

The Γ -limit of traveling waves in the FitzHugh-Nagumo system

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Abstract: Patterns and waves are basic and important phenomena that govern the dynamics of physical and biological systems. A common theme in investigating such systems is to identify the intrinsic factors responsible for such self-organization. The Γ -convergence is a well-known technique applicable to variational formulations of concentration phenomena of stable patterns. Recently a geometric variational functional associated with the Γ -limit of standing waves of the FitzHugh-Nagumo system has been built. This article studies the Γ -limit of traveling waves. To the best of our knowledge, this is the first attempt to expand the scope of applicability of Γ -convergence to cover non-stationary problems.

Key words: Γ -convergence, FitzHugh-Nagumo, geometric variational problem, traveling front, traveling pulse.

AMS subject classification:

1 Introduction

Patterns and waves are basic and important phenomena [1, 12, 13, 17, 25, 28, 30, 34, 35] that govern the dynamics of physical and biological systems. This can be seen, for instance, in morphological phases in diblock copolymers, skin pigmentation in cell development and semiconductor gas-discharge systems. In the investigation of such systems, a common theme is to identify the intrinsic factors responsible for such self-organization. For the reaction-diffusion systems, self-organized patterns have not only been found in the neighborhoods of Turing's instability [31], but recent works

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[6, 7, 8, 9, 14, 15, 16, 18, 19, 29, 32, 33] showed that some patterns and waves possess localized spatial structures. In fact localized waves in reaction-diffusion systems are commonly observed, referred to as dissipative solitons [4, 21, 24, 27] in physical literature.

The FitzHugh-Nagumo model, which was originally derived as an excitable system for studying nerve impulse propagation, is now of great interest to the scientific community as breeding ground for patterns, traveling waves, and other localized structures. It has been extensively studied as a paradigmatic activator-inhibitor system for patterns generated from homogeneous media destabilized by a spatial modulation. These patterns are robust in the sense that they are stable and exist for a wide range of parameters.

The Γ -convergence [5] is a well-known technique applicable to variational formulations of concentration phenomena of stable patterns. When a stationary FitzHugh-Nagumo system is equipped with an appropriate scaling of the parameters, we are led to study a geometric variational problem [10, 11] with a Γ -limit energy functional defined by

$$\mathcal{J}_D(\Omega) = \mathcal{P}_D(\Omega) - \alpha|\Omega| + \frac{\sigma}{2} \int_{\Omega} \mathcal{N}_D(\Omega) dx, \quad (1.1)$$

where Ω is a measurable subset of the domain $D \subset \mathbb{R}^N$, $|\Omega|$ stands for the Lebesgue measure of Ω , and $\mathcal{P}_D(\Omega)$ denotes the perimeter of Ω in D which coincides with the measure of the part of the boundary of Ω inside D when Ω is sufficiently smooth. The integral in (1.1) is a nonlocal term where $\mathcal{N}_D(\Omega)$ is the solution of

$$-\Delta \mathcal{N}_D(\Omega) + \mathcal{N}_D(\Omega) = \chi_{\Omega}, \quad \partial_{\nu} \mathcal{N}_D(\Omega) = 0,$$

and ∂_{ν} is the outward normal derivative.

In this paper we derive a geometric variational functional associated with the study of traveling waves. To understand this derivation as opposed to that of the stationary problem, we give a brief review of the connection between (1.1) and the FitzHugh-Nagumo system. Consider

$$u_t = \epsilon^2 \Delta u - u \left(u - \frac{1}{2} \right) (u - 1) + \epsilon \alpha - \epsilon \sigma v, \quad (1.2)$$

$$v_t = \Delta v - v + u. \quad (1.3)$$

With $\alpha > 0$ and $\sigma > 0$ being fixed, the (small) parameter ϵ identifies a range where a singular limit will emerge. Recall that u acts as an activator and v is the inhibitor. Physically α measures the driving force towards a non-trivial state while σ represents the stabilizing inhibition mechanism. Their competition leads to interesting dynamics and the emergence of patterns. In dealing with stationary solutions of (1.2)-(1.3), both u_t and

v_t vanish. Solving (1.3) for v in terms of u and denoting this solution by $v = \mathcal{L}_D u$, we see that (1.2) becomes

$$-\epsilon^2 \Delta u + u \left(u - \frac{1}{2} \right) (u - 1) - \epsilon \alpha + \epsilon \sigma \mathcal{L}_D u = 0. \quad (1.4)$$

The solutions of (1.4) are the critical points of

$$\mathcal{I}_{D,\epsilon}(u) = \int_D \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{u^2(u-1)^2}{4} - \epsilon \alpha u + \frac{\epsilon \sigma}{2} u \mathcal{L}_D u \right) dx.$$

When D is bounded, the functionals $\epsilon^{-1} \mathcal{I}_{D,\epsilon}$ Γ -converge in $L^1(D)$ to

$$\frac{\sqrt{2}}{12} \mathcal{P}_D(\Omega) - \alpha |\Omega| + \frac{\sigma}{2} \int_{\Omega} \mathcal{N}_D(\Omega) dx, \quad (1.5)$$

a functional equivalent to (1.1). In case $D = \mathbb{R}^N$, a ball shaped stationary set of $\mathcal{J}_{\mathbb{R}^N}$ is referred as a bubble or an entire solution.

Front and pulse are localized waves, the latter appearing as small spots. In the past Γ -convergence has been employed to establish many interesting results for stable patterns; however to the best of our knowledge, this tool has not been utilized to treat traveling waves. We make an attempt in this direction, starting with the investigation of planar traveling wave solutions of the following FitzHugh-Nagumo system:

$$\begin{cases} u_t = \Delta u + \frac{1}{d}(f_\epsilon(u) - \epsilon \sigma v), \\ v_t = \Delta v + u - \gamma v, \end{cases} \quad (1.6)$$

where

$$f_\epsilon(\xi) = -\xi(\xi - \beta_\epsilon)(\xi - 1), \quad \beta_\epsilon = \frac{1}{2} - \frac{\alpha \epsilon}{\sqrt{2}} \quad (1.7)$$

and $\alpha, \gamma, \sigma > 0$. Note that when $d = \epsilon^2$ and $\gamma = 1$ it can be shown that the corresponding stationary problem leads, as $\epsilon \rightarrow 0$, to the same geometric variational functional (1.5), with α replaced by $\alpha/6\sqrt{2}$.

A planar traveling wave solution is of the form $(u(x-ct), v(x-ct))$; that is, the wave moves with speed c and keeps the same profile along the moving coordinates. To treat traveling waves using Γ -convergence, the ansatz $(u(c(x-ct)), v(c(x-ct)))$, proposed in [22], is more appropriate and will be adopted in the paper. We are thus led to deal with

$$dc^2 u_{xx} + dc^2 u_x + f_\epsilon(u) - \epsilon \sigma v = 0, \quad (1.8)$$

$$c^2 v_{xx} + c^2 v_x - \gamma v + u = 0, \quad (1.9)$$

with the value of c to be determined.

Let $L_e^2 = \{w : \int_{-\infty}^{\infty} e^x w^2 dx < \infty\}$. In studying the Γ -convergence in the L_e^2 topology of the traveling wave functional associated with (1.8)-(1.9), the quantity $\sqrt{dc^2}$ will be set equal to the parameter ϵ . Note that $0 < \beta_\epsilon < 1/2$ and that $\int_0^1 f_\epsilon(\xi) d\xi > 0$ for all ϵ sufficiently small. Then, given the parameters γ, σ and α , we look for traveling waves with speed $c(\epsilon)$, $d = \epsilon^2/c^2(\epsilon)$ and $c(\epsilon) \rightarrow c_0$ as $\epsilon \rightarrow 0$, where c_0 depending on the given parameters.

To be precise, in order to prove the existence of traveling waves whose L_e^2 -limit is a front, we shall assume that the following condition holds:

$$(A1) \quad \alpha > \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0.$$

Then, setting $h_* = 1 - \frac{(\alpha-1)\gamma}{3\sqrt{2}\sigma}$, we define

$$c_f = \frac{2h_*\sqrt{\gamma}}{\sqrt{1-h_*^2}}. \quad (1.10)$$

Since a translation of a traveling wave solution remains a solution, when looking for such waves we impose the additional constraint $\|u_\epsilon\|_{L_e^2} = 1$.

Theorem 1.1. *Let σ, γ and α be positive numbers satisfying (A1). There exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then there exists $c_\epsilon > 0$ for which, setting $d_\epsilon = \epsilon^2/c_\epsilon^2$, a traveling wave solution $(c_\epsilon, u_\epsilon, v_\epsilon)$ of (1.8)-(1.9) exists with $\|u_\epsilon\|_{L_e^2} = 1$. Moreover $c_\epsilon \rightarrow c_f$ and $u_\epsilon \rightarrow \chi_{(-\infty, 0]}$ in L_e^2 as $\epsilon \rightarrow 0$, where c_f is given by (1.10).*

In Theorem 1.1, c_f is uniquely determined when σ, γ and α are given, and for small ϵ the speeds c_ϵ of the traveling wave solutions are known up to leading order. Since the limit is unique, the convergence takes place along the whole family of solutions u_ϵ .

Next we investigate when the L_e^2 -limit is a traveling pulse. In this case we assume that the parameters satisfy the following condition:

$$(A2) \quad \frac{3\sqrt{2}\sigma}{\gamma} > \alpha > 1.$$

Theorem 1.2. *Let σ, γ and α be positive numbers satisfying (A2). There exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then there exists $c_\epsilon > 0$ for which, setting $d_\epsilon = \epsilon^2/c_\epsilon^2$, a traveling wave solution $(c_\epsilon, u_\epsilon, v_\epsilon)$ of (1.8)-(1.9) exists with $\|u_\epsilon\|_{L_e^2} = 1$. Moreover if $\epsilon \rightarrow 0$ then $c_\epsilon \rightarrow c_p$ and $u_\epsilon \rightarrow \chi_{[a, b]}$ in L_e^2 , where c_p, a and b are uniquely determined by the given parameters σ, γ and α .*

It was shown [10, 11] that when $N = 1$ there exists always a single bubble; when $N = 2$ there may exist zero, one, two, or even three bubble profiles, depending on the

values of α and σ , while if $N \geq 3$, there can be no more than two bubble profiles. The Γ -limit of higher dimensional traveling waves of (1.6) will be addressed in a work in progress.

We now give an outline of how the paper is organized. In Section 2 we introduce a class of functions with weighted bounded variation which fit well in the Γ -convergence scheme for the traveling wave problem. The variational formulation for the FitzHugh-Nagumo system is given in Section 3 and the representation formula for the Γ -limit is proved in Section 4. In Section 5 we prove that the limit functional has always a minimizer that can be either a traveling front or a pulse. The occurrence of a front or a pulse for the limit problem, depending on the different physical parameter regimes, will be discussed in Sections 6 and 7. Moreover we shall also show that the existence of these fronts and pulses in the limit case implies the existence of traveling waves for the approximating problems for ϵ sufficiently small.

2 Weighted BV functions

Let $\Omega \subset \mathbb{R}$ be an open set and let $p \geq 1$. We denote by $L^p_e(\Omega)$ the set of functions $u \in L^1_{loc}(\Omega)$ such that $\|u\|_{L^p_e(\Omega)} = \left(\int_{\Omega} e^x |u|^p dx\right)^{1/p} < \infty$. We shall denote by $H^1_e(\Omega)$ the space of functions $u \in L^2_e(\Omega)$ with derivative in $L^2_e(\Omega)$ equipped with the norm $\|u\|_{H^1_e(\Omega)} = \|u\|_{L^2_e(\Omega)} + \|u'\|_{L^2_e(\Omega)}$. Clearly, $L^2_e(\Omega)$ and $H^1_e(\Omega)$ are Hilbert spaces with the inner product defined in the obvious way.

Given a function $u \in L^1_{loc}(\Omega)$, the total variation of u in Ω with respect to the measure $e^x dx$ is defined as

$$\|Du\|_e(\Omega) = \sup \left\{ \int_{\Omega} u (e^x \varphi)' dx : \varphi \in C^1_0(\Omega), |\varphi| \leq 1 \right\}. \quad (2.1)$$

If $\|Du\|_e(\Omega) < \infty$, it is easily checked, see [3], that for any open set $\Omega' \subset\subset \Omega$ the total variation of u in Ω'

$$\|Du\|(\Omega') = \sup \left\{ \int_{\Omega'} u \varphi' dx : \varphi \in C^1_0(\Omega'), |\varphi| \leq 1 \right\}$$

is also finite. Therefore if $u \in L^1_{loc}(\Omega)$ and $\|Du\|_e(\Omega) < \infty$, then u has bounded variation on any $\Omega' \subset\subset \Omega$. By the Riesz representation theorem there exist a Radon measure μ and a μ -measurable function σ with $|\sigma| = 1$ such that for any $\varphi \in C^1_0(\Omega)$

$$\int_{\Omega} u \varphi' dx = - \int_{\Omega} \varphi \sigma d\mu. \quad (2.2)$$

It follows from (2.2) that the measure $\sigma d\mu$ coincides with the distributional derivative Du of u . Hence if u is smooth then $\sigma d\mu = u' dx$ and $d\mu = |u'| dx$.

