GEOMETRIC CONTROL OF THE ROBIN LAPLACIAN EIGENVALUES: THE CASE OF NEGATIVE BOUNDARY PARAMETER

DORIN BUCUR, SIMONE CITO

Abstract. This paper is motivated by the study of the existence of optimal domains maximizing the k-th Robin Laplacian eigenvalue among sets of prescribed measure, in the case of a negative boundary parameter. We answer positively to this question and prove an existence result in the class of measurable sets and for quite general spectral functionals. The key tools of our analysis rely on tight isodiametric and isoperimetric geometric controls of the eigenvalues. In two dimensions of the space, under simply connectedness assumptions, further qualitative properties are obtained on the optimal sets.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\beta > 0$ be a fixed positive real number. A number $\lambda \in \mathbb{R}$ is an *eigenvalue of the Robin problem for the Laplace operator with boundary parameter* $-\beta$ if there exists a non-zero function $u \in H^1(\Omega)$ solving, in the weak sense, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \beta u & \text{on } \partial\Omega \end{cases} \tag{1}$$

(here n is the outer normal on $\partial\Omega$), i.e.

$$u \in H^1(\Omega), u \neq 0, \forall\, v \in H^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v \, dx - \beta \int_{\partial\Omega} uv \, d\mathcal{H}^{d-1} = \lambda \int_\Omega uv \, dx.$$

The eigenvalues, counted with their multiplicity are ordered and denoted

$$\lambda_{1,\beta}(\Omega) \leq \lambda_{2,\beta}(\Omega) \leq \ldots \to +\infty.$$

In general, $\lambda_{1,\beta}(\Omega) < 0$ and, if Ω is connected, $\lambda_{1,\beta}(\Omega)$ is simple. Starting with some index, all eigenvalues become positive.

Precisely, for every $k \in \mathbb{N}$, the k-th eigenvalue, is given by the Rayleigh min-max formula

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2 \, dx}, \tag{2}$$

where $\mathcal{S}_k$ denotes the family of all k-dimensional subspaces of $H^1(\Omega)$.

We are motivated by the following shape optimization problem

$$\sup\{\lambda_{k,\beta}(\Omega) : \Omega \subset \mathbb{R}^d,\ |\Omega| = m\}. \tag{3}$$

It has been conjectured by Bareket in 1977 that for $k = 1$ the solution to the problem above is the ball. In 2015, Freitas and Krejcirik proved in [13] that even in $\mathbb{R}^2$ the solution is, in general, not the disc. Precisely, if the boundary parameter β is larger than a fixed threshold

Dorin Bucur is member of the Institut Universitaire de France.

β_1 (depending on the area m), then the first eigenvalue of a suitably chosen annulus is strictly greater than the same eigenvalue computed for the disk of the same measure. On the opposite sense, if β is smaller than another threshold β_2, then the ball is the only maximizer. A natural conjecture, still unsolved, is that the solution is the disc in the class of simply connected two dimensional sets. For higher order eigenvalues, several aspects of the problem have been investigated by Antunes, Freitas and Krejcirik in 2017 in [3].

The existence of an optimal solution to problem (3) (independent of the precise knowledge of its shape), or to the two dimensional problem in the class of simply connected sets, remained open questions. The purpose of this paper is to bring positive answers to both questions. Precisely, we will prove an existence result for (3) in a relaxed sense, obtaining also some information about the structure of the optimal sets. For this purpose, we extend the variational definition of $\lambda_{k,\beta}(\Omega)$ to measurable sets with finite perimeter in the sense of De Giorgi and to arbitrary simply connected sets in $\mathbb{R}^2$ having a topological boundary of finite length. We are not able to prove any regularity of those optimal sets, but we prove they are bounded with a controlled diameter, they have a controlled perimeter and not more than a number of "connected" components depending on m, k, d. Surprisingly, one could expect that this number is not larger than k, but we are not able to prove it, since a strange phenomenon due to the uncontrolled behaviour of the eigenvalues to rescaling occurs. In two dimensions of the space, we prove also existence of a solution in the class of unions of pairwise disjoint open, simply connected sets. As expected for Robin boundary conditions, the geometry of optimal sets will depend on the mass m.

The existence result is a consequence of a tight geometric control of the specturm, which may be of independent interest. For the Steklov spectrum, such results have been proved by Colbois, Girouard and El Soufi [14] where they get upper bounds for the eigenvalues by a quantity involving the isoperimetric ratio of the set and by Bogosel, Bucur and Giacomini [6] where such bounds are obtained in terms of diameter. The case of Robin boundary conditions is more tricky. Contrary to the Steklov problem, we have simultaneously both negative and positive eigenvalues and they do not obey any homogeneity law. Consequently, the control of the spectrum by homogeneous geometric quantities is more involved and less explicit. Nevertheless, roughly speaking, both results state that larger is either the isoperimetric ratio or the diameter of a connected set, then lower is its k-th Robin eigenvalue. We point out that for positive boundary parameter, the isoperimetric ratio and the diameter do not play any role on the control of the spectrum.

Here we summarize, in a simplified way, the main results of the paper.

Theorem 1 (Isoperimetric control of the spectrum). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz set. There exist two positive constants C_1, C_2 depending only on the dimension of the space, such that*

$$\lambda_{k,\beta}(\Omega) \le -\frac{C_2}{|\Omega|}\left(k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1}\mathcal{H}^{d-1}(\partial\Omega)\right)^- + \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}\left(k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1}\mathcal{H}^{d-1}(\partial\Omega)\right)^+$$

Above x^+, x^- denote the positive and negative parts of the real number x.

Theorem 2 (Isodiametric control of the spectrum). *Given $A > 0$ there exists $m^* = m^*(A, \beta, d)$, $L = L(A, \beta, d)$ such that for every $\Omega \subset \mathbb{R}^d$ open, bounded, connected Lipschitz set*

$$\text{if} \qquad \operatorname{diam}(\Omega) > 2\left(k + \left\lfloor \frac{|\Omega|}{m^*} \right\rfloor + 1\right) L,$$

$$\text{then} \qquad \lambda_{k,\beta}(\Omega) \le -A. \tag{4}$$

Theorem 3 (Existence of optimal shapes)**.** *The shape optimization problem* (3) *has a solution in any of the following settings:*

1. *In the class of measurable sets with finite perimeter of* $\mathbb{R}^d$*, for* $d \geq 2$*.*
2. *In the class of finite unions of pairwise disjoint open, simply connected sets with boundary of finite length in* $\mathbb{R}^2$*.*

The structure of the paper is the following. In Section 2 we recall some known facts about sets of finite perimeter, convergences of sets and properties of the Robin eigenvalues. In Sections 3 and 4 we prove Theorem 1 and Theorem 2, respectively, for general domains, not necessarily smooth. The existence result, Theorem 3, is proved in Section 5 for general spectral functionals depending on a finite number of low eigenvalues, which are increasing and upper semicontinuous in each variable.

An important distinction is made between the general setting of measurable sets in $\mathbb{R}^d$ and the two dimensional setting of open, simply connected sets. The two dimensional setting requires some technicalities related to the pointwise behaviour of SBV functions, so that we isolate this topic it in a specific subsection at the end of the paper, for the interested reader. We point out that this section contains a generalization of [6, Proposition 2.6], which is a sort of Golab like lower semicontinuity result involving the L^2 norms of the double traces of Sobolev functions on moving, connected, not necessarily smooth, cracks of uniformly bounded Hausdorff measure.

2. Notation and preliminaries

In this section we fix the notation and recall some definitions and tools. For $x \in \mathbb{R}^d$ and and $r > 0$, $B_r(x)$ denotes the ball of radius r centered in x. We use the term *annulus* to denote a spherical shell of $\mathbb{R}^d$ for any dimension d. We denote the annulus of radii $r < R$ and centered in x with the symbol $A_{r,R}(x)$. By ω_d we denote the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

For every measurable set $E \subseteq \mathbb{R}^d$, $|E|$ and $\mathcal{H}^{d-1}(E)$ stand respectively for the Lebesgue measure and the Hausdorff $(d-1)$-dimensional measure of E and we use the symbols χ_E for its characteristic function, E^c for its complement, tE ($t \in \mathbb{R}$) for the rescaled set $\{tx : x \in E\}$. If μ is a Radon measure on $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ is a μ-measurable set, we denote by $\mu\lfloor A$ the restriction of μ to A.

Throughout the paper, the following convention is used: if $u : \Omega \to \mathbb{R}$ is a function defined on $\Omega \subseteq \mathbb{R}^d$, we still denote by u the function $u : \mathbb{R}^d \to \mathbb{R}$ equal to u on Ω and extended by 0 on $\mathbb{R}^d \setminus \Omega$. This convention allows us to handle easily situations of varying domains carrying their own eigenfunctions and gradients of eigenfunctions. Whenever we discuss about a pointwise behaviour of a Sobolev H^1-function, we implicitly consider its quasi-continuous representative and if the function belongs to SBV we consider its approximate limits (if they are different, we denote them u^+ and u^-).

Sets of finite perimeter. We refer to [1, Chapter 3] for details about sets of finite perimeter. Below we only recall the main definition and some important properties.

Definition 4 (Sets of finite perimeter)**.** Let $E \subseteq \mathbb{R}^d$ be measurable with finite measure and let $\Omega \subseteq \mathbb{R}^d$ be open. The perimeter of E in Ω is

$$P(E, \Omega) := sup \left\{ \int_E \operatorname{div}(\varphi)\, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

It is said that E is of finite perimeter in Ω if $P(E,\Omega)<+\infty$. If $\Omega=\mathbb{R}^d$ we simply say that E is of finite perimeter and denote its perimeter by $P(E)$.

