

ON A NOTION OF UNILATERAL SLOPE FOR THE MUMFORD-SHAH FUNCTIONAL

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ABSTRACT. In this paper we introduce a notion of unilateral slope for the Mumford-Shah functional, and provide an explicit formula in the case of smooth cracks. We show that the slope is not lower semicontinuous and study the corresponding relaxed functional.

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1. INTRODUCTION

In the study of the gradient flow for a nonsmooth functional \mathcal{F} on a metric space (X, d) , it is useful to introduce a notion of slope $|\partial\mathcal{F}|$, which coincides with the norm of the gradient $\nabla\mathcal{F}$ in the case of a smooth functional on a Hilbert space. For every $u \in X$ with $\mathcal{F}(u) < +\infty$ the slope $|\partial\mathcal{F}|(u)$ is defined by

$$|\partial\mathcal{F}|(u) := \limsup_{v \rightarrow u} \frac{(\mathcal{F}(u) - \mathcal{F}(v))^+}{d(v, u)},$$

where $(\cdot)^+$ denotes the positive part. For the general properties of the slope and for the comparison with other classical notions we refer to [19], [20], [25], [2], [4] and to the forthcoming book [5].

In this paper we begin the study of this notion for the Mumford-Shah functional defined in the space $SBV(\Omega)$ of special functions with bounded variation (see [3]) on a bounded open set $\Omega \subset \mathbb{R}^2$ with C^1 boundary.

In view of the applications to irreversible crack growth in fracture mechanics, it is convenient to write this functional in the form

$$\mathcal{F}(u, S) := \|\nabla u\|^2 + \mathcal{H}^1(S), \tag{1.1}$$

using two independent variables: the function $u \in SBV(\Omega)$, which plays the role of the displacement, and the set S , which plays the role of the crack. Here and henceforth $\|\cdot\|$ denotes the L^2 norm, ∇ denotes the (approximate) gradient, and \mathcal{H}^1 is the one dimensional Hausdorff measure. Since the displacement must be (approximately) continuous out of the crack, the domain of the functional $\mathcal{F}(u, S)$ satisfies the constraint $S(u) \tilde{\subset} S$, where $S(u)$ is the jump set of u and $\tilde{\subset}$ denotes inclusion up to an \mathcal{H}^1 -negligible set. For the precise definitions of all these notions we refer to [3].

The irreversibility of crack growth leads to the following unilateral variant of the notion of slope:

$$|\partial\mathcal{F}|(u, S) := \limsup_{v \rightarrow u} \frac{(\mathcal{F}(u, S) - \mathcal{F}(v, S \cup S(v)))^+}{\|v - u\|}, \tag{1.2}$$

where $v \rightarrow u$ in $L^2(\Omega)$. We consider also the case with fixed boundary conditions on $\partial\Omega$

$$|\partial\mathcal{F}|_b(u, S) := \limsup_{\substack{v \rightarrow u \\ v=u \text{ on } \partial\Omega}} \frac{(\mathcal{F}(u, S) - \mathcal{F}(v, S \cup S(v)))^+}{\|v - u\|}, \quad (1.3)$$

where the equality $v = u$ on $\partial\Omega$ means that the traces of v and u agree \mathcal{H}^1 -a.e. on $\partial\Omega$. It is easy to see that $|\partial\mathcal{F}|_b(u, S) \leq |\partial\mathcal{F}|(u, S)$.

For every $S \subset \Omega$ with $\mathcal{H}^1(S) < +\infty$ we define $SBV^2(\Omega, S)$ as the set of functions $u \in SBV(\Omega) \cap L^2(\Omega)$ with $\nabla u \in L^2(\Omega; \mathbb{R}^2)$ and $S(u) \tilde{\subset} S$; we define $SBV_0^2(\Omega, S)$ as the set of functions $v \in SBV^2(\Omega, S)$ whose trace vanishes \mathcal{H}^1 -a.e. on $\partial\Omega$. We use the symbol $(\cdot | \cdot)$ to denote the scalar product in $L^2(\Omega)$ or $L^2(\Omega; \mathbb{R}^2)$, according to the context.

A necessary condition for the finiteness of the slope is given by the following proposition.

Proposition 1.1. *Let S be a subset of Ω with $\mathcal{H}^1(S) < +\infty$ and let $u \in SBV^2(\Omega, S)$. If $|\partial\mathcal{F}|_b(u, S) < +\infty$, then there exists $f \in L^2(\Omega)$ such that*

$$(\nabla u | \nabla \varphi) = (f | \varphi) \quad \forall \varphi \in SBV_0^2(\Omega, S). \quad (1.4)$$

Moreover, $2\|f\| \leq |\partial\mathcal{F}|_b(u, S)$.

If, in addition, $|\partial\mathcal{F}|(u, S) < +\infty$, then we have also

$$(\nabla u | \nabla \varphi) = (f | \varphi) \quad \forall \varphi \in SBV^2(\Omega, S). \quad (1.5)$$

If S and u are sufficiently smooth, condition (1.4) is equivalent to say that $-\Delta u = f$ in $\Omega \setminus S$ and $\partial u / \partial n = 0$ on S , while (1.5) implies also that $\partial u / \partial n = 0$ on $\partial\Omega$.

For every open set $U \subset \mathbb{R}^2$ the space of distributions on U is denoted by $\mathcal{D}'(U)$. The following theorem shows that the slope in the previous proposition is given exactly by $2\|f\|$ when S and u are sufficiently smooth, and u satisfies the Neumann boundary condition.

Theorem 1.2. *Let S be a one dimensional C^1 manifold without boundary contained in Ω and let u be a function such $u|_{\Omega_i} \in C^1(\overline{\Omega_i})$ for every connected component Ω_i of $\Omega \setminus S$. Assume that $-\Delta u = f$ in $\mathcal{D}'(\Omega \setminus S)$, with $f \in L^2(\Omega)$, and that $\partial u / \partial n = 0$ on S . Then $|\partial\mathcal{F}|_b(u, S) = 2\|f\|$. If, in addition, $\partial u / \partial n = 0$ on $\partial\Omega$, then $|\partial\mathcal{F}|(u, S) = 2\|f\|$.*

The hypothesis that S is a manifold without boundary is crucial in the previous theorem. In Example 2.2 we will show that, if S is a closed segment in Ω and u is harmonic on $\Omega \setminus S$, satisfies the Neumann boundary condition on S , and has a large stress intensity factor at one of the crack tips, then $|\partial\mathcal{F}|_b(u, S) > 0$. On the other hand we will show in Example 2.3 that, in this case, there exists a sequence S_k of C^1 manifolds without boundary, which converges to S in any reasonable sense, and a sequence u_k of harmonic functions on $\Omega \setminus S_k$, satisfying the Neumann boundary condition on S_k , and such that u_k converges to u strongly in $L^2(\Omega)$ and ∇u_k converges to ∇u strongly in $L^2(\Omega; \mathbb{R}^2)$. By Theorem 1.2 this implies that $|\partial\mathcal{F}|_b(u_k, S_k) = 0$ and shows that $|\partial\mathcal{F}|_b$ is not lower semicontinuous.

In the general theory of gradient flows, when the slope is not lower semicontinuous, its lower semicontinuous envelope plays an important role. In the case of the Mumford-Shah functional, for every $S \subset \Omega$ with $\mathcal{H}^1(S) < +\infty$ and every $u \in SBV^2(\Omega, S)$, the natural notion of lower semicontinuous envelope is defined by

$$|\overline{\partial\mathcal{F}}|(u, S) := \inf \left\{ \liminf_{k \rightarrow \infty} |\partial\mathcal{F}|(u_k, S_k) \right\},$$

where the infimum is taken over all sequences (u_k, S_k) such that $u_k \rightarrow u$ strongly in $L^2(\Omega)$, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$, and S_k σ^2 -converges to S , according to Definition 3.1 below. Similarly we define

$$|\overline{\partial\mathcal{F}}|_b(u, S) := \inf \left\{ \liminf_{k \rightarrow \infty} |\partial\mathcal{F}|_b(u_k, S_k) \right\},$$

where the infimum is taken over the same set of sequences.

We prove the following general result on the relaxed slope.

Proposition 1.3. *Let S be a subset of Ω with $\mathcal{H}^1(S) < +\infty$ and let $u \in SBV^2(\Omega, S)$. If $|\overline{\partial\mathcal{F}}|_b(u, S) < +\infty$, then there exists $f \in L^2(\Omega)$ such that*

$$-\operatorname{div}(\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega), \quad (1.6)$$

$$|\nabla u|^2 - \operatorname{div}(u\nabla u) \leq fu \quad \text{in } \mathcal{D}'(\Omega). \quad (1.7)$$

Moreover, $2\|f\| \leq |\overline{\partial\mathcal{F}}|_b(u, S)$.

If, in addition, $|\partial\mathcal{F}|(u, S) < +\infty$, then we have also

$$-\operatorname{div}(\nabla \tilde{u}) = \tilde{f} \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \quad (1.8)$$

where the tilde denotes the zero extension to \mathbb{R}^2 .

