LOCAL MINIMALITY RESULTS FOR THE MUMFORD-SHAH FUNCTIONAL VIA MONOTONICITY

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ABSTRACT. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded piecewise $C^{1,1}$ open set with convex corners, and let

$$MS(u) := \int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}^1(J_u) + \beta\int_\Omega |u-g|^2\,dx$$

be the Mumford-Shah functional on the space $SBV(\Omega)$, where $g \in L^\infty(\Omega)$ and $\alpha, \beta > 0$. We prove that the function $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + \beta u = \beta g & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

is a local minimizer of MS with respect to the L^1-topology. This is obtained as an application of interior and boundary monotonicity formulas for a weak notion of quasi minimizers of the Mumford-Shah energy. The local minimality result is then extended to more general free discontinuity problems taking into account also boundary conditions.

KEYWORDS: Local minimality, Monotonicity formulas, free discontinuity functionals.

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CONTENTS

1. INTRODUCTION

The Mumford-Shah functional has been introduced in [13, 14] in the context of image segmentation, and then has found important applications in several other fields, among all variational theories in fracture mechanics (see [11, 4]). It can be considered the typical example of *free discontinuity functional*, characterized by the coupling of bulk and surface energies.

The *weak* formulation of the functional due to De Giorgi and Ambrosio [9] takes the form

$$MS(u) := \int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}^1(J_u) + \beta\int_\Omega |u-g|^2\,dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set, $\alpha, \beta > 0$, $g \in L^\infty(\Omega)$. Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, u belongs to the class of *special functions of bounded variation* $SBV(\Omega)$ (see

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Section 2), and J_u is the jump set of u. The last term involving g is usually called the *fidelity term*: in image segmentation, g is the level of grey of the picture, which has to be approximated by choosing conveniently the jump set J_u (the *edges*) and the function u outside it (*regularized image*). When $\beta = 0$, we speak of the homogeneous version of MS.

Within this framework, and in general dimension N, existence of minimizers can be proved easily through the direct method of the Calculus of Variations, as a direct application of Ambrosio's compactness and lower semicontinuity theorem. Moreover, thanks to the regularity result of De Giorgi, Carriero and Leaci [10], minimizers have a topologically closed jump set and are regular outside, yielding an admissible configuration of the original formulation of [13, 14], in which the discontinuity set was considered as an independent variable. The regularity of the discontinuity set has then been improved by Ambrosio, Fusco and Pallara, who proved that up to a $\mathcal{H}^{N-1}$-negligible set, J_u is a manifold of class $C^{1,\delta}$ for any $\delta < 1$ and of class $C^{1,1}$ if $N = 2$.

The issue of detecting minimizers, or more generally *local minimizers* of the Mumford-Shah functional is very delicate. Since MS is only lower semicontinuous (with respect to the natural L^1-topology), necessary conditions for minimality cannot be obtained by "standard" differentiation.

First order necessary conditions are established for jump sets $\Gamma = J_u$ sufficiently regular by considering *inner variations* (see [2, Chapter 7]): they yield that u satisfies an elliptic PDE outside Γ, while its (mean) curvature H_Γ is involved in a transmission condition coupling the values of u and ∇u on both sides of Γ.

For the homogeneous version of MS, a second order necessary condition for minimality has been proposed by Cagnetti, Mora and Morini [6], involving the positive semidefinitness of a suitable quadratic form defined on $H^1_0(\Gamma)$. Under strict positivity, the authors prove that u is a "local" minimizer among those function $v \in SBV(\Omega)$ such that $J_v \subseteq \Phi(\Gamma)$ and $v = u$ on $\partial\Omega$, where Φ is any diffeomorphism of $\mathbb{R}^N$ which is sufficiently C^2-close to the identity and such that $Id - \Phi$ is compactly supported in Ω. This local minimality has then been extended to a full local minimality in the L^1 topology in dimension $N = 2$ by Bonacini and Morini [3], employing a penalization/regularization technique together with results from the regularity theory of the area and the Mumford-Shah functional.

The aim of this paper is to show that, in dimension $N = 2$ and under mild regularity assumptions on the geometry of Ω, the Mumford-Shah functional (1.1) admits a natural local minimizer with no jumps . More precisely, the main result of the paper is the following.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, piecewise $C^{1,1}$-domain with convex corners. Then the function $u \in H^1(\Omega)$ such that*

$$\begin{cases} -\Delta u + \beta u = \beta g & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

is a local minimizer of MS with respect to the L^1-topology.

In other words, the minimizer of MS within the class of Sobolev functions (which satisfies (1.2)) turns out to be a local minimizer for the natural L^1-topology in the full class of SBV competitors.

A similar result has been obtained by Chambolle, Ponsiglione and the third author in [8] for a generalization of the homogeneous version of the Mumford Shah functional under boundary conditions, motivated by the study of the issue of *crack initiation* in brittle materials within variational theories of crack propagation (see [11, 4]). The technique of [8] is based essentially on the maximum principle through a truncation argument, and can be used to deal with nonlinear energies for the gradient (with p-growth for example) and more general geometries for Ω, but does not apply to the Mumford-Shah functional since it fails when the fidelity term is present.

Our proof of Theorem 1.1 is based on the use of the monotonicity formula introduced by Luckhaus and the first author in [5] for quasi-minimizers of MS, which, in dimension two, we extend up to boundary points and establish for a quite weak notion of quasi-minimizers.

Namely we consider functions $u \in SBV(\Omega)$ which are *weak local almost quasi-minimizers* of the Mumford-Shah energy *with respect to their own jump set*, i.e., such that

$$\int_{B_\rho(x)\cap\Omega} |\nabla u|^2\,dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \le \int_{B_\rho(x)\cap\Omega} |\nabla v|^2\,dx + \Lambda\mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma} \tag{1.3}$$

for every $v \in SBV(\Omega)$ with $\{v \neq u\} \subseteq B_\rho(x)$ and $J_v \subseteq J_u$, where $x \in \overline{\Omega}$, $\rho < \rho_0$ are such that

$$\int_{B_\rho(x)\cap\Omega} |\nabla u|^2\,dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \le \rho,$$

while $\Lambda \ge 1$ and $\gamma, c_\gamma > 0$. We then prove that if $x \in \Omega$, then the quantity

$$E_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma}\rho^\gamma \tag{1.4}$$

is non decreasing on $(0, \rho_0 \wedge dist(x, \partial\Omega))$, where

$$\mathcal{E}_x(\rho) := \int_{\Omega\cap B_\rho(x)} |\nabla u|^2\,dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)),$$

while if $x \in \partial\Omega$, monotonicity holds true for the modification

$$\tilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma}\rho^\gamma + k_\Omega \int_0^\rho \left(\frac{\mathcal{E}_x(r)}{r} \wedge 1\right) dr \tag{1.5}$$

on the interval $(0, \rho_0 \wedge r_\Omega \wedge dist(x, S \setminus \{x\}))$, where S denotes the set of corners of $\partial\Omega$ and $r_\Omega, k_\Omega > 0$.

The proof of these monotonicities follows that of [5], which is based on a precise harmonic extension estimate on a ball (see Proposition 3.1). The dimension $N = 2$ entails drastic simplifications in the arguments, and this is the main reason for which we can deal with a weaker class of quasi-minimizers (with respect to their own jump set, see also Remark 4.2). The extension of the monotonicity to the boundary requires a generalization of the key harmonic extension estimate to domains of the form $\Omega \cap B_\rho(x)$ (see Proposition 3.7): this motivates the piecewise $C^{1,1}$-regularity assumption for $\partial\Omega$ and the restriction to *convex* angles. We also refer the reader to the paper [7] by Chambolle, Séré and Zanini, where a boundary monotonicity formula is proved on flat boundaries, under Neumann conditions.

The use of monotonicity in the proof of Theorem 1.1 is roughly as follows. Assuming by contradiction that there exists $u_n \to u$ strongly in $L^1(\Omega)$ such that $MS(u_n) < MS(u)$, it is easily seen (through Ambrosio's theorem) that $\mathcal{H}^1(J_{u_n}) = \varepsilon_n \to 0$. We then consider the *constrained* minimization problem

$$\min_{\substack{w \in SBV(\Omega)\\ \mathcal{H}^1(J_w) \le \varepsilon_n}} MS(w)$$

whose minimizers w_n are such that

$$MS(w_n) \le MS(u_n) < MS(u). \tag{1.6}$$

Writing $w_n = u + v_n$, we can show that v_n satisfy the weak minimality property (1.3), so that monotonicity is available, and that

$$\int_\Omega |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) \to 0. \tag{1.7}$$

Since $J_{w_n} = J_{v_n}$, inequality (1.6) can hold only for $J_{v_n} \neq \emptyset$. If $x_n \in J_{v_n}$ is a point of density one with $x_n \to x \in \Omega$, monotonicity entails a strictly positive uniform lower bound for $E_{x_n}(\rho)$ defined in (1.4). But (1.7) yields the existence of ρ_n such that $E_{x_n}(\rho_n) \to 0$, a contradiction. If x_n approaches $\partial\Omega$, boundary monotonicity for (1.5) can be employed to get the same conclusion.

In Section 6 we extend the minimality result to a free discontinuity functional of the type

$$F(u) := \int_\Omega A(x)\nabla u \cdot \nabla u\,dx + \int_{J_u} b(x, u^+, u^-, \nu_u)\,d\mathcal{H}^1 + \beta \int_\Omega |u - g|^2\,dx \tag{1.8}$$

under suitable assumptions on the coefficient A and b (here ν_u denotes the normal vector to J_u, $u^\pm$ are the two traces of u on the jump set, see Section 2), and considering also boundary condition of Dirichlet type on a part of the boundary. Again, provided that Ω satisfies some geometric assumptions (see Theorem 6.7), the minimizer of F within the class $H^1(\Omega)$ under boundary conditions is a local minimizer with respect to the L^1 topology in $SBV(\Omega)$.

Finally, as a numerical consequence of our result, it is worth to observe that for any iterative local descent method the regularized image given by (1.2) can not be used as initial point. In particular, the topological derivative of the Mumford-Shah functional will be non-negative at any point, hence no jump can be naturally detected in this way.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts of the space SBV used in the main proofs. In Section 3 we prove some preliminary results concerning harmonic extensions in corner domains which are pivotal to extend the monotonicity formula up to boundary points in Section 4. Section 5 is devoted to the proof of the main local minimality result, while the extension to the case with boundary conditions with the general form (1.8) is contained in Section 6.

2. Notation and preliminaries

In this section we fix the notation and recall some basic facts concerning the space SBV.

General notation. Throughout the paper $B_\rho(x)$ will denote the open ball of center $x \in \mathbb{R}^2$ and radius $\rho > 0$. We will write B_1 for the ball of center 0 and radius 1, and set $S^1 := \partial B_1$. If $E \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}^2$, $dist(x, E)$ will stand for the distance between x and E. $\mathcal{H}^1$ will denote the one dimensional Hausdorff measure, which coincides with the usual length measure on sufficiently regular curves.

If $\Omega \subseteq \mathbb{R}^2$ is open, $L^p(\Omega)$ will denote the usual Lebesgue space of measurable functions which are p-summable, while $H^1(\Omega)$ will stand for the Sobolev space of functions in $L^2(\Omega)$ with square integrable gradient.

A function $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1,1}$ if it belongs to $C^1(\mathbb{R})$ and f' is Lipschitz continuous. We say that f is piecewise $C^{1,1}$ if it is continuous and there exists a (locally finite) subdivision of $\mathbb{R}$ such that on every associated open subinterval f is of class $C^{1,1}$. An open bounded domain $\Omega \subseteq \mathbb{R}^2$ has a piecewise $C^{1,1}$-boundary if for every $x \in \partial\Omega$ there exists a neighborhood U and a piecewise $C^{1,1}$ function f on $\mathbb{R}$ such that, up to a rotation, $\Omega \cap U = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > f(x_1)\} \cap U$.

Finally, for $a.b \in \mathbb{R}$ we set

$$a \wedge b := \min\{a, b\} \qquad \text{and} \qquad a \vee b := \max\{a, b\}.$$

Special functions of bounded variation. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The space $SBV(\Omega)$ of special functions of bounded variation is given by all functions $u \in L^1(\Omega)$ such that the distributional derivative Du of u can be represented as a vector valued bounded Radon measure of the form

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Here $\mathcal{L}^N$ is the Lebesgue measure, $\nabla u \in L^1(\Omega; \mathbb{R}^N)$ is the approximate gradient of u, and J_u is the jump set of u. J_u turns out to be countably $\mathcal{H}^{N-1}$-rectifiable, *i.e.*, it is contained up to a set of $\mathcal{H}^{N-1}$-measure zero in the union of C^1-submanifolds of $\mathbb{R}^N$. It is possible to define $\mathcal{H}^{N-1}$-a.e. on J_u an approximate normal denoted by ν_u, as well as traces $u^\pm$. We refer to [2] for a detailed account of this topic.

The following result is fundamental when dealing with the analysis of the Mumford-Shah functional.

Theorem 2.1 (Ambrosio's Theorem). *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(\Omega)$ such that*

$$\|\nabla u_n\|_p + \mathcal{H}^{N-1}(J_{u_n}) + \|u_n\|_\infty \leq C,$$

where $p > 1$ and $C \geq 0$. Then, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $u \in SBV(\Omega)$ such that

$$\nabla u_{n_k} \rightharpoonup \nabla u \qquad \text{weakly in } L^p(\Omega;\mathbb{R}^N),$$
$$\mathcal{H}^{N-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{N-1}(J_{u_{n_k}}).$$

and

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Omega).$$

3. Harmonic extension results

In this section we collect some harmonic extension estimates on different kinds of domains.

We start by recalling the interior harmonic extension estimate for a disk, which was already exploited in [5] to obtain the monotonicity formula. As a straightforward extension, we generalize the estimate to the case of circular sectors. Then, through a careful deformation procedure, we handle the case of what we call "corner domains", namely the epigraphs of functions with a corner type singularity which obey suitable estimates. Finally, via a localization argument, we cover the case of admissible $C^{1,1}$ domains, which will be a crucial tool to extend the monotonicity formula up to the boundary. Below and throughout this section, we denote by ∇_τ the tangential gradient.

