TRAVELLING WAVES IN THE DISCRETE STOCHASTIC NAGUMO EQUATION

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Abstract. Many physical, chemical and biological systems have an inherent discrete spatial structure that strongly influences their dynamical behaviour. Similar remarks apply to internal or external noise, as well as to nonlocal coupling. In this paper we study the combined effect of nonlocal spatial discretization and stochastic perturbations on travelling waves in the Nagumo equation. We prove that under suitable parameter conditions, various discrete-stochastic variants of the Nagumo equation have solutions, which stay close on long time scales to the classical monotone Nagumo front with high probability if the noise level and spatial discretization are sufficiently small.

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1. Introduction

The Nagumo [33] partial differential equation (PDE) for $V = V(t,x) \in \mathbb{R}$ is given by

$$\partial_t V = \nu \partial_x^2 V + f(V), \qquad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{Nag$\mathbb{R}$}$$

where $f(V) = f(V;a) = V(1-V)(V-a)$, where $a \in (0,1/2)$ and $\nu > 0$ are parameters. The Nagumo equation is bistable in the sense that $V \equiv 0$ and $V \equiv 1$ are locally asymptotically steady states, while $V \equiv a$ is unstable. For any $a \in (0,1/2)$ the PDE (Nag$\mathbb{R}$) admits travelling front solutions $V(t,x) = V^{\mathrm{TW}}(x-ct) = V^{\mathrm{TW}}(\zeta)$ connecting the two locally stable states, i.e., $V^{\mathrm{TW}}(-\infty) = 0$ and $V^{\mathrm{TW}}(\infty) = 1$. The front is monotonic $(V^{\mathrm{TW}})'(\zeta) > 0$, left-moving with a unique wave speed satisfying $c = c(a) < 0$, unique up to translation, and (locally) nonlinearly stable. Extensions to the standing wave for $a = 1/2$, and to right-moving waves for $a \in (1/2,1)$ are easily obtained from symmetry arguments.

The Nagumo equation plays an important role in neuroscience [13] as the simplest toy model of signal propagation through axons. It is very actively studied also outside neuroscience applications as an amplitude equation [10], in population dynamics modelling [6], and in materials science [2]. When modelling signal propagation in neurons, several effects are not taken into account in (Nag$\mathbb{R}$):

- (I) The electric signals travelling through a myelinated nerve fiber do not move continuously. The signal jumps from one gap in the myeline coating of the nerve fiber to the next [26]. This suggests the use of a *spatially discrete* setting.
- (II) The propagation of the electric signal along the axon is influenced by many internal and/or external biophysical processes. Since modelling every process microscopically is usually impossible, this leads naturally to a *stochastic* version of the Nagumo equation.
- (III) The precise coupling distance of diffusion between myeline coating gaps is not easy to measure. This implies we should also allow *nonlocal* coupling terms.
- (IV) The axon does not have infinite length. Hence, one should consider *bounded domains* instead. Furthermore, propagation takes place on a *finite time scale.*
- (V) The propagation of fronts is an idealization of the eletrical signal as usually we would expect localized pulses. This requires *systems* of reaction-diffusion equations.

Here we shall not cover the case (V), which is usually modelled using the Hodgkin-Huxley [19] or FitzHugh-Nagumo [14] PDEs. However, all the arguments we present can be carried over, in principle, to these cases. Instead, we focus on a model to cover the *combined* effects (I)-(IV). In fact, each of the individual aspects (I)-(IV) have received some attention recently. We briefly review some background and introduce the relevant PDEs.

The space-discrete setting will be modeled via a lattice differential equation (LDE), whose solution at node i, called $V_i = V_i(t)$, represents the potential at the i-th myeline gap. The discrete Nagumo equation with nonlocal diffusive coupling, reads at each node i for some fixed coupling range $R \in \mathbb{N}$ as follows

$$\partial_t V_i = \frac{\nu}{Rh^2}\left(\sum_{j=-R}^{R} J(j)(V_j - V_i)\right) + f(V_i), \qquad i \in \mathbb{Z}, \tag{1.1}$$

where h is a parameter controlling the discretization and $J(j) \in \mathbb{R}$ are weights. The classical case of local diffusive coupling is given by

$$\partial_t V_i = \frac{\nu}{h^2}(V_{i+1} - 2V_i + V_{i-1}) + f(V_i), \qquad i \in \mathbb{Z}, \tag{1.2}$$

The equation (1.2) is the nearest-neighbor discretization of (Nag$\mathbb{R}$). The Nagumo LDE can be interpreted as being posed on an infinite lattice $\mathbb{Z}$ with lattice spacing h so that V_i corresponds to $V(ih)$. We write

$$V^h := (\dots, V_{-2}, V_{-1}, V_0, V_1, V_2, \dots) \tag{1.3}$$

to emphasize that V^h solves the discrete Nagumo equation. The LDE (1.2) also admits travelling wave solutions for sufficiently strong diffusion strength ν, i.e., for sufficiently large coupling; for small coupling, propagation failure may occur [26, 20, 32]. More generally, the type of the discrete model may have substantial impact on the existence and uniqueness of travelling waves of the Nagumo and FitzHugh-Nagumo PDEs [11, 12, 22] as well as on the numerical analysis of discretization schemes for travelling waves [16].

Notice that in (1.1) the general difference stencil involves $2R$ nodes, and R may diverge with N, so it can be viewed as nonlocal. In fact, nonlocal variants of the Nagumo equation have been studied in the LDE/PDE setting in several analytical and numerical works; see e.g. [1, 3, 4, 8, 11] and references therein. A similar difference stencil as used here was studied in [3, 21], where existence of travelling wave solutions was proven for unbalanced nonlinearities and under certain conditions on the weights.

Another important variation of the Nagumo equation is the stochastic PDE (SPDE) version for $U = U(t,x)$ given by

$$\partial_t U = \nu \partial_x^2 U + f(U) + g(U)\xi, \qquad (t,x) \in [0,\infty) \times \mathbb{R} \tag{SNag$\mathbb{R}$}$$

where $\xi = \xi(t,x)$ is a space-time dependent stochastic process and g arises as a suitable mapping from modelling considerations. Although there is a detailed existence theory for many SPDEs [35, 9, 29] going back to at least the late 1970s, and good physical understanding of many noisy pattern phenomena going back at least to the 1990s [15], the rigorous mathematical study of noisy (Nagumo) waves has just started to develop recently; see e.g. [18, 23, 28, 38]. These studies have been driven by numerical observations [31, 37, 39] revealing that travelling wave solutions may persist under stochastic forcing, but their speed and form may change with varying noise strength. Of course, these results are also connected to recent advances in the numerical analysis of classical numerical schemes for the Nagumo SPDE [36]; see also e.g. [17, 24, 30].

In this paper we are interested in the combined influence of (I)-(IV) on the finite-time evolution of travelling fronts. In this context, the key object is the stochastic LDE (SLDE)

$$\mathrm{d}u_i = \left(\frac{\nu}{Rh^2}\left(\sum_{j=-R}^{R} J(j)(u_j - u_i)\right) + f(u_i)\right)\,\mathrm{d}t + \sqrt{\mu_i}\, g(u_i(t))\,\mathrm{d}B_i(t), \tag{dSNag$N$}$$

with $i \in \{1, 2, \dots, N\}$, $u_i = u_i(t)$, independent Brownian Motions $B_i(t)$, constants $\nu > 0$, $\mu_i \geq 0$, $a \in (0, 1/2)$, and $R \leq N$ with $R \in \mathbb{N}$, where Neumann boundary conditions are used. In addition to viewing the solution as a vector

$$u^h := (u_1, u_2, \dots, u_N) \tag{1.4}$$

we may interpret the solution u^h, say via piecewise linear interpolation, as a function on the domain $\mathcal{D} := [-L, L]$ with u_1 and u_N corresponding to the values at the left and right endpoints. Despite its evident importance for applications, particularly in the context of neuroscience, there seems to be no study available regarding the *dynamics* of (dSNag$N$) although some first study without reference to dynamics is [5]. One potential reason could be that physical intuition would lead us to believe that the effects coming from (I)-(IV) are somehow "small" so that we can neglect them. To make this intuition mathematically precise is a key contribution of our study. For parameters for which travelling waves to the deterministic PDE are known to exist, we prove in the stochastic setting that for sufficiently small h and sufficiently small noise that the solution to the Nagumo SLDE (dSNag$N$) is close to a phase-adapted travelling front solution of the Nagumo PDE (Nag$\mathbb{R}$) over finite time scales. Our main result can informally be stated as follows:

Theorem 1.1. *Let $V^{\mathrm{TW}} = V^{\mathrm{TW}}(t,x)$ the travelling front solution to* (Nag$\mathbb{R}$). *Suppose $\delta > 0$ and $\tilde{\varepsilon} > 0$ are given. Then there exists $\varepsilon > 0$ such that if $u^h(0)$ is deterministic with*

$$\|u^h(0) - V^{\mathrm{TW}}(0,\cdot)\|^2_{L^2(\mathbb{R})} < \varepsilon. \tag{1.5}$$

and for sufficiently small noise and sufficiently small h, we have for the solution $u^h = u^h(t)$ of (dSNag$N$) *the estimate*

$$\mathbb{P}\left[\sup_{t\in[0,t_*]} \|u^h(t) - V^{\mathrm{TW}}(t,\cdot)\|_{L^2(\mathbb{R})} > \delta\right] \le \tilde{\varepsilon} \tag{1.6}$$

for some $t_* > 0$.

The precise formulation of "sufficiently small noise" will be stated in Theorem 3.13, it mainly deals with a sufficiently small covariance of the underlying Wiener process, and the growth of g. The time t_* will be made more precise in Lemma 3.12. In summary, Theorem 1.1 confirms our intuition from biophysics/neuroscience, i.e., the wave propagation mechanism must be robust against structural perturbations to make the Nagumo equation a good model.

