+1}, satisfy

$$\mathcal{E}_u(1) := \sum_{i=1}^k \Big[\int_{\partial B^{d+1}} |\nabla_\tau u_i|^2 \, d\mathcal{H}^d + \mathcal{H}^{d-1}(J_{u_i}) \Big] \leq c\Lambda^{1-d}; \tag{17}$$

the value of the constant c will be fixed at the end of the proof.

We consider the auxiliary problem

$$\min \Big\{ \Phi(w) \ : \ w \in (SBV(\partial B^{d+1}))^k, w_i \cdot w_j = 0 \text{ on } \partial B^{d+1} \Big\}, \tag{18}$$

with

$$\Phi(w) := \sum_{i=1}^k \left[\int_{\partial B^{d+1}} |\nabla_\tau w_i|^2 d\mathcal{H}^d + \mathcal{H}^{d-1}(J_{w_i}) + \Lambda d \mathcal{H}^d\big(\{u_i \neq w_i\}\big) \right].$$

In order to achieve the proof it is enough to show that, if the constant c in (17) is suitably chosen, a solution to problem (18) belongs to $(H^1(\partial B^{d+1}))^k$ (in fact, such solution will provide the function required by the lemma).

Assume by contradiction that this is not the case. Let $w := (w_1, \ldots, w_k)$ be a solution to problem (18) such that, for some fixed index $h \in \{1, \ldots, k\}$, the set J_{w_h} has positive $\mathcal{H}^{d-1}$ measure. With no loss of generality, we can assume that the north pole $e_{d+1} := (0, \ldots, 0, 1)$ is a point of density 1 for J_{w_h}. Let Π denote the orthogonal projection

$$\Pi := \partial B^{d+1} \cap B^{d+1}_\lambda(e_{d+1}) \to B^d_{\frac{1}{2}}(0) \times \{0\} \subset \mathbb{R}^d \times \{0\}$$

(where the radius λ is chosen so that Π is a bijection onto $B^d_{\frac{1}{2}}(0)\times\{0\}$), and consider the function in $\mathcal{U}(B^d_{\frac{1}{2}}(0))$ defined by

$$\overline{w}(x) := w\circ\Pi^{-1}(x)\,,\qquad x\in B^d_{\frac{1}{2}}(0)\,.$$

We now prove the following

Claim: the function $\overline{w}$ is a local $(\overline{\Lambda}_d, 1, \alpha_d\Lambda)$-almost-quasi minimizer of a multiphase free discontinuity problem at 0 in $B^d_{\frac{1}{2}}(0)$, where $\overline{\Lambda}_d$ and α_d are dimensional constants.

We take a competitor $\overline{\xi}\in\mathcal{U}(B^d_{\frac{1}{2}}(0))$ with $\bigcup_i\{\overline{\xi}_i\neq\overline{w}_i\}\subset B^d_\rho(0)\subset B^d_{\frac{1}{2}}(0)$, and we consider the function $\xi\in\mathcal{U}(\partial B^{d+1})$ defined by

$$\xi(x) := \begin{cases} w(x) & \text{for } x\in\partial B^{d+1}\setminus B^{d+1}_\lambda(e_{d+1})\\ \overline{\xi}\circ\Pi(x) & \text{for } x\in\partial B^{d+1}\cap B^{d+1}_\lambda(e_{d+1})\,.\end{cases}$$

Hereafter, we set $B_\rho := B^d_\rho(0)$, and we denote by γ_j dimensional constants.

Since w is a solution to problem (18), we know that $\Phi(w)\leq\Phi(\xi)$. Next we observe that

$$(19)\qquad (1-\gamma_1\rho)\int_{B_\rho}|\nabla\overline{w}_i|^2\,dx \leq \int_{\Pi^{-1}(B_\rho)}|\nabla_\tau w_i|^2\,dx \leq (1+\gamma_2\rho)\int_{B_\rho}|\nabla\overline{w}|^2\,dx$$

$$(20)\qquad \mathcal{H}^{d-1}(J_{\overline{w}_i}\cap\overline{B}_\rho)) \leq \mathcal{H}^{d-1}(J_{w_i}\cap\Pi^{-1}(\overline{B}_\rho))) \leq \gamma_3\mathcal{H}^{d-1}(J_{\overline{w}_i}\cap\overline{B}_\rho)$$

$$(21)\ \mathcal{H}^d\big(\{\overline{u}_i\neq\overline{w}_i\}\cap B_\rho\big) \leq \mathcal{H}^d\big(\{u_i\neq w_i\}\cap\Pi^{-1}(B_\rho)\big) \leq \gamma_4\mathcal{H}^d\big(\{\overline{u}_i\neq\overline{w}_i\}\cap B_\rho\big) \leq \gamma_5\rho^d\,,$$

where we have set $\overline{u}(x) := u\circ\Pi^{-1}(x)$ for $x\in B^d_{\frac{1}{2}}(0)$.

Using (19)-(20)-(21) (also applied with ξ in place of w), the inequality $\Phi(w)\leq\Phi(\xi)$ gives

$$(22)\quad \sum_{i=1}^k\left(\int_{B_\rho}|\nabla\overline{w}_i|^2\,dx + \mathcal{H}^{d-1}(J_{\overline{w}_i}\cap\overline{B}_\rho)\right) \leq (1+\gamma_6\rho)^2\sum_{i=1}^k\left(\int_{B_\rho}|\nabla\overline{\xi}_i|^2\,dx + \gamma_3\mathcal{H}^{d-1}(J_{\overline{\xi}_i}\cap\overline{B}_\rho)\right) + \gamma_7\Lambda\rho^d.$$

Now we observe that, from the optimality of w in problem (18), the sum at the l.h.s. of (22) is bounded from above by a dimensional constant times ρ^{d-1}; hence we may assume the same for the sum at the r.h.s. of (22) (the conclusion being otherwise immediate). After this remark, (22) implies

$$\sum_{i=1}^k\left(\int_{B_\rho}|\nabla\overline{w}_i|^2\,dx + \mathcal{H}^{d-1}(J_{\overline{w}_i}\cap\overline{B}_\rho)\right) \leq \sum_{i=1}^k\left(\int_{B_\rho}|\nabla\overline{\xi}_i|^2\,dx + \gamma_3\mathcal{H}^{d-1}(J_{\overline{\xi}_i}\cap\overline{B}_\rho)\right) + (\gamma_8+\gamma_9\Lambda)\rho^d\,.$$

The claim follows by choosing $\overline{\Lambda}_d := \gamma_3$ and $\alpha_d := \gamma_8+\gamma_9$ (recall that $\Lambda\geq 1$).

