Variational Problems on Classes of Convex Domains

Nicolas Van Goethem

Abstract

We prove the existence of a minimizer for a large class of functionals defined over all convex domains of given volume included in a bounded subspace D of $\mathbb{R}^N$. Some applications are given, in particular we shall see that the eigenvalues of a class of second and fourth order operators with non-constant coefficients can be minimized over this class of domains, as well as integral functionals depending on the solution of an elliptic equation. Moreover, the third application of this result is related to the famous Newton's problem of minimal resistance. In general, all the results we shall develop are valid for elliptic operators of any order $2p, p \geq 1$.

1 Introduction

Minimizing functionals over classes of domains is a very broad area. We can mention the Newton's problem of minimal resistance (see [8], [14]) as a very classical problem where an optimal body Ω is searched among a class of sets, for instance among convex domains of $\mathbb{R}^N$. In this case, the natural functional to be associated to Ω is the so called total resistance of the body, i.e. an integral on Ω involving a suitable function of the normal vector at each point of the surface of the body. The integrand may be related to the solution of a simple PDE (see [14] for the Newton's problem) but sometimes an optimization problem is considered in a very global approach, for instance minimizing on a space of measures (see [5] for the shape optimization problem). In this latter case, one is forced to use tools from the geometric measure theory, which may become very complicated.

This paper is based on some simple concepts related to convexity. One of the concepts needed in an optimization problem is the space on which one seeks to minimize the given functional: the larger the class of candidates for the minimizer is, the harder the proof of an existence result will be. Often, one is forced to introduce constraints on the space, and this constraints should be natural enough, otherwise one loses some of the interest of the problem. For example, the concept of compact sets is needed in the direct methods of the calculus of variations but since compactness is a very strong condition, how can it be realized for general spaces? As usual, one has to pay for the generality of a result, and the cost is precisely the complexity of the topology on the chosen space. This is a reason why the heavy artillery of geometric measure theory is used if one seeks to minimize shape optimization's type functionals on the space of all open sets of $\mathbb{R}^N$ (see [5]). As soon as the domains are "too much" general, the candidates to the minimum may have very different kinds of behaviors. This variety of candidates can be compensated either by the stiffness of the class of functionals, i.e. only a narrow class of functionals is proved to assume their minima, or by the choice of the "most suitable" topology adapted to the problem. Eventually, the chosen topology on the space should give some kind of compactness. Alternatively, one can also decide to concentrate on the minimizing sequences to prove existence results.

In a recent work of Kawhol (see [14]), this latter approach is used for proving an existence result. In this work, a minimum is proved to exist on a class of convex domains, provided the functional is bounded below, homogeneous of degree $-s$, monotone decreasing with respect to the set inclusion and coercive. The two latter conditions allow the author to consider domains which are not a priori bounded. We decided to consider bounded domains, since often one is lead anyhow to consider this case. Since we wish to consider elliptic equations with non-constant coefficients, the homogeneity and the shift invariance of the functionals are too hard constraints. Moreover, it seems that assuming both convexity of the domains and monotonicity of the functional is redundant, since sometimes (see for instance [4]) it can be showed monotonicity could be compensated by convexity.

Our point of view is to attempt to have a result for a rather general class of functionals; in fact, the only conditions required are the boundedness and the semi-continuity of the functional. To obtain such a result and to successfully apply it to a large class of problems, we decided to define and use the simplest topology on the selected class of domains, the so-called uniform convergence. This topology seems to be the "most appropriate" to the class of convex domains of $\mathbb{R}^N$ of prescribed volume and induces a rather strong convergence. By "most appropriate", we mean for instance that our abstract result can be adapted to PDE of arbitrary order, and not only of order two.

This paper is inspired by the papers of G. Buttazzo and B. Kawohl and was realized with G. Buttazzo after being awared of [14]. I take the opportunity to deeply thank professor G. Buttazzo for introducing me in his research team during this year.

2 Preliminary results

Before proving the abstract existence result (Theorem 3.1) we introduce some of the tools we will need. In particular, we intend to make light over some properties of convex domains.

Let $D \subset \mathbb{R}^N$ be a compact set and let $m > 0$; we consider the following class of domains

$$C_m(D) = \{\Omega \subset D \text{ open, convex and s.t. } |\Omega| = m\}.$$

Definition 2.1. We say that a sequence $(\Omega_n)_{n\in\mathbb{N}} \subset C_m(D)$ uniformly converges to $\Omega \in C_m(D)$ if for every $\epsilon > 0$ one has

(d1) $\Omega_n \subset \Omega + B(\epsilon)$,

(d2) $\Omega \subset \Omega_n + B(\epsilon)$,

for n large enough and with $B(\epsilon)$ the open ball of radius ϵ in $\mathbb{R}^N$, centered at the origin.

In the following, a functional $J : C_m(D) \to \overline{\mathbb{R}}$ is said to be continuous if it is continuous according to this definition of convergence.

Remark 2.2. We remark that the definition of the uniform convergence is equivalent to the convergence induced by the Hausdorff distance,

$$d(\Omega_1, \Omega_2) = \max\{ \sup_{x\in\Omega_1} d(x, \Omega_2), \sup_{x\in\Omega_2} d(x, \Omega_1)\}$$

with $d(x, A) = \inf_{y\in A}\{|x - y|\}$. It could also be useful to define the complementary Hausdorff distance (see [4]),

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x\in\mathbb{R}^N} \left|d(x, \Omega_1^c) - d(x, \Omega_2^c\right|.$$

Similarly, for $(\Omega_n)_{n\in\mathbb{N}} \subset C_m(D)$ and $\Omega \subset C_m(D)$, we have, as $n \to \infty$,

$$\Omega_n \to \Omega \quad \text{uniformly} \quad \Longleftrightarrow \quad d_{H^c}(\Omega_n, \Omega) \to 0.$$

Moreover, it is well-known (see for instance [15] or [20]) that the class of closed subset of D is compact for the Hausdorff convergence.

Let us give some properties of convex domains. First of all, convex domains have some kind of regular boundary. More precisely, according to [13] (section 1.2) we have the following lemma.

Lemma 2.3. *If Ω is a convex domain $\subset \mathbb{R}^N$ then $\partial\Omega$ is locally Lipschitz continuous. In particular, we have following properties of the boundary :*

- *(i) there exists a tangent plane $\mathcal{H}^{N-1}$-almost everywhere;*
- *(ii) there are no zero interior angles.*

Note that points (i) and (ii) follow respectively from Rademacher's theorem (see [11]) and from the definition of convexity. According to [11], the boundary $\partial\Omega$ can by described by a $W^{1,\infty}$ function of some parameters.