Let us start with the simple case of the unit disk B_1.

Proposition 3.1 (Harmonic extension estimate in a disk). *Let $w \in H^1(\partial B_1)$, and let $h_w \in H^1(B_1)$ be its harmonic extension. Then*

$$\int_{B_1} |\nabla h_w|^2\,dx \leq \int_{\partial B_1} |\nabla_\tau w|^2\,d\mathcal{H}^1, \tag{3.1}$$

Proof. Several proofs of this inequality are available in the literature (see for instance [5] for a proof in N-dimensions). For further needs, we give below a short two dimensional proof employing Fourier expansions. Let us write

$$w(\vartheta) = \sum_n \left[a_n \cos(n\vartheta) + b_n \sin(n\vartheta)\right].$$

The harmonic extension on B_1 is given in polar coordinates by

$$h_w(r,\vartheta) = \sum_n r^n \left[a_n \cos(n\vartheta) + b_n \sin(n\vartheta)\right].$$

A direct computation shows that

$$\int_{B_1} |\nabla h_w|^2\,dx = \pi \sum_n n(a_n^2 + b_n^2) \qquad \text{and} \qquad \int_{\partial B_1} |\nabla_\tau w|^2\,d\mathcal{H}^1 = \pi \sum_n n^2(a_n^2 + b_n^2),$$

so that the estimate easily follows. □

Let us generalize the estimate (3.1) to the case of circular sectors. Given $\vartheta_0 \in [0, 2\pi]$, we consider the circular unit sector and associated arc given in polar coordinates by

$$S_{\vartheta_0} := \Big\{(r,\vartheta) \,:\, r \in [0,1], \vartheta \in \Big[-\frac{\vartheta_0}{2}, \frac{\vartheta_0}{2}\Big]\Big\} \quad \text{and} \quad \Gamma_{\vartheta_0} := \Big\{(r,\vartheta) \,:\, r = 1, \vartheta \in \Big[-\frac{\vartheta_0}{2}, \frac{\vartheta_0}{2}\Big]\Big\}.$$

The following extension of Proposition 3.1 holds true.

Proposition 3.2 (Harmonic extension estimate in a sector). *Let $w \in H^1(\Gamma_{\vartheta_0})$, and let $h_w \in H^1(S_{\vartheta_0})$ denote a harmonic extension of w. Then the inequality*

$$\int_{S_{\vartheta_0}} |\nabla h_w|^2\,dx \leq \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w|^2\,d\mathcal{H}^1,$$

holds true provided that one of the following assumptions is fulfilled:

(a) *$\vartheta_0 \in]0,\pi]$ and h_w satisfies Neumann conditions or Dirichlet homogeneous conditions on $\partial S_{\vartheta_0} \setminus \Gamma_{\vartheta_0}$.*

(b) *$\vartheta_0 \in]0,\pi/2]$ and h_w satisfies homogeneous Dirichlet conditions for $\vartheta = -\vartheta_0/2$ and Neumann conditions for $\vartheta = \vartheta_0/2$.*

Above, a harmonic function $h \in H^1(S_{\vartheta_0})$ is said to satisfy Neumann conditions on a portion of the boundary $\Gamma_N \subset \partial S_{\vartheta_0}$ and Dirichlet conditions g on the complement $\Gamma_D = \partial S_{\vartheta_0} \setminus \Gamma_N$ for some function $g \in H^1(S_{\vartheta_0})$, if h is a minimizer of

$$\min \left\{ \int_{S_{\vartheta_0}} |\nabla v|^2 dx : v \in H^1(S_{\vartheta_0}), v = g \text{ on } \Gamma_D \right\}.$$

Proof. Let us consider the case of homogeneous Dirichlet conditions. We may develop w as

$$w = \sum_{j=1}^{\infty} c_j \sin \left[j \frac{\pi}{\vartheta_0} \left(\vartheta + \frac{\vartheta_0}{2} \right) \right]$$

so that the associated harmonic extension takes the form

$$h_w = \sum_{j=1}^{\infty} c_j r^{j \frac{\pi}{\vartheta_0}} \sin \left[j \frac{\pi}{\vartheta_0} \left(\vartheta + \frac{\vartheta_0}{2} \right) \right].$$

A straightforward computation for the energies of the gradients in polar coordinates shows that

$$\frac{\pi}{\vartheta_0} \int_{S_{\vartheta_0}} |\nabla h_w|^2 \, dx \leq \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w|^2 \, d\mathcal{H}^1,$$

so that the result follows.

Let us come to the case of Neumann conditions. The case of the semicircle, i.e., $\vartheta_0 = \pi$ is easily obtained through an even extension of w to ∂B_1 and employing Proposition 3.1. If $\vartheta_0 \in]0, \pi[$, we pass from w to a function $\tilde{w}$ by prolonging the constant values of the extremes. Applying the inequality on the semicircle we may write

$$\int_{S_{\vartheta_0}} |\nabla h_w|^2 \, dx \leq \int_{S_{\vartheta_0}} |\nabla h_{\tilde{w}}|^2 \, dx \leq \int_{S_\pi} |\nabla h_{\tilde{w}}|^2 \, dx \leq \int_{\Gamma_\pi} |\nabla_\tau \tilde{w}|^2 \, d\mathcal{H}^1 = \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w|^2 \, d\mathcal{H}^1,$$

so that the result follows, and point (a) is proved.

Let us come finally to point (b) which follows by combing the idea used above. If $\vartheta_0 = \pi/2$, we can perform an even extension to a semicircle and employ the harmonic estimate with Dirichlet conditions of point (a), and then restrict to $\Gamma_{\pi/2}$ to get the result. If $\vartheta_0 \in]0, \pi/2[$, we can proceed by extension to $\Gamma_{\pi/2}$ and using the associated inequality. □

We are now going to consider the case of *corner domains*. By corner domain, we mean the epigraph (see Figure 1)

$$\Omega_f := \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 > f(x_2)\}. \tag{3.2}$$

of a given function f of the form

$$f(x) := \begin{cases} f_1(x) & \text{if } x < 0 \\ f_2(x) & \text{if } x \geq 0 \, , \end{cases} \tag{3.3}$$

with $f_1 \in C^1(]-\infty, 0])$ and $f_2 \in C^1([0, +\infty[)$ nonnegative functions satisfying

$$f_1(0) = f_2(0) = 0 \, , \qquad f_2'(0) = -f_1'(0) \, . \tag{3.4}$$

We define

$$\vartheta_0(\Omega_f) := \text{angle of } \Omega_f \text{ at the origin} \, , \tag{3.5}$$

meant as the angle at the origin of the sector $\{x_1 \geq f_1'(0)x_2\} \cap \{x_1 \geq f_2'(0)x_2\}$. In particular, $\theta_0(\Omega_f) = \pi$ in case $\partial\Omega_f$ is smooth ($f_1'(0) = f_2'(0) = 0$). We also set

$$\partial_1\Omega_f := \{(f_1(x), x) \, : \, x < 0\} \cap \bar{B}_1(0) \qquad \text{and} \qquad \partial_2\Omega_f := \{(f_2(x), x) \, : \, x \geq 0\} \cap \bar{B}_1(0) \, . \tag{3.6}$$

The following result holds true.

Proposition 3.3 (**Harmonic extension estimate in corner domains**). *For $\rho > 0$, let Ω_{f_ρ} be a family of corner domains as in* (3.2)*, where each function f_ρ is of the kind* (3.3)-(3.4)*. Assume further that:*

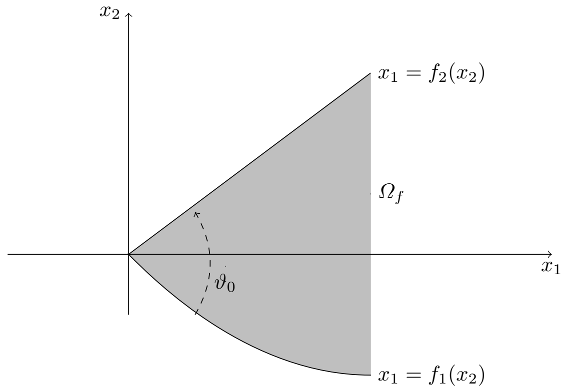

FIGURE 1. The domain Ω_f.

– *the left and right derivatives of f_ρ at the origin are independent of ρ, i.e.*

$$f'_{\rho,2}(0) = -f'_{\rho,1}(0) =: \lambda \geq 0\,, \tag{3.7}$$

– *$f'_{\rho,1}$, $f'_{\rho,2}$ satisfy the following Lipschitz type estimates for some constant $c > 0$:*

$$\forall x \in [-1,0]\,:\, |f'_{\rho,1}(x) + \lambda| \leq c\rho|x| \qquad \text{and} \qquad \forall x \in [0,1]\,:\, |f'_{\rho,2}(x) - \lambda| \leq c\rho x\,. \tag{3.8}$$

Then there exist $\rho_0 > 0$ and $c_0 > 0$ such that, for every $\rho \in (0, \rho_0]$ and $w \in H^1(\Omega_{f_\rho} \cap \partial B_1(0))$, denoting by h_w a harmonic extension of w to $\Omega_{f_\rho} \cap B_1(0)$, it holds

$$(1 - c_0\rho) \int_{\Omega_{f_\rho} \cap B_1(0)} |\nabla h_w|^2\, dx \leq \int_{\Omega_{f_\rho} \cap \partial B_1(0)} |\nabla_\tau w|^2\, d\mathcal{H}^1, \tag{3.9}$$

provided that $\theta_0 := \vartheta_0(\Omega_{f_\rho})$ (the inner angle at the origin of f_ρ) and h_w fulfill one of the following assumptions:

(a) *$\vartheta_0 \in]0, \pi]$ and h_w satisfies Neumann conditions or Dirichlet homogeneous conditions on $\partial_1\Omega_{f_\rho} \cup \partial_2\Omega_{f_\rho}$.*
(b) *$\vartheta_0 \in]0, \pi/2]$ and h_w satisfies homogeneous Dirichlet conditions on $\partial_1\Omega_{f_\rho}$ and Neumann conditions on $\partial_2\Omega_{f_\rho}$.*

In order to prove the previous Proposition, we need some preliminary work. Let us consider the planar domain given in polar coordinates (r, ϑ) by

$$C_{\theta_0,g} := \left\{(r,\vartheta) \;:\; r \in [0,1],\ \vartheta \in \left[0, \frac{\vartheta_0}{2} + g(r)\right]\right\}\,, \tag{3.10}$$

where

$$\theta_0 \in]0,\pi] \qquad \text{and} \qquad g \in C^1([0,1])\,,\ g(0) = 0\,,\ g(r) \in \left[-\frac{\vartheta_0}{2}, -\frac{\vartheta_0}{2} + \frac{\pi}{2}\right]. \tag{3.11}$$

The domain $C_{\theta_0,g}$ can be mapped to the unit sector

$$S^+_{\vartheta_0} := \left\{(r,\vartheta) \;:\; r \in [0,1],\ \vartheta \in \left[0, \frac{\vartheta_0}{2}\right]\right\}$$

by means of the transformation defined by

$$T : C_{\theta_0,g} \to S^+_{\vartheta_0}\,, \qquad T(r,\vartheta) := \left(r, \frac{\vartheta_0}{2} \frac{\vartheta}{\frac{\vartheta_0}{2} + g(r)}\right)\,, \tag{3.12}$$

with inverse

$$T^{-1}(r',\vartheta') := \left(r', \frac{2}{\vartheta_0}\left(\frac{\vartheta_0}{2} + g(r')\right)\vartheta'\right)\,.$$

Notice that T maps $C_{\theta_0,g} \cap \partial B_1(0)$ onto $\Gamma^+_{\theta_0} := \{(r,\vartheta) \,:\, r=1, \vartheta \in [0,\vartheta_0/2]\}$. Accordingly, with a given function v defined on $C_{\theta_0,g}$ (resp. on $C_{\theta_0,g} \cap \partial B_1(0)$), we can associate the function $v^\sharp$ defined on $S^+_{\vartheta_0}$ (resp. $\Gamma^+_{\theta_0}$) by

$$v^\sharp := v \circ T^{-1}\,. \tag{3.13}$$

Lemma 3.4. *For $\rho > 0$, let C_{θ_0,g_ρ} be a family of domains as in* (3.10)*, with θ_0 (independent of ρ) and g_ρ as in* (3.11)*. Assume further that g'_ρ satisfy the following L^∞ estimate for some constant $c > 0$:*

$$|g'_\rho(r)| \le c\rho \qquad \text{for every } r \in [0,1]\,. \tag{3.14}$$

Let $T : C_{\theta_0,g_\rho} \to S^+_{\vartheta_0}$ be defined as in (3.12)*. There exist $\rho_0 > 0$ and $c_0 > 0$ such that, for every $\rho \in (0,\rho_0]$ the following items hold true.*

(a) *If $v^\sharp \in H^1(S^+_{\vartheta_0})$ is associated with $v \in H^1(C_{\theta_0,g_\rho})$ as in* (3.13)*, it holds*
$$\int_{S^+_{\vartheta_0}} |\nabla v^\sharp|^2\,dx' \ge (1-c_0\rho)\int_{C_{\theta_0,g_\rho}} |\nabla v|^2\,dx.$$

(b) *If $w^\sharp \in H^1(\Gamma^+_{\theta_0})$ is associated with $w \in H^1(C_{\theta_0,g} \cap \partial B_1(0))$ as in* (3.13)*, it holds*
$$\int_{\Gamma^+_{\vartheta_0}} |\nabla_\tau w^\sharp|^2\,d\mathcal{H}^1 \le (1+c_0\rho)\int_{C_{\theta_0,g_\rho}\cap\partial B_1(0)} |\nabla_\tau w|^2\,d\mathcal{H}^1.$$