Note that (2.2) implies in particular that for any $\varphi \in C_0^1(\Omega)$

$$\int_{\Omega} e^x u \varphi dx = - \int_{\Omega} (e^x u \varphi' dx + e^x \varphi \sigma d\mu) . \quad (2.3)$$

We shall denote by $BV_e(\Omega)$ the set of functions $u \in L_e^1(\Omega)$ such that $\|Du\|_e(\Omega) < \infty$. This function space is equipped with the norm

$$\|u\|_{BV_e(\Omega)} = \|u\|_{L_e^1(\Omega)} + \|Du\|_e(\Omega) .$$

Also, if $u \in BV_e(\Omega)$ then

$$\|Du\|_e(\Omega) = \int_{\Omega} e^x d\mu . \quad (2.4)$$

In the following, when $\Omega = \mathbb{R}$ we shall omit the explicit dependence on Ω of the above spaces, thus writing BV_e , L_e^p and H_e^1 . Similarly, we shall write $\|Du\|_e$ instead of $\|Du\|_e(\Omega)$.

The first of the next two lemmas is a straightforward consequence of (2.1), while the second one can be proved exactly as for the standard BV functions, see [20, p.172, Th. 2].

Lemma 2.1. (*Lower semicontinuity of the weighted variation measure*)

If $u_k \rightarrow u_0$ in $L_{e,loc}^1(\Omega)$, then $\|Du_0\|_e(\Omega) \leq \liminf_{k \rightarrow \infty} \|Du_k\|_e(\Omega)$.

Lemma 2.2. (*Local approximation by smooth functions*)

Suppose $u \in BV_e(\Omega)$. Then there exists a sequence $\{u_k\} \subset BV_e(\Omega) \cap C^\infty(\Omega)$ such that

- (i) $u_k \rightarrow u$ in $L_e^1(\Omega)$ and
- (ii) $\|Du_k\|_e(\Omega) \rightarrow \|Du\|_e(\Omega)$ as $k \rightarrow \infty$.

If $u = \chi_{[a,b]}$ with $-\infty \leq a < b < \infty$, an easy calculation gives

$$\|D\chi_{[a,b]}\|_e = \sup_{|\varphi| \leq 1} \int_a^b (e^x \varphi)' dx = \sup_{|\varphi| \leq 1} \{e^x \varphi|_{x=a}^b\} = e^b + e^a . \quad (2.5)$$

The following lemma will be useful in the sequel.

Lemma 2.3. If $E = \cup_i [a_i, b_i]$ is a union of countably many disjoint intervals, then

$$\|D\chi_E\|_e = \sum_i (e^{a_i} + e^{b_i}) . \quad (2.6)$$

Moreover $\chi_E \in BV_e$ if and only if E is the union of countably many disjoint intervals $[a_i, b_i]$ and the right hand side of (2.6) is finite.

Proof. It is clear that (2.6) is an immediate consequence of (2.5). Furthermore, if $E = \cup_i [a_i, b_i]$ is the union of countably many disjoint intervals and the right hand side of (2.6) is finite, then

$$\int_{-\infty}^{+\infty} e^x \chi_E(x) dx \leq \sum_i e^{b_i} < \infty .$$

This implies $\chi_E \in L_e^1$ and thus $\chi_E \in BV_e$.

Conversely, if $\chi_E \in BV_e$, then, as observed before, χ_E has finite total variation in any bounded interval. It follows that E is the union of countably many disjoint intervals $[a_i, b_i]$, see, e.g. [2, Prop. 3.52], and the conclusion is immediate from (2.6). \square

Lemma 2.4. *Let $u \in BV_e$ and $h \in \mathbb{R}$. Then*

- (i) $\|u(\cdot + h)\|_{L_e^1} = e^{-h}\|u\|_{L_e^1}$;
- (ii) $\|D(u(\cdot + h))\|_e = e^{-h}\|Du\|_e$;
- (iii) u^+ and u^- are in BV_e with $\|Du^+\|_e \leq \|Du\|_e$ and $\|Du^-\|_e \leq \|Du\|_e$;
- (iv) $\|u\|_{L_e^1} \leq \|Du\|_e$;
- (v) $|u(x)|e^x \leq 2\|Du\|_e$ a.e..

Proof. (i) and (ii) follow directly from the definitions.

If $u \in BV_e$ then u has locally finite variation in \mathbb{R} . This implies that u^+ has locally finite variation in \mathbb{R} and $d(Du^+) = \chi_{\{u>0\}}d(Du) = \chi_{\{u>0\}}\sigma d\mu$ (see [2, Ex. 3.100]). It follows from (2.4) that

$$\|Du^+\|_e = \int_{\{u>0\}} e^x d\mu \leq \|Du\|_e.$$

A similar calculation works for u^- , so (iii) follows.

By Lemma 2.2 it suffices to verify (iv) and (v) for smooth u only. Consider the case $u \geq 0$ first. For any $\delta > 0$, there exists a large $R > 0$ such that $0 \leq \int_{|x|>R} e^x u dx \leq \delta$. Take $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 1$ on $[-R, R]$, $\varphi = 0$ for $|x| \geq R + 2$ and $0 \leq \varphi \leq 1$ on $[R, R + 2]$ with $|\varphi'| \leq 1$. Then, using (2.3) and (2.4), we have

$$\begin{aligned} \|u\|_{L_e^1} - \delta &\leq - \int_{-\infty}^{\infty} e^x u \varphi dx = - \int_{-\infty}^{\infty} e^x \varphi \sigma d\mu - \int_{-\infty}^{\infty} e^x u \varphi' dx \\ &\leq \|Du\|_e + \int_{R \leq |x| \leq R+2} e^x |u| dx \leq \|Du\|_e + \delta. \end{aligned}$$

This gives $\|u\|_{L_e^1} \leq \|Du\|_e$ if $u \geq 0$. In the general case, setting $u = u^+ - u^-$ yields

$$\|u\|_{L_e^1} = \|u^+\|_{L_e^1} + \|u^-\|_{L_e^1} \leq \|Du^+\|_e + \|Du^-\|_e = \|Du\|_e,$$

which completes the proof of (iv).

Suppose u is smooth and with compact support. Then $-u(x)e^x = \int_x^\infty (ue^t)' dt = \int_x^\infty e^t(u + u') dt$. Therefore

$$|u(x)|e^x \leq \int_x^\infty e^t(|u| + |u'|) dt \leq \|u\|_{L_e^1} + \|Du\|_e \leq 2\|Du\|_e,$$

which yields (v) in this special case. This inequality can now be extended to $u \in C^\infty$ by using $u\varphi$ as an approximation, where φ is the cut-off function we used above. \square

3 Variational formulation

We now turn to the variational formulation needed for studying the traveling waves of (1.6). Set $F_\epsilon(w) \equiv -\int_0^w f_\epsilon(\xi) d\xi$, where f_ϵ is defined as in (1.7).

Given $u \in L_e^2$, we denote by $v = \mathcal{L}_c u$ the unique solution in H_e^1 of (1.9). Note that $\mathcal{L}_c : L_e^2 \rightarrow H_e^1$ is a self-adjoint operator with respect to the inner product of L_e^2 . Given $c, d > 0$, let $\mathcal{I}_{c,d} : H_e^1 \rightarrow \mathbb{R}$ be defined as

$$\mathcal{I}_{c,d}(w) = \int_{-\infty}^{\infty} e^x \left(\frac{dc^2}{2} w'^2 + F_\epsilon(w) + \frac{\epsilon\sigma}{2} w \mathcal{L}_c w \right) dx. \quad (3.1)$$

A standard variational argument shows that (u, v, c) solves (1.8)-(1.9) provided $v = \mathcal{L}_c u$ and u is a critical point of $\mathcal{I}_{c,d}$. We shall refer to the terms on the right hand side of (3.1) as to the gradient energy, the F -integral (or potential) and the nonlocal energy, respectively. The nonlocal energy is always non-negative since $\int_{-\infty}^{\infty} e^x w \mathcal{L}_c w dx \geq 0$ for all $w \in L_e^2$.

Let $\epsilon_1 > 0$ be such that $\max\{\frac{1}{\sigma}, \frac{1}{2\sigma}\} \geq \epsilon_1 > 0$. Observe that there exist $\beta_2 > 1$, with $\beta_2 - 1$ small if ϵ_1 is small, and $\widetilde{M}_1 = \widetilde{M}_1(\gamma) > 0$ satisfying $f_\epsilon(-\widetilde{M}_1) \geq \frac{\beta_2}{\gamma}$ for all $\epsilon \in (0, \epsilon_1)$. In the following we shall consider the restriction of $\mathcal{I}_{c,d}$ to the admissible set Y defined by

$$Y \equiv \left\{ w \in H_e^1 : \int_{-\infty}^{\infty} e^x w^2 dx = 1, -\widetilde{M}_1 - 1 \leq w \leq \beta_2 \right\}. \quad (3.2)$$

In the definition of Y the constraint $\|w\|_{L_e^2} = 1$ is imposed in order to eliminate a continuum of critical points due to translation. Suppose that $u \in Y$ is a constrained minimizer of $\mathcal{I}_{c,d}$, then u is the sought-after traveling wave solution provided $\mathcal{I}_{c,d}(u) = 0$ and $-\widetilde{M}_1 - 1 < u < \beta_2$. We refer to [8] for the detailed argument.

The constrained variational approach has been employed in [6, 8, 9] to establish the existence of traveling wave solutions of FitzHugh-Nagumo system. There all the parameters are fixed and of order $O(1)$, except d which has to be sufficiently small. In the situation studied therein when $d \rightarrow 0$ the wave speed c tends to infinity and dc^2 approaches a positive number, which depends only on $\beta \in (0, 1/2)$, where $f(\xi) = \xi(\xi - \beta)(1 - \xi)$. This is a case when the Γ -limit of $\mathcal{I}_{c,d}$ does not exist when $d \rightarrow 0$. Instead, we shall see here a situation where the Γ -limit exists and can be used to prove the existence of traveling waves. As far as we know, this is one of the first instances where Γ -convergence is applied to non-stationary problems. To this end we will require that the other parameters vary in a suitable way depending on d .

Remark 3.1. *Observe that in [6, 8, 9] the constraint $\|w'\|_{L_e^2} = \sqrt{2}$ was imposed in order to prevent translations. Here instead, to simplify the Γ -convergence analysis, we have chosen in the definition of Y the constraint $\|w\|_{L_e^2} = 1$.*

The Γ -limit of $\mathcal{I}_{c,d}$ and its minimizers will be studied later. The next two lemmas enable us to recover the traveling wave solutions from the minimizers of the limit functional of $\mathcal{I}_{c,d}$.

Lemma 3.2. *Fix γ, σ, α and ϵ_1 as above. If $c, d > 0$ and $\epsilon \in (0, \epsilon_1)$, then $\inf_{w \in Y} \mathcal{I}_{c,d}(w)$ is uniformly bounded from below with respect to ϵ . Moreover, if $\inf_{w \in Y} \mathcal{I}_{c,d}(w) \leq 0$, then the infimum is attained at some $u \in Y$.*

Proof. Note that there exists a constant $M_2 > 0$ such that $-M_2\xi^2 \leq F_\epsilon(\xi)$ for all $\xi \in \mathbb{R}$ and $0 < \epsilon < \epsilon_1$ then. Then, for $w \in Y$ we have

$$\mathcal{I}_{c,d}(w) \geq \int_{-\infty}^{\infty} e^x F_\epsilon(w) dx \geq -M_2 \int_{-\infty}^{\infty} e^x w^2 dx = -M_2,$$

hence $\inf_Y \mathcal{I}_{c,d}$ is bounded from below uniformly with respect to ϵ .

