

ON THE FINE STRUCTURE OF THE FREE BOUNDARY FOR THE CLASSICAL OBSTACLE PROBLEM

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ABSTRACT. In the classical obstacle problem, the free boundary can be decomposed into “regular” and “singular” points. As shown by Caffarelli in his seminal papers [C77, C98], regular points consist of smooth hypersurfaces, while singular points are contained in a stratified union of C^1 manifolds of varying dimension. In two dimensions, this C^1 result has been improved to $C^{1,\alpha}$ by Weiss [W99].

In this paper we prove that, for $n = 2$ singular points are locally contained in a C^2 curve. In higher dimension $n \geq 3$, we show that the same result holds with $C^{1,1}$ manifolds (or with countably many C^2 manifolds), up to the presence of some “anomalous” points of higher codimension. In addition, we prove that the higher dimensional stratum is always contained in a $C^{1,\alpha}$ manifold, thus extending to every dimension the result in [W99].

We note that, in terms of density decay estimates for the contact set, our result is optimal. In addition, for $n \geq 3$ we construct examples of very symmetric solutions exhibiting linear spaces of anomalous points, proving that our bound on their Hausdorff dimension is sharp.

1. INTRODUCTION

The classical obstacle problem consists in studying the regularity of solutions to the minimization problem

$$\min_v \left\{ \int_{B_1} \frac{|\nabla v|^2}{2} : v \geq \psi \text{ in } B_1, v|_{\partial B_1} = g \right\},$$

where $g : \partial B_1 \rightarrow \mathbb{R}$ is some prescribed boundary condition, and the “obstacle” $\psi : B_1 \rightarrow \mathbb{R}$ satisfies $\psi|_{\partial B_1} < g$.

Assuming that ψ is smooth, it is well-known that this problem has a unique solution v of class $C_{\text{loc}}^{1,1}$ [BK74], and that $u := v - \psi$ satisfies the Euler-Lagrange equation

$$\Delta u = -\Delta \psi \chi_{\{u>0\}} \quad \text{in } B_1.$$

As already observed in [C77, C98], in order to prove some regularity results for the *free boundary* $\partial\{u > 0\}$ it is necessary to assume that $\Delta \psi < 0$. In addition, as also noticed in [C98, W99, M03, PSU12], from the point of view of the local structure it suffices to understand the model case $\Delta \psi \equiv -1$. For this reason, from now on, we shall focus on the problem

$$(1.1) \quad \Delta u = \chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_1 \subset \mathbb{R}^n.$$

As shown by Caffarelli in his seminal papers [C77, C98], points of the free boundary $\partial\{u > 0\}$ are divided into two classes: regular points and singular points. A free

boundary point x_o is either regular or singular depending on the type of blow-up of u at that point. More precisely:

$$(1.2) \quad x_o \text{ is called } \textit{regular} \text{ point} \quad \Leftrightarrow \quad r^{-2}u(x_o + rx) \xrightarrow{r \downarrow 0} \frac{1}{2} \max\{\mathbf{e} \cdot x, 0\}^2$$

for some $\mathbf{e} = \mathbf{e}_{x_o} \in \mathbb{S}^{n-1}$, and

$$(1.3) \quad x_o \text{ is called } \textit{singular} \text{ point} \quad \Leftrightarrow \quad r^{-2}u(x_o + rx) \xrightarrow{r \downarrow 0} p_{*,x_o}(x) := \frac{1}{2}x \cdot Ax$$

for some symmetric nonnegative definite matrix $A = A_{x_o} \in \mathbb{R}^{n \times n}$ with $\text{tr}(A) = 1$. The existence of the previous limits in (1.2) and (1.3), as well as the classification of possible blow-ups are well-known results; see [C98, W99, M03, PSU12].

By the theory in [KN77, C77] (see also [CR76, CR77, Sak91, Sak93, C98, M00, PSU12]), the free boundary is an analytic hypersurface near regular points. On the other hand, near singular points the *contact set* $\{u = 0\}$ forms cups and can be pretty wild —see for instance the examples given in [Sch76] and [KN77]. Moreover, as shown in [Sch76], even C^∞ strictly superharmonic obstacles in the plane ($n = 2$) may lead to contact sets with Cantor set like structures. In particular, in such examples, the contact set has (locally) an infinite number of connected components, each containing singular points.