Our proof relies on a discrete version of the monotone operator theory approach to SPDEs, as presented in [34] and described in the monographs [9, 29]. We also make use results obtained in the pathwise stability analysis for the continuous Nagumo SPDE from [38]. Our proof essentially decomposes the different error terms [27], e.g., the dynamical stochastic approximation error is treated separately from the discretization error in the stencil. Therefore, it is natural to consider several intermediate evolution equations, e.g., the Nagumo PDE on a bounded domain

$$\partial_t v = \nu\partial_x^2 v + f(v), \qquad (t,x) \in [0,\infty)\times\mathcal{D}, \tag{Nag$\mathcal{D}$}$$

and the Nagumo SPDE on a bounded domain

$$\partial_t u = \nu\partial_x^2 u + f(u) + g(u)\xi, \qquad (t,x) \in [0,\infty)\times\mathcal{D}. \tag{SNag$\mathcal{D}$}$$

Hopefully, our notation conventions are by now already evident to the reader but let us stress again that we use v, V for the deterministic PDE solutions whereas u, U are SPDE solutions, while small letter solutions u, v are based on the bounded domain $\mathcal{D}$ and capital letter solutions U, V on the unbounded domain $\mathbb{R}$. Furthermore, discrete solutions will be treated as vectors u^h, U^h, v^h, V^h or indicated by subindices.

2. Notation and Setting

We discretize the bounded domain $\mathcal{D} \subset \mathbb{R}$ into N intervals of size h and enumerate the respective grid points with the index i. The set of grid points is denoted by $\mathcal{D}^h$.

We work with the Gelfand triple of Banach spaces $H_0^1(\mathcal{D}) \cong W_0^{1,2}(\mathcal{D}) \subset L^2(\mathcal{D}) \subset H^{-1}(\mathcal{D})$.

Note that functions in the Sobolev spaces such as $L^2(\mathcal{D})$ or $W_0^{1,2}(\mathcal{D})$ evaluated on the grid $\mathcal{D}^h$ are N-dimensional vectors. Extending these functions in a piecewise linear manner, we can work with them also in the original Sobolev spaces. We choose a orthonormal basis $\{e_k\} \subset L^2(\mathcal{D})$, consisting of elements in $W_0^{1,2}(\mathcal{D})$, and span $\mathbb{R}^N$ with the first N of these basis vectors. The projections of the Sobolev spaces on their first N basis vectors can then be identified with $\mathbb{R}^N$, e.g. $P_N H_0^1(\mathcal{D}) \cong \mathbb{R}^N$.

To simplify notation, we denote both the scalar product on $L^2(\mathcal{D})$ and on $\mathbb{R}^N = P_N L^2(\mathcal{D})$ by $(\cdot,\cdot)$ and the norm by $\|\cdot\|$ and by $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{H^{-1}(\mathcal{D}),H_0^1(\mathcal{D})}$ both the dual product and the scalar product on $\mathbb{R}^n \cong P_N H_0^1(\mathcal{D})$, the projection to $\mathbb{R}^N$ spanned by the first N basis vectors of $H_0^1(\mathcal{D})$. Using the representation

$$w = \sum_{k=1}^{N} \langle w, e_k\rangle e_k \qquad \text{for all elements } w \in H^{-1}(\mathcal{D}^h), \tag{2.1}$$

we can work with the same basis vectors also in the space $H_0^1(\mathcal{D})$ and its dual.

2.1. **Operations with discrete-in-space functions.** For $u^h(.,t)$ a piecewise linear function on the grid $\mathcal{D}^h$, there are several ways to define a (discrete) gradient. Using only two nodal values, we can identify $\nabla^h u^h(ih,t)$ either with the backward difference $D^- u_i(t) = \frac{1}{h}\left(u_i(t) - u_{i-1}(t)\right)$ or its adjoint, the forward difference $D^+ u_i(t) = \frac{1}{h}\left(u_{i+1}(t) - u_i(t)\right)$. This choice leads to the discrete nearest-neighbour Laplacian as $\Delta^h u_i = D^+ D^- v_i := h^{-2}\left(u_{i+1} - 2u_i + u_{i-1}\right)$.

We would like to use more general discrete stencils, which involve up to R neighbours of u_i in each direction, in other words involving the nodal values $u_{i-R}\dots u_{i+R}$. We introduce coefficients $J(j) \in \mathbb{R}$ to attribute a weight of the j-th right neighbouring nodal value u_{i+j}. Such a general second-order stencil then reads

$$\begin{aligned}
\Delta_R^h u_i &= \frac{1}{h^2}\sum_{j=-R}^{R} J(j)(u_j - u_i)\\
&= \frac{1}{h^2}\left(-2\sum_{k=1}^{R} J(k)u_i \;+\; \sum_{k=1}^{R} J(k)u_{i+k} \;+\; \sum_{k=1}^{R} J(k)u_{i-k}\right)\\
&= \frac{1}{h^2}\sum_{k=1}^{R} J(k)\left(-2u_i \;+\; u_{i+k} \;+\; u_{i-k}\right)\\
&= \frac{1}{h}\left(\frac{1}{h}\sum_{k=1}^{R} J(k)(u_{i+k} - u_i) \;-\; \frac{1}{h}\sum_{k=1}^{R} J(k)(u_i - u_{i-k})\right).
\end{aligned} \tag{2.2}$$

Note that $J(j)$ are fixed numbers and do not change with time, therefore the central difference operator Δ_R^h it is deterministic and time-independent. It is therefore natural to define

$$\nabla_R^- u_i := (\Delta_R^h)^{1/2} u_i = \frac{1}{h}\sum_{k=1}^{R} J(k)(u_i - u_{i-k}) \tag{2.3}$$

as the long-range analogues of the difference operators D^-. Using the adjoint operator to ∇_R^-, denoted by ∇_R^+, we can write (2.2) as

$$\Delta_R^h u_i = \nabla_R^+(\nabla_R^- u_i). \tag{2.4}$$

We need to impose conditions on the coefficients $J(j)$ to ensure that (2.2) approximates a Laplacian. To this aim, notice first of all that by construction $J(0) = -\sum_{j=-R,j\neq 0}^{R} J(j)$, which is a special case of diagonal dominance, from which we immediately follow $\langle \Delta_R^h u^h, u^h\rangle \leq 0$.

Assumption 2.1. *The weights $J(j) \in \mathbb{R}$ satisfy*

(1) $J(j) = J(-j)$
(2) $\sum_{j=-R}^{R} J(j)j^2 = 1$
(3) $\sum_{j=-R}^{R} J(j)j^4 < \infty$ *or at least* $h^4\sum_{j=-R}^{R} J(j)j^4 \sim o(h^3)$

The symmetry condition assures that the operator Δ_R^h is self-adjoint in ℓ^2, it is not strictly necessary mathematically, but it simplifies computations and is moreover very natural considering the real-world phenomena from which the model was derived. The moment conditions ensure that we approximate a Laplacian, as can be seen immediately from the construction of finite difference operators via Taylor approximation at nodal distance jh, which gives

$$\sum_{j=-R}^{R} J(j)(v_j - v_i) \;=\; h^2\Big(\sum_{j=-R}^{R} j^2 J(j)\Big)\partial_{xx} v_i \;+\; \frac{h^4}{12}\Big(\sum_{j=-R}^{R} j^4 J(j)\Big)\partial_{xxxx} v_i \;+\; O(h^6). \tag{2.5}$$

Note that, in contrast to [5], the assumptions on the moments imply a certain decay in the coefficients $J(j)$. This is because we allow for arbitrarily diverging stencil range R, especially also for $R = N$, while in the semigroup approach used in [5], the range was limited to $R \sim N^{1/2}$.

2.2. **The probabilistic setting.** Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ a filtered probability space.

We denote by $W : \Omega \times [0,T] \to L^2(\mathcal{D})$ a Q-Wiener process with values in $L^2(\mathcal{D})$ and we assume that $W(t)$ is adapted to the filtration $\mathcal{F}_t$. A function $u : \mathcal{D} \times [0,T] \times \Omega \to \mathbb{R}$, which is evaluated on the grid $\mathcal{D}^h$ will be denoted by $u^h(\cdot, t, \omega)$ and at each node identified with a stochastic process $X_t^i(\omega)$, which takes values in $\mathbb{R}$.

We assume that the covariance operator Q is linear, bounded, self-adjoint, positive semidefinite. Moreover, it is convenient to assume that Q has a common set of eigenfunctions with Δ. We fix notation as

$$Qe_k = \mu_k e_k. \tag{2.6}$$

We assume that Q is of trace class, i.e. $M := \operatorname{Tr} Q < +\infty$, which implies that the sum of the eigenvalues of Q is bounded $\sum_{k=1}^{\infty} \mu_k < \infty$. It is well known that a Q-Wiener process in $L^2(\mathcal{D})$ can be approximated in $L^2(\Omega, C([0,T], L^2(\mathcal{D})))$ by a sequence of i.i.d. Brownian motions $\{B_j\}_{j\in\mathbb{N}}$ as

$$W(x,t) = \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) B_k(t). \tag{2.7}$$

By means of an exponential inequality and Borel-Cantelli Lemma, the convergence can be obtained uniformly with probability one. Thus, the sample paths of $W(t)$ belong to $C([0,T], L^2(\mathcal{D}))$ almost surely, and we may therefore choose a continuous version.

2.3. **The stochastic Nagumo equation.** The stochastic Nagumo equation we are using in this work is a perturbation of the (deterministic) Nagumo equation (Nag$\mathbb{R}$). As stochastic perturbation, we choose a Q-Wiener Process $W(t)$ on $L^2(\mathcal{D})$ with covariance operator Q being positive semi-definite, symmetric and of trace class. Moreover, we take a multiplicative noise term called $g(u) : L^2(\mathcal{D}) \to \mathcal{H}$, where we denote by $\mathcal{H}$ the space of Hilbert-Schmidt operators, and assume it is Lipschitz continuous and satifies linear growth conditions.

More precisely, we assume

$$\|g(u)\|_{\mathcal{H}}^2 \;\leq\; c(1 + \|u\|^2) \qquad \text{a. e. } (t,\omega) \in \Omega \times [0,T] \tag{2.8}$$

and

$$\|g(u) - g(v)\|_{\mathcal{H}}^2 \;\leq\; c(\|u - v\|^2) \qquad \text{a. e. } (t,\omega) \in \Omega \times [0,T] \tag{2.9}$$

for all $u, v \in L^2(\mathcal{D})$.