Now, since $\overline{w}$ is a local $(\overline{\Lambda}_d, 1, \alpha_d\Lambda)$-almost-quasi minimizer of a multiphase free boundary problem at 0, and we are assuming the validity of Theorem 2 in $\mathbb{R}^d$, we can write

$$F_{\overline{w}}(\rho) \geq \lim_{\rho\to 0^+} F_{\overline{w}}(\rho)$$

where the functional $F_{\overline{w}} = F_{\overline{w}}(\rho)$ is defined according to Theorem 2.

In particular, exploiting the assumption that e_{d+1} is a point of density 1 for J_{w_h}, it follows that for every $\rho < 1/2$

$$\frac{E_{\overline{w}}(\rho)}{\rho^{d-1}} \wedge \frac{c_d\overline{\Lambda}_d^{\,2-d}}{d-1} + (d-1)\alpha_d\Lambda\rho \geq \omega_{d-1} \wedge \frac{c_d\overline{\Lambda}_d^{\,2-d}}{d-1} =: \eta_d\,. \tag{23}$$

Next we notice that, for every $\rho < 1/2$ and a suitable dimensional constant γ_d, we have

$$E_{\overline{w}}(\rho) \leq \gamma_d\Phi(w) \leq \gamma_d\Phi(u) = \gamma_d\mathcal{E}_u(1) \leq c\,\gamma_d\Lambda^{1-d}\,, \tag{24}$$

where the first inequality can be obtained by arguing as in the proof of the claim above, the second inequality holds by definition of w, the third equality by definition of Φ, and the last inequality by the initial assumption on u.

Let us choose (recall $\Lambda \geq 1$)

$$\rho_c := \frac{c^{1/d}}{\Lambda} \leq c^{1/d} \tag{25}$$

with $c < 1/2^d$ so that $\rho_c < 1/2$. With this choice of ρ_c we have for c small enough (depending only on d)

$$\frac{c\,\gamma_d\Lambda^{1-d}}{{\rho_c}^{d-1}} = c^{1/d}\gamma_d \leq \frac{c_d\overline{\Lambda}_d^{\,2-d}}{d-1}\,, \tag{26}$$

and consequently, in view of (24), (25) and (26), inequality (23) entails

$$c^{1/d}\gamma_d + (d-1)\alpha_d c^{1/d} \geq \frac{c\,\gamma_d\Lambda^{1-d}}{{\rho_c}^{d-1}} + (d-1)\alpha_d\Lambda\rho_c \geq \eta_d\,,$$

which yields a contradiction if c is small enough (depending only on d).

□

4. Proof of Theorem 3

For any $u := (u_1, \ldots, u_k) \in \mathcal{U}(\Omega)$, $A \subseteq \Omega$, and $c \geq 0$, we define

$$\Phi(u, c, A) := \sum_{i=1}^{k}\left[\int_A |\nabla u_i|^2\,dx + c\mathcal{H}^{d-1}\left(J_{u_i} \cap A\right)\right],$$

and we set $Dev(u, c, A)$ the minimum $\lambda \geq 0$ such that

$$\Phi(u, c, A) \leq \Phi(v, c, A) + \lambda$$

for every competitor $v \in \mathcal{U}(\Omega)$ such that $\{u \neq v\} \subset\subset A$. For $c = 1$, as done in the Introduction, we write simply $\Phi(u, A)$ and $Dev(u, A)$.

In the following, $u = (u_1, \ldots, u_m) \in H^1(B_1; \mathbb{R}^m)$ with $u_i \cdot u_j = 0$ for $i \neq j$ is said to be a *multiphase local minimizer* of the Dirichlet energy provided that

$$\sum_{i=1}^{m}\int_{B_1} |\nabla u_i|^2\,dx \leq \sum_{i=1}^{m}\int_{B_1} |\nabla v_i|^2\,dx$$

for every $v = (v_1, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$ with $v_i \cdot v_j = 0$ and $\{v \neq u\} \subset\subset B_1$. In the case $m = 1$, the definition reduces to the usual one of local minimizer of the Dirichlet energy.

We proceed as follows:

- in Section 4.1, we describe the asymptotic behaviour of a sequence of elements in $\mathcal{U}(B_1)$ for which the deviation from minimality and the total length of the jumps go to zero; this is a multiphase adaptation of Theorem 7.7 in [1], and is carried over separately in the cases where more than one or just one phase are prevailing in the limit (see Propositions 15 and 16);
- in Section 4.2 (Proposition 17), we prove an auxiliary result about a subharmonicity property of local multiphase minimizers for the Dirichlet energy on B_1;
- finally in Section 4.3 we give the proof of Theorem 3.

4.1. **Asymptotic behaviour of sequences with vanishing jumps and deviation.** We begin with a useful lemma:

Lemma 14. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ with $u_n \geq 0$,*

$$\int_{B_1} |\nabla u_n|^2 \, dx \leq C, \qquad \mathcal{H}^{d-1}(J_{u_n}) \to 0$$

and such that

$$\liminf_n |\{u_n = 0\}| > 0. \tag{27}$$

Then there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, $\tau_{n_k} \geq 0$, $u \in H^1(B_1)$ and $c > 0$ such that setting

$$\bar{u}_{n_k} := u_{n_k} \wedge \tau_{n_k}$$

we have

$$|\{u_{n_k} \neq \bar{u}_{n_k}\}| \leq c \left(\mathcal{H}^{d-1}(J_{u_{n_k}})\right)^{d/d-1},$$

$$\bar{u}_{n_k} \to u \qquad \text{strongly in } L^2(B_1),$$

and

$$\int_{B_1} |\nabla u|^2 \, dx \leq \liminf_k \int_{B_1} |\nabla \bar{u}_{n_k}|^2 \, dx.$$

Proof. The result is a variant of [1, Proposition 7.5], where the conclusion is proved for

$$[(u_n \wedge \tau^+(u_n, B_1) \vee \tau^-(u_n, B_1)] - m_n,$$

where m_n is a median for u_n, while $\tau^\pm(u_n, B_1)$ are truncation levels associated to the Poincaré inequality in SBV (see [1, Theorem 4.14]).