If E has finite perimeter, its reduced boundary is denoted $\partial^* E$ and consists $\mathcal{H}^{d-1}$-a.e. on points of density $\frac{1}{2}$. Then $P(E,\Omega)=\mathcal{H}^{d-1}(\partial^* E\cap\Omega)$ for every open set Ω. Moreover, the reduced boundary is countably $\mathcal{H}^{d-1}$-rectifiable, i.e. there exists a sequence $(M_n)_n$ of C^1 manifolds in $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial^* E\setminus\bigcup_n M_n)=0$.

The following properties hold.

Proposition 5. *Let $\Omega\subseteq\mathbb{R}^d$ be open and bounded and let $(E_n)_n$ be a sequence of measurable subsets of Ω such that $\sup_n P(E_n,\Omega)<+\infty$. Then, there exists a set of finite perimeter $E\subseteq\Omega$ such that, up to subsequences,*

$$\chi_{E_n}\to\chi_E\quad \textit{strongly in } L^1(\Omega)$$

and

$$P(E,\Omega)\le\liminf_{n\to+\infty}P(E_n,\Omega).$$

The following two results have been proved in [6, Proposition 2.3, Lemma 2.2].

Proposition 6. *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of sets of finite perimeter of $\mathbb{R}^d$ and let $E\subset\mathbb{R}^d$ such that*

$$\limsup_{n\to\infty}\mathcal{H}^{d-1}(\partial^* E_n)<+\infty\qquad \textit{and}\qquad \chi_{E_n}\xrightarrow{L^1(\mathbb{R}^d)}\chi_E.$$

Let $(u_n)_{n\in\mathbb{N}}\subset H^1(\mathbb{R}^d)$ and $u\in H^1(\mathbb{R}^d)$ such that $u_n\rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$. Then E has finite perimeter and

$$\int_{\partial^* E}u^2\,d\mathcal{H}^{d-1}\le\liminf_{n\to\infty}\int_{\partial^* E_n}u_n^2\,d\mathcal{H}^{d-1}.$$

Lemma 7 (Uniform relative isoperimetric inequality in annuli)**.** *Let $m>0$ be given. Then there exist two positive constants $c=c(d)$ and $w=w(m,d)$ such that, for every $r\ge 0$, $l\ge w$ and every measurable set $E\subseteq A_{r,r+l}(0)$ with $|E|\le m$, we have*

$$|E|^{\frac{d-1}{d}}\le cP(E,A_{r,r+l}(0)).$$

The argument of Colbois, Girouard and El Soufi to obtain isoperimetric control of the Steklov spectrum [14] used a crucial tool due to Grigor'yan, Netrusov and Yau, [16, Corollary 3.12]. We recall it below, using the following notation: if A is the annulus $A_{r,R}(x)$, we will denote with $2{\cdot}A$ the annulus $A_{\frac{r}{2},2R}(x)$.

Lemma 8. *Let ν be a finite, non negative, non atomic Radon measure on $\mathbb{R}^d$. Then, for every $k\in\mathbb{N}$, there exist a family $\mathcal{A}$ of k annuli in $\mathbb{R}^d$ such that*

(a) there exists a positive dimensional constant γ_d for which

$$\nu(A)\ge\gamma_d\frac{\nu(\mathbb{R}^d)}{k};$$

(b) the annuli $\{2{\cdot}A\}_{A\in\mathcal{A}}$ are disjoint.

Hausdorff convergence. The Hausdorff distance on the family of nonempty compact subsets of $\mathbb{R}^d$ is defined by

$$d_H(K_1,K_2):=\max\left\{\sup_{x\in K_1}\operatorname{dist}(x,K_2),\sup_{y\in K_2}\operatorname{dist}(y,K_1)\right\}. \tag{5}$$

The following compactness property holds: if F is a compact set of $\mathbb{R}^d$ and $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets contained in F, there exists a compact set $K \subseteq F$ such that up to a subsequence

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Moreover, if K_n are connected then K is also connected.

Eigenvalues of the Robin Laplacian. For a recent survey on Robin eigenvalues, we refer to [8] (see also [15]). Assume $\Omega \subset \mathbb{R}^d$ open, bounded and Lipschitz. Then

$$\lambda_{1,\beta}(\Omega) \leq -\beta \frac{\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|}$$

is negative while $\lambda_{k,\beta}(\Omega)$ for $k \geq 2$ may be positive or negative. Of course, for a fixed Ω, starting from some index, all eigenvalues will be positive. If Ω is the union of two connected components, Ω_1, Ω_2 the eigenvalues of Ω is the union of the eigenvalues of Ω_1, Ω_2, multiplicity being counted. Assume that Ω is the union of p connected components. Then, $\lambda_{p,\beta}(\Omega) < 0$. Taking, for instance, Ω to be the union of n disjoint balls of measure $\frac{m}{n}$, we notice that for every $k \leq n$

$$\lambda_{k,\beta}(\Omega) \leq -\beta C_d \Big(\frac{m}{n}\Big)^{\text{-1/d}},$$

where C_d is a dimensional constant. Making $n \to +\infty$, the infimum of $\lambda_{k,\beta}(\Omega)$ in the class of sets with prescribed measure is $-\infty$.

The function $\beta \mapsto \lambda_{k,\beta}(\Omega)$ is decreasing on $]0, +\infty[$ and for every $t > 0$, we have

$$\lambda_{k,\beta}(t\Omega) = \frac{1}{t^2}\lambda_{k,t\beta}(\Omega). \tag{6}$$

The Robin eigenvalues do not behave monotonically under dilations, unlike the Robin problem with positive boundary parameter (see [8] for details). As well, there is no monotonicity under inclusions in general.

In the spirit of [6] we define the relaxed eigenvalues associated to a set with finite perimeter.

Definition 9. Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite perimeter. We denote

$$\tilde{\lambda}_{k,\beta}(\Omega) := \inf_{S\in\mathcal{S}_k} \max_{u\in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx - \beta\int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\,dx},$$

where $\mathcal{S}_k$ denotes the family of all k-dimensional subspaces of $H^1(\mathbb{R}^d)|_\Omega \subseteq L^2(\Omega)$ and call it the k-th relaxed eigenvalue associated to Ω.

We have the following.

Proposition 10 (properties of the relaxed eigenvalues)**.** *Let $\Omega \subset \mathbb{R}^d$ be a set of finite perimeter and $\beta > 0$ be fixed.*

(a) If Ω is open, bounded and Lipschitz, then for every $k \in \mathbb{N}$ it holds

$$\tilde{\lambda}_{k,\beta}(\Omega) = \lambda_{k,\beta}(\Omega)$$

(classical setting).

(b) For every $k \in \mathbb{N}$ and for every $t > 0$ one has

$$\tilde{\lambda}_{k,\beta}(t\Omega) = \frac{1}{t^2}\tilde{\lambda}_{k,t\beta}(\Omega)$$

(scaling property).

(c) *For every Ω of finite perimeter and for every $\beta > 0$ one has*

$$\tilde{\lambda}_{1,\beta}(\Omega) \leq -\beta \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|} < 0$$

(strict negativity of the first relaxed eigenvalue).

(d) *For every Ω of finite perimeter given by a union of $N \geq k$ sets, pairwise at positive distance, (i.e. well separated sets, see Definition 12) with positive Lebesgue measure and for every $\beta > 0$ one has*

$$\tilde{\lambda}_{k,\beta}(\Omega) < -\frac{\beta}{\omega_d}\left(\frac{N-k}{|\Omega|}\right)^{1/d} \leq 0.$$

Proof. Item (a) is a consequence of the choice of the admissible subspaces for the computation of $\tilde{\lambda}_{k,\beta}(\Omega)$; items (b) and (c) follow by the same computations made in the classical setting in (6), replacing the topological boundary with the reduced one. To prove item (d), we choose k well separated sets each one with measure not larger than $|\Omega|/(N-k)$ (if $N = k$ we just take the k sets) and consider the test space for $\lambda_{k,\beta}$ spanned by their k characteristic functions. □

3. Isoperimetric control of the spectrum: proof of Theorem 1

In this section we prove Theorem 1 in a more general setting, namely for measurable sets. We follow the strategy employed by Girouard, Colbois end El Soufi, the new difficulties coming from the presence of the L^2 norms, which do not play any role in the Steklov spectrum. We rephrase Theorem 1 as follows.

Proposition 11. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite perimeter. Then, there exist two positive constants C_1, C_2 depending only on the dimension of the space, such that*

$$\tilde{\lambda}_{k,\beta}(\Omega) \leq -\frac{C_2}{|\Omega|}\left(k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1}\mathcal{H}^{d-1}(\partial^*\Omega)\right)^- + \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}\left(k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1}\mathcal{H}^{d-1}(\partial^*\Omega)\right)^+$$

This inequality can be interpreted in the following way: there exists two positive constants C_1 and C_2, depending only on d, such that if

$$\mathcal{H}^{d-1}(\partial^*\Omega) > \frac{C_1}{\beta}k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}},$$

then

$$\tilde{\lambda}_{k,\beta}(\Omega) \leq \frac{C_2 k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta\mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|},$$

otherwise, if

$$\mathcal{H}^{d-1}(\partial^*\Omega) \leq \frac{C_1}{\beta}k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}},$$

then

$$\tilde{\lambda}_{k,\beta}(\Omega) \leq \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}.$$

In other words, if $\mathcal{H}^{d-1}(\partial^*\Omega)$ is very large, then $\tilde{\lambda}_{k,\beta}(\Omega)$ becomes negative and very small.