The stochastic Nagumo equation then reads

$$du(t) = [\nu \partial_{xx}^2 u(t) + bf(u(t))]dt + g(u(t))dW(t) \qquad \text{on } \mathcal{D} \times [0,T]. \tag{2.10}$$

The existence of mild solutions to (2.10) for Lipschitz nonlinearities is classical, see e.g. [35]. Using a localization and truncation argument, see e.g. [7, 5], the results can be carried over to the polynomial nonlinearity f with one-sided lipschitz condition such as in (2.10). Via monotone operator theory, we see furthermore [34] that (2.10) admits a variational solution in $L^2(\Omega, C([0,T], L^2)) \cap L^2(\Omega \times [0,T], H_0^1)$. In particular, we have that almost surely $u \in L^\infty([0,T]; H) \cap L^2([0,T]; V)$.

Due to Itô's formula, the stochastic Nagumo equation satisfies an energy equation of the form

$$\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] = \|u(0)\|_{L^2}^2 + 2\nu\mathbb{E}\left[\int_0^t (\Delta u(s), u(s))\,ds\right] + 2b\,\mathbb{E}\left[\int_0^t (f(u(s)), u(s))\,ds\right] + \mathbb{E}\left[\int_0^t g(u(s))^2 ds\right]. \tag{2.11}$$

The stochastic LDE. we consider in this work is the discrete-in-space evolution

$$du_i(t) = \frac{\nu}{Rh^2}\sum_{j=-R}^{R} J(j)(u_j - u_i)dt + bf(u_i(t))dt + g(u_i(t))dW_i(t) \qquad i = 1 \dots N, \tag{2.12}$$

where we introduced a general difference stencil with coefficients $J(j) \in \mathbb{R}$ satisfying Assumption 2.1 and denoted by $W_i(t) = \sqrt{\mu_i}B_i(t)$. The discrete multiplicative noise operator is defined by projection of the continuous Hilbert-Schmidt operator g (without changing the notation). It then obviously satisfies (2.9) and (2.8).

We may sometimes call (2.12) the "generalized Nagumo LDE" and will often use the convenient abbreviation $\Delta_R^h u_i := \frac{1}{Rh^2}\sum_{j=-R}^{R} J(j)(u_j - u_i)$ to indicate its property as a (generalized) discretization of the Laplace operator, a fact which will be justified below.

2.4. **Monotone operators.** The following paragraph recalls that, thanks to the properties of f, the sum $\nu\Delta + bf(u)$ defines a monotone operator. In the continuous context, this is well-understood: a concise treatment of the theory of monotone operators can be found for example, in [40], or, including local monotonicity, in [29]. We will briefly state those precise properties which will be used in the proofs.

First, we note that the nonlinear term $f(u)$ is Lipschitz continuous w.r.t u on bounded subsets of $H_0^1(\mathcal{D})$ with lipschitz constant independent of t.More precisely, we have the following Lemma:

Lemma 2.2. *For any $M > 0$, there exists a constant $K_M > 0$ such that the local Lipschitz continuity condition holds:*

$$\|f(v_1) - f(v_2)\|^2 \le K_M\|v_1 - v_2\|^2_{H_0^1(\mathcal{D})} \qquad \text{a. e. } (t,\omega) \in [0,T]\times\Omega \tag{2.13}$$

for any $v_1, v_2 \in H_0^1(\mathcal{D})$ with $\|v_1\|^2_{H_0^1(\mathcal{D})} < M$ and $\|v_2\|^2_{H_0^1(\mathcal{D})} < M$

Proof. We have

$$\|v_1^3 - v_2^3\|^2 = \|(v_1^2 + v_1v_2 + v_2^2)(v_1 - v_2)\|^2 \le 8\left(\|v_1^2(v_1 - v_2)\|^2 + \|v_2^2(v_1 - v_2)\|^2\right) \tag{2.14}$$

By Sobolev embedding, we can get for some constants $C_1, C_2 > 0$ the estimates $\|v_1\|_{L^4} \le C_1\|v_1\|_{H_0^1}$ and $\|v_1^2 v_2\|^2_{L^2} \le C_2\|v_1\|^4_{H_0^1}\|v_2\|^2_{H_0^1}$. Hence, there exists a constant $C_3 > 0$ such that

$$\|v_1^3 - v_2^3\|^2 \le C_3\left(\|v_1\|^4_{H_0^1} + \|v_2\|^4_{H_0^1}\right)\|v_1 - v_2\|^2_{H_0^1} \tag{2.15}$$

which satisfies (2.13). ☐

The following estimates, derived similar to 2.2, give us linear growth estimates in the dual norm of the sum $A := \nu\Delta + b$:

$$\|f(v)\|_{H^{-1}(\mathcal{D})} \le c_1\|v\|_{H_0^1(\mathcal{D})}\left(1 + \|v\|^2_{L^2(\mathcal{D})}\right) \tag{2.16}$$

$$\|f(v_1) - f(v_2)\|_{H^{-1}(\mathcal{D})} \le c_2\left(1 + \|v_1\|^2_{H_0^1(\mathcal{D})} + \|v_2\|^2_{H_0^1(\mathcal{D})}\right)\|v_1 - v_2\|_{L^2(\mathcal{D})} \tag{2.17}$$

Moreover, the combined operator $A := \nu\Delta + bf$ is obviously hemicontinuous in $H_0^1(\mathcal{D})$, in the sense that for all $v_1, v_2, v_3 \in H_0^1(\mathcal{D})$ and $t \in [0,T]$ the mapping

$$\theta \mapsto \langle A(v_1 + \theta v_2), v_3\rangle \tag{2.18}$$

is continuous from $\mathbb{R}$ into $\mathbb{R}$.

Thanks to (2.9) and (2.8), we know [34, 9, 29, 38] that the sum of operators satisfies for all $v \in H_0^1(\mathcal{D})$, $t \in [0,T]$ the coercivity condition

$$\langle \nu\Delta v + bf(v), v\rangle + \|g(v)\|_{\mathcal{H}} \leq -\nu\|v\|^2_{H_0^1(\mathcal{D})} + (bc_a + \nu)\|v\|^2_{L^2(\mathcal{D})} \tag{2.19}$$

with the quantity $c_a = \sup_{\xi\in\mathbb{R}} f'(\xi) = \frac{1}{3}\big(a^2 + a + 1\big)$ holds.

Finally, the sum of operators satisfies for all $v_1, v_2 \in H_0^1(\mathcal{D})$ the monotonicity condition

$$\langle \nu\Delta v_1 + bf(v_1) - \nu\Delta v_2 - bf(v_2), v_1 - v_2\rangle + \|g(v_1) - g(v_2)\|^2_{\mathcal{H}} \leq c_a \cdot b\|v_1 - v_2\|^2 \tag{2.20}$$

on $[0,T] \times \Omega$.

We will now verify that analogues properties hold in the discrete setting of our LDE (2.12). The proof is elementary, yet we will write it in detail to make our strategy transparent to the reader.

Lemma 2.3. *Let the conditions of Assumption 2.1 on the general stencil* Δ_R^h *be satisfied. Then the discrete operators appearing in* (2.12) *satisfy the following estimates:*

(1) coercivity

$$\sum_{i=1}^N \big(\nu\Delta_R^h u_i + bf(u_i)\big)\, u_i + \|g(u_i)\|^2 \leq -\nu\|\nabla_R^- u^h\|^2 + bc_a\|u^h\|^2 \tag{2.21}$$

(2) monotonicity

$$\sum_{i=1}^N \big(\nu\Delta_R^h u_i - \nu\Delta_R^h v_i + bf(u_i) - bf(v_i)\big)\,(u_i - v_i) + \|g(u_i) - g(v_i)\|^2 \leq c_a \cdot b\|u^h - v^h\|^2. \tag{2.22}$$

Proof. First, note that

$$\sum_{i=1}^N f(u_i)\cdot u_i = \sum_{i=1}^N (f(u_i) - f(0))\,(u_i - 0) = \sum_{i=1}^N \left(\frac{f(u_i) - f(0)}{u_i - 0}\right)(u_i - 0)^2 \leq c_a \sum_{i=1}^N u_i^2 \tag{2.23}$$

where we used the mean-value theorem in the last inequality. Second, we look at the discrete integration by parts formula, which reads in its standard form

$$\sum_{i=1}^N \Delta^h u_i \cdot u_i = \sum_{i=1}^N D^+(D^- u_i)\cdot u_i = -\sum_{i=1}^N (D^- u_i)\cdot D^- u_i = -\sum_{i=1}^N (D^- u_i)^2. \tag{2.24}$$

and can be extended to the long-range case via the operators ∇_R^- and ∇_R^+. Using these two simple tools, we can derive easily, in the special case of the nearest-neighbour stencil, the coercivity

$$\sum_{i=1}^N \big(\nu D^+(D^- u_i) + bf(u_i)\big)\, u_i + \|g(u_i)\|^2 \leq -\nu\|D^- u^h\|^2 + bc_a\|u^h\|^2 \tag{2.25}$$

where we used (2.8). Using (2.9), we get the monotonicity of the sum of operators

$$\sum_{i=1}^N \big(\nu D^+(D^- u_i) - \nu D^+(D^- v_i) + bf(u_i) - bf(v_i)\big)\,(u_i - v_i) + \|g(u_i) - g(v_i)\|^2 \leq c_a \cdot b\|u^h - v^h\|^2. \tag{2.26}$$

For the general case, it was already noted (without proof) by Bates, Chen, Chmai [3], that general stencils of the form (2.2) satisfy the monotonicity condition with $c = 0$. Indeed, testing with u^h and using the summation by parts formula we obtain

$$\begin{aligned} \langle \Delta_R^h u^h, u^h\rangle &= \frac{1}{h^2}\sum_{i=1}^N \left(\sum_{j=-R}^R J(j)(u_j - u_i)\right)\cdot u_i \\ &= \sum_{i=1}^N \nabla_R^+(\nabla_R^- u_i)\cdot u_i \\ &= -\langle \nabla_R^- u^h, \nabla_R^- u^h\rangle = -\|\nabla_R^- u^h\|^2 \leq 0. \end{aligned} \tag{2.27}$$

Repeating the strategy of the nearest neighbour case, using again (2.23), (2.8), (2.9) and (2.27) gives us immediately (2.21) and (2.22), which concludes the proof. □

3. Existence and stability of travelling wave fronts

Our goal is show that the stochastic LDE (2.12) admits, for sufficiently small h, travelling front solutions, in the sense that its solutions are very likely to be close to deterministic travelling fronts.