We proceed in two steps.

Step 1. Up to a subsequence we may assume that for every $n \in \mathbb{N}$

$$|\{u_n = 0\}| \geq \delta > 0. \tag{28}$$

Let us set following [16]

$$\tau_n := \tau^+(u_n, B_1) := \inf\left\{t \geq 0 \,:\, |\{u < t\}| > \omega_d - [2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n})]^{\frac{d}{d-1}}\right\},$$

where γ_δ is the constant for which the isoperimetric inequality

$$\gamma_\delta Per(E, B_1) \geq |E|^{\frac{d-1}{d}} \tag{29}$$

holds true for every set $E \subset B_1$ such that $|E| \leq \omega_d - \delta$: here $Per(E, B_1)$ denotes the perimeter of E in B_1.

Let us set $\bar u_n := u_n \wedge \tau_n$ so that

$$|\{u_n \neq \bar u_n\}| \le [2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n})]^{\frac{d}{d-1}}. \tag{30}$$

We may write

$$|D\bar u_n|(B_1) \le \int_{B_1} |\nabla \bar u_n|\,dx + \tau_n \mathcal{H}^{d-1}(J_{u_n}), \tag{31}$$

while using the coarea formula and the isoperimetric inequality (29) we have

$$|D\bar u_n|(B_1) = \int_0^{+\infty} Per(\{\bar u_n > t\}, B_1)\,dt = \int_0^{\tau_n} Per(\{u_n > t\}, B_1)\,dt$$
$$\ge \frac{1}{\gamma_\delta}\int_0^{\tau_n} |\{u_n > t\}|^{\frac{d-1}{d}}\,dt.$$

Since by the very definition of τ_n we have

$$|\{u_n > t\}|^{\frac{d-1}{d}} \ge 2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n}) \qquad \text{for every } 0 \le t < \tau_n,$$

we conclude

$$|D\bar u_n|(B_1) \ge 2\tau_n \mathcal{H}^{d-1}(J_{u_n})$$

which together with (31) yields

$$|D\bar u_n|(B_1) \le 2\int_{B_1} |\nabla \bar u_n|\,dx. \tag{32}$$

In view of (28), Poincaré-Sobolev inequality in BV holds true for $\bar u_n$ (without subtracting a median), so that we may write

$$\left(\int_{B_1} \bar u_n^{\frac{d}{d-1}}\,dx\right)^{\frac{d-1}{d}} \le \eta_\delta |D\bar u_n|(B_1) \le 2\eta_\delta \int_{B_1} |\nabla \bar u_n|\,dx \le 2\eta_\delta \int_{B_1} |\nabla u_n|\,dx \tag{33}$$

for some $\eta_\delta > 0$.

Let $d \ge 3$. Applying the previous inequality to $v_n := u_n^{\frac{2(d-1)}{d-2}}$, being $\tau^+(v_n, B_1) = [\tau^+(u_n, B_1)]^{\frac{2(d-1)}{d-2}}$ we deduce by a straightforward calculation

$$\|\bar u_n\|_{L^{2^*}} \le c_\delta \|\nabla u_n\|_{L^2}. \tag{34}$$

for some $c_\delta > 0$, where as usual $2^* := \frac{2d}{d-2}$.

If $d = 2$, for any $q > 2$ Poincaré-Sobolev inequality together with (32) yields

$$\|\bar u_n\|_{L^q} \le \eta_{\delta,q} |D\bar u_n|(B_1) \le 2\eta_{\delta,q}\int_{B_1} |\nabla \bar u_n|\,dx \le 2\eta_{\delta,q}|B_1|^{1/2}\|\nabla u_n\|_{L^2}. \tag{35}$$

Step 2. Thanks to (32), (33) and (34) ((35) if $d = 2$), in view of the compact embedding of BV into L^1, there exist $u \in BV(B_1) \cap L^2(B_1)$ and $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\bar u_{n_k} \to u \qquad \text{strongly in } L^2(B_1).$$

Using Ambrosio's theorem on $\bar u_n \wedge M$ for any $M > 0$ as in [1, Proposition 7.5], we infer that $u \in H^1(B_1)$, so that the conclusion follows (take into account (30)).

□

Proposition 15. *Let $u_n := (u_1^n, u_2^n, \dots, u_k^n) \in \mathcal{U}(B_1)$ and $c_n > 0$ be such that*

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}\left(J_{u_i^n}\right) \to 0, \qquad \Phi(u_n, c_n, B_1) \le C, \qquad Dev(u_n, c_n, B_1) \to 0. \tag{36}$$

Assume that, for some $2 \le m \le k$,

$$\liminf_n |\{u_i^n \neq 0\}| > 0 \qquad i = 1, \dots, m \tag{37}$$

and

$$\lim_n |\{u_i^n \neq 0\}| = 0 \qquad i = m+1, \dots, k. \tag{38}$$

Then up to a subsequence

$$(u_1^n, \dots, u_m^n) \to (u_1, \dots, u_m) \in H^1(B_1; \mathbb{R}^m) \qquad a.e.\ in\ B_1,$$

where $(u_1, \dots, u_m)$ is a multiphase local minimizer for the Dirichlet energy with

$$\sum_{i=1}^{m} \int_{B_\rho} |\nabla u_i|^2 \, dx = \lim_n \Phi(u_n, c_n, B_\rho) \tag{39}$$

for every $0 \le \rho < 1$.

Proof. We follow [1, Theorem 7.7] adapting the arguments to our multiphase setting. We divide the proof in several steps.