If S and u are sufficiently smooth, condition (1.6) is equivalent to say that $-\Delta u = f$ in $\Omega \setminus S$ and $\partial u / \partial n$ is continuous across S , while (1.8) implies also that $\partial u / \partial n = 0$ on $\partial\Omega$. Note that (1.4) implies the stronger condition $\partial u / \partial n = 0$ on S . Condition (1.7) says that $[u] \partial u / \partial n \geq 0$ on S , where $[u]$ denotes the jump of u on S .

The following theorem shows that in the previous proposition we have $|\overline{\partial\mathcal{F}}|_b(u, S) = 2\|f\|$ if S is partially smooth, and u satisfies (1.6) and (1.7) with an $f \in L^p(\Omega)$ with $p > 2$.

Theorem 1.4. *Assume that Ω has a C^2 boundary and let S be a compact subset of Ω . Suppose that there exists a finite set $F \subset S$ such that $S \setminus F$ is a one dimensional C^1 manifold, and that for every $x \in F$ there exists an open neighbourhood U of x such that $U \setminus S$ can be expressed as the union of a finite number of open sets with Lipschitz boundary and of a set of Lebesgue measure zero. Let $u \in SBV^2(\Omega, S) \cap L^\infty(\Omega)$ be a function which satisfies (1.6) and (1.7) with $f \in L^p(\Omega)$, $p > 2$. Assume that $[u] \neq 0$ \mathcal{H}^1 a.e. on S . Then $|\overline{\partial\mathcal{F}}|_b(u, S) = 2\|f\|$. If, in addition, (1.8) holds, then $|\partial\mathcal{F}|(u, S) = 2\|f\|$.*

The main difference between Theorems 1.2 and 1.4 is that in Theorem 1.2 the set S is assumed to be smooth, while in Theorem 1.4 it is smooth except possibly for a finite number of points. The comparison between these results shows that the slope is sensitive to the behaviour of u near the crack tips, while this is not the case for the relaxed slope. Another difference is the fact that in Theorem 1.2 we assume the Neumann condition $\partial u / \partial n = 0$ on S , which is replaced in Theorem 1.4 by the weaker assumptions (1.6) and (1.7). This is due to the fact that the set S can be approximated by a sequence S_k of sets with an increasing number of connected components, and this leads to a homogenization process (known as the sieve problem, see, e.g., [7], [18], and [27]), where the Neumann condition is replaced in the limit by a transmission condition.

2. PROOF OF THE PROPERTIES OF THE SLOPE

We shall use the following compactness and lower semicontinuity theorem.

Theorem 2.1. *Let $E \subset \Omega$ be an \mathcal{H}^1 -measurable set with $\mathcal{H}^1(E) < +\infty$, and let u_k be a sequence in $SBV(\Omega) \cap L^\infty(\Omega)$ such that u_k is bounded in $L^\infty(\Omega)$, $\mathcal{H}^1(S(u_k))$ is bounded, and ∇u_k is bounded in $L^2(\Omega; \mathbb{R}^2)$. Then there exist a subsequence, still denoted u_k , and a function $u \in SBV(\Omega) \cap L^\infty(\Omega)$, such that $u_k \rightarrow u$ a.e. in Ω , $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$, and $\mathcal{H}^1(S(u) \setminus E) \leq \liminf_k \mathcal{H}^1(S(u_k) \setminus E)$.*

If, in addition, there exists a function $\psi \in H^1(\Omega)$ such that $u_k - \psi \in SBV_0^2(\Omega, E)$ for every k , then $u - \psi \in SBV_0^2(\Omega, E)$.

Proof. The former statement is proved in [1] (see also [3]) when $E = \emptyset$. The proof of the general case can be found in [14, Theorem 2.8]. The latter statement can be obtained by considering an extension of all functions to a larger domain. \square

We begin by proving Proposition 1.1.

Proof of Proposition 1.1. Let $\varphi \in SBV_0^2(\Omega, S)$. By the definition of slope (see (1.3)) and of the functional \mathcal{F} (see (1.1)) we have

$$|\partial\mathcal{F}|_b(u, S) \geq \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathcal{F}(u, S) - \mathcal{F}(u + \varepsilon\varphi, S)}{\varepsilon\|\varphi\|} = -2 \frac{(\nabla u | \nabla \varphi)}{\|\varphi\|}. \quad (2.1)$$

Therefore the linear functional $\varphi \mapsto (\nabla u | \nabla \varphi)$ is continuous on $SBV_0^2(\Omega, S)$ with respect to the L^2 norm. Thus there exists $f \in L^2(\Omega)$ such that (1.4) holds. As $SBV_0^2(\Omega, S)$ is dense in $L^2(\Omega)$, (2.1) implies that $2\|f\| \leq |\partial\mathcal{F}|_b(u, S)$.

The proof of (1.5) is similar. \square

We now prove Theorem 1.2.

Proof of Theorem 1.2. We prove the theorem only for $|\partial\mathcal{F}|_b(u, S)$. By Proposition 1.1 we have only to prove that $|\partial\mathcal{F}|_b(u, S) \leq 2\|f\|$. Assume, by contradiction, that $|\partial\mathcal{F}|_b(u, S) > 2\|f\|$. Then there exist a constant α , with $0 < \alpha \leq +\infty$, and a sequence $v_k \in SBV(\Omega)$ such that $v_k \rightarrow u$ in $L^2(\Omega)$, the traces of v_k and u agree on $\partial\Omega$, and

$$\lim_{k \rightarrow \infty} \frac{\|\nabla u\|^2 - \|\nabla v_k\|^2 - \mathcal{H}^1(S(v_k) \setminus S)}{\|u - v_k\|} = \alpha + 2\|f\|. \quad (2.2)$$

This implies that $\liminf_k (\|\nabla u\|^2 - \|\nabla v_k\|^2) \geq 0$ and $\limsup_k \mathcal{H}^1(S(v_k) \setminus S) \leq 0$. Indeed, if this is not the case the numerator in (2.2) would have a negative limit (along a suitable subsequence), which contradicts $\alpha > 0$. These inequalities show that $\|\nabla v_k\|$ and $\mathcal{H}^1(S(v_k))$ are bounded uniformly with respect to k . By lower semicontinuity (Theorem 2.1) we have also $\limsup_k (\|\nabla u\|^2 - \|\nabla v_k\|^2) \leq 0$. Therefore

$$\|\nabla v_k\| \rightarrow \|\nabla u\| \quad \text{and} \quad \mathcal{H}^1(S(v_k) \setminus S) \rightarrow 0. \quad (2.3)$$

As $\Omega \setminus S$ has a finite number of connected components, the function u belongs to $L^\infty(\Omega)$. By a truncation argument (changing, if needed, the value of α), it is not restrictive to assume that

$$\|v_k\|_\infty \leq \|u\|_\infty, \quad (2.4)$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm.

Under our hypotheses on u there exists a constant L , with $0 < L < +\infty$, such that the restriction of u to each connected component U of $\Omega \setminus S$ has Lipschitz constant L . Let us fix $\varepsilon > 0$, with $2L^2\varepsilon + 2L\varepsilon < 1$, such that for every $y \in \partial U$ and every $\rho \in]0, \varepsilon[$ the set $\partial B(y, \rho) \cap U$ is connected and $\mathcal{H}^1(\partial B(y, \rho) \cap U) \geq 2\rho$. Let $\Phi_\varepsilon(w) := \|\nabla w\|^2 + \frac{1}{\varepsilon}((|f| + \varepsilon)(w - u)|w - u) - 2(f|w)$, and let W_k be the set of all functions w such that $w - u \in SBV_0^2(\Omega, S \cup S(v_k))$. It is clear that w_k is a solution of the minimum problem

$$\min_{w \in W_k} \Phi_\varepsilon(w) \quad (2.5)$$

if and only if it minimizes

$$\min_{w \in W_k} \left\{ \|\nabla w\|^2 + \frac{1}{\varepsilon}((|f| + \varepsilon)(w - u - \varepsilon g_\varepsilon)|w - u - \varepsilon g_\varepsilon) \right\}, \quad (2.6)$$

where $g_\varepsilon := f/(|f| + \varepsilon)$. By a truncation argument, we can find a minimizing sequence of (2.6) whose L^∞ norm is bounded by $\|u\|_\infty + \varepsilon$. Since the approximate gradients in the minimizing sequence are uniformly bounded in $L^2(\Omega; \mathbb{R}^2)$, we can apply the compactness and lower semicontinuity theorem (Theorem 2.1) and we obtain that the minimum problems (2.5) and (2.6) have a solution w_k , which is unique by strict convexity.

