WEIGHTED BECKMANN PROBLEM WITH BOUNDARY COSTS

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Abstract. We show that a solution to a variant of the Beckmann problem can be obtained by studying the limit of some weighted $p-$Laplacian problems. More precisely, we find a solution to the following minimization problem:

$$\min\left\{\int_\Omega k\,\mathrm{d}|w| + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \,:\, w \in \mathcal{M}^d(\Omega),\, \nu \in \mathcal{M}(\partial\Omega),\, -\nabla\cdot w = f+\nu\right\}$$

where f, k and $g^\pm$ are given. In addition, we connect this problem to a formulation with Kantorovich potentials with Dirichlet boundary conditions.

1. Introduction

In this paper we consider a variant of the flow-minimization problem introduced by Beckmann in 1950 [2] as a particular case of a wider class of convex optimization problem, of the form $\min\{\int H(w)\,\mathrm{d}x \,:\, -\nabla\cdot w = f^+ - f^-\}$, for convex H. The case $H(z) = |z|$ is very interesting because of its equivalence with the Monge problem which deals with the optimal way of moving points from one mass distribution to another so that the total work done is minimized. In his work, the cost of moving one unit of mass from x to y is measured with the Euclidean distance $|x-y|$, even though many other cost functions have been studied later on.

Given two finite positive Borel measures f^+ and f^- on a compact convex domain $\Omega \subset \mathbb{R}^d$, satisfying the mass balance condition $f^+(\Omega) = f^-(\Omega)$, then, the classical Monge optimal transportation problem [12] is the following:

$$\text{(MP)} \qquad \inf\left\{\int_\Omega |x-T(x)|\,\mathrm{d}f^+ \,:\, T_\# f^+ = f^-\right\}$$

where $T_\# f^+ = f^- \Leftrightarrow f^-(A) = f^+(T^{-1}(A))$ for every Borel set $A \subset \Omega$. The existence of optimal maps was addressed by many authors [1], [5], [8], [14] and [17]. Although this problem may have no solutions, its relaxed setting (which is the Kantorovich problem [13]) always has one. The relaxed problem is the following

$$\text{(KP)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma :\ \gamma \in \Pi(f^+, f^-)\right\}$$

where

$$\Pi(f^+, f^-) := \left\{\gamma \in \mathcal{M}^+(\Omega\times\Omega) :\ (\Pi_x)_\#\gamma = f^+\ ,\ (\Pi_y)_\#\gamma = f^-\ \right\}$$

and Π_x, Π_y are the two projections of $\Omega\times\Omega$ onto Ω. The authors of [15, 16] prove that the dual of (KP) is the following:

$$\text{(DP)} \qquad \max\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-) \,:\, u \in \mathrm{Lip}_1(\Omega)\right\}.$$

The equality of the two optimal values implies that optimal γ and u satisfy $u(x)-u(y)=|x-y|$ on the support of γ, which means that the potential u decreases at the rate one as we move along the transport ray $[x,y]$ (note that the gradient of u gives the direction of these transport rays). It is well-known that there exists a non-negative Borel measure σ over Ω (which is called *transport density*) such that (σ, u) solves a particular PDE system, called Monge-Kantorovich system [15]:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f := f^+ - f^- & \text{in } \Omega \\ \sigma\nabla u\cdot n = 0 & \text{on } \partial\Omega \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma-\text{a.e.} \end{cases} \tag{1.1}$$

This measure σ represents the amount of transport taking place in each region of Ω, i.e. for a given Borel set A, $\sigma(A)$ stands for "how much" the transport takes place in A, if particles move from their origin x to their destination y on transport rays.

In addition, the flow $w := \sigma\nabla u$ solves the Beckmann problem (see [15]), which is the following:

$$\text{(BP)} \qquad \min\left\{\int_\Omega \mathrm{d}|w| \,:\, w\in\mathcal{M}^d(\Omega),\ -\nabla\cdot w = f^+ - f^-\right\}$$

and, we have the following equalities:

$$\min(\mathrm{BP}) = \sup(\mathrm{DP}) = \min(\mathrm{KP}).$$

An interesting variant of (KP), which is already present in [6, 7, 11], is to transport the mass f^+ to another one f^- (which do not have a priori the same total mass) with the possibility of transporting some mass to/from the boundary, paying the transport cost that is assumed to be given by the Euclidean distance $|x-y|$ plus an extra cost $g^-(y)$ for each unit of mass that comes out from a point $y\in\partial\Omega$ or $-g^+(x)$ for each unit of mass that enters at the point $x\in\partial\Omega$. Yet, it is raisonnable to consider a distance d_k associated with a Riemannian metric k (where k is supposed to be positive and continuous), instead of the Euclidean distance, when we want to model a non-uniform cost for the movement (due to geographical obstacles or configurations). Recall that this distance d_k is defined as follows

$$d_k(x,y) := \inf\left\{\int_0^1 k(\omega(t))|\omega'(t)|\,\mathrm{d}t \,:\, \omega\in\mathrm{Lip}([0,1],\Omega),\ \omega(0)=x,\ \omega(1)=y\right\},\ \forall\ x,\ y\in\Omega.$$

First of all, we assume that $g^\pm\in C(\partial\Omega)$ with

$$g^+(x) - g^-(y) \le d_k(x,y), \text{ for all } x,\ y\in\partial\Omega. \tag{1.2}$$

Set

$$\Pi b(f^+,f^-) := \left\{\gamma\in\mathcal{M}^+(\Omega\times\Omega) \,:\, ((\Pi_x)_\#\gamma)_{|\mathring{\Omega}} = f^+,\ ((\Pi_y)_\#\gamma)_{|\mathring{\Omega}} = f^-\right\},$$

we minimize the quantity

$$\text{(KPb)} \qquad \min\left\{\int_{\Omega\times\Omega} d_k(x,y)\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma\in\Pi b(f^+,f^-)\right\}.$$

In this paper, we will prove that the problem (KPb) has a dual formulation, which is the following

$$\sup\left\{\int_\Omega \varphi\,\mathrm{d}(f^+-f^-) \,:\, |\nabla\varphi|\le k,\; g^+\le\varphi\le g^- \text{ on } \partial\Omega\right\} \qquad \text{(DPb)}.$$

Note that, for this optimal transportation problem with boundary costs, the system (1.1) becomes

$$\text{(1.3)} \qquad \begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega,\\ \frac{\partial u}{\partial \mathbf{n}} \ge 0 & \text{on } \{u\ne g^-\},\\ \frac{\partial u}{\partial \mathbf{n}} \le 0 & \text{on } \{u\ne g^+\},\\ g^+\le u\le g^- & \text{on } \partial\Omega,\\ |\nabla u|\le k & \text{in } \Omega,\\ |\nabla u| = k & \sigma-\text{a.e.} \end{cases}$$

and, the problem (BP) becomes (BPb):

$$\min\left\{\int_\Omega k\,\mathrm{d}|w| + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \,:\, (w,\nu)\,\in\,\mathcal{M}^d(\Omega)\times\mathcal{M}(\partial\Omega),\; -\nabla\cdot w = f+\nu\right\}.$$

In [8], the authors prove that a solution to (1.1) can be constructed by studying the $p-$Laplacian equation

$$-\nabla\cdot(|\nabla u_p|^{p-2}\nabla u_p) = f$$

in the limit as $p\to\infty$. In this paper, we prove that a solution to (1.3) (or equivalently, to (BPb)) can be constructed by studying the limit as $p\to\infty$ of the following weighted $p-$Laplacian problem:

$$\text{(1.4)} \qquad \begin{cases} -\nabla\cdot(k^{-p}|\nabla u_p|^{p-2}\nabla u_p) = f & \text{in } \Omega,\\ \frac{\partial u_p}{\partial \mathbf{n}} = 0 & \text{on } \{g^+ < u_p < g^-\},\\ \frac{\partial u_p}{\partial \mathbf{n}} \ge 0 & \text{on } \{u_p = g^+\},\\ \frac{\partial u_p}{\partial \mathbf{n}} \le 0 & \text{on } \{u_p = g^-\},\\ g^+\le u_p\le g^- & \text{on } \partial\Omega. \end{cases}$$

Using this approach, we get finally the following

$$\min(\text{BPb}) = \sup(\text{DPb}) = \min(\text{KPb}).$$

This paper is organized as follows. In Section 2, we introduce a novel proof for the duality of (KPb). In Section 3, we introduce the weighted $p-$Laplacian problems that we use to approximate a maximizer u_∞ of (DPb). Then, we prove existence of a solution to (1.3), this means that we want to find a non-negative Borel measure σ such that (σ, u_∞) solves (1.3). Finally, in Section 4, we prove that min(BPb)= sup(DPb) and we find a minimizer to (BPb).