Step 1: Compactness. By Helly's theorem we may assume that up to a subsequence for every $\rho \in [0, 1]$

$$\lim_n \Phi(u_n, c_n, B_\rho) = \alpha(\rho),$$

where $\alpha : [0, 1] \to [0, +\infty[$ is non decreasing. Moreover we may assume $c_n \to c_\infty \in [0, +\infty]$.

In view of (36), (37) and (38), by Lemma 14 we get that up to a subsequence, for $i = 1, \dots, m$

$$\bar{u}_i^n \to u_i \in H^1(B_1) \qquad \text{strongly in } L^2(B_1),$$

while for $i = m+1, \dots, k$

$$\bar{u}_i^n \to 0 \qquad \text{strongly in } L^2(B_1).$$

Here $\bar{u}_i^n := (u_n^i)^+ \wedge \tau_{i,+}^n - (u_n^i)^- \wedge \tau_{i,-}^n$ with $\tau_{i,\pm}^n > 0$ and

$$|\{u_i^n \neq \bar{u}_i^n\}| \le c_d \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1} \to 0.$$

This means that

$$u_n \to (u_1, \dots, u_m, 0, \dots, 0) \qquad \text{a.e. in } B_1,$$

with $u_i \cdot u_j = 0$ a.e. in B_1 for $i \neq j$, while

$$\bar{u}_n := (\bar{u}_1^n, \dots, \bar{u}_k^n) \to (u_1, \dots, u_m, 0, \dots, 0) \qquad \text{strongly in } L^2(B_1; \mathbb{R}^k).$$

Finally, thanks again to Lemma 14 for every $\rho \in [0, 1]$

$$\sum_{i=1}^{m} \int_{B_\rho} |\nabla u_i|^2 \, dx \le \liminf_n \sum_{i=1}^{m} \int_{B_\rho} |\nabla \bar{u}_i^n|^2 \, dx \le \alpha(\rho). \tag{40}$$

Step 2: Local minimality for $\bar u_n$. We may write thanks to (36)

$$c_n \int_0^1 \mathcal{H}^{d-1}(\{u_i^n \neq \bar u_i^n\} \cap \partial B_\rho)\, d\rho = c_n |\{u_i^n \neq \bar u_i^n\} \cap B_1| \le c c_n \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1}$$
$$= c c_n \mathcal{H}^{d-1}(J_{u_i^n}) \left[\mathcal{H}^{d-1}(J_{u_i^n})\right]^{1/d-1} \to 0.$$

Thus up to a further subsequence, we may assume that for every $i = 1, \dots, k$ and for a.e. $\rho \in [0,1]$

$$c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar u_i^n\} \cap \partial B_\rho) \to 0.$$

Since $\mathcal{H}^{d-1}(J_{\bar u_i^n} \cap \partial B_\rho) = 0$ for a.e. $\rho \in [0,1]$, by comparing u_n with

$$v_n := (v_1^n, \dots, v_k^n)$$

where

$$v_i^n := \bar u_i^n 1_{B_\rho} + u_i^n 1_{B_1 \setminus B_\rho}, \qquad \rho \in [0,1[,$$

we get

$$\Phi(\bar u_n, c_n, B_\rho) \le \Phi(u_n, c_n, B_\rho)$$
$$\le \Phi(\bar u_n, c_n, B_\rho) + \sum_{i=1}^k c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar u_i^n\} \cap \partial B_\rho) + Dev(u_n, c_n, B_1).$$

We conclude that for a.e. $\rho \in [0,1[$

$$\Phi(\bar u_n, c_n, B_\rho) \to \alpha(\rho), \tag{41}$$

and since (see e.g. [1, Lemma 7.3])

$$Dev(\bar u_n, c_n, B_\rho) \le \Phi(\bar u_n, c_n, B_\rho) - \Phi(u_n, c_n, B_\rho)$$
$$+ \sum_{i=1}^k c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar u_i^n\} \cap \partial B_\rho) + Dev(u_n, c_n, B_1),$$

we infer

$$Dev(\bar u_n, c_n, B_\rho) \to 0 \qquad \text{for a.e. } \rho \in [0,1[. \tag{42}$$

Step 3: Multiphase local minimality in the limit. Let us consider

$$(v_1, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$$

admissible multiphase competitor for $(u_1, \dots, u_m)$. Let us choose $\rho < \rho' < 1$ such that

$$\{(v_1, \dots, v_m) \neq (u_1, \dots, u_m)\} \subset\subset B_\rho,$$

the function α is continuous at ρ', the convergences (41) and (42) hold true at ρ and ρ'.

Let us set

$$w_i^n := \varphi v_i + (1-\varphi)\bar u_i^n \qquad i = 1, \dots, m$$

and

$$w_j^n := (1-\varphi)\bar u_j^n \qquad j = m+1, \dots, k,$$

where φ is a smooth cut-off function such that $\varphi = 1$ on B_ρ, the support of φ is contained in $B_{\rho'}$, and such that $|\nabla \varphi| \le C/(\rho' - \rho)$.

Notice that $w_n := (w_1^n, \dots, w_k^n)$ is not a priori a good competitor for $\bar{u}_n$ since we have no control on the supports of the components. Following [13] we set

$$v_i^n := \left((w_i^n)^+ - \sum_{j \neq i} |w_j^n| \right)^+ - \left((w_i^n)^- - \sum_{j \neq i} |w_j^n| \right)^+ .$$