Let $\{w_n\} \subset Y$ be a minimizing sequence with $\mathcal{I}_{c,d}(w_n) \leq \inf_Y \mathcal{I}_{c,d} + 1$. Then

$$\begin{aligned} \frac{dc^2}{2} \int_{-\infty}^{\infty} e^x w_n'^2 dx &\leq \mathcal{I}_{c,d}(w_n) - \int_{-\infty}^{\infty} e^x F_\epsilon(w_n) dx \\ &\leq \inf_Y \mathcal{I}_{c,d} + 1 + M_2. \end{aligned}$$

Using the following Poincaré type inequality for $w \in H_e^1$

$$\int_{\mathbb{R}} e^x w'^2 dx \geq \frac{1}{4} \int_{\mathbb{R}} e^x w^2 dx,$$

we get that the sequence $\{w_n\}$ is bounded in H_e^1 . Therefore, we may assume that there exists $w \in H_e^1$ such that, up to a (not relabelled) subsequence, $w_n \rightharpoonup w$ weakly in H_e^1 and strongly in $L_{loc}^\infty(\mathbb{R}) \cap L_{loc}^2(\mathbb{R})$. Then, $-\widetilde{M}_1 - 1 \leq w \leq \beta_2$. Arguing as in the proof of Lemma 4.2 in [8] we obtain

$$\begin{aligned} \int_{\mathbb{R}} e^x w'^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} e^x w_n'^2 dx, \\ \int_{\mathbb{R}} e^x F_\epsilon(w) dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} e^x F_\epsilon(w_n) dx, \\ \int_{\mathbb{R}} e^x w \mathcal{L}_c w dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} e^x w_n \mathcal{L}_c w_n dx \end{aligned}$$

and thus $\mathcal{I}_{c,d}(w) \leq \liminf_{n \rightarrow \infty} \mathcal{I}_{c,d}(w_n)$. Moreover $\int_{\mathbb{R}} e^x w^2 dx \leq 1$, since $\int_{-l}^l e^x w^2 dx = \lim_{n \rightarrow \infty} \int_{-l}^l e^x w_n^2 dx \leq 1$ for any $l > 0$.

Suppose now that $\inf_Y \mathcal{I}_{c,d} \leq 0$, hence $\mathcal{I}_{c,d}(w) \leq 0$. We claim that $w \not\equiv 0$. Indeed, otherwise we would get

$$\begin{aligned} 0 &\geq \liminf_{n \rightarrow \infty} \mathcal{I}_{c,d}(w_n) \geq \frac{dc^2}{8} \int_{\mathbb{R}} e^x w_n^2 dx + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} e^x \left\{ F_\epsilon(w_n) + \frac{1}{2} w_n \mathcal{L}_c w_n \right\} dx \\ &\geq \frac{dc^2}{8} + \int_{\mathbb{R}} e^x \left\{ F_\epsilon(w) + \frac{1}{2} w \mathcal{L}_c w \right\} dx = \frac{dc^2}{8}, \end{aligned}$$

which is absurd. Consequently $1 \geq \int_{\mathbb{R}} e^x w^2 dx > 0$.

Take $a \geq 0$ such that $e^a \int_{\mathbb{R}} e^x w^2 dx = 1$. Setting $u(x) \equiv w(x - a)$, then $u \in Y$ and

$$\mathcal{I}_{c,d}(u) = e^a \mathcal{I}_{c,d}(w) \leq \mathcal{I}_{c,d}(w) \leq \inf_{w \in Y} \mathcal{I}_{c,d}(w) \leq \mathcal{I}_{c,d}(u) .$$

Hence u is a minimizer of $\mathcal{I}_{c,d}$ in Y . Note also that if $\inf_Y \mathcal{I}_{c,d} < 0$, then $a = 0$ and $u = w$. \square

Lemma 3.3. *Let u be a global minimizer of $\mathcal{I}_{c,d}$ in Y and $v = \mathcal{L}_c u$. Suppose that $\mathcal{I}_{c,d}(u) = 0$, then (u, v) satisfies (1.8)-(1.9); that is, a traveling wave solution of (1.6) with c as wave speed.*

Proof. We only give a sketch of the proof, since the argument is similar to that of [8, 9]. Arguing as in those papers, we first show that $-\widetilde{M}_1 - 1 < u < \beta_2$. Following a simplified version of [8, Lemma 4.5] (originally dealt with additional oscillation constraints), we next prove that $u \geq -\widetilde{M}_1$. When d is sufficiently small, invoking [8, Lemma 6.2] yields $v > 0$. This in turn leads to $u_0 < 1 < \beta_2$ by making use of [8, Lemma 7.1].

Since $\mathcal{I}_{c,d}(u) = 0$, with a slight modification, the argument used in [8] shows that there is no Lagrange multiplier associated with the constraint $\int_{\mathbb{R}} e^x w^2 dx = 1$. Consequently (u, v) satisfies (1.8)-(1.9). \square

The condition $\inf_{w \in Y} \mathcal{I}_{c,d}(w) = 0$ depends on the fact that the parameters in the governing equations change in a coordinate manner. This will be achieved by using the tool of Γ -convergence.

4 Γ -convergence

To investigate the Γ -convergence for the traveling wave functional, we rewrite (1.8)-(1.9) choosing d such that $\epsilon = \sqrt{dc^2}$

$$\epsilon^2 u_{xx} + \epsilon^2 u_x + f_\epsilon(u) - \epsilon \sigma v = 0 , \tag{4.1}$$

$$c^2 v_{xx} + c^2 v_x - \gamma v + u = 0 . \tag{4.2}$$

Then, in order to obtain a nontrivial geometric variational functional as a Γ -limit when $\epsilon \rightarrow 0$, we need to choose a suitable dependence of c , hence of d , on the parameter ϵ .

Note that $F_\epsilon = F_0 + \alpha \epsilon G$, where $F_0(u) \equiv \frac{1}{4} u^2 (u - 1)^2$ and $G(u) \equiv \frac{1}{\sqrt{2}} (\frac{u^3}{3} - \frac{u^2}{2})$. In this decomposition of F_ϵ , F_0 is a balanced bistable nonlinearity and $F_0(0) = F_0(1) = \min F_0 = 0$. G has a local maximum at 0 and a local minimum at 1 with $G(0) = 0$ and $G(1) = -1/6\sqrt{2}$. Their sum F_ϵ has a local maximum at β_ϵ and $F_\epsilon(\beta_\epsilon) > 0$. $F_\epsilon(0) = 0$ is

a local minimum and $F_\epsilon(1) = -\frac{1-2\beta_\epsilon}{12} = -\frac{1}{6\sqrt{2}}\alpha\epsilon$ is the global minimum.

To evaluate the Γ -limit in L^2_ϵ as $\epsilon \rightarrow 0$ of the functional $\mathcal{I}_{c,d}/\epsilon$, we set $J_{c(\epsilon)} \equiv \mathcal{I}_{c,d}/\epsilon$, that is, for $w \in Y$

$$J_{c(\epsilon)}(w) = \int_{-\infty}^{\infty} e^x \left\{ \frac{\epsilon w'^2}{2} + \frac{F_0(w)}{\epsilon} + \alpha G(w) + \frac{\sigma}{2} w \mathcal{L}_{c(\epsilon)} w \right\} dx. \quad (4.3)$$

In (4.3) c is not necessarily a constant but a function of ϵ with the property that $c(\epsilon) \rightarrow c_0$ as $\epsilon \rightarrow 0$ for some positive constant c_0 . If u is a minimizer of $J_{c(\epsilon)}$ in Y and

$$J_{c(\epsilon)}(u) = 0,$$

then by Lemma 3.3 we obtain a solution of (4.1)-(4.2) with $v = \mathcal{L}_{c(\epsilon)}u$. Our goal is to find traveling waves with speed close to c_0 . A traveling wave solution of (4.1)-(4.2) will be denoted by $(c_\epsilon, u_\epsilon, v_\epsilon)$, where c_ϵ is the wave speed and $d = \epsilon^2/c_\epsilon^2$.

Next we study the Γ -convergence of $J_{c(\epsilon)}$ as $\epsilon \rightarrow 0$. As an application of the Γ -convergence Theorem 4.3 below, we shall prove the existence of traveling wave solutions of the original problems by seeking suitable conditions on the parameters α , γ and σ .

Let us now define the function $\phi(\xi) = \int_0^\xi \sqrt{2F_0(\eta)} d\eta$, which is commonly used in the framework of phase transition problems, see for instance [23, 26]. Although the situation considered in these papers is similar to ours, in dealing with traveling waves instead of stationary solutions additional complications arise, due to the presence of an unbounded domain, of a nonlocal term and of the weight e^x . Observe that ϕ is a strictly increasing function, $\phi(0) = 0$ and $\phi(1) = \frac{\sqrt{2}}{12}$.

Throughout this paper, to simplify the notation, we shall denote with a slight abuse of notation by $\{w_\epsilon\}$ a sequence $\{w_{\epsilon_h}\}$, with $\epsilon_h \rightarrow 0^+$. Moreover, in order to avoid the use of double indices, a subsequence of such a sequence will be also denoted by $\{w_\epsilon\}$.

Lemma 4.1. (*Compactness*) *Let $\{w_\epsilon\} \subset Y$ be a sequence such that $\liminf_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) \leq C_0$ for some $C_0 > 0$. Then, setting $C_1 = \frac{1}{\sqrt{2}}\left(\frac{1+\widetilde{M}_1}{3} + \frac{1}{2}\right)$, the following hold:*

- (i) $\phi(w_\epsilon) \in BV_\epsilon$;
- (ii) *there exist a (not relabelled) subsequence $\{w_\epsilon\}$ and a set E such that $w_\epsilon \rightarrow \chi_E$ in L^2_ϵ and $\|D\chi_E\|_\epsilon \leq 6\sqrt{2}(C_0 + C_1\alpha)$;*
- (iii) $\int_{-\infty}^{\infty} e^x \chi_E = 1$ and $E \subset (-\infty, \log 6\sqrt{2}(C_0 + C_1\alpha)]$.

Proof. From the definition of Y in (3.2) we have that if $\epsilon \in (0, \epsilon_1)$, then $\|w_\epsilon\|_{L^\infty} \leq 1 + \widetilde{M}_1$ and $\|w_\epsilon\|_{L^p_\epsilon} \leq \|w_\epsilon\|_{L^2_\epsilon}^{2/p} \|w_\epsilon\|_{L^\infty}^{(p-2)/p} \leq (1 + \widetilde{M}_1)^{(p-2)/p}$ for all $p > 2$. Thus, since $w_\epsilon \in H^1_\epsilon$

and the nonlocal energy is non-negative, we obtain

$$\begin{aligned}
\|D(\phi(w_\epsilon))\|_e &= \int_{-\infty}^{\infty} e^x |\sqrt{2F_0(w_\epsilon)}| |w'_\epsilon| dx \leq \int_{-\infty}^{\infty} e^x \left(\epsilon \frac{w_\epsilon'^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon} \right) dx \\
&\leq J_{c(\epsilon)}(w_\epsilon) + \alpha \int_{-\infty}^{\infty} e^x |G(w_\epsilon)| dx \\
&\leq C_0 + \frac{\alpha}{\sqrt{2}} \left(\frac{1}{3} \int_{-\infty}^{\infty} e^x |w_\epsilon|^3 dx + \frac{1}{2} \int_{-\infty}^{\infty} e^x |w_\epsilon|^2 dx \right) + o(1) \\
&\leq C_0 + \frac{\alpha}{\sqrt{2}} \left(\frac{1 + \widetilde{M}_1}{3} + \frac{1}{2} \right) + o(1) = C_0 + C_1 \alpha + o(1). \tag{4.4}
\end{aligned}$$

This shows that the functions $\phi(w_\epsilon)$ are uniformly bounded in BV_e . By applying on each bounded interval $[-k, k]$, with $k \in \mathbb{N}$, the compactness theorem for BV functions on bounded domains and then using a standard diagonalization argument, we easily obtain that there exist a function $\Phi_0 \in L^1_{loc}(\mathbb{R})$ and a (not relabelled) subsequence $\{\phi(w_\epsilon)\}$ converging to Φ_0 in $L^1_{loc}(\mathbb{R})$ and pointwise a.e.. Thus, from Lemmas 2.1 and 2.4 we have that $\Phi_0 \in BV_e$ and that

$$\|\Phi_0\|_{L^1_e} \leq \|D\Phi_0\|_e \leq \liminf_{\epsilon \rightarrow 0} \|D\phi(w_\epsilon)\|_e \leq C_0 + C_1 \alpha.$$

Since ϕ is a strictly increasing function, setting $w_0 \equiv \phi^{-1}(\Phi_0)$, yields $w_\epsilon \rightarrow w_0$ pointwise a.e. and thus $-\widetilde{M}_1 - 1 \leq w_0 \leq \beta_2$ a.e.. Observe that

$$\int_{-\infty}^{\infty} e^x F_0(w_\epsilon) dx \leq \epsilon \left\{ J_{c(\epsilon)}(w_\epsilon) + \alpha \int_{-\infty}^{\infty} e^x |G(w_\epsilon)| dx \right\} \leq \epsilon(C_0 + C_1 \alpha + o(1)). \tag{4.5}$$

An application of Fatou's lemma gives

$$0 \leq \int_{-\infty}^{\infty} e^x F_0(w_0) dx \leq \liminf_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^x F_0(w_\epsilon) dx \leq \liminf_{\epsilon \rightarrow 0} \epsilon(C_0 + C_1 \alpha + o(1)) = 0,$$

which implies $F_0(w_0) = 0$ a.e.. In turn this implies that $w_0(x) \in \{0, 1\}$ a.e., that is $w_0 = \chi_E$ for some measurable set $E \subset \mathbb{R}$. As a consequence, $\phi(w_\epsilon) \rightarrow \Phi_0 = \phi(w_0) = \phi(1)\chi_E$ in $L^1_{loc}(\mathbb{R})$ and pointwise a.e.. Using Lemmas 2.1 and 2.4 again, we get

$$\phi(1)\|\chi_E\|_{L^1_e} \leq \phi(1)\|D\chi_E\|_e = \|D\phi(w_0)\|_e \leq \liminf_{\epsilon \rightarrow 0} \|D\phi(w_\epsilon)\|_e \leq C_0 + C_1 \alpha. \tag{4.6}$$

Hence $\chi_E \in BV_e$ and $\|D\chi_E\|_e \leq \frac{C_0 + C_1 \alpha}{\phi(1)}$, which in turn implies $E \subset (-\infty, \log \frac{C_0 + C_1 \alpha}{\phi(1)}]$.

