

Equivalence of two BV classes of functions in metric spaces, and existence of a Semmes family of curves under a 1-Poincaré inequality

Estibalitz Durand-Cartagena, Sylvester Eriksson-Bique,
Riikka Korte, Nageswari Shanmugalingam*

September 11, 2018

Abstract

We consider two notions of functions of bounded variation in complete metric measure spaces, one due to Martio [M1, M2] and the other due to Miranda Jr. [Mi]. We show that these two notions coincide, if the measure is doubling and supports a 1-Poincaré inequality. In doing so, we also prove that if the measure is doubling and supports a 1-Poincaré inequality, then the metric space supports a *Semmes family of curves* structure.

Key words: AM-modulus, bounded variation, 1-Poincaré inequality, metric measure space, Semmes pencil of curves.

MSC classification: 26A45, 30L99, 31E05.

1 Introduction

Given $1 \leq p < \infty$, a function u in $L^p(\mathbb{R}^n)$ is in $W^{1,p}(\mathbb{R}^n)$ if and only if u has an L^p -representative that is absolutely continuous on almost every non-constant compact rectifiable curve in \mathbb{R}^n with derivative in $L^p(\mathbb{R}^n)$, see [Vä] for an in-depth discussion on this. Equivalently, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and there is a non-negative Borel function $g \in L^p(\mathbb{R}^n)$ such that for all

*S.E.-B. was partially supported by grant DMS#-1704215 of NSF(U.S.), R.K. was supported by Academy of Finland grant number 308063, N.S. was partially supported by grant DMS#-1500440 of NSF(U.S.A.), and E. D-C. 's research is partially supported by the grants MTM2015-65825-P (MINECO of Spain) and 2018-MAT14 (ETSI Industriales, UNED). Part of the research was done during the visit of the second and fourth authors to Aalto University and Linköping University, and during the visit of the third and fourth authors to Universidad Complutense de Madrid and UNED; the authors wish to thank these institutions for their kind hospitality. We also thank Olli Martio for many valuable discussions related to this subject and for sharing his early manuscript [HMM2].

non-constant compact rectifiable curves γ in \mathbb{R}^n ,

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g \circ \gamma(t) |\gamma'(t)| dt. \quad (1)$$

On the other hand, the class of BV functions on \mathbb{R}^n has a more complicated analog; there should be a sequence $f_k \in W^{1,1}(\mathbb{R}^n)$, with $f_k \rightarrow u$ in $L^1(\mathbb{R}^n)$ and function g_k associated with f_k as in the inequality above, such that $\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k dx$ is finite. Thus to verify that a function u belongs to the class $BV(\mathbb{R}^n)$ we need a sequence of *pairs* of functions (f_k, g_k) satisfying (1), where f_k approximates u in $L^1(\mathbb{R}^n)$, whereas to define a function in $W^{1,1}(\mathbb{R}^n)$ we only need a single energy function g that satisfies (1).

The above complication carries through from \mathbb{R}^n to more general metric measure spaces X , and so while we need only the energy function g in order to know that u is in the Sobolev class, to know that u is in the BV class we need both, the approximating sequence f_k as well as the corresponding energy functions g_k . To avoid this discrepancy, the recent work of Martio [M1, M2] proposed a new definition of BV functions in the Euclidean and general metric measure setting, denoted in the current paper by $BV_{AM}(X)$, see Definition 2.5. In this notion one needs a single sequence of “energy” functions g_k associated with the function u in a specific manner in order to determine whether $u \in BV_{AM}(X)$. The backbone of the construction of $BV_{AM}(X)$ is the notion of AM-modulus, and it appears that this modulus is better suited to the study of sets of finite perimeter than the standard 1-modulus. It is shown in [HMM2, Theorem 11] that Euclidean Borel sets E are of finite perimeter if and only if the AM-modulus of the collection of all curves that cross the measure-theoretic boundary $\partial_* E$ of E is finite; and in this case the perimeter measure of E is precisely the AM-modulus of that collection of curves. This is a variant of the Federer characterization of sets of finite perimeter. Federer proved that a measurable set $E \subset \mathbb{R}^n$ is of finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_* E)$ is finite; a new, potential-theoretic proof of this characterization, valid even in the metric setting, can be found in [L].

The goal of this paper is to show that if the metric measure space X is of controlled geometry, that is, if X is complete, the measure μ is doubling and supports a 1-Poincaré inequality, then the notion of $BV_{AM}(X)$ from [M1, M2] gives the same function space as the BV class $BV(X)$ as defined by Miranda Jr. in [Mi]. To do so we also prove that if μ is doubling, then X supports a 1-Poincaré inequality if and only if X supports a *Semmes family of curves* corresponding to each pair $x, y \in X$ of points, that is, there is a family Γ_{xy} of quasiconvex curves connecting x to y and a probability measure σ_{xy} on Γ_{xy} satisfying a Riesz-type inequality, see Definition 3.6 below. This auxiliary result is of independent interest. The notion of Semmes family of curves, first proposed in [Se] (where clearly it was not termed a “Semmes family”), is known to imply the support of a 1-Poincaré inequality, see the discussion in [He, page 29]. In this paper we show that the converse also holds true, that is, if the measure is doubling and supports a 1-Poincaré inequality, then it supports a Semmes family of curves structure. Thus, our paper also characterizes the support of a 1-Poincaré inequality (in doubling complete metric measure spaces) via the existence of a Semmes family of curves. A recent preprint [FO] gives another characterization of the support of a 1-Poincaré inequality in terms of the existence of normal 1-currents for each pair of points $x, y \in X$, in the sense of Ambrosio and Kirchheim, such that the mass of the current is controlled by the Riesz measure R_{xy} , see (6) below. For the study

comparing BV-AM spaces with BV classes of functions, a Semmes family of curves seems to be more useful.

The equality of $BV(X)$ with $BV_{AM}(X)$ and the equivalence between the Semmes family of curves structure and the 1-Poincaré inequality form the two main results in this paper, see Theorem 3.10.

2 Two definitions of BV functions

In the rest of the paper, (X, d, μ) is a *metric measure space*, where (X, d) is a complete metric space and μ is a Borel measure. We denote by B an open ball in X and by λB the ball with the same center as B and radius λ times the radius of B . Recall that the measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that $\mu(2B) \leq C\mu(B)$ for every ball B in X .

Given a compact interval $I \subset \mathbb{R}$, a *curve* $\gamma : I \rightarrow X$ is a continuous mapping. We only consider curves that are non-constant and rectifiable. A curve γ , connecting two points $x, y \in X$, is *C-quasiconvex* if its length is at most $C d(x, y)$.