In addition to this property, any convex domain can be approximated by a sequence of convex domains with smooth boundary.

Lemma 2.4. *Let Ω be a convex, bounded and open set of $\mathbb{R}^N$. Then for every $\epsilon > 0$, there exist two convex open subset Ω_1 and Ω_2 in R^N such that*

- *(a) $\Omega_1 \subset \Omega \subset \Omega_2$,*
- *(b) Ω_1 and Ω_2 have C^2 boundaries $\partial\Omega_1$ and $\partial\Omega_2$,*
- *(c) $d(\partial\Omega_1, \partial\Omega_2) < \epsilon$,*

where $d(\partial\Omega_1, \partial\Omega_2)$ denotes the Hausdorff distance between $\partial\Omega_1$ and $\partial\Omega_2$.

Proof. We refer to [13], lemma 3.2.1.1, p.147 for a proof.

■

In the following lemmas, $|\Omega| = \mathcal{H}^N(\Omega)$ denotes the $\mathcal{H}^N$-measure of Ω and $|\partial\Omega| = \mathcal{H}^{N-1}(\partial\Omega)$ denotes the $\mathcal{H}^{N-1}$-measure of its boundary (see [1] for an introduction to the Hausdorff measure).

Lemma 2.5. *The following properties hold for convex domains:*

(i) If $\Omega_1 \subset \Omega_2$ are two convex bodies then we also have $|\partial\Omega_1| \leq |\partial\Omega_2|$.

(ii) If $(\Omega_n)_n$ converges uniformly to Ω, then $|\Omega_n| \to |\Omega|$ and $|\partial\Omega_n| \to |\partial\Omega|$.

Proof. We refer to [7] for a proof.

■

In the second statement, the convergence of the measures of the domains and of their boudaries is related to the convexity and to the Hausdorff convergence of the domains. However, the convexity is not a necessary condition for this property to hold. In fact, a more general class for this property would be the class of star-shaped domains w.r.t a ball. Let us mention that in general, for open domains, if $d_{H^c}(\Omega_n, \Omega) \to 0$ as $n \to \infty$ then $\liminf_{n\to\infty} |\Omega_n| \geq |\Omega|$. Moreover, concerning the second part of statement *(ii)*, let us remark that for general open subsets of $\mathbb{R}^N$ even a relationship such as $\liminf_{n\to\infty} |\partial\Omega_n| \geq |\partial\Omega|$ is false in general. In fact, we only have the lower semicontinuity of the one-dimensional measure of the boundary w.r.t. the Hausdorff convergence of one-dimensional connected sets (see [1] for more results of this kind).

Let us continue these preliminary results by a fundamental isoperimetric inequality, i.e. a relationship between volume and area of subsets of R^N. Such inequalities, called "Bonnessens inequalities" are summarized and proved in [19].

Lemma 2.6. *For convex bodies $\Omega \subset R^N, N \geq 2$, one has*

$$\mathcal{H}^N(\Omega) < \rho\, \mathcal{H}^{N-1}(\partial\Omega),$$

where ρ is the radius of the largest ball included in Ω.

Proof. See [19] for a proof.

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Let us compare this Bonnesen's inequality to the following isoperimetric inequality (see [12])

$$\mathcal{H}^N(\Omega) < C\, \left|\left|\partial\Omega\right|\right|^{\frac{N}{N-1}},$$

valid in the more general context of sets of finite perimeter. Instead of the Haudorff $N-1$-dimensional measure on the right hand-side, one has the 'measure' perimeter. For convex domains, we remark that the relationship $\mathcal{H}^{N-1}(\partial\Omega) = \left|\left|\partial\Omega\right|\right|$ holds (see [7]) and consequently the constant C is related to ρ, that is, C has a geometrical meaning.

As announced, we seek to minimize the eigenvalues of some elliptic operator L of order $2p, p \geq 1$ over $C_m(D)$. To do this we first need to show each eigenvalue is a continuous functional on $C_m(D)$. To achieve this aim, an important result is the well-known min-max characterization of the k-th eigenvalue of L. In this paper, we mean by ellipticity of an operator that the associated weak form is coercive (sometimes this is called "strong ellipticity"). Let $\Omega \in C_m(D)$ and let $H(\Omega) \subset L^2(\Omega)$ be a Hilbert space. If the operator L induces a symmetric and coercive weak form (Lu, u) in $H(\Omega)$, then the eigenvalues $\lambda_k \in \mathbb{R}$ and the eigenfunctions $u_k \in H(\Omega)$ satisfy $Lu_k = \lambda_k u_k$ $(k \in \mathbb{N})$ with

$$\lambda_k = \min_{H_k} \max_{\substack{u \in H_k,\\ (u,u)=1}} (Lu, u),$$

where H_k is any k-dimensional subspace of $H(\Omega)$ and $(.,.)$ is the scalar product in $L^2(\Omega)$. Similarly, if the eigenvalues λ_k and the eigenfunctions u_k are such that $Lu_k = \lambda_k Bu_k$, for an elliptic operator B of order $2q, q < p$ (B is related to a symmetric and coercive weak form (Bu, u)), then we would have

$$\lambda_k = \min_{H_k} \max_{\substack{u \in H_k, \\ (Bu,u)=1}} (Lu, u).$$

Another important characterization reads as

$$\lambda_k = \min_{\substack{u \in H_{k-1}, \\ (Bu,u)=1}} (Lu, u) \tag{2.1}$$

being H_{k-1} the following subspace of H defined with the $(k-1)$ first eigenvectors e_i :

$$H_{k-1} = \{u \in H, (Bu, e_i) = 0, 1 \leq i \leq k-1\}.$$

See [18],[21] for a review on this topic. This characterization will be very useful in proving the following monotonicity result. Let us define the space $\mathcal{A}(D) = \{\Omega \subset D,\ \Omega \text{ is open}\}$.