Let us prove that

$$w_k \rightarrow u \quad \text{strongly in } L^2(\Omega), \quad (2.7)$$

$$\nabla w_k \rightarrow \nabla u \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2). \quad (2.8)$$

Indeed, $\|w_k\|_\infty$, $\|\nabla w_k\|$, and $\mathcal{H}^1(S(w_k))$ are uniformly bounded, so that by Ambrosio's compactness theorem (Theorem 2.1) there exist a subsequence, not relabelled, and a function $w \in SBV(\Omega)$ such that $w_k \rightarrow w$ in $L^2(\Omega)$, $\nabla w_k \rightharpoonup \nabla w$ weakly in $L^2(\Omega; \mathbb{R}^2)$, and $\mathcal{H}^1(S(w) \setminus S) \leq \liminf_k \mathcal{H}^1(S(w_k) \setminus S)$. Since $S(w_k) \tilde{\subset} S \cup S(v_k)$ and $\mathcal{H}^1(S(v_k) \setminus S) \rightarrow 0$ we have that $S(w) \tilde{\subset} S$, hence $w \in SBV^2(\Omega, S)$. As $w_k - u \in SBV_0^2(\Omega, S \cup S(v_k))$, we deduce also that $w - u \in SBV_0^2(\Omega, S)$. Let v be a function such that $v - u \in SBV_0^2(\Omega, S)$. As $v \in W_k$, by the minimality of w_k in (2.5) we get $\Phi_\varepsilon(w_k) \leq \Phi_\varepsilon(v)$. Passing to the limit as $k \rightarrow \infty$, the lower semicontinuity implies that w minimizes Φ_ε on the set W of all functions v with $v - u \in SBV_0^2(\Omega, S)$. Since, by hypothesis, $-\Delta u = f$ in $\mathcal{D}'(\Omega)$ and $\partial u / \partial n = 0$ on S , the function u minimizes $\|\nabla v\|^2 - 2(f|v)$ on W . Therefore

$$\Phi_\varepsilon(u) = \|\nabla u\|^2 - 2(f|u) \leq \|\nabla w\|^2 - 2(f|w) \leq \Phi_\varepsilon(w),$$

so that $u = w$ by the uniqueness of the minimizer of Φ_ε on W . To prove that the convergence of ∇w_k to ∇u is strong, we observe that $u \in W_k$, so that, by the minimality of w_k in (2.5), we have $\Phi_\varepsilon(w_k) \leq \Phi_\varepsilon(u)$. This implies that $\|\nabla u\|^2 \geq \limsup_k \|\nabla w_k\|^2$, which gives the strong convergence.

Since $\Phi_\varepsilon(w_k) \leq \Phi_\varepsilon(v_k)$ by (2.5), we have

$$\begin{aligned} \|\nabla u\|^2 - \|\nabla v_k\|^2 - \mathcal{H}^1(S(v_k) \setminus S) &\leq \|\nabla u\|^2 - \|\nabla w_k\|^2 - \\ &- \frac{1}{\varepsilon}((|f| + \varepsilon)(w_k - u)|w_k - u) + \frac{1}{\varepsilon}((|f| + \varepsilon)(v_k - u)|v_k - u) + \\ &+ 2(f|w_k - u) - 2(f|v_k - u) - \mathcal{H}^1(S(v_k) \setminus S). \end{aligned}$$

We shall prove that for k large enough

$$\|\nabla u\|^2 - \|\nabla w_k\|^2 - \frac{1}{\varepsilon}((|f| + \varepsilon)(w_k - u)|w_k - u) + 2(f|w_k - u) \leq \mathcal{H}^1(S(v_k) \setminus S), \quad (2.9)$$

hence

$$\|\nabla u\|^2 - \|\nabla v_k\|^2 - \mathcal{H}^1(S(v_k) \setminus S) \leq \frac{1}{\varepsilon}((|f| + \varepsilon)(v_k - u)|v_k - u) - 2(f|v_k - u). \quad (2.10)$$

Since $\frac{1}{\varepsilon}((|f| + \varepsilon)(v_k - u)|v_k - u) \rightarrow 0$ strongly in $L^2(\Omega)$ as $k \rightarrow \infty$ by the dominated convergence theorem, and $-2(f|v_k - u) \leq 2\|f\| \|v_k - u\|$, inequality (2.10) contradicts the fact that $\alpha > 0$ in (2.2).

It remains to prove (2.9). The Euler condition for the minimum problem (2.5) implies that

$$(\nabla w_k | \nabla \varphi) + \frac{1}{\varepsilon}((|f| + \varepsilon)(w_k - u)|\varphi) - (f|\varphi) = 0$$

for every $\varphi \in SBV_0^2(\Omega, S(v_k) \cup S)$. Taking $\varphi = u - w_k$, and using the identity $\|\nabla u\|^2 - \|\nabla w_k\|^2 = (\nabla u | \nabla u - \nabla w_k) + (\nabla w_k | \nabla u - \nabla w_k)$, we obtain

$$\begin{aligned} \|\nabla u\|^2 - \|\nabla w_k\|^2 - \frac{1}{\varepsilon}((|f| + \varepsilon)(w_k - u)|w_k - u) + 2(f|w_k - u) &= \\ &= (\nabla u | \nabla u - \nabla w_k) + (f|w_k - u). \end{aligned}$$

Integrating by parts on each connected component U of $\Omega \setminus S$ (for a justification under our regularity assumptions see, e.g., the proof of (2.39) in [15]) we obtain

$$(\nabla u | \nabla u - \nabla w_k) + (f|w_k - u) = \int_{S(v_k) \setminus S} \frac{\partial u}{\partial n} [w_k] d\mathcal{H}^1,$$

so that (2.9) becomes

$$\int_{S(v_k) \setminus S} \frac{\partial u}{\partial n} [w_k] d\mathcal{H}^1 \leq \mathcal{H}^1(S(v_k) \setminus S).$$

Since $|\partial u / \partial n| \leq L$, it is enough to show that

$$\mathcal{H}^1\text{-ess sup}_{S(v_k) \setminus S} |[w_k]| \leq \frac{1}{L} \quad (2.11)$$

for k large enough.

To do this, we fix a connected component U of $\Omega \setminus S$. We will prove that for k large enough there exists a finite number of balls $B(y_i, \rho_i)$, depending possibly on k , with $0 < \rho_i < \varepsilon$, such that

$$\bar{U} \subset \bigcup_i B(y_i, \rho_i), \quad (2.12)$$

$$\partial B(y_i, \rho_i) \cap U \cap S(v_k) = \emptyset, \quad (2.13)$$

$$\mathcal{H}^1\text{-ess sup}_{U \cap \partial B(y_i, \rho_i)} |w_k - u| \leq \varepsilon. \quad (2.14)$$

Using an argument related to the maximum principle we will then show that (2.14) implies that

$$\operatorname{osc}_{U \cap B(y_i, \rho_i)} w_k \leq 2L\varepsilon + 2\varepsilon. \quad (2.15)$$

This gives immediately

$$\mathcal{H}^1\text{-ess sup}_{U \cap B(y_i, \rho_i) \cap S(v_k)} |[w_k]| \leq 2L\varepsilon + 2\varepsilon \leq \frac{1}{L}.$$

Since this estimate does not depend on i nor on the connected component U , we obtain (2.11).

We begin by proving that (2.14) implies (2.15). Let $x_i \in U \cap B(y_i, \rho_i)$ and let $m_i := u(x_i) - L\varepsilon - \varepsilon$ and $M_i := u(x_i) + L\varepsilon + \varepsilon$. Since u is L -Lipschitz on U we have

$$m_i + \varepsilon \leq u \leq M_i - \varepsilon \quad \text{on } U \cap B(y_i, \rho_i). \quad (2.16)$$

By (2.14) we have

$$m_i \leq w_k \leq M_i \quad \mathcal{H}^1 \text{ a.e. on } U \cap \partial B(y_i, \rho_i). \quad (2.17)$$

Moreover, the function g_ε which appears in (2.6) satisfies

$$m_i \leq u + \varepsilon g_\varepsilon \leq M_i \quad \text{a.e. on } U \cap B(y_i, \rho_i), \quad (2.18)$$

Let $z_k \in SBV(\Omega)$ be defined by

$$z_k := \begin{cases} (m_i \vee w_k) \wedge M_i & \text{on } U \cap B(y_i, \rho_i), \\ w_k & \text{elsewhere.} \end{cases}$$

By (2.17) we have $z_k \in SBV^2(\Omega, S \cup S(v_k))$ and by (2.16) the traces of z_k and u agree on $\partial\Omega$. Therefore $z_k \in W_k$. By (2.18) we have

$$\begin{aligned} \|\nabla z_k\|^2 &+ \frac{1}{\varepsilon} (|f| + \varepsilon) (z_k - u - \varepsilon g_\varepsilon) |z_k - u - \varepsilon g_\varepsilon| \leq \\ &\leq \|\nabla w_k\|^2 + \frac{1}{\varepsilon} (|f| + \varepsilon) (w_k - u - \varepsilon g_\varepsilon) |w_k - u - \varepsilon g_\varepsilon|. \end{aligned}$$

Since the solution of (2.6) is unique we have $z_k = w_k$, hence $m_i \leq w_k \leq M_i$ a.e. on $U \cap B(y_i, \rho_i)$, which implies (2.15).