2.1 p -Modulus and AM-modulus of a family of curves

Definition 2.1 Given a family Γ of curves in X , set $\mathcal{A}(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \rightarrow [0, \infty]$ such that

$$\int_{\gamma} \rho ds \geq 1 \quad \text{for all } \gamma \in \Gamma,$$

and set $\mathcal{A}_{seq}(\Gamma)$ to be the family of all sequences (ρ_i) of non-negative Borel measurable functions ρ_i on X such that

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

The integral $\int_{\gamma} \rho ds$ denotes the path-integral of γ against the arc-length re-parametrization of γ , see for example the description in [He]. We define the ∞ -modulus of Γ by

$$\text{Mod}_{\infty}(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \|\rho\|_{L^{\infty}(X)},$$

and for $1 \leq p < \infty$ the p -modulus of Γ is

$$\text{Mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p d\mu.$$

Following [M1, M2], we define the *approximate modulus* (AM-modulus) of Γ by

$$\text{AM}(\Gamma) = \inf_{(\rho_i) \in \mathcal{A}_{seq}(\Gamma)} \left\{ \liminf_{i \rightarrow \infty} \int_X \rho_i d\mu \right\}.$$

The notion of $\text{AM}_p(\Gamma)$ is defined analogously, with $\int_X \rho_i d\mu$ replaced by $\int_X \rho_i^p d\mu$. If a property holds for all except for a family Γ of curves with $\text{Mod}_p \Gamma = 0$ (respectively with $\text{AM}(\Gamma) = 0$), then we say that the property holds for *p-a.e. curve* (respectively for *AM-a.e. curve*).

Note that $\text{AM}(\Gamma) \leq \text{Mod}_1(\Gamma)$. Thanks to Mazur's lemma, it is a trivial consequence of the reflexivity of $L^p(X)$ that $\text{AM}_p(\Gamma) = \text{Mod}_p(\Gamma)$ when $1 < p < \infty$, see [HMM, Theorem 1]. It is also easy to see that for any family of curves Γ we have $\text{AM}_\infty(\Gamma) = \text{Mod}_\infty(\Gamma)$. Indeed, let $\tau = \text{AM}_\infty(\Gamma)$. If $\tau = \infty$ there is nothing to prove, so let us assume that $\tau < \infty$. By definition, there is a sequence of non-negative Borel functions $(g_i^\varepsilon) \in \mathcal{A}_{\text{seq}}(\Gamma)$ such that

$$\liminf_{i \rightarrow \infty} \|g_i^\varepsilon\|_{L^\infty(X)} < \tau + \varepsilon \quad \text{and} \quad \liminf_{i \rightarrow \infty} \int_\gamma g_i^\varepsilon ds \geq 1 \text{ for each } \gamma \in \Gamma.$$

Let $\rho_\varepsilon := \sup_i g_i^\varepsilon$. As $\rho_\varepsilon \geq g_i^\varepsilon$ for each $i \in \mathbb{N}$, it follows that

$$1 \leq \liminf_{i \rightarrow \infty} \int_\gamma g_i^\varepsilon ds \leq \int_\gamma \rho_\varepsilon ds,$$

and so $\text{Mod}_\infty(\Gamma) \leq \|\rho_\varepsilon\|_{L^\infty(X)} \leq \tau + \varepsilon$ and the result follows.

Note that if every curve in Γ is contained in a fixed ball B , then

$$\text{AM}(\Gamma) \leq \text{Mod}_1(\Gamma) \leq \mu(B)^{1-1/p} \text{Mod}_p(\Gamma)^{1/p} \leq \mu(B) \text{Mod}_\infty(\Gamma),$$

and therefore

$$\limsup_{p \rightarrow \infty} [\text{Mod}_p(\Gamma)]^{1/p} \leq \text{Mod}_\infty(\Gamma).$$

The next example shows that it is possible to have $\text{Mod}_1(\Gamma) = \infty$ but $\text{AM}(\Gamma) = 1$. Further examples can be found in [HMM, Section 9]. The examples found there are families of curves that tangentially approach a smooth co-dimension one sub-manifold of \mathbb{R}^n .

Example 2.2 Let Γ be the collection of all rectifiable curves of length at most 1 in the plane, and start from the x -axis with $0 \leq x \leq 1$ and are parallel to the y -axis. Then there is no acceptable $\rho \in L^1(X)$ for computing $\text{Mod}_1(\Gamma)$, and hence $\text{Mod}_1(\Gamma) = \infty$. On the other hand, $\text{AM}(\Gamma)$ is finite but positive. To see this, for each positive integer let $\rho_n = n \chi_{[0,1] \times [0,1/n]}$. Then $\int_\gamma \rho_n ds \geq 1$ whenever γ is in Γ with length at least $1/n$, and as every curve in Γ has positive length, we have that

$$\lim_{n \rightarrow \infty} \int_\gamma \rho_n ds \geq 1.$$

So the sequence (ρ_n) is admissible for Γ , and thus

$$\text{AM}(\Gamma) \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2} \rho_n d\mathcal{L}^2 = \limsup_{n \rightarrow \infty} n \left(\frac{1}{n} \times 1 \right) = 1.$$

To see that $\text{AM}(\Gamma) > 0$, we consider the sub-family $\Gamma_{1/2}$ of all line segments in Γ with length $1/2$, and let $(\rho_i) \in \mathcal{A}_{\text{seq}}(\Gamma_{1/2})$. Then by Fubini's theorem, for each $i \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^2} \rho_i d\mathcal{L}^2 \geq \int_0^1 \int_0^{1/2} \rho_i(x, y) dy dx = \int_0^1 \left(\int_0^{1/2} \rho_i(x, y) dy \right) dx.$$

Now by Fatou's lemma,

$$\liminf_{i \rightarrow \infty} \int_{\mathbb{R}^2} \rho_i d\mathcal{L}^2 \geq \int_0^1 \left(\liminf_{i \rightarrow \infty} \int_0^{1/2} \rho_i(x, y) dy \right) dx \geq 1.$$

It follows that

$$\text{AM}(\Gamma) \geq \text{AM}(\Gamma_{1/2}) \geq 1.$$

2.2 BV functions based on the notion of AM-modulus.

Definition 2.3 A nonnegative Borel function g on X is a *1-weak upper gradient* of an extended real-valued function u on X if for 1-a.e. curve $\gamma : [a, b] \rightarrow X$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g ds.$$

Given a function u that has a 1-weak upper gradient in $L^1(X)$, there is a *minimal* 1-weak upper gradient of u , denoted g_u , in the sense that whenever g is a 1-weak upper gradient of u , we have $g_u \leq g$ almost everywhere in X .

The following notion of BV functions on X is due to Miranda Jr. [Mi].

Definition 2.4 (BV functions) For $u \in L^1_{\text{loc}}(X)$, we define the total variation of u as

$$\|Du\|(X) := \inf \left\{ \liminf_{i \rightarrow \infty} \inf_{g_{u_i}} \int_X g_{u_i} d\mu : u_i \in \text{LIP}_{\text{loc}}(X), u_i \rightarrow u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where the second infimum is over all 1-weak upper gradients g_{u_i} of u_i . We say that a function $u \in L^1_{\text{loc}}(X)$ is of *bounded variation*, $u \in \text{BV}(X)$ if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said of *finite perimeter* if $\|D\chi_E\|(X) < \infty$.