Lemma 2.7. *If $(\lambda_k)_k$ is the sequence of all eigenvalues of the problem $Lu = \lambda Bu$, $u \in H_0^p(\Omega)$, for elliptic symmetric operators L (of order $2p, p \geq 1$) and B (of order $2q, q < p$), then the functional $\lambda_k : \mathcal{A}(D) \to \overline{\mathbb{R}}$ is monotone decreasing with respect to the set inclusion, i.e. if $\Omega_1 \subset \Omega_2$, then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$.*

Proof. Since $H_0^p(\Omega)$ is the usual Sobolev space with all derivatives of order $\leq p-1$ being 0 on the boundary $\partial\Omega$, we remark that any competing function for $\lambda_k(\Omega_1)$ extended by 0 in $\Omega_2 \setminus \Omega_1$, is also a competing function for $\lambda_k(\Omega_2)$. Then the proof follows from the characterization (2.1) with $H = H_0^p$, since we have

$$\lambda_k(\Omega_2) = \min_{\substack{v \in H_{k-1}(\Omega_2), \\ (Bv,v)=1}} (Lv, v) \leq (L\tilde{u}, \tilde{u}) = (Lu, u),$$

for any $u \in H_0^p(\Omega_1)$ and its extension $\tilde{u} \in H_0^p(\Omega_2)$. Passing on the right hand-side to the minimum over all normalized functions $u \in H_{k-1}(\Omega_1)$, we obtain

$$\lambda_k(\Omega_2) = \min_{\substack{v \in H_{k-1}(\Omega_2), \\ (Bv,v)=1}} (Lv, v) \leq \lambda_k(\Omega_1) = \min_{\substack{u \in H_{k-1}(\Omega_1), \\ (Bu,u)=1}} (Lu, u).$$

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For the existence of eigenvalues and eigenvectors to the spectral problem $Lu = \lambda Bu$ we refer to spectral theorems, see for instance [18] or [21] for a review.

This monotonicity result applies for instance for second order operators of type

$$Lu = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c_0(x)u,$$

with symmetric and positive defined $a_{ij}(\cdot) \in \mathcal{C}(D)$ and nonnegative $c_0 \in L^\infty(D)$.

3 The Existence Result

The abstract existence result we shall apply later is the following.

Theorem 3.1. *Let $D \subset \mathbb{R}^N$ be bounded and consider a functional $J : C_m(D) \to \overline{\mathbb{R}}$ such that*

- *J is bounded from below*
- *J is lower semicontinuous , i.e. for each sequence $(\Omega_n)_n$ such that $\Omega_n \to \Omega$ as $n \to \infty$, we have $\liminf_{n\to\infty} J(\Omega_n) \geq J(\Omega)$.*

Then the infimum of J over $C_m(D)$ is achieved, i.e. there exist $\Omega^* \in C_m(D)$ such that

$$\inf_{C_m(D)} J(\Omega) = \min_{C_m(D)} J(\Omega) = J(\Omega^*).$$

Proof. Since the functional is bounded below, there exist a finite infimum in $C_m(D)$ and the result follows from the direct method of calculus of variations, as soon as we prove the set $C_m(D)$ is (sequentially) compact with respect to the uniform convergence. We then consider any minimizing sequence, and extract a subsequence which converges to the minimizer Ω^*, the functional being lower semicontinuous.

Let us show the compactness of $C_m(D)$. If we consider any sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $C_m(D)$, then a ball of fixed radius $\rho > 0$ fits every Ω_n, since, according to Lemma 2.5 and Lemma 2.6, one has $\rho_n > \frac{|\Omega_n|}{|\partial\Omega_n|} > \frac{m}{S} > 0$, with S the $\mathcal{H}^{N-1}$-measure of the boundary of D, and ρ_n the largest inner radius associated to Ω_n. We define $\rho \doteq \frac{m}{S}$ and consider the sequence of inner balls $(B_n)_{n\in\mathbb{N}} \subset D, B_n = B(x_n, \rho)$, with $(x_n)_{n\in\mathbb{N}}$ being the sequence of all the corresponding centers. The sequence $(x_n)_{n\in\mathbb{N}}$ is contained in the compact set D, thus we may extract a subsequence converging to a point $x^*$of D.

Following an idea found in [10], we shall associate the above sequence $(x_n)_{n\in\mathbb{N}}$ to the sequence of functions $(r_n)_n$, each radius r_n describing parametrically the boundary $\partial\Omega_n$ (this kind of parametric description make sense, since the domains are convex). Since $B(x_n, \rho) \subset \Omega_n \subset B[0, R_m]$, R_m large enough such that $D \subset B[0, R_m]$, then by convexity of the Ω_n, the sequence $(\partial\Omega_n)_{n\in\mathbb{N}}$ is equi-Lipschitz continuous, i.e. the Lipschitz constants of all r_n are uniformly bounded: $L_n(x) < L$, $x \in \partial\Omega_n$. With S^{N-1} being the unit ball of $\mathbb{R}^{N-1}$, the functions $r_n : S^{N-1} \to [\rho, 2R_m]$ are such that $(r_n)_n$ is bounded in $W^{1,\infty}(S^{N-1})$. Then, we have by Rellich-Kondrachov's compact embedding theorem, that up to a subsequence, $(r_n(\cdot))_n$ converges to a continuous function $r^*(\cdot)$ in $\mathcal{C}(S^{N-1})$. Since there is a sequence $((x_n, r_n(\cdot)))_n$ converging to $(x^*, r^*(\cdot))$, then for given $\epsilon > 0$, we have, n being large enough, $|x^* - x_n| < \epsilon$ and $|r^* - r_n|_\infty < \epsilon$. This is exactly, by definition 2.1 the uniform convergence. Thus $\Omega_n \to \Omega^*$, with $(\Omega_n)_{n\in\mathbb{N}}$ is a subsequence of the initial minimizing sequence and $B(x^*, \rho) \subset \Omega^* \subset D$. This convergence is strong enough to force Ω^* to be convex and, according to Lemma 2.5, the volume is preserved, $|\Omega^*| = m$. Moreover, since r^* is continuous, its epigraph is closed and thus Ω^* is open. Since Ω^* is an admissible domain, this ends the proof.

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Remark 3.2. Clearly, the supremum of J is achieved as soon as J is bounded from above and upper semicontinuous, i.e. for each sequence $(\Omega_n)_n$ such that $\Omega_n \to \Omega$ as

$n \to \infty$, we have $\limsup_{n\to\infty} J(\Omega_n) \leq J(\Omega)$. Moreover if the functional is bounded and continuous, then the infimum and the maximum over $C_m(D)$ are both achieved.

Remark 3.3. Instead of the constraint $|\Omega| = m$ we could prove the same existence result with the weaker condition that $c \leq |\Omega| \leq m$, for any $c > 0$ and $m \leq |D|$.