To this end, we use several ingredients: first, results on the existence and properties of solutions to the LDE (2.12), second, the convergence of the solutions of the h approximations to the solution of (2.10) and third, results on stability of traveling wave fronts for (2.10).

The rest of this paper is organized along these three ingredients: In section 3.1, we investigate existence and properties of solutions to the LDE (2.12), which is followed by the convergence result in section 3.2. In section 3.4, we finally obtain the main result, which can be summarized in an informal way as follows:

Theorem 3.1 (Main Theorem, informal version)**.** *Let v^{TW} the deterministic travelling front solution to* (Nag$\mathbb{R}$)*. Then, starting with initial data sufficiently close to the traveling wave, the solution to the discrete generalized stochastic Nagumo equation* (2.12) *are very likely to be close to deterministic travelling fronts.*

3.1. A priori estimates.

For the rest of the paper, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathcal{D})}$. Rigorous results on the existence and properties of solutions to equation (2.12) are obtained in the classical framework of strong solutions of stochastic differential equations (SDE). We will not repeat the entire theory here, but summarize the line of arguments in the first part of Lemma 3.3 below. The main part of this section is the following a priori estimate:

Proposition 3.2. *Let the initial data $u_0 \in C^4(\mathcal{D})$ be deterministic. Then solution u^h of* (2.12) *satisfies for any h*

$$\mathbb{E}\left[\int_0^T \|\nabla_R^- u^h(t)\|^2 dt \; + \; \sup_{t\le T} \|u^h(t)\|^2\right] < \infty. \tag{3.1}$$

The proof of Proposition 3.2 is split into three parts, which are Lemmata 3.3, 3.4 and 3.5.

Lemma 3.3. *For any $h > 0$, the solution u^h of* (2.12) *exists and satisfies an energy equality*

$$\begin{aligned} \|u^h(t)\|_{L^2}^2 \;&=\; \|u^h(0)\|_{L^2}^2 \;+\; 2\nu \int_0^t \left\langle \Delta_R^h u^h(s), u^h(s)\right\rangle ds \\ &+\; 2b \int_0^t \big(f(u^h(s)), u^h(s)\big)\, ds \;+\; 2\int_0^t g(u^h(s))\, \big(u^h(s), dW^h(s)\big) + \int_0^t g(u^h(s))^2 ds \end{aligned} \tag{3.2}$$

Proof. The stochastic LDE (2.12) is just a system of SDEs, which is rigorously stated in integral form

$$u_i(t) = u_0 + \frac{\nu}{h^2}\int_0^t \sum_{j=-R}^{R} J(j)(u_j - u_i)ds \;+\; b\int_0^t f(u_i(s))ds + \int_0^t g(u_i)dW_i(s)\;, \quad i = 1 \dots N. \tag{3.3}$$

From (3.3) it is easily seen that for each single i, the stochastic LDE is in fact an Itô equation, for which it is well known (see e.g. [25]) that there exists a solution, which is an adapted process.

Moreover, as u_i are the coefficients of u^h in the basis of $\mathbb{R}^N \cong P_N H_0^1(\mathcal{D})$, (3.3) is a finite dimensional Itô equation, which therefore has a solution as an adapted process. This adapted process also has a continuous version, i.e., $u^h \in C([0,T], V^h)$.

To derive the energy equation, for some fixed $h = \frac{1}{N}$ we build a solution vector u^h via the integral form of the stochastic LDE (2.12), equation (3.3). We can write, using an orthonormal basis e_i of

$V^h = P_N H_0^1$, in the following form

$$(3.4)\qquad \begin{aligned} \big(u^h(t), e_i\big) \;=\; \big(u_0^h, e_i\big) \;+\; 2\nu \int_0^t \big\langle \Delta_R^h u^h, e_i\big\rangle\, ds + 2b \int_0^t \big(f(u^h), e_i\big)\, ds \\ +\; \int_0^t g(u^h)\Big(e_i, dW^h(s)\Big). \end{aligned}$$

Next, in (3.4) we sum up from $i = 1 \dots N$ and apply Itô's formula in finite dimensions to get

$$(3.5)\qquad \begin{aligned} \|u^h(t)\|^2 \;=\; \|u^h(0)\|^2 \;+\; 2\nu \int_0^t \big\langle \Delta_R^h u^h(s), u^h(s)\big\rangle\, ds \\ +\; 2b \int_0^t \big(f(u^h(s)), u^h(s)\big)\, ds \;+\; 2\int_0^t g(u^h(s))\,\big(u^h(s), dW^h(s)\big) \;+\; \int_0^t g(u^h(s))^2 ds \end{aligned}$$

which is exactly (3.2). □

With the way of writing (3.4) we already point to the fact that we consider the stochastic LDE as a generalized Galerkin approximation of the stochastic Nagumo equation. Indeed, recalling that (2.10) has a solution whose trajectory is in $L^2([0,T], H_1(\mathcal{D}))$, $\Delta u(\cdot) \in L^2([0,T], H^{-1}(\mathcal{D}))$ and we may write the stochastic Nagumo equation in weak form, using the scalar product $\big(\cdot,\cdot\big)_{L^2(\mathcal{D})}$ and the dual product $\big\langle \cdot,\cdot\big\rangle$

$$(3.6)\qquad \begin{aligned} (u(t), \varphi) \;=\; \big(u_0^h, \varphi\big) \;+\; \nu \int_0^t \langle \Delta u(s), \varphi\rangle\, ds \;+\; b \int_0^t (f(u(s)), \varphi)\, ds \\ +\; \int_0^t (g(u(s)), \varphi)\, dW(s). \end{aligned}$$

Recalling (2.7), we can interpret the stochastic integral term as

$$(3.7)\qquad \int_0^t \Big(g(u(s)), \varphi\Big) dW(s) = \sum_{k=1}^{\infty} \int_0^t \Big(g_k(u(s)), \varphi\Big) dW_k(s) = \int_0^t \Big(\varphi, g(u(s)) dW(s)\Big)$$

and relate the notation of the stochastic integral term in (3.5) to the weak form (3.6).

Lemma 3.4. *For initial data $u_0 \in C^4(\mathcal{D})$ deterministic, the discrete solution u^h of (2.12) is uniformly bounded in $L^2(\Omega \times [0,T], H_0^1(\mathcal{D}))$, i.e.*

$$(3.8)\qquad \sup_h \mathbb{E}\left[\int_0^T \|\nabla_R^- u^h(t)\|_{L^2(\mathcal{D})}^2 dt\right] \;<\; \infty.$$

Proof. We start with the energy equation (3.2). Taking the expectation, the stochastic integral is zero and we arrive at

$$(3.9)\qquad \begin{aligned} \mathbb{E}\left[\|u^h(t)\|^2\right] -\; \mathbb{E}\left[\|u^h(0)\|^2\right] \;=\; 2\nu \mathbb{E}\left[\int_0^t \big(\Delta_R^h u^h(s), u^h(s)\big)\, ds\right] \\ +\; 2b\, \mathbb{E}\left[\int_0^t \big(f(u^h(s)), u^h(s)\big)\, ds\right] \;+ \mathbb{E}\left[\int_0^t g(u^h(s))^2 ds\right] \end{aligned}$$

Abbreviate now the right hand side of (3.9) by $RHS \;:=\; 2\nu\mathbb{E}\left[\int_0^t \big(\Delta_R^h u^h(s), u^h(s)\big)\, ds\right] + 2b\mathbb{E}\left[\int_0^t \big(f(u^h(s)), u^h(s)\big)\, ds\right] + \mathbb{E}\left[\int_0^t g(u^h(s))^2 ds\right]$. We use the coercivity estimate (2.21) to get that the right hand side of (3.9) satisfies

$$(3.10)\qquad RHS \;\leq -\; 2\nu \mathbb{E}\left[\int_0^t \|\nabla_R^- u^h(s)\|^2\, ds\right] + 2(bc_a + \nu)\mathbb{E}\left[\int_0^t \|u^h(s)\|^2 ds\right]$$

which is, as the initial data is deterministic,

$$\begin{aligned}\mathbb{E}\left[\|u^h(t)\|^2\right] + 2\nu\mathbb{E}\left[\int_0^t \|\nabla_R^- u^h(s)\|^2\, ds\right] \;\leq\; & \|u_0\|^2 \\ & + 2(bc_a+\nu)\mathbb{E}\left[\int_0^t \|u^h(s)\|^2 ds\right]\end{aligned} \tag{3.11}$$

Now we apply Gronwall's Lemma to $\mathbb{E}\left[\|u^h(t)\|^2\right]$ to get

$$\mathbb{E}\left[\|u^h(t)\|^2\right] \;\leq\; e^{2(bc_a+\nu)t}\, \|u_0\|^2. \tag{3.12}$$

Furthermore, as the RHS is independent of h for $t \in [0,T]$ we get

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\|u^h(t)\|^2\right] \leq c(b,a,\nu,T)\, \|u_0\|^2 \tag{3.13}$$

with a constant c which is independent of h. Going back to (3.11), we see that the term $\mathbb{E}\left[\|u^h(t)\|^2\right]$ on the LHS is estimated against a constant by (3.13), so we remain with the second term and get therefore its boundedness

$$\mathbb{E}\left[\int_0^T \|\nabla_R^- u^h(t)\|^2\, dt\right] \;\leq\; c(b,a,\nu,T,u_0), \tag{3.14}$$

which is the desired estimate. □

Note that the constant in the Gronwall estimate grows exponentially with t, therefore (3.13) diverges for $T \to \infty$. Moreover, we did not directly estimate the discrete gradient $\|\nabla_R^- u^h(t)\|^2$, but we made use of the energy equation, which is a consequence of Itô's formula. Therefore, the exact range R of the discrete stencil does not directly affect the a priori estimates.