The following definition of BV_{AM} class is from [M1].

Definition 2.5 (BV-AM functions) A function $u \in L^1(X)$ is in the $\text{BV}_{\text{AM}}(X)$ class if there is a family Γ of rectifiable curves in X with $\text{AM}(\Gamma) = 0$, and a sequence (g_i) of non-negative Borel measurable functions in $L^1(X)$ such that whenever $\gamma : [a, b] \rightarrow X$ is a non-constant compact rectifiable curve that does not belong to Γ , we have that

$$|u(\gamma(t)) - u(\gamma(s))| \leq \liminf_{i \rightarrow \infty} \int_{\gamma|_{[s,t]}} g_i ds \tag{2}$$

for \mathcal{H}^1 -a.e. $s, t \in [a, b]$ with $s < t$, and

$$\liminf_{i \rightarrow \infty} \int_X g_i d\mu < \infty.$$

Such a sequence (g_i) is said to be a BV_{AM} -upper bound of u . We set

$$\|D_{AM}u\|(X) := \inf_{(g_i)} \liminf_{i \rightarrow \infty} \int_X g_i d\mu,$$

where the infimum is over all BV_{AM} -upper bounds of u .

Notice that by [M2, Theorem 4.1], $BV(X) \subseteq BV_{AM}(X)$. This also follows from the next lemma. The following lemma holds even if μ is not doubling or does not support a 1-Poincaré inequality.

Lemma 2.6 *Assume that $u \in BV(X)$. Then there is a set $N \subset X$ with $\mu(N) = 0$ and a sequence (g_i) of non-negative Borel measurable functions in $L^1(X)$ such that whenever γ is a non-constant compact rectifiable curve with end-points $x, y \in X \setminus N$,*

$$|u(y) - u(x)| \leq \liminf_{i \rightarrow \infty} \int_\gamma g_i ds$$

(that is, (2) holds) and

$$\liminf_{i \rightarrow \infty} \int_X g_i d\mu < \infty.$$

Note that the lemma gives a stronger control of u than allowed by the BV_{AM} -control. For functions in $BV_{AM}(X)$, we know that given a path γ there is a set N_γ with $\mathcal{H}^1(\gamma^{-1}(N_\gamma)) = 0$ so that whenever x, y lie in the trajectory of γ with $x, y \notin N_\gamma$, inequality (2) holds. Here we show that we can choose N_γ to be independent of γ and in addition with μ -measure zero.

Proof. Given $u \in BV(X)$ there is a sequence $u_i \in \text{LIP}_{\text{loc}}(X)$ such that $u_i \rightarrow u$ in $L^1(X)$ and $\lim_{i \rightarrow \infty} \int_X g_i d\mu \leq M < \infty$ for a choice of upper gradients g_i of u_i . By passing to a subsequence if necessary, we may also assume that $u_i \rightarrow u$ pointwise μ -a.e. in X . Let N be the set of all points $x \in X$ for which $\lim_{i \rightarrow \infty} u_i(x) \neq u(x)$. Then $\mu(N) = 0$. Let γ be a non-constant compact rectifiable curve in X with end points $x, y \in X \setminus N$. Then

$$|u(x) - u(y)| = \lim_{i \rightarrow \infty} |u_i(x) - u_i(y)| \leq \liminf_{i \rightarrow \infty} \int_\gamma g_i ds.$$

□

The main focus of this paper is to show that $BV_{AM}(X) = BV(X)$ when the measure on X is doubling and supports a 1-Poincaré inequality.

2.3 The spaces $N^{1,1}(X)$ and $N_{\text{AM}}^{1,1}(X)$

Let $\tilde{N}^{1,1}(X, d, \mu)$, where $1 \leq p < \infty$, be the class of all L^1 -integrable Borel functions on X for which there exists a 1-weak upper gradient in $L^1(X)$. For $u \in \tilde{N}^{1,1}(X, d, \mu)$ we define

$$\|u\|_{\tilde{N}^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all 1-weak upper gradients g of u . As usual, we can now define $N^{1,1}(X, d, \mu)$ equipped with the norm $\|u\|_{N^{1,1}(X)} = \|u\|_{\tilde{N}^{1,1}(X)}$.

Once we have the new concept of AM-a.e. curve, it is natural to define an upper gradient and a Sobolev class related to this notion.

Definition 2.7 (Weak AM-upper gradient) A nonnegative Borel function g on X is a *weak AM-upper gradient* of u on X if $|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds$ for AM-a.e. curve $\gamma : [a, b] \rightarrow X$.

Definition 2.8 ($N_{\text{AM}}^{1,1}$ functions) Let $\tilde{N}_{\text{AM}}^{1,1}(X, d, \mu)$, be the class of all Borel functions in $L^1(X)$ for which there exists a weak AM-upper gradient in $L^1(X)$. For $u \in \tilde{N}_{\text{AM}}^{1,1}(X)$ we define

$$\|u\|_{\tilde{N}_{\text{AM}}^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all weak AM-upper gradient g of u . We can now define $N_{\text{AM}}^{1,1}(X)$ to be the class $\tilde{N}_{\text{AM}}^{1,1}(X, d, \mu)$, equipped with the norm $\|u\|_{N_{\text{AM}}^{1,1}(X)} = \|u\|_{\tilde{N}_{\text{AM}}^{1,1}(X)}$.

The following lemma proves that the first definition implies the second one. In some sense, the first definition is related to the Sobolev class $N^{1,1}$ while the second is related to the BV class.

Lemma 2.9 *If a function u on X has g as a weak AM-upper gradient, then there exists a BV_{AM} -upper bound of u .*

Proof. Assume that

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds \tag{3}$$

for AM-a.e. curve $\gamma : [a, b] \rightarrow X$. Let Γ be the collection of curves for which (3) does not hold. By definition $\text{AM}(\Gamma) = 0$ and so by [HMM, Theorem 7] there is a sequence of non-negative Borel functions \tilde{g}_i such that

$$\liminf_{i \rightarrow \infty} \|\tilde{g}_i\|_{L^1} < \infty \quad \text{and} \quad \liminf_{i \rightarrow \infty} \int_{\gamma} \tilde{g}_i \, ds = \infty \quad \text{for all } \gamma \in \Gamma.$$

Let Γ_0 be the collection of all non-constant compact rectifiable curves γ in X for which

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \tilde{g}_i \, ds = \infty;$$

then $\text{AM}(\Gamma_0) = 0$. Observe that if γ is a non-constant compact rectifiable curve in X such that $\gamma \notin \Gamma_0$, then every sub-curve of γ also does not belong to Γ_0 . Now, for each $\varepsilon > 0$ the sequence of functions $g_i = g + \varepsilon \tilde{g}_i$ has the property that for $\gamma \notin \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds \leq \liminf_{i \rightarrow \infty} \int_{\gamma} (g + \varepsilon \tilde{g}_i) \, ds,$$

and for $\gamma \in \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \infty = \int_{\gamma} (g + \varepsilon \tilde{g}_i) \, ds,$$

so $|u(\gamma(a)) - u(\gamma(b))| \leq \liminf_{i \rightarrow \infty} \int_{\gamma} g_i \, ds$ holds for every curve γ . \square

Note that we have more than just that the sequence (g_i) forms a BV_{AM} -upper bound of u ; the inequality holds for *every* subcurve of γ , not merely for \mathcal{H}^1 -almost every pair of points in the domain of γ .