Remark 3.4. We could prove this result with other assumptions. According to the work of J. Cox and M. Ross (see [10]), another suitable class of domains in $\mathbb{R}^2$ are the domains which are starshaped, which contain a disk, occupy a given area and do not exceed a prescribed parameter. It can be shown, by some counter examples, that this results for planar domains may not be extended to any space dimension $N > 2$.

4 First Application: Minimum Problems for Eigenvalues.

The monotonicity of the eigenvalues (see Lemma 2.7) is fundamental because it allows us to prove the continuity of the spectrum of a certain class of problems. For example, for an operator L of order $2p$ with no lower order terms, the continuity follows from the monotonicity property and from the homogeneity of the eigenvalues, i.e. $\lambda_k(\alpha\Omega) = \alpha^{-2p}\lambda_k(\Omega)$ $(\alpha \in \mathbb{R})$. The homogeneity holds if the a_{ij} are constants. Unfortunately, the homogeneity fails at soon as the a_{ij} are non-constant functions but we will show in the following that the continuity of the eigenvalues still holds if the $a_{ij}(\cdot)$ are continuous functions on D. Moreover, we will mostly consider that the operator B is the identity.

Let us consider as first step the second order elliptic operator of following type:

$$Lu = \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right), \tag{4.1}$$

where $(a_{ij})_{i,j}$ is a constant positive defined symmetric matrix of order $N \times N$. Since Ω is bounded, then by Poincaré's inequality the Sobolev space $H_0^1(\Omega)$ is endowed with the norm $||u||^2_{H_0^1(\Omega)} = \int_\Omega u_{x_i}^2 dx$ $(= \int_\Omega (Du)^2 dx)$ and thus the operator L admits a spectral decomposition. If $\Lambda(\Omega)$ denotes the whole spectrum of L over $H_0^1(\Omega)$, we wish to prove that the problem

$$\min\{\Phi(\Lambda(\Omega)), \Omega \in C_m(D)\} \tag{4.2}$$

admits at least a solution provided the functional $\Phi : \mathbb{R}^{\mathbb{N}} \to \overline{\mathbb{R}}$ is bounded from below and is lower semicontinuous in the sense that

$$\text{if} \quad \lambda_k(\Omega_n) \to \lambda_k(\Omega) \quad \text{for every } k, \text{ as } n \to \infty \quad \text{then} \quad \liminf_{n\to\infty} \Phi(\Lambda(\Omega_n)) \geq \Phi(\Lambda(\Omega)),$$

being $\Lambda(\Omega) = (\lambda_k(\Omega))_{k\in\mathbb{N}}$ and $\Lambda(\Omega_n) = (\lambda_k(\Omega_n))_{k\in\mathbb{N}}$. In order to prove this result, it is sufficient to prove the following lemma.

Lemma 4.1. *For an operator of type (4.1) and for every $k \in \mathbb{N}$, the functional $\lambda_k : C_m(D) \to \overline{\mathbb{R}}$ is bounded and continuous.*

Proof. The boundedness of λ_k simply follows from the monotonicity of the eigenvalues (see Lemma 2.7) because

$$\lambda_k(D) \leq \lambda_k(\Omega) \leq \lambda_k(B_\rho) \qquad \forall \Omega \in C_m(D),$$

where B_ρ is an inner ball of radius ρ (see proof of the Theorem 3.1) included in Ω. We can consider, in this case, that B_ρ is centered at the origin. To show the continuity we shall use the monotonicity and the homogeneity of the eigenvalues (since the a_{ij} are constants, then by simply changing variable it is easy to show the homogeneity of degree -2). Since the domains are convex, we have the existence of $t \geq 1$, such that for all $\epsilon > 0$:

$$\Omega + B(0,\epsilon) \subset (1 + t\epsilon)\Omega, \tag{4.3}$$

the expansion being performed with respect to $x_\Omega \in \Omega$ (for constant a_{ij} we can consider x_Ω to be the origin). Let us denote $\Omega + B(0,\epsilon)$ by Ω_ϵ. By the relationship (4.3), by the homogeneity and by Lemma 2.7 we have $(1 + t\epsilon)^{-2}\lambda_k(\Omega) = \lambda_k((1 + t\epsilon)\Omega) \leq \lambda_k(\Omega_\epsilon)$. By definition of the uniform convergence of the domains, $(d1)$ leads, for n large enough, to $(1+t\epsilon)^{-2}\lambda_k(\Omega) \leq \lambda_k(\Omega_n)$ and $(d2)$ to $(1+t\epsilon)^{2}\lambda_k(\Omega) \geq \lambda_k(\Omega_n)$. Hence, as $\epsilon \to 0$ we have proved the continuity of the k-th eigenvalue for every $k \in \mathbb{N}$.

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Corollary 4.2. *Problem (4.2) admits at least a solution.*

Proof. The existence result follows now easily from Theorem 3.1, since Lemma 4.1 allows us to show that $\Phi(\Lambda(\Omega))$ is l.s.c. with respect to the uniform convergence as soon as Φ is l.s.c on $\mathbb{R}^{\mathbb{N}}$.

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We consider now the problem (4.2) associated to the spectral problem (4.1) where the $a_{ij}(\cdot)$ are not constants any more but still continuous functions on D. In this case, we deal with second order operators of type

$$Lu = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right). \tag{4.4}$$

We already pointed out that we do not have the homogeneity of the eigenvalues

$$\lambda_k(\Omega) = \min_{\substack{u \in H_{k-1},\\ \int_\Omega u^2(x)dx = 1}} \int_\Omega a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx, \tag{4.5}$$

with H_{k-1} as defined in (2.1) and the sum is made on repeated indices.

Once again, our goal, is to prove an existence result to problem (4.2) for an operator L as (4.4). As before the existence will easily follow from the following lemma.