Proof of Proposition 11. We apply Lemma 8 to the finite non atomic measure $\nu := \mathcal{H}^{d-1}\lfloor\partial^*\Omega$: there exist $2k$ annuli $A_1, \dots, A_{2k}$ in $\mathbb{R}^d$ and a positive constant γ_d depending only on the d such that, for every $i = 1, \dots, 2k$

$$\mathcal{H}^{d-1}(\partial^*\Omega \cap A_i) \geq \gamma_d \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k} \tag{7}$$

and, if $i \neq j$,

$$2{\cdot}A_i \cap 2{\cdot}A_j = \emptyset. \tag{8}$$

In particular, we can order the $2k$ annuli in such a way that

$$|\Omega \cap 2{\cdot}A_i| \leq \frac{|\Omega|}{k}. \tag{9}$$

Now let us write each annulus as

$$A_1 = \{r_{1,i} < |x - x_i| < r_{2,i}\}$$

and consider the functions h_i defined as follows

$$h_i(x) := \begin{cases} \dfrac{1}{r_{2,i}}\mathrm{dist}(x, \mathbb{R}^d \setminus 2{\cdot}A_i) & \text{if } x \notin B_{r_{2,i}}(x_i), \\ 1 & \text{if } x \in A_i, \\ \dfrac{1}{r_{1,i}}\mathrm{dist}(x, \mathbb{R}^d \setminus 2{\cdot}A_i) & \text{if } x \in B_{r_{1,i}}(x_i). \end{cases} \tag{10}$$

Observe that $h_i \in H^1(\mathbb{R}^d)$, $h_i = 0$ on $\mathbb{R}^d \setminus 2{\cdot}A_i$ and

$$|\nabla h_i| = \begin{cases} \dfrac{1}{r_{2,i}} & \text{in } B_{2r_{2,i}}(x_i) \setminus B_{r_{2,i}}(x_i), \\ 0 & \text{in } A_i \cup (2{\cdot}A_i)^c, \\ \dfrac{1}{r_{1,i}} & \text{in } B_{r_{1,i}}(x_i) \setminus B_{\frac{r_{1,i}}{2}}(x_i). \end{cases}$$

Denoting by $R(u)$ the Rayleigh coefficient for any admissible function u, we estimate for every $i = 1, \dots, k$ the quantity $R(h_i)$ as follows:

$$\begin{aligned} R(h_i) &= \frac{\displaystyle\int_\Omega |\nabla h_i|^2\,dx - \beta \int_{\partial^*\Omega} h_i^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega h_i^2\,dx} \\ &\leq \frac{\left(\displaystyle\int_{2A_i} |\nabla h_i|^d\,dx\right)^{\frac{2}{d}} |\Omega \cap 2{\cdot}A_i|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap A_i)}{\displaystyle\int_\Omega h_i^2\,dx}, \end{aligned} \tag{11}$$

where we used the Hölder inequality. Observe that the quantity

$$\left(\int_{2A_i} |\nabla h_i|^d\,dx\right)^{\frac{2}{d}}$$

is a positive constant depending only on d. So, using estimates (7) and (8), we obtain by (11)

$$R(h_i) \leq \frac{\left(\int_{2A_i} |\nabla h_i|^d \, dx\right)^{\frac{2}{d}} \left(\frac{|\Omega|}{k}\right)^{\frac{d-2}{d}} - \beta\gamma_d \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{2k}}{\int_\Omega h_i^2 \, dx}. \tag{12}$$

Let us consider now the positive constants

$$C_2 := \left(\int_{2A_i} |\nabla h_i|^d \, dx\right)^{\frac{2}{d}}, C_1 := \frac{2C_2}{\gamma_d},$$

which depend only on the dimension d. Observe that, if

$$\mathcal{H}^{d-1}(\partial^*\Omega) > \frac{C_1}{\beta} k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}},$$

then the numerator of the right hand side of (12) is negative, then we can continue the estimate using again (9):

$$\begin{aligned} R(h_i) &\leq \frac{C_2 \left(\frac{|\Omega|}{k}\right)^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{k}}{|\Omega \cap 2{\cdot}A_i|} \\ &\leq \frac{C_2 \left(\frac{|\Omega|}{k}\right)^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta \frac{\mathcal{H}^{d-1}(\partial^*\Omega)}{k}}{\frac{|\Omega|}{k}} \qquad (13) \\ &\leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta \mathcal{H}^{d-1}(\partial^*\Omega)}{|\Omega|}. \end{aligned}$$

Since $h_1, \dots, h_k$ have disjoint supports in view of (8), we achieve the thesis in the first case by considering the space $S_k := \operatorname{span}\{h_1, \dots, h_k\}$.

Now, let $\mathcal{H}^{d-1}(\partial^*\Omega) \leq \frac{C_1}{\beta} k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}}$. In view of Lemma 8 applied to the finite non atomic measure $\mathcal{L}^d \lfloor \Omega$, there exists a family of k annuli $\tilde{A}_1, \dots, \tilde{A}_k$ such that

$$|\Omega \cap \tilde{A}_i| \geq \gamma_d \frac{|\Omega|}{k} \tag{14}$$

and, if $i \neq j$,

$$2{\cdot}\tilde{A}_i \cap 2{\cdot}\tilde{A}_j = \emptyset. \tag{15}$$

Consider the test functions $l_1, \dots, l_k$ for the annuli $\tilde{A}_1, \dots, \tilde{A}_k$, constructed in the same way as the h_i in (10), and observe that their supports are disjoint, thanks to (15). Now, let us estimate $R(l_i)$:

$$R(l_i) = \frac{\displaystyle\int_\Omega |\nabla l_i|^2\, dx - \beta \int_{\partial^*\Omega} l_i^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega l_i^2\, dx} \le \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap \tilde{A}_i)}{\displaystyle\int_\Omega l_i^2\, dx}. \tag{16}$$

If the numerator of the last fraction is negative, then $R(l_i) < 0$. Otherwise we can estimate $R(l_i)$ using (14):

$$R(l_i) \le \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap \tilde{A}_i)}{|\Omega \cap \tilde{A}_i|} \le \frac{C_2|\Omega|^{\frac{d-2}{d}} - \beta\mathcal{H}^{d-1}(\partial^*\Omega \cap \tilde{A}_i)}{\gamma_d \frac{|\Omega|}{k}} \le \frac{C_2 k}{\gamma_d |\Omega|^{\frac{2}{d}}} = \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}.$$

In both cases $R(l_i) \le \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}}$; reasoning as above on span $\{l_1, \ldots, l_k\}$, we complete the proof. □

We conclude this section with the following.

Proof of Theorem 1. This is a consequence of Proposition 11. □

4. Isodiametric control of the spectrum: proof of Theorem 2

We shall prove Theorem 2 in the context of measurable sets. For that reason, we start with the following.

Definition 12 (Well separated sets)**.** Let $A, B \subseteq \mathbb{R}^d$. We say that A and B are well separated if there exist two open sets E_A, E_B such that, up to negligible sets, $A \subseteq E_A$, $B \subseteq E_B$ and $\operatorname{dist}(E_A, E_B) > 0$.

Let $A > 0$. Below, we denote $c = c(d), w = w(m^*, d)$ the constants from Lemma 7, for $m^* = \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$. As well, by $A_{r,r+l}$ we denote the annulus $A_{r,r+l}(0)$ centered at the origin.

Lemma 13. *Then, there exists $L = L(d, \beta, A) > w$ such that, for every $r \ge 0$, $l \ge L$ and for every measurable set $E \subseteq A_{r,r+l}(0)$ with finite perimeter and with $|E| \le m^*$, at least one of the following possibilities occurs.*

(a) There exists $\varphi \in H^1_0(A_{r,r+l})$ such that

$$\int_{\partial^* E} \varphi^2\, d\mathcal{H}^{d-1} > 0, \qquad \int_E \varphi^2\, dx > 0$$

and

$$\frac{\displaystyle\int_E |\nabla\varphi|^2\, dx - \beta \int_{\partial^* E} \varphi^2\, d\mathcal{H}^{d-1}}{\displaystyle\int_E \varphi^2\, dx} \le -A.$$

(b) We have

$$\left| E \cap A_{r+\frac{l-w}{2}, r+\frac{l+w}{2}} \right| = 0,$$

i.e., up to negligible sets, E lies outside an annulus of width w.

Proof. Let $L>0$ such that

$$\frac{L-w}{2}>\sqrt{\frac{2c}{\beta}}(m^*)^{\frac{1}{2d}}\sum_{j=1}^{\infty}\frac{1}{\left(2^{\frac{1}{2d}}\right)^k}. \tag{17}$$

Let $r\geq 0$, $l\geq L$ and $E\subseteq A_{r,r+l}$ measurable, with finite perimeter and such that $|E|\leq m^*$. Moreover, let us assume that $|E|>0$, otherwise situation (b) would occur trivially.