From the above we know that for $1 < p < \infty$, $N^{1,1}(X) \subseteq N_{\text{AM}}^{1,1}(X) \subsetneq \text{BV}_{\text{AM}}(X)$ and

$$\text{LIP}^{\infty}(X) \subseteq N^{1,\infty}(X) \subseteq N^{1,p}(X) \subseteq N^{1,1}(X) \subsetneq \text{BV}(X) \subseteq \text{BV}_{\text{AM}}(X).$$

In Section 3 we will show that if X supports a 1-Poincaré inequality then $\text{BV}_{\text{AM}}(X) = \text{BV}(X)$ and that $N_{\text{AM}}^{1,1}(X) = N^{1,1}(X)$.

Remark 2.10 For $u \in \text{BV}_{\text{AM}}(X)$ and a sequence (g_i) such that $\lim_{i \rightarrow \infty} \int_X g_i \, d\mu < \infty$, the sequence of measures $(g_i \, d\mu)$ is a bounded sequence. We can assume (by localizing the argument if need be) that X is compact as well. Then there is a subsequence, also denoted $(g_i \, d\mu)$, and a Radon measure ν on X such that the sequence of measures $(g_i \, d\mu)$ converges weakly* to $d\nu$ in X . As X is compact, we see that $\|D_{\text{AM}}u\|(X) \leq \nu(X)$.

3 Equivalence of BV and AM-BV classes under Poincaré inequality

The aim of this section is to show the equivalence of the functional spaces $\text{BV}(X)$ and $\text{BV}_{\text{AM}}(X)$, under the additional hypothesis that the metric space supports a doubling measure and a 1-Poincaré inequality.

Definition 3.1 The metric measure space X supports a 1-Poincaré inequality if there are positive constants C, λ such that whenever B is a ball in X and g is an upper gradient of u ,

$$\int_B |u - u_B| \, d\mu \leq C \, \text{rad}(B) \int_{\lambda B} g \, d\mu.$$

Here $u_B := \mu(B)^{-1} \int_B u \, d\mu = \int_B u \, d\mu$ is the average of u on the ball B .

With the notion of BV_{AM} class, one could even define a stronger version of 1-Poincaré inequality.

Definition 3.2 We say that X supports an *AM-Poincaré inequality* if there exist constants $C > 0$, $\lambda \geq 1$ such that for each measurable function u on X , each BV-upper bound (g_i) of u , and each ball $B \subset X$, we have

$$\int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \liminf_{i \rightarrow \infty} \int_{\lambda B} g_i d\mu.$$

This should imply that

$$\int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \frac{\|D_{AM}u\|(\lambda B)}{\mu(\lambda B)}.$$

On the other hand, notice that 1-Poincaré inequality implies

$$\int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \frac{\|Du\|(\tau B)}{\mu(\tau B)}.$$

As a first step, in the following proposition we prove the equivalence of $BV(X)$ and $BV_{AM}(X)$ under the hypotheses that the measure is doubling and the space supports an AM-Poincaré inequality. We will see in Theorem 3.10 that the support of an AM-Poincaré inequality is equivalent to the support of a 1-Poincaré inequality.

Proposition 3.3 *If X supports a AM-Poincaré inequality and μ is doubling, then the two classes $BV_{AM}(X)$ and $BV(X)$ are equal, with comparable norms.*

Proof. Note first that $BV(X) \subset BV_{AM}(X)$, see Lemma 2.6.

Now let us prove that if $u \in BV_{AM}(X)$, then $u \in BV(X)$. By the doubling property of μ , for $\varepsilon > 0$ we can cover X by balls $B_i = B(x_i, \varepsilon)$ such that the balls $5\lambda B_i$ have bounded overlap. Let φ_i^ε be a partition of unity subordinate to the cover $2B_i$. For $u \in BV_{AM}(X)$ let

$$u_\varepsilon = \sum_i u_{B_i} \varphi_i^\varepsilon.$$

Recall that we have bounded overlap of the collection $5B_i$ with $X = \bigcup_j B_j$, μ is doubling, and that if $2B_i$ intersects B_j then $5B_j \supset 2B_i$. Then we have for $x \in B_j \subset X$,

$$\begin{aligned} |u(x) - u_\varepsilon(x)| &= \left| \sum_i [u_{B_i} - u(x)] \varphi_i^\varepsilon(x) \right| \leq \sum_i |u_{B_i} - u(x)| \varphi_i^\varepsilon(x) \\ &\leq \sum_{i: x \in 2B_i} |u_{B_i} - u(x)| \\ &\leq \sum_{i: 2B_i \cap B_j \neq \emptyset} |u_{B_i} - u(x)| \\ &\leq C C_D^3 |u_{5B_j} - u(x)|. \end{aligned}$$

Therefore, by the AM-Poincaré inequality,

$$\begin{aligned}
\int_X |u - u_\varepsilon| d\mu &\leq \sum_j \int_{B_j} |u - u_\varepsilon| d\mu \leq C \sum_j \int_{B_j} |u_{5B_j} - u| d\mu \\
&\leq C \sum_j \int_{5B_j} |u - u_{5B_j}| d\mu \\
&\leq C\varepsilon \sum_j \|D_{AM}u\|(5\lambda B_j).
\end{aligned}$$

Since $\|D_{AM}u\|$ is a Radon measure ([M1, Theorem 3.4]) and $5\lambda B_j$ have bounded overlap, we have

$$\int_X |u - u_\varepsilon| d\mu \leq C\varepsilon \|D_{AM}u\|(X) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+.$$

Thus $u_\varepsilon \rightarrow u$ in $L^1(X)$, and we also know from the definition of u_ε that each u_ε is locally Lipschitz and hence in $N_{loc}^{1,1}(X)$. Next, for $x, y \in B_j$,

$$\begin{aligned}
|u_\varepsilon(x) - u_\varepsilon(y)| &= \left| \sum_i u_{B_i} [\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)] \right| = \left| \sum_i [u_{B_i} - u_{B_j}] [\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)] \right| \\
&\leq \frac{d(x, y)}{\varepsilon} \sum_{i: 2B_i \cap B_j \neq \emptyset} |u_{B_i} - u_{B_j}| \\
&\leq \frac{C d(x, y)}{\varepsilon} \sum_{i: 2B_i \cap B_j \neq \emptyset} \int_{5B_j} |u - u_{5B_j}| d\mu.
\end{aligned}$$