Lemma 4.3. *For every $k \in \mathbb{N}$, the functional $\lambda_k : C_m(D) \to [-M,+M]$ associated to the operator L defined in (4.4), is continuous, provided the a_{ij} are continuous functions on D.*

Proof. Since we have the characterization (4.5), $\lambda_n(\cdot)$ is monotone decreasing (see Lemma 2.7) and the boundedness of $\lambda_k(\cdot)$ follows by the same arguments as in Lemma 4.1. In fact, we have

$$\lambda_k(\Omega) \leq \lambda_k\big(B(x_\Omega, \rho)\big) \leq \max_{i,j} \big|\big|a_{ij}\big|\big|_{L^\infty(D)} \hat{\lambda}_k\big(B(0,\rho)\big),$$

with $\hat{\lambda}_k$ standing for the k-th eigenvalue of the problem with every $a_{ij}(\cdot)$ set to 1. To prove the continuity, let us fix $\epsilon > 0$ and consider a sequence $(\Omega_n)_{n\in\mathbb{N}}$ converging uniformly to Ω, then we have, by convexity of the domains, for n large enough,

$$\lambda_k(\Omega_n) \leq \lambda_k(\Omega + B_\epsilon) \leq \lambda_k((1+\epsilon)\Omega) \qquad \text{(A)}$$

and

$$\lambda_k(\Omega) \leq \lambda_k(\Omega_n + B_\epsilon) \leq \lambda_k((1+\epsilon)\Omega_n) \qquad \text{(B)}.$$

Moreover, by (4.5),

$$\lambda_k(G) = \min_{\substack{u\in H_{k-1}, \\ \int_G u^2(x)dx=1}} \int_G a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx = \int_G a_{ij}(x)\frac{\partial u_G(x)}{\partial x_i}\frac{\partial u_G(x)}{\partial x_j}dx,$$

with the generic symbol G used indifferently for Ω_n or Ω and $u_G \in H_{k-1}$, a solution of $Lu(x) = \lambda_k(G)u(x)$, $x \in G$. Hence, dealing first with Ω, we have

$$\lambda_k((1+\epsilon)\Omega) \leq \frac{\displaystyle\int_{(1+\epsilon)\Omega} a_{ij}(x)\frac{\partial u^\epsilon_\Omega(x)}{\partial x_i}\frac{\partial u^\epsilon_\Omega(x)}{\partial x_j}dx}{\displaystyle\int_{(1+\epsilon)\Omega} |u^\epsilon_\Omega(x)|^2 dx},$$

with the function $u^\epsilon_\Omega(.) = u_\Omega(\frac{\cdot}{1+\epsilon})$ being a competing function for the infimum on $(1+\epsilon)\Omega$. Then, by simply changing variable in the right hand-side (i.e integrating in $y = (1+\epsilon)x$), it follows that

$$\lambda_k((1+\epsilon)\Omega) \leq (1+\epsilon)^{-2}\frac{\displaystyle\int_\Omega a_{ij}((1+\epsilon)x)\frac{\partial u_\Omega(x)}{\partial x_i}\frac{\partial u_\Omega(x)}{\partial x_j}dx}{\displaystyle\int_\Omega |u_\Omega(x)|^2 dx} \tag{4.6}$$

Whence, by continuity of the $a_{ij}(.)$,

$$\text{(C)} \qquad \liminf_{\epsilon\to 0} \lambda_k((1+\epsilon)\Omega) \leq \lambda_k(\Omega) \qquad \text{and by (A)} \qquad \liminf_{n\to\infty} \lambda_k(\Omega_n) \leq \lambda_k(\Omega) \qquad \text{(D)}.$$

Inequality (C) holds with Ω_n instead of Ω, so from(B) we have

$$\lambda_k(\Omega) \leq \liminf_{n\to\infty} \liminf_{\epsilon\to 0} \lambda_k((1+\epsilon)\Omega_n) \leq \liminf_{n\to\infty} \lambda_k(\Omega_n).$$

Then, by this last inequality and by (D), it follows that

$$\liminf_{n\to\infty} \lambda_k(\Omega_n) = \lambda_k(\Omega) = \limsup_{n\to\infty} \lambda_k(\Omega_n) = \lim_{n\to\infty} \lambda_k(\Omega_n),$$

since the lim sup could be used instead of the lim inf for the limit process above.

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Corollary 4.4. *Problem (4.2) associated to the spectral decomposition of (4.4) admits at least a solution.*

Proof. See the proof of Corollary 4.2.

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Our existence result allows us to consider any symmetric (strongly) elliptic operator of order $2p, p \geq 1$, since the uniform convergence is a priori adapted to any Sobolev space $H_0^p(\Omega), p \geq 1$. For example, let us consider a 4-th order operator L of the form

$$Lu = \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}(x)\frac{\partial^2 u}{\partial x_k \partial x_l}\right). \tag{4.7}$$

with $a_{ijkl}(\cdot)$, a continuous tensor on D, verifying the symmetry assumption $a_{ijkl} = a_{ijlk} = a_{jikl} = a_{klij}$ and such that L is elliptic, i.e. there exists $\nu > 0$ such that for every $\xi \in \mathbb{R}^{N\times N}$:

$$\nu\xi_{ij}^2 \leq a_{ijkl}(x)\xi_{ij}\xi_{kl} \qquad \text{a.e. in } D.$$

Since Ω is bounded, then by Poincaré's inequality the Sobolev space $H_0^2(\Omega)$ is endowed with the norm $||u||^2_{H_0^2(\Omega)} = \int_\Omega u^2_{x_ix_j}dx$ $(= \int_\Omega |D^2u|^2dx)$ and thus the operator L admits a spectral decomposition in $H_0^2(\Omega)$.

The proof of a solution for problem (4.2) relies again on the continuity of each eigenvalue

$$\gamma_k(\Omega) = \min_{\substack{u\in H_{k-1},\\ \int_\Omega u^2(x)dx=1}} \int_\Omega a_{ijkl}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_k \partial x_l}dx. \tag{4.8}$$

This can be easily proved as in Lemma 4.1 or Lemma 4.3.

We point out that the method used for proving Corollary 4.2 and Corollary 4.4 can be used in a similar way to easily prove such existence results for any linear, symmetric and strongly elliptic operators L of order $2p$, provided we have the continuity of the coefficient of L on D and seek an optimal domain over the class of all bounded convex domains with prescribed measure.

Remark 4.5. The continuity property also holds for a class of operators such as (4.1) or (4.7) with an additional zero order coefficient $c_0(x)u$, provided c_0 is nonnegative and is a continuous function on D. In fact, for a second order operator, relation (4.6) becomes

$$\lambda_k((1+\epsilon)\Omega) \leq \qquad \ldots \qquad + (1+\epsilon)\frac{\int_\Omega c_0((1+\epsilon)x)(u_\Omega(x))^2dx}{\int_\Omega |u_\Omega(x)|^2dx}$$

and the result follows from the continuity of $c_0(\cdot)$.