Let us suppose that assertion (a) does not hold and let us show that situation (b) occurs. Let us consider the functions m_1 and p_1 defined, for every $t\in\left[0,\frac{l-w}{2}\right]$, by

$$m_1(t):=|E\cap(A_{r,r+t}\cup A_{r+l-t,r+l})|$$

and

$$p_1(t):=P\left(E,A_{r+t,r+l-t}\right).$$

If $p_1(t)=0$ for some $t\in\left[0,\frac{l-w}{2}\right]$, situation (b) takes place trivially. Then, we can assume $p_1(t)>0$ and consider, for every $\frac{l-w}{2}\geq t>0$, the function $\varphi_{1,t}\in H^1_0\left(A_{r,r+l}\right)$ defined by

$$\varphi_{1,t}(x):=\left[\frac{1}{t}\mathrm{dist}\left(x,A^c_{r,r+l}\right)\right]\wedge 1.$$

Notice that $\int_E\varphi^2_{1,t}\,dx>0$ ($|E|>0$ and $\varphi_{1,t}$ is not the zero function) and that

$$\int_{\partial^*E}\varphi^2_{1,t}\,d\mathcal{H}^{d-1}\geq p_1(t)>0$$

for any t. Then, since situation (a) cannot occur, we have that

$$-A<\frac{\displaystyle\int_E|\nabla\varphi_{1,t}|^2\,dx-\beta\int_{\partial^*E}\varphi^2_{1,t}\,d\mathcal{H}^{d-1}}{\displaystyle\int_E\varphi^2_{1,t}\,dx}\leq\frac{\displaystyle\frac{1}{t^2}m_1(t)-\beta p_1(t)}{\displaystyle\int_E\varphi^2_{1,t}\,dx}.$$

Observe that there exists $t_1\in\left]0,\frac{l-w}{2}\right]$ such that $0<m(t_1)=\frac{|E|}{2}$. We claim that $t_1<\frac{l-w}{2}$. To prove this fact, consider $\varphi_1:=\varphi_{1,t_1}$; we obtain

$$-A<\frac{\displaystyle\frac{1}{t_1^2}\frac{|E|}{2}-\beta p_1(t_1)}{\displaystyle\int_E\varphi_1^2\,dx}.$$

If the numerator is nonnegative we have

$$\frac{1}{t_1^2}\frac{|E|}{2}\geq\beta p_1(t_1)\geq\frac{\beta}{c}\left(\frac{|E|}{2}\right)^{\frac{d-1}{d}},$$

where we used the uniform relative isoperimetric inequality in annuli and the fact that both $E\cap(A_{r,r+t_1}\cup A_{r+l-t_1,r+l})$ and $E\cap A_{r+t_1,r+l-t_1}$ have measure $\frac{|E|}{2}$. Last inequality yields the estimate

$$t_1\leq\sqrt{\frac{c}{\beta}}|E|^{\frac{1}{2d}}\frac{1}{2^{\frac{1}{2d}}}<\sqrt{\frac{2c}{\beta}}(m^*)^{\frac{1}{2d}}\frac{1}{2^{\frac{1}{2d}}}<\frac{l-w}{2}.$$

On the other hand, if $\frac{1}{t_1^2}\frac{|E|}{2}-\beta p_1(t_1)<0$, it holds

$$-A<\frac{\frac{1}{t_1^2}\frac{|E|}{2}-\beta p_1(t_1)}{m^*}\leq\frac{\frac{1}{t_1^2}\frac{m^*}{2}-\frac{\beta}{c}\left(\frac{m^*}{2}\right)^{\frac{d-1}{d}}}{m^*}.$$

By an easy computation we have

$$\frac{1}{t_1^2}\frac{m^*}{2} - \frac{\beta}{2c}\left(\frac{m^*}{2}\right)^{\frac{d-1}{d}} > \frac{\beta}{2c}\left(\frac{m^*}{2}\right)^{\frac{d-1}{d}} - Am^* \geq 0,$$

since $m^* \leq \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$. Thus we obtain the estimate

$$t_1 < \sqrt{\frac{2c}{\beta}}(m^*)^{\frac{1}{2d}}\frac{1}{2^{\frac{1}{2d}}} < \frac{l-w}{2}$$

and this completely proves the claim on t_1.

We now proceed as above, reasoning on the annulus $A_{r+t_1,r+l-t_1}$ and the set $E\cap A_{r+t_1,r+l-t_1}$, whose measure is $\frac{|E|}{2}$. For every $t \in \left[0, \frac{l-w}{2} - t_1\right]$ we define the quantities

$$m_2(t) := |E \cap (A_{r+t_1,r+t_1+t}(0) \cup A_{r+l-t_1-t,r+l-t_1})|$$

and

$$p_2(t) := P\left(E, A_{r+t_1+t,r+l-t_1-t}\right).$$

As a test function, we consider

$$\varphi_{2,t}(x) := \left[\frac{1}{t}\text{dist}\left(x, A^c_{r+t_1,r+l-t_1}\right)\right] \wedge 1.$$

Using the same arguments as above, we can find $t_2 \in \left]0, \frac{l-w}{2} - t_1\right[$ such that

$$|E \cap (A_{r+t_1,r+t_1+t_2} \cup A_{r+l-t_1-t_2,r+l-t_1})| = |E \cap A_{r+t_1,r+l-t_1}| = \frac{|E|}{4}$$

and such that the following estimate is satisied:

$$t_2 < \sqrt{\frac{2c}{\beta}}(m^*)^{\frac{1}{2d}}\frac{1}{\left(2^{\frac{1}{2d}}\right)^2} < \frac{l-w}{2} - t_1.$$

Thanks to the choice of L, we can carry out the argument infinitely many times, obtaining a sequence $(t_n)_n$ such that $\sum_n t_n \leq \frac{l-w}{2}$ and

$$|E \cap A_{r+t_1+\dots+t_n,r+l-t_1-\dots-t_n}| = \frac{|E|}{2^n}.$$

Consequently

$$|E \cap A_{r+\frac{l-w}{2},r+\frac{l+w}{2}}| = |E|,$$

which implies that situation (b) occurs, completing the proof. □

Proposition 14. *Let $m, A > 0$. There exists a constant $D = D(m, \beta, d, k, A)$ such that if $\Omega \subset \mathbb{R}^d$ is a set of finite perimeter, $|\Omega| = m$ with $\tilde{\lambda}_{k,\beta}(\Omega) > -A$ then Ω can be decomposed as a union of N well separated and bounded sets of finite perimeter*

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_N,$$

where $N < \frac{mA^d\omega^d}{\beta^d} + k$ and diam$(\Omega_j) \leq D(m, \beta, d, k, A)$.

Proof. Without loss of generality, we can consider the origin as a point of density one for Ω. Let $c = c(d) > 0$ be the dimensional constant in the relative isoperimetric inequality for annuli and set $m^* := \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$. Let us consider the constants $w = w(m^*, d)$ and $L = L(d, \beta, A)$ of Lemma 13 define the annuli centered at the origin

$$A_j := A_{jL,(j+1)L}(0) := \left\{x \in \mathbb{R}^d : jL < |x| < (j+1)L\right\}$$

for $j = 0, \dots, \left\lfloor \frac{m}{m^*} \right\rfloor + k$. By construction, there exist k of this annuli, say $A_{n_1}, \dots, A_{n_k}$, such that $|\Omega \cap A_{n_h}| \leq m^*$. For each $h = 1, \dots, k$, let us apply Lemma 13 to each set $\Omega \cap A_{n_h}$: either there exist $\varphi_h \in H^1_0(A_{n_h})$ such that

$$\int_{\partial^*(\Omega\cap A_{n_h})} \varphi_h^2 \, d\mathcal{H}^{d-1} > 0, \quad \int_{\Omega\cap A_{n_h}} \varphi_h^2 \, dx > 0$$

and

$$\frac{\displaystyle\int_{\Omega\cap A_{n_h}} |\nabla\varphi_h|^2 \, dx - \beta \int_{\partial^*(\Omega\cap A_{n_h})} \varphi_h^2 \, d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega\cap A_{n_h}} \varphi_h^2 \, dx} \leq -A$$

or

$$\left|\Omega \cap A_{n_h} \cap A_{n_hL+\frac{n_hL-w}{2}, n_hL+\frac{n_hL+w}{2}}\right| = 0.$$

Observe that there exists one of the sets $\Omega \cap A_{n_h}$ in which the first case is not satisfied otherwise, building k test functions with mutually disjoint support, we trivially get $\tilde{\lambda}_{k,\beta}(\Omega) \leq -A$, in contradiction with the hypoteses.

Let now be $p \in \{1, \dots, k\}$ such that A_{n_p} does not satisfy the first alternative and let us set

$$\Omega_1 := \Omega \cap B_{n_pL+\frac{L}{2}}(0).$$

Observe that Ω_1 is bounded, with finite perimeter and well separated from $\Omega \setminus \Omega_1$, with $\mathrm{dist}(\Omega_1, \Omega \setminus \Omega_1) \geq w$. Moreover

$$\mathrm{diam}(\Omega_1) \leq 2\left(\left\lfloor \frac{m}{m^*} \right\rfloor + k + 1\right) L.$$

Now consider the set $\Omega \setminus \Omega_1$, whose measure is less than m, translate the set in such a way that the origin is a point of density one for $\Omega \setminus \Omega_1$ and repeat the same arguments as above to build Ω_2; moreover, we find the same bound on the diameter. The argument could be carried on to build the following well separated parts of Ω, whose diameters are still uniformly bounded by $2\left(\left\lfloor \frac{m}{m^*} \right\rfloor + k + 1\right) L$. Notice that we cannot repeat the argument too many times. In fact, for each set Ω_n, the first eigenvalue is less than $-\frac{\beta}{\omega_d}|\Omega_n|^{-\frac{1}{d}}$ so as soon as the measure of Ω_n becomes lower than a threshold for k sets, we get that $\tilde{\lambda}_{k,\beta}(\Omega) \leq -A$.

Assume indeed that $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ with $N > k$. Then, there exists k among them having the volume not larger than $\frac{m}{N-k}$. Consequently, by Proposition 10, item (d),

$$-A < \tilde{\lambda}_{k,\beta}(\Omega) \leq -\frac{\beta}{\omega_d}\left(\frac{m}{N-k}\right)^{-\frac{1}{d}}$$

and this yields

$$N < \frac{mA^d\omega^d}{\beta^d} + k,$$

completing the proof. □

Observe that as A decreases to 0, the bound on the number of well separated components decreases to k: this fact suggests that, if $\tilde{\lambda}_{k,\beta}(\Omega)$ is nonnegative, then Ω is union of at most k well separated parts (up to negligible sets).

We give below a slightly more general version than Theorem 2. Given $m, A > 0$, below we denote $C_1 = C_1(d), C_2 = C_2(d) > 0$ the dimensional constants in Lemma 11, $c = c(d)$ from Lemma 7, $m^* = \left(\frac{\beta}{Ac}\right)^d 2^{1-2d}$, and $L = L(d, \beta, A)$ from Lemma 13.