It remains to prove that we can find a finite number of balls $B(y_i, \rho_i)$, with $0 < \rho_i < \varepsilon$, satisfying (2.12)–(2.14). Choose k large enough to have

$$\mathcal{H}^1(S(v_k) \setminus S) < \frac{\varepsilon}{8}, \quad (2.19)$$

$$64 \left(\pi + \frac{1}{\varepsilon^2} \right) (\|w_k - u\|^2 + \|\nabla w_k - \nabla u\|^2) < \varepsilon^2, \quad (2.20)$$

and define $H_\varepsilon := \{x \in U : d(x, U^c) \geq \varepsilon/2\}$. Since H_ε is compact, it can be covered by a finite number of balls $B(y_i, \varepsilon/4)$ with $y_i \in H_\varepsilon$. Note that the balls $B(y_i, \varepsilon/2)$ are contained in U . Let $E_i := \{\rho \in]\varepsilon/4, \varepsilon/2[: \partial B(y_i, \rho) \cap S(v_k) = \emptyset\}$. By (2.19) we have $\mathcal{H}^1(E_i) \geq \varepsilon/8$. Moreover

$$\int_{E_i} d\rho \int_{\partial B(y_i, \rho)} |\nabla_\tau w_k - \nabla_\tau u|^2 d\mathcal{H}^1 \leq \|\nabla w_k - \nabla u\|^2$$

and

$$\int_{E_i} d\rho \int_{\partial B(y_i, \rho)} |w_k - u|^2 d\mathcal{H}^1 \leq \|w_k - u\|^2,$$

where ∇_τ denotes the tangential gradient. As $\mathcal{H}^1(E_i) \geq \varepsilon/8$, we may choose $\rho_i \in E_i$ such that

$$\int_{\partial B(y_i, \rho_i)} |\nabla_\tau w_k - \nabla_\tau u|^2 d\mathcal{H}^1 \leq \frac{16}{\varepsilon} \|\nabla w_k - \nabla u\|^2 \quad (2.21)$$

$$\int_{\partial B(y_i, \rho_i)} |w_k - u|^2 d\mathcal{H}^1 \leq \frac{16}{\varepsilon} \|w_k - u\|^2. \quad (2.22)$$

Using the one-dimensional estimate

$$\sup_{t \in]0, a[} |\zeta(t)|^2 \leq \frac{2}{a} \int_0^a |\zeta(t)|^2 dt + 2a \int_0^a |\zeta'(t)|^2 dt, \quad (2.23)$$

from (2.20), (2.21), and (2.22) we obtain (2.14).

Let us cover now $K_\varepsilon := \{x \in \bar{U} : d(x, U^c) \leq \varepsilon/2\}$. Since K_ε is compact we can cover it with a finite number of balls $B(y_i, 3\varepsilon/4)$ with $y_i \in \partial U$. By our choice of ε the set $\partial B(y, \rho) \cap U$ is connected and $\mathcal{H}^1(\partial B(y_i, \rho) \cap U) \geq \varepsilon$ for $3\varepsilon/4 < \rho < \varepsilon$. As in the case of balls centered in interior points of U there exist radii ρ_i with $3\varepsilon/4 < \rho_i < \varepsilon$ such that (2.13), (2.21), and (2.22) hold with $\partial B(y_i, \rho_i)$ replaced by $\partial B(y_i, \rho_i) \cap U$. Using again (2.23) and the lower bound $\mathcal{H}^1(\partial B(y_i, \rho_i) \cap U) \geq \varepsilon$ we obtain (2.14). \square

The following example shows that in the previous theorem it is not enough to assume that S is a manifold with boundary.

Example 2.2. Let S be a closed segment contained in Ω with endpoints a, b , and let $u \in H^1(\Omega \setminus S)$ be a harmonic function which satisfies the Neumann boundary condition on S . It is well known that there exists a constant κ such that in a neighbourhood U of a we have

$$u - \kappa \sqrt{2\rho/\pi} \sin(\theta/2) \in H^2(U \setminus S) \cap H^{1,\infty}(U \setminus S),$$

where ρ and θ are the polar coordinates around a with $\theta = \pi$ on S (see, e.g., [22, Theorem 4.4.3.7 and Section 5.2] or [26, Appendix 1]). Moreover, any constant κ can be obtained by a suitable choice of the boundary conditions satisfied by u on $\partial\Omega$.

For every $s \geq 0$, let S_s be the closed segment obtained by adding to S a collinear segment of length s starting from a , and let $u(s)$ be the solution of the problem $\Delta u(s) = 0$ on $\Omega \setminus S_s$, $\partial u(s)/\partial n = 0$ on S_s , and $u(s) = u$ on $\partial\Omega$. Then

$$|\partial\mathcal{F}|_b(u, S) \geq \limsup_{s \rightarrow 0} \frac{(\mathcal{F}(u(0), S) - \mathcal{F}(u(s), S_s))^+}{\|u(s) - u(0)\|}. \quad (2.24)$$

By [23, Theorem 6.4.1] the derivative with respect to s of $\mathcal{F}(u(s), S_s)$ at $s = 0$ exists and is equal to $1 - \kappa^2$. Moreover, a similar argument shows that the function $s \mapsto u(s)$ has a derivative $\dot{u}(0)$ (in the L^2 -sense) at $s = 0$. Therefore (2.24) gives

$$|\partial\mathcal{F}|_b(u, S) \geq (\kappa^2 - 1)^+ \frac{1}{\|\dot{u}(0)\|}.$$

In particular, if $|\kappa| > 1$, we conclude that $|\partial\mathcal{F}|_b(u, S) > 0$.

The next example shows that $|\partial\mathcal{F}|_b$ is not lower semicontinuous.

Example 2.3. Under the hypotheses of the previous example, assume in addition that Ω has a C^2 boundary. Let $A_k := \{x \in \Omega : d(x, S) < 1/k\}$, $S_k = \partial A_k$, and let ψ_k be a sequence of functions in $C^2(\bar{\Omega})$ which converge to u strongly in H^1 near $\partial\Omega$. Let u_k be the solution of $\Delta u_k = 0$ in $\Omega \setminus S_k$, which satisfies the Neumann boundary condition $\partial u_k/\partial n = 0$ on S_k , the Dirichlet boundary condition $u_k = \psi_k$ on $\partial\Omega$, and vanishes on A_k . Since S_k is connected and converges to S in the Hausdorff metric (and also in the

sense of σ^2 -convergence, see Definition 3.1), we deduce that u_k converges to u strongly in $L^2(\Omega)$ and ∇u_k converges to ∇u strongly in $L^2(\Omega; \mathbf{R}^2)$ (see, e.g., [17, Theorem 5.1]). By Theorem 1.2 we have $|\partial\mathcal{F}|_b(u_k, S_k) = 0$, hence $|\overline{\partial\mathcal{F}}|_b(u, S) = 0$, while $|\partial\mathcal{F}|_b(u, S) > 0$ (see Example 2.2). This shows that $|\partial\mathcal{F}|_b$ is not lower semicontinuous.

3. PROOF OF THE PROPERTIES OF THE RELAXED SLOPE

The definition of relaxed slope is based on the following notion of convergence of sets, introduced in [14] and used in the study of quasi-static crack growth.

Definition 3.1. We say that a sequence S_k σ^2 -converges to S if $S_k, S \subset \Omega$, $\mathcal{H}^1(S_k)$ is bounded uniformly with respect to k , and the following two conditions are satisfied:

- (a) if $u_j \in SBV^2(\Omega, S_{k_j}) \cap L^\infty(\Omega)$ for some sequence $k_j \rightarrow \infty$, u_j is bounded in $L^\infty(\Omega)$, $u_j \rightarrow u$ a.e. in Ω , and $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbf{R}^2)$, then $u \in SBV^2(\Omega, S) \cap L^\infty(\Omega)$;
- (b) there exist a function $u \in SBV^2(\Omega, S) \cap L^\infty(\Omega)$, with $S(u) \cong S$, and a sequence u_k , bounded in $L^\infty(\Omega)$, such that $u_k \in SBV^2(\Omega, S_k)$ for every k , $u_k \rightarrow u$ a.e. in Ω , and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbf{R}^2)$.

We begin by proving Proposition 1.3.

Proof of Proposition 1.3. Assume that $|\overline{\partial\mathcal{F}}|_b(u, S) < +\infty$. Then there exist (u_k, S_k) such that $u_k \rightarrow u$ strongly in $L^2(\Omega)$, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbf{R}^2)$, S_k σ^2 -converges to S , and $\lim_k |\partial\mathcal{F}|_b(u_k, S_k) = |\overline{\partial\mathcal{F}}|_b(u, S)$. By Proposition 1.1 there exist $f_k \in L^2(\Omega)$, with $2\|f_k\| \leq |\partial\mathcal{F}|_b(u_k, S_k)$, such that $(\nabla u_k | \nabla \varphi) = (f_k | \varphi)$ for every $\varphi \in SBV_0^2(\Omega, S_k)$. It follows that, up to a subsequence, f_k converges weakly in $L^2(\Omega)$ to some function f . By lower semicontinuity

$$2\|f\| \leq 2 \liminf_k \|f_k\| \leq \lim_k |\partial\mathcal{F}|_b(u_k, S_k) = |\overline{\partial\mathcal{F}}|_b(u, S).$$

Let $\varphi \in C_0^\infty(\Omega)$. As $\varphi \in SBV_0^2(\Omega, S_k)$, we have $(\nabla u_k | \nabla \varphi) = (f_k | \varphi)$, and passing to the limit as $k \rightarrow \infty$ we get (1.6). As $\varphi u_k \in SBV_0^2(\Omega, S_k)$ we have also

$$(\nabla u_k | \varphi \nabla u_k) + (\nabla u_k | u_k \nabla \varphi) = (f_k | \varphi u_k).$$

Passing to the limit as $k \rightarrow \infty$, when $\varphi \geq 0$ we obtain (1.7) using lower semicontinuity with respect to weak convergence in the first term.