Proof. Let us prove item (a). We write for simplicity g in place of g_ρ. By (3.13), we have
$$v(r,\vartheta) = v^\sharp(T(r,\vartheta)) = v^\sharp\left(r, \frac{\vartheta_0}{2}\frac{\vartheta}{\frac{\vartheta_0}{2}+g(r)}\right).$$
Computing the partial derivatives of the function v, we obtain
$$\partial_r v(r,\vartheta) = \partial_{r'}v^\sharp(T(r,\vartheta)) + \partial_{\vartheta'}v^\sharp(T(r,\vartheta))\frac{\vartheta_0}{2}\frac{-\vartheta g'(r)}{(\frac{\vartheta_0}{2}+g(r))^2}$$
and
$$\partial_\vartheta v(r,\vartheta) = \partial_{\vartheta'}v^\sharp(T(r,\vartheta))\frac{\vartheta_0}{2}\frac{1}{\frac{\vartheta_0}{2}+g(r)},$$
so that
$$\begin{cases} \partial_{r'}v^\sharp(T(r,\vartheta)) = \partial_r v(r,\vartheta) + \dfrac{\vartheta g'(r)}{\frac{\vartheta_0}{2}+g(r)}\partial_\vartheta v(r,\vartheta) \\[2ex] \partial_{\vartheta'}v^\sharp(T(r,\vartheta)) = \frac{2}{\vartheta_0}\left(\frac{\vartheta_0}{2}+g(r)\right)\partial_\vartheta v(r,\vartheta). \end{cases} \tag{3.15}$$
Since the jacobian of T is given by
$$J_T(r,\vartheta) = \frac{\vartheta_0}{2}\frac{1}{\frac{\vartheta_0}{2}+g(r)},$$
the change of variable formula together with (3.15) yields
$$\begin{aligned} &\int_{S^+_{\vartheta_0}} \left[(\partial_{r'}v^\sharp)^2 + \left(\frac{\partial_{\vartheta'}v^\sharp}{r'}\right)^2\right] r'dr'd\vartheta' \\ &\qquad = \int_{C_{\theta_0,g}} \left[(\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2 + e(\rho)\right]\frac{\vartheta_0}{2}\frac{1}{\frac{\vartheta_0}{2}+g(r)} r dr d\vartheta, \end{aligned} \tag{3.16}$$
where
$$e(\rho) := 2\frac{\vartheta g'(r)}{\frac{\vartheta_0}{2}+g(r)}\partial_r v \partial_\vartheta v + \left(\frac{\vartheta g'(r)}{\frac{\vartheta_0}{2}+g(r)}\right)^2(\partial_\vartheta v)^2 + \left[\left(\frac{2}{\vartheta_0}\left(\frac{\vartheta_0}{2}+g(r)\right)\right)^2 - 1\right]\left(\frac{\partial_\vartheta v}{r}\right)^2. \tag{3.17}$$

It is easy to check that there exist $\rho_0, \tilde c > 0$ (depending only on the constant c in (3.14) and on ϑ_0), such that for every $\rho \in (0, \rho_0]$

$$-\tilde c\rho\Big[(\partial_r v)^2 + \Big(\frac{\partial_\vartheta v}{r}\Big)^2\Big] \le e(\rho) \le \tilde c\rho\Big[(\partial_r v)^2 + \Big(\frac{\partial_\vartheta v}{r}\Big)^2\Big]. \tag{3.18}$$

Indeed, let us show for instance how the first term on the right hand side of (3.17) can be handled (the other terms being similar). Notice first that the assumption (3.14) entails also also $|g(r)| \le c\rho$ for every $r \in [0,1]$. Then, taking ρ sufficiently small we obtain, for a suitable constant $\tilde c > 0$,

$$\begin{aligned}\left|2\frac{\vartheta g'(r)}{\frac{\vartheta_0}{2}+g(r)}\partial_r v\partial_\vartheta v\right| &\le \left|\frac{\vartheta g'(r)}{\frac{\vartheta_0}{2}+g(r)}\right|\left[r^2(\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2\right] \\ &\le \frac{(\frac{\vartheta_0}{2}+c\rho)c\rho}{\frac{\vartheta_0}{2}-c\rho}\left[(\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2\right] \le \tilde c\rho\left((\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2\right).\end{aligned}$$

In view of (3.18), coming back to (3.16) we get

$$\begin{aligned}\int_{S^+_{\vartheta_0}}\left[(\partial_{r'}v^\sharp)^2 + \left(\frac{\partial_{\vartheta'}v^\sharp}{r'}\right)^2\right]r'dr'd\vartheta' &\ge (1-\tilde c\rho)\int_{C_{\theta_0}}\left[(\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2\right]\frac{\vartheta_0}{2}\frac{1}{\frac{\vartheta_0}{2}+g(r)}rdrd\vartheta \\ &\ge \frac{1-\tilde c\rho}{1-\frac{2}{\vartheta_0}c\rho}\int_{C_{\theta_0,g}}\left[(\partial_r v)^2 + \left(\frac{\partial_\vartheta v}{r}\right)^2\right]rdrd\vartheta,\end{aligned}$$

from which part (a) of the statement follows.

The proof of part (b) is analogous, by using the lower bound inequality in (3.18) in place of the upper bound one. □

We are now in a position to prove Proposition 3.3.

Proof of Proposition 3.3. Throughout the proof we write for simplicity f in place of f_ρ and g in place of g_ρ. We claim that

$$\Omega_f^+ := \Omega_f \cap B_1(0) \cap \{x_2 > 0\}$$

corresponds to the domain $C_{\theta_0,g}$ defined in (3.10), with θ_0 and g satisfying (3.11) and (3.14), if we take

$$\theta_0 = \theta_0(\Omega_f)\,, \qquad g(r) := \arccos\left(\frac{f_2(x_2)}{\sqrt{x_2^2 + f_2(x_2)^2}}\right) - \frac{\theta_0}{2}, \tag{3.19}$$

where $\theta_0(\Omega_f)$ is the angle of Ω_f at the origin defined according to (3.5), and $x_2 = x_2(r)$ is obtained by inverting the relation

$$r = \sqrt{x_2^2 + f_2(x_2)^2}. \tag{3.20}$$

To prove the claim notice firstly that $x_2(r)$ is well-defined because (3.20) defines a bijection if ρ is sufficiently small: if $f_2'(0) \neq 0$, this follows immediately from (3.8); if $f_2'(0) = 0$, setting $h(x_2) := x_2^2 + f_2(x_2)^2$ and using again (3.8), we get

$$h'(x_2) = 2x_2 + 2f_2(x_1)f_2'(x_2) \ge 2x_2(1 - c_2^2\rho^2) > 0 \qquad \text{for } x_2 > 0 \text{ and } \rho \ll 1.$$

Next observe that, since by assumption f_2 is nonnegative, the domain Ω_f^+ can be written in polar coordinates (r, θ) as

$$0 \le r \le 1\,, \qquad 0 \le \theta \le \varphi(r) := \arccos\left(\frac{f_2(x_2)}{\sqrt{x_2^2 + f_2(x_2)^2}}\right), \tag{3.21}$$

(with $x_2 = x_2(r)$ as above). In view of (3.21), and noticing that the function φ satisfies

$$\varphi(0^+) = \arccos\left(\frac{f_2'(0)}{\sqrt{1 + f_2'(0)^2}}\right) = \frac{\theta_0(\Omega_f)}{2}\,,$$

we see that Ω_f^+ corresponds to $C_{\theta_0,g}$ with θ_0 and g as in (3.19). To achieve the proof of the claim, it remains to check that θ_0 and g satisfy (3.11) and (3.14). Since θ_0 is the angle of Ω_f at the origin, recalling that f_1 and f_2 are nonnegative by assumption, it is clear that $\theta_0 \in [0,\pi[$. It is also immediate from the above definition of g that $g \in C^1([0,1])$ and that $g(0) = \varphi(0) - \theta_0/2 = 0$. Moreover, since the function φ in (3.21) takes values into $[0,\pi/2]$, g satisfies the bounds in (3.11). Finally, taking into account (3.8), a straightforward computation shows that there exists $c > 0$ such that (3.14) is fulfilled.

A similar change of variable can be operated on the set $\Omega_f^- := \Omega_f \cap B_1(0) \cap \{x_2 < 0\}$. We conclude that we can map $\Omega_f \cap B_1(0)$ onto the sector S_{ϑ_0}, with an estimate on the L^2-norm of the gradients according to Lemma 3.4.

We are now ready to prove the estimate (3.9). Let us consider firstly the case of Neumann conditions. Given $w \in H^1(\Omega_f \cap \partial B_1(0))$, let $w^\sharp \in H^1(\Gamma_{\theta_0})$ be associated with w according to (3.13). If $h_{w^\sharp}$ is the harmonic extension of $w^\sharp$ to S_{ϑ_0} with Neumann conditions on $\partial S_{\vartheta_0} \setminus \Gamma_{\vartheta_0}$, thanks to Proposition 3.2 (a) we have

$$\int_{S_{\vartheta_0}} |\nabla h_{w^\sharp}|^2\,dx \le \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w^\sharp|^2\,d\mathcal{H}^1. \tag{3.22}$$

Coming back to the original domain, if we consider the function $h \in H^1(\Omega_f \cap B_1(0))$ defined by $h := h_{w^\sharp} \circ T$ (so that $h^\sharp = h_{w^\sharp}$), by using in the order Lemma 3.4 (a), (3.22), and Lemma 3.4 (b), we infer

$$\begin{aligned}(1-c_0\rho)\int_{\Omega_f\cap B_1(0)} |\nabla h|^2\,dx &\le \int_{S_{\vartheta_0}} |\nabla h_{w^\sharp}|^2\,dx\\ &\le \int_{\Gamma_{\vartheta_0}} |\nabla_\tau w^\sharp|^2\,d\mathcal{H}^1\\ &\le (1+c_0\rho)\int_{\Omega_f\cap\partial B_1(0)} |\nabla_\tau w|^2\,d\mathcal{H}^1\,.\end{aligned}$$

Since the harmonic extension h_w of w which satisfies Neumann conditions on $\partial_1\Omega_f \cup \partial_2\Omega_f$ verifies the inequality

$$\int_{\Omega_f\cap B_1(0)} |\nabla h_w|^2\,dx \le \int_{\Omega_f\cap B_1(0)} |\nabla h|^2\,dx,$$

the result follows.

The case of Dirichlet homogeneous conditions or mixed Dirichlet/Neumann conditions can be handled in a similar way, by employing the corresponding inequalities on sectors established in Proposition 3.2.

□

In order to handle more general geometries and also boundary conditions, the following definition will be useful.

Definition 3.5 (Admissible domains). *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a piecewise $C^{1,1}$-boundary. Let S denote the set of corners of $\partial\Omega$, and let $\gamma_1,\dots,\gamma_k$ be the open arcs (connected components) of $\partial\Omega \setminus S$. We will say that Ω is admissible if $\partial\Omega$ can be decomposed as*

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup S, \tag{3.23}$$

where $\Gamma_D \cup \Gamma_N$ is a partition of $\{\gamma_1,\dots,\gamma_k\}$ and satisfies the following conditions

- (i) *the angle formed (on the side of Ω) by any pair of consecutive arcs in Γ_D or in Γ_N is less than or equal to π;*
- (ii) *the angle formed (on the side of Ω) by any arc of Γ_D adjacent to an arc of Γ_N is less than or equal to $\pi/2$.*

Remark 3.6. In our further considerations Γ_D and Γ_N stand for the arcs with Dirichlet and Neumann boundary conditions respectively For $\Gamma_D = \partial\Omega$ or $\Gamma_N = \partial\Omega$, admissibility according to the above definition reduces to the assumption that Ω has convex corners. In particular, smooth $C^{1,1}$ domains are admissible.

Proposition 3.7 (Harmonic extension estimate in admissible domains). *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a piecewise $C^{1,1}$-boundary, which is admissible according to Definition 3.5. Let $\partial\Omega$ be decomposed as in (3.23). There exist $\rho_\Omega > 0$ and $c_\Omega > 0$ such that, for every $x \in \partial\Omega$ and every $\rho < \rho_\Omega \wedge dist(x, S \setminus \{x\})$, if $w \in H^1(\Omega \cap \partial B_\rho(x))$ and h_w denotes a harmonic extension of w to $\Omega \cap B_\rho(x)$, it holds*

$$\frac{(1 - c_\Omega \rho)}{\rho} \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2 \, dx \leq \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2 \, d\mathcal{H}^1 \,, \tag{3.24}$$

provided one of the following assumptions is fulfilled:

(a) *h_w satisfies Neumann conditions or homogeneous Dirichlet conditions on $\partial\Omega \cap B_\rho(x)$;*

(b) *h_w satisfies homogeneous Dirichlet conditions on $\Gamma_D \cap B_\rho(x)$ and Neumann conditions on $\Gamma_N \cap B_\rho(x)$.*

Proof. Let us first consider the case when h_w satisfies Neumann conditions on $\partial\Omega \cap B_\rho(x)$. Since Ω is compact and with a piecewise $C^{1,1}$-boundary, there exists $\rho_\Omega > 0$ such that, for every $x \in \partial\Omega$ and $\rho < \rho_\Omega$, we have that $\Omega \cap B_\rho(x)$ is given in a suitable coordinate system with center in x by the intersection with $B_\rho(x)$ of the epigraph of a piecewise $C^{1,1}$-function $f_x : \mathbb{R} \to \mathbb{R}$. In view of the compactness of $\partial\Omega$, it is not restrictive to assume

$$\sup_{x \in \partial\Omega} \|f_x\|_{C^{1,1}_{pw}} = C < +\infty. \tag{3.25}$$

Let $x \in \partial\Omega$ be a smooth point, and let

$$\rho < \rho_\Omega \wedge dist(x, S).$$

Up to a translation and a rotation, we may assume that $x = 0$ and that $\Omega \cap B_\rho(x)$ is given by

$$\Omega_f \cap B_\rho(0)$$

where

$$\Omega_f := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 > f(x_2)\}$$

for a suitable $f \in C^{1,1}(\mathbb{R})$ with $f(0) = f'(0) = 0$ and $\|f\|_{C^{1,1}} \leq C$. If we rescale to unit size, i.e., we consider the map

$$x \mapsto x/\rho$$

then $\Omega_f \cap B_\rho(0)$ is transformed into the set $\Omega_{f_\rho} \cap B_1(0)$ where

$$f_\rho(s) := \frac{1}{\rho} f(\rho s).$$

Since $f'_\rho(s) = f'(\rho s)$ and $f(0) = f'(0) = 0$, in view of (3.25) we deduce that for every $s \in [-1, 1]$

$$|f'_\rho(s)| \leq c\rho|s|.$$

Hence, the family of functions f_ρ satisfy conditions (3.7) and (3.8). Therefore, up to reducing ρ_Ω, we can apply Proposition 3.3 (under assumption (a), case of Neumann conditions) to the function

$$\tilde{w}(y) := w(\rho y) \in H^1(\Omega_{f_\rho} \cap \partial B_1(0)) \,.$$

Rescaling back to size ρ, we obtain easily the result.