Finally, the same kind of arguments can also be repeated for a larger class of operators, having the general form

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}(x)\frac{\partial^2 u}{\partial x_k \partial x_l}\right) - \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c_0(x)u, \tag{4.9}$$

with nonnegative, continuous and symmetric b_{ij} and nonnegative, continuous c_0. If, for every (i,j), b_{ij} and c_0 are not nonnegative on D, then these low order terms may create some troubles concerning the coerciveness of the weak form, unless $|b_{ij}|$, $|c_0|$ and the measure of Ω are "small enough" compared to the ellipticity constant ν (see for instance [16]).

Remark 4.6. The following is known (see [4]) when the domains are not required to be convex but are all quasi-open sets included in D: for $L = -\Delta$, any lower semi-continuous functional of the variable $\big(\lambda_1(\Omega), \lambda_2(\Omega)\big)$ achieve its infimum. Moreover, if the functional is also monotone decreasing with respect to the set inclusion, then the latter result holds for the whole spectrum of $-\Delta$.

Remark 4.7. We considered the first application for the general Sobolev spaces $H_0^p(\Omega)$. For an operator L such as 4.7, in the space $H_0^1(\Omega) \cap H^2(\Omega)$, the previous results do not hold, since the monotonicity property of the eigenvalues fails.

5 Second Application: Minimizing Functionals given by an Integral.

In this second application let us consider functionals of following type:

$$J : C_m(D) \to \mathbb{R} \qquad \text{is such that} \qquad J(\Omega) = \int_D j(x, u_\Omega, Du_\Omega(x), ..., D^{p-1}u_\Omega(x))dx,$$

with $j : \Omega \times \mathbb{R}^{p-1} \to \overline{\mathbb{R}}$ and u_Ω is the unique solution, extended by zero outside Ω, of the problem

$$u \in H_0^p(\Omega), \qquad Lu = f, \qquad f \in L^2(D),$$

with the operator L of order $2p, p \geq 1$. Let us write the solution as $u_\Omega = L^{-1}(\Omega, f)$. We will consider the problem

$$\min_{\substack{\Omega \in C_m(D) \\ u_\Omega = L^{-1}(\Omega, f)}} \int_D j(x, u_\Omega(x), Du_\Omega(x), ..., D^{p-1}u_\Omega(x))dx. \tag{5.1}$$

We are interested in searching for a solution to the problem (5.1) using the abstract Theorem 3.1. We remark that without the convexity constraint on the domains, problem (5.1) could have its infimum achieved outside the set we have chosen for the minimizers. In this case, one should relax the problem, i.e. minimize a generalized functional over a more general class, for instance a class of measures, and try to link this latter problem to the original one, as done in [5] for a class of open but not necessarily convex domains of $\mathbb{R}^N$ and for $p = 1$. We shall prove that, on the contrary, thanks to convexity of admissible domains, problem (5.1) admits at least a solution, provided the function $j : \Omega \times \mathbb{R}^{p-1} \to \mathbb{R}$ is bounded from below and lower semicontinuous in the $(p-1)$ last variables. We remark that no growth condition on j is required in our assumptions. We shall consider second order operators of type (4.4) and fourth order operators of type (4.7) and finally apply the existence result to the biharmonic operator Δ^2. Let us start with L as in (4.7). For more general operators L of higher order, the proof is similar to the following one, but for convenience we restricted ourself, for the proof, to the fourth-order situation.

Corollary 5.1. *The optimization problem (5.1) with the 4-th order operator L as in (4.7) admits a solution.*

Proof. The weak form of the equation $Lu = f$ on Ω is, for $u \in H_0^2(\Omega)$

$$(Lu, \phi) = \int_\Omega a_{ijkl}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \frac{\partial^2 \phi(x)}{\partial x_k \partial x_l} dx = \int_\Omega f(x)\phi(x)dx \qquad \forall \phi \in C_0^\infty(\Omega) \tag{5.2}$$

The tensor $a_{ijkl}(\cdot)$ is continuous on the compact D and thus it is also bounded. Since it is also positive defined on D, we have the bilinear and symmetric form (Lu, v) defines a scalar product on $H_0^2(\Omega)$, i.e.

$$\nu \int_\Omega u_{x_i x_j}^2(x)dx \leq (Lu, u) \leq \alpha \int_\Omega u_{x_i x_j}^2(x)dx, \tag{5.3}$$

with $\nu, \alpha > 0$. The existence of a unique weak solution u_Ω in $H_0^2(\Omega)$ for the equation $Lu = f$ on Ω follows then by Lax-Milgram's theorem. Problem 5.1 is well defined and we chose a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}}$. Let us show now the solutions $u_{\Omega_n} \to u_\Omega$ in $H_0^1(D)$, provided Ω_n uniformly converges to Ω. Since we have (5.2) and (5.3), we also have

$$\nu ||u_{\Omega_n}||_{H_0^2(D)}^2 = \nu \int_D u_{x_i x_j}^2(x)dx \leq (Lu_{\Omega_n}, u_{\Omega_n}) = (f, u_{\Omega_n})$$

$$\leq ||f||_{L^2(\Omega_n)} ||u_{\Omega_n}||_{L^2(\Omega_n)} \leq ||f||_{L^2(D)} ||u_{\Omega_n}||_{H_0^2(D)}$$

and finally $||u_{\Omega_n}||_{H_0^2(D)} \leq \frac{||f||_{L^2(D)}}{\nu}$, the solutions u_{Ω_n} being extended by 0 outside Ω_n. Thus $u_{\Omega_n} \to u$ weakly in $H_0^2(D)$ and by Rellich's theorem, $u_{\Omega_n} \to u$ strongly in $H_0^1(D)$. It remains to show $u = u_\Omega$. Since the domains are convex, the uniform convergence of the domains implies that any test function $\phi \in \mathcal{C}_0^\infty(\Omega)$ is a test function in $\mathcal{C}_0^\infty(\Omega_n)$, provided n is large enough. Hence, for n large enough and $\phi \in \mathcal{C}_0^\infty(\Omega)$,

$$(f, \phi) = (Lu_{\Omega_n}, \phi) = \int_D a_{ijkl}(x) \frac{\partial^2 u_{\Omega_n}(x)}{\partial x_i \partial x_j} \frac{\partial^2 \phi(x)}{\partial x_k \partial x_l} dx,$$

by the above property of the test function. This integral converges to

$$\int_D a_{ijkl}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \frac{\partial^2 \phi(x)}{\partial x_k \partial x_l} dx = (Lu, \phi)$$

as $n \to \infty$, by the weak convergence of the u_{Ω_n} in $H_0^2(D)$. Hence

$$(Lu, \phi) = (f, \phi) \qquad \forall \phi \in C_0^\infty(\Omega).$$

Moreover, we have the $\mathcal{H}^N$-almost everywhere pointwise convergence of $u_{\Omega_n}(x)$ to $u(x)$ and of $(Du)_{\Omega_n}(x)$ to $(Du)(x)$ as $n \to 0$, that is, $u(x) = (Du)(x) = 0$ $\mathcal{H}^N$-a.e. on $\partial\Omega$ by the classical trace theorems. Since the weak solution of $Lu = f$ is unique in $H_0^2(\Omega)$, it follows that $u = u_\Omega$.