The proof for $|\partial\mathcal{F}|(u, S)$ is analogous. \square

To prove Theorem 1.4 we shall use some properties of the Newtonian capacity and of quasicontinuous representatives of functions in Sobolev spaces, for which we refer to [21] and [24]. Given an orientation of the C^1 manifold $S \setminus F$, for every $v \in H^1(\Omega \setminus S)$ the traces v^+ and v^- on the positive and negative faces of $S \setminus F$ are defined cap-q.e. on $S \setminus F$, hence the jump $[v] := v^+ - v^-$ is defined cap-q.e. on S and is cap-quasicontinuous on $S \setminus F$. If $v \in H^1(\Omega \setminus S) \cap L^\infty(\Omega \setminus S)$, by using cut-off functions which vanish in a neighbourhood of F , it is easy to prove that $v \in SBV^2(\Omega, S)$ and that $Dv \llcorner S = [v] n \mathcal{H}^1 \llcorner S$, where n is the oriented unit normal to $S \setminus F$. Conversely, if $v \in SBV^2(\Omega, S)$ its restriction to $\Omega \setminus S$ belongs to $H^1(\Omega \setminus S)$. Let

$$H_{0, \partial\Omega}^1(\Omega \setminus S) := \{v \in H^1(\Omega \setminus S) : v = 0 \text{ on } \partial\Omega\}.$$

The previous remarks show that $SBV_0^2(\Omega, S) \cap L^\infty(\Omega)$ can be identified with $H_{0, \partial\Omega}^1(\Omega \setminus S) \cap L^\infty(\Omega \setminus S)$.

Under the assumptions of Theorem 1.4, let ν be the nonnegative Radon measure on Ω defined by

$$|\nabla u|^2 - \operatorname{div}(u \nabla u) + \nu = f u \quad \text{in } \mathcal{D}'(\Omega). \quad (3.1)$$

Since $|\nabla u|^2$ and $f u$ belong to $L^1(\Omega)$, while $\operatorname{div}(u \nabla u)$ belongs to $H^{-1}(\Omega)$ it turns out that ν vanishes on all sets of capacity zero. As ∇u is the distributional gradient of u on

$\Omega \setminus S$, from (1.6) we get $(\nabla u | \nabla(\varphi u)) = (f | \varphi u)$ for every $\varphi \in C_c^\infty(\Omega \setminus S)$, which implies that $\text{supp } \nu \subset S$. Using a standard approximation argument we can prove that

$$(\nabla u | \nabla u \varphi) + (u \nabla u | \nabla \varphi) + \int_S \varphi d\nu = (f | u | \varphi) \quad (3.2)$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let μ be the nonnegative Borel measure defined by

$$\mu(B) := \begin{cases} \int_B \frac{d\nu}{[u]^2} & \text{if } \text{cap}(B \cap \{[u] = 0\}) = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.3)$$

We note that μ vanishes on all sets of capacity zero.

Lemma 3.2. *Under the assumptions of Theorem 1.4, let μ and ν be defined by (3.3) and (3.1). Then $[u] \in L^2(S, \mu)$, $\nu = [u]^2 \mu$, and*

$$(\nabla u | \nabla v) + ([u] | [v])_{S, \mu} = (f | v) \quad (3.4)$$

for every $v \in H_{0, \partial\Omega}^1(\Omega \setminus S)$ with $[v] \in L^2(S, \mu)$, where $(\cdot | \cdot)_{S, \mu}$ denotes the scalar product in $L^2(S, \mu)$.

Proof. We have

$$\int_S [u]^2 d\mu = \int_{S \cap \{[u] \neq 0\}} [u]^2 d\mu = \nu(S \cap \{[u] \neq 0\}) \leq \nu(S) < +\infty,$$

hence $[u] \in L^2(S, \mu)$.

By (3.3), in order to prove that $\nu = [u]^2 \mu$ it is enough to show that $\nu(S \cap \{[u] = 0\}) = 0$. Since $\nu(F) = 0$ it is enough to show that for every $y \in S \setminus F$ there exists an open neighbourhood U of y such that

$$\nu(U \cap S \cap \{[u] = 0\}) = 0. \quad (3.5)$$

To this aim we consider a neighbourhood U such that $U \cap S$ is a C^1 manifold and $U \setminus S$ has two connected components U^\oplus and U^\ominus , corresponding to the positive and negative faces of $U \cap S$. By possibly reducing U we may assume that there exist two functions $u^\oplus, u^\ominus \in H^1(U) \cap L^\infty(U)$ such that $u^\oplus = u$ a.e. on U^\oplus and $u^\ominus = u$ a.e. on U^\ominus , so that $[u] = u^\oplus - u^\ominus$ cap-q.e. on $U \cap S$.

Let $\varphi \in C_c^\infty(U)$ and let $F_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the Lipschitz function defined by $F_\varepsilon(t) := 1 + t/\varepsilon$ for $-\varepsilon \leq t \leq 0$, $F_\varepsilon(t) := 1 - t/\varepsilon$ for $0 \leq t \leq \varepsilon$, and $F_\varepsilon(t) := 0$ for $|t| > \varepsilon$. We will prove that

$$\int_S F_\varepsilon([u]) \varphi d\nu \rightarrow 0. \quad (3.6)$$

Since $F_\varepsilon([u]) \rightarrow 1_{\{[u]=0\}}$ pointwise on S , by the dominated convergence theorem (3.6) implies

$$\int_{S \cap \{[u]=0\}} \varphi d\nu = 0,$$

which gives (3.5) by the arbitrariness of φ .

To prove (3.6) we use $v := u^\oplus F_\varepsilon(u^\oplus - u^\ominus) \varphi$ and $w := F_\varepsilon(u^\oplus - u^\ominus) \varphi$ as test functions in (1.6) and (3.2), respectively. We obtain

$$\begin{aligned} & (\nabla u | \nabla u^\oplus F_\varepsilon(u^\oplus - u^\ominus) \varphi) + (u^\oplus \nabla u | F_\varepsilon'(u^\oplus - u^\ominus) (\nabla u^\oplus - \nabla u^\ominus) \varphi) + \\ & + (u^\oplus \nabla u | \nabla \varphi F_\varepsilon(u^\oplus - u^\ominus)) = (f | u^\oplus F_\varepsilon(u^\oplus - u^\ominus) \varphi) \end{aligned}$$

and

$$\begin{aligned} & (\nabla u |\nabla u F_\varepsilon(u^\oplus - u^\ominus) \varphi) + (u \nabla u | F'_\varepsilon(u^\oplus - u^\ominus) (\nabla u^\oplus - \nabla u^\ominus) \varphi) + \\ & + (u \nabla u |\nabla \varphi F_\varepsilon(u^\oplus - u^\ominus)) + \int_S F_\varepsilon([u]) \varphi d\nu = (f | u F_\varepsilon(u^\oplus - u^\ominus) \varphi). \end{aligned}$$

Subtracting term by term we get

$$\begin{aligned} & (\nabla u | (\nabla u^\oplus - \nabla u) F_\varepsilon(u^\oplus - u^\ominus) \varphi) + ((u^\oplus - u) \nabla u | F'_\varepsilon(u^\oplus - u^\ominus) (\nabla u^\oplus - \nabla u^\ominus) \varphi) + \\ & + ((u^\oplus - u) \nabla u |\nabla \varphi F_\varepsilon(u^\oplus - u^\ominus)) - \int_S F_\varepsilon([u]) \varphi d\nu = (f | (u^\oplus - u) F_\varepsilon(u^\oplus - u^\ominus) \varphi). \end{aligned} \quad (3.7)$$

Since $u^\oplus = u$ and $\nabla u^\oplus = \nabla u$ a.e. on the set $\{u^\oplus = u^\ominus\}$, the first, the third term and the last term in (3.7) tend to 0 by the dominated convergence theorem. As $|(u^\oplus - u) F'_\varepsilon(u^\oplus - u^\ominus)| < 1$ and the measure of the set $\{0 < |u^\oplus - u^\ominus| < \varepsilon\}$ tends to 0, the second term tends to 0 too. This proves (3.6), which concludes the proof of the fact that $\nu = [u]^2 \mu$.