2. Duality

The proof of the duality formula of (KPb), introduced in [11], concerns only the Euclidean case and, it is based on the Fenchel-Rocafellar duality Theorem and it is decomposed into two steps: firstly, the authors suppose that the inequality in (1.2) is strict and secondly, they use an approximation argument to cover the other case. Here, we want to give an alternative proof for this duality formula, based on a simple convex analysis trick. Before that, let us introduce the following existence result.

Proposition 2.1. *(KPb) reaches a minimum.*

Proof. Set

$$K(\gamma) := \int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma \;+ \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma,\;\;\forall\,\gamma\in\mathcal{M}(\bar\Omega\times\bar\Omega).$$

Then, K is continuous with respect to the weak convergence of measures in $\Pi b(f^+,f^-)$. Indeed, if $(\gamma_n)_n$ is a sequence in $\Pi b(f^+,f^-)$ such that $\gamma_n \rightharpoonup \gamma$, then, for every n, there exists $\chi_n^\pm \in \mathcal{M}^+(\partial\Omega)$ such that

$$(\Pi_x)_\#\gamma_n = f^+ + \chi_n^+\,,\;\;(\Pi_y)_\#\gamma_n = f^- + \chi_n^-$$

and

$$\chi_n^\pm \rightharpoonup \chi^\pm,$$

where $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. As $g^\pm$ are continuous, then

$$K(\gamma_n)\to K(\gamma).$$

On the other hand, we observe that if $\gamma\in\Pi b(f^+,f^-)$ and $\tilde\gamma := \gamma_{|(\partial\Omega\times\partial\Omega)^c}$, then $\tilde\gamma$ also belongs to $\Pi b(f^+,f^-)$. In addition, we have

$$\int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma \;+ \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma$$

$$= \int_{\partial\Omega\times\partial\Omega}(d_k(x,y)+g^-(y)-g^+(x))\mathrm{d}\gamma + \int_{(\partial\Omega\times\partial\Omega)^c} d_k(x,y)\mathrm{d}\gamma + \int_{\Omega^\circ\times\partial\Omega} g^-(y)\mathrm{d}\gamma - \int_{\partial\Omega\times\Omega^\circ} g^+(x)\mathrm{d}\gamma.$$

As

$$d_k(x,y)+g^-(y)-g^+(x)\geq 0,$$

we get

$$\int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma \;+ \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma$$
$$\geq \int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\tilde\gamma \;+ \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\tilde\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\tilde\gamma.$$

Now, let $(\gamma_n)_n \subset \Pi b(f^+,f^-)$ be a minimizing sequence. Then, we can suppose that
$$\gamma_n(\partial\Omega \times \partial\Omega) = 0.$$
In this case, we get
$$\begin{array}{rcl}\gamma_n(\bar\Omega \times \bar\Omega) & \leq & \gamma_n(\Omega^0 \times \bar\Omega) + \gamma_n(\bar\Omega \times \Omega^0)\\ & = & f^+(\bar\Omega) + f^-(\bar\Omega).\end{array}$$

Hence, there exist a subsequence $(\gamma_{n_k})_{n_k}$ and a plan $\gamma \in \Pi b(f^+,f^-)$ such that $\gamma_{n_k} \rightharpoonup \gamma$. But, the continuity of K implies that this γ is a minimizer for (KPb). $\square$

Proposition 2.2. *Let $g^\pm$ be in $C(\partial\Omega)$. Then under the assumption (1.2), we have the following equality*
$$\min\left\{\int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma \in \Pi b(f^+,f^-)\right\} \quad \textit{(KPb)}$$
$$= \sup\left\{\int_\Omega \varphi\,\mathrm{d}(f^+-f^-) \,:\, |\nabla\varphi| \leq k,\ g^+ \leq \varphi \leq g^- \text{ on } \partial\Omega\right\} \quad \textit{(DPb).}$$
Notice that if (1.2) is not satisfied, then both sides of this equality are $-\infty$.