Now $v_n := (v_1^n, \dots, v_k^n)$ is an admissible competitor for $\bar{u}_n$, with

$$v_n \to (v_1, \dots, v_m, 0, \dots, 0) \qquad \text{strongly in } L^2(B_1)$$

and

$$v_n = (v_1, \dots, v_m, 0, \dots, 0) \qquad \text{in } B_\rho.$$

Comparing v_n with $\bar{u}_n$ we get easily for some $c > 0$

$$\begin{aligned} \Phi(\bar{u}_n, c_n, B_\rho) \le & \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \\ & + c \left[\Phi(\bar{u}_n, c_n, B_{\rho'} \setminus B_\rho) + \sum_{i=1}^m \int_{B_{\rho'} \setminus B_\rho} |\nabla v_i|^2 \, dx + \frac{1}{(\rho' - \rho)^2} \sum_{i=1}^k \int_{B_{\rho'} \setminus B_\rho} |\bar{u}_i^n - v_i^n|^2 \, dx \right] \\ & \qquad + Dev(\bar{u}_n, c_n, B_{\rho'}), \end{aligned}$$

so that, in the limit as $n \to +\infty$,

$$\begin{aligned} \alpha(\rho) \le & \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \\ & + c \left[\alpha(\rho') - \alpha(\rho) + \sum_{i=1}^m \int_{B_{\rho'} \setminus B_\rho} |\nabla v_i|^2 \, dx + \frac{1}{(\rho' - \rho)^2} \sum_{i=1}^m \int_{B_{\rho'} \setminus B_\rho} |u_i - v_i|^2 \, dx \right]. \end{aligned}$$

Letting $\rho \to \rho'$, and since $u_i = v_i$ on $B_{\rho'} \setminus B_\rho$, we infer

$$\alpha(\rho') \le \sum_{i=1}^m \int_{B_{\rho'}} |\nabla v_i|^2 \, dx.$$

Choosing $v_i = u_i$ and recalling (40) we deduce that

$$\sum_{i=1}^m \int_{B_{\rho'}} |\nabla u_i|^2 \, dx = \alpha(\rho'),$$

which yields in particular the multiphase local minimality of the limit configuration $(u_1, \dots, u_m)$.

If $\rho \in [0, 1[$, we can choose $\rho' > \rho$ satisfying the relation above, so that by monotonicity

$$\alpha(\rho) \le \alpha(\rho') = \sum_{i=1}^m \int_{B_{\rho'}} |\nabla u_i|^2 \, dx.$$

For $\rho' \to \rho$ we deduce relation (39) and the proof is concluded. □

Proposition 16. *Let $u_n := (u_1^n, u_2^n, \dots, u_k^n) \in \mathcal{U}(B_1)$ and $c_n > 0$ be such that*

$$\sum_{i=1}^k \mathcal{H}^{d-1} \left(J_{u_i^n} \right) \to 0, \qquad \Phi(u_n, c_n, B_1) \le C, \qquad Dev(u_n, c_n, B_1) \to 0.$$

Assume that for every $2 \leq i \leq k$

$$\lim_n |\{u_i^n \neq 0\}| = 0.$$

Then up to a subsequence

$$u_1^n - m_n \to u_1 \in H^1(B_1) \qquad a.e.\ in\ B_1,$$

where m_n *is a median of* u_1^n*, and* u_1 *is a local minimizer for the Dirichlet energy with*

$$\int_{B_\rho} |\nabla u_1|^2\, dx = \lim_n \Phi(u_n, c_n, B_\rho)$$

for every $0 \leq \rho < 1$.

Proof. The proof is a variant of that of Proposition 15. We need to employ a median for u_1^n since we have no control on the size of its zero set.

By Helly's theorem we may assume that up to a subsequence for every $\rho \in [0,1]$

$$\lim_n \Phi(u_n, c_n, B_\rho) = \alpha(\rho),$$

where $\alpha : [0,1] \to [0,+\infty[$ is non decreasing. Moreover we may assume $c_n \to c_\infty \in [0,+\infty]$.

We divide the proof in several steps.

Step 1: Truncation for $(u_2^n, \dots, u_k^n)$**.** We can repeat Step 1 and Step 2 of the proof of Proposition 15 working on $(u_2^n, \dots, u_k^n)$ for which we know that the corresponding zero set is covering in measure the entire B_1. We infer that

$$\bar{u}_n := (u_1^n, \bar{u}_2^n, \dots, \bar{u}_k^n)$$

is such that for a.e. $\rho \in [0,1[$

$$\Phi(\bar{u}_n, c_n, B_\rho) \to \alpha(\rho) \tag{43}$$

and

$$Dev(\bar{u}_n, c_n, B_\rho) \to 0. \tag{44}$$

Here $\bar{u}_i^n := (u_n^i)^+ \wedge \tau_{i,+}^n - (u_n^i)^- \wedge \tau_{i,-}^n$ with $\tau_{i,\pm}^n > 0$,

$$\bar{u}_i^n \to 0 \qquad \text{strongly in } L^2(B_1)$$

and

$$|\{u_i^n \neq \bar{u}_i^n\}| \leq c_d \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1} \to 0.$$

Step 2: Local minimality for u_1^n**.** Let us fix $\rho \in [0,1[$ satisfying (43), (44) and such that α is continuous at ρ. Let $v_n \in SBV(B_1)$ be such that

$$\{v_n \neq u_1^n\} \subset\subset B_\rho.$$

Let us consider $\rho' > \rho$ satisfying (43), (44), and let us compare $\bar{u}_n$ with $(v_n, \varphi\bar{u}_2^n, \dots, \varphi\bar{u}_k^n)$, where φ is a smooth cut-off function such that $\varphi = 0$ on B_ρ, $\varphi = 1$ on $B_1 \setminus B_{\rho'}$, and $|\nabla\varphi| \leq 2/(\rho' - \rho)$. We get

$$\begin{aligned}\Phi(\bar{u}_n, c_n, B_{\rho'}) \ &\leq \int_{B_{\rho'}} |\nabla v_n|^2\, dx + 2\sum_{i=2}^k \int_{B_{\rho'}} [\varphi^2 |\nabla \bar{u}_i^n|^2 + (\bar{u}_i^n)^2 |\nabla\varphi|^2]\, dx \\ &\quad + c_n \mathcal{H}^{d-1}(J_{v_n} \cap B_{\rho'}) + c_n \sum_{i=2}^k \mathcal{H}^{d-1}(J_{\varphi \bar{u}_i^n} \cap B_{\rho'}) + Dev(\bar{u}_n, c_n, B_{\rho'}),\end{aligned}$$

so that

$$\Phi(\bar{u}_n, c_n, B_\rho) \le \int_{B_\rho} |\nabla v_n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + e_n(\rho, \rho'), \tag{45}$$

where

$$e_n(\rho,\rho') := 2\Phi(\bar{u}_n, c_n, B_{\rho'} \setminus B_\rho) + \frac{8}{(\rho'-\rho)^2} \sum_{i=2}^k \int_{B_1} |\bar{u}_i^n|^2 \, dx + Dev(\bar{u}_n, c_n, B_{\rho'}).$$