Despite these “negative” results showing that singular points could be rather bad, it is still possible to prove some nice structure. More precisely, singular points are naturally stratified according to the dimension of the linear space

$$L_{x_o} := \{p_{*,x_o} = 0\} = \ker(A_{x_o}).$$

For $m \in \{0, 1, 2, \dots, n-1\}$ we define the m -th stratum as

$$\Sigma_m := \{x_o : \text{singular point with } \dim(L_{x_o}) = m\}.$$

As shown by Caffarelli in [C98], each stratum Σ_m is locally contained in a m -dimensional manifold of class C^1 (see also [M03] for an alternative proof). This result has been improved in dimension $n = 2$ by Weiss [W99]: using an epiperimetric-type approach, he has been able to prove that Σ_1 is locally contained in a $C^{1,\alpha}$ curve, for some universal exponent $\alpha > 0$. Along the same lines, in a recent paper Colombo, Spolaor, and Velichkov [CSV17] have obtained a logarithmic epiperimetric inequality at singular points in any dimension $n \geq 3$, thus improving the known C^1 regularity to a more quantitative C^{1,\log^ϵ} one.

The aim of this paper is to improve the previous known results by showing that, up to the presence of some “anomalous” points of higher codimension, singular points can be covered by $C^{1,1}$ (and in some cases C^2) manifolds. As we shall discuss in Remark 1.2, this result provides the optimal decay estimate for the contact set. In addition, anomalous points may exist and our bound on their Hausdorff dimension is optimal.

Before stating our result we note that, as a consequence of [C98], points in Σ_0 are isolated and u is strictly positive in a neighborhood of them. In particular u solves $\Delta u = 1$ in a neighborhood of Σ_0 , hence it is analytic there. Thus, it is enough to understand the structure of Σ_m for $m = 1, \dots, n-1$.

Here and in the sequel, $\dim_{\mathcal{H}}(E)$ denotes the Hausdorff dimension of a set E (see (3.16) for a definition). Our main result is the following:

Theorem 1.1. *Let $u \in C^{1,1}(B_1)$ be a solution of (1.1), and let $\Sigma := \cup_{m=0}^{n-1} \Sigma_m$ denote the set of singular points. Then:*

- ($n = 2$) Σ_1 is locally contained in a C^2 curve.
- ($n \geq 3$) (a) *The higher dimensional stratum Σ_{n-1} can be written as the disjoint union of “generic points” Σ_{n-1}^g and “anomalous points” Σ_{n-1}^a , where:*
- Σ_{n-1}^g is locally contained in a $C^{1,1}$ $(n-1)$ -dimensional manifold;
 - Σ_{n-1}^a is a relatively open subset of Σ_{n-1} satisfying $\dim_{\mathcal{H}}(\Sigma_{n-1}^a) \leq n-3$ (actually, Σ_{n-1}^a is discrete when $n = 3$).
- Furthermore, Σ_{n-1} can be locally covered by a C^{1,α_0} $(n-1)$ -dimensional manifold, for some dimensional exponent $\alpha_0 > 0$.*
- (b) *For all $m = 1, \dots, n-2$ we can write $\Sigma_m = \Sigma_m^g \cup \Sigma_m^a$, where:*
- Σ_m^g can be locally covered by a $C^{1,1}$ m -dimensional manifold;
 - Σ_m^a is a relatively open subset of Σ_m satisfying $\dim_{\mathcal{H}}(\Sigma_m^a) \leq m-1$ (actually, Σ_m^a is discrete when $m = 1$).
- In addition, Σ_m can be locally covered by a $C^{1,\log \epsilon_0}$ m -dimensional manifold, for some dimensional exponent $\epsilon_0 > 0$.*

Remark 1.2. We first discuss the optimality of the above theorem.

- (1) Our $C^{1,1}$ regularity provides the optimal control on the contact set in terms of the density decay. Indeed our result implies that, at all singular points up to a $(n-3)$ -dimensional set (in particular at all singular points when $n = 2$, and at all singular points up to a discrete set when $n = 3$), the following bound holds:

$$\frac{|\{u = 0\} \cap B_r(x_o)|}{|B_r(x_o)|} \leq Cr \quad \forall r > 0$$

(see Proposition 2.13, Definition (3.13), and Lemmas 3.4, 3.6, 3.7, and 3.9). In view of the two dimensional Example 1 in [Sch76, Section 1], this estimate is optimal.