From Lemma 2.4 we have $e^x |\phi(w_\epsilon(x))| \leq 2\|D\phi(w_\epsilon)\|_e \leq 2(C_0 + C_1 \alpha + o(1))$ and thus $|\phi(w_\epsilon(x))| \leq 2(C_0 + C_1 \alpha + 1)e^{-x}$. Since ϕ^{-1} is continuous and $\phi^{-1}(0) = 0$, there exists y_0 such that $|w_\epsilon(x)| \leq 1/2$ if $x \geq y_0$. This together with (4.5) gives

$$\int_{y_0}^{\infty} e^x w_\epsilon^2 dx \leq 4 \int_{y_0}^{\infty} e^x w_\epsilon^2 (w_\epsilon - 1)^2 dx \leq 16 \int_{-\infty}^{\infty} e^x F_0(w_\epsilon) dx \leq 16\epsilon(C_0 + C_1 \alpha + o(1)).$$

We claim now that the subsequence $\{w_\epsilon\}$ is a Cauchy sequence in L_e^2 . To this end we fix $\delta > 0$, and estimate the norm of $w_\epsilon - w_\eta$, for $\epsilon, \eta > 0$, as follows

$$\begin{aligned}
& \int_{-\infty}^{\infty} e^x |w_\epsilon - w_\eta|^2 dx \\
& \leq \int_{-\infty}^{-y_\delta} e^x |w_\epsilon - w_\eta|^2 dx + \int_{-y_\delta}^{y_0} e^x |w_\epsilon - w_\eta|^2 dx + \int_{y_0}^{\infty} e^x |w_\epsilon - w_\eta|^2 dx \\
& \leq 4(1 + \widetilde{M}_1)^2 \int_{-\infty}^{-y_\delta} e^x dx + \int_{-y_\delta}^{y_0} e^x |w_\epsilon - w_\eta|^2 dx + 2 \int_{y_0}^{\infty} e^x (w_\epsilon^2 + w_\eta^2) dx \\
& \leq 4(1 + \widetilde{M}_1)^2 e^{-y_\delta} + \int_{-y_\delta}^{y_0} e^x |w_\epsilon - w_\eta|^2 dx + 64(C_0 + C_1\alpha + o(1)) \max\{\epsilon, \eta\}.
\end{aligned}$$

Note that the first and the third addend in the last line of the above formula can be made smaller than δ by fixing y_δ large and by choosing ϵ and η smaller than a suitable ϵ_δ . On the other hand, since the functions w_ϵ are uniformly bounded and converge to χ_E pointwise a.e., they converge to χ_E also in $L_e^2(-y_\delta, y_0)$. Therefore also the last integral in the above formula is smaller than δ , provided ϵ and η are small enough. Thus $\{w_\epsilon\}$ is a Cauchy sequence in L_e^2 and thus converges to χ_E also in L_e^2 . Hence

$$\int_{-\infty}^{\infty} e^x \chi_E dx = \int_{-\infty}^{\infty} e^x \chi_E^2 dx = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^x w_\epsilon^2 dx = 1.$$

This completes the proof. \square

In order to use the Γ -convergence for studying the existence and the qualitative behavior of traveling wave solutions as $\epsilon \rightarrow 0$, we extend the domain of $J_{c(\epsilon)}$ to L_e^2 by setting

$$J_{c(\epsilon)}(w) = \begin{cases} \text{the right hand side of (4.3)} & \text{if } w \in Y, \\ \infty, & \text{if } w \in L_e^2 \setminus Y. \end{cases}$$

Next we define a functional that will turn out to be the Γ -limit of $J_{c(\epsilon)}$ with respect to the L_e^2 convergence. Let $E = \cup_i [a_i, b_i]$ be the union of countably many disjoint intervals such that $b_1 < \infty$, $a_i > b_{i+1}$ for all i and $a_i \geq -\infty$. If $a_i = -\infty$ for some i , then E reduces to the union of finitely many intervals, while if E is the union of infinitely many intervals and $\chi_E \in BV_e$ then $\lim_{i \rightarrow \infty} a_i = -\infty$.

We introduce the functional $J_c^* : L_e^2 \rightarrow \mathbb{R}$ defined as

$$J_c^*(w) \equiv \begin{cases} \frac{\sqrt{2}}{12} \sum_i (e^{a_i} + e^{b_i}) - \frac{\sqrt{2}\alpha}{12} \sum_i (e^{b_i} - e^{a_i}) + \frac{\sigma}{2} \int_{-\infty}^{\infty} e^x \chi_E \mathcal{L}_c \chi_E dx, & \text{if } w = \chi_E \in BV_e \text{ and } \int_{-\infty}^{\infty} e^x \chi_E dx = 1, \\ \infty, & \text{otherwise.} \end{cases} \quad (4.7)$$

Remark 4.2. *Since $\sum (e^{b_i} - e^{a_i}) \leq e^{b_1} < \infty$, if $J_c^*(\chi_E) < \infty$, then both the first and the third terms of $J_c^*(\chi_E)$ are positive and bounded from above.*

We can now state the main result of this section.

Theorem 4.3. *For given $\gamma, \sigma, \alpha > 0$, let $0 < \epsilon_1 < \max\{\frac{1}{\sigma}, \frac{1}{2\alpha}\}$. Assume that $c : (0, \epsilon_1) \rightarrow (0, \infty)$ is a bounded function such that $\lim_{\epsilon \rightarrow 0} c(\epsilon) = c_0 > 0$. Then $J_{c_0}^* = \Gamma\text{-}\lim_{\epsilon \rightarrow 0} J_{c(\epsilon)}$ with respect to the L_e^2 convergence; that is, the following two properties hold:*

(i) *if $\{w_\epsilon\} \subset L_e^2$ and this sequence converges to w_0 in L_e^2 then*

$$J_{c_0}^*(w_0) \leq \liminf_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) ;$$

(ii) *for any $w_0 \in L_e^2$, there exists a sequence $\{w_\epsilon\} \subset L_e^2$, converging to w_0 in L_e^2 and*

$$J_{c_0}^*(w_0) \geq \limsup_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) .$$

The proof of Theorem 4.3 will follow from Lemma 4.4 and Lemma 4.5, as to be derived below. To this end, we start with (4.2) and using integration by parts to obtain

$$c^2(\epsilon) \|(\mathcal{L}_{c(\epsilon)} w)'\|_{L_e^2}^2 + \gamma \|\mathcal{L}_{c(\epsilon)} w\|_{L_e^2}^2 = \int_{-\infty}^{\infty} e^x w \mathcal{L}_{c(\epsilon)} w \, dx .$$

Since $c(\epsilon) \rightarrow c_0$, it follows that $\|\mathcal{L}_{c(\epsilon)} w\|_{H_e^1} \leq C \|w\|_{L_e^2}$ for some positive constant C , independent of ϵ . Observe also that if $w_\epsilon \rightarrow w_0$ in L_e^2 , then it is easily verified that $\mathcal{L}_{c(\epsilon)} w_\epsilon \rightarrow \mathcal{L}_{c_0} w_0$ in H_e^1 and $\int_{-\infty}^{\infty} e^x w_\epsilon \mathcal{L}_{c(\epsilon)} w_\epsilon \, dx \rightarrow \int_{-\infty}^{\infty} e^x w_0 \mathcal{L}_{c_0} w_0 \, dx$ as $\epsilon \rightarrow 0$. In other words the nonlocal term in J_{c_ϵ} is continuous with respect to the L_e^2 convergence. This fact will be used in the proof of Theorem 4.3.

Lemma 4.4. *(liminf inequality) If $w_0 \in L_e^2$ then*

$$J_{c_0}^*(w_0) \leq \liminf_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) \tag{4.8}$$

for any sequence $w_\epsilon \rightarrow w_0$ in L_e^2 .

Proof. It is enough to assume that $\liminf_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) < \infty$ since otherwise there is nothing to prove. Thus $\|w_\epsilon\|_{L_e^2} = 1$ and $-\widetilde{M}_1 - 1 \leq w_\epsilon \leq \beta_2$. Moreover, from Lemmas 4.1 and 2.3 we have that $w_0 = \chi_E \in BV_e$, $\int_{-\infty}^{\infty} e^x \chi_E \, dx = 1$ and that $E = \cup_i [a_i, b_i]$ is the union of countably many disjoint intervals.

As we observed before, the nonlocal term is continuous with respect to the L_e^2 convergence. Note that the same is true for the integral of $e^x G$. In fact, since the sequence w_ϵ is bounded in L^∞ and converges to χ_E in L_e^2 , we have $w_\epsilon \rightarrow \chi_E$ in L_e^3 as well. As a consequence,

$$\alpha \int_{-\infty}^{\infty} e^x G(w_\epsilon) \, dx \rightarrow \alpha \int_{-\infty}^{\infty} e^x G(\chi_E) \, dx = \alpha G(1) \sum_i (e^{b_i} - e^{a_i}) = -\frac{\sqrt{2}\alpha}{12} \sum_i (e^{b_i} - e^{a_i}).$$

Thus it remains to verify the liminf inequality for the gradient term and the integral of $e^x F_0$ only. Arguing as in the proof of (4.4) and (4.6), we get

$$\begin{aligned} \frac{\sqrt{2}}{12} \sum_i (e^{a_i} + e^{b_i}) &= \phi(1) \|D\chi_E\|_e \leq \liminf_{\epsilon \rightarrow 0} \|D\phi(w_\epsilon)\|_e \\ &\leq \liminf_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^x \left(\frac{\epsilon w_\epsilon'^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon} \right) dx . \end{aligned}$$

From this inequality (4.8) follows. \square

To prove the limsup inequality, the first step is to construct auxiliary functions similar to those used in phase transition problems. To this end, setting $f_0(\xi) = -F_0'(\xi) = -\xi(\xi - 1/2)(\xi - 1)$, we consider the equation

$$\epsilon^2 U_\epsilon'' + f_0(U_\epsilon) = 0 .$$

Integrating this equation we get

$$\epsilon^2 U_\epsilon'^2 / 2 - F_0(U_\epsilon) = \text{constant} .$$

Taking the constant on the right hand side equal to $\epsilon/2$ gives

$$U_\epsilon' = \frac{\sqrt{\epsilon + 2F_0(U_\epsilon)}}{\epsilon} . \quad (4.9)$$

Adding the condition $U_\epsilon(0) = 0$, we get a solution $U_\epsilon(x)$, strictly increasing and such that

$$\int_0^{U_\epsilon} \frac{\epsilon}{\sqrt{\epsilon + 2F_0(s)}} ds = x .$$

Note that $U_\epsilon(\rho_\epsilon) = 1$, where

$$\rho_\epsilon = \int_0^1 \frac{\epsilon}{\sqrt{\epsilon + 2F_0(s)}} ds \leq \sqrt{\epsilon} .$$

Setting $\tilde{U}_\epsilon = 1 - U_\epsilon$ we get a strictly decreasing function satisfying

$$\tilde{U}_\epsilon' = -\frac{\sqrt{\epsilon + 2F_0(\tilde{U}_\epsilon)}}{\epsilon}$$

and such that $\tilde{U}_\epsilon(0) = 1$ and $\tilde{U}_\epsilon(\rho_\epsilon) = 0$.