If x is a convex corner for $\partial\Omega$, the proof is similar, using again Proposition 3.3 (under assumption (a), case of Neumann conditions) .

The cases when h_w satisfies homogeneous Dirichlet conditions on $\partial\Omega \cap B_\rho(x)$, or homogeneous Dirichlet conditions on $\Gamma_D \cap B_\rho(x)$ and Neumann conditions on $\Gamma_N \cap B_\rho(x)$, can be settled in the analogous way, by using the parts of Proposition 3.3 in which the corresponding boundary conditions are considered. □

Remark 3.8. An inspection in the proof of Proposition 3.7 shows that the constants c_Ω, ρ_Ω remain bounded if Ω is replaced by the domain $L(\Omega)$ where $L : \mathbb{R}^2 \to \mathbb{R}^2$ varies in a family of linear transformations with bounded norm. This observation will be useful in Section 6.

4. The monotonicity formula up to the boundary

In this section we prove a monotonicity formula up to boundary points for a suitable notion of quasi-minimizers of the Mumford-Shah functional in dimension two. We point out that this notion is much weaker than the classical one employed in [5], since the family of test functions is much smaller. The results of this section are typically two dimensional and can not, a priori, be extended in N dimensions.

Let us start with the following definition.

Definition 4.1. (Weak minimizers) *Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and $u \in SBV(\Omega)$. We say that u is a weak local almost quasi-minimizer of the Mumford-Shah energy with respect to its own jump set at the point $x \in \overline{\Omega}$ if there exist $\rho_x > 0$, $\gamma > 0$, $c_\gamma > 0$ and $\Lambda \geq 1$ such that for every $\rho < \rho_x$ with*

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \rho \tag{4.1}$$

and for every $v \in SBV(\Omega)$ with $\{v \neq u\} \subseteq B_\rho(x)$ and $J_v \subseteq J_u$, we have

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 \, dx + \Lambda \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma}. \tag{4.2}$$

Remark 4.2. Notice that local almost quasi minimizers of the Mumford-Shah functional considered in [5] (in particular absolute minimizers) are weak local almost quasi minimizers with respect to their own jump set: indeed they satisfy the minimality property (4.2) on *every* ball $B_\rho(x)$ (with ρ sufficiently small) not necessarily satisfying the energy bound (4.1), and for every competitor v such that $\{v \neq u\} \subseteq B_\rho(x)$, without the restriction $J_v \subseteq J_u$.

With a given weak local almost quasi-minimizer u according to the previous definition, we associate for every $x \in \overline{\Omega}$ and $\rho > 0$ the quantity

$$\mathcal{E}_x(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)). \tag{4.3}$$

Our monotonicity formula at interior points reads as follows.

Theorem 4.3 (Interior monotonicity). *Let $\Omega \subset \mathbb{R}^2$ be open, and let $u \in SBV(\Omega)$ be a weak local almost quasi-minimizer for the Mumford-Shah energy with respect to its own jump set at the point $x \in \Omega$ according to Definition 4.1. Let $\mathcal{E}_x(\rho)$ be associated with u as in* (4.3).

Then the quantity

$$E_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma} \rho^\gamma \tag{4.4}$$

is non decreasing on $(0, \rho_x \wedge dist(x, \partial\Omega))$.

This formula appears to have a similar expression as the one in [5], but applies to a much weaker notion of minimizer. In particular there is no natural upper bound for $\mathcal{E}_x(\rho)/\rho$ as in [5] (such bound is usually obtained by using as a test for minimality the function $u 1_{\Omega \setminus B_\rho(x)}$, which is not a priori an admissible competitor for *weak* local almost quasi minimizers with respect to their own jump set).

Coming to boundary points, we are going to state a monotonicity formula for domains which are admissible according to Definition 3.5. First, let us consider the case of $C^{1,1}$-domains with convex corners (cf. Remark 3.6), with no imposition of Dirichlet boundary condition. In the case of a flat boundary, such a monotonicity formula has been obtained in [7], in which case $k_\Omega = 0$.

Theorem 4.4 (Boundary monotonicity). *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, piecewise $C^{1,1}$-domain with convex corners, and let $u \in SBV(\Omega)$ be a weak local almost quasi-minimizer for the Mumford-Shah energy with respect to its own jump set at the point $x \in \partial\Omega$ according to Definition 4.1. Let $\mathcal{E}_x(\rho)$ be associated with u as in* (4.3).

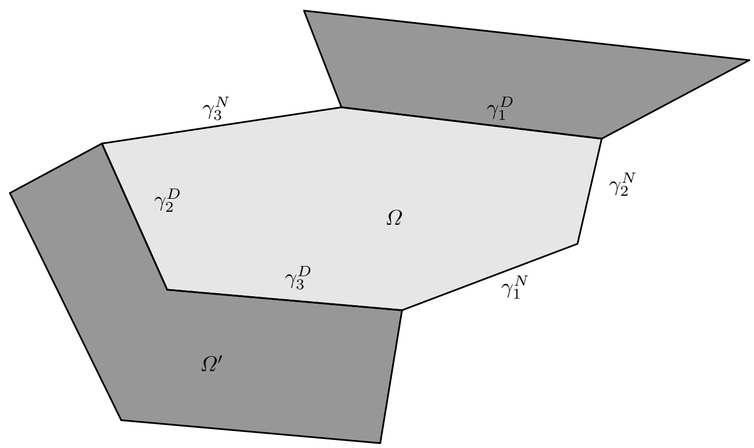

FIGURE 2. The domain Ω'.

Then there exist $r_\Omega > 0$ *and* $k_\Omega \geq 0$ *such that the quantity*

$$\widetilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma}\rho^\gamma + k_\Omega \int_0^\rho \left(\frac{\mathcal{E}_x(r)}{r} \wedge 1 \right) dr \tag{4.5}$$

is non decreasing on $(0, \rho_x \wedge r_\Omega \wedge dist(x, S \setminus \{x\}))$, *where* S *denotes the set of corners of* $\partial\Omega$.

To formulate a monotonicity result which takes into account also boundary conditions, we consider the case when Ω is an admissible domain according to Definition 3.5, and homogeneous Dirichlet boundary conditions are imposed on the (nonempty) portion Γ_D of its boundary. To this aim, let us consider an open bounded domain Ω' such that $\Omega \subset\subset \Omega' \subset \mathbb{R}^2$ and with (see Figure 2).

$$\partial\Omega \cap \Omega' = \Gamma_D \tag{4.6}$$

Setting

$$\mathcal{A}_0 := \{u \in SBV(\Omega') \,:\, u = 0 \text{ on } \Omega' \setminus \overline{\Omega}\}. \tag{4.7}$$

we adapt the notion of weak local almost quasi-minimizers as follows.

Definition 4.5. (Weak minimizers in $\mathcal{A}_0$) *We say that* $u \in \mathcal{A}_0$ *is a weak local almost quasi-minimizer in* $\mathcal{A}_0$ *of the Mumford-Shah energy with respect to its own jump set at the point* $x \in \overline{\Omega}$ *if there exist* $\rho_x > 0$, $\gamma > 0$, $c_\gamma > 0$ *and* $\Lambda \geq 1$ *such that for every* $\rho < \rho_x$ *with*

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \rho$$

and for every $v \in \mathcal{A}_0$ *with* $\{v \neq u\} \subseteq B_\rho(x)$ *and* $J_v \subseteq J_u$, *we have*

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}^1(J_u \cap \bar{B}_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 \, dx + \Lambda \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\gamma \rho^{1+\gamma}.$$

The following variant of Theorem 4.4 holds true.

Theorem 4.6 (Boundary monotonicity in $\mathcal{A}_0$). *Let* $\Omega \subset \mathbb{R}^2$ *be an admissible domain according to Definition 3.5, and let* $u \in \mathcal{A}_0$ *be a weak local almost quasi-minimizer in* $\mathcal{A}_0$ *for the Mumford-Shah energy with respect to its own jump set at the point* $x \in \partial\Omega$ *according to Definition 4.5. Let* $\mathcal{E}_x(\rho)$ *be associated with* u *as in* (4.3).

Then there exist $r_\Omega > 0$ and $k_\Omega > 0$ such that the quantity

$$\widetilde{E}_x(\rho) := \frac{\mathcal{E}_x(\rho)}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma}\rho^\gamma + k_\Omega \int_0^\rho \left(\frac{\mathcal{E}_x(r)}{r} \wedge 1 \right) dr \tag{4.8}$$

is non decreasing on $(0, \rho_x \wedge r_\Omega \wedge dist(x, S \setminus \{x\}))$, where S denotes the set of corners of $\partial\Omega$.

In order to establish these results, we need to revisit the proof of the monotonicity formula given in [5]. The new ingredients here are the jump constraint (for which the two-dimensional setting is crucial) and the fact that the point x can belong to the boundary. We start with the boundary case, which contains the relevant modifications with respect to the case treated in [5], and then we go back to the (simpler) interior case.

Proof of Theorems 4.4 and 4.6. Let us consider firstly the case of Theorem 4.4. Following [5], to prove the result it is enough to show that there exist $r_\Omega > 0$ and $k_\Omega > 0$ such that, for $\rho \in (0, \rho_x \wedge r_\Omega \wedge dist(x, S \setminus \{x\}))$, it holds $\widetilde{E}'_x(\rho) \geq 0$ at almost every differentiability point ρ of $\widetilde{E}_x$ such that $\mathcal{E}_x(\rho) < \rho$. Let ρ_Ω and c_Ω be as in Proposition 3.7, and let us choose $k_\Omega = 2c_\Omega$.

We argue by contradiction. Assume that

$$\widetilde{E}'_x(\rho) = \frac{\mathcal{E}'_x(\rho)}{\rho} - \frac{\mathcal{E}_x(\rho)}{\rho^2} + c_\gamma \rho^{\gamma-1} + k_\Omega \frac{\mathcal{E}_x(\rho)}{\rho} < 0,$$

so that

$$\rho\, \mathcal{E}'_x(\rho) + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4.9}$$

In particular, we infer $\mathcal{E}'_x(\rho) < 1$ so that from

$$\int_{\Omega \cap \partial B_\rho(x)} |\nabla u|^2\, dx + \mathcal{H}^0(J_u \cap \partial B_\rho(x)) \leq \mathcal{E}'_x(\rho) < 1$$

we get $J_u \cap \partial B_\rho(x) = \emptyset$. This means that the restriction of the SBV function u on $\partial B_\rho(x) \cap \Omega$ is a Sobolev function which we denote by w. Notice that (4.9) entails

$$\rho \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2\, d\mathcal{H}^1 + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4.10}$$

Let h_w denote a harmonic extension h_w of w to $\Omega \cap B_\rho(x)$ which satisfies Neumann conditions on $\partial\Omega \cap B_\rho(x)$. By Proposition 3.7, we have

$$\frac{(1 - c_\Omega \rho)}{\rho} \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2\, dx \leq \int_{\Omega \cap \partial B_\rho(x)} |\nabla_\tau w|^2\, d\mathcal{H}^1.$$

From (4.10) we deduce that

$$(1 - c_\Omega \rho) \int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2\, dx + c_\gamma \rho^{\gamma+1} + k_\Omega \rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho).$$

Up to replacing ρ_Ω by a smaller r_Ω if necessary, we infer in particular

$$\int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2\, dx \leq 2\mathcal{E}_x(\rho)$$

so that for every $\rho < r_\Omega \wedge dist(x, S \setminus \{x\})$

$$\int_{\Omega \cap B_\rho(x)} |\nabla h_w|^2\, dx + c_\gamma \rho^{\gamma+1} + (k_\Omega - 2c_\Omega)\rho \mathcal{E}_x(\rho) < \mathcal{E}_x(\rho). \tag{4.11}$$

We now consider the admissible competitor for u given by

$$v(y) := \begin{cases} h_w(y) & \text{if } y \in \Omega \cap B_\rho(x) \\ u(y) & \text{otherwise.} \end{cases} \tag{4.12}$$

Notice that $J_v \subseteq J_u$, since v has no jumps inside $\Omega \cap B_\rho(x)$ and coincides with u on $\Omega \cap \partial B_\rho(x)$. Then, recalling that we have chosen $k_\Omega = 2c_\Omega$, inequality (4.11) for $\rho < \rho_x \wedge r_\Omega \wedge dist(x, S \setminus \{x\})$ contradicts the weak local almost quasi-minimality property of u according to Definitions 4.1 (recall that $\mathcal{E}_x(\rho) < \rho$).

Coming to Theorem 4.6, we can follow the previous arguments by considering the harmonic extension h_w which satisfy homogeneous Dirichlet conditions on $\Gamma_D \cap B_\rho(x)$ and Neumann conditions on $\Gamma_N \cap B_\rho(x)$, and using again Proposition 3.7. The contradiction then follows by noting that the competitor v in (4.12) belongs to $\mathcal{A}_0$. □

Remark 4.7. Notice that the uniformity property of Remark 3.8 holds also for the constants r_Ω and k_Ω.