Lemma 3.5. *For initial data $u_0 \in C^4(\mathcal{D})$ deterministic, the discrete solution u^h of* (2.12) *is $L^2(\Omega, C([0,T], L^2(\mathcal{D})))$ independently of h, i.e.*

$$\mathbb{E}\left[\sup_{t\leq T} \|u^h(t)\|^2_{L^2(\mathcal{D})}\right] \;\leq\; c(b,a,\nu,T,u_0)$$

Proof. We start with the energy equation (3.2) over which we take the supremum in t and the expectation

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq T}\|u^h(t)\|^2\right] \;=\; & \|u^h(0)\|^2 \;+\; 2\nu\mathbb{E}\left[\int_0^T \left(\Delta_R^h u^h(s), u^h(s)\right) ds\right] \\ & +\; 2b\,\mathbb{E}\left[\int_0^T \left(f(u^h(s)), u^h(s)\right) ds\right] \;+\; 2\mathbb{E}\left[\sup_{t\leq T}\int_0^t g(u^h(s))\left(u^h(s), dW^h(s)\right)\right] \\ & +\; \mathbb{E}\left[\int_0^T g(u^h(s))^2 ds\right]\end{aligned}$$

The noise term can be analyzed using the Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t g(u^h(s))\left(u^h(s), dW^h(s)\right)\right|\right] \;\leq\; & c\,\mathbb{E}\left[\left(\int_0^T \left(g(u^h(s)), u^h(s)\right)^2_{L^2(\mathcal{D})} ds\right)^{1/2}\right] \\ \leq\; & c\,\mathbb{E}\left[\sup_{t\leq T}\|u^h(t)\|\left(\int_0^T g(u^h(s))^2 ds\right)^{1/2}\right] \\ \leq\; & \frac{1}{2}\mathbb{E}\left[\sup_{t\leq T}\|u^h(t)\|^2\right] \;+\; \frac{c^2}{2}\mathbb{E}\left[\int_0^T g(u^h(s))^2 ds\right]\end{aligned}$$

and estimating the other terms by coercivity, we get

$$\frac{1}{2}\mathbb{E}\left[\sup_{t\leq T}\|u^h(t)\|^2\right] \;\leq\; \|u_0\|^2 \;-\; 2\nu\mathbb{E}\left[\int_0^T \|\nabla_R^- u^h(s)\|^2\,ds\right] \;+\; c(b,c_a,\nu)\mathbb{E}\left[\int_0^T \|u^h(s)\|^2 ds\right]$$

Notice now that we can estimate, thanks to the last proposition, in particular (3.14),

$$\mathbb{E}\left[\int_0^T \|u^h(s)\|^2 ds\right] \;\leq\; \mathbb{E}\left[\int_0^T \|\nabla_R^- u^h(s)\|^2\,ds\right] \leq c(b,a,\nu,T) \tag{3.15}$$

and as $\|u_0\|^2 \leq c$ by assumption, so

$$\mathbb{E}\left[\sup_{t\leq T}\|u^h(t)\|^2_{L^2(\mathcal{D})}\right] \;\leq\; c(b,a,\nu,T,u_0)$$

which means that u^h is bounded in $L^2(\Omega, C([0,T],L^2(\mathcal{D})))$ independently of h. □

Note that we used here again the estimate (3.13), which comes from Gronwall's inequality, so this result holds only for finite t, or better, the constant may diverge for $T\to\infty$.

3.2. **Convergence and identification of the limit.** We begin with the proof of a simple lower semicontinuity statement on $\mathbb{E}\left[\|u(T)\|^2_{L^2}\right]$, which we will use in the proof of the convergence theorem, precisely in equation (3.25).

Lemma 3.6. *Let $u\in L^2\left(\Omega\times[0,T];H^1_0(\mathcal{D})\right)\cap L^2\left(\Omega;L^\infty([0,T],L^2(\mathcal{D})\right)$. For $u^h(t)\longrightarrow u(t)$ weakly in $L^2(\Omega,L^2(\mathcal{D}))$ it holds that*

$$\mathbb{E}\left[\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}\right] \;\leq\; \liminf_{h\to 0}\mathbb{E}\left[\|u^h(T)\|^2_{L^2} - \|u^h_0\|^2_{L^2}\right] \tag{3.16}$$

Proof. First, as $u^h \rightharpoonup u$ in $L^2(\mathcal{D})$, the lower semicontinuity of the L^2-norm gives $\|u(t)\|_{L^2}\leq \liminf_{h\to 0}\|u^h(t)\|_{L^2}$. As the mapping $u(t)\to\mathbb{E}\left[\|u(t)\|^2_{L^2}\right]$ is convex as a map from $L^2(\Omega,L^2(\mathcal{D}))$ to $\mathbb{R}$, and by the same argument, for any $t\in[0,T]$ also convex as a map from $L^2(\Omega\times[0,T];L^2(\mathcal{D}))$ to $\mathbb{R}$, we get furthermore,

$$\mathbb{E}\left[\|u(T)\|^2_{L^2}\right] \;\leq\; \liminf_{h\to 0}\mathbb{E}\left[\|u^h(T)\|^2_{L^2}\right]. \tag{3.17}$$

By strong convergence of the initial condition in $L^2(\mathcal{D})$, we have $u^h(0)=\sum_{i=1}^\infty (u_0,e_i)e_i \longrightarrow u_0$ and so

$$\begin{aligned}\mathbb{E}\left[\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}\right] \;&=\; \mathbb{E}\left[\|u(T)\|^2_{L^2}\right] \;-\; \mathbb{E}\left[\|u_0\|^2_{L^2}\right]\\ &\leq\; \liminf_{h\to 0}\mathbb{E}\left[\|u^h(T)\|^2_{L^2}\right] \;-\; \liminf_{h\to 0}\|u^h_0\|^2_{L^2}\end{aligned}$$

which finally leads to

$$\mathbb{E}\left[\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}\right] \;\leq\; \liminf_{h\to 0}\mathbb{E}\left[\|u^h(T)\|^2_{L^2} - \|u^h_0\|^2_{L^2}\right] \tag{3.18}$$

finishing the proof. □

Theorem 3.7. *Let the initial data $u_0\in C^4(\mathcal{D})$ be deterministic and let Assumption 2.1 and the conditions* (2.8) *and* (2.9) *be satisfied. Then the solution u^h of* (2.12) *converges in $L^2(\Omega;L^2([0,T];H^1_0(\mathcal{D})))$ to the solution u of* (2.10) *as $h\to 0$.*

Proof. Recall that we can write the discrete problem in integral form (3.3) in a suggestive way, using an orthonormal bais $\{e_i\}_{i=1\ldots N}$ of $P_N H^1_0$, as

$$\begin{aligned}\left(u^h(T),e_i\right) \;&=\; \left(u^h_0,e_i\right) \;+\; 2\nu\int_0^T \left\langle\Delta^h_R u^h, e_i\right\rangle dt + 2b\int_0^T \left(f(u^h),e_i\right) dt\\ &\quad+\int_0^T g(u^h)\left(e_i, dW^h(t)\right) \qquad i=1\ldots N.\end{aligned} \tag{3.19}$$

The above priori estimates, Proposition 3.2, imply the boundedness of the sequence $\Delta_R^h u^h$ in $L^2(\Omega \times [0,T]; H^{-1}(\mathcal{D}))$ and the boundedness of the sequence $g(u^h)$ in $L^2(\Omega \times [0,T]; \mathcal{H})$. Hence there exists a subsequence, which we do not relabel, such that

$$
\begin{aligned}
u^h &\rightharpoonup u \quad \text{in } L^2(\Omega; L^2([0,T]; H_0^1(\mathcal{D}))) \cap L^2\left(\Omega; L^\infty([0,T], L^2(\mathcal{D})\right) \\
\Delta_R^h u^h &\rightharpoonup \zeta_1 \quad \text{in } L^2(\Omega \times [0,T]; H^{-1}(\mathcal{D})) \\
f(u^h) &\rightharpoonup \zeta_2 \quad \text{in } L^2(\Omega \times [0,T]; L^2(\mathcal{D})) \\
g(u^h) &\rightharpoonup \widetilde{g} \quad \text{in } L^2(\Omega \times [0,T]; \mathcal{H})
\end{aligned} \tag{3.20}
$$

We pass to the weak limit in (3.19) and get that for all $t \geq 0$

$$
\begin{aligned}
(u(T), e_i) \;=\; (u_0, e_i) \;+\; 2\nu \int_0^T \langle \zeta_1(t), e_i \rangle \, dt \\
+\; 2b \int_0^T (\zeta_2(t), e_i) \, dt \;+\; \int_0^T \widetilde{g}(t) \, (e_i, dW(t)) \qquad i = 1 \dots N.
\end{aligned} \tag{3.21}
$$

It remains to identify the weak limit objects in (3.21) with the objects in the stochastic Nagumo equation. Set in the rest of the proof $\nu, b = 1$ for convenience, as it does not change the argument.