Lemma 4.5. (*limsup inequality*) For every $w_0 \in L_e^2$ there exists a sequence $w_\epsilon \rightarrow w_0$ in L_e^2 such that

$$J_{c_0}^*(w) \geq \limsup_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) . \quad (4.10)$$

Proof. It is clear that is not restrictive to assume that $J_{c_0}^*(w_0) < \infty$. In this case, by (4.7) $w_0 = \chi_E \in BV_e$, where $E = \cup_i [a_i, b_i]$ is the union of countably many intervals with $a_i > b_{i+1}$ for all i .

Let us start by assuming that E is the union of finitely many intervals, say $E = \cup_i^k [a_i, b_i]$ with a_k possibly equal to $-\infty$. In this case we set $x_{2i-1} = b_i$ and $x_{2i} = a_i$ for all $i = 1, \dots, k$. Note that as x increases, $\chi_E(x)$ jumps from 0 to 1 at x_{2i} and from 1 to 0 at x_{2i-1} . Then, we fix $\Delta > 0$ such that $\min_{1 \leq j \leq 2k-1} (x_j - x_{j+1}) \geq \Delta$ and take ϵ so small that $\rho_\epsilon \leq \sqrt{\epsilon} < \Delta/2$. Finally, for all $j = 1, \dots, 2k$ we set $q_j = x_j - \theta_j \rho_\epsilon$, with $\theta_j \in [0, 1]$. We now define

$$w_\epsilon(x) = \begin{cases} U_\epsilon(x - q_{2i}) & \text{if } x \in [q_{2i}, q_{2i} + \rho_\epsilon] \text{ and } i \leq k, \\ \tilde{U}_\epsilon(x - q_{2i-1}) & \text{if } x \in [q_{2i-1}, q_{2i-1} + \rho_\epsilon] \text{ and } i \leq k, \\ \chi_E(x) & \text{otherwise.} \end{cases}$$

Note that if $a_k = -\infty$, then $w_\epsilon(x) = 1$ for $x \leq q_{2k-1}$. At a point x_{2i} , we have that

$$\begin{aligned} \text{if } \theta_{2i} = 0, \text{ then } & \int_{x_{2i}-\rho_\epsilon}^{x_{2i}+\rho_\epsilon} e^x w_\epsilon^2 dx \leq \int_{x_{2i}-\rho_\epsilon}^{x_{2i}+\rho_\epsilon} e^x \chi_E dx , \\ \text{if } \theta_{2i} = 1, \text{ then } & \int_{x_{2i}-\rho_\epsilon}^{x_{2i}+\rho_\epsilon} e^x w_\epsilon^2 dx \geq \int_{x_{2i}-\rho_\epsilon}^{x_{2i}+\rho_\epsilon} e^x \chi_E dx . \end{aligned}$$

Similar inequalities hold at the points x_{2i-1} . Therefore, using the intermediate value theorem, for all j we can always choose $\theta_j \in [0, 1]$ so that $\int_{-\infty}^{\infty} e^x w_\epsilon^2 dx = 1$. Observe that $w_\epsilon \in Y$ and $w_\epsilon(x) = 0$ when $x \geq q_1 + \rho_\epsilon$. Note also that by construction $w_\epsilon \rightarrow \chi_E$ in L_e^2

as $\epsilon \rightarrow 0$. We now evaluate

$$\begin{aligned}
\int_{-\infty}^{\infty} e^x \left(\frac{\epsilon w_\epsilon'^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon} \right) dx &\leq \sum_{j=1}^{2k} \int_{q_j}^{q_j + \rho_\epsilon} e^x \left(\frac{\epsilon w_\epsilon'^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon} \right) dx \\
&= \sum_{i=1}^k e^{q_{2i}} \int_0^{\rho_\epsilon} e^x \left(\frac{\epsilon U_\epsilon'^2}{2} + \frac{F_0(U_\epsilon)}{\epsilon} \right) dx + \sum_{i=1}^k e^{q_{2i-1}} \int_0^{\rho_\epsilon} e^x \left(\frac{\epsilon \tilde{U}_\epsilon'^2}{2} + \frac{F_0(\tilde{U}_\epsilon)}{\epsilon} \right) dx \\
&\leq \sum_{i=1}^k e^{q_{2i}} e^{\rho_\epsilon} \int_0^{\rho_\epsilon} \left(\frac{\epsilon U_\epsilon'^2}{2} + \frac{F_0(U_\epsilon)}{\epsilon} \right) dx + \sum_{i=1}^k e^{q_{2i-1}} e^{\rho_\epsilon} \int_0^{\rho_\epsilon} \left(\frac{\epsilon \tilde{U}_\epsilon'^2}{2} + \frac{F_0(\tilde{U}_\epsilon)}{\epsilon} \right) dx \\
&= \sum_{j=1}^{2k} e^{q_j} e^{\rho_\epsilon} \int_0^{\rho_\epsilon} \left(\frac{\epsilon U_\epsilon'^2}{2} + \frac{F_0(U_\epsilon)}{\epsilon} \right) dx \quad (\text{using symmetry}) \\
&\leq \sum_{j=1}^{2k} e^{q_j} e^{\rho_\epsilon} \int_0^{\rho_\epsilon} \frac{\epsilon + 2F_0(U_\epsilon)}{\epsilon} dx \quad (\text{using (4.9)}) \\
&= \sum_{j=1}^{2k} e^{q_j} e^{\rho_\epsilon} \int_0^{\rho_\epsilon} \sqrt{\epsilon + 2F_0(U_\epsilon)} U_\epsilon' dx = \sum_{j=1}^{2k} e^{q_j} e^{\rho_\epsilon} \int_0^1 \sqrt{\epsilon + 2F_0(s)} ds.
\end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$ yields

$$\limsup_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^x \left(\frac{w_\epsilon'^2}{2} + \frac{F_0(w_\epsilon)}{\epsilon} \right) dx \leq \phi(1) \sum_{j=1}^{2k} e^{x_j} = \frac{\sqrt{2}}{12} \sum_{i=1}^k (e^{a_i} + e^{b_i}).$$

From this inequality, recalling that both the integral term containing G and the nonlocal term are continuous with respect to the L_e^2 convergence, (4.10) immediately follows.

If E is the union of infinitely many intervals, since $\chi_E \in BV_e$, then $\lim_{i \rightarrow \infty} a_i = -\infty$. In this case, for any integer $h \geq 1$ we set $E_h = \cup_{i=1}^h [a_i, b_i] \cup (-\infty, l_h]$, where $l_h < a_h$ is chosen so that $\int_{-\infty}^{\infty} e^x \chi_{E_h} dx = 1$. Observe that $\lim_{h \rightarrow \infty} l_h = -\infty$. Therefore $\chi_{E_h} \rightarrow \chi_E$ in L_e^2 and $J_{c_0}^*(\chi_{E_h}) \rightarrow J_{c_0}^*(\chi_E)$. By applying the limsup inequality to each E_h for any h we may find $\epsilon_h < 1/h$ and a function $w_{\epsilon_h} \in H_e^1$ such that $\|w_{\epsilon_h} - \chi_{E_h}\|_{L_e^2} < 1/h$ and

$$J_{c(\epsilon_h)}(w_{\epsilon_h}) \leq J_{c_0}^*(\chi_{E_h}) + \frac{1}{h}.$$

Then the conclusion follows, since $w_{\epsilon_h} \rightarrow \chi_E$ in L_e^2 and $J_{c_0}^*(\chi_{E_h}) \rightarrow J_{c_0}^*(\chi_E)$. \square

To conclude this section, let us observe that $\liminf_{\epsilon \rightarrow 0} (\inf_{w \in Y} J_{c(\epsilon)}(w)) < \infty$ when the assumptions of Theorem 4.3 hold. Indeed setting

$$w_\epsilon(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 1 - \frac{x}{\epsilon} & \text{if } 0 \leq x \leq \epsilon, \\ 0 & \text{if } x \geq \epsilon, \end{cases}$$

we obtain $w_\epsilon(\cdot - \theta_\epsilon) \in Y$ for some $\theta_\epsilon \rightarrow 0$. Then a direct calculation gives yields $\liminf_{\epsilon \rightarrow 0} J_{c(\epsilon)}(w_\epsilon) < \infty$.

Since there exists a $C_2 > 0$, not depending on ϵ , such that $G(\xi) \geq -C_2\xi^2$ for $\xi \in [-\widetilde{M}_1 - 1, \beta_2]$, it follows that

$$J_{c(\epsilon)}(w) \geq \alpha \int_{-\infty}^{\infty} e^x G(w) dx \geq -\alpha C_2 \int_{-\infty}^{\infty} e^x w^2 dx = -\alpha C_2$$

for all $w \in Y$. Hence

$$\infty > \liminf_{\epsilon \rightarrow 0} \left(\inf_{w \in Y} J_{c(\epsilon)}(w) \right) \geq -\alpha C_2$$

and consequently there exists a sequence $\{\zeta_\epsilon\} \subset Y$ such that

$$\lim_{\epsilon \rightarrow 0} J_{c(\epsilon)}(\zeta_\epsilon) = \liminf_{\epsilon \rightarrow 0} \left(\inf_{w \in Y} J_{c(\epsilon)}(w) \right). \quad (4.11)$$

Such a minimizing sequence will be used in the next section to establish the existence of minimizers for $J_{c_0}^*$. Note that neither $c(\epsilon)$ nor c_0 is constrained to be a wave speed.

5 Minimizer for J_c^*

Throughout this section, in order to simply the notation, we assume that $c(\epsilon) \rightarrow c > 0$ as $\epsilon \rightarrow 0$. Moreover, if $\chi_E \in BV_e$ we denote by $\widetilde{J}_c^*(\chi_E)$ the right hand side of (4.7), which is a finite quantity, regardless that $\|\chi_E\|_{L_e^1}$ is equal to 1 or not.

Lemma 5.1. *Let $c > 0$, then J_c^* has a minimizer χ_{E_c} . Moreover*

- (i) $\int_{-\infty}^{\infty} e^x \chi_{E_c} dx = 1$;
- (ii) $E_c = [a, b]$ for some $-\infty \leq a < b < \infty$.

Proof. Let $c(\epsilon) \rightarrow c$. Since $\liminf_{\epsilon \rightarrow 0} (\inf_{w \in Y} J_{c(\epsilon)}(w)) \leq C_0$ for some $C_0 > 0$, from Lemma 4.1 it follows that there exists a set E_c such that, up to a not relabelled subsequence, the sequence $\{\zeta_\epsilon\}$ in (4.11) converges to χ_{E_c} in L_e^2 and that $\|\chi_{E_c}\|_{L_e^1} = 1$, $\|D\chi_{E_c}\|_e \leq 6\sqrt{2}(C_0 + C_1\alpha)$. This shows that $\chi_{E_c} \in BV_e$ and that (i) holds. Moreover, a fundamental theorem of Γ -convergence, see [5, Th. 2.1], yields that χ_{E_c} is a minimizer of J_c^* and that

$$J_c^*(\chi_{E_c}) = \lim_{\epsilon \rightarrow 0} J_{c(\epsilon)}(\zeta_\epsilon).$$

We argue by contradiction assuming that E_c is the union of two or more disjoint intervals $[a_i, b_i]$. Then, we set $E^1 = [a_1, b_1]$ and $E^2 = E_c \setminus E^1$. For $i = 1, 2$, let $\|\chi_{E^i}\|_{L_e^1} = e^{-h_i}$ for some $h_i > 0$. Thus

$$1 = \|\chi_{E_c}\|_{L_e^1} = e^{-h_1} + e^{-h_2} \quad (5.1)$$

and $\|\chi_{E^i}(\cdot - h_i)\|_{L^1_e} = 1$. Since χ_{E_c} is a minimizer of J_c^* , it is clear that $J_c^*(\chi_{E_c}) \leq J_c^*(\chi_{E^i}(\cdot - h_i)) = e^{h_i} \tilde{J}_c^*(\chi_{E^i})$. A simple calculation gives

$$J_c^*(\chi_{E_c}) = \tilde{J}_c^*(\chi_{E^1}) + \tilde{J}_c^*(\chi_{E^2}) + \sigma \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} dx .$$

Then

$$J_c^*(\chi_{E_c}) \geq e^{-h_1} J_c^*(\chi_{E_c}) + e^{-h_2} J_c^*(\chi_{E_c}) + \sigma \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} dx .$$

This inequality, together with (5.1), implies that

$$0 \geq \sigma_0 \int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} dx .$$

But this is absurd, since $\mathcal{L}_c \chi_{E^2}(x) > 0$ for all x , hence $\int_{-\infty}^{\infty} e^x \chi_{E^1} \mathcal{L}_c \chi_{E^2} dx > 0$. As a conclusion, $E_c = [a, b]$ or $(-\infty, 0]$. \square

Now that we know that E_c can be either a finite or a semi-infinite interval, we want to understand when each of these two cases may occur. To this end we need to perform a preliminary analysis.