Proof of Theorem 4.3. The proof reduces essentially to the original case of [5], by noting that, since we work in dimension two, the key competitors involved in the arguments turn out to have a jump set contained in J_u. More precisely, we can follow the proof of Theorem 4.4, by using the estimate of Proposition 3.1 in place of that of Proposition 3.7: in this way we can choose $k_\Omega = 0$ and obtain the monotonicity for the energy in the simpler form (4.4). □

5. The local minimality result

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set, and let

$$MS(u) := \int_\Omega |\nabla u|^2\,dx + \alpha \mathcal{H}^1(J_u) + \beta \int_\Omega |u-g|^2\,dx$$

denote the Mumford-Shah functional on $SBV(\Omega)$, where $g \in L^\infty(\Omega)$, $\alpha, \beta > 0$.

The following result holds true.

Theorem 5.1 (**Small jump sets are not convenient).** *Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, piecewise $C^{1,1}$-domain with convex corners and let $g \in L^\infty(\Omega)$. Let $U \in H^1(\Omega)$ be the solution to*

$$\min_{v \in H^1(\Omega)} \int_\Omega |\nabla v|^2\,dx + \beta \int_\Omega |v-g|^2\,dx,$$

i.e., such that

$$\begin{cases} -\Delta U + \beta U = \beta g & \text{in } \Omega \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Then there exists $\varepsilon > 0$ such that for every $u \in SBV(\Omega)$ with $\mathcal{H}^1(J_u) < \varepsilon$ we have

$$MS(U) < MS(u).$$

The main result of the paper, already stated in the Introduction as Theorem 1.1, is a simple consequence of the previous theorem. For convenience, we restate it hereafter:

Theorem 5.2 (**Local minimality in L^1).** *Under the assumptions of Theorem 5.1, the function U is a local minimizer for the Mumford-Shah functional in $SBV(\Omega)$ with respect to the L^1-topology.*

Proof. Assume by contradiction that there exists $v_n \in SBV(\Omega)$ such that

$$v_n \to U \qquad \text{strongly in } L^1(\Omega)$$

with

$$MS(v_n) < MS(U). \tag{5.2}$$

By truncation we may assume that $\|v_n\|_\infty \le \|g\|_\infty$, so that the convergence holds also in $L^2(\Omega)$. By (5.2), we may apply Ambrosio's theorem (see Theorem 2.1) and deduce

$$\int_\Omega |\nabla U|^2\,dx \le \liminf_n \int_\Omega |\nabla v_n|^2\,dx,$$

so that, since the fidelity terms are converging, we infer $\lim_n \mathcal{H}^1(J_{v_n}) = 0$: but then (5.2) is in contradiction with Theorem 5.1. □

Remark 5.3. Under suitable regularity assumptions on g, it has been proved in [1, Sections 5.1 and 5.3] by using the *calibration method*, that U is a *global minimizer* for MS if β sufficiently small or if β sufficiently large.

The proof of Theorem 5.1 rests on a suitable use of the monotonicity formulas up to the boundary developed in Section 4.

Proof of Theorem 5.1. First of all, by considering the change of variable $x \mapsto \sqrt{\alpha}x$, it is not restrictive to assume $\alpha = 1$.

Let $u_\varepsilon \in SBV(\Omega)$ be a minimizer of

$$\min_{\substack{u \in SBV(\Omega) \\ \mathcal{H}^1(J_u) \le \varepsilon}} MS(u).$$

We shall prove by a contradiction argument that $J_{u_\varepsilon} = \emptyset$ for ε small enough, so that $u_\varepsilon = U$ and the proof follows.

Let ε_n be an infinitesimal sequence, and let us denote by u_n the corresponding functions u_{ε_n}. By truncation we may assume that

$$\|u_n\|_\infty \le \|g\|_\infty \qquad \text{and} \qquad \|U\|_\infty \le \|g\|_\infty. \tag{5.3}$$

Comparing u_n with the zero function we get

$$\int_\Omega |\nabla u_n|^2\,dx + \mathcal{H}^1(J_{u_n}) + \beta \int_\Omega |u_n - g|^2\,dx \le \beta \int_\Omega g^2\,dx. \tag{5.4}$$

We will concentrate on

$$v_n := u_n - U.$$

We divide the proof in several steps.

Step 1: Regularity for U. In view of the bound (5.3) and of the regularity of $\partial\Omega$, we infer from the elliptic problem (5.1) satisfied by U that $U \in H^2(\Omega)$ (see [12, Theorem 3.2.1.3 and Remark 3.2.4.6]). In particular we deduce that $\nabla U \in L^p(\Omega)$ for every $p > 1$. Then we may write for every $x \in \overline{\Omega}$, $\rho > 0$ and $p > 4$

$$\int_{\Omega\cap B_\rho(x)} |\nabla U|^2\,dx \le \left(\int_{\Omega\cap B_\rho(x)} |\nabla U|^p\,dx\right)^{\frac{2}{p}} |\Omega \cap B_\rho(x)|^{1-\frac{2}{p}}.$$

As a consequence, for every $\gamma \in (0,1)$ we get the estimate

$$\int_{\Omega\cap B_\rho(x)} |\nabla U|^2\,dx \le c_1 \rho^{1+\gamma}, \tag{5.5}$$

for some $c_1 > 0$ (here, c_1 depends on γ, Ω, β, g, but not on ρ).

Step 2: Weak local almost quasi-minimality of v_n with respect to it own jump set. Let us show that there exist $\delta > 0$ and $c_\delta > 0$ such that for every $x \in \overline{\Omega}$, for every $\rho \le 1$ with

$$\int_{\Omega\cap B_\rho(x)} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \le \rho, \tag{5.6}$$

and for every $v \in SBV(\Omega)$ with $\{v \ne v_n\} \subseteq B_\rho(x)$ and $\mathcal{H}^1(J_v) \le \varepsilon_n$ we have

$$\int_{\Omega\cap B_\rho(x)} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \le \int_{\Omega\cap B_\rho(x)} |\nabla v|^2\,dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c_\delta \rho^{1+\delta}. \tag{5.7}$$

In particular we get that v_n is a weak local almost quasi-minimizer of the Mumford-Shah energy with respect to its own jump set at any point $x \in \overline{\Omega}$ according to Definition 4.1.

Recalling (5.3) we have $\|v_n\|_\infty \le 2\|g\|_\infty$, so that it is not restrictive to assume that also

$$\|v\|_\infty \le 2\|g\|_\infty. \tag{5.8}$$

Moreover, we may assume

$$\int_{\Omega\cap B_\rho(x)} |\nabla v|^2\,dx < \int_{\Omega\cap B_\rho(x)} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)), \tag{5.9}$$

since otherwise (5.7) is immediately satisfied.

Since

$$\{v + U \neq u_n\} \subseteq B_\rho(x)$$

by minimality of u_n we get

$$\int_{\Omega\cap B_\rho(x)} |\nabla v_n + \nabla U|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) + \beta \int_{\Omega\cap B_\rho(x)} |u_n + U - g|^2\,dx$$
$$\leq \int_{\Omega\cap B_\rho(x)} |\nabla v + \nabla U|^2\,dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + \beta \int_{\Omega\cap B_\rho(x)} |v + U - g|^2\,dx$$

so that

$$\int_{\Omega\cap B_\rho(x)} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \bar{B}_\rho(x)) \leq \int_{\Omega\cap B_\rho(x)} |\nabla v|^2\,dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)) + c(\rho),$$

where

$$c(\rho) := 2\int_{\Omega\cap B_\rho(x)} \nabla U \cdot (\nabla v - \nabla v_n)\,dx$$
$$+ \beta \int_{\Omega\cap B_\rho(x)} |v + U - g|^2\,dx - \beta \int_{\Omega\cap B_\rho(x)} |v_n + U - g|^2\,dx.$$

Recalling (5.3) and (5.8), we have

$$\int_{\Omega\cap B_\rho(x)} |v + U - g|^2\,dx - \int_{\Omega\cap B_\rho(x)} |v_n + U - g|^2\,dx \leq c_2\rho^2 \tag{5.10}$$

for some $c_2 > 0$. Thanks to the estimate (5.5) for ∇U obtained in Step 1, and taking into account (5.9) and (5.6), we have

$$\left|\int_{\Omega\cap B_\rho(x)} \nabla U \cdot (\nabla v - \nabla v_n)\,dx\right| \tag{5.11}$$
$$\leq \|\nabla U\|_{L^2(\Omega\cap B_\rho(x);\mathbb{R}^2)}(\|\nabla v\|_{L^2(\Omega\cap B_\rho(x);\mathbb{R}^2)} + \|\nabla v_n\|_{L^2(\Omega\cap B_\rho(x);\mathbb{R}^2)})$$
$$\leq c_3\rho^{\frac{1+\gamma}{2}}\rho^{\frac{1}{2}} = c_3\rho^{1+\frac{\gamma}{2}}$$

for some $c_3 > 0$.

Collecting (5.10) and (5.11), we get for every $\rho \leq 1$

$$c(\rho) \leq c_\delta \rho^{1+\delta}$$

for $\delta = \gamma/2$ and $c_\delta > 0$, so that inequality (5.7) is proved.

Step 3: Vanishing energy. We claim that

$$\int_\Omega |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) \to 0. \tag{5.12}$$

Indeed, in view of (5.4) and of (5.3) we may apply Ambrosio's theorem (see Theorem 2.1) to the sequence $(u_n)_{n\in\mathbb{N}}$: there exists $u \in SBV(\Omega)$ such that up to a subsequence

$$u_n \to u \qquad \text{strongly in } L^2(\Omega)$$

$$\nabla u_n \rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^2) \tag{5.13}$$

and

$$\mathcal{H}^1(J_u) \leq \liminf_n \mathcal{H}^1(J_{u_n}) = 0. \tag{5.14}$$

In particular, $u \in H^1(\Omega)$. Moreover, by the minimality of u_n, we deduce that in the limit

$$\int_\Omega |\nabla u|^2\,dx + \beta\int_\Omega |u - g|^2\,dx \leq \int_\Omega |\nabla\varphi|^2\,dx + \beta\int_\Omega |\varphi - g|^2\,dx$$

for every $\varphi \in H^1(\Omega)$, which yields $u = U$. Passing to the limit in the inequality

$$\int_\Omega |\nabla u_n|^2\,dx + \mathcal{H}^1(J_{u_n}) + \beta \int_\Omega |u_n - g|^2\,dx \le \int_\Omega |\nabla U|^2\,dx + \beta \int_\Omega |U - g|^2\,dx,$$

we deduce

$$\limsup_n \int_\Omega |\nabla u_n|^2\,dx \le \int_\Omega |\nabla U|^2\,dx$$

which together with the weak convergence (5.13) yields

$$\nabla u_n \to \nabla U \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^2).$$

Recalling that $v_n = u_n - U$ and that $\mathcal{H}^1(J_{v_n}) = \mathcal{H}^1(J_{u_n}) = \varepsilon_n$, claim (5.12) follows.

Step 4: Conclusion. We can now conclude the proof via a contradiction argument. Assume that $\mathcal{H}^1(J_{u_n}) > 0$ for every n. Since $J_{v_n} = J_{u_n}$, this implies that for every n

$$\mathcal{H}^1(J_{v_n}) > 0. \tag{5.15}$$

To derive a contradiction from (5.15) when n is large enough, we consider for every n a point $x_n \in J_{v_n}$ of density one, i.e., such that

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^1(J_{v_n} \cap B_\rho(x_n))}{2\rho} = 1\,. \tag{5.16}$$

Let $y_n \in \partial\Omega$ be a projection of x_n on $\partial\Omega$; since Ω has convex corners, y_n is a smooth point of $\partial\Omega$. Let us set

$$d_n := dist(x_n, \partial\Omega) = |x_n - y_n| \qquad \text{and} \qquad d'_n := dist(y_n, S),$$

where S is the set of corners of Ω.

For $x \in \overline{\Omega}$, we set

$$\mathcal{E}^n_x(\rho) := \int_{\Omega\cap B_\rho(x)} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n} \cap \overline{B}_\rho(x)).$$

Notice that, in view of Step 3,

$$\eta_n := \int_\Omega |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) \to 0 \tag{5.17}$$

and that

$$\mathcal{E}^n_x(\rho) \le \eta_n\,. \tag{5.18}$$

We distinguish four cases.

Case 1. Assume that

$$\limsup_n \frac{d_n}{\sqrt{\eta_n}} > 0. \tag{5.19}$$

Thanks to Step 2 and the interior monotonicity formula of Theorem 4.3 we infer that the map

$$\rho \mapsto \frac{\mathcal{E}^n_{x_n}(\rho)}{\rho} \wedge 1 + c_\gamma \rho^\gamma \tag{5.20}$$

is non decreasing on $(0, d_n \wedge 1)$. Thanks to (5.19), possibly passing to a subsequence, for n large we may choose as an admissible radius $\rho_n = C\sqrt{\eta_n}$ for some $C > 0$, and write

$$1 \le \frac{\mathcal{E}^n_{x_n}(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma \le \frac{\mathcal{E}^n_{x_n}(\rho_n)}{\rho_n} + c_\gamma \rho_n^\gamma \le \frac{\eta_n}{C\sqrt{\eta_n}} + c_\gamma (C\sqrt{\eta_n})^\gamma,$$

where the first inequality comes from monotonicity at x_n and (5.16), the last one by (5.18). In view of (5.17), the above relation gives a contradiction for n large enough.

Case 2. Assume that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0 \qquad \text{and} \qquad \limsup_n \frac{d'_n}{\sqrt{\eta_n}} > 0. \tag{5.21}$$

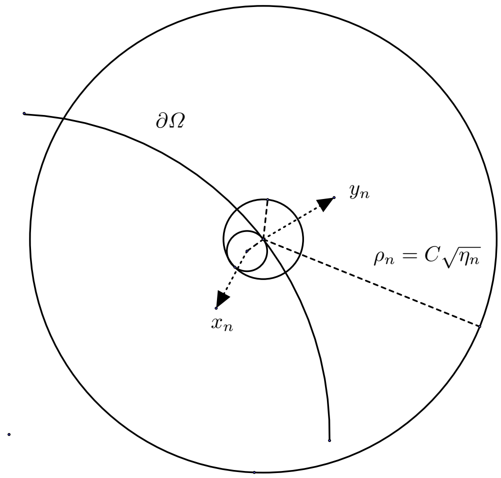

FIGURE 3. Illustration of Case 2.