It follows from the bounded overlap of $5B_i$ that

$$\text{Lip}u_\varepsilon(x) \leq \frac{C}{\varepsilon} \int_{5B_j} |u - u_{5B_j}| d\mu$$

whenever $x \in B_j$. Integrating the above inequality over $X = \bigcup_j B_j$, we obtain

$$\begin{aligned}
\int_X \text{Lip}u_\varepsilon d\mu &\leq \frac{C}{\varepsilon} \sum_j \mu(B_j) \int_{5B_j} |u - u_{5B_j}| d\mu \\
&\leq \frac{C}{\varepsilon} \sum_j \int_{5B_j} |u - u_{5B_j}| d\mu \\
&\leq \frac{C}{\varepsilon} \sum_j \varepsilon \|D_{AM}u\|(5\lambda B_j) \\
&\leq C \|D_{AM}u\|(X).
\end{aligned} \tag{4}$$

Thus

$$\liminf_{\varepsilon \rightarrow 0^+} \int_X g_{u_\varepsilon} d\mu \leq C \|D_{AM}u\|(X) < \infty,$$

and as $u_\varepsilon \rightarrow u$ in $L^1(X)$, it follows that $u \in \text{BV}(X)$ with $\|Du\|(X) \leq C \|D_{AM}u\|(X)$.

We also have $\|D_{\text{AM}}u\|(X) \leq \|Du\|(X)$, as we now show. Suppose now that $\|Du\|(X)$ is finite, and let $u_k \in \text{BV}(X)$ be such that $u_k \rightarrow u$ in $L^1(X)$ and $\lim_{k \rightarrow \infty} \int_X g_{u_k} d\mu = \|Du\|(X)$. By passing to a subsequence if necessary, we may also assume that $u_k \rightarrow u$ pointwise almost everywhere in X as well. For each $k \in \mathbb{N}$ we choose an upper gradient g_k of u_k such that $\int_X g_k d\mu \leq \int_X g_{u_k} d\mu + \varepsilon/2^k$. We set N to be the collection of all points $x \in X$ at which $u_k(x)$ does not converge to $u(x)$. Then $\mu(N) = 0$, and so the 1-modulus of the collection $\widehat{\Gamma}_N^+$ of non-constant compact rectifiable curves γ in X for which $\mathcal{H}^1(\gamma^{-1}(N)) > 0$ is zero. Using [He, (7.8)], we know that the collection Γ_N^+ of all non-constant compact rectifiable curves in X with a subcurve in $\widehat{\Gamma}_N^+$ is also of 1-modulus zero. Let γ be a non-constant compact rectifiable curve in X with $\gamma \notin \Gamma_N^+$. By re-parametrizing if necessary, we now assume that $\gamma : [a, b] \rightarrow X$ is arc-length parametrized; then $\mathcal{H}^1([a, b] \setminus \gamma^{-1}(N)) = 0$. For $s, t \in [a, b] \setminus \gamma^{-1}(N)$ with $s > t$ we have that

$$|u(\gamma(t)) - u(\gamma(s))| = \lim_{k \rightarrow \infty} |u_k(\gamma(t)) - u_k(\gamma(s))| \leq \liminf_{k \rightarrow \infty} \int_{\gamma|_{[t,s]}} g_k ds \leq \liminf_{k \rightarrow \infty} \int_{\gamma} g_k ds.$$

This verifies that (g_k) is a BV_{AM} -upper bound for u in the sense of Definition 2.5. \square

Proposition 3.4 *If X supports an AM-Poincaré inequality and μ is doubling, then $N_{\text{AM}}^{1,1}(X) = N^{1,1}(X)$ with comparable norms.*

Proof. Note that $N^{1,1}(X) \subset N_{\text{AM}}^{1,1}(X)$. Thus it suffices to prove the reverse inclusion. Let $u \in N_{\text{AM}}^{1,1}(X)$, and let $g \in L^1(X)$ be a weak AM-upper gradient of u . Let Γ be the corresponding exceptional family; then $\text{AM}(\Gamma) = 0$. Then by the proof of Lemma 2.9 there is a sequence (ρ_i) of non-negative Borel functions in $L^1(X)$ such that $\int_X \rho_i d\mu \leq M < \infty$ for each $i \in \mathbb{N}$ and for each $\gamma \in \Gamma$ we have

$$\lim_{i \rightarrow \infty} \int_{\gamma} \rho_i ds = \infty.$$

Then for each $\varepsilon > 0$ we have that $(g + \varepsilon\rho_i)$ forms a BV_{AM} -upper bound of u , and so as X supports an AM-Poincaré inequality, whenever B is a ball in X we have

$$\int_B |u - u_B| d\mu \leq \frac{C \text{rad}(B)}{\mu(B)} \liminf_{i \rightarrow \infty} \int_{\lambda B} [g + \varepsilon\rho_i] d\mu.$$

As before, by passing to a subsequence if necessary, we may assume that $\rho_i d\mu$ converges weakly to a Radon measure ν on X , and so the above turns into

$$\int_B |u - u_B| d\mu \leq C \text{rad}(B) \left(\int_{\lambda B} g d\mu + \varepsilon \frac{\nu(2\lambda B)}{\mu(2\lambda B)} \right).$$

Letting $\varepsilon \rightarrow 0$ we get

$$\int_B |u - u_B| d\mu \leq C \text{rad}(B) \int_{\lambda B} g d\mu.$$

We now know from Proposition 3.3 that $u \in \text{BV}(X)$. Now an argument as in the proof of Proposition 3.3, up to and including (4), applied to open sets $U \subset X$ with $\mu(\partial U) = 0$, we obtain that

$$\|Du\|(U) \leq C \int_{\overline{U}} g \, d\mu = \int_U g \, d\mu.$$

Note that $g \in L^1(X)$, and hence for each $\eta > 0$ there is some $\varepsilon > 0$ such that whenever $K \subset X$ is measurable with $\mu(K) < \varepsilon$, we have $\int_K g \, d\mu < \eta$. Since whenever $E \subset X$ with $\mu(E) = 0$, for each $\varepsilon > 0$ we can find an open set $U_\varepsilon \supset E$ such that $\mu(U_\varepsilon) < \varepsilon$ and $\mu(\partial U_\varepsilon) = 0$, it follows that $\|Du\| \ll \mu$, and hence $u \in N^{1,1}(X)$ by [HKLL, Theorem 4.6]. \square

Note that if X does not support a 1-Poincaré inequality, we do not know the equivalence of $N^{1,1}(X)$ with $N_{\text{AM}}^{1,1}(X)$. Similar difficulties show up in comparing other alternative notions of $N^{1,1}(X)$ as well, see for example [ADiM, Section 8]. We will prove in Theorem 3.10 that X supports a 1-Poincaré inequality if and only if it supports the a priori stronger AM-Poincaré inequality.