We start with identifying $\widetilde{g} = g(u)$. For this, note first that, by using the monotonicity property (2.20) with $c = 0$, we infer that for any $\varphi \in L^2(\Omega \times [0,T]; H_0^1(\mathcal{D}))$ holds

$$
2\mathbb{E}\left[\int_0^T \left\langle \Delta_R^h u^h + f(u^h) - \Delta_R^h \varphi - f(\varphi), u^h - \varphi \right\rangle \, dt\right] + \mathbb{E}\left[\int_0^T \|g(u^h) - g(\varphi)\|_{\mathcal{H}}^2 \, dt\right] \leq 0 \tag{3.22}
$$

We can split the first term into 4 terms and use the postivitiy of $\mathbb{E}\left[\int_0^T \|g(u^h) - g(\varphi)\|_{\mathcal{H}}^2 \, dt\right]$ to get

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\langle \Delta_R^h u^h + f(u^h) - \Delta_R^h \varphi - f(\varphi), u^h - \varphi \right\rangle \, dt\right] \\
&= \mathbb{E}\left[\int_0^T \left\langle \Delta_R^h u^h + f(u^h), u^h \right\rangle dt\right] \;+\; \mathbb{E}\left[\int_0^T \left\langle \Delta_R^h \varphi + f(\varphi), \varphi \right\rangle dt\right] \\
&-\; \mathbb{E}\left[\int_0^T \left\langle \Delta_R^h u^h + f(u^h), \varphi \right\rangle dt\right] \;-\; \mathbb{E}\left[\int_0^T \left\langle \Delta_R^h \varphi + f(\varphi), u^h \right\rangle dt\right] \;\leq\; 0.
\end{aligned} \tag{3.23}
$$

By weak convergence,

$$
\begin{aligned}
\int_0^T \left\langle \Delta_R^h \varphi + f(\varphi), \varphi \right\rangle dt \;&\longrightarrow\; \int_0^T \langle \Delta \varphi + f(\varphi), \varphi \rangle \, dt \\
\int_0^T \left\langle \Delta_R^h u^h + f(u^h), \varphi \right\rangle dt \;&\longrightarrow\; \int_0^T \langle \zeta_1(t) + \zeta_2(t), \varphi \rangle \, dt \\
\int_0^T \left\langle \Delta_R^h \varphi + f(\varphi), u^h \right\rangle dt \;&\longrightarrow\; \int_0^T \langle \Delta \varphi + f(\varphi), u \rangle \, dt \\
\int_0^T \left(g(u^h), g(\varphi)\right)_{\mathcal{H}} dt \;&\longrightarrow\; \int_0^T \left(\tilde{g}(t), g(\varphi)\right)_{\mathcal{H}} dt
\end{aligned} \tag{3.24}
$$

so the last 3 terms in (3.23) pass to the limit and preserve the sign in (3.22).

For the first term of (3.23), we employ that by semicontinuity of the norm, Lemma 3.6, we can relate solutions u^h to (2.12) with solutions u to (2.10) as

$$
\begin{aligned}
&2\mathbb{E}\left[\int_0^T \langle \Delta u(t), u(t)\rangle\, dt\right] + 2\,\mathbb{E}\left[\int_0^T (f(u(t)), u(t))\, dt\right] + \mathbb{E}\left[\int_0^T g(u(t))^2 dt\right] \\
&\le 2\liminf_{h\to 0}\mathbb{E}\left[\int_0^T \left\langle \Delta_R^h u^h(t), u^h(t)\right\rangle dt\right] + 2\liminf_{h\to 0}\mathbb{E}\left[\int_0^T \left(f(u^h(t)), u^h(t)\right) dt\right] \\
&+ \liminf_{h\to 0}\mathbb{E}\left[\int_0^T g(u^h(t))^2 dt\right].
\end{aligned} \tag{3.25}
$$

Consequently also for the first term in (3.23) the sign is preserved in the limit. Passing to the limit in (3.22), we get

$$
2\mathbb{E}\left[\int_0^T \langle \zeta_1 + \zeta_2 - \Delta\varphi - f(\varphi), u - \varphi\rangle\ dt\right] + \mathbb{E}\left[\int_0^T \|\widetilde{g} - g(u)\|_{\mathcal{H}}^2\, dt\right] \ \le\ 0. \tag{3.26}
$$

Choosing $u = \varphi$ in (3.26), we deduce $\widetilde{g} = g(u)$. It remains to identify the limit objects ζ_1 and ζ_2 to prove that $u := \lim_{h\to 0} u^h$ is indeed a solution to the stochastic Nagumo equation. To this aim, notice first that (3.26) implies

$$
\mathbb{E}\left[\int_0^T \langle \zeta_1(t) + \zeta_2(t) - \Delta\varphi(t) - f(\varphi(t)), u(t) - \varphi(t)\rangle\ dt\right] \ \le\ 0. \tag{3.27}
$$

Now we take $\theta > 0$ and define another testfunction w via

$$
\theta w(t) = u(t) - \varphi(t) \tag{3.28}
$$

with $\varphi(t)$ the testfunction used in (3.27). As w is an admissible test function in $L^2\left(\Omega \times [0,T]; H_0^1(\mathcal{D})\right)$, we can employ it in (3.27) instead of φ. We obtain

$$
\mathbb{E}\left[\int_0^T \langle \zeta_1(t) + \zeta_2(t) - \Delta\left(u(t) - \theta w(t)\right) - f\left(u(t) - \theta w(t)\right), w(t)\rangle\, dt\right] \le 0 \tag{3.29}
$$

As $\theta \mapsto \langle \Delta(u - \theta w), w\rangle$ and $\theta \mapsto \langle f(u - \theta w), w\rangle$ are continuous from $\mathbb{R} \to \mathbb{R}$, it is admissible to pass to the limit $\theta \to 0$ and we reach

$$
\mathbb{E}\left[\int_0^T \langle \zeta_1(t) + \zeta_2(t) - \Delta u(t) - f(u(t)), w(t)\rangle\, dt\right] \le 0 \qquad \text{for any } w \in L^2\left(\Omega \times [0,T]; H_0^1(\mathcal{D})\right). \tag{3.30}
$$

Since w is arbitrary, the left hand side must vanish, hence $\zeta_1 + \zeta_2 = \Delta u + f(u)$. Setting now $w = u$ we identify $\zeta_2 = f(u)$. Plugging this result into (3.27) gives $\zeta_1 = \Delta u$. ☐

3.3. **The cutt-off error.** The next Proposition deals with an extension of Theorem 3.7, which dealt with the error in the bounded domain $\mathcal{D}$, to the whole of $\mathbb{R}$.

Proposition 3.8. *Let v^{TW} the deterministic travelling front solution to* (Nag$\mathbb{R}$). *Let the initial data u_0 be such a traveling wave, i.e. let u_0 satisfy $u_0 = v^{\mathrm{TW}}(0)$. Let u^h solve* (dSNag$N$) *and U solve* (SNag$\mathbb{R}$). *Assume furthermore that $g \in L^2(\Omega, L^2([0,T] \times \mathbb{R}))$. Then, for sufficiently small noise (in the sense that $\|Q\|_{HS}^2 \le \tilde{\epsilon}$) and $t \le T_0$*

$$
\mathbb{E}\left[\|u^h(t) - U(t)\|_{L^2(\mathbb{R})}^2\right] \le \epsilon. \tag{3.31}
$$

Proof. The error between the solution u^h of the discrete Nagumo equation (dSNag$N$) and the solution u of (SNag$\mathcal{D}$), the stochastic Nagumo equation on a bounded domain $\mathcal{D} = [-L, L]$, was controlled in the previous subsection, Theorem 3.7. It remains to control the cut-off error

$\mathbb{E}\left[\|u(T)-U(T)\|^2_{L^2(\mathbb{R})}\right]$. We do this by splitting the cut-off error in several parts, making use of the deterministic solutions v and V:

$$\begin{aligned}\mathbb{E}\left[\|u^h(t)-u(t)\|^2_{L^2(\mathbb{R})}\right] \le \mathbb{E}\left[\|u^h(t)-u(t)\|^2_{L^2(\mathcal{D})}\right] + \mathbb{E}\left[\|u(t)-v(t)\|^2_{L^2(\mathcal{D})}\right] \\ + \|v(t)-V(t)\|_{L^2(\mathbb{R})} + \mathbb{E}\left[\|V(t)-U(t)\|^2_{L^2(\mathbb{R})}\right]\end{aligned} \tag{3.32}$$

The control of $\|v(t)-V(t)\|_{L^2(\mathbb{R})}$ is established in Lemma 3.9 below. Lemma 3.10 estimates $\mathbb{E}\left[\|u(t)-v(t)\|^2_{L^2(\mathcal{D})}\right] \le \epsilon$ and Lemma 3.10 shows that $\mathbb{E}\left[\|V(t)-U(t)\|^2_{L^2(\mathcal{D})}\right] \le \epsilon$, which yields the desired result, equation (3.31). □

Control of the truncation error in the deterministic equation.

Lemma 3.9. *Let V solve the Nagumo PDE on the real line* (Nag$\mathbb{R}$) *and let v^{TW} be the deterministic travelling front solution to* (Nag$\mathbb{R}$). *Let v be the solution to a finite-domain Nagumo PDE* (Nag$\mathcal{D}$) *with Dirichlet boundary conditions*

$$\begin{aligned}\partial_t v &= \nu\partial_{xx}v + bf(v) \qquad (t,x)\in\mathbb{R}_+\times\mathcal{D}\\ v(-L,t) &= 0\\ v(L,t) &= 1\end{aligned} \tag{3.33}$$

where the domain $\mathcal{D}=[-L,L]\subset\mathbb{R}$ is a finite interval of length $2L$. Let the initial data v_0 be a traveling wave around zero. Then, for all $t<T_0$, where T_0 depends on the size of the domain and the speed of the wave, we have

$$\|V(t)-v(t)\|_{L^2(\mathbb{R})} \le \epsilon. \tag{3.34}$$

Proof. We focus on such v which are travelling front solutions, i.e. which satisfy the boundary conditions $v^{\text{TW}}(-\infty)=0$ and $v^{\text{TW}}(\infty)=1$.

For our analysis, we formally extend v to $\pm\infty$ according to the boundary conditions, so $v(x,t)=1$ for all $x\in[L,\infty)$ and $v(x,t)=0$ for all $x\in(-\infty,-L]$. See equation (3.35) for the choice of L.

Let us now start both equations with the same initial data v_0, which forms a traveling wave around zero.