By Step 2 and the boundary monotonicity formula of Theorem 4.4 we infer that the map

$$\rho \mapsto \frac{\mathcal{E}^n_{y_n}(\rho)}{\rho} \wedge 1 + c_\gamma \rho^\gamma + k_\Omega \int_0^\rho \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr \tag{5.22}$$

is non decreasing on $(0, 1 \wedge r_\Omega \wedge d'_n)$, where $r_\Omega, k_\Omega > 0$. Thanks to (5.21), possibly passing to a subsequence, for n large we may choose as an admissible radius $\rho_n = C\sqrt{\eta_n}$, assume that $\rho_n > 2d_n$, and write

$$\begin{aligned} \frac{\mathcal{E}^n_{y_n}(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr & \\ \geq \frac{\mathcal{E}^n_{y_n}(2d_n)}{2d_n} \wedge 1 + c_\gamma (2d_n)^\gamma + k_\Omega \int_0^{2d_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr & \\ \geq \frac{\mathcal{E}^n_{x_n}(d_n)}{2d_n} \wedge 1 + c_\gamma d_n^\gamma \geq \frac{1}{2} \left[\frac{\mathcal{E}^n_{x_n}(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{2}, & \end{aligned}$$

where we have used in the first inequality monotonicity at y_n, then the inclusion $B_{d_n}(x_n) \subseteq B_{2d_n}(y_n)$ (see Figure 3), and finally monotonicity at x_n combined with (5.16). We infer

$$\frac{1}{2} \leq \frac{\mathcal{E}^n_{y_n}(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr \leq \frac{\eta_n}{C\sqrt{\eta_n}} + c_\gamma (C\sqrt{\eta_n})^\gamma + k_\Omega C\sqrt{\eta_n}, \tag{5.23}$$

which is a contradiction for n large in view of (5.17).

Case 3. Assume that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0, \qquad \lim_n \frac{d'_n}{\sqrt{\eta_n}} = 0 \qquad \text{and} \qquad \lim_n \frac{d_n}{d'_n} = 0. \tag{5.24}$$

Then there exists a vertex $z \in S$ such that, possibly passing to a subsequence, $x_n, y_n \to z$. Set

$$d'_n = dist(y_n, z).$$

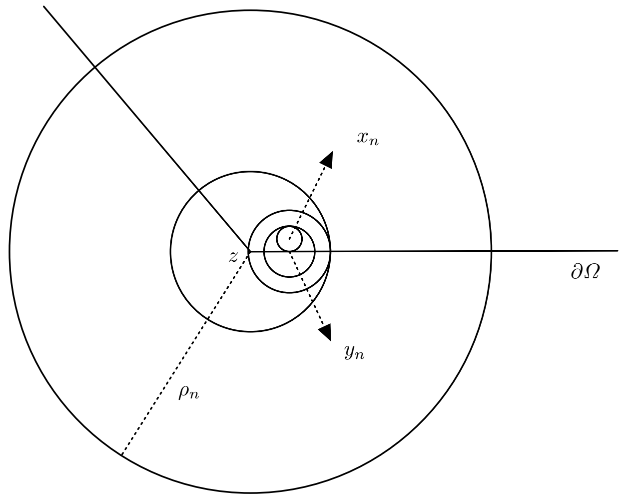

FIGURE 4. Illustration of Case 3.

We choose $\rho_n := \sqrt{\eta_n}$. Since for n large we have $\rho_n \geq 2d'_n$, we may write using the monotonicity at z and the inclusion $B_{d'_n}(y_n) \subseteq B_{2d'_n}(z)$ (see Figure 4)

$$\frac{\mathcal{E}^n_z(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \\ \geq \frac{\mathcal{E}^n_z(2d'_n)}{2d'_n} \wedge 1 + c_\gamma (2d'_n)^\gamma + k_\Omega \int_0^{2d'_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \\ \geq \frac{\mathcal{E}^n_{y_n}(d'_n)}{2d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma,$$

On the other hand, since for n large we also have $d'_n \geq 2d_n$, by monotonicity at y_n and the inclusion $B_{d_n}(x_n) \subseteq B_{2d_n}(y_n)$ (see again Figure 4), we have

$$\frac{\mathcal{E}^n_{y_n}(d'_n)}{2d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma + k_\Omega \int_0^{d'_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr \\ \geq \frac{1}{2} \left[\frac{\mathcal{E}^n_{y_n}(d'_n)}{d'_n} \wedge 1 + c_\gamma (d'_n)^\gamma + k_\Omega \int_0^{d'_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr \right] \\ \geq \frac{1}{2} \left[\frac{\mathcal{E}^n_{y_n}(2d_n)}{2d_n} \wedge 1 + c_\gamma (2d_n)^\gamma + k_\Omega \int_0^{2d_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr \right] \\ \geq \frac{1}{2} \left[\frac{\mathcal{E}^n_{x_n}(d_n)}{2d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{4} \left[\frac{\mathcal{E}^n_{x_n}(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{4},$$

the last inequality coming from monotonicity at x_n combined with (5.16).

Collecting the previous inequalities we obtain

$$\frac{1}{4} \leq k_\Omega \int_0^{d'_n} \left(\frac{\mathcal{E}^n_{y_n}(r)}{r} \wedge 1 \right) dr + \frac{\mathcal{E}^n_z(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \\ \leq k_\Omega d'_n + \frac{\eta_n}{\sqrt{\eta_n}} + c_\gamma (\sqrt{\eta_n})^\gamma + k_\Omega \sqrt{\eta_n},$$

which yields a contradiction for n large.

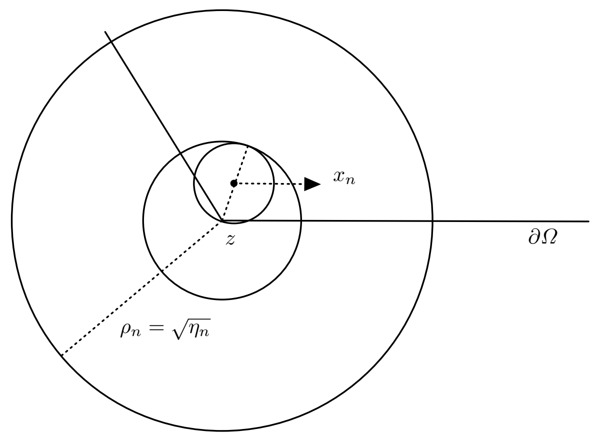

FIGURE 5. Illustration of Case 4.

Case 4. Assume finally that

$$\lim_n \frac{d_n}{\sqrt{\eta_n}} = 0, \qquad \lim_n \frac{d'_n}{\sqrt{\eta_n}} = 0 \qquad \text{and} \qquad \limsup_n \frac{d_n}{d'_n} > 0. \tag{5.25}$$

Let $z \in S$ be a vertex such that, up to a subsequence if necessary, $x_n, y_n \to z$, so that $d'_n = dist(y_n, z)$. Set $d''_n := dist(x_n, z)$. Up to a subsequence we may assume that, for n large, $d'_n \leq C d_n$ with $C > 0$, so that from the inequalities

$$d_n = dist(x_n, \partial\Omega) \leq d''_n = dist(x_n, z) \leq dist(x_n, y_n) + dist(y_n, z) = d_n + d'_n \leq (1 + C)d_n \tag{5.26}$$

we infer that d_n and d''_n are comparable.

Then, if we choose $\rho_n := \sqrt{\eta_n}$, in view of (5.25) and (5.26), for n large we have $\rho_n \geq 2d''_n$. By monotonicity at z and the inclusion $B_{d''_n}(x_n) \subseteq B_{2d''_n}(z)$ (see Figure 5), we obtain

$$\begin{aligned}
&\frac{\mathcal{E}^n_z(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \\
&\qquad \geq \frac{\mathcal{E}^n_z(2d''_n)}{2d''_n} \wedge 1 + c_\gamma (2d''_n)^\gamma + k_\Omega \int_0^{2d''_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \\
&\qquad \geq \frac{\mathcal{E}^n_{x_n}(d''_n)}{2d''_n} \wedge 1 + c_\gamma (d''_n)^\gamma \geq \frac{\mathcal{E}^n_{x_n}(d_n)}{2(1+C)d_n} \wedge 1 + c_\gamma d_n^\gamma \\
&\qquad \geq \frac{1}{2(1+C)} \left[\frac{\mathcal{E}^n_{x_n}(d_n)}{d_n} \wedge 1 + c_\gamma d_n^\gamma \right] \geq \frac{1}{2(1+C)},
\end{aligned}$$

where as above the last inequality comes from monotonicity at x_n combined with (5.16). Then,

$$\frac{1}{2(1+C)} \leq \frac{\mathcal{E}^n_z(\rho_n)}{\rho_n} \wedge 1 + c_\gamma \rho_n^\gamma + k_\Omega \int_0^{\rho_n} \left(\frac{\mathcal{E}^n_z(r)}{r} \wedge 1 \right) dr \leq \frac{\eta_n}{\sqrt{\eta_n}} + c_\gamma (\sqrt{\eta_n})^\gamma + k_\Omega \sqrt{\eta_n},$$

which yields a contradiction for n large. The proof is now concluded. □

Remark 5.4. For later use, let us notice that in order to carry out Step 4 in the previous proof, the monotonicity properties given by Theorems 4.3 and 4.4 can be replaced respectively by *interior and boundary quasi-monotonicity properties* of the following type: if u is a weak local almost quasi-minimizer for the Mumford-Shah energy with respect to its own jump set at every point of $\overline{\Omega}$, and

E_x and $\widetilde{E}_x$ are defined respectively by (4.4) and (4.5), there exist $c>1$ and $\tilde r_\Omega>0$ such that

$$\begin{cases} E_x(\rho_2) \geq \dfrac{1}{c} E_x\left(\dfrac{\rho_1}{c}\right) & \text{if } x\in\Omega \text{ and } \rho_1\leq\rho_2\leq \tilde r_\Omega \wedge dist(x,\partial\Omega) \\ \widetilde{E}_x(\rho_2) \geq \dfrac{1}{c} \widetilde{E}_x\left(\dfrac{\rho_1}{c}\right) & \text{if } x\in\partial\Omega \text{ and } \rho_1\leq\rho_2\leq \tilde r_\Omega \wedge dist(x,S\setminus\{x\}). \end{cases}$$

Indeed, considering for example Case 2, we can choose again $\rho_n := C\sqrt{\eta_n}$ for some $C>0$, and use the *boundary quasi-monotonicity* at y_n for the radii $\rho_n \geq 2cd_n$ to write

$$\begin{aligned} \frac{\mathcal{E}^n_{y_n}(\rho_n)}{\rho_n}\wedge 1 + c_\gamma\rho_n^\gamma + k_\Omega\int_0^{\rho_n}\left(\frac{\mathcal{E}^n_{y_n}(r)}{r}\wedge 1\right)dr \\ \geq \frac{1}{c}\left[\frac{\mathcal{E}^n_{y_n}(2d_n)}{2d_n}\wedge 1 + c_\gamma(2d_n)^\gamma + k_\Omega\int_0^{2d_n}\left(\frac{\mathcal{E}^n_{y_n}(r)}{r}\wedge 1\right)dr\right] \\ \geq \frac{1}{2c}\left[\frac{\mathcal{E}^n_{x_n}(d_n)}{d_n}\wedge 1 + c_\gamma d_n^\gamma\right]. \end{aligned} \tag{5.27}$$

Since x_n is a point of density one for J_{v_n}, there exists r_n such that $cr_n\leq d_n$ and

$$\frac{\mathcal{H}^1(J_{v_n}\cap \bar B_{r_n}(x_n))}{r_n} \geq \frac{3}{2}.$$

Then, by using the *interior quasi-monotonicity* at x_n for the radii $cr_n\leq d_n$, we get

$$\frac{1}{2c}\left[\frac{\mathcal{E}^n_{x_n}(d_n)}{d_n}\wedge 1 + c_\gamma d_n^\gamma\right] \geq \frac{1}{2c^2}\left[\frac{\mathcal{E}^n_{x_n}(r_n)}{r_n}\wedge 1 + c_\gamma r_n^\gamma\right] \geq \frac{1}{2c^2}.$$

But this lower bound is incompatible with (5.27) since the first line of (5.27) is going to zero as $n\to\infty$ (cf. (5.23)). The other cases can be treated similarly.

6. The case of general energies with boundary conditions

In this section we deal with the local minimality of the Sobolev minimizer of a generalization of the Mumford-Shah functional under prescribed boundary conditions.

6.1. **Setting of the problem and the local minimality result.** Given an open bounded set $\Omega\subset\mathbb{R}^2$, which is admissible according to Definition 3.5, we want to impose Dirichlet boundary conditions on the (nonempty) portion Γ_D of its boundary. To this aim, similarly as in Section 4 we consider an open bounded domain Ω' such that $\Omega\subset\subset\Omega'\subset\mathbb{R}^2$ and with $\partial\Omega\cap\Omega'=\Gamma_D$ (see Figure 2). Then, given $w\in H^1(\Omega')\cap L^\infty(\Omega')$, we set

$$\mathcal{A}_w := \{u\in SBV(\Omega')\,:\, u=w \text{ on } \Omega'\setminus\overline{\Omega}\}. \tag{6.1}$$

We are interested in the minimization on the class $\mathcal{A}_w$ of the Mumford-Shah type functional

$$F(u) := \int_{\Omega'} A(x)\nabla u\cdot\nabla u\,dx + \int_{J_u} b(x,u^+,u^-,\nu_u)\,d\mathcal{H}^1 + \beta\int_{\Omega'}|u-g|^2\,dx\,. \tag{6.2}$$

Here ν_u denotes a normal along J_u, and $u^\pm$ the associated traces of u (see Section 2), while the assumptions satisfied by the matrix A, the function b, and the datum g are specified below.