The key point in the above proof is that if $u \in \text{BV}(X)$ and $\|Du\| \ll \mu$, then $u \in N^{1,1}(X)$; the validity of this point requires a doubling measure supporting a 1-Poincaré inequality. The following counterexample is from [ADiM, Example 7.4]. We do not have a counterexample for the statement “ $\|Du\| \ll \mu$ implies $u \in N^{1,1}(X)$ ” in the case μ is doubling, but the measure μ in the following example is asymptotically doubling.

Example 3.5 Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric and the measure $\mu = \mathcal{L}^2 + \mathcal{H}^1|_C$ where C is the boundary of the unit disk D in \mathbb{R}^2 centered at the origin. Let $u = \chi_D$. Then, by the approximations $f_\varepsilon(x) = (1 - \varepsilon^{-1} \text{dist}(x, D))_+$ of u we see that $u \in \text{BV}(X)$ with $\|Du\| \equiv \mathcal{H}^1|_C$. It follows that $\|Du\| \ll \mu$. However, $u \notin N^{1,1}(X)$: with Γ the collection of all line segments γ_x , $-1 < x < 1$, given by $\gamma_x : [-2, 2] \rightarrow X$ where $\gamma_x(t) = (x, t)$, we have that $u \circ \gamma_x$ is not absolutely continuous on $[-2, 2]$, and furthermore, $\text{Mod}_1(\Gamma) > 0$.

The existence of a Semmes family of curves provides a key tool for the proof that the AM-Poincaré inequality and the standard 1-Poincaré inequality are equivalent, which in turn allows us to prove equivalence of the two classes $\text{BV}(X)$ and $\text{BV}_{\text{AM}}(X)$ with just the assumption of a 1-Poincaré inequality in addition to the doubling property of μ . Thus we next prove that the existence of 1-Poincaré inequality in the doubling complete metric measure space X is equivalent to the existence of the following Semmes pencil of curves. See [FO] for a closely related characterization of the 1-Poincaré inequality in terms of 1-currents in the sense of Ambrosio and Kirchheim [AK].

If A is a Borel subset of X and γ is a rectifiable curve, we define $\ell(\gamma \cap A) := \mathcal{H}^1(\gamma \cap A)$.

Definition 3.6 ([Se, He]) A space X supports a *Semmes pencil of curves* if there exists a constant $C > 0$ such that for each pair of points $x, y \in X$ with $x \neq y$ there is a family Γ_{xy} of rectifiable curves in X equipped with a probability measure $d\sigma = d\sigma_{x,y}$ so that each $\gamma \in \Gamma_{xy}$ is

a C -quasiconvex curve joining x to y , and for each Borel set $A \subset X$, the map $\gamma \mapsto \ell(\gamma \cap A)$ is σ -measurable and satisfies

$$\int_{\Gamma_{xy}} \ell(\gamma \cap A) d\sigma(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z) d\mu(z). \quad (5)$$

In the previous inequality, for $C > 0$, $CB_{x,y} := B(x, Cd(x,y)) \cup B(y, Cd(x,y))$ and

$$R_{xy}(z) := \frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))}. \quad (6)$$

We next show that if the measure on X is doubling and supports a 1-Poincaré inequality, then it supports a Semmes pencil of curves.

Denote

$$\Gamma_{xy}^C := \{(\gamma, I) : \text{curve } \gamma : I \rightarrow X \text{ is 1-Lipschitz, } \gamma(0) = x, \gamma(\max(I)) = y\}, \quad (7)$$

where I are intervals contained in $[0, Cd(x,y)]$ with left-hand end point 0. We equip Γ_{xy}^C with the following metric. The elements of Γ_{xy}^C can be identified with their graphs

$$\Gamma_\gamma = \{(t, \gamma(t)) : t \in I\} \subset \mathbb{R} \times X.$$

We define a metric on Γ_{xy}^C by setting

$$d(\gamma, \gamma') := d_H(\Gamma_\gamma, \Gamma_{\gamma'}),$$

where d_H is the Hausdorff metric. Thanks to the Arzela-Ascoli theorem, this metric makes Γ_{xy}^C into a complete and compact metric space because X is complete and doubling and hence closed bounded subsets of X are compact. For $f \in C(X)$, the functional $\Phi_f : \Gamma_{xy}^C \rightarrow \mathbb{R}$ given by

$$\Phi_f((\gamma, I)) := \int_I f \circ \gamma dt,$$

is continuous on Γ_{xy}^C .

We denote the Riesz measure by

$$d\overline{\mu}_{xy}^C(z) = R_{xy} d\mu|_{CB_{x,y}}.$$

Theorem 3.7 *If (X, d, μ) satisfies a 1-Poincaré inequality, then there exists $C \geq 1$ such that for every $x, y \in X$ with $x \neq y$, there exist a compact family of curves Γ_{xy} and a Radon probability measure α_{xy} on Γ_{xy} which constitutes a Semmes family of curves, i.e. for every Borel set A ,*

$$\int_{\Gamma_{xy}^C} \int_\gamma \chi_A ds d\alpha_{xy}(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z) d\mu(z) = \overline{\mu}_{xy}^C(A).$$

The proof of the above statement could be derived by a careful application of the techniques in [AMS] combined with the modulus estimates of [K]. However, the method in [AMS] directly works only for $p > 1$, and some additional care is necessary for $p = 1$. Further, the following proof is somewhat more direct than theirs. Our proof is more in line with the approaches in [B, S] in combination with the estimates from [K] to construct probability measures on the space of curve fragments. The papers [B, S] employ the Rainwater lemma from [R2, Theorem 9.4.3]. However, we are able to avoid this lemma by directly using the Min-Max theorem [R2, Theorem 9.4], restated below for the reader's convenience.

Proposition 3.8 (Min-Max Theorem [R2, Theorem 9.4.2]) *Suppose that*

- (i) G is a convex subset of some vector space,
- (ii) K is a compact convex subset of some topological vector space, and
- (iii) $F : G \times K \rightarrow \mathbb{R}$ satisfies
 - (a) $F(\cdot, y)$ is convex on G for every $y \in K$,
 - (b) $F(x, \cdot)$ is concave and continuous on K for every $x \in G$.

Then

$$\sup_{y \in K} \inf_{x \in G} F(x, y) = \inf_{x \in G} \sup_{y \in K} F(x, y).$$

Proof. [Proof of Theorem 3.7] Denote $d(x, y) = r$. By the 1-Poincaré inequality and [K, Theorem 2], there exists a C such that we have

$$\text{Mod}_{1, \overline{\mu}_{xy}^C}(\Gamma_{xy}^C) = \inf \int_X \rho d\overline{\mu}_{xy}^C > \frac{1}{C},$$

where the infimum is over non-negative Borel functions ρ with $\int_\gamma \rho \geq 1$ for every $\gamma \in \Gamma_{xy}^C$. Note that the estimates in [K] give the modulus bound for the set of all rectifiable curves between x, y , but the collection of curves that are longer than $4C^2d(x, y)$ has modulus less than $1/(2C)$, and can be excluded using the subadditivity of the modulus.