To prove (3.4) we need the following density result. \square

Lemma 3.3. *Under the assumptions of Theorem 1.4 for every $v \in H_{0,\partial\Omega}^1(\Omega \setminus S)$ with $[v] \in L^2(S, \mu)$, there exist $\varphi_k, \psi_k \in C_c^\infty(\Omega)$ such that, setting $v_k := \varphi_k + \psi_k u$, we have*

$$v_k \rightarrow v \quad \text{strongly in } H^1(\Omega \setminus S), \quad (3.8)$$

$$[v_k] \rightarrow [v] \quad \text{strongly in } L^2(S, \mu). \quad (3.9)$$

Proof of Lemma 3.2 (continuation). It remains to prove (3.4). Let $v \in H_{0,\partial\Omega}^1(\Omega \setminus S)$ with $[v] \in L^2(S, \mu)$ and let φ_k, ψ_k be as in Lemma 3.3. Taking φ_k and ψ_k as test function in (1.6) and (3.2), respectively, and using the equality $\nu = [u]^2 \mu$, we obtain

$$\begin{aligned} & (\nabla u |\nabla \varphi_k) = (f | \varphi_k) \\ & (\nabla u |\psi_k \nabla u) + (\nabla u | u \nabla \psi_k) + ([u] |\psi_k [u])_{S,\mu} = (f | \psi_k u). \end{aligned}$$

Adding term by term we get

$$(\nabla u |\nabla v_k) + ([u] |[v_k])_{S,\mu} = (f | v_k),$$

where $v_k := \varphi_k + \psi_k u$. Passing to the limit thanks to Lemma 3.3 we obtain (3.4). \square

Proof of Lemma 3.3. Since every function $v \in H_{0,\partial\Omega}^1(\Omega \setminus S)$ with $[v] \in L^2(S, \mu)$ can be approximated by truncations, it is enough to prove the lemma when v is bounded. Since F has capacity zero, we may also assume that v vanishes a.e. in a neighbourhood of F . By using a partition of unity, it is enough to prove the lemma in one of the following cases:

- (a) $\text{supp } v \cap S = \emptyset$;
- (b) $\text{supp } v$ is contained in an open set U such that $U \cap S$ is a C^1 manifold and $U \setminus S$ has two connected components U^\oplus and U^\ominus , corresponding to the positive and negative faces of $U \cap S$.

In the former case it is enough to take $\psi_k = 0$, and the result follows from the density of $C_c^\infty(\Omega \setminus S)$ in $H_0^1(\Omega \setminus S)$.

In the latter case there exist two functions $v^\oplus, v^\ominus \in H_0^1(U) \cap L^\infty(U)$, with compact support in U , such that $v^\oplus = v$ a.e. on U^\oplus and $v^\ominus = v$ a.e. on U^\ominus , so that $[v] = v^\oplus - v^\ominus$ cap-q.e. on $U \cap S$.

Since $[v] \in L^2(S, \mu)$, from the definition of μ (see (3.3)) we obtain $\text{cap}(\{[v] \neq 0\} \cap \{[u] = 0\}) = 0$. It is not restrictive to assume that there exists $\varepsilon > 0$ such that $\{[v] \neq 0\} \subset \{|[u]| > \varepsilon\}$ (see Lemma 3.4 below).

Let u^\oplus and u^\ominus be the functions defined in the proof of Lemma 3.2, and let $G_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the Lipschitz function defined by $G_\varepsilon(t) := 1/t$ for $|t| \geq \varepsilon$ and $G_\varepsilon(t) := t/\varepsilon^2$ for $|t| \leq \varepsilon$. Since $G_\varepsilon(u^\oplus - u^\ominus) \in H^1(U) \cap L^\infty(U)$, the function $(v^\oplus - v^\ominus) G_\varepsilon(u^\oplus - u^\ominus)$ belongs to $H_0^1(U) \cap L^\infty(U)$ and coincides with $[v] G_\varepsilon([u])$ cap-q.e. on S . Since $\{[v] \neq 0\} \subset \{|[u]| > \varepsilon\}$, we have $[u] [v] G_\varepsilon([u]) = [v]$ cap-q.e. on S .

By the density of $C_c^\infty(U)$ in $H_0^1(U)$ there exists a sequence $\psi_k \in C_c^\infty(U)$ which is bounded in $L^\infty(U)$ and converges to $(v^\oplus - v^\ominus)G_\varepsilon(u^\oplus - u^\ominus)$ strongly in $H_0^1(U)$. In particular, up to a subsequence, $\psi_k \rightarrow [v]G_\varepsilon([u])$ cap-q.e. on S . Hence $\psi_k[u] \rightarrow [v]$ strongly in $L^2(S, \mu)$. Let $w_k \in H^1(U \setminus S)$ be defined by $w_k := (u^\oplus - u^\ominus)(v^\oplus - v^\ominus)G_\varepsilon(u^\oplus - u^\ominus) - \psi_k(u^\oplus - u^\ominus)$ on U^\oplus , and $w_k := 0$ on U^\ominus . Note that $w_k \rightarrow 0$ strongly in $H^1(U \setminus S)$ and $\text{supp } w_k \subset\subset U$. As $[w_k] = [v] - \psi_k[u]$ cap-q.e. on S , the function $v - \psi_k u - w_k$ has no jump on S , hence it belongs to $H^1(U)$. Since $\text{supp}(v - \psi_k u - w_k) \subset\subset U$, we have $v - \psi_k u - w_k \in H_0^1(U)$, thus there exists a sequence $\varphi_k \in C_c^\infty(U)$ such that $v - \psi_k u - w_k - \varphi_k \rightarrow 0$ strongly in $H_0^1(U)$. It is then clear that the sequence v_k defined in the statement of the lemma satisfies (3.8) and (3.9). \square

Lemma 3.4. *Let u be as in Theorem 1.4, and let U, U^\oplus, U^\ominus be as in Lemma 3.3. Let $v \in H^1(U \setminus S) \cap L^\infty(U)$ with $\text{supp } v \subset\subset U$ and $[v] \in L^2(S, \mu)$. Then there exist a sequence $v_k \in H^1(U \setminus S) \cap L^\infty(U)$ and a sequence $\varepsilon_k > 0$ such that*

$$v_k \rightarrow v \quad \text{strongly in } H^1(U \setminus S), \quad (3.10)$$

$$[v_k] \rightarrow [v] \quad \text{strongly in } L^2(S \cap U, \mu), \quad (3.11)$$

$$\{[v_k] \neq 0\} \subset \{[u] > \varepsilon_k\}. \quad (3.12)$$

Proof. As we noticed in the proof of Lemma 3.3, the definition of μ (see (3.3)) implies that $\text{cap}(\{[v] \neq 0\} \cap \{[u] = 0\}) = 0$.

Let us prove that there exists a sequence $w_k \in H^1(U \setminus S) \cap L^\infty(U)$ such that

$$w_k \rightarrow v \quad \text{strongly in } H^1(U \setminus S), \quad (3.13)$$

$$[w_k] \rightarrow [v] \quad \text{strongly in } L^2(S \cap U, \mu), \quad (3.14)$$

$$|[w_k]| \leq k|[u]| \quad \text{cap-q.e. on } S. \quad (3.15)$$

Let $g_k: U \rightarrow \mathbb{R}$ be the function defined by $g_k := (v^\oplus - v^\ominus) \vee 0$ cap-q.e. on $U \setminus S$ and $g_k := [v] \vee 0 \wedge k|[u]|$ cap-q.e. on S . The functions g_k are cap-quasi lower semicontinuous, the sequence g_k is increasing and converges to (the cap-quasi continuous representative of) $(v^\oplus - v^\ominus) \vee 0$ cap-q.e. on U . By [11, Lemma 1.6] there exists a sequence $w_k^{(1)} \in H_0^1(U)$ such that $0 \leq w_k^{(1)} \leq [v] \wedge k|[u]|$ cap-q.e. on $S \cap U$ and $w_k^{(1)} \rightarrow (v^\oplus - v^\ominus) \vee 0$ strongly in $H_0^1(U)$. Similarly, there exists a sequence $w_k^{(2)} \in H_0^1(U)$ such that $-([v] \wedge k|[u]|) \leq w_k^{(2)} \leq 0$ and $w_k^{(2)} \rightarrow (v^\oplus - v^\ominus) \wedge 0$ strongly in $H_0^1(U)$. Then $w_k^{(1)} + w_k^{(2)} \rightarrow v^\oplus - v^\ominus$ strongly in $H_0^1(U)$, hence, up to a subsequence, $w_k^{(1)} + w_k^{(2)} \rightarrow [v]$ cap-q.e. on $S \cap U$. Moreover $|w_k^{(1)} + w_k^{(2)}| \leq [v] \wedge k|[u]|$ cap-q.e. on $S \cap U$. Let us define $w_k^\oplus := v^\oplus + w_k^{(1)} + w_k^{(2)}$ and $w_k^\ominus := v^\ominus$. It is then clear that the function w_k defined by $w_k := w_k^\oplus$ in U^\oplus and $w_k := w_k^\ominus$ in U^\ominus satisfies (3.13)–(3.15).

For every $\varepsilon > 0$ let $T_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ be the Lipschitz function defined by $T_\varepsilon(t) := t - \varepsilon$, for $t \geq \varepsilon$, $T_\varepsilon(t) := t + \varepsilon$ for $t \leq -\varepsilon$, and $T_\varepsilon(t) := 0$ for $|t| \leq \varepsilon$. Let $w_{k,\varepsilon} \in H_{0,\partial U}^1(U \setminus S)$ be the function defined by

$$w_{k,\varepsilon} := \begin{cases} w_k^\oplus + T_{k\varepsilon}(w_k^\oplus - w_k^\ominus) & \text{in } U^\oplus, \\ w_k^\ominus & \text{in } U^\ominus. \end{cases}$$

Then $[w_{k,\varepsilon}] = T_{k\varepsilon}([w_k])$ cap-q.e. on $S \cap U$, $w_{k,\varepsilon} \rightarrow w_k$ strongly in $H^1(U \setminus S)$, and $[w_{k,\varepsilon}] \rightarrow [w_k]$ strongly in $L^2(S \cap U, \mu)$.