Thus, given $f \in L^2(D)$, we proved that both $u_{\Omega_n} \to u_\Omega$ and $(Du)_{\Omega_n} \to (Du)_\Omega$ in $L^2(D)$. It follows that for almost every $x \in D$ there are two subsequences $u_{\Omega_n}(x) \to u_\Omega(x)$ and $(Du)_{\Omega_n}(x) \to (Du)_\Omega(x)$. By Fatou's lemma and by the semicontinuity of j,

$$\liminf_{n\to\infty} \int_D j(x, u_{\Omega_n}(x), Du_{\Omega_n}(x))dx \geq \int_D \liminf_{n\to\infty} j(x, u_{\Omega_n}(x), Du_{\Omega_n}(x))dx$$

$$\geq \int_D j(x, u_\Omega(x), Du_\Omega(x))dx,$$

whence

$$\liminf_{n\to\infty} J(\Omega_n) \geq J(\Omega).$$

The functional being bounded below, the existence result follows from the lower semicontinuity of the functional and from the existence Theorem 3.1.

■

Again, we easily obtain a generalization of Corollary 5.1 in the case the operator L is of order $2p$, is linear, symmetric and strongly elliptic. In additioon to these assumptions, required conditions for the proof are the boundedness of the coefficient of L on D and the convexity of the domains. Note that the latter condition may be weakened (see for instance the definition of the "compactivorous property" for connected domains in the paper [9]).

Remark 5.2. If $j(x,0,\cdots,0) = 0$ a.e. in D, then the previous existence result holds for $J(\Omega) = \int_\Omega j(x, u_\Omega, Du_\Omega(x), ..., D_\Omega^{p-1}(x))dx$ since the solutions are, as usual, extended by 0 outside Ω.

Remark 5.3. Corollary 5.1 still holds for a generalized second order operator L with first order and zero order terms (such as (4.9)). In fact, we can follow the proof of Corollary 5.1 provided the low order coefficients and the size of Ω are "small enough" compared to the ellipticity constant ν.

Let us apply Corollary 5.1 to the biharmonic operator, i.e let us consider optimization problem (5.1) associated to the equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

with $\Omega \in C_m(D)$ and $f \in L^2(D)$.

It is enough to show that the operator Δ^2 verifies (5.3) (i.e. that Δ^2 is strongly elliptic) in Corollary 5.1 (here we have $a_{ijkl} = \delta_{ij}\delta_{kl}$). In fact, in [16], the authors proved (easily) that $||u||_{H_0^2(\Omega)} \leq \beta||\Delta u||_{L^2(\Omega)} \leq \beta||u||_{H_0^2(\Omega)}$ and the thesis of Corollary 5.1 then follows for the biharmonic operator.

Remark 5.4. We could wonder whether the second application holds for spectral problems, i.e. for

$$u \in H_0^p(\Omega), \qquad Lu = \lambda u, \qquad \lambda \in \mathbb{R},$$

with the operator L of order $2p, p \geq 1$. The question is once again the existence of a minimizer of the functional $J(\Omega) = \int_D j(x, u_\Omega^k, Du_\Omega^k(x), ..., D^{p-1}u_\Omega^k(x))dx$ for the k-th eigenfunction u^k. Actually, we could follow the previous proof if we knew the k-th eigenvalue $\lambda_k(\Omega)$ is single, or equivalently if the associated eigenfunction u_Ω^k is unique modulo a constant (for a strongly elliptic and symmetric L, the k-th eigenfunction on $H_0^p(\Omega_n)$ is such that $||u_k||^2_{H_0^p(\Omega_n)} \leq \frac{\lambda_k(\Omega_n)}{\nu} \leq \frac{\lambda_k(B(\rho))}{\nu}$, for n large enough). Thus the previous existence result holds for instance for $L = \Delta$ in $H_0^1(\Omega)$ and $k = 1$. For other problems, for instance for the Laplacian and $k \geq 2$, for the Neumann problem involving the Laplacian, for the bilaplacian in $H_0^2(\Omega)$ or for the Lamé system of linear elasticity, the existence of a minimizer still holds for the following problem:

$$\min_{\substack{\Omega \in C_m(D) \\ u_\Omega^k \in L^{-1}(\Omega,k)}} \int_D j(x, u_\Omega^k(x), Du_\Omega^k(x), ..., D^{p-1}u_\Omega^k(x))dx,$$

with $L^{-1}(\Omega, k)$ being a finite dimensional subset of $H_0^p(\Omega)$.

6 Third Application: On Newton's Problem of Minimal Resistance.

We are interested in proving the existence of a convex body $\Omega \subset \mathbb{R}^3$ which minimizes the so-called total resistance (see [8] and [14] and the references therein). According to Newton's model, given a domain $G \subset \mathbb{R}^2$, the total resistance is given by

$$R(G, u) = \int_G \frac{1}{1 + |Du|^2} dx,$$

with $u : G \to \mathbb{R}$. For a convex G and a concave u, a 3-dimensional convex body Ω can be built with G and the graph of u. In [8], the authors prove that the functional R achieves its infimum over the class $C_M = \{u \text{ concave on } G : 0 \leq u \leq M\}$.

In [7], the authors introduced another formalism for writing the total resistance in $\mathbb{R}^N$. They showed that $R(G, u)$ could be re-written as

$$F(\Omega) = \int_{\partial\Omega} ((\nu \cdot A)^+)^3 d\mathcal{H}^{N-1}, \tag{6.1}$$

with $(\nu \cdot A)^+$ denoting the positive part of the projection of the normal ν on the direction A. With this notations, the authors proved the existence of a minimizer of the functional F on the class

$$C'_m(D) = \{\Omega \subset D, \text{open and convex and s.t. } m \leq |\Omega|\}.$$

For proving this result, they made use of tools from the geometric measure theory, starting with an abstract theorem of Reshetnyak. Our goal is to prove this result using the tools and the results developed above. Once again, the convexity will be the most relevant property of the problem.