Proof. For every $p^\pm \in C(\partial\Omega)$, set
$$H(p^+,p^-) := -\sup\left\{\int_\Omega \varphi\,\mathrm{d}(f^+-f^-) \,:\, |\nabla\varphi| \leq k,\ g^+ + p^+ \leq \varphi \leq g^- - p^- \text{ on } \partial\Omega\right\}.$$
It is easy to see that $H(p^+,p^-) \in \mathbb{R} \cup \{+\infty\}$. Indeed, if $(\varphi_n)_n$ is a maximizing sequence, then φ_n are equicontinuous since $|\nabla\varphi_n| \leq k$ and they are also equibounded thanks to the fact that $g^+ + p^+ \leq \varphi_n \leq g^- - p^-$ on $\partial\Omega$ and so, we can apply Ascoli-Arzelà's Theorem. In addition, we claim that H is convex and l.s.c.

For convexity: take $t \in (0,1)$ and (p_0^+,p_0^-), $(p_1^+,p_1^-) \in C(\partial\Omega)\times C(\partial\Omega)$ and let φ_0, φ_1 be their optimal potentials. Set
$$p_t^+ := (1-t)p_0^+ + tp_1^+,\ p_t^- := (1-t)p_0^- + tp_1^-$$
and
$$\varphi_t := (1-t)\varphi_0 + t\varphi_1.$$
As
$$g^+ + p_0^+ \leq \varphi_0 \leq g^- - p_0^- \quad\text{and}\quad g^+ + p_1^+ \leq \varphi_1 \leq g^- - p_1^- \quad\text{on}\quad \partial\Omega,$$
then
$$g^+ + p_t^+ \leq \varphi_t \leq g^- - p_t^- \ \text{ on } \partial\Omega.$$

In addition, $|\nabla\varphi_t| \leq k$. Consequently, φ_t is admissible in the max defining $-H(p_t^+,p_t^-)$ and then,
$$H(p_t^+,p_t^-) \leq -\int_\Omega \varphi_t\,\mathrm{d}(f^+-f^-) = (1-t)H(p_0^+,p_0^-) + tH(p_1^+,p_1^-).$$

For semi-continuity: take $p_n^+ \to p^+$ and $p_n^- \to p^-$ uniformly on $\partial\Omega$. Let $(p_{n_k}^+, p_{n_k}^-)_{n_k}$ be a subsequence such that $\liminf_n H(p_n^+, p_n^-) = \lim_{n_k} H(p_{n_k}^+, p_{n_k}^-)$ (for simplicity of notation, we still denote this subsequence by $(p_n^+, p_n^-)_n)$ and let $(\varphi_n)_n$ be their corresponding optimal potentials. As $|\nabla\varphi_n| \leq k$ and $(p_n^+)_n$, $(p_n^-)_n$ are equibounded, then, by Ascoli-Arzelà's Theorem, there exist a function φ with $|\nabla\varphi| \leq k$ and a subsequence $(\varphi_{n_k})_{n_k}$ such that $\varphi_{n_k} \to \varphi$ uniformly. As

$$g^+ + p_{n_k}^+ \leq \varphi_{n_k} \leq g^- - p_{n_k}^- \text{ on } \partial\Omega,$$

then

$$g^+ + p^+ \leq \varphi \leq g^- - p^- \text{ on } \partial\Omega.$$

Consequently, φ is admissible in the max defining $-H(p^+, p^-)$ and one has

$$H(p^+, p^-) \leq -\int_\Omega \varphi \,\mathrm{d}(f^+ - f^-) = \lim_{n_k} H(p_{n_k}^+, p_{n_k}^-) = \liminf_n H(p_n^+, p_n^-).$$

Hence, we get that $H^{\star\star} = H$ and in particular, $H^{\star\star}(0,0) = H(0,0)$. But by the definition of H, we have

$$H(0,0) = -\sup\left\{\int_\Omega \varphi \,\mathrm{d}(f^+ - f^-) \,:\, |\nabla\varphi| \leq k,\, g^+ \leq \varphi \leq g^- \text{ on } \partial\Omega\right\}.$$

On the other hand, let us compute $H^{\star\star}(0,0)$. Take $\chi^\pm$ in $\mathcal{M}(\partial\Omega)$, then we have

$$H^\star(\chi^+, \chi^-) = \sup_{p^\pm \in C(\partial\Omega)} \left\{\int_{\partial\Omega} p^+ \,\mathrm{d}\chi^+ + \int_{\partial\Omega} p^- \,\mathrm{d}\chi^- - H(p^+, p^-)\right\}$$

$$= \sup_{p^\pm \in C(\partial\Omega),\, |\nabla\varphi| \leq k} \left\{\int p^+ \,\mathrm{d}\chi^+ + \int p^- \,\mathrm{d}\chi^- + \int \varphi \,\mathrm{d}(f^+ - f^-) : g^+ + p^+ \leq \varphi \leq g^- - p^- \text{ on } \partial\Omega\right\}.$$