We choose now $\varepsilon_k > 0$ such that $\|w_{k,\varepsilon_k} - w_k\| + \|\nabla w_{k,\varepsilon_k} - \nabla w_k\| < 1/k$ and $\|[w_{k,\varepsilon_k}] - [w_k]\|_\mu < 1/k$. Then the function $v_k := w_{k,\varepsilon_k}$ satisfies (3.10) and (3.11). Since $[v_k] = T_{k\varepsilon_k}([w_k])$ and $|[w_k]| \leq k|[u]|$ cap-q.e. on $S \cap U$, we obtain that (3.12) is satisfied. \square

Let $w \in H^1(\Omega)$. For every compact subset M of Ω we consider the functional $\mathcal{G}_M^w : L^2(\Omega) \rightarrow [0, +\infty]$ defined by

$$\mathcal{G}_M^w(v) := \begin{cases} \|\nabla v\|^2 + \|v\|^2 & \text{if } v - w \in H_{0,\partial\Omega}^1(\Omega \setminus M), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.16)$$

For every nonnegative Borel measure μ on Ω , vanishing on all sets of capacity zero and with $\text{supp } \mu \subset S$, we consider the functional $\mathcal{F}_\mu : L^2(\Omega) \rightarrow [0, +\infty]$ defined by

$$\mathcal{F}_\mu^w(v) := \begin{cases} \|\nabla v\|^2 + \|v\|^2 + \|[v]\|_{S,\mu}^2 & \text{if } v - w \in H_{0,\partial\Omega}^1(\Omega \setminus S), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.17)$$

For the definition and properties of Γ -convergence we refer to [12] and [8].

Lemma 3.5. *Let S be as in Theorem 1.4, and let μ be a nonnegative Borel measure on Ω vanishing on all sets of capacity zero and with $\text{supp } \mu \subset S$. Then there exists a sequence M_k of C^1 manifolds with boundary such that $M_k \subset S$ and $\mathcal{G}_{M_k}^{w_k}$ Γ -converges to \mathcal{F}_μ^w in $L^2(\Omega)$ for every sequence w_k converging to w strongly in $H^1(\Omega)$.*

Proof. This proof is obtained by adapting [16, Theorem 4.16] and [6, Theorem 2.38]. First of all we observe that it is enough to prove the lemma when $w_k = w = 0$. The general case can be obtained by modifying the functions near the boundary in order to match the boundary conditions.

Let \mathcal{X} be the set of lower semicontinuous functionals $\mathcal{G} : L^2(\Omega) \rightarrow [0, +\infty]$ with $\mathcal{G} \geq \mathcal{G}_S^0$. Under our regularity assumptions on S we can apply Rellich's Theorem and we obtain that the set $\{v \in L^2(\Omega) : \mathcal{G}_S^0(v) \leq t\}$ is compact in $L^2(\Omega)$ for every $t < +\infty$. Therefore Γ -convergence in $L^2(\Omega)$ is metrizable on \mathcal{X} by [12, Theorem 10.22].

Let \mathcal{Y} be the set of functionals \mathcal{G}_M^0 , where M runs over all C^1 manifolds with boundary contained in S . Since $\mathcal{Y} \subset \mathcal{X}$, to prove the lemma for $w_k = w = 0$ we have to show that $\mathcal{F}_\mu^0 \in \overline{\mathcal{Y}}$, where $\overline{\mathcal{Y}}$ denotes the closure of \mathcal{Y} with respect to the metric which induces Γ -convergence on \mathcal{X} . Indeed, in this case there exists a sequence M_k of C^1 manifolds with boundary such that $M_k \subset S$ and $\mathcal{G}_{M_k}^0$ Γ -converges to \mathcal{F}_μ^0 in $L^2(\Omega)$.

Using the techniques developed in [10] it is possible to prove that $\mathcal{F}_\mu^0 \in \overline{\mathcal{Y}}$ whenever $\mu = \mathcal{H}^1 \llcorner E$ and $E \subset S \setminus F$ is an arc, i.e., a connected C^1 manifold with or without boundary. More in general, the same techniques show that $\mathcal{F}_\mu^0 \in \overline{\mathcal{Y}}$ whenever $\mu = g\mathcal{H}^1 \llcorner S$ and g is a step-function, i.e., $g = \sum c_i 1_{E_i}$ where $c_i \geq 0$ and E_i are disjoint arcs contained in $S \setminus F$.

To continue the proof we need the following lemma. \square

Lemma 3.6. *Let S be as in Theorem 1.4, and let μ_k, μ be positive measures in $H^{-1}(\Omega)$ with support in S . If $\mu_k \rightarrow \mu$ strongly in $H^{-1}(\Omega)$ then $\mathcal{F}_{\mu_k}^0$ Γ -converges to \mathcal{F}_μ^0 in $L^2(\Omega)$.*

Proof. Let v_k be a sequence of functions such that $v_k \rightarrow v$ strongly in $L^2(\Omega)$ and $\mathcal{F}_{\mu_k}^0(v_k) \leq C$ for some constant $C < +\infty$. Then v_k converges to v weakly in $H_{0,\partial\Omega}^1(\Omega \setminus S)$. For every $\varepsilon > 0$ let S_ε be the difference between S and the union of the closed balls with centers in F and radius ε . As S_ε is a C^1 manifold, arguing as in the proof of Lemma 3.3, for every ε we can construct $z_k, z \in H_0^1(\Omega)$ (possibly depending on ε) such that $z_k = [v_k]$ cap-q.e. on S_ε , $z = [v]$ cap-q.e. on S_ε , and $z_k \rightharpoonup z$ weakly in $H_0^1(\Omega)$. For every $t \in \mathbb{R}$ the sequence $(|z_k| \wedge t)^2$ converges to $(|z| \wedge t)^2$ weakly in $H_0^1(\Omega)$. Therefore

$$\int_{S_\varepsilon} (|z| \wedge t)^2 d\mu \leq \liminf_{k \rightarrow \infty} \int_{S_\varepsilon} (|z_k| \wedge t)^2 d\mu_k \leq \liminf_{k \rightarrow \infty} \int_{S_\varepsilon} z_k^2 d\mu_k.$$

Passing to the limit first as $t \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$ we obtain

$$\int_S [v]^2 d\mu \leq \liminf_{k \rightarrow \infty} \int_S [v_k]^2 d\mu_k,$$

which implies

$$\mathcal{F}_\mu^0(v) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_{\mu_k}^0(v_k).$$

Arguing as in the beginning of the proof of Lemma 3.3 we see that it is enough to prove the Γ -limsup inequality when $v \in H_{0, \partial\Omega}^1(\Omega \setminus S)$ is bounded and vanishes a.e. in a neighbourhood of F . In this case there exists $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $z = [v]$ cap-q.e. on S . As $z^2 \in H_0^1(\Omega)$ we have

$$\|[v]\|_{S, \mu}^2 = \int_S z^2 d\mu = \lim_{k \rightarrow \infty} \int_S z^2 d\mu_k = \lim_{k \rightarrow \infty} \|[v]\|_{S, \mu_k}^2,$$

which implies

$$\mathcal{F}_\mu^0(v) = \lim_{k \rightarrow \infty} \mathcal{F}_{\mu_k}^0(v).$$

This concludes the proof of the Γ -limsup inequality. \square

Proof of Lemma 3.5 (continuation). Since every positive measure in $H^{-1}(\Omega)$ with support in S can be approximated strongly in $H^{-1}(\Omega)$ by measures of the form $\mu_k = g_k \mathcal{H}^1 \llcorner S$, with g_k step-functions, from the previous step of the proof and from Lemma 3.6 we deduce that $\mathcal{F}_\mu^0 \in \overline{\mathcal{Y}}$ for every positive measure μ in $H^{-1}(\Omega)$ with support in S .

Let μ be a nonnegative Borel measure on Ω vanishing on all sets of capacity zero and with $\text{supp } \mu \subset S$. By [16, Lemma 4.15] there exist a positive measure μ_0 in $H^{-1}(\Omega)$ with support in S , and a Borel function $g : S \rightarrow [0, +\infty]$ such that

$$\int_S v^2 d\mu = \int_S v^2 g d\mu_0$$

for every $v \in H^1(\Omega)$.