Proposition 15. *Let $\Omega \subset \mathbb{R}^d$ an open, bounded, Lipschitz and connected set of measure m. If*

$$\mathrm{diam}(\Omega) > 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L,$$

then

$$\lambda_{k,\beta}(\Omega) \leq -A. \tag{18}$$

Otherwise,

(19)
$$\begin{aligned}\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) \leq \omega\left(\frac{2C_2}{(\omega m)^{1/d}} k^{2/d} - \frac{2C_2\beta d}{C_1}\right)\chi_{]0,+\infty[}\left(\mathcal{H}^{d-1}(\partial\Omega) - \frac{C_1}{\beta} m^{\frac{d-2}{d}} k^{2/d}\right) \\ + \frac{C_1 L}{m^{2/d}}\left[2k^2 + \left(\frac{2^{2d}c^d m + 2\beta^d}{\beta^d}\right)k\right]\left[1 - \chi_{]0,+\infty[}\left(\mathcal{H}^{d-1}(\partial\Omega) - \frac{C_1}{\beta} m^{\frac{d-2}{d}} k^{2/d}\right)\right].\end{aligned}$$

Proof. Without loss of generality, let us consider the origin as a point of density 1 for Ω. Let us suppose that

$$\mathrm{diam}(\Omega) > 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L$$

and define the $k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1$ concentric annuli of width L

$$A_j := A_{jL,(j+1)L}(0) = \left\{x \in \mathbb{R}^d : jL < |x| < (j+1)L\right\}, \quad j = 0, \dots, \left\lfloor \frac{m}{m^*} \right\rfloor + k.$$

By construction, there exist k of this annuli, say $A_{n_1}, \dots, A_{n_k}$, such that $|\Omega \cap A_{n_h}| \leq m^*$. For each $h = 1, \dots, k$, let us apply Lemma 13 to each set $\Omega \cap A_{n_h}$ and observe that, as Ω is connected, in each annulus the first alternative takes place. Then,

$$\lambda_{k,\beta}(\Omega) \leq -A.$$

Let now be

$$\mathrm{diam}(\Omega) \leq 2\left(k + \left\lfloor \frac{m}{m^*} \right\rfloor + 1\right) L$$

and suppose, in view Lemma 11, that

$$\mathcal{H}^{d-1}(\partial\Omega) < \frac{C_1}{\beta} m^{\frac{d-2}{d}} k^{2/d}.$$

Then,

$$\lambda_{k,\beta}(\Omega) \leq \frac{C_2 k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \dfrac{C_2}{C_1}\beta\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|} < 0,$$

i.e. $\lambda_{k,\beta}(\Omega)$ is necessarily strictly negative. Using the isoperimetric inequality and the isodiametric inequality

$$|\Omega| \leq \omega_d \left(\frac{\mathrm{diam}(\Omega)}{2}\right)^d$$

we obtain

$$\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) \le \frac{C_2 k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta\mathcal{H}^{d-1}(\partial\Omega)}{|\Omega|}\frac{2|\Omega|^{1/d}}{\omega^{1/d}}$$
$$\le \frac{C_2 k^{\frac{2}{d}}|\Omega|^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta d\omega^{1/d}|\Omega|^{\frac{d-1}{d}}}{|\Omega|^{\frac{d-1}{d}}}\frac{2}{\omega^{1/d}} = \frac{2C_2}{(\omega m)^{1/d}}k^{2/d} - \frac{2C_2\beta d}{C_1}.$$

Let us suppose now, again in view of Lemma 11 that,

$$\mathcal{H}^{d-1}(\partial\Omega) \ge \frac{C_1}{\beta}m^{\frac{d-2}{d}}k^{2/d}.$$

In that case, if $\lambda_{k,\beta}(\Omega) < 0$, we trivially have $\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) < 0$; otherwise, if $\lambda_{k,\beta}(\Omega) \ge 0$, using the upper bound on $\mathrm{diam}(\Omega)$ we have

$$\lambda_{k,\beta}(\Omega)\mathrm{diam}(\Omega) \le \frac{C_1 L}{m^{2/d}}\left[2k^2 + \left(\frac{2^{2d}c^d m + 2\beta^d}{\beta^d}\right)k\right],$$

getting (19) and completing the proof of the proposition. □

We can now conclude.

Proof of Theorem 2. This is a consequence of Proposition 15. □

5. Proof of Theorem 3

5.1. **Existence of optimal shapes in the class of measurable sets.** In this section we prove the first assertion of Theorem 3, in a more general form, in the spirit of the existence result for spectral problems with Dirichlet boundary conditions by Buttazzo and Dal Maso [11]. Precisely, given $F : \mathbb{R}^k \to \mathbb{R}$, non decreasing in each variable and upper semicontinuous, we consider the problem

$$\max\left\{F(\tilde\lambda_{1,\beta}(\Omega), \dots, \tilde\lambda_{k,\beta}(\Omega)) : \Omega \subset \mathbb{R}^d \text{ measurable with finite perimeter, } |\Omega| = m\right\}. \tag{20}$$

To avoid trivial situations, we assume some coercivity on F, namely that

$$\lim_{A\to+\infty} F(-A, \dots, -A) = -\infty.$$

We prove the following.

Proposition 16 (Existence of optimal domains)**.** *Problem* (20) *has at least a solution. Moreover, each optimal set is bounded and can be written as the union of a finite number of well separated sets of finite perimeter. The number of connected components does not exceed* $\frac{m(A^*)^d\omega^d}{\beta^d} + k$, *where* A^* *is such that* $F(-A^*, \dots, -A^*) < F(\lambda_{1,\beta}(B^*), \dots, \lambda_{k,\beta}(B^*))$, B^* *being a ball of measure* m.

Proof. Let $(\Omega_n)_n$ be a maximizing sequence for (20). In view of the coercivity of F, we observe that any admissible domain E such that $\tilde\lambda_{j,\beta}(E) \le -A^*$ for every $j = 1, \dots, k$ can not be optimal, for some value A^* large enough. Then, it is not restrictive to assume that $\tilde\lambda_{k,\beta}(\Omega_n) > -A^*$. Moreover, from Proposition 11 we have that either the perimeter of Ω_n is less than a constant (depending on m, d, β, k) or

$$-A^* \le \tilde\lambda_{k,\beta}(\Omega_n) \le \frac{C_2 k^{\frac{2}{d}} m^{\frac{d-2}{d}} - \frac{C_2}{C_1}\beta\mathcal{H}^{d-1}(\partial^*\Omega_n)}{m}.$$

Via a straightforward computation, we obtain that also in this second case it holds $\mathcal{H}^{d-1}(\partial^*\Omega_n) \leq C = C(m, d, \beta, k, A^*)$. Hence, we deduce that

$$\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\partial^*\Omega_n) < +\infty. \tag{21}$$

Thanks to Proposition 14, we can write

$$\Omega_n = \Omega_n^1 \cup \ldots \cup \Omega_n^{N_n}, \quad N_n \leq \frac{m(A^*)^d\omega^d}{\beta^d} + k,$$

with $\Omega_n^1, \ldots, \Omega_n^{N_n}$ equibounded and well separated. Up to translations, we can assume that the Ω_n are contained in a fixed ball of $\mathbb{R}^d$; so, by (21) and Proposition 5, we deduce that there exists $\Omega \subset \mathbb{R}^d$ of finite perimeter such that, up to subsequences, that $\chi_{\Omega_n} \to \chi_\Omega$ strongly in L^1, $|\Omega| = m$ and $P(\Omega) \leq \liminf_n P(\Omega_n)$.

Let us show that such admissible set Ω is a solution of the problem (20). We claim that

$$\limsup_{n\to\infty} \tilde{\lambda}_{h,\beta}(\Omega_n) \leq \tilde{\lambda}_{h,\beta}(\Omega). \tag{22}$$

To prove it, let us fix $\varepsilon > 0$ and let $V_h := \operatorname{span}\{u_1, \ldots, u_h\} \subset H^1(\mathbb{R}^d)$ be an admissible subspace for the computation of $\tilde{\lambda}_{h,\beta}(\Omega)$ such that

$$\tilde{\lambda}_{h,\beta}(\Omega) \leq \max_{u\in V_h\setminus\{0\}} R(u) - \varepsilon.$$

For each $n \in \mathbb{N}$, let

$$u_n := \sum_{j=1}^h \alpha_j^n u_j \text{ with } \sum_{j=1}^h (\alpha_j^n)^2 = 1$$

attaining the maximum

$$\max_{v\in V_h\setminus\{0\}} \frac{\displaystyle\int_{\Omega_n} |\nabla v|^2\,dx - \beta \int_{\partial^*\Omega_n} v^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega_n} v^2\,dx}.$$

So, up to subsequences, there exist $\alpha_1, \ldots, \alpha_h$ such that α_j^n converges to α_j as n goes to infinity, with $\sum_{j=1}^h \alpha_j^2 = 1$. Hence, set

$$u := \sum_{j=1}^h \alpha_j u_j,$$

we have that $u_n \to u$ strongly in $H^1(\mathbb{R}^d)$. By the covergence of χ_{Ω_n} to χ_Ω and Proposition 6 we obtain that

$$\begin{gathered}\lim_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2\,dx = \int_\Omega |\nabla u|^2\,dx \\ \lim_{n\to\infty} \int_{\Omega_n} u_n^2\,dx = \int_\Omega u^2\,dx \\ \liminf_{n\to\infty} \int_{\partial^*\Omega_n} u_n^2\,d\mathcal{H}^{d-1} \geq \int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}.\end{gathered} \tag{23}$$

We finally have

$$\limsup_{n\to\infty} \tilde\lambda_{h,\beta}(\Omega_n) \le \limsup_{n\to\infty} \frac{\displaystyle\int_{\Omega_n} |\nabla u_n|^2\,dx - \beta\int_{\partial^*\Omega_n} u_n^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega_n} u_n^2\,dx}$$
$$\le \frac{\displaystyle\int_{\Omega} |\nabla u|^2\,dx - \beta\int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_{\Omega} u^2\,dx} \le \tilde\lambda_{h,\beta}(\Omega)+\varepsilon.$$

Letting $\varepsilon \to 0$ we get the required upper semicontinuity (22).