If $\chi^+ \notin \mathcal{M}^+(\partial\Omega)$, i.e there exists $p_0^+ \in C(\partial\Omega)$ such that $p_0^+ \geq 0$ and $\int_{\partial\Omega} p_0^+ \,\mathrm{d}\chi^+ < 0$, we may see that

$$H^\star(\chi^+, \chi^-) \geq -n \int_{\partial\Omega} p_0^+ \,\mathrm{d}\chi^+ + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+ \underset{n \to +\infty}{\longrightarrow} +\infty.$$

Similarly if $\chi^- \notin \mathcal{M}^+(\partial\Omega)$. Now, suppose that $\chi^\pm \in \mathcal{M}^+(\partial\Omega)$. As $g^+ + p^+ \leq \varphi \leq g^- - p^-$ on $\partial\Omega$, we should choose the largest possible $p^\pm$, i.e $p^+(x) = \varphi(x) - g^+(x)$ and $p^-(y) = g^-(y) - \varphi(y)$ for all x, $y \in \partial\Omega$. Hence, we have

$$H^\star(\chi^+, \chi^-) = \sup\left\{\int_{\bar\Omega} \varphi \,\mathrm{d}(f + \chi) \,:\, |\nabla\varphi| \leq k\right\} + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+.$$

By [16, Theorem 1.14], we get

$$H^\star(\chi^+, \chi^-) = \min\left\{\int_{\bar\Omega \times \bar\Omega} d_k(x,y) \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+ + \chi^+, f^- + \chi^-)\right\} + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+$$

$$= \min \left\{ \int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma\in\Pi(f^+ + \chi^+, f^- + \chi^-) \right\}.$$

Finally, we have

$$H^{\star\star}(0,0) = \sup\left\{ -H^\star(\chi^+,\chi^-) \,:\, \chi^+,\, \chi^- \in \mathcal{M}^+(\partial\Omega) \right\}$$

$$= -\min \left\{ \int_{\bar\Omega\times\bar\Omega} d_k(x,y)\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma\in\Pi b(f^+, f^-) \right\}. \quad \square$$

3. The limit of the weighted $p-$Laplacian problems

In this section, the aim is to obtain estimates, independent of p, on solution of (1.4), similar to those in [8, 11]. First of all, we note that the unique (may be up to a constant) weak solution u_p of (1.4) is found as the minimizer of the functional

$$\mathcal{J}_p(u) := \frac{1}{p}\int_\Omega k^{-p}|\nabla u|^p\,\mathrm{d}x - \int_\Omega uf\,\mathrm{d}x$$

over all $u\in W^{1,p}(\Omega)$, $g^+\le u\le g^-$ on $\partial\Omega$. Under the assumption (1.2), we have the following

Proposition 3.1. *Let u_p be the solution of (1.4). Then, up to a subsequence, $u_p\to u_\infty$ uniformly as $p\to\infty$, where u_∞ solves (DPb).*

Proof. Set

$$v(x) := \min_{y\in\partial\Omega}\{g^-(y) + d_k(x,y)\}, \text{ for all } x\in\Omega.$$

Then, it is easy to see that v is Lip_1 according to the distance d_k and then, $|\nabla v|\le k$. In addition, (1.2) gives that

$$g^+\le v\le g^- \text{ on } \partial\Omega.$$

From the optimality of u_p, we have

$$\mathcal{J}_p(u_p)\le \mathcal{J}_p(v)\le \frac{|\Omega|}{p} + C$$

where C is a constant independent of p. As

$$g^+\le u_p\le g^- \text{ on } \partial\Omega,$$

then, it is easy to check that

$$||u_p||_{L^\infty(\Omega)} \le C(d,\mathrm{diam}(\Omega))\,||\nabla u_p||_{L^p(\Omega,\mathbb{R}^d)} + ||g||_{L^\infty(\partial\Omega)}.$$