Another way of stating this estimate is that if f is a non-negative continuous function, and $\int_X f d\overline{\mu}_{xy}^C < \infty$, then for every $\epsilon > 0$ there exists a $\gamma \in \Gamma_{xy}^C$ such that

$$\int_\gamma f ds \leq (C + \epsilon) \int_X f d\overline{\mu}_{xy}^C,$$

for otherwise, $\frac{f}{(C+\epsilon) \int_X f d\overline{\mu}_{xy}^C}$ would be admissible with a too small a norm. In particular,

$$\inf_{(\gamma, I) \in \Gamma_{xy}^C} \int_\gamma f ds \leq C \int_X f d\overline{\mu}_{xy}^C. \quad (8)$$

Since f is continuous and Γ_{xy}^C is a compact family, the above infimum is a minimum. Parametrizing the curves γ by length we also get

$$\int_{(\gamma, I) \in \Gamma_{xy}^C} \int_I f \circ \gamma(t) dt d\beta(\gamma, I) \leq C \int_X f d\bar{\mu}_{xy}^C, \quad (9)$$

where β is the Dirac measure on Γ_{xy}^C based at any of the optimal choices (γ, I) that achieves the infimum in (8).

Let K be the set of probability measures α on Γ_{xy}^C ; thus K is a compact and convex set of measures with respect to weak* convergence. Set

$$G = \{ f : X \rightarrow [0, 1] \mid f \text{ is continuous} \} \subset C(X).$$

Here $C(X)$ is the set of all continuous functions equipped with the uniform topology and G is a closed convex subset thereof. Then define $F : G \times K \rightarrow \mathbb{R}$ by

$$F(f, \alpha) = C \int_X f d\bar{\mu}_{xy}^C - \int_{\Gamma_{xy}^C} \int_I f(\gamma(t)) dt d\alpha(\gamma, I).$$

Clearly F is continuous in α , since $\Phi_f((\gamma, I)) = \int_I f(\gamma(t)) dt$ is continuous in γ . Also, $F(\cdot, \alpha)$ is convex for every $\alpha \in K$, and $F(f, \cdot)$ is affine and *a fortiori* concave for any $f \in G$. Thus, we can apply Proposition 3.8 to obtain

$$\sup_{\alpha \in K} \inf_{f \in G} F(f, \alpha) = \inf_{f \in G} \sup_{\alpha \in K} F(f, \alpha).$$

Now, for $f \in G$, by estimate (8) we have $\sup_{\alpha \in K} F(f, \alpha) \geq 0$. Thus, we get

$$\sup_{\alpha \in K} \inf_{f \in G} F(f, \alpha) \geq 0.$$

In particular, for every $\epsilon > 0$ and every $f \in G$ there exists a $\alpha_\epsilon \in K$, such that

$$F(f, \alpha_\epsilon) \geq -\epsilon.$$

Since for each $f \in G$ the map $K \ni \alpha \mapsto F(f, \alpha)$ is continuous, we can extract a weakly convergent sequence $\alpha_{\epsilon_i} \rightarrow \alpha_{xy} \in K$ (with $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$), such that for every $f \in G$

$$F(f, \alpha_{xy}) \geq 0.$$

Now, recalling the definition of F , for every $f \in G$,

$$\int_{\Gamma_{xy}^C} \int_I f(\gamma(t)) dt d\alpha_{xy}(\gamma, I) \leq C \int_X f d\bar{\mu}_{xy}^C.$$

Also, since the curves γ are 1-Lipschitz, it follows that $\int_\gamma f ds \leq \int_I f(\gamma(t)) dt$, and α_{xy} induces a measure (which we denote by the same symbol) on $\Gamma_{xy} = \{\gamma : (\gamma, I) \in \Gamma_{xy}^C \text{ for some } I\}$. With respect to this measure, we have for every $f \in C(X)$ with $0 \leq f \leq 1$ that

$$\int_{\Gamma_{xy}} \int_\gamma f ds d\alpha_{xy}(\gamma) \leq C \int_X f d\bar{\mu}_{xy}^C.$$

By a limiting argument we obtain the same inequality for $f = \chi_A$ corresponding to Borel sets $A \subset X$, and thus the measure $\sigma_{xy} = \alpha_{xy}$, which is supported on the compact set Γ_{xy} , constitutes a Semmes family of curves in the sense of Definition 3.6, and the proof is complete. \square

Each Borel function in $L^1_{loc}(X)$ can be approximated by simple Borel functions. Hence it follows from (5) that

$$\int_{\Gamma_{xy}} \int_{\gamma} g ds d\sigma(\gamma) \leq C \int_{CB_{x,y}} g(z) R_{xy}(z) d\mu(z), \quad (10)$$

for Borel functions $g : CB_{x,y} \rightarrow \mathbb{R}$. Doubling metric measure spaces supporting a Semmes pencil curves support a 1-Poincaré inequality (see e.g. the discussion following [Se, Definition 14.2.4]). In what follows we prove that they also support the AM-Poincaré inequality. Recall that

$$I_A(u)(x) = \int_A \frac{u(z) d(x,z)}{\mu(B(x, d(x,z)))} d\mu(z)$$

denotes the *Riesz potential* of a non-negative function u defined on X on a subset $A \subset X$.

Proposition 3.9 *If X supports a Semmes pencil of curves, then X supports the AM-Poincaré inequality.*

Proof. Let $u \in L^1_{loc}(X)$ and let (g_i) be a BV-upper bound of u , and let N be the collection of all points $x \in X$ for which

$$\limsup_{r \rightarrow 0^+} \int_{B(x,r)} |u - u(x)| d\mu > 0;$$

Then $\mu(N) = 0$. We focus on points $x, y \in X \setminus N$. Then for each $\varepsilon > 0$ we know that the sets

$$E_\varepsilon(x) := \{z \in X : |u(z) - u(x)| > \varepsilon\}, \quad E_\varepsilon(y) = \{z \in X : |u(z) - u(y)| > \varepsilon\}$$

satisfy

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x,r) \cap E_\varepsilon(x))}{\mu(B(x,r))} = 0 = \limsup_{r \rightarrow 0^+} \frac{\mu(B(y,r) \cap E_\varepsilon(y))}{\mu(B(y,r))}.$$

We can inductively choose a strictly decreasing sequence $r_i > 0$ such that $r_1 < d(x,y)/4$, $r_{i+1} < r_i/4$, and

$$\frac{\mu(B(x,r_i) \cap E_\varepsilon(x))}{\mu(B(x,r_i))} < \frac{2^{-i}}{2C_d}, \quad \frac{\mu(B(y,r_i) \cap E_\varepsilon(y))}{\mu(B(y,r_i))} < \frac{2^{-i}}{2C_d}.$$