In particular we may write

$$\int_{B_\rho} |\nabla u_1^n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \le \int_{B_\rho} |\nabla v_n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + e_n(\rho,\rho').$$

Notice that in view of (43) and (44) we have

$$\limsup_n e_n(\rho,\rho') \le 2[\alpha(\rho') - \alpha(\rho)].$$

Choosing ρ' of the form $\rho' := (1 + a_n)\rho$ with a suitable $a_n > 0$, we deduce that for a.e. $\rho \in [0, 1[$

$$\int_{B_\rho} |\nabla u_1^n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \le \int_{B_\rho} |\nabla v_n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + \hat{e}_n(\rho)$$

with $\hat{e}_n(\rho) \to 0$ and (choose $v_n = u_1^n$ in (45))

$$\int_{B_\rho} |\nabla u_1^n|^2 \, dx + c_n \mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \to \alpha(\rho).$$

Step 3: Conclusion. In view of Step 2, the function u_1^n enjoys a local minimality property for the Mumford Shah energy with constant c_n which is independent of the other phases $(u_2^n, \dots, u_k^n)$. We are thus in the classical setting of [1, Theorem 7.7]: truncating (from above and below) and translating with a median m_n, we get the convergence almost everywhere to some function $u \in H^1(B_1)$ which is a local minimizer for the Dirichlet energy. The proof is thus concluded. □

4.2. **An auxiliary subharmonicity result.**

Proposition 17. *Let $(v_1, v_2, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$ be a local multiphase minimizer of the Dirichlet energy. Then the function $\sum_{i=1}^m |\nabla v_i|^2$ is subharmonic in B_1.*

Proof. It is not restrictive to assume that the functions are positive (considering positive and negative parts, we obtain a $(2k)$-multiphase local minimizer of the Dirichlet energy). To get the conclusion, it suffices to check that

$$-\Delta \left(\sum_{i=1}^m |\nabla v_i|^2 \right) \le 0 \qquad \text{in the sense of distributions on } B_1,$$

i.e.,

$$\sum_{i=1}^m \int_{B_1} \Delta\varphi |\nabla v_i|^2 \, dx \ge 0 \tag{46}$$

for every $\varphi \in C_c^\infty(B_1)$ with $\varphi \geq 0$. Denoting with Γ the interfaces in B_1 associated to $(v_1, v_2, \ldots, v_m)$ we may write integrating by parts

$$\sum_{i=1}^m \int_{B_1} \Delta\varphi |\nabla v_i|^2\, dx = \sum_{i=1}^m \sum_{k,j=1}^d \int_{B_1} |\partial_k v_i|^2 \partial_{jj}\varphi\, dx$$
$$= \sum_{i=1}^m \sum_{k,j=1}^d \left[\int_\Gamma |\partial_k v_i|^2 \partial_j \varphi n_j\, d\mathcal{H}^{d-1} - 2\int_{B_1} \partial_k v_i \partial_{kj} v_i \partial_j \varphi\, dx \right].$$

Notice that

$$\sum_{i=1}^m \sum_{k,j=1}^d \int_\Gamma |\partial_k v_i|^2 \partial_j \varphi n_j\, d\mathcal{H}^{d-1} = \sum_{i=1}^m \int_\Gamma |\nabla v_i|^2 \frac{\partial \varphi}{\partial n}\, d\mathcal{H}^{d-1} = 0$$

since at the interface between v_i and v_j we have $\nabla v_i = -\nabla v_j$ (see Proposition 6), while the normals are oppositely oriented. We infer integrating again by parts, taking into account that each v_i is harmonic on its phase

$$\sum_{i=1}^m \int_{B_1} \Delta\varphi |\nabla v_i|^2\, dx = -2\sum_{i=1}^m \sum_{k,j=1}^d \left[\int_\Gamma \partial_k v_i \partial_{kj} v_i \varphi n_j\, d\mathcal{H}^{d-1} + \int_{B_1} |\partial_{kj} v_i|^2 \varphi\, dx \right]. \tag{47}$$

Since $v_i = 0$ on Γ, we have

$$\partial_k v_i = \frac{\partial v_i}{\partial n} n_k \qquad \text{and} \qquad \sum_{k,j=1}^d \partial_{kj} v_i n_k n_j = -(d-1)\frac{\partial v_i}{\partial n} H_\Gamma \qquad \text{on } \Gamma,$$

where H_Γ stands for the mean curvature. Again by cancellation due to the different sign of H_Γ on the two sides of the interfaces we deduce

$$\sum_{i=1}^m \sum_{k,j=1}^d \int_\Gamma \partial_k v_i \partial_{kj} v_i \varphi n_j\, d\mathcal{H}^{d-1} = \sum_{i=1}^m \sum_{k,j=1}^d \int_\Gamma \frac{\partial v_i}{\partial n} n_k \partial_{kj} v_j \varphi n_j\, d\mathcal{H}^{d-1}$$
$$= -(d-1)\sum_{i=1}^m \int_\Gamma H_\Gamma \left(\frac{\partial v_i}{\partial n}\right)^2 \varphi\, d\mathcal{H}^{d-1} = 0.$$

Equality (47) thus yields

$$\sum_{i=1}^m \int_{B_1} \Delta\varphi |\nabla v_i|^2\, dx = -2\sum_{i=1}^m \sum_{k,j=1}^d \int_{B_1} |\partial_{kj} v_i|^2 \varphi\, dx \leq 0$$

so that (46) follows, and the proof is complete. □

4.3. **Proof of Theorem 3.** If $1/2 \leq \tau < 1$, the result follows by choosing $C_d \geq 2^d$. Let us thus consider the case $0 < \tau < 1/2$, and let $C_d > 0$ to be fixed below.