Hence,

$$\begin{aligned}\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x &\le p\int_\Omega u_p f\,\mathrm{d}x + Cp\\ &\le Cp\Big(\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} + Cp.\end{aligned}$$

Yet, this implies that

$$\left(\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq (Cp)^{\frac{1}{p}}$$

and then, for $m < p$,

$$\left(\int_\Omega k^{-m}|\nabla u_p|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \leq (Cp)^{\frac{1}{p}}\,|\Omega|^{\frac{1}{m}-\frac{1}{p}}.$$

Hence, up to a subsequence, $u_p \rightharpoonup u_\infty$ in $W^{1,m}(\Omega)$, for all $m \in \mathbb{N}^\star$, and then, $u_p \to u_\infty$ uniformly in Ω. In addition, we have

$$\left(\int_\Omega k^{-m}|\nabla u_\infty|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \leq |\Omega|^{\frac{1}{m}}, \text{ for all } m \in \mathbb{N}^\star$$

and then,

$$|\nabla u_\infty| \leq k.$$

On the other hand, for any admissible function φ in (DPb), we have, from the optimality of u_p, that

$$-\int_\Omega u_p f \,\mathrm{d}x \leq \mathcal{J}_p(u_p) \leq \mathcal{J}_p(\varphi) \leq \frac{|\Omega|}{p} - \int_\Omega \varphi f \,\mathrm{d}x.$$

When $p \to \infty$, we infer that u_∞ solves (DPb). $\square$

For all $p > d$, set

$$w_p := k^{-p}\,|\nabla u_p|^{p-2}\,\nabla u_p,$$

where u_p is the solution of (1.4). So, the aim now is to study the limit as $p \to \infty$ of $(w_p)_p$. In particular, we show that $w_p \rightharpoonup w$ in the sense of measures and that (σ, u_∞) solves (1.3) with $\sigma := k^{-1}\,|w|$.

Lemma 3.2. *For all $p > d$, there exists a measure ν_p, which is concentrated on the boundary of Ω, such that*

$$\int_\Omega w_p \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi f \,\mathrm{d}x + \int_{\partial\Omega} \varphi \,\mathrm{d}\nu_p\,, \textit{ for every } \varphi \in W^{1,p}(\Omega).$$

In addition, we have

$$\mathrm{spt}\,\nu_p^\pm \subset \{u_p = g^\pm\}.$$

Proof. Take $\varphi \in C^\infty(\Omega)$ with

$$\mathrm{spt}(\varphi) \cap \{u_p = g^\pm\} = \emptyset.$$

As $u_p \in C(\Omega)$, then there exists $\varepsilon_0 > 0$ such that $g^+ \leq u_p + \varepsilon\varphi \leq g^-$ on $\partial\Omega$, for all $|\varepsilon| < \varepsilon_0$. Yet, from the optimality of u_p, we have

$$\mathcal{J}_p(u_p) \leq \mathcal{J}_p(u_p + \varepsilon\varphi)$$

and when $\varepsilon \to 0$, we get

$$\int_\Omega w_p \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi f \,\mathrm{d}x.$$

Let $\varphi^\pm \geq 0$ be in $C^\infty(\Omega)$ with

$$\operatorname{spt}(\varphi^+) \cap \{u_p = g^-\} = \emptyset \ \text{ and } \ \operatorname{spt}(\varphi^-) \cap \{u_p = g^+\} = \emptyset.$$

Working as above, we get

$$\int_\Omega w_p \cdot \nabla\varphi^+ \,\mathrm{d}x \ \geq \int_\Omega \varphi^+ f \,\mathrm{d}x \ \text{ and } \int_\Omega w_p \cdot \nabla\varphi^- \,\mathrm{d}x \ \leq \int_\Omega \varphi^- f \,\mathrm{d}x.$$

□

From now on, we assume that the inequality in (1.2) is strict, this means that

$$g^+(x) - g^-(y) < d_k(x,y), \ \forall \ \ x,\, y \in \partial\Omega.$$

Then, we have the following

Proposition 3.3. *$w_p \rightharpoonup w$ and $\nu_p \rightharpoonup \nu$ in the sense of measures.*

Proof. Set again

$$v(x) := \min_{y\in\partial\Omega}\{g^-(y) + d_k(x,y)\}, \text{ for all } x \in \Omega.$$