By localizing the problem to an open set U satisfying condition (b) in the proof of Lemma 3.3, and using the functions v^\oplus, v^\ominus introduced in that proof we obtain that $\int_S [v]^2 d\mu = \int_S [v]^2 g d\mu_0$ for every $v \in H^1(\Omega \setminus S)$. Therefore $\mathcal{F}_\mu^0 = \mathcal{F}_{g\mu_0}^0$. Let $g_k := g \vee k$. Then $g_k \mu_0 \in H^{-1}(\Omega)$, hence $\mathcal{F}_{g_k \mu_0}^0 \in \overline{\mathcal{Y}}$. As $\mathcal{F}_{g_k \mu_0}^0$ is increasing and converges pointwise to $\mathcal{F}_{g\mu_0}^0 = \mathcal{F}_\mu^0$ we conclude that $\mathcal{F}_{g_k \mu_0}^0$ Γ -converges to \mathcal{F}_μ^0 in $L^2(\Omega)$, hence $\mathcal{F}_\mu^0 \in \overline{\mathcal{Y}}$. \square

Proof of Theorem 1.4. By Proposition 1.3 it is enough to prove that $|\overline{\partial\mathcal{F}}|_b(u, S) \leq 2\|f\|$. To do this we show that, under the regularity hypotheses of the theorem, there exist (u_k, S_k) with S_k smooth, such that S_k σ^2 -converges to S , $u_k \rightarrow u$ strongly in $L^2(\Omega)$, $\nabla u_k \rightarrow \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$, $|\partial\mathcal{F}|_b(u_k, S_k) = 2\|f_k\|$, and $f_k \rightarrow f$ strongly in $L^2(\Omega)$.

Let w be a function in $H^1(\Omega)$ having the same trace as u on $\partial\Omega$, and let w_k be a sequence in $C^\infty(\overline{\Omega})$ such that $w_k \rightarrow w$ strongly in $H^1(\Omega)$. Using Lemma 3.5 we obtain that there exists a sequence M_k of C^1 manifolds with boundary such that $M_k \subset S$ and $\mathcal{G}_{M_k}^{w_k}$ Γ -converges to \mathcal{F}_μ^w in $L^2(\Omega)$, where μ is the measure given by (3.3).

As (3.4) is the Euler equation for the problem

$$\min_{v \in L^2(\Omega)} \left\{ \mathcal{F}_\mu^w(v) - 2(f + u|v) \right\}, \quad (3.18)$$

the function u is the solution of (3.18). Let v_k be the minimizer of

$$\min_{v \in L^2(\Omega)} \left\{ \mathcal{G}_{M_k}^{w_k}(v) - 2(f + u|v) \right\}. \quad (3.19)$$

By Γ -convergence $v_k \rightarrow u$ strongly in $L^2(\Omega)$. As ∇v_k are bounded in $L^2(\Omega; \mathbb{R}^2)$, we have that $\nabla v_k \rightarrow \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$.

We now approximate each manifold M_k in the Hausdorff metric by a sequence M_k^ε , $\varepsilon > 0$, of C^∞ manifolds without boundary having the same number of connected components as M_k , and we consider the solutions v_k^ε of the minimum problems

$$\min_{v \in L^2(\Omega)} \left\{ \mathcal{G}_{M_k^\varepsilon}^{w_k}(v) - 2(f + u|v) \right\}. \quad (3.20)$$

By [13, Theorems 2.3 and 4.2] $v_k^\varepsilon \rightarrow v_k$ strongly in $L^2(\Omega)$ and $\nabla v_k^\varepsilon \rightarrow \nabla v_k$ strongly in $L^2(\Omega; \mathbb{R}^2)$. Therefore we can choose $\varepsilon_k > 0$ such that setting $u_k := v_k^{\varepsilon_k}$ and $S_k := M_k^{\varepsilon_k}$ we have $\|u_k - v_k\| < 1/k$, $\|\nabla u_k - \nabla v_k\| < 1/k$, and the Hausdorff distance between S_k and M_k is less than $1/k$.

Then $u_k \rightarrow u$ strongly in $L^2(\Omega)$, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$. The Euler equation of (3.20) implies that

$$\begin{cases} -\Delta u_k = f + u - u_k & \text{on } \Omega \setminus S_k, \\ \frac{\partial u_k}{\partial n} = 0 & \text{on } S_k, \\ u_k = w_k & \text{on } \partial\Omega. \end{cases} \quad (3.21)$$

The regularity theory for (3.21) (see, e.g., [28, Theorem 3.17]) gives that $u_k \in W^{2,p}(\Omega \setminus S_k)$. In particular, for every connected component U of $\Omega \setminus S_k$ we have that $u_k \in C^1(\bar{U})$. By Theorem 1.2 we have $|\partial\mathcal{F}|_b(u_k, S_k) = 2\|f + u - u_k\|$.

Therefore it remains to prove that S_k σ^2 -converges to S . The condition on the Hausdorff distance between S_k and M_k , together with the inclusion $M_k \subset S$, implies that S_k is contained in the closed ε -neighbourhood $S^{(\varepsilon)}$ of S for k large enough. If $z_k \in SBV^2(\Omega) \cap L^\infty(\Omega)$ is a sequence as in condition (a) in Definition 3.1 for S_k , then z_k is bounded in $H^1(\Omega \setminus S^{(\varepsilon)})$. This implies that its limit z belongs to $H^1(\Omega \setminus S^{(\varepsilon)})$, hence $S(z) \tilde{\subset} S^{(\varepsilon)}$. As ε is arbitrary, we deduce that $S(z) \tilde{\subset} S$, so that condition (a) is satisfied. As for condition (b), it is enough to take u and a suitable truncation of u_k . \square

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REFERENCES

- [1] Ambrosio L.: A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital. (7)* **3-B** (1989), 857-881.
- [2] Ambrosio L.: Movimenti minimizzanti. *Rend. Accad. Naz. Sci. XL Mem. Mat.* **19** (1995), 191-246.
- [3] Ambrosio L., Fusco N., Pallara D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford University Press, Oxford, 2000.
- [4] Ambrosio L., Gigli N., Savaré G.: Gradient flows with metric and differentiable structures, and applications to the Wasserstein space. Preprint, 2003.
- [5] Ambrosio L., Gigli N., Savaré G.: Gradient Flows in Metric Spaces and in the Wasserstein Space of Probability Measures. Birkhäuser, Boston, to appear.
- [6] Attouch H.: Variational Convergence for Functions and Operators. Pitman, London, 1984.
- [7] Attouch H., Picard C.: Comportement limite de problèmes de transmission unilatéraux à travers des grilles de forme quelconque. *Rend. Sem. Mat. Politec. Torino* **45** (1987), 71-85.
- [8] Braides A.: Γ -Convergence for Beginners. Oxford Lecture Series in Mathematics and its Applications, 22. Oxford University Press, Oxford, 2002.
- [9] Brezis H.: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland, Amsterdam-London; American Elsevier, New York, 1973.
- [10] Cortesani G.: Asymptotic behaviour of a sequence of Neumann problems. *Comm. Partial Differential Equations* **22** (1997), 1691-1729.
- [11] Dal Maso G.: On the integral representation of certain local functionals. *Ricerche Mat.* **32** (1983), 85-113.
- [12] Dal Maso G.: An Introduction to Γ -Convergence. Birkhäuser, Boston, 1992.
- [13] Dal Maso G., Ebobisse F., Ponsiglione M.: A stability result for nonlinear Neumann problems under boundary variations. *J. Math. Pures Appl.* **82** (2003), 503-532.
- [14] Dal Maso G., Francfort G.A., Toader R.: Quasistatic crack growth in nonlinear elasticity. *Arch. Rational Mech. Anal.*, to appear.
- [15] Dal Maso G., Morel J.M., Solimini S.: A variational method in image segmentation: existence and approximation results. *Acta Math.* **168** (1992), 89-151.
- [16] Dal Maso G., Mosco: Wiener’s criterion and Γ -convergence. *Appl. Math. Optim.* **15** (1987), 15-63.
- [17] Dal Maso G., Toader R.: A model for the quasi-static growth of brittle fractures: existence and approximation results. *Arch. Rational Mech. Anal.* **162** (2002), 101-135.

- [18] Damlamian A.: Le problème de la passoire de Neumann. *Rend. Sem. Mat. Univ. Politec. Torino* **43** (1985), 427-450.
- [19] De Giorgi E., Marino A., Tosques M.: Problemi di evoluzione in spazi metrici e curve di massima pendenza. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **68** (1980), 180-187.
- [20] Degiovanni M., Marino A., Tosques M.: Evolution equations with lack of convexity. *Nonlinear Anal.* **9** (1985), 1401-1443.
- [21] Evans L.C., Gariepy R.F.: *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, 1992.
- [22] Grisvard P.: *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.
- [23] Grisvard P.: *Singularities in Boundary Value Problems*. Masson, Paris, 1992.
- [24] Heinonen J., Kilpeläinen T., Martio O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Clarendon Press, Oxford, 1993.
- [25] Marino A., Saccon C., Tosques M.: Curves of maximal slope and parabolic variational inequalities with nonconvex constraints. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **16** (1989), 281-330.
- [26] Mumford D., Shah A.: Optimal approximation by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (1989), 577-685.
- [27] Murat F.: The Neumann sieve. *Non Linear Variational Problems (Isola d'Elba, 1983)*, 27-32, *Res. Notes in Math. 127*, Pitman, London, 1985.
- [28] Troianiello G.M.: *Elliptic Differential Equations and Obstacle Problems*. The University Series in Mathematics. Plenum Press, New York, 1987.

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