Define $\ell = b - a$. Since $\|\chi_{[a,b]}\|_{L^1_e} = e^b - e^a = 1$, from (4.7) we get

$$\begin{aligned} e^{-b} J_c^*(\chi_{[a,b]}) &= J_c^*(\chi_{[-\ell,0]}) \\ &= \frac{\sqrt{2}}{12} (e^{-\ell} + 1) - \frac{\sqrt{2}\alpha}{12} (1 - e^{-\ell}) + \frac{\sigma}{2} \int_{-\ell}^0 e^x \mathcal{L}_c \chi_{[-\ell,0]} dx . \end{aligned} \quad (5.2)$$

Let us now calculate the nonlocal term. Let us denote by $r_1 < r_2$ the solutions of the characteristic equation $c^2 r^2 + c^2 r - \gamma = 0$,

$$r = \frac{1}{2c} (-c \pm \sqrt{c^2 + 4\gamma}) . \quad (5.3)$$

Note that $r_1 < -1 < 0 < r_2$ and that $r_1 + r_2 = -1$. Let us assume $\ell < \infty$. The general solution of $c^2 v'' + c^2 v' - \gamma v = 0$ is an element of $\text{span}\{e^{r_1 x}, e^{r_2 x}\}$. Solving (4.2), we obtain

$$\mathcal{L}_c \chi_{[-\ell,0]} = \begin{cases} A_4 e^{r_2 x} & \text{for } x \leq -\ell, \\ \frac{1}{\gamma} + A_3 e^{r_1 x} + A_2 e^{r_2 x} & \text{for } -\ell \leq x \leq 0, \\ A_1 e^{r_1 x} & \text{for } x \geq 0, \end{cases}$$

where

$$A_1 = \frac{r_2(1 - e^{r_1 \ell})}{\gamma(r_2 - r_1)}, \quad A_2 = \frac{r_1}{\gamma(r_2 - r_1)}, \quad A_3 = \frac{r_2 e^{r_1 \ell}}{\gamma(r_1 - r_2)}, \quad A_4 = \frac{r_1(1 - e^{r_2 \ell})}{\gamma(r_2 - r_1)} .$$

Evaluating the nonlocal term in (5.2) and recalling that $r_1 + r_2 = -1$, we get

$$\begin{aligned} \int_{-\ell}^0 e^x \mathcal{L}_c \chi_{[-\ell, 0]} dx &= \int_{-\ell}^0 e^x \left(\frac{1}{\gamma} + A_3 e^{r_1 x} + A_2 e^{r_2 x} \right) dx \\ &= \frac{2}{\gamma(r_2 - r_1)} (r_2 + r_1 e^{-\ell} + e^{r_1 \ell}). \end{aligned}$$

Substituting into (5.2) gives

$$J_c^*(\chi_{[a, b]}) = e^b \mathcal{J}(\ell, c),$$

where we have defined

$$\mathcal{J}(\ell, c) \equiv \frac{\sqrt{2}}{12}(1 - \alpha) + \frac{\sqrt{2}}{12}(1 + \alpha)e^{-\ell} + \frac{\sigma}{\gamma(r_2 - r_1)}(r_2 + r_1 e^{-\ell} + e^{r_1 \ell}). \quad (5.4)$$

Recall that $\int_{-\ell}^0 e^x \mathcal{L}_c \chi_{[-\ell, 0]} dx > 0$, thus the last term on the right hand side of (5.4) is also positive.

Since $E_c = [a, b]$ and $1 = \|\chi_{[a, b]}\|_{L_c^1} = e^b(1 - e^{-\ell})$, we have $b = -\log(1 - e^{-\ell})$ and

$$J_c^*(\chi_{E_c}) = \frac{1}{1 - e^{-\ell}} \mathcal{J}(\ell, c). \quad (5.5)$$

To study the dependence of the minimizer E_c on c we introduce the auxiliary function $H(c) = \frac{c}{\sqrt{c^2 + 4\gamma}}$. Clearly, H is strictly increasing, $H(0) = 0$ and $H(c) \rightarrow 1$ as $c \rightarrow \infty$. Note also the following useful identities

$$\frac{r_2}{r_2 - r_1} = \frac{1}{2}(1 - H(c)), \quad \frac{r_1}{r_2 - r_1} = -\frac{1}{2}(1 + H(c)), \quad (5.6)$$

$$\frac{dr_1}{dc} > 0, \quad \frac{dr_2}{dc} < 0,$$

$$r_2 - r_1 = \frac{1}{H(c)}, \quad \frac{1}{(r_2 - r_1)^2} \frac{d(r_2 - r_1)}{dc} = -H'(c). \quad (5.7)$$

Given $c > 0$, a simple argument shows that there exists always $\ell_c \in (0, \infty]$ such that

$$\frac{1}{1 - e^{-\ell_c}} \mathcal{J}(\ell_c, c) \leq \frac{1}{1 - e^{-\ell}} \mathcal{J}(\ell, c) \quad \text{for all } \ell > 0. \quad (5.8)$$

Note that when c is the wave speed, the case $\ell_c = \infty$ corresponds to a traveling front for the limit problem, while a finite ℓ_c corresponds to a traveling pulse.

6 Traveling front

In this section we examine the case $\ell_c = \infty$. To this end we define

$$\mathcal{F}(c) \equiv \lim_{\ell \rightarrow \infty} \mathcal{J}(\ell, c) = \frac{\sqrt{2}}{12}(1 - \alpha) + \frac{\sigma}{2\gamma}(1 - H(c)). \quad (6.1)$$

It is clear that

$$\mathcal{F}'(c) = -\frac{\sigma}{2\gamma}H'(c) < 0.$$

Lemma 6.1. *Let $c > 0$ and $\ell_c \in (0, \infty]$ satisfying (5.8). Then $\ell_c = \infty$ if and only if*

$$\frac{\sqrt{2}\gamma}{6\sigma} \geq H(c). \quad (6.2)$$

Moreover, if (6.2) holds, $\chi_{(-\infty, 0]}$ is the unique minimizer of J_c^* .

Proof. A direct calculation gives

$$\frac{1}{1 - e^{-\ell}}\mathcal{J}(\ell, c) - \mathcal{F}(c) = \frac{1}{1 - e^{-\ell}} \left\{ \frac{\sqrt{2}}{6}e^{-\ell} - \frac{\sigma H(c)}{\gamma}e^{-\ell} + \frac{\sigma}{\gamma(r_2 - r_1)}e^{r_1\ell} \right\}. \quad (6.3)$$

If $\ell_c = \infty$ then $\frac{1}{1 - e^{-\ell}}\mathcal{J}(\ell, c) - \mathcal{F}(c) \geq 0$ for all $\ell > 0$. In particular for large ℓ , it follows from (6.3) and $r_1 < -1$ that (6.2) holds.

Conversely, if (6.2) holds, it is clear from (6.3) that $\frac{1}{1 - e^{-\ell}}\mathcal{J}(\ell, c) - \mathcal{F}(c) > 0$ for all ℓ . Since the last inequality is strict, $\chi_{(-\infty, 0]}$ is the unique minimizer of J_c^* . \square

Having characterized the case $\ell_c = \infty$, we now study under which additional constraints on the physical parameters one has also $\mathcal{F}(c) = 0$. To this end we introduce the following condition:

$$(A1)^* \quad \alpha \geq \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0.$$

Lemma 6.2. *Given $\gamma, \sigma, \alpha > 0$, (A1)* is equivalent to the following two conditions:*

(i) *there is a unique $c_f > 0$ such that $\mathcal{F}(c_f) = 0$.*

(ii) *$\chi_{(-\infty, 0]}$ is the unique global minimizer of $J_{c_f}^*$ and $J_{c_f}^*(\chi_{(-\infty, 0]}) = 0$.*

Moreover, c_f is given by (1.10).

Proof. First we prove the sufficiency of (A1)*. If $\alpha > 1$ and $\frac{3\sqrt{2}\sigma}{\gamma} > (\alpha - 1)$, then one has $\mathcal{F}(0) = \frac{\sqrt{2}}{12}(1 - \alpha) + \frac{\sigma}{2\gamma} > 0$ and $\mathcal{F}(\infty) = \frac{\sqrt{2}}{12}(1 - \alpha) < 0$. Since \mathcal{F} is strictly decreasing in c , there exists a unique c_f such that $\mathcal{F}(c_f) = 0$. Moreover, from (6.1) we obtain

$$H(c_f) = 1 - \frac{(\alpha - 1)\gamma}{3\sqrt{2}\sigma}.$$

Thus, recalling that $h_* = 1 - \frac{(\alpha-1)\gamma}{3\sqrt{2}\sigma}$, we have (1.10). To prove (ii) it is enough to show that $\ell_c = \infty$. By Lemma 6.1 this is equivalent to show that (6.2) holds with $c = c_f$. In turn, this follows from the inequality $\alpha \geq \frac{3\sqrt{2}\sigma}{\gamma}$.

Conversely, note that by Lemma 6.1 condition (ii) implies (6.2). Moreover, if (i) holds, then c_f is given by (1.10). Since $\mathcal{F}(c_f) = 0$ and \mathcal{F} is strictly decreasing, it follows that $\mathcal{F}(0) > 0$ and $\mathcal{F}(\infty) < 0$. These two inequalities, together with (6.2), immediately yield (A1)*. \square

Remark 6.3. (a) From (1.10) it follows that c_f is an increasing function of h_* . Thus c_f is a decreasing function of α and an increasing function of σ . In fact c_f can approach 0 or ∞ depending on the values of the parameters γ, σ and α .

(b) Note that (1.10) gives a formula to calculate the speed for the traveling front solutions of the Γ -limit.

In Theorem 1.1, the stronger condition (A1) is imposed; that is, we require that (A1)* is satisfied with strict inequalities. To give a physical interpretation of the assumption (A1), we take a look at the nullclines. Denote by $\gamma_* = \gamma_*(\epsilon)$ a positive constant such that the two regions enclosed by the line $v = u/\gamma_*$ and the curve $v = f_\epsilon(u)/(\epsilon\sigma)$ have the same area with opposite signs.

Lemma 6.4. *If ϵ is sufficiently small then $\gamma_* = \frac{3\sqrt{2}\sigma}{\alpha}(1 + O(\epsilon))$.*

Proof. The nullclines $v = u/\gamma_*$ and $v = f_\epsilon(u)/(\epsilon\sigma)$ intersect at $u = 0, \mu_2^*, \mu_3^*$, where μ_2^*, μ_3^* are the roots of the quadratic equation $u^2 - (1 + \beta_\epsilon)u + \beta_\epsilon + \frac{\epsilon\sigma}{\gamma_*} = 0$. As both nullclines are anti-symmetric about $(u, v) = (\mu_2^*, \mu_2^*/\gamma_*)$, it can be easily checked that $\mu_2^* = \frac{1+\beta_\epsilon}{3}$ and $\mu_3^* = \frac{2(1+\beta_\epsilon)}{3}$. Using the fact that $\mu_2^*\mu_3^* = \beta_\epsilon + \frac{\epsilon\sigma}{\gamma_*}$, we obtain

$$\frac{\epsilon\sigma}{\gamma_*} = \frac{1}{9}(1 - 2\beta_\epsilon)(2 - \beta_\epsilon). \quad (6.4)$$

The lemma follows by substituting $\beta_\epsilon = \frac{1}{2} - \frac{\epsilon\alpha}{\sqrt{2}}$ into (6.4). \square

For sufficiently small ϵ , Lemma 6.4 shows that (A1) is equivalent to

$$\alpha > 1 \quad \text{and} \quad \frac{\alpha}{\alpha - 1}\gamma_* > \gamma > \gamma_*,$$

which can be viewed as the constraints imposed on γ to generate a wave front. We next prove the existence of traveling wave solutions of (4.1)-(4.2) when ϵ is small, as stated in Theorem 1.1.