Then, it is clear that $g^+ < v \leq g^-$ on $\partial\Omega$ and $|\nabla v| \leq k$. In addition, we have the following equality

$$\int_\Omega w_p \cdot \nabla(u_p - v) \,\mathrm{d}x = \int_\Omega (u_p - v) f \,\mathrm{d}x + \int_{\partial\Omega} (u_p - v) \,\mathrm{d}\nu_p.$$

Hence,

$$\begin{aligned} \int_{\partial\Omega} (v - u_p) \,\mathrm{d}\nu_p + \int_\Omega k^{-p} |\nabla u_p|^p \,\mathrm{d}x \ &= \ \int_\Omega w_p \cdot \nabla v \,\mathrm{d}x + \int_\Omega (u_p - v) f \,\mathrm{d}x \\ &\leq \ \int_\Omega w_p \cdot \nabla v \,\mathrm{d}x \,+\, C \end{aligned}$$

where C is a constant independent of p. As $v - g^+ \geq c > 0$ on $\partial\Omega$, then, by Lemma 3.2, we get

$$\begin{aligned} c\int_{\partial\Omega} \mathrm{d}\nu_p^+ + \int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x \ &\leq \ \int_\Omega k^{-(p-1)} |\nabla u_p|^{p-2} \nabla u_p \cdot k^{-1} \nabla v \,\mathrm{d}x \,+\, C \\ &\leq \ |\Omega|^{\frac{1}{p}} \left(\int_\Omega k^{-p} |\nabla u_p|^p \,\mathrm{d}x \right)^{1-\frac{1}{p}} \,+\, C \\ &\leq \ \left(1 - \frac{1}{p}\right) \int_\Omega k^{-p} |\nabla u_p|^p \,\mathrm{d}x \,+\, C. \end{aligned}$$

Finally, we infer that

$$c\int_{\partial\Omega} \mathrm{d}\nu_p^+ \;+\; \frac{1}{p}\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x \le C.$$

Therefore,

$$\int_{\partial\Omega} \mathrm{d}\nu_p^\pm \le C.$$

Yet, we have

$$\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x = \int_\Omega u_p f\,\mathrm{d}x \;+\; \int_{\partial\Omega} u_p\,\mathrm{d}\nu_p.$$

Hence, the sequence $(w_p)_p$ (resp. $(\nu_p)_p$) is bounded in $\mathcal{M}^d(\Omega)$ (resp. $\mathcal{M}(\partial\Omega)$) and so, there exists a vector measure w (resp. a measure ν supported on $\partial\Omega$) such that $w_p \rightharpoonup w$ (resp. $\nu_p \rightharpoonup \nu$) in the sense of measures. $\square$

We conclude this section by proving existence of a solution to (1.3).

Proposition 3.4. *There exists a non-negative Borel measure σ over Ω such that (σ, u_∞), where u_∞ is a maximizer for (DPb), is a solution to (1.3).*

Proof. By Proposition 3.3, as $w_p \rightharpoonup w$ and $\nu_p \rightharpoonup \nu$, we get, using Lemma 3.2, that for all $\varphi \in C^1(\Omega)$,

$$\int_\Omega \nabla\varphi\cdot\mathrm{d}w = \int_\Omega \varphi f\,\mathrm{d}x + \int_{\partial\Omega}\varphi\,\mathrm{d}\nu. \tag{3.1}$$

Set

$$\sigma := k^{-1}|w|.$$

Now, consider a sequence $(\varphi_n)_n \subset C^\infty(\Omega)$ such that $\varphi_n \to u_\infty$ uniformly and $\nabla\varphi_n \to \nabla_\sigma u_\infty$ in $L^2_\sigma(\Omega,\mathbb{R}^d)$, where ∇_σ is the tangential gradient operator with respect to σ defined in [4]. By (3.1), we get

$$\begin{aligned}\int_\Omega \nabla_\sigma u_\infty\cdot\mathrm{d}w &= \int_\Omega u_\infty f\,\mathrm{d}x \;+\; \int_{\partial\Omega} u_\infty\,\mathrm{d}\nu\\ &= \int_\Omega u_\infty f\,\mathrm{d}x \;+\; \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \;-\; \int_{\partial\Omega} g^-\,\mathrm{d}\nu^-.\end{aligned}$$