Remark 6.1. Notice that working on the set $\mathcal{A}_w$ the boundary condition

$$u=w \qquad \text{on } \Gamma_D$$

is taken into account in a relaxed sense: indeed, the parts of Γ_D on which $u\neq w$ are contained in J_u, so that they turn out to be penalized by the functional F. This is usual in variational problems for functions of bounded variation (like for example the graph area problem). Finally, observe that the bulk terms provide a fixed contribute on $\Omega'\setminus\Omega$, since $u=w$ on this set, so that for the minimization it suffices simply to integrate on Ω.

Assumptions on the functional. Concerning the volume terms in (6.2), we require

$$A \in C^{0,\delta}(\overline{\Omega'}; M^{2\times 2}_{sym}), \qquad c_1^A|\eta|^2 \le A(x)\eta\cdot\eta \le c_2^A|\eta|^2 \quad \text{for every } x\in\Omega' \text{ and } \eta\in\mathbb{R}^2, \tag{6.3}$$

and

$$g \in L^\infty(\Omega'), \qquad \beta \ge 0 \tag{6.4}$$

for suitable constants $c_1^A, c_2^A > 0$ and $\delta \in (0,1)$.

Concerning the surface term

$$\Psi(u) := \int_{J_u} b(x, u^+, u^-, \nu_u)\, d\mathcal{H}^1,$$

we require for $b : \Omega' \times \mathbb{R} \times \mathbb{R} \times S^1 \to [0,+\infty[$ to be such that

$$c_1^b \le b(x, s_1, s_2, \nu) \le c_2^b + \Phi(s_1, s_2) \tag{6.5}$$

where $c_1^b, c_2^b > 0$ and $\Phi : \mathbb{R}^2 \to [0,+\infty[$. In addition we ask that

$$u \mapsto \Psi(u) \text{ is l.s.c with respect to the weak convergence in } SBV(\Omega), \tag{6.6}$$

and the *monotonicity under truncation*

$$\Psi((u\wedge c_2)\vee c_1) \le \Psi(u) \qquad \text{for every } c_1 \le c_2. \tag{6.7}$$

Remark 6.2 (**Example of admissible surface energies).** An admissible surface term could be given for example by

$$b(x, s_1, s_2, \nu) := \varphi(x,\nu) + |s_1 - s_2|$$

where $\varphi : \Omega' \times \mathbb{R}^2 \to]0,+\infty[$ is such that

$$x \mapsto \varphi(x,\nu) \quad \text{is lower semicontinuous on } \Omega' \text{ for every } \nu \in S^1,$$

$$\nu \mapsto \varphi(x,\nu) \quad \text{is convex, positively one homogeneous on } \mathbb{R}^2 \text{ for every } x \in \Omega',$$

and

$$0 < c_1^b \le \varphi(x,\nu) \le c_2^b.$$

Requirements (6.5) and (6.7) are immediately fulfilled. The lower semicontinuity (6.6) is a consequence of Reshetnyak Theorem (see [2, Theorem 2.38]) and of the lower semicontinuity result in SBV [2, Theorem 5.22].

A-admissible domains. To control the interaction between Neumann and Dirichlet conditions in connection with monotonicity, we need to introduce the following property concerning the interplay between the geometry of Ω and the bulk energy.

Definition 6.3 (**A-Admissible domains).** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a piecewise $C^{1,1}$-boundary, decomposed as in* (3.23), *and let A satisfy* (6.3). *We say that Ω is A-admissible if, for every $x \in \overline{\Omega}$, the domain $\Omega_x := A(x)^{-1/2}\Omega$ is admissible according to Definition 3.5.*

Remark 6.4. Notice that admissibility according to Definition 3.5 is equivalent to Id-admissibility according to Definition 6.3. In particular, when $\Gamma_D = \partial\Omega$ or $\Gamma_N = \partial\Omega$, A-admissibility reduces to the assumption of convex corners, and smooth $C^{1,1}$ domains are always A-admissible.

The local minimality result under boundary conditions. The analogue of Theorem 5.1 under boundary conditions is the following.

Theorem 6.5. (Small jump sets are not convenient under boundary conditions) *Assume the functional F satisfies assumptions* (6.3), (6.4), (6.5), (6.6), *and* (6.7). *Let $\Omega \subseteq \mathbb{R}^2$ be A-admissible according to Definition 6.3, and let $w \in H^1(\Omega') \cap L^\infty(\Omega')$. Denoting with $U_w \in H^1(\Omega')$ the solution to*

$$\min_{v \in \mathcal{A}_w \cap H^1(\Omega')} F(v),$$

assume that U_w admits at most uniformly weak singularities in $\overline{\Omega}$, i.e., there exist $C > 0$ and $\alpha > 0$ such that for every $x \in \overline{\Omega}$ and $\rho \le 1$

$$\int_{\Omega \cap B_\rho(x)} |\nabla U_w|^2 \, dx \le C\rho^{1+\alpha}. \tag{6.8}$$

Then there exists $\varepsilon > 0$ such that for every $u \in \mathcal{A}_w$ with $\mathcal{H}^1(J_u) < \varepsilon$ we have

$$F(U_w) < F(u).$$

Remark 6.6. Assumptions on the kind of *singularities* which the function U_w can exhibit are essential to exclude that small jump sets are not convenient. Indeed, assume for example that

$$\int_{\Omega \cap B_\rho(x)} |\nabla U_w|^2 \, dx \approx c\rho^\gamma \tag{6.9}$$

with $0 < \gamma < 1$ and $c > 0$ at some point $x \in \overline{\Omega}$: this means that for small ρ, the singularity has an energy content in $\Omega \cap B_\rho(x)$ which is higher than the length of $\partial(\Omega \cap B_\rho(x))$. It turns out that the admissible competitor

$$v := \begin{cases} 0 & \text{in } \Omega \cap B_\rho(x) \\ U_w & \text{otherwise} \end{cases}$$

is such that $F(v) < F(U_w)$ if ρ is sufficiently small, so that the minimality property of Theorem 6.5 cannot hold.

Theorem 6.5 states that the energy content given by (6.8) is not enough to destroy the minimality of U_w, which is preserved even if small jumps sets are allowed. In the case $\Gamma_D = \partial\Omega$, inequality (6.8) holds for example if $w \in H^2(\Omega')$ and A is Lipschitz continuous on $\overline{\Omega'}$, since elliptic regularity entails $U_w \in H^2(\Omega)$ (see e.g. [12, 3.2.1.2]).

In the language of fracture mechanics, the situation (6.9) is denoted as a *strong singularity*: the "elastic energy" stored in $\Omega \cap B_\rho(x)$ is much higher than the energy required to create a crack along $\partial(\Omega \cap B_\rho(x))$ so that the elastic configuration is not in equilibrium, and the creation of a small crack is energetically convenient. "*Weak singularities*" are on the contrary compatible with local equilibrium.

As mentioned in the Introduction, the minimality property of Theorem 6.5 (without fidelity term but with nonlinear bulk energies and general Lipschitz boundaries) was derived in [8, Theorem 1] under the additional assumption that the competitors have a closed jump set with a preset number of connected components, while the full SBV case was derived under the stronger assumption $U_w \in C^1(\overline{\Omega})$ (see [8, Theorem 6]).

As in Section 5, we can draw the following local minimality result with respect to the L^1 topology under boundary conditions.

Theorem 6.7 (Local minimality in L^1 under boundary conditions). *Under the assumptions of Theorem 6.5, the function U_w is a local minimizer for the Mumford-Shah type energy F on $\mathcal{A}_w$ with respect to the L^1-topology.*

Proof. Assume by contradiction that there exists $v_n \in \mathcal{A}_w$ such that

$$v_n \to U_w \qquad \text{strongly in } L^1(\Omega')$$

with

$$F(v_n) < F(U_w). \tag{6.10}$$

It is not restrictive to assume that $\|v_n\|_\infty \le \|g\|_\infty + \|w\|_\infty$, so that the convergence holds also in $L^2(\Omega')$ and is weak in $SBV(\Omega')$ (thanks to the coercivities assumptions on A and b). By Ambrosio's Theorem we deduce

$$\liminf_n F(v_n) \ge F(U_w),$$

so that from (6.10) we infer $\lim_n \mathcal{H}^1(J_{v_n}) = 0$: but then (6.10) is in contrast with Theorem 6.5. $\square$

6.2. **Proof of Theorem 6.5.** Let us start by deriving some properties of a solution $v \in \mathcal{A}_0$ to

$$\min_{\substack{h\in\mathcal{A}_0\\ \mathcal{H}^1(J_h)\le\varepsilon}} F(U_w+h). \tag{6.11}$$

They are stated in two separate lemmas below, and concern respectively a uniform almost-quasi minimality property, and a crucial quasi-monotonicity property. Some preliminary notation and remarks are in order.

Firstly notice that existence of minimizers to (6.11) is guaranteed by the application of Ambrosio's theorem in view of the assumptions (6.3)-(6.7) on the terms appearing in F. Notice that we may assume (thanks in particular to the truncation assumption (6.7) for the surface energy)

$$\|U_w\|_\infty \le \|w\|_\infty + \|g\|_\infty \qquad \text{and} \qquad \|v\|_\infty \le 2\|w\|_\infty + \|g\|_\infty. \tag{6.12}$$

For every $\xi \in \overline{\Omega}$, let us consider the matrix

$$L_\xi := A(\xi)^{1/2}$$

and the sets

$$\widetilde{\Omega}_\xi := L_\xi^{-1}\Omega \qquad \text{and} \qquad \widetilde{\Omega}'_\xi := L_\xi^{-1}\Omega'.$$

To every function $u \in SBV(\Omega')$ let us associate the function $\tilde{u}_\xi \in SBV(\widetilde{\Omega}'_\xi)$ given by

$$\tilde{u}_\xi(y) := \sqrt{\frac{2detL_\xi}{c_1^b\sqrt{c_1^A}}}\,u(L_\xi y), \tag{6.13}$$

where c_1^A and c_1^b are the constants appearing in (6.3) and (6.5). Let us denote by $\widetilde{\mathcal{A}}_0$ the space of functions associated to $\mathcal{A}_0$ under the previous transformation, and let $\tilde{\xi}$ be the point corresponding to ξ.

Lemma 6.8 (**Uniform almost quasi-minimality**). *Under the assumptions of Theorem 6.5, let $v \in \mathcal{A}_0$ be a solution to the minimization problem* (6.11). *There exist $\rho_0 > 0$, $\Lambda \ge 1$, $\gamma > 0$, $c_\gamma > 0$ such that for every $\xi \in \overline{\Omega}$, for every $\rho < \rho_0$ with*

$$\int_{B_\rho(\tilde{\xi})\cap\widetilde{\Omega}_\xi} |\nabla\tilde{v}_\xi|^2\,dx + \mathcal{H}^1(J_{\tilde{v}_\xi}\cap\bar{B}_\rho(\tilde{\xi})) \le \rho, \tag{6.14}$$

and for every $\tilde{h} \in \widetilde{\mathcal{A}}_0$ with $\{\tilde{h} \ne \tilde{v}_\xi\} \subseteq B_\rho(\tilde{x})$ and $J_{\tilde{h}} \subseteq J_{\tilde{v}_\xi}$ we have

$$\int_{B_\rho(\tilde{\xi})\cap\widetilde{\Omega}_\xi} |\nabla\tilde{v}_\xi|^2\,dx + \mathcal{H}^1(J_{\tilde{v}_\xi}\cap\bar{B}_\rho(\tilde{\xi})) \le \int_{B_\rho(\tilde{\xi})\cap\widetilde{\Omega}_\xi} |\nabla\tilde{h}|^2\,dx + \Lambda\mathcal{H}^1(J_{\tilde{h}}\cap\bar{B}_\rho(\tilde{\xi})) + c_\gamma\rho^{1+\gamma}.$$

In other words, uniformly in $\xi \in \overline{\Omega}$, the function $\tilde{v}_\xi$ is a weak local almost quasi-minimizer in $\tilde{\mathcal{A}}_0$ of the Mumford-Shah energy with respect to its own jump set at the point $\tilde{\xi}$.

Proof. Let us divide the proof in two steps.

Step 1. Assume that $A(\xi) = Id$. Let us show that if $h \in \mathcal{A}_0$ with $\{h \ne v\} \subseteq B_\rho(\xi)$, $\rho \le 1$ and $J_h \subseteq J_v$, we have

$$\begin{aligned}\int_{\Omega\cap B_\rho(\xi)} |\nabla v|^2\,dx &+ c_1^b\mathcal{H}^1(J_v\cap\bar{B}_\rho(\xi))\\ &\le \int_{\Omega\cap B_\rho(\xi)} |\nabla h|^2\,dx + C_1\mathcal{H}^1(J_h\cap\bar{B}_\rho(\xi)) + 2c_A(1+c_1^b)\rho^{1+\delta} + C_2\rho^{(1+\frac{\alpha}{2})\wedge 2},\end{aligned} \tag{6.15}$$

where c_1^b is the constant appearing in the estimates (6.5) for the surface energy b, c_A is the Holder constant of $A \in C^{0,\delta}(\overline{\Omega'}, M^{2\times 2}_{sym})$, α is the constant appearing in (6.8), and $C_1, C_2 > 0$ are suitable constants depending only on the data of the problem.

Indeed, recalling (6.12) and using a truncation argument, it is not restrictive to assume that

$$\|h\|_\infty \le 2\|w\|_\infty + \|g\|_\infty. \tag{6.16}$$

Moreover, we may assume

$$\int_{\Omega\cap B_\rho(\xi)} |\nabla h|^2\,dx < \int_{\Omega\cap B_\rho(\xi)} |\nabla v|^2\,dx + c_1^b\mathcal{H}^1(J_v\cap \bar{B}_\rho(\xi)) \tag{6.17}$$

since otherwise (6.15) is immediately satisfied.