Thanks to the monotonicity and upper semicontinuity assumptions on F we deduce that

$$F(\tilde\lambda_{1,\beta}(\Omega),\dots,\tilde\lambda_{k,\beta}(\Omega)) \ge \limsup_{n\to\infty} F(\tilde\lambda_{1,\beta}(\Omega_n),\dots,\tilde\lambda_{k,\beta}(\Omega_n))$$
$$= \sup_{\substack{|E|=m\\ \mathcal{H}^{d-1}(\partial^* E)<+\infty}} F(\tilde\lambda_{1,\beta}(E),\dots,\tilde\lambda_{k,\beta}(E)),$$

concluding the proof. □

Proof of Theorem 3, assertion 1. Proposition 16 applies to $F(\tilde\lambda_{1,\beta}(\Omega),\dots,\tilde\lambda_{k,\beta}(\Omega)) = \tilde\lambda_{k,\beta}(\Omega)$. □

5.2. **The two dimensional case: open simply connected sets.** A natural setting in the case of the Robin eigenvalues with negative boundary parameter, is to perform the optimization problem the class of open sets which are simply connected in $\mathbb{R}^2$. In fact, as we deal with higher order eigenvalues, the natural framework is to work in the class of unions of mutually disjoint, open simply connected sets. Precisely, let us denote

$$\mathcal{O}(\mathbb{R}^2) := \{\Omega \subseteq \mathbb{R}^2 : \Omega \text{ open}, \Omega^c \text{ connected}, \mathcal{H}^1(\partial\Omega) < +\infty\}.$$

A set $\Omega \in \mathcal{O}(\mathbb{R}^2)$ can be decomposed on an at most countable union of mutually disjoint, open simply connected sets. Each such a set has a connected boundary with finite length, so that $\partial\Omega$ is rectifiable. We introduce below the associated relaxed eigenvalues.

Definition 17 (Relaxed eigenvalues)**.** Let $\Omega \in \mathcal{O}(\mathbb{R}^2)$. For every $k \in \mathbb{N}$ we set

$$\bar\lambda_{k,\beta}(\Omega) := \inf_{S\in\mathcal{S}_k} \sup_{u\in S\setminus\{0\}} \frac{\displaystyle\int_{\Omega} |\nabla u|^2\,dx - \beta\int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2\right]\,d\mathcal{H}^1}{\displaystyle\int_{\Omega} u^2\,dx}, \tag{24}$$

where $\mathcal{S}_k$ denotes the set of all k-dimensional subspaces of $H^1(\Omega)\cap L^\infty(\Omega)$.

Note that we use above the Sobolev space $H^1(\Omega)$ and not $H^1(\mathbb{R}^2)|_\Omega$. This allows to deal with domains having inner cracks.

The definition above is correct. Indeed, as $\partial\Omega$ is $\mathcal{H}^1$-rectifiable and $u \in H^1(\Omega)\cap L^\infty(\Omega)$, if we consider the extension by zero on $\mathbb{R}^2\setminus\Omega$ (still denoted by u), we obtain a function in $SBV(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ which has the jumps on $\partial\Omega$. In other words, we can speak about u^+ and u^-, the upper and lower approximate limits of u pointwisely $\mathcal{H}^1$-a.e on $\partial\Omega$. In general $u^+(x), u^-(x)$ are not necessarily different, the set $\partial\Omega$ being possibly larger than the jump set J_u of u. If Ω has a smooth boundary, then $\{u^+(x), u^-(x)\} = \{u(x), 0\}$ where $u(x)$ is the value of the trace of u from the inside at the point x of the boundary and 0 is the trace of the zero function from outside Ω. In this case we recover the classical setting

$$\int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2\right]\, d\mathcal{H}^1 = \int_{\partial\Omega} u^2\, d\mathcal{H}^1.$$

We refer the reader to [10] for details.

The problem we are going to study is the following:

$$\max\Big\{F(\bar\lambda_{1,\beta}(\Omega), \dots, \bar\lambda_{k,\beta}(\Omega)) : \Omega \in \mathcal{O}(\mathbb{R}^2),\ |\Omega| = m\Big\}, \tag{25}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ satisfies the hypotheses of Proposition 16.

Proposition 18. *Problem* (25) *admits at least a solution.*

Proof. The streamline of the proof is similar to the case of measurable sets, but there is a quite technical point concerning the behaviour of the boundary integral on moving sets, that we isolate at the end of the section, as a technical Lemma. This is due to the fact that the space of work, which is $H^1(\Omega)$, is moving with the geometric domain and the Sobolev functions, we deal with, may have two different traces on the same portion of the boundary of Ω, for instance when this is a crack. In such a case, there is no extension operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^2)$.

Let $(\Omega_n)_n$ be a maximizing sequence in $\mathcal{O}(\mathbb{R}^2)$. Similarly to the previous setting, the number of connected components is at most $N(A, k, \beta, m)$ and $\limsup \mathcal{H}^1(\partial\Omega_n) < +\infty$. In particular, the number of connected components of $\partial\Omega_n$ is uniformly bounded. The isoperimetric control follows in the same way as for measurable sets. Possibly translating some of the connected components of Ω_n, we can assume that all Ω_n are contained in a ball B_R. By compactness of the Hausdorff topology, there exists $\Omega \subset B_R$ such that, up to subsequences, $\overline{B}_R \setminus \Omega_n \to \overline{B}_R \setminus \Omega$ in the Hausdorff metric. Moreover, $\overline{B}_R \setminus \Omega$ is connected, so that $\Omega \in \mathcal{O}(\mathbb{R}^2)$ and $\partial\Omega$ is a subset of any Hausdorff limit of the sequence $\{\partial\Omega_n\}$. Consequently, by Golab's theorem we have that $\mathcal{H}^1(\partial\Omega) \le \liminf \mathcal{H}^1(\partial\Omega_n)$ and $|\Omega| = m$. We have to prove that for every $1 \le h \le k$ it holds

$$\bar\lambda_{h,\beta}(\Omega) \ge \limsup_{n\in\mathbb{N}} \bar\lambda_{h,\beta}(\Omega_n). \tag{26}$$

Following [7, Theorem 7.2.1], we have that $H^1(\Omega_n) \to H^1(\Omega)$ in the sense of Mosco, in particular, for every function $v \in H^1(\Omega)$ there exists a sequence $v_n \in H^1(\Omega_n)$ such that $v_n \to v$ in $L^2(\mathbb{R}^2)$ and $\nabla v_n \to \nabla v$ in $L^2(\mathbb{R}^2; \mathbb{R}^2)$ (here the convention of extension by 0 of functions outside their natural domain is in force).

Let now $\varepsilon > 0$ and let S an admissible vector space in the min-max formula for $\bar\lambda_{h,\beta}(\Omega)$ such that

$$\bar\lambda_{h,\beta}(\Omega) \ge \max_{u \in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx - \beta \int_{\partial\Omega} \left[(u^+)^2 + (u^-)^2\right]\, d\mathcal{H}^1}{\displaystyle\int_\Omega u^2\, dx} - \varepsilon. \tag{27}$$

Let $\{u_j : j = 1, \dots, h\}$ an $L^2(\Omega)$-orthonormal basis for S. Then, for every $j = 1, \dots, h$, there exists $v_j^n \in H^1(\Omega_n)$ such that $v_j^n \to u_j$ in $L^2(\mathbb{R}^2)$ and $\nabla v_j^n \to \nabla u_j$ in $L^2(\mathbb{R}^2; \mathbb{R}^2)$. Now, since $\{u_1, \dots, u_h\}$ is $L^2(\Omega)$-orthonormal and $\Omega_n \to \Omega$ in $L^1(\mathbb{R}^2)$, we deduce that $\{v_1^n, \dots, v_h^n\}$

can be choosen linearly independent in $L^2(\Omega_n)$, for $n \in \mathbb{N}$ sufficiently large. Let $S_n := \operatorname{span}\{v_1^n, \dots, v_h^n\}$; it is an admissible subspace for the computation of $\bar{\lambda}_{h,\beta}(\Omega)$. Let

$$v^n = \sum_{j=1}^h \alpha_j^n v_j^n \in S_n \text{ with } \sum_{j=1}^h (\alpha_j^n)^2 = 1$$

such that the associated Rayleigh quotient is maximal

$$\max_{w \in S_n} R(w) = R(v^n).$$

Then, up to subsequences, $\alpha_j^n \to \alpha_j$ in $\mathbb{R}$, with $\sum_{j=1}^h (\alpha_j)^2 = 1$. Setting

$$v = \sum_{j=1}^h \alpha_j u_j,$$

we have that $v \in S \setminus \{0\}$, $v^n \to v$ in $L^2(\mathbb{R}^2)$ and $\nabla v^n \to \nabla v$ in $L^2(\mathbb{R}^2; \mathbb{R}^2)$.

The following chain of inequalities

$$\begin{aligned}\limsup_{n\to+\infty} \bar{\lambda}_{h,\beta}(\Omega_n) &\le \limsup_{n\to+\infty} \sup_{w\in S_n} R(w) \le \limsup_{n\to+\infty} R(v^n) + \varepsilon \\ &\le R(v) + \varepsilon \le \max_{u \in S\setminus\{0\}} R(u) \le \bar{\lambda}_{h,\beta}(\Omega) + \varepsilon.\end{aligned}$$

holds true, as a consequence of the continuity of the volume integrals (from the choice of v_n^j via the Mosco convergence) and of the lower semicontinuity of the boundary integral :

$$\int_{\partial\Omega} \left[(v^+)^2 + (v^-)^2\right] \, d\mathcal{H}^1 \le \liminf \int_{\partial\Omega_n} \left[((v^n)^+)^2 + ((v^n)^-)^2\right] \, d\mathcal{H}^1. \tag{28}$$

This last point is neither a consequence of Proposition 6 nor of [6, Proposition 2.6], but is much related to the last one. As it is quite technical, we prove it separately, in Lemma 19 below.