Proof of Theorem 1.1. Since c_f satisfies (6.2) with a strict inequality, there exists $\eta > 0$ such that if $c \in [c_f - \eta, c_f + \eta]$ then (6.2) still holds and thus $\inf_{L_c^2} J_c^* = \mathcal{F}(c)$. Furthermore,

since \mathcal{F} is a strictly decreasing function of c , then $\mathcal{F}(c_f - \eta) > 0 = \mathcal{F}(c_f) > \mathcal{F}(c_f + \eta)$. For $c > 0$, $\epsilon \in (0, \epsilon_1)$ and $w \in Y$, we define

$$I_{\epsilon,c}(w) = \int_{-\infty}^{\infty} e^x \left\{ \epsilon \frac{w'^2}{2} + \frac{F_0(w)}{\epsilon} + \alpha G(w) + \frac{\sigma}{2} w \mathcal{L}_c w \right\} dx .$$

Applying Theorem 4.3 with $c(\epsilon) = c$ (so that $J_{c(\epsilon)} = I_{\epsilon,c}$), we get

$$\Gamma\text{-}\lim_{\epsilon \rightarrow 0} I_{\epsilon,c} = J_c^* .$$

Set $c^\pm = c_f \pm \eta$. By the fundamental theorem of Γ -convergence [5, Theorem 2.1], there exists $\epsilon_0 > 0$ such that $\inf_Y I_{\epsilon,c^-} > 0$ and $\inf_Y I_{\epsilon,c^+} < 0$ for all $0 < \epsilon < \epsilon_0$. Observe also that a slight modification of the argument of Lemma 3.5 of [7] shows that $\inf_Y I_{\epsilon,c}$ is a continuous function of c . Thus, by the intermediate value theorem, there exists $c_\epsilon \in (c_f - \eta, c_f + \eta)$ such that $\inf_Y I_{\epsilon,c_\epsilon} = 0$. Furthermore, by Lemmas 3.2 and 3.3, I_{ϵ,c_ϵ} has a minimizer $u_\epsilon \in Y$ and, setting $v_\epsilon = \mathcal{L}_{c_\epsilon} u_\epsilon$, (u_ϵ, v_ϵ) is a traveling wave solution of (4.1)-(4.2) with speed c_ϵ . Since $c_\epsilon \in (c_f - \eta, c_f + \eta)$, from Lemma 4.1 we get that there exists a subsequence such that $c_\epsilon \rightarrow \hat{c}_f$ and $u_\epsilon \rightarrow \chi_E$ in L_e^2 for some $\hat{c}_f \in \mathbb{R}$ and $\chi_E \in BV_e$. Moreover, by Theorem 4.3 we may also assume that I_{ϵ,c_ϵ} Γ -converges in L_e^2 to $J_{\hat{c}_f}^*$ and thus that χ_E is a minimizer of $J_{\hat{c}_f}^*$.

By Lemma 6.2, $\hat{c}_f = c_f$ and $\chi_E = \chi_{(-\infty, 0]}$. Therefore, since these two limits do not depend on the particular subsequence, we may conclude that $c_\epsilon \rightarrow c_f$ and that $u_\epsilon \rightarrow \chi_{(-\infty, 0]}$ in L_e^2 as $\epsilon \rightarrow 0$. \square

7 Traveling pulse

In this section our attention turns to the existence of a traveling pulse for the limit problem. Let $c > 0$ and suppose that there exists $\ell_c \in (0, \infty)$ such that

$$\frac{1}{1 - e^{-\ell_c}} \mathcal{J}(\ell_c, c) \leq \frac{1}{1 - e^{-\ell}} \mathcal{J}(\ell, c) \quad \text{for all } \ell > 0 .$$

It follows that

$$\frac{\partial}{\partial \ell} \left(\frac{\mathcal{J}(\ell, c)}{1 - e^{-\ell}} \right) \Big|_{\ell=\ell_c} = 0 . \tag{7.1}$$

To prove the existence of traveling pulses beside (7.1) we have to look for additional conditions on the parameters insuring that there exists $c > 0$ satisfying

$$\mathcal{J}(\ell_c, c) = 0 . \tag{7.2}$$

If this happens, then $\chi_{[a,b]}$ is a minimizer of J_c^* , where $b = -\log(1 - e^{-\ell_c})$ and $a = b - \ell_c$. A lower bound on ℓ_c is given in the next lemma.

Lemma 7.1. *If $\chi_{[a,b]}$ is a minimizer of J_c^* satisfying $J_c^*(\chi_{[a,b]}) = 0$, then $\alpha > 1$ and $\ell_c = b - a > \log \frac{\alpha+1}{\alpha-1}$.*

Proof. The result is a straightforward consequence of the equality $\mathcal{J}(\ell_c, c) = 0$ and of the fact that the last term in (5.4) is positive. \square

Lemma 7.2. *Let $c > 0$ and assume that J_c^* has a minimizer $\chi_{[a,b]}$ with $\|\chi_{[a,b]}\|_{L_e^1} = 1$ and $J_c^*(\chi_{[a,b]}) = 0$. If $\ell_c = b - a$, then (ℓ_c, c) satisfies*

$$\frac{\sigma}{2\gamma}(1 + H(c))(1 - e^{-r_2\ell_c}) = \frac{\sqrt{2}}{12}(\alpha + 1) \quad (7.3)$$

and

$$\frac{\sigma}{2\gamma}(1 - H(c))(1 - e^{r_1\ell_c}) = \frac{\sqrt{2}}{12}(\alpha - 1). \quad (7.4)$$

Proof. Using (5.6), we get by a direct calculation

$$\frac{\partial}{\partial \ell} \mathcal{J}(\ell, c) = -\frac{\sqrt{2}}{12}(1 + \alpha)e^{-\ell} - (1 + H(c))\frac{\sigma}{2\gamma}(-e^{-\ell} + e^{r_1\ell}). \quad (7.5)$$

Since both (7.1) and (7.2) must hold, we obtain

$$\frac{\partial \mathcal{J}}{\partial \ell}(\ell_c, c) = 0.$$

This gives (7.3). Using this equation together with (7.2) yields

$$\frac{\sqrt{2}}{12}(1 - \alpha) + \frac{\sigma}{\gamma(r_2 - r_1)}(r_2 + e^{r_1\ell}) + \frac{\sigma r_1}{\gamma(r_2 - r_1)}e^{r_1\ell} = 0,$$

from which, recalling (5.7) and (5.6) again, (7.4) follows. \square

Lemma 7.3. *Let $\sigma, \gamma > 0$ be given. If J_c^* has a minimizer $\chi_{[a,b]}$ with $\|\chi_{[a,b]}\|_{L_e^1} = 1$ and $J_c^*(\chi_{[a,b]}) = 0$, then*

$$\frac{3\sqrt{2}\sigma}{\gamma} > \alpha > 1. \quad (7.6)$$

Proof. By Lemma 7.2, (ℓ_c, c) satisfies (7.3) and (7.4). Clearly the left hand side of (7.4) is positive, hence $\alpha > 1$, a condition already given by Lemma 7.1. Define

$$Q(\ell, c) \equiv \frac{\sqrt{2}\gamma}{12\sigma} \left\{ \frac{1 + \alpha}{1 - e^{-r_2\ell}} + \frac{\alpha - 1}{1 - e^{r_1\ell}} \right\} - 1.$$

Note that, eliminating $H(c)$, from (7.3) and (7.4) we have $Q(\ell_c, c) = 0$.

By a direct calculation

$$\frac{\partial Q}{\partial \ell} = \frac{\sqrt{2}\gamma}{12\sigma} \left\{ -\frac{(1+\alpha)r_2}{(1-e^{-r_2\ell})^2} e^{-r_2\ell} + \frac{(\alpha-1)r_1}{(1-e^{r_1\ell})^2} e^{r_1\ell} \right\} < 0. \quad (7.7)$$

Since $\frac{\partial Q}{\partial \ell} < 0$ for all ℓ and $Q(\ell, c) = 0$, it follows that $0 > \lim_{\ell \rightarrow \infty} Q(\ell, c) = \frac{\sqrt{2}\alpha\gamma}{6\sigma} - 1$, which completes the proof. \square

Remark 7.4. If $\ell > 0$ is a solution of $\mathcal{J}(\ell, c) = \frac{\partial \mathcal{J}}{\partial \ell}(\ell, c) = 0$, then it satisfies (7.3)-(7.4) and is also a solution of $\frac{\partial \mathcal{J}}{\partial \ell}(\ell, c) = Q(\ell, c) = 0$. In turn, since Q is a linear combination of \mathcal{J} and $\frac{\partial \mathcal{J}}{\partial \ell}$, we expect that if $Q(\ell, c) = \mathcal{J}(\ell, c) = 0$, then also $\frac{\partial \mathcal{J}}{\partial \ell}(\ell, c) = 0$ holds.

Lemma 7.5. If $\mathcal{J}(\ell, c) = 0$ and $\frac{\partial \mathcal{J}}{\partial \ell}(\ell, c) = 0$, then $\frac{\partial^2 \mathcal{J}}{\partial \ell^2}(\ell, c) > 0$.

Proof. A direct calculation gives

$$\frac{\partial^2}{\partial \ell^2} \mathcal{J}(\ell, c) = \frac{\sqrt{2}}{12} e^{-\ell} \left[(1+\alpha) - (1+H(c)) \frac{3\sqrt{2}\sigma}{\gamma} (1+r_1 e^{-r_2\ell}) \right].$$

Invoking (7.3) yields

$$\frac{\partial^2 \mathcal{J}}{\partial \ell^2}(\ell, c) = -\frac{\sqrt{2}}{12} e^{-\ell} \left[\frac{3\sqrt{2}\sigma}{\gamma} (1+H(c))(1+r_1) e^{-r_2\ell} \right] > 0.$$

\square

Lemma 7.3 shows that (7.6) is a necessary condition for the existence of the required minimizer. We are now going to show that it is also a sufficient condition. Given σ and γ , if $\alpha \in (1, \frac{3\sqrt{2}\sigma}{\gamma})$, it follows with straightforward calculations that

$$\frac{\partial Q}{\partial c} = \frac{\sqrt{2}\gamma}{12\sigma} \left\{ -\frac{(1+\alpha)\ell}{(1-e^{-r_2\ell})^2} e^{-r_2\ell} \frac{dr_2}{dc} + \frac{(\alpha-1)\ell}{(1-e^{r_1\ell})^2} e^{r_1\ell} \frac{dr_1}{dc} \right\} > 0.$$

For $c > 0$ (7.7) gives $\frac{\partial Q}{\partial \ell} < 0$. Moreover, $\lim_{\ell \rightarrow 0^+} Q(\ell, c) = \infty$ and $\lim_{\ell \rightarrow \infty} Q(\ell, c) = \frac{\sqrt{2}\alpha\gamma}{6\sigma} - 1 < 0$, thanks to (7.6). Hence there exists a unique $L = L(c) > 0$ satisfying $Q(L(c), c) = 0$. Note that

$$L'(c) = -\frac{\partial Q/\partial c}{\partial Q/\partial \ell} > 0. \quad (7.8)$$

Next we derive the asymptotic properties of $L(c)$. As $c \rightarrow 0^+$, it can be easily checked that $r_1 \sim -\frac{\sqrt{\gamma}}{c}(1+O(c))$ and $r_2 \sim \frac{\sqrt{\gamma}}{c}(1+O(c))$. Plugging these relations into the equation $Q(L(c), c) = 0$ we have

$$\frac{\sqrt{2}\gamma}{12\sigma} \left(\frac{1+\alpha}{1-e^{-\sqrt{\gamma}L/c}} + \frac{\alpha-1}{1-e^{-\sqrt{\gamma}L/c}} \right) = 1 + o(1),$$

which can be simplified as

$$1 - e^{-\sqrt{\gamma}L/c} = \frac{\sqrt{2}\gamma\alpha}{6\sigma}(1 + o(1)).$$

This shows $\frac{L}{c} \rightarrow k_1 \equiv -\frac{1}{\sqrt{\gamma}} \log(1 - \frac{\sqrt{2}\gamma\alpha}{6\sigma})$ as $c \rightarrow 0^+$. Then from (5.4),

$$\lim_{c \rightarrow 0^+} \mathcal{J}(L(c), c) = \frac{\sqrt{2}}{6}.$$

Next, if $c \rightarrow \infty$ then $r_2 \sim \frac{\gamma}{c^2}$ and $r_1 \sim -1$. As $L(c)$ is an increasing function, $\lim_{c \rightarrow \infty} L(c)$ exists. However it cannot be a finite number, since the equality $Q(L(c), c) = 0$ yields

$$\frac{1 + \alpha}{1 - \lim_{c \rightarrow \infty} e^{-r_2 L}} + \frac{\alpha - 1}{1 - \lim_{c \rightarrow \infty} e^{r_1 L}} = \frac{6\sqrt{2}\sigma}{\gamma}. \quad (7.9)$$