Assume by contradiction that there exist $\varepsilon_n, \vartheta_n \to 0$, $B_{\rho_n}(x_n) \subset \Omega$ and $u_n = (u_1^n, \ldots, u_k^n) \in \mathcal{U}(\Omega)$ such that

$$\sum_{i=1}^k \mathcal{H}^{d-1}(J_{u_i^n} \cap B_{\rho_n}(x_n)) = \varepsilon_n \rho_n^{d-1}, \qquad Dev(u_n, B_{\rho_n}(x_n)) = \vartheta_n \Phi(u_n, B_{\rho_n}(x_n)),$$

and

$$\Phi(u_n, B_{\tau\rho_n}(x_n)) > C_d \tau^d \Phi(u_n, B_{\rho_n}(x_n)).$$

Setting

$$v_n(y) := \sqrt{\frac{c_n}{\rho_n}} u_n(x_n + \rho_n y), \qquad y \in B_1, \qquad c_n := \frac{\rho_n^{d-1}}{\Phi(u_n, B_{\rho_n}(x_n))},$$

we obtain $v_n = (v_1^n, \ldots, v_k^n) \in \mathcal{U}(B_1)$ with

$$\Phi(v_n, c_n, B_1) = 1, \qquad \sum_{i=1}^k \mathcal{H}^{d-1}(J_{v_i^n}) = \varepsilon_n, \qquad Dev(v_n, c_n, B_1) = \vartheta_n$$

and

$$\Phi(v_n, c_n, B_\tau) > C_d \tau^d. \tag{48}$$

If the phases vary according to Proposition 15, up to a subsequence we have

$$(v_1^n, \ldots, v_m^n) \to (v_1, \ldots, v_m) \in H^1(B_1; \mathbb{R}^m) \qquad \text{a.e. in } B_1$$

with

$$\sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx = \lim_n \Phi(v_n, c_n, B_\rho) \le 1$$

for every $\rho \in [0, 1[$, where $(v_1, \ldots, v_m)$ is a local multiphase minimizer of the Dirichlet energy. Since in view of Proposition 17

$$\sum_{i=1}^m |\nabla v_i|^2 \text{ is subharmonic on } B_1,$$

we deduce that for every $\rho \in [0, 1/2]$

$$\frac{1}{\omega_d \rho^d} \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \le \sum_{i=1}^m \int_{B_1} |\nabla v_i|^2 \, dx \le 1$$

so that

$$\sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \le \omega_d \rho^d.$$

Passing to the limit in (48) we get

$$\sum_{i=1}^m \int_{B_\tau} |\nabla v_i|^2 \, dx \ge C_d \tau^d$$

which yields a contradiction if we choose $C_d > \omega_d$.

If the phases vary according to Proposition 16, up to a subsequence we have

$$u_1^n - m_n \to u_1 \in H^1(B_1) \qquad \text{a.e. in } B_1,$$

with

$$\int_{B_\rho} |\nabla u_1|^2 \, dx = \lim_n \Phi(v_n, c_n, B_\rho) \le 1$$

for every $\rho \in [0, 1[$, where m_n is a median of u_1^n, and u_1 is a local minimizer for the Dirichlet energy. This means that u_1 is harmonic in B_1, so that the function $|\nabla u_1|^2$ is subharmonic in B_1. We can thus adapt the previous arguments to get again a contradiction provided that $C_d > \omega_d$.

□

5. Proof of Theorem 4

Let $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ be a local almost-quasi minimizer of a multiphase free discontinuity problem at every point $x \in \Omega$.

We proceed in three steps.

Step 1: Hölder continuity of u and openness of the phases. We denote by $J^r_{u_i}$ the set of regular points of J_{u_i} (namely points of density 1), and we set

$$C := \overline{\bigcup_i J^r_{u_i}}\,.$$

Since C is a closed set such that $\mathcal{H}^{d-1}(J_{u_h} \setminus C) = 0$ for every $h = 1, \dots, k$, we deduce that $u_i \in H^1_{loc}(\Omega \setminus C)$. Moreover, thanks to the local minimality, we get that, for every $B_\rho(x) \subset \Omega \setminus C$,

$$\int_{B_\rho(x)} |\nabla u_i|^2\, dx \le \Lambda d \omega_d \rho^{d-1} + c_\alpha \rho^{d-1+\alpha}.$$

By Poincaré inequality we infer

$$\int_{B_\rho(x)} |u_i - (u_i)_{x,\rho}|^2\, dx \le C_d \rho^2 \int_{B_\rho(x)} |\nabla u_i|^2\, dx \le C_d(\Lambda d \omega_d \rho^{d+1} + c_\alpha \rho^{d+1+\alpha}),$$

where $(u_i)_{x,\rho}$ denotes the integral mean of u_i on $B_\rho(x)$. Thanks to Campanato's criterion (see e.g. [1, Theorem 7.51]), we deduce that u_i is Hölder continuous (with exponent 1/2) in $\Omega \setminus C$. In particular we obtain $J_{u_i} \setminus C = \emptyset$, so that

$$\bigcup_i J_{u_i} \subseteq C \subseteq \overline{\bigcup_i J_{u_i}},$$

which entails (C is closed)

$$\overline{\bigcup_i J_{u_i}} = C. \tag{49}$$

From the Hölder continuity of u in $\Omega \setminus \overline{\bigcup_i J_{u_i}}$, we infer that the phases $\Omega_i := \{u_i \neq 0\}$ are open sets, whose boundary is composed either of jump points of u_i or of regular points for which $u_i = 0$.

Step 2: Essential closedness of the union of jump sets. In view of (49), in order to obtain the essential closedness of $\bigcup_i J_{u_i}$, it is enough to show that

$$\mathcal{H}^{d-1}\Big(\overline{\bigcup_i J^r_{u_i}} \setminus \bigcup_i J_{u_i}\Big) = 0\,. \tag{50}$$

Let $x \in \bigcup_i J^r_{u_i}$, and let $F_u(\rho)$ be defined as in Theorem 2. By Theorem 2, comparing $F_u(\rho)$ with its behaviour as $\rho \to 0^+$, we obtain, for $\rho \in (0, dist(x, \partial\Omega))$

$$\left[\frac{1}{\rho^{d-1}} \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x))\right)\right] + (d-1)\frac{c_\alpha}{\alpha}\rho^\alpha \ge \omega_{d-1} \wedge \frac{c_d \Lambda^{2-d}}{d-1}\,.$$

Hence there exists $\rho_0 > 0$ and $C_0 > 0$ (independent of x) such that for every $\rho < dist(x, \partial\Omega) \wedge \rho_0$

$$\sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x))\right) \ge C_0 \rho^{d-1}. \tag{51}$$

By continuity, we can extend the above inequality to points $\overline{x} \in \overline{\bigcup_i J^r_{u_i}}$.