Letting $\varepsilon \to 0^+$ we obtain that $\bar{\lambda}_{h,\beta}(\cdot)$ is upper semicontinuous for maximizing sequences, which are also compact; then reasoning as in Proposition 16, we conclude that (25) admits a solution. □

Proof of Theorem 3, assertion 2. This is a consequence of Proposition 18. □

We prove now the lower semicontinuity result (28), which is a more general version of [6, Proposition 2.6]. We only detail the differences with respect to [6, Proposition 2.6], many steps being identical.

Lemma 19. *Let $k \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $(K_n)_n, K \subseteq \Omega$ be compact sets with at most k connected components, such that $K_n \to K$ in the Hausdorff metric and $\limsup \mathcal{H}^1(K_n) < +\infty$. Let $u_n \in H^1(\Omega \setminus K_n)$ be such that*

$$\limsup \|u_n\|_{H^1(\Omega\setminus K_n)} + \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1 < +\infty. \tag{29}$$

Then, there exists $u \in H^1(\Omega \setminus K)$ such that for a subsequence (still denoted with the same index) we have

$$u_n \to u \text{ strongly in } L^2_{loc}(\Omega),$$

$$\nabla u_n \to \nabla u \text{ weakly in } L^2(\Omega, \mathbb{R}^2),$$

(30) $$\int_K \left[(u^+)^2 + (u^-)^2\right]\, d\mathcal{H}^1 \le \liminf_{n\to+\infty} \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right]\, d\mathcal{H}^1.$$

Proof. We can assume without restricting generality that $u_n \ge 0$ and $k = 1$. By weak compactness in $L^2(\Omega)$ and from the geometric properties of the Hausdorff convergence, we can build a function $u \in H^1(\mathbb{R}^2 \setminus K)$ such that

$$u_n \to u \text{ weakly in } L^2(\Omega),$$

$$\nabla u_n \to \nabla u \text{ weakly in } L^2(\Omega, \mathbb{R}^2).$$

As a consequence of (29) and of $\limsup \mathcal{H}^1(K_n) < +\infty$ we get that $u_n^2 \in SBV(\Omega)$ are bounded so can use [9, Theorem 3.3] to get that $u_n \to u$ strongly in $L^2_{loc}(\Omega)$. We point out, that Theorem 3.3 in [9] does not give (30), since the sets of integration K, K_n are in general larger than the jump sets of u^2, u_n^2, respectively.

It remains to prove (30). By truncation, we can also simplify and assume that there exists $M > 0$ such that for all $n \in \mathbb{N}$, $u_n \le M$ a.e. Moreover, we can assume that Ω is bounded and smooth (i.e. restrict our discussion to a smooth neighborhood of K) and, by standard cut off, that u_n are vanishing outside Ω.

We follow the ideas of the proof of [10, Proposition 2.6] where the result was proved for functions u_n which belong to $H^1(\mathbb{R}^2)$, without jumps on K_n. The proof is generally the same, with some adaptation of the slicing argument to the SBV context. Let us divide it in several steps.

Step 1. In view of the hypotheses on u_n, u, we deduce that $u_n, u, u_n^2, u^2 \in SBV(\Omega)$

$$J_{u_n^2} = J_{u_n} \subseteq K_n, J_{u^2} = J_u \subseteq K$$

and that the SBV-traces $u_n^\pm, u^\pm$ are well defined, as the functions are in L^∞ and K_n, K are $\mathcal{H}^1$-countably rectifiable. Let us define the sequence of positive Radon measures $(\mu_n)_n \subset \mathcal{M}_b(\Omega)$ by setting, for any Borel set $A \subseteq \Omega$

$$\mu_n(A) := \int_{A\cap K_n} \left[(u_n^+)^2 + (u_n^-)^2\right]\, d\mathcal{H}^1.$$

The sequence $(|\mu_n|(\Omega))_n$ is equibounded, so we can assume that there exists $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$. Let now be $\nu := \mathcal{H}^1 \lfloor K$. Since K is $\mathcal{H}^1$-countably rectifiable and $u \in SBV(\Omega)$, for $\mathcal{H}^1$-a.e. point $x \in K$, the following facts hold true:

(a) K admits an approximate tangent line l_x at x;
(b) x is either an approximate jump point or a an approximate continuity point for u;
(c) the Radon-Nikodym derivative $d\mu/d\nu$ is given by

$$\frac{d\mu}{d\nu}(x) = \lim_{\rho\to 0^+} \frac{\mu\left(\overline{Q_{2\rho}(x)}\right)}{\nu\left(\overline{Q_{2\rho}(x)}\right)}.$$

Let us prove that

(31) $$\text{for } \mathcal{H}^1\text{-a.e. } x \in K,\ \frac{d\mu}{d\nu}(x) \ge u^+(x)^2 + u^-(x)^2.$$

Indeed, if (31) holds, denoting by μ^a and μ^s respectively the absolutely continuous and the singular part of μ with respect to ν we have

$$\int_K \left[(u^+)^2 + (u^-)^2\right] \, d\mathcal{H}^1 \leq \int_K \frac{d\mu}{d\nu} \, d\mathcal{H}^1 = \mu^a(\Omega) \leq \mu^a(\Omega) + \mu^s(\Omega)$$
$$= \mu(\Omega) \leq \liminf_{n\to+\infty} \mu_n(\Omega) = \liminf_{n\to+\infty} \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1$$

i.e. the required lower semicontinuity in (30).

Step 2. Let $x \in K$ satisfying the previous properties (a), (b), (c). Without loss of generality, we can suppose $x = 0$ (we will prove (31) in the form $\frac{d\mu}{d\nu}(0) \geq u^+(0)^2 + u^-(0)^2$) and that the approximate tangent line to K at the point 0, say l, is horizontal (i.e. $l \subset \{x_2 = 0\}$). Moreover, after a possible rotation by π, we can assume that the approximate limit associated to the upper half plane is u^+. For every $\varepsilon > 0$, let us set

$$K_\varepsilon := \frac{1}{\varepsilon} K;$$

by the definition of approximate tangent line, we obtain that

$$\mathcal{H}^1 \lfloor K_\varepsilon \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l \tag{32}$$

weakly* in $\mathcal{M}_b(\Omega)$ as ε goes to 0. Then, for every $r > 0$

$$K_\varepsilon \cap \overline{Q_{2r}(0)} \to l \cap \overline{Q_{2r}(0)} \tag{33}$$

in the Hausdorff topology as ε goes to 0. This blow up argument is proved in [6, Proposition 2.6, Step 2].

Let us observe that

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap K)}{2\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap K)}{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap l)} \cdot \frac{\mathcal{H}^1(\overline{Q_{2\varepsilon}(0)} \cap l)}{2\varepsilon} = 1.$$

Then, for every positive, decreasing, infinitesimal sequence $(\varepsilon_m)_m$, it holds

$$\frac{d\mu}{d\nu}(0) = \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{\nu\left(\overline{Q_{2\varepsilon_m}(0)}\right)} = \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{\mathcal{H}^1\left(\overline{Q_{2\varepsilon_m}(0)} \cap K\right)}$$
$$= \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m}.$$

Moreover, by the weak* convergence $\mu_n \overset{*}{\rightharpoonup} \mu$ we have

$$\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right) \geq \limsup_{m\to+\infty} \mu_n\left(\overline{Q_{2\varepsilon_m}(0)}\right).$$

Hence, there exists a sequence of index $(n_m)_m$ such that

$$\varepsilon_m^2 + \mu\left(\overline{Q_{2\varepsilon_m}(0)}\right) \geq \mu_{n_m}\left(\overline{Q_{2\varepsilon_m}(0)}\right)$$

and that, setting

$$\hat{K}_m := \frac{1}{\varepsilon_m} K_{n_m} \cap \overline{Q_2(0)},$$

we have the convergence

$$\hat{K}_m \to l \cap \overline{Q_2(0)}$$

in the Hausdorff metric.

Let us define the function v_m by

$$v_m(y) := u_{n_m}(\varepsilon_m y) \quad (y \in Q_2(0)),$$

where n_m is possibly adjusted such that v_m has a uniformy bounded $H^1(Q_2(0)\backslash \hat{K}_m)$-norm and converges strongly in $L^2(Q_2(0))$. To perform this extraction we follow the same arguments as in [6, Proposition 2.6, Step 3].

It turns out that the L^2-limit of v_m equals $u^\pm$, where $u^\pm$ is the piecewise constant function given by

$$u^\pm(x_1, x_2) := \begin{cases} u^+(0) & \text{if } x_2 > 0, \\ u^-(0) & \text{if } x_2 < 0. \end{cases}$$

In view of the choice of $(n_m)_m$, we obtain

$$\begin{aligned} \frac{d\mu}{d\nu}(0) &= \lim_{m\to+\infty} \frac{\mu\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m} \geq \liminf_{m\to+\infty} \frac{\mu_{n_m}\left(\overline{Q_{2\varepsilon_m}(0)}\right)}{2\varepsilon_m} \\ &= \liminf_{m\to+\infty} \frac{1}{2\varepsilon_m} \int_{K_{n_m}\cap\overline{Q_{2\varepsilon_m}(0)}} \left[(u_{n_m}^+)^2 + (u_{n_m}^-)^2\right] \, d\mathcal{H}^1 \\ &= \frac{1}{2} \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1. \end{aligned}$$

Hence, if we prove that

$$\frac{1}{2} \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1 \geq u^+(0)^2 + u^-(0)^2, \tag{34}$$

we will get estimate (31), concluding the proof of the theorem.