Proposition 6.1. *Let Ω_n be a sequence of convex domains converging to Ω^* uniformly. Then for all $x \in \partial\Omega \setminus S$ and all $x_n \in \partial\Omega_n \setminus S_n$ we have*

$$\text{if} \qquad x_n \to x \qquad \text{then} \qquad \nu_n(x_n) \to \nu(x),$$

where S_n and S respectively denote the sets of points where $\partial\Omega_n$ and $\partial\Omega$ are not differentiable.

Proof. Since $\Omega_n \to \Omega^*$ as $n \to \infty$, then for $n \geq n_0$ we have a ball $B(x^*, \rho)$ is included in every $\Omega_n, n \geq n_0$. Let us consider a point $x \in \partial\Omega^*$ where the boundary is differentiable, i.e. where the tangent plane $T(x)$ exists. By Lemma 2.3 there is a tangent plane almost everywhere, i.e. the sets S_n and S are $\mathcal{H}^{N-1}$ - negligible. Since all bodies are convex, the half line starting at x^* and passing by x crosses each Ω_n at only one point x_n. If we discard all x_n where the boundary $\partial\Omega_n$ has a corner, it remains a sequence $(\hat{x}_n)_{n \geq n_0}$ of points of differentiability converging to x as $n \to \infty$. If we can not extract from $(\hat{x}_n)_{n \geq n_0}$ a subsequence of points of differentiability converging to the point x, then let us discard the point x. The points we discarded remain anyway $\mathcal{H}^{N-1}$ - negligible. In the following, $\hat{\Omega}$ and $\partial\hat{\Omega}$, etc, are related to the sequence $(\hat{x}_n)_{n \in \mathbb{N}}$.

Let us define the tangent plane $\hat{T}_n(\hat{x}_n)$ to $\partial\hat{\Omega}_n$ at $\hat{x}_n$. We have defined a sequence of points of differentiability which all lie on a straight line, then by passing to a subsequence if necessary, we can consider either they all lie between x^* and x or all above x. We first consider the second situation occurs. If $\hat{r}_n$ is the parametric description of the $\partial\hat{\Omega}_n$ then, since $B(x^*, \rho) \subset \hat{\Omega}_n \subset D$, we know the sequence $(\nabla\hat{r}_n(\cdot))_{n\geq n_0}$ is equi-bounded and belongs then to a bounded subspace of $L^p(S^{N-1})$, for some $1 \leq p < \infty$. Moreover, for $n \geq n_0$, given $\epsilon > 0$, then for every small enough $h \geq 0$, we have by Lebesgue's theorem $|\nabla\hat{r}_n(\cdot + h) - \nabla\hat{r}_n(\cdot)|_{L^p} \leq \epsilon$. Then, by Riesz-Kolmogorov's compactness theorem the existence of a converging subsequence in $L^p(S^{N-1})$ follows (see [2]). Let us write $\nabla\hat{r}_n(\cdot) \to \gamma(\cdot)$ in $L^p(S^{N-1})$. Then, up to a subsequence, $(\nabla\hat{r}_n(\cdot))_{n\geq n_0}$ converges almost everywhere on S^{N-1}: $\nabla\hat{r}_n(\hat{x}_n) \to \gamma(x)$ as $n \to \infty$ or, in terms of normal vectors, $\hat{\nu}_n(\hat{x}_n) \to \hat{\nu}(x)$ as $n \to \infty$. It remains to prove that $\nu^*(x) = \hat{\nu}(x)$, where $\nu^*(x)$ is the normal to Ω^* at x.

Indeed, assume $\nu^*(x) \neq \hat{\nu}(x)$, then the normal plane to $\hat{\nu}(x)$ at x cuts the domain Ω^* in two parts and lies below a portion of the boundary $\partial\Omega^*$ of area δ. Since the converging domains $\hat{\Omega}_n$ are convex, the tangent plane at $\hat{x}_n$ (normal to $\hat{\nu}_n$) lies above the boundary $\partial\hat{\Omega}_n$. The contradiction follows easily, since the boundaries $\partial\hat{\Omega}_n$ converge to $\partial\Omega^*$ uniformly, while the tangent planes $\hat{T}_n$ also converge uniformly to $\hat{T}$ ($\hat{T}$ is normal to $\hat{\nu}$ at x). The same kind of arguments holds if the sequence of points $(\hat{x}_n)_{n\geq n_0}$ lie above x. This ends the proof.

■

With this result it is now very easy to prove the main result of this section.

Proposition 6.2. *Among all convex bodies of $D \subset \mathbb{R}^N$ of prescribed volume, there exists at least one body for which*

$$F(\Omega) = \int_{\partial\Omega} f(x, \nu(x)) d\mathcal{H}^{N-1}$$

is minimized, provided the function f is measurable and lower semicontinuous in the second variable.

Proof. Since for every n, there exists a one-to-one application $\phi_n : \partial\Omega^* \to \partial\Omega_n$ almost everywhere differentiable and s.t. ϕ_n^{-1} is a.e. differentiable,then

$$F(\Omega_n) = \int_{\partial\Omega_n} f(x_n, \nu_n(x)) d\mathcal{H}^{N-1} = \int_{\partial\Omega^*} f(x_n, \nu_n(x_n)) |J_{\phi_n}| d\mathcal{H}^{N-1}.$$

Then, according to Proposition 6.1 and by Fatou's lemma, the functional $F : C'_m(D) \to \overline{\mathbb{R}}$ is lower semicontinuous with respect to the uniform convergence. Thus, thanks to the existence result of section 3 and to remark 3.3 we may conclude the proof.

■

Corollary 6.3. *Among all convex bodies of $D \subset \mathbb{R}^N$ of prescribed volume, there exists at least one body for which the total resistance (6.1) is minimum.*

Proof. We apply Proposition 6.2 to $f(x, \nu(x)) = \left((\nu(x) \cdot A(x))^-\right)^3$ with the unit vector field A being the stream direction (we consider only the negative part of the inner product, i.e. $u^- \triangleq \frac{1}{2}\left(|u| - u\right)$ not to take into account the points that are not relevant for the total resistance).

■

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author's adress : vangoeth@mema.ucl.ac.be

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