Note that this last equation implies that we cannot have $\lim_{c \rightarrow \infty} r_2 L = 0$. Furthermore, if $\limsup_{c \rightarrow \infty} r_2 L = \infty$ we would get $\frac{3\sqrt{2}\sigma}{\gamma} = \alpha$, which violates (7.6). Thus the only possibility left is that $\limsup_{c \rightarrow \infty} r_2 L \equiv R_L$. Using (7.9), R_L can be solved from

$$\frac{1 + \alpha}{1 - R_L} + \alpha - 1 = \frac{6\sqrt{2}\sigma}{\gamma}.$$

Consequently there exists a sequence $c_k \rightarrow \infty$ such that along this sequence

$$L \sim \frac{R_L c_k^2}{\gamma}$$

and then

$$\lim_{k \rightarrow \infty} \mathcal{J}(L(c_k), c_k) = \frac{\sqrt{2}}{12}(1 - \alpha) < 0.$$

By the intermediate value theorem there exists $c > 0$ such that $\mathcal{J}(L(c), c) = 0$. As observed in Remark 7.4, if (ℓ, c) satisfies $\mathcal{J}(\ell, c) = Q(\ell, c) = 0$, then $\frac{\partial \mathcal{J}}{\partial \ell}(\ell, c) = 0$ should be true as well. Indeed,

$$\frac{\partial \mathcal{J}}{\partial \ell}(L(c), c) = -\frac{\sqrt{2}}{12}(1 + \alpha)e^{-L} + \frac{\sigma r_1}{\gamma(r_2 - r_1)}(-e^{-L} + e^{r_1 L}) \quad (7.10)$$

$$\begin{aligned} &= -\frac{\sqrt{2}}{12}(1 + \alpha)e^{-L} + \frac{\sigma r_1}{\gamma(r_2 - r_1)}(-e^{-L} + e^{r_1 L}) + \mathcal{J}(L(c), c) \\ &= \frac{\sqrt{2}}{12}(1 - \alpha) + \frac{\sigma}{\gamma(r_2 - r_1)}(r_2 + e^{r_1 L}) + \frac{\sigma r_1}{\gamma(r_2 - r_1)}e^{r_1 L} \\ &= \frac{\sigma}{2\gamma}(1 - H(c))(1 - e^{r_1 L}) - \frac{\sqrt{2}}{12}(\alpha - 1). \end{aligned} \quad (7.11)$$

From (7.10) we have also

$$e^L \frac{\partial \mathcal{J}}{\partial \ell}(L(c), c) = \frac{\sigma}{2\gamma}(1 + H(c))(1 - e^{-r_2 L}) - \frac{\sqrt{2}}{12}(\alpha + 1). \quad (7.12)$$

Eliminating $H(c)$ from (7.11)-(7.12) yields

$$\frac{\partial \mathcal{J}}{\partial \ell}(L(c), c) = -\frac{\sigma}{\gamma}[(1 - e^{r_1 L})^{-1} + e^L(1 - e^{-r_2 L})^{-1}]^{-1} Q(L(c), c) = 0.$$

By (5.5), a direct calculation gives

$$\frac{\partial^2 J_c^*}{\partial \ell^2} \Big|_{(L(c), c)} = \frac{1}{1 - e^{-\ell}} \frac{\partial^2 \mathcal{J}}{\partial \ell^2} \Big|_{(L(c), c)}.$$

Let us now denote by c_p the value of c such that $\mathcal{J}(L(c_p), c_p) = \frac{\partial \mathcal{J}}{\partial c}(L(c_p), c_p) = 0$. From Lemma 7.5 $L(c_p)$ is a local minimizer of $\mathcal{J}(\cdot, c_p)$. Moreover from (7.3) and (7.5) we have

$$\frac{\partial \mathcal{J}}{\partial \ell} = e^{-\ell} \left(-\frac{\sqrt{2}}{12}(1 + \alpha) + (1 + H(c_p)) \frac{\sigma}{2\gamma}(1 - e^{-r_2 \ell}) \right) \begin{cases} > 0, & \text{if } \ell > L(c_p), \\ < 0, & \text{if } \ell < L(c_p), \end{cases}$$

from which we deduce that $L(c_p)$ is a global minimizer of $\mathcal{J}(\cdot, c_p)$. Setting $\ell_p = L(c_p)$ and $b = -\log(1 - e^{-\ell_p})$, we conclude that J_c^* has a minimizer $\chi_{[b-\ell_p, b]}$ with $J_c^*(\chi_{[b-\ell_p, b]}) = 0$ and $\|\chi_{[b-\ell_p, b]}\|_{L_e^1} = 1$. From Lemma 7.1 it follows that $\ell_p > \log \frac{\alpha+1}{\alpha-1}$.

In conclusion, we have proved the following.

Lemma 7.6. *Let σ and γ be given. If (A2) is satisfied, there exists $c_p > 0$ such that $J_{c_p}^*$ has a minimizer $\chi_{[a, b]}$ with $\|\chi_{[a, b]}\|_{L_e^1} = 1$ and $J_{c_p}^*(\chi_{[a, b]}) = 0$. Thus, c_p is the speed of the traveling pulse solutions for the limit equation and $b - a > \log \frac{\alpha+1}{\alpha-1}$.*

We next study the uniqueness of the Γ -limit speed c_p . This will in turn imply the uniqueness of both $L(c_p)$ and the minimizer of $J_{c_p}^*$. As a first step, we prove the following lemma.

Lemma 7.7. *Let $\ell, c > 0$. Then*

$$\frac{\partial \mathcal{J}}{\partial c}(\ell, c) < 0.$$

Proof. If $\ell, c > 0$, it follows from (5.4), the first equation in (5.7) and (5.3) that

$$\begin{aligned}
\frac{\gamma}{\sigma} \frac{\partial \mathcal{J}}{\partial c}(\ell, c) &= \frac{\partial}{\partial c} \left[\frac{1}{r_2 - r_1} (r_2 + r_1 e^{-\ell} + e^{r_1 \ell}) \right] = \frac{\partial}{\partial c} \left[\frac{1}{r_2 - r_1} (r_1(1 + e^{-\ell}) + e^{r_1 \ell}) \right] \\
&= \frac{\partial}{\partial c} \left[H(c) (r_1(1 + e^{-\ell}) + e^{r_1 \ell}) \right] \\
&= H'(c) (r_1(1 + e^{-\ell}) + e^{r_1 \ell}) + H(c) \frac{\partial r_1}{\partial c} (1 + e^{-\ell} + \ell e^{r_1 \ell}) \\
&= \frac{4\gamma}{(c^2 + 4\gamma)^{3/2}} (r_1(1 + e^{-\ell}) + e^{r_1 \ell}) + \frac{2\gamma}{c(c^2 + 4\gamma)} (1 + e^{-\ell} + \ell e^{r_1 \ell}) \\
&= \frac{4\gamma}{(c^2 + 4\gamma)^{3/2}} \left[r_1(1 + e^{-\ell}) + e^{r_1 \ell} + \frac{1 + e^{-\ell} + \ell e^{r_1 \ell}}{2H(c)} \right].
\end{aligned}$$

Since $2r_1 H(c) = -1 - H(c)$, we obtain

$$\begin{aligned}
\frac{\gamma}{\sigma} \frac{\partial \mathcal{J}}{\partial c} &= \frac{2\gamma}{H(c)(c^2 + 4\gamma)^{3/2}} \left[-(1 + H(c))(1 + e^{-\ell}) + 2H(c)e^{r_1 \ell} + 1 + e^{-\ell} + \ell e^{r_1 \ell} \right] \\
&= \frac{2\gamma}{(c^2 + 4\gamma)^{3/2}} \mathcal{K},
\end{aligned}$$

where

$$\mathcal{K}(\ell, c) \equiv -1 - e^{-\ell} + 2e^{r_1 \ell} + \frac{\ell}{H(c)} e^{r_1 \ell}.$$

Recalling the first equality in (5.7), a direct calculation yields

$$\frac{\partial \mathcal{K}}{\partial \ell} = e^{-\ell} + 2r_1 e^{r_1 \ell} + \frac{1}{H(c)} (e^{r_1 \ell} + r_1 \ell e^{r_1 \ell}) = e^{-\ell} - e^{r_1 \ell} + \frac{r_1 \ell}{H(c)} e^{r_1 \ell}.$$

It suffices to show $\mathcal{K} < 0$ for all $\ell, c > 0$. Note that $\mathcal{K}(0, c) = 0$. For small ℓ , a Taylor's expansion gives

$$\begin{aligned}
\mathcal{K} &= -1 - (1 - \ell + \frac{\ell^2}{2}) + 2(1 + r_1 \ell + \frac{r_1^2 \ell^2}{2}) + \frac{\ell}{H(c)} (1 + r_1 \ell) + O(\ell^3) \\
&= \ell^2 \left(-\frac{1}{2} + r_1^2 + \frac{r_1}{H(c)} \right) + O(\ell^3) = \ell^2 \left(-\frac{1}{2} + r_1 r_2 \right) + O(\ell^3) \\
&= \ell^2 \left(-\frac{1}{2} - \frac{\gamma}{c^2} \right) + O(\ell^3) < 0.
\end{aligned}$$

Therefore $\mathcal{K} = \frac{\partial \mathcal{K}}{\partial \ell} = 0$ and $\frac{\partial^2 \mathcal{K}}{\partial \ell^2} < 0$ at $\ell = 0$, which implies $\frac{\partial \mathcal{K}}{\partial \ell} < 0$ for small positive ℓ . Besides $\ell = 0$, we claim that there exists only another non-negative root of $\frac{\partial \mathcal{K}}{\partial \ell}$. Indeed such a root satisfies

$$e^{r_2 \ell} = 1 - \frac{r_1}{H(c)} \ell,$$

which corresponds to an intersection point of the exponential function $y = e^{r_2 \ell}$ and the straight line $y = 1 - \frac{r_1}{H(c)} \ell$ in the (ℓ, y) plane.

As $\frac{\partial \mathcal{K}}{\partial \ell} < 0$ for small positive ℓ , the straight line lies above the graph of exponential function near $\ell = 0$. On the other hand the exponential function will dominate the straight line when ℓ is large, hence there exists one strictly positive intersection point, that we denote by ℓ_1 . Clearly $\frac{\partial \mathcal{K}}{\partial \ell} < 0$ for $\ell < \ell_1$ and $\frac{\partial \mathcal{K}}{\partial \ell} > 0$ for $\ell > \ell_1$. As $\ell \rightarrow \infty$, it is easily checked that $\mathcal{K} \rightarrow -1$. We can now conclude that \mathcal{K} dips below 0 near $\ell = 0$, reaching a negative minimum at $\ell = \ell_1$, then increases again and converges to -1 as $\ell \rightarrow \infty$. Hence for any given $c > 0$, we have $\mathcal{K} < 0$ for $\ell > 0$. This concludes the proof of the lemma. \square

Lemma 7.8. *Under the assumptions of Lemma 7.6, the Γ -limit speed c_p is unique.*

Proof. We argue by contradiction. Suppose that there exist two minimizers $\chi_{[a_i, b_i]}$ of $J_{c_i}^*$, for $i = 1, 2$, with $\ell_i = b_i - a_i$ and $\ell_2 > \ell_1$. By (7.8) we have that $c_2 > c_1$. Then $\mathcal{J}(\ell, c_i) > \mathcal{J}(\ell_i, c_i) = 0$ for all $\ell \neq \ell_i$. From these inequalities, recalling that $\frac{\partial \mathcal{J}}{\partial c} < 0$, we obtain $0 = \mathcal{J}(\ell_1, c_1) > \mathcal{J}(\ell_1, c_2) > \mathcal{J}(\ell_2, c_2) = 0$, which is absurd. \square

Remark 7.9. *Under the assumptions of Lemma 7.6 and Lemma 7.8, the Γ -limit has a unique traveling pulse, whose speed is c_p . Moreover, J_{c_p} has the unique minimizer $\chi_{[a, b]}$.*

The proof of Theorem 1.2 is analogous to that of Theorem 1.1, we omit it. Finally recall γ_* from Lemma 6.4 and note that when ϵ is sufficiently small, (A2) is equivalent to

$$\alpha > 1 \quad \text{and} \quad \gamma_* > \gamma.$$

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