Now we recall the well-known fact that, since $u_i \in SBV(\Omega)$, it holds

$$\lim_{\rho \to 0^+} \rho^{1-d} \sum_{i=1}^{k} \Big(\int_{B_\rho(y)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(y)) \Big) = 0$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \Omega \setminus \bigcup_i J_{u_i}$ (see for instance [16, Theorem 3.6]). In view of this fact, the validity of inequality (51) for points $\overline{x} \in \overline{\bigcup_i J^r_{u_i}}$ tells us that

$$\overline{\bigcup_i J^r_{u_i}} \subseteq \bigcup_i J_{u_i} \cup A \,,$$

for some set A with $\mathcal{H}^{d-1}(A) = 0$, which implies (50).

Step 3: Ahlfors regularity of the union of jump sets. Firstly observe that there exist $c' > 0$ and $\rho'_0 > 0$ such that, for every $x \in \bigcup_i J_{u_i}$ and every $B_\rho(x) \subset \Omega$ with $\rho < \rho'_0$, it holds

$$c' \rho^{d-1} \leq \sum_{i=1}^{k} \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \leq \frac{1}{c'} \rho^{d-1} \,.$$

Namely, the estimate from above follows immediately from the almost-quasi minimality of u, whereas the estimate from below follows by arguing as in Step 2 (*cf.* (49) and (51)). A simple monotonicity by inclusion argument yields also

$$c' \rho^{d-1} \leq \sum_{i=1}^{k} \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \right) \leq \frac{1}{c'} \rho^{d-1} \,. \tag{52}$$

As a consequence of (52), we immediately obtain the upper bound inequality in (4). In order to show the lower bound inequality in (4), we argue by contradiction. Assume there exist sequences $\{x_n\} \subset \bigcup_i J_{u_i}$, $\rho_n \to 0$, $c_n \to 0$ such that

$$\mathcal{H}^{d-1}\Big(\bigcup_i J_{u_i} \cap B_{\rho_n}(x_n) \Big) \leq c_n \rho_n^{d-1}$$

and hence

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}\Big(J_{u_i} \cap B_{\rho_n}(x_n) \Big) \leq k c_n \rho_n^{d-1} \,. \tag{53}$$

In view of (52), this implies that there exists a sequence $\varepsilon_n \to 0$ such that

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho_n}(x_n)) \leq \varepsilon_n \sum_{i=1}^{k} \int_{B_{\rho_n}(x_n)} |\nabla u_i|^2 \, dx \leq \frac{\varepsilon_n}{c'} \rho_n^{d-1}.$$

Using the above inequality and the almost-quasi minimality of u, we get, for every $(v_1,\dots,v_k)\in\mathcal{U}(\Omega)$ such that $\bigcup_i\{v_i\neq u_i\}\subset\subset B_{\rho_n}(x_n)$

$$
\begin{aligned}
&\sum_{i=1}^k\Big(\int_{B_{\rho_n}(x_n)}|\nabla u_i|^2\,dx+\Lambda\mathcal{H}^{d-1}(J_{u_i}\cap B_{\rho_n}(x_n))\Big)\\
&=\sum_{i=1}^k\Big(\int_{B_{\rho_n}(x_n)}|\nabla u_i|^2\,dx+\mathcal{H}^{d-1}(J_{u_i}\cap B_{\rho_n}(x_n))\Big)+\sum_{i=1}^k(\Lambda-1)\mathcal{H}^{d-1}(J_{u_i}\cap B_{\rho_n}(x_n))\Big)\\
&\quad\le\sum_{i=1}^k\Big(\int_{B_{\rho_n}(x_n)}|\nabla v_i|^2\,dx+\Lambda\mathcal{H}^{d-1}(J_{v_i}\cap B_{\rho_n}(x_n))\Big)+c_\alpha\rho_n^{d-1+\alpha}+(\Lambda-1)\frac{\varepsilon_n}{c'}\rho_n^{d-1}.
\end{aligned}
$$

Setting

$$
\Phi(u,\Lambda,A):=\sum_{i=1}^k\left[\int_A|\nabla u_i|^2\,dx+\Lambda\mathcal{H}^{d-1}(J_{u_i}\cap A)\right],
$$

the previous inequality reads

$$
\Phi(u,\Lambda,B_{\rho_n}(x_n))\le\Phi(v,\Lambda,B_{\rho_n}(x_n))+c_\alpha\rho_n^{d-1+\alpha}+(\Lambda-1)\frac{\varepsilon_n}{c'}\rho_n^{d-1},
$$

which means that the associated deviation from minimality (with coefficient Λ) of u in $B_{\rho_n}(x_n)$ (see (3)) satisfies

$$
Dev(u,\Lambda,B_{\rho_n}(x_n))\le\Big[(\Lambda-1)\frac{\varepsilon_n}{c'}+c_\alpha\rho_n^\alpha\Big]\rho_n^{d-1}.
$$

Recalling (52), this implies

$$
Dev(u,\Lambda,B_{\rho_n}(x_n))\le\theta_n\Phi(u,\Lambda,B_{\rho_n}(x_n)),\tag{54}
$$

for an infinitesimal sequence θ_n. By (53) and (54), applying Theorem 3 (Λ is fixed), for every $\tau\in(0,1)$ and n large enough, we obtain that

$$
\Phi(u,\Lambda,B_{\tau\rho_n}(x_n))\le C_d\tau^d\Phi(u,\Lambda,B_{\rho_n}(x_n)).
$$

This contradicts the energy estimate (52) as soon as τ is chosen small enough.

□

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