By minimality of v (since the inclusion $J_h \subseteq J_v$ ensures that $\mathcal{H}^1(J_h)\le\varepsilon$), we get

$$\begin{aligned}\int_{\Omega\cap B_\rho(\xi)} A(x)\nabla v\cdot\nabla v\,dx + c_1^b\mathcal{H}^1(J_v\cap\bar{B}_\rho(\xi))\\ \le \int_{\Omega\cap B_\rho(\xi)} A(x)\nabla h\cdot\nabla h\,dx + C_1\mathcal{H}^1(J_h\cap\bar{B}_\rho(\xi)) + c(\rho),\end{aligned} \tag{6.18}$$

where

$$c(\rho) := 2\int_{\Omega\cap B_\rho(\xi)} A(x)\nabla U_w\cdot(\nabla h-\nabla v)\,dx + \beta\int_{\Omega\cap B_\rho(\xi)}(h-v)(2U_w+h+v-2g)\,dx$$

and

$$C_1 := c_2^b + \Phi(2\|w\|_\infty + \|g\|_\infty, -(2\|w\|_\infty+\|g\|_\infty)),$$

where c_2^b and Φ appear in (6.5).

In view of the energy bounds (6.8), (6.14), (6.17) and of the L^∞-bounds (6.12) and (6.16) we deduce for every $\rho\le 1$

$$\begin{aligned}c(\rho) &\le 2c_2^A\int_{\Omega\cap B_\rho(\xi)}|\nabla U_w||\nabla h-\nabla v|\,dx + \widetilde{C}\rho^2\\ &\le 2c_2^A\|\nabla U_w\|_{L^2(\Omega\cap B_\rho(\xi))}(\|\nabla h\|_{L^2(\Omega\cap B_\rho(\xi))}+\|\nabla v\|_{L^2(\Omega\cap B_\rho(\xi))}) + \widetilde{C}\rho^2\\ &\le 2c_2^A\sqrt{C}\rho^{\frac{1+\alpha}{2}}\left(\sqrt{\left(1+c_1^b\right)\rho}+\sqrt{\rho}\right) + \widetilde{C}\rho^2 \le C_2\rho^{(1+\frac{\alpha}{2})\wedge 2},\end{aligned}$$

for a suitable $C_2>0$.

Since $A(\xi)=Id$ and $A\in C^{0,\delta}(\overline{\Omega'};M^{2\times 2}_{sym})$, coming back to (6.18) we may write

$$\begin{aligned}\int_{\Omega\cap B_\rho(\xi)}|\nabla v|^2\,dx - c_A\rho^\delta\int_{\Omega\cap B_\rho(\xi)}|\nabla v|^2\,dx + c_1^b\mathcal{H}^1(J_v\cap\bar{B}_\rho(x))\\ \le \int_{\Omega\cap B_\rho(\xi)}|\nabla h|^2\,dx + c_A\rho^\delta\int_{\Omega\cap B_\rho(\xi)}|\nabla h|^2\,dx + C_1\mathcal{H}^1(J_h\cap\bar{B}_\rho(\xi)) + C_2\rho^{(1+\frac{\alpha}{2})\wedge 2},\end{aligned}$$

where $c_A>0$ is the Hölder constant of A, so that taking into account (6.17) and (6.14) we infer

$$\begin{aligned}\int_{\Omega\cap B_\rho(\xi)}|\nabla v|^2\,dx + c_1^b\mathcal{H}^1(J_v\cap\bar{B}_\rho(\xi))\\ \le \int_{\Omega\cap B_\rho(\xi)}|\nabla h|^2\,dx + C_1\mathcal{H}^1(J_h\cap\bar{B}_\rho(\xi)) + 2c_A(1+c_1^b)\rho^{1+\delta} + C_2\rho^{(1+\frac{\alpha}{2})\wedge 2}.\end{aligned}$$

so that (6.15) follows.

Step 2. Let us come to the general case. Under the transformation (6.13), a direct computation shows that the functional F is transformed up to a multiplicative constant into the functional on $SBV(\widetilde{\Omega}'_\xi)$ given by

$$\widetilde{F}(\tilde{u}) := \int_{\widetilde{\Omega}'_\xi}\widetilde{A}(y)\nabla\tilde{u}\cdot\nabla\tilde{u}\,dy + \int_{J_{\tilde{u}}}\tilde{b}(y,\tilde{u}^+,\tilde{u}^-,\nu_{\tilde{u}})\,d\mathcal{H}^1(y) + \beta\int_{\widetilde{\Omega}'_\xi}|\tilde{u}-\tilde{g}_\xi|^2\,dy,$$

with

$$\widetilde{A}(y) := L_\xi^{-1}A(L_\xi y)L_\xi^{-1}$$

and

$$\tilde{b}(y,s_1,s_2,\nu) := \frac{2}{c_1^b\sqrt{c_1^A}}b\left(L_\xi y, \sqrt{\frac{2detL_\xi}{c_1^b}}s_1, \sqrt{\frac{2detL_\xi}{c_1^b}}s_2, L_\xi\nu\right)|L_\xi\nu^\perp|,$$

where $\nu^\perp$ is a unit vector orthogonal to ν (so that $|L_\xi \nu_{\tilde{u}}^\perp|$ turns out to be the one dimensional jacobian of the transformation $y \mapsto L_\xi y$ involved in the change of variable to pass from an integral on J_u to that on $J_{\tilde{u}}$, see [2, Theorem 2.91]). Notice that by construction

$$\tilde{b}(y, s_1, s_2, \nu) \geq 2. \tag{6.19}$$

Clearly the function

$$\tilde{u} := \widetilde{(U_w)}_\xi + \tilde{v}_\xi$$

is a minimizer of the functional $\widetilde{F}$ among the functions in $SBV(\widetilde{\Omega}'_\xi)$ such that $\tilde{u} = \tilde{w}_\xi$ on $\widetilde{\Omega}'_\xi \setminus \widetilde{\Omega}_\xi$ and $\mathcal{H}^1(L_\xi J_{\tilde{v}}) \leq \varepsilon$.

Notice that $\widetilde{A}(\tilde{\xi}) = Id$. Therefore, taking into account (6.19), inequality (6.15) of Step 1 entails that, for every $\tilde{h} \in \widetilde{\mathcal{A}}_0$ with $\{\tilde{h} \neq \tilde{v}_\xi\} \subseteq B_\rho(\tilde{\xi})$ and $J_{\tilde{h}} \subseteq J_{\tilde{v}_\xi}$ (which is an admissible competitor for $\tilde{v}_\xi$ since automatically $\mathcal{H}^1(L_\xi J_{\tilde{h}}) \leq \varepsilon$),

$$\begin{aligned} \int_{\widetilde{\Omega}_\xi \cap B_\rho(\tilde{\xi})} |\nabla \tilde{v}_\xi|^2 \, dx + \mathcal{H}^1(J_{\tilde{v}_\xi} \cap \bar{B}_\rho(\tilde{\xi})) \\ \leq \int_{\widetilde{\Omega}_\xi \cap B_\rho(\tilde{\xi})} |\nabla \tilde{h}|^2 \, dx + \widetilde{C}_1 \mathcal{H}^1(J_{\tilde{h}} \cap \bar{B}_\rho(\tilde{\xi})) + 2c_{\tilde{A}}(1 + c_1^{\tilde{b}})\rho^{1+\delta} + \widetilde{C}_2 \rho^{(1+\frac{\alpha}{2})\wedge 2}, \end{aligned} \tag{6.20}$$

if ρ is sufficiently small. The statement follows thanks to the uniform estimates on L_ξ coming from (6.3). □

Lemma 6.9 (**Quasi-monotonicity**). *Under the assumptions of Theorem 6.5, let $v \in \mathcal{A}_0$ be a solution to the minimization problem* (6.11).

There exist $c > 1$, $\tilde{r}_\Omega, \tilde{k}_\Omega > 0$ such that, if $E_x(\rho)$, $\widetilde{E}_x(\rho)$ are defined respectively as in (4.4) (4.5), *being γ, c_γ the constants of Lemma 6.8, and*

$$\mathcal{E}_x(\rho) := \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 \, dx + \mathcal{H}^1(J_v \cap \bar{B}_\rho(x)),$$

the following properties hold true:

(a) *For every $x \in \Omega$ and for every $\rho_1 < \rho_2 < \tilde{r}_\Omega \wedge dist(x, \partial\Omega)$*

$$E_x(\rho_2) \geq \frac{1}{c} E_x\left(\frac{\rho_1}{c}\right). \tag{6.21}$$

(b) *For every $x \in \partial\Omega$ and for every $\rho_1 < \rho_2 < \tilde{r}_\Omega \wedge dist(x, S \setminus \{x\})$, where S denotes the set of corners of $\partial\Omega$,*

$$\widetilde{E}_x(\rho_2) \geq \frac{1}{c} \widetilde{E}_x\left(\frac{\rho_1}{c}\right).$$

Proof. Let us start with point (a). Thanks to Lemma 6.8, we have that $\tilde{v}_x \in \widetilde{\mathcal{A}}_0$ is a weak local almost quasi-minimizer (in $\tilde{\mathcal{A}}_0$) for the Mumford-Shah energy with respect to its own jump set at the point $\tilde{x} = L_x^{-1}x$, so that the quantity

$$\rho \mapsto \frac{\int_{\widetilde{\Omega}_x \cap B_\rho(\tilde{x})} |\nabla \tilde{v}_x|^2 \, dy + \mathcal{H}^1(J_{\tilde{v}_x} \cap B_\rho(\tilde{x}))}{\rho} \wedge 1 + \frac{c_\gamma}{\gamma}\rho^\gamma \tag{6.22}$$

is non decreasing on $(0, \rho_0 \wedge r_{\widetilde{\Omega}_x} \wedge dist(\tilde{x}, \partial\widetilde{\Omega}_x))$ in view of the interior monotonicity formula of Theorem 4.4. Coming back to the domain Ω, thanks to the bounds on A, we find universal constants $c_i > 0$ such that

$$c_3 \mathcal{E}_x(c_4\rho) \leq \int_{\widetilde{\Omega}_x \cap B_\rho(\tilde{x})} |\nabla \tilde{v}_x|^2 \, dy + \mathcal{H}^1(J_{\tilde{v}_x} \cap \bar{B}_\rho(\tilde{x})) \leq c_1 \mathcal{E}_x(c_2\rho).$$

This estimate, together with the monotonicity of (6.22), Remarks 3.8 and 4.7, yields easily point (a) of the statement for a suitable $\tilde{r}_\Omega > 0$.

The proof of point (b) is similar: we need to invoke, in place of Theorem 4.4, the boundary monotonicity formula of Theorem 4.6, which is in force since $\widetilde{\Omega}_x$ is admissibile, being Ω A-admissible by hypothesis, together with Remarks 3.8 and 4.7. □

We are now in a position to prove Theorem 6.5.

Proof of Theorem 6.5. It suffices to show that, for ε small enough, a minimizer $v_\varepsilon \in SBV(\Omega)$ of

$$\min_{\substack{v\in\mathcal{A}_0\\ \mathcal{H}^1(J_v)\le\varepsilon}} F(U_w+v)$$

is such that $J_{v_\varepsilon}=\emptyset$, which entails $v_\varepsilon=0$, and the proof follows.

We proceed by contradiction, assuming that there exists $\varepsilon_n\to 0$ such that, for the corresponding $v_n := v_{\varepsilon_n}$, it holds $\mathcal{H}^1(J_{v_n})>0$. Let us divide the proof in two steps.

Step 1: Vanishing energy. We claim that

$$\int_{\Omega'} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) \to 0. \tag{6.23}$$

Notice indeed that

$$F(U_w+v_n)\le F(U_w)$$

so that we get easily, taking into account (6.3)-(6.7) and (6.12)

$$\int_{\Omega'} |\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) + \|v_n\|_\infty \le C.$$

By applying Ambrosio's theorem to the sequence $(v_n)_{n\in\mathbb{N}}$, there exists $v\in\mathcal{A}_0$ such that, up to a subsequence,

$$v_n \to v \qquad \text{strongly in } L^2(\Omega')$$

$$\nabla v_n \rightharpoonup \nabla v \qquad \text{weakly in } L^2(\Omega';\mathbb{R}^2) \tag{6.24}$$

and

$$\mathcal{H}^1(J_v) \le \liminf_n \mathcal{H}^1(J_{v_n}) \le \lim_n \varepsilon_n = 0. \tag{6.25}$$

In particular $v\in H^1(\Omega')$. Moreover, by the minimality of v_n, in the limit we deduce that

$$F(U_w+v)\le F(U_w+\varphi)$$

for every $\varphi\in H^1(\Omega')$ with $\varphi=0$ on $\Omega'\setminus\overline{\Omega}$, which yields $v=0$ in view of the definition of U_w. Passing to the limit in the inequality

$$\int_{\Omega'} A(x)[\nabla U_w+\nabla v_n][\nabla U_w+\nabla v_n]\,dx + \int_{J_{v_n}} b(x,v_n^+,v_n^-,\nu_{v_n})\,d\mathcal{H}^1 + \beta\int_{\Omega'}|U_w+v_n-g|^2$$
$$\le \int_{\Omega'} A(x)\nabla U_w\nabla U_w\,dx + \beta\int_{\Omega'}|U_w-g|^2$$

and taking into account (6.24), (6.25) and the coercivity of A and b, we deduce that

$$\nabla v_n \to 0 \qquad \text{strongly in } L^2(\Omega';\mathbb{R}^2),$$

so that the claim follows.

Step 2: Conclusion. By Step 1 we have

$$\eta_n := \int_{\Omega'}|\nabla v_n|^2\,dx + \mathcal{H}^1(J_{v_n}) \to 0.$$

In view of the *quasi-monotonicity* properties enjoyed by v by Lemma 6.9, and taking into account Remark 5.4, we can repeat the arguments of Step 4 in the proof of Theorem 5.1 and get a contradiction, so that the conclusion follows. □

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