Yet,

$$\begin{aligned}\int_\Omega k\,\mathrm{d}|w| &\le \liminf_p \int_\Omega k\,|w_p|\,\mathrm{d}x\\ &= \liminf_p \int_\Omega k^{-(p-1)}|\nabla u_p|^{p-1}\,\mathrm{d}x\\ &\le \liminf_p |\Omega|^{\frac{1}{p}}\left(\int_\Omega k^{-p}|\nabla u_p|^p\,\mathrm{d}x\right)^{1-\frac{1}{p}}.\end{aligned}$$

In addition, we have

$$\begin{aligned}\int_\Omega k^{-p}|\nabla u_p|^p \,\mathrm{d}x &= \int_\Omega u_p f \,\mathrm{d}x + \int_{\partial\Omega} u_p \,\mathrm{d}\nu_p \\ &= \int_\Omega u_p f \,\mathrm{d}x + \int_{\partial\Omega} g^+ \,\mathrm{d}\nu_p^+ - \int_{\partial\Omega} g^- \,\mathrm{d}\nu_p^- \\ &\to \int_\Omega u_\infty f \,\mathrm{d}x \;+\; \int_{\partial\Omega} g^+ \,\mathrm{d}\nu^+ \;-\; \int_{\partial\Omega} g^- \,\mathrm{d}\nu^- = \int_\Omega \nabla_\sigma u_\infty \cdot \mathrm{d}w.\end{aligned}$$

Finally, we get

$$\int_\Omega k \,\mathrm{d}|w| \leq \int_\Omega \nabla_\sigma u_\infty \cdot \mathrm{d}w.$$

Since $|\nabla u_\infty| \leq k$, hence

$$\int_\Omega \nabla_\sigma u_\infty \cdot \mathrm{d}w = \int_\Omega k \,\mathrm{d}|w|$$

and

$$w = \sigma\, \nabla_\sigma u_\infty \,,\; |\nabla_\sigma u_\infty| = k \quad \sigma - \text{a.e.}$$

□

4. Producing a solution to a variant of the Beckmann Problem

Now, we are ready to find a solution to (BPb). Let w (resp. ν) be the limit of $(w_p)_p$ (resp. $(\nu_p)_p$) as in the proposition 3.3. Then, we have the following

Proposition 4.1. *(w, ν) solves the problem (BPb). Moreover, the minimal value of (BPb) equals the maximal value of (DPb).*

Proof. We start from min(BPb) $\geq$ sup(DPb). In order to do so, take an arbitrary function $\varphi \in C^1(\Omega)$ with $|\nabla\varphi| \leq k$ and $g^+ \leq \varphi \leq g^-$ on $\partial\Omega$. Consider that for any $(v, \chi) \in \mathcal{M}^d(\Omega) \times \mathcal{M}(\partial\Omega)$ with $-\nabla \cdot v = f + \chi$, we have

$$\int_\Omega k \,\mathrm{d}|v| \geq \int_\Omega \nabla\varphi \cdot \mathrm{d}v = \int_\Omega \varphi \,\mathrm{d}(f + \chi) \geq \int_\Omega \varphi f \,\mathrm{d}x + \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+ - \int_{\partial\Omega} g^- \,\mathrm{d}\chi^-.$$

By an approximation argument, we can infer that

$$\int_\Omega k \,\mathrm{d}|v| + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+ \geq \sup\,(\mathrm{DPb}) = \min\,(\mathrm{KPb})$$

for any admissible (v, χ), i.e., min(BPb) $\geq$ sup(DPb). Yet, by Proposition 3.4, we have

$$\int_\Omega k \,\mathrm{d}|w| + \int_{\partial\Omega} g^- \,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\nu^+ = \int_\Omega u_\infty f \,\mathrm{d}x.$$

Hence, (w, ν) solves (BPb) and, recalling Proposition 2.2, we get min(BPb) = sup(DPb) = min(KPb). □

Remark 4.1. *Note that, from* [10]*, we have* $\sigma \in L^1$ *as soon as* $f \in L^1$ *and* $k \in C^{1,1}$*, and then, the following problem*

$$\min\left\{ \int_\Omega k|w|\,\mathrm{d}x + \int_{\partial\Omega} g^-\,\mathrm{d}\nu^- - \int_{\partial\Omega} g^+\,\mathrm{d}\nu^+ \,:\, w \in L^1(\Omega,\mathbb{R}^d),\, \nu \in \mathcal{M}(\partial\Omega),\, -\nabla\cdot w = f+\nu \right\}$$

reaches a minimum.

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