Due to the boundary conditions at infinity, we know that $V(-\infty,t)=0$ and $V(\infty,t)=1$. By classical theory, 0 and 1 are hyperbolic steady states of (Nag$\mathbb{R}$). Therefore, the decay of the traveling waves is exponential as it approaches the steady states, in other words, there exists ϵ such that $e^{\varepsilon x}\cdot V(x,t)\to 0$ for $x\to-\infty$, and analogously for $x\to+\infty$. Therefore, as long as the transition part of the traveling wave is far away from the boundary of $\mathcal{D}$,

$$\exists L_0\in\mathbb{R}\ :\forall L\ge L_0\ :\quad \|V(x,t)-1\|_{L^2((L,\infty))}\le\epsilon\quad,\qquad \|V(x,t)-0\|_{L^2((-\infty,-L))}\le\epsilon. \tag{3.35}$$

Consequently, by the boundary conditions of the finite-domain Nagumo PDE (3.33), we conclude that up to a time T_0 when the transition part of the traveling waves has not yet reached the boundary of the domain $\mathcal{D}$,

$$\|V(x,t)-v(x,t)\|_{L^2(\mathbb{R}\setminus[-L-\delta,L+\delta])}\le 2\epsilon. \tag{3.36}$$

Second, we investigate the error at the boundary of $\mathcal{D}$. First look at the positive boundary point of $\mathcal{D}$, i.e. the point $L\in\mathbb{R}$. In a neighborhood $B_\delta(L)$ we have $f(1-\delta)=(1-\delta-a)(1-\delta)\delta\le\delta$ and $V(x,t)$ is almost constant so by spatial regularity, we have $\max_x\partial_{xx}V(x,t)\le\tilde{\epsilon}$, and as $V\equiv 1$

in $[L, L+\delta]$,
(3.37)

$$\begin{aligned}\|V(x,t)-v(x,t)\|^2_{L^2(B_\delta(L))} &= \int_L^{L+\delta} \left(\nu\partial_{xx}V(x,t)+bf(V(x,t))-0\right)^2 dx \\ &+ \int_{L-\delta}^{L} \left(\nu\partial_{xx}V(x,t)+bf(V(x,t))-\nu\partial_{xx}v(x,t)-bf(v(x,t))\right)^2 dx \\ &\leq \int_L^{L+\delta} \left(\nu \max_{x\in[L,L+\delta]} \partial_{xx}V(x,t)+b\max_{x\in[L-\delta,L]} f(V(x,t))-0\right)^2 dx \\ &+ \delta\cdot\left(\nu\max_{x\in[L-\delta,L]}(\partial_{xx}V(x,t)-\partial_{xx}v(x,t))+b\max_{x\in[L-\delta,L]}(f(V)-f(v))\right)^2 \\ &\leq \delta\nu^2\tilde{\epsilon}^2+\delta^3b^2+\ 3\delta\nu^2\left(\max_{x\in[L-\delta,L]}(\partial_{xx}V(x,t)-\partial_{xx}v(x,t))\right)^2 +\ 3b^2\delta^2 \\ &\leq \epsilon\end{aligned}$$

where we employed the continuity in space of solutions to (Nag$\mathbb{R}$) and (3.33) again to infer that $\max_{x\in[L-\delta,L]}(\partial_{xx}V(x,t)-\partial_{xx}v(x,t)) \leq c$ is a finite quantity. Analoguous reasoning holds for the δ-neighbourhood around $-L$. Therefore

$$\|V(x,t)-v(x,t)\|_{L^2(B_\delta(L))} \leq \epsilon \quad , \qquad \|V(x,t)-v(x,t)\|_{L^2(B_\delta(-L))} \leq \epsilon. \tag{3.38}$$

Third, we look at the difference between the two solutions in the interior of the domain $\mathcal{D}$, $\|V(x,t)-v(x,t)\|_{L^2([-L+\delta,L-\delta])}$. Note first of all that this difference is zero at time $t=0$, as we started with the same wave as initial condition. Moreover, as
(3.39)

$$\begin{aligned}\|\partial_t V(x,t)-\partial_t v(x,t)\|^2_{L^2([-L+\delta,L-\delta])} &= \int_{-L+\delta}^{L-\delta} \left(\nu\partial_{xx}V(x,t)+bf(V(x,t))-\nu\partial_{xx}v(x,t)-bf(v(x,t))\right)^2 dx \\ &\leq 3\nu\int_{-L+\delta}^{L-\delta} \left(\partial_{xx}(V(x,t)-v(x,t))\right)^2 dx \\ &+\ 3b\int_{-L+\delta}^{L-\delta} \left(f(V(x,t))-f(v(x,t))\right)^2 dx\end{aligned}$$

so $\|\partial_t(V(x,t)-v(x,t))\|_{L^2([-L+\delta,L-\delta])}=0$ for times $t\leq T_0$, i.e. t small enough that the boundary conditions are not propagated in the interior of the domain. As we have already controlled the error at the boundary by ϵ, by continuity of the solutions V and v, we can conclude $\|\partial_t(V(x,t)-v(x,t))\|_{L^2([-L+\delta,L-\delta])}\leq\epsilon$.

We summarize, using (3.36), (3.38) and (3.39),
(3.40)

$$\begin{aligned}\|V(t,x)-v(t,x)\|_{L^2(\mathbb{R})} &= \|V(t,x)-v(t,x)\|_{L^2([-L+\delta,L-\delta])}+\|V(t,x)-v(t,x)\|_{L^2(\mathbb{R}\setminus[-L-\delta,L+\delta])} \\ &+\ \|V(t,x)-v(t,x)\|_{L^2(B_\delta(L))}+\|V(t,x)-v(t,x)\|_{L^2(B_\delta(-L))} \\ &\leq 4\epsilon.\end{aligned}$$

□

Control of the error caused by noise. To complete the proof of Proposition 3.8, it remains to estimate the remaining two terms in (3.32), $\mathbb{E}\left[\|u(t)-v(t)\|_{L^2(\mathcal{D})}\right]$ and $\mathbb{E}\left[\|V(t)-U(t)\|_{L^2(\mathbb{R})}\right]$, which both deal with the error between deterministic and stochastic solution. First we look at the difference between the deterministic solution on a bounded domain (Nag$\mathcal{D}$) and the stochastic solution on a bounded domain (SNag$\mathcal{D}$).

Lemma 3.10. *Let v be a solution to* (Nag$\mathcal{D}$) *and u a solution to* (SNag$\mathcal{D}$). *Assume furthermore that $g\in L^2(\Omega, L^2([0,T]\times\mathcal{D}))$. Then, for sufficiently small noise (in the sense that $\|Q\|^2_{HS}\leq\tilde{\epsilon}$),*

$$\mathbb{E}\left[\|v(t)-u(t)\|^2_{L^2(\mathcal{D})}\right]\leq\epsilon \tag{3.41}$$

Proof. Notice first that both equations satisfy the same boundary conditions, namely $v(-L,t)=0$ and $v(L,t)=1$, and we assume that they start with the same deterministic initial data u_0. We write the error between the deterministic and the stochastic solution via the mild solution expression as

$$\begin{aligned} u(t)-v(t) = S(t)(u_0-u_0) + \ b\int_0^t S(t-s)\left(f(u(s))-f(v(s))\right)ds \\ + \int_0^t S(t-s)\tilde{g}(u(s))d\tilde{W}(s). \end{aligned} \tag{3.42}$$

with $\tilde{W}$ a cylindrical Wiener Process on $L^2(\mathcal{D})$ and

$$\tilde{g}(u)\phi = g(u(s))\sqrt{Q}\phi \tag{3.43}$$

where $\sqrt{Q}$ is the positive definite square root of the the covariance operator Q of the Wiener Process. We use the boundedness of the heat semigroup, the local lipschitz continuity of f from Lemma 2.2, Itô's Isometry and Gronwall's equality to get

(3.44)

$$\begin{aligned} \mathbb{E}\left[\|u(t)-v(t)\|^2_{L^2(\mathcal{D})}\right] &= b\mathbb{E}\left[\int_{\mathcal{D}}\left(\int_0^t S(t-s)\left(f(u(s))-f(v(s))\right)ds\right)^2 dx\right] \\ &\quad +\mathbb{E}\left[\int_{\mathcal{D}}\left(\int_0^t S(t-s)g(u(s))dW(s)\right)^2 dx\right] \\ &\leq bc_a^2K_M\|S(t-s)\|_\infty\mathbb{E}\left[\int_{\mathcal{D}}\left(\int_0^t |u(s)-v(s)|ds\right)^2 dx\right] \\ &\quad + \ \mathbb{E}\left[\int_{\mathcal{D}}\int_0^t S(t-s)^2g(u(s))^2\sigma ds dx\right] \\ &\leq \sigma\cdot\exp\left(\int_0^t bc_aK_M\|S(t-s)\|_\infty ds\right)\mathbb{E}\left[\int_{\mathcal{D}}\int_0^t S(t-s)^2g(u(s))^2 ds dx\right] \\ &=\sigma\cdot c\,\mathbb{E}\left[\int_{\mathcal{D}}\int_0^t S(t-s)^2g(u(s))^2 ds dx\right] \end{aligned}$$

where we used the notation $\sigma := \|Q\|^2_{\mathcal{H}}$. and $c = c\left(t,b,c_a,\|u\|_{H_0^1(\mathcal{D})},\|v\|_{H_0^1(\mathcal{D})},\|S(t-s)\|_\infty\right)$. Under the assumption $g(u(s))\in L^2(\Omega,L^2([0,T]\times D))$, we can conclude that for finite times $t\leq T$,

$$\mathbb{E}\left[\|u(t)-v(t)\|^2_{L^2(\mathcal{D})}\right] \leq \|Q\|^2_{\mathcal{H}}\cdot c\left(T,b,c_a,\|u\|_{H_0^1(\mathcal{D})},\|v\|_{H_0^1(\mathcal{D})},\|S(t-s)\|_\infty\right)) \tag{3.45}$$

The choice of $\|Q\|^2_{\mathcal{H}}\leq\frac{\epsilon}{c}$ with the c from (3.45) finally gives $\mathbb{E}\left[\|u(T)-v(T)\|^2_{L^2(\mathcal{D})}\right]\leq\epsilon$, finishing the proof. ☐

Lemma 3.11. *Let* v^{TW} *the deterministic travelling front solution to* (Nag$\mathbb{R}$). *Let* V *be a solution to* (Nag$\mathbb{R}$) *and* U *a solution to* (SNag$\mathbb{R}$). *Assume furthermore that* $g\in L^2(\Omega,L^2([0,T]\times\mathbb{R}))$. *Then, for sufficiently small noise (in the sense that* $\|Q\|^2_{HS}\leq\tilde{\epsilon}$ *),*

$$\mathbb{E}\left[\|V(t)-U(t)\|^2_{L^2(\mathbb{R})}\right]\leq\epsilon. \tag{3.46}$$

Proof. Recall that the noise acts only on the wave part of the solution and in particular $g(0)=g(1)=0$. Therefore, as the initial data is a traveling wave, $g(v^{TW})\xi=0$ in $\mathbb{R}\setminus\mathcal{D}$, and

$$\partial_t V = \partial_t U \qquad \text{in } \mathbb{R}\setminus\mathcal{D} \tag{3.47}$$

As we initialized both equations with the same travelling wave v^{TW}, we conclude that for finite times $t\leq T$

$$\partial_t(V-U)=0 \qquad \text{in } \mathbb{R}\setminus\mathcal{D} \tag{3.48}$$

and we are back in the situation of Lemma 3.10. ☐

3.4. **Proof of the main theorem.** The last step is to make a statement on the stability of the traveling wave solution. For this we prove morally that it is very likely that the solution stays close to a deterministic traveling wave if we start with an initial data close to the traveling wave and if the covariance of the noise is small.