We can assume

$$\limsup \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1 < +\infty. \tag{35}$$

Step 3. To prove (34), we consider the one-dimensional sections of the functions v_m and u. In view of the countably $\mathcal{H}^1$-rectifiability of $\hat{K}_m$, we can apply the area formula, obtaining the inequality

$$\begin{aligned} &\liminf_{m\to+\infty} \int_{-1}^{1} \int_{(\hat{K}_m)_{x_1}} \left[v_m^+(x_1, s)^2 + v_m^-(x_1, s)^2\right] \, d\mathcal{H}^0(s) \, dx_1 \\ &\quad \leq \liminf_{m\to+\infty} \int_{\hat{K}_m} \left[(v_m^+)^2 + (v_m^-)^2\right] \, d\mathcal{H}^1, \end{aligned} \tag{36}$$

where

$$(\hat{K}_m)_{x_1} := \left\{ s \in [-1, 1] : (x_1, s) \in \hat{K}_m \right\}.$$

Let us relabel the sequence $(m_k)_k$ to realize the liminf in the left hand side above.

We deduce that, for a.e. $x_1 \in [-1, 1]$, $v_{m_k}(x_1, \cdot)$ is a $SBV(-1, 1)$ function, with a finite number of jumps (so that it is piecewise H^1 on $(-1, 1)$) and it holds

$$v_{m_k}(x_1, \cdot) \to u^\pm \quad \text{strongly in } L^2(-1, 1) \tag{37}$$

where we denoted for brievity again by $u^\pm$ the one dimensional section of the piecewise constant function $u^\pm$. For almost every $x_1 \in]-1,1[$, the slices $(\hat K_{m_k})_{x_1}$ are non-empty, i.e. there exists $N(x_1) \in \mathbb{N}$ such that

$$(\hat K_{m_k})_{x_1} \neq \emptyset \tag{38}$$

for every $k \geq N(x_1)$ (see [6, Proposition 2.6, Step 4]) and, moreover,

$$(\hat K_{m_k})_{x_1} \to (x_1, 0) \tag{39}$$

in the Hausdorff metric (this is a consequence of the fact that $\hat K_{m_k} \to l \cap \overline{Q_2(0)}$).

For every such x_1, let us define $i(x_1,k) = \inf\{y, y \in (\hat K_{m_k})_{x_1}\}$, $s(x_1,k) = \sup\{y, y \in (\hat K_{m_k})_{x_1}\}$. Then, $i(x_1,k), s(x_1,k) \to 0$ and $v_{m_k}(x_1,\cdot)$ belongs to H^1 on both $(-1, i(x_1,k))$ and $(s(x_1,k),1)$. Note that the domains (which are intervals) are also moving, but they converge to a segment of length 1. For the argument below, it is convenient to see all the functions as defined on segments of length 1, posisbly rescaling the segments if necessary. These rescaled functions converge weakly to u^-, u^+, respectively, and also uniformly.

Consequently, using the same arguments as in [6, Proposition 2.6], we get

$$u^+(0)^2 + u^-(0)^2 \leq \liminf \int_{(\hat K_{m_{k_h}})_{x_1}} v^+_{m_{k_h}}(x_1,s)^2 + v^-_{m_{k_h}}(x_1,s)^2 \, d\mathcal{H}^0(s)$$

Summing in x_1 on $[-1,1]$, we conclude the proof thanks to (36) and the choice of (m_k). □

6. Furhter remarks

The smoothness of the boundary of optimal sets is an unsolved problem, which probably is very difficult. For now, progress in this direction was done only for the spectral problems involving Dirichlet boundary conditions or Robin boundary conditions with positive parameter (but in this last case only for local minimizers of energy type functionals). Provided the optimal set was smooth and $\lambda_{k,\beta}$ was simple, the optimality condition reads (see [3])

$$\int_{\partial\Omega} \Big(|\nabla_{\partial\Omega} u|^2 - (\lambda_{k,\beta}(\Omega) + \beta^2 + \beta\mathcal{H})u^2\Big) V\cdot n \, d\mathcal{H}^{d-1} = 0, \tag{40}$$

for every smooth vector field satisfying $\int_{\partial\Omega} V\cdot n\, d\mathcal{H}^{d-1} = 0$. Above, $\mathcal{H}$ stands for the mean curvature of $\partial\Omega$.

The following inequality holds true

$$\lambda_{1,\beta}(\Omega) < -\beta^2.$$

We refer the reader to Giorgi and Smits [15, Theorem 2.3] and to Daners and Kennedy [12, Lemma 2.1] in the context of Lipschitz sets, but it can naturally be extended to the first relaxed eigenvalue in both frameworks of measurable or simply connected sets. For smooth planar domains the following inequality was proved in [3]

$$\lambda_{1,\beta}(\Omega) < -\beta^2 - \frac{2\pi}{\mathcal{H}^1(\partial\Omega)}\beta.$$

It remains unclear how many connected components (or well separated sets) should have the solution of problem (3). Even for $k=1$ the connectedness of the optimal shape is not straightforward. Assume that the optimal set consists on two well separated sets. One of them will give the first eigenvalue, and the second one could possibly be cancelled. Nevertheless, erasing some connected component would make that the measure of the set is not anymore satisfying the constraint. Contrary to the case of positive boundary parameter, the behavior of the eigenvalues to dilations is not controlled.

Assuming the smoothness of the boundary of the optimal set for $\lambda_{1,\beta}$ in two dimensions of the space, one can prove its connectedness [1]. Indeed, if the connected component Ω_1 giving the first eigenvalue uses less measure than allowed, then the constraint $\int_{\partial\Omega_1} V \cdot n d\mathcal{H}^{d-1} = 0$ on the admissible vector fields can be removed, leading to $|\nabla_{\partial\Omega_1} u|^2 - (\lambda_{k,\beta}(\Omega_1) + \beta^2 + \beta\mathcal{H})u^2 = 0$ on $\partial\Omega_1$. Consequently,

$$\lambda_{1,\beta}(\Omega_1) + \beta^2 + \beta\mathcal{H} \geq 0$$

at almost every point of the boundary. This would contradict $\lambda_{1,\beta}(\Omega_1) < -\beta^2 - \frac{2\pi}{\mathcal{H}^1(\partial\Omega_1)}\beta$, after summation over $\partial\Omega_1$.

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References

[1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press (2000)

[2] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces*, MPS-SIAM (2006)

[3] P. R. S. Antunes, P. Freitas, D. Krejcirik, *Bounds and extremal domains for Robin eigenvalues with negative boundary parameter.* Adv. Calc. Var. 10, no. 4, 357–379 (2017).

[4] C. Bandle, A. Wagner, *Isoperimetric inequalities for the principal eigenvalue of a membrane and energy of problems with Robin boundary conditions*, Reports des Instituts f ur Mathematik der RWTH Aachen, No. 71 (2014)

[5] B. Bogosel, *Regularity result for a shape optimization problem under perimeter constraint*, to appear in Communications in Analysis and Geometry (2018)

[6] B. Bogosel, D. Bucur, A. Giacomini: *Optimal shapes maximizing the Steklov eigenvalues*, SIAM J. Math. Analysis 49, no. 2, 1645–1680 (2017)

[7] D. Bucur, G. Buttazzo: *Variational Methods in Shape Optimization Problems*, Birkhauser (2005)

[8] D. Bucur, P. Freitas, J. Kennedy: *The Robin Problem*, in A. Henrot (Ed.): *Shape optimization and spectral theory*, pp. 78-119, De Gruyter Open (2017)

[9] D. Bucur, A. Giacomini: *Arch. Ration. Mech. Anal.* Volume 198, Number 3, 927–961 (2010).

[10] D. Bucur, A. Giacomini, *Shape optimization problems with Robin conditions on the free boundary.* Ann. Inst. H. Poincar Anal. Non Linaire 33 (2016), no. 6, 1539–1568.

[11] G. Buttazzo, Giuseppe, G. Dal Maso, *An existence result for a class of shape optimization problems.* Arch. Rational Mech. Anal. 122, no. 2, 183–195 (1993)

[12] D. Daners, J. Kennedy, *On the asymptotic behaviour of the eigenvalues of a Robin problem.* Differential Integral Equations 2, no. 7-8, 659–669 (2010)

[13] P. Freitas, D. Krejcirik,*The first Robin eigenvalue with negative boundary parameter*, Adv. Math. 280 (2015), 322-339.

[14] B. Colbois, A. El Soufi, A. Girouard: *Isoperimetric control of the Steklov spectrum*, J. Funct. Anal. 261 (2011), no. 5, 1384–1399.

[15] T. Giorgi, R. Smits, *Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity.* Z. Angew. Math. Phys. 58 (2007), no. 2, 224–245.

[16] A. Grigor'yan, Y. Netrutsov, S.-T. Yau: Eigenvalues of elliptic operators and geometric applications, in *Surveys in differential geometry. Vol. IX*, Surv. Differ. Geom., IX, pages 147-217. Int. Press, Somerville, MA (2004)

[17] A. Henrot, M. Pierre: *Variation et optimisation de formes. Une analyse gomtrique*, Springer-Verlag Berlin Heidelberg (2005)

[1]This remark is due to James Kennedy

[18] F. John: Extremum problems with inequalities as subsidiary conditions, in *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, Interscience Publishers, Inc., New York, N. Y., pp. 187-204 (1948)

(Dorin Bucur) Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

E-mail address: dorin.bucur@univ-savoie.fr

(Simone Cito) Università del Salento, Dipartimento di Matematica e Fisica "Ennio De Giorgi", 73100 Lecce, Italy

E-mail address: simone.cito@unisalento.it