For each i let $\Gamma_i(x)$ denote the collection of all $\gamma \in \Gamma_{xy}$ such that

$$\mathcal{H}^1(\gamma^{-1}([B(x,r_i) \setminus B(x,r_i/2)] \setminus E_\varepsilon(x))) = 0,$$

and $\Gamma_i(y)$ the analogous family with y playing the role of x . By the fact that Γ_{xy} is a Semmes family and by the fact that μ is doubling, we have that

$$\frac{r_i}{2} \sigma_{xy}(\Gamma_i(x)) \leq \int_{\Gamma_{xy}} \ell(\gamma \cap E_\varepsilon(x) \cap B(x, r_i) \setminus B(x, r_i/2)) d\sigma_{xy}(\gamma) \leq C_d \frac{r_i}{\mu(B(x, r_i))} \mu(E_\varepsilon(x) \cap B(x, r_i)),$$

and so by the choice of r_i we have

$$\sigma_{xy}(\Gamma_i(x)) \leq 2^{-i}.$$

Hence for each positive integer n ,

$$\sigma_{xy} \left(\bigcup_{i=n}^{\infty} \Gamma_i(x) \right) \leq 2^{1-n},$$

and so with

$$\Gamma(x) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=n}^{\infty} \Gamma_i(x),$$

we have that $\sigma_{xy}(\Gamma(x)) = 0$. Note that if $\gamma \in \Gamma_{xy} \setminus \Gamma(x)$, then whenever N_γ is a subset of the domain of γ with $\mathcal{H}^1(N_\gamma) = 0$, we can find a sequence of points $x_i \in \gamma \setminus [E_\varepsilon(x) \cup \gamma(N_\gamma)]$ such that $x_i \rightarrow x$ as $i \rightarrow \infty$. Let $\Gamma(y)$ be the analogous subfamily of curves with respect to the point y ; then $\sigma_{xy}(\Gamma(x) \cup \Gamma(y)) = 0$. Let (g_i) be a BV_{AM} -upper bound for u . For $\gamma \in \Gamma_{xy} \setminus [\Gamma(x) \cup \Gamma(y)]$, we set N_γ to be the set of points in the domain of γ that forms the exceptional set in the condition (2), and we select the sequences x_i, y_i as above. Then we have that

$$|u(x) - u(y)| - 2\varepsilon \leq \liminf_{i \rightarrow \infty} |u(x_i) - u(y_i)| \leq \liminf_{i \rightarrow \infty} \int_\gamma g_i ds.$$

Therefore, for $x, y \in X \setminus N$ and for each $\gamma \in \Gamma_{xy} \setminus (\Gamma(x) \cup \Gamma(y))$, we have

$$|u(x) - u(y)| - 2\varepsilon \leq \liminf_{i \rightarrow \infty} \int_\gamma g_i ds.$$

We then have by Fatou's lemma and (10) that

$$\begin{aligned} |u(x) - u(y)| - 2\varepsilon &\leq \int_{\Gamma_{xy}} \liminf_{i \rightarrow \infty} \int_\gamma g_i ds d\sigma_{xy}(\gamma) \\ &\leq \liminf_{i \rightarrow \infty} \int_{\Gamma_{xy}} \int_\gamma g_i ds d\sigma_{xy}(\gamma) \\ &\leq C \liminf_{i \rightarrow \infty} \int_{CB_{x,y}} g_i(z) R_{xy}(z) d\mu(z) \\ &\leq \int_{CB_{x,y}} R_{xy}(z) d\nu(z) \\ &\leq C(I_{CB_{x,y}}\nu(x) + I_{CB_{x,y}}\nu(y)), \end{aligned}$$

where ν is the Radon measure as in Remark 2.10. The constant C in the above does not depend on ε ; hence, by letting $\varepsilon \rightarrow 0^+$ we obtain

$$|u(x) - u(y)| \leq C(I_{CB_{x,y}}\nu(x) + I_{CB_{x,y}}\nu(y))$$

whenever $x, y \in X \setminus N$. For $x, y \in B$ with R the radius of B , setting $B_i = B(x, 2^{-i}Cd(x, y))$ for $i \in \mathbb{N} \cup \{0\}$, we see that

$$\begin{aligned} I_{CB_{x,y}}\nu(x) &\leq \int_{B(x, Cd(x,y))} \frac{d(x, z)}{\mu(B(x, d(z, x)))} d\mu(z) \leq C \sum_{i=0}^{\infty} \frac{2^{-i}Cd(x, y)}{\mu(B_i)} \nu(B_i) \\ &\leq C d(x, y) h_B(x) \sum_{i=0}^{\infty} 2^{-i}, \end{aligned}$$

where

$$h_B(x) = \sup_{0 < r \leq CR} \frac{\nu(B(x, r))}{\mu(B(x, r))}.$$

Thus h_B is a Hajlasz gradient of u in B , that is,

$$|u(x) - u(y)| \leq Cd(x, y)[h_B(x) + h_B(y)]$$

for μ -a.e. $x, y \in B$, and we have the weak inequality

$$\mu(\{x \in B : h_B(x) > t\}) \leq C \frac{\nu(B)}{t} \text{ for } t > 0.$$

Thus $h_B \in L^q(B)$ for $0 < q < 1$, and hence $u \in M^{1,q}(B)$ in the sense of [HajC], and so by [HajC, Corollary 8.9 of page 202] or by [KLS, Proposition 2.4], we know that

$$\int_B |u - u_B| d\mu \leq CR \frac{\nu(B)}{\mu(B)}.$$

The proof is then completed by taking a sequence of sequences $(g_i^j)_i$ that are BV_{AM} -upper bound of u with corresponding measures ν_j such that $\lim_j \nu_j(2B) = \|D_{AM}u\|(2B)$. \square

From Proposition 3.9, Theorem 3.7, and Proposition 3.3 we have the following.

Theorem 3.10 *Let μ be a doubling measure on X . Then the following are equivalent:*

1. X supports a 1-Poincaré inequality.
2. X supports a Semmes pencil of curves.
3. X supports an AM-Poincaré inequality.

In any (and therefore all) of the above, we have $BV(X) = BV_{AM}(X)$ and $N^{1,1}(X) = N_{AM}^{1,1}(X)$.

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Addresses:

E.D-C.: Departamento de Matemática Aplicada, ETSI Industriales, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain.
E-mail: edurand@ind.uned.es

S.E.-B.: UCLA Mathematics Department, P.O. 951555, Los Angeles, CA 90095–1555, U.S.A.
E-mail: syerikss@math.ucla.edu

R.K.: Aalto University, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland
E-mail: riikka.korte@aalto.fi

N.S.: Department of Mathematical Sciences, P.O.Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: shanmun@uc.edu