For this last step we use the following result presented in [38], where a global bound on the error between the solution u of the stochastic Nagumo equation and the dynamically phase-shifted deterministic travelling wave $\tilde{v}^{\mathrm{TW}} := v^{\mathrm{TW}}(\cdot + C(t) + ct)$ of the deterministic Nagumo equation was proven.

Lemma 3.12 ([38], Theorem 3.1.)**.** *Let u be the unique solution of the stochastic Nagumo equation on the real line* (SNag$\mathbb{R}$) *and v^{TW} the traveling wave front solution of* (Nag$\mathbb{R}$) *and $\tilde{v}^{\mathrm{TW}} := v^{\mathrm{TW}}(\cdot + C(t) + ct)$. Let g be a Hilbert-Schmidt operator satisfying global Lipschitz and linear growth conditions in L^2. Then, for all $t \leq t_*$*

(3.49)

$$\mathbb{E}\left(\|u(t) - \tilde{v}^{\mathrm{TW}}(t)\|^2_{L^2(\mathbb{R})}\right) \;\leq\; \|u_0 - v^{\mathrm{TW}}(0)\|^2_{L^2(\mathbb{R})} + \; c(\nu, b, a) \cdot M_{\sqrt{Q}} Lip_g^2 \;\|v^{\mathrm{TW}} \wedge (1 - v^{\mathrm{TW}})\|^2_{L^2(\mathbb{R})}$$

with $M_{\sqrt{Q}} := \sup_x \int k_{\sqrt{Q}}(x, y)^2 dy$, where $k_{\sqrt{Q}}(x, y)$ is the representing integral kernel of the (positive semidefinite square root of the) covariance operator. In particular,

$$\mathbb{P}(t_* < \infty) \;\leq\; \frac{1}{\tilde{c}}\left(\|u_0 - v^{\mathrm{TW}}(0)\|^2_{L^2(\mathbb{R})} + \; c(\nu, b, a) \cdot M_{\sqrt{Q}} Lip_g^2 \;\|v^{\mathrm{TW}} \wedge (1 - v^{\mathrm{TW}})\|^2_{L^2(\mathbb{R})}\right) \tag{3.50}$$

for some $c(\nu, b, a)$ finite as long as $\nu, b \in (0, \infty)$ and $a \in (0, 1)$

Summary of the proof of Lemma 3.12. Equation (3.49) is part of Theorem 3.1. of [38]. Its proof goes along the following lines:

The solution to the stochastic Nagumo equation is dynamically phase-shifted in order to match the position of the traveling wave. This procedure produces an extra term in the equation, but the author showed that an energy equality still holds and linear growth estimates on the multiplicative noise term can be established. By linearization of $f(u)$ around zero additional estimates are obtained, which, together with Itô's formula, lead to (3.49). This reasoning uses in a crucial way a "local dissipativity" estimate, which holds only for $a \in (0, 1)$. The constant from this local dissipativity estimate also appears in (3.49) and blows up for $a = 0$ and $a = 1$.

In the second part of the statement of our Lemma, we summarized relevant findings from [38] concerning the long-time behaviour of the error between the solution of the stochastic Nagumo equation and the phase-shifted travelling wave. This is the final part of Theorem 3.1. of [38], which continues with an application of Markov's inequality and a stopping time argument to get a precise estimate on the probability that the L^2-norm of the phase-shifted solution exceeds a certain threshold value. □

In particular, the result of [38] tells that the probability that t_* is infinite depends on the initial error $\|v_0 - v^{\mathrm{TW}}(0)\|$ and on the covariance operator of the noise term. The smaller the covariance, the smaller the probability for t_* being finite.

Remark 3.1 (On covariance-related quantities)**.** *A technicality is that different quantities related to the representing integral kernel of the covariance operator appear in our proofs: Lemma 3.12 used the quantity $M_{\sqrt{Q}} := \sup_x \int k_{\sqrt{Q}}(x, y)^2 dy$, where the supremum over x is unavoidable as in the proof this quantity needed to be taken out of a spatial integral. In Lemma 3.10, however, and L^2 argument (Itô Isometry) involved the quantity*

$$\sigma := \|Q\|^2_{HS} = \sum_{k=1}^{\infty} \mu_k^2 = \int_{\mathcal{D}} \int_{\mathcal{D}} k(x, y) dy dx. \tag{3.51}$$

Though we assume enough regularity of $k(x, y)$ to ensure the finiteness of each quantity, they might become small at a different rate. We prefer a more clumsy statement of the main theorem in order to keep this transparent.

With Theorem 3.7 and Lemma 3.12 at hand, we can now finally prove that the solution of the stochastic LDE (dSNag$N$) is likely to be close to the traveling wave.

Theorem 3.13. *Let v^{TW} the deterministic travelling front solution to* (Nag$\mathbb{R}$). *Assume $g \in L^2(\Omega, L^2([0,T] \times D))$. Let u_0 be deterministic and satisfy $\|u_0 - v^{\mathrm{TW}}(0)\|^2_{L^2(\mathbb{R})} < \epsilon$. Then, for sufficiently small noise in the sense that $M_{\sqrt{Q}} \leq \delta_*$ and $\|Q\|^2_{HS} \leq \tilde{\epsilon}$, there exists a solution u^h to* (3.3) *for which holds*

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathbb{R})} > \delta\right] \leq \tilde{\varepsilon} \quad \text{for sufficiently small } h \tag{3.52}$$

for $T < t_*$ with t_* as in Lemma 3.12.

Proof. As we have outlined in the beginning of this section, there exists a solution to equation (2.12), which is an adapted process with a continuous version. The question is whether this (discrete-in-space) solution is likely to be a travelling wave. We will show that this is indeed true by comparing u^h to solutions to intermediary problems to which traveling waves solutions exist.

To set up notation, let us now denote by $u^h(t)$ be the piecewise linear extension of the solution to the stochastic LDE (2.12) to the whole domain $\mathcal{D}$. Moreover we extend the solution as $u^h(t) \equiv 1$ and $u^h(t) \equiv 0$ from the two boundary points of $\mathcal{D}$ to $\pm\infty$. We the adapted stochastic process $e_t := \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathcal{D})}$. By the properties of u^h, e_t defines a martingale whose trajectories are continuous almost surely. We can then estimate by Doob's inequality

$$\begin{aligned}\mathbb{P}\left[\sup_{t\in[0,T]} \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathbb{R})} > \delta\right] &\leq \frac{1}{\delta^2}\mathbb{E}\left[\|u^h(T) - v^{TW}(T)\|^2_{L^2(\mathbb{R})}\right] \\ &\leq \frac{1}{\delta^2}\mathbb{E}\left[\|u^h(T) - u(T)\|^2_{L^2(\mathbb{R})}\right] + \mathbb{E}\left[\|u(T) - v^{\mathrm{TW}}(T)\|^2_{L^2(\mathbb{R})}\right]\end{aligned} \tag{3.53}$$

where we split the error between the stochastic LDE and the travelling wave front into two parts, using the linearity of the expectation. By Proposition 3.8, the first term goes to zero as $h \to 0$, so in particular there exists h_* such that $\mathbb{E}\left[\|u^h(T) - u(T)\|_{L^2(\mathbb{R})}\right] \leq \epsilon$ for all $h \leq h_*$. For the second term, $\mathbb{E}\left[\|u(T) - v^{\mathrm{TW}}(T)\|_{L^2(\mathbb{R})}\right]$, we use the estimate (3.49) of [38]. Combining these two estimates, we get

$$\begin{aligned}\mathbb{P}\left[\sup_{t\in[0,T]} \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathbb{R})} > \delta\right] &\leq \frac{1}{\delta^2}\Big(\epsilon \,+\, \|u_0 - v^{\mathrm{TW}}(0)\|^2_{L^2(\mathbb{R})} \\ &\quad + c(\nu, b, a) \cdot M_{\sqrt{Q}} Lip_g^2\, \|v^{\mathrm{TW}} \wedge (1 - v^{\mathrm{TW}})\|^2_{L^2(\mathbb{R})}\Big).\end{aligned} \tag{3.54}$$

As $\|v^{\mathrm{TW}} \wedge (1 - v^{\mathrm{TW}})\|^2_{L^2(\mathbb{R})} \leq c$, employing the assumption on the initial data and the observation that for sufficiently small δ_* such that $M_{\sqrt{Q}} \leq \delta_*$ the second term is also small, in formula

$$M_{\sqrt{Q}} \cdot c(\nu, b, a) Lip_g^2\, \|v^{\mathrm{TW}} \wedge (1 - v^{\mathrm{TW}})\|^2_{L^2(\mathbb{R})} \leq \bar{\epsilon} \tag{3.55}$$

and therefore (3.54) becomes

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathbb{R})} > \delta\right] \leq \frac{1}{\delta^2}\left(2\epsilon + \bar{\epsilon}\right) \tag{3.56}$$

with $\bar{\epsilon} = \bar{\epsilon}(\nu, b, a, M_{\sqrt{Q}})$.

To summarize, for $h \leq h_*$ and $M_{\sqrt{Q}} \leq \delta_*$, the choice of $\tilde{\varepsilon} := \frac{2\epsilon + \bar{\epsilon}}{\delta^2}$ gives

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^h(t) - v^{\mathrm{TW}}(t)\|_{L^2(\mathbb{R})} > \delta\right] \leq \tilde{\varepsilon} \tag{3.57}$$

which concludes the proof. □

Note that zero covariance does not imply that the noise strength is zero, but just that it is constant, and so it affects the solution only by a shift of $c \cdot t$, which does not destroy the traveling wave property.

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