- (2) The possible presence of anomalous points comes from different reasons depending on the dimension of the stratum. More precisely, as the reader will see from the proof (see also the description of the strategy of the proof given below), the following holds:
- (a) The possible presence of points in Σ_{n-1}^a comes from the potential existence, in dimension $n \geq 3$, of λ -homogeneous solutions to the Signorini problem with $\lambda \in (2, 3)$. Whether this set is empty or not is an interesting open problem.
- (b) The anomalous points in the strata Σ_m^a for $m \leq n-2$ come from the possibility that, around a singular point x_o , the function $(u - p_{*,x_o})|_{B_r(x_o)}$ behaves as $\varepsilon_r q$, where:
- ε_r is infinitesimal as $r \rightarrow 0^+$, but $\varepsilon_r \gg r^\alpha$ for any $\alpha > 0$;
 - q is a nontrivial second order harmonic polynomial.
- Although this behavior may look strange, it can actually happen and our estimate on the size of Σ_m^a is optimal. Indeed, in the Appendix we construct examples of solutions for which $\dim(\Sigma_m^a) = m-1$.

We now make some general comments on Theorem 1.1.

- Remark 1.3.** (1) Our result on the higher dimensional stratum Σ_{n-1} extends the result of [W99] to every dimension, and improves it in terms of the regularity. Actually, as shown in Theorem 4.7, for any $m = 1, \dots, n-1$ we can cover Σ_m with countably many C^2 m -dimensional manifolds, up to a set of dimension at most $m-1$.
- (2) The last part of the statement in the case $(n \geq 3)$ -(b) was recently proved in [CSV17]. Here we reobtain the same result as a simple consequence of our analysis (see the proof of Theorem 1.1).
- (3) As we shall see, the higher regularity of the free boundary stated in the previous theorem comes with a higher regularity of the solution u around singular points. More precisely, Σ being of class $C^{k,\alpha}$ at some singular point x_o corresponds to u being of class $C^{k+1,\alpha}$ at such point.
- (4) The fact that Σ_m^a is relatively open implies that if $x_o \in \Sigma_m^a$ then $B_\rho(x_o) \cap \Sigma_m^a = B_\rho(x_o) \cap \Sigma_m$ for $\rho > 0$ small. In particular $\dim_{\mathcal{H}}(B_\rho(x_o) \cap \Sigma_m) \leq m-1$ ($\leq n-3$ if $m = n-1$). In other words, the whole stratum Σ_m is lower dimensional near anomalous points.
- (5) In [Sak91, Sak93], Sakai proved very strong structural results for the free boundary in dimension $n = 2$. However, his results are very specific to the two dimensional case with analytic right hand side, as they rely on complex analysis techniques. On the other hand, all the results mentioned before [C77, C98, W99, CSV17] are very robust and apply to more general right hand sides. Analogously, also our techniques are robust and can be extended to general right hand sides. In addition, our methods can be applied to the study of the regularity of the free boundary in the parabolic case (the so-called Stefan problem), a problem that cannot be studied with complex variable techniques even in dimension two.

Strategy of the proof of Theorem 1.1. The idea of the proof is the following: let 0 be a singular free boundary point. As shown in [C98, M03] u is C^2 at 0, namely there exists a second order homogeneous polynomial p_* , with $D^2 p_* \geq 0$ and $\Delta p_* = 1$, such that $u(x) = p_*(x) + o(|x|^2)$. In order to obtain our result, our goal is to improve the convergence rate $o(|x|^2)$ into a quantitative bound of the form $O(|x|^{2+\gamma})$ for some $\gamma > 0$. In particular, to obtain $C^{1,1}$ regularity of the singular set we would like to show that $\gamma \geq 1$.

Using monotonicity formulae due to Weiss and Monneau, we are able to prove that Almgren frequency function is monotone on $w := u - p_*$ (this result came as a complete surprise to us, as the Almgren frequency formula has never been used in the classical obstacle problem). This allows us to perform blow-ups around 0 by considering limits of

$$\tilde{w}_r(x) := \frac{w(rx)}{\|w(r \cdot)\|_{L^2(\partial B_1)}} \quad \text{as } r \rightarrow 0,$$

and prove that if λ_* is the value of the frequency at 0 then $u(x) = p_*(x) + O(|x|^{\lambda_*})$. Although it is easy to see that $\lambda_* \geq 2$, it is actually pretty delicate—and actually sometimes impossible—to exclude that $\lambda_* = 2$ (note that, in such a case, we would

get no new informations with respect to what was already known). Hence our goal is to understand the possible value of λ_* .

To this aim, we consider q a limit of \tilde{w}_r and, exploiting the monotonicity of the frequency, we prove that $q \not\equiv 0$, q is λ_* -homogeneous, and $q\Delta q \equiv 0$.

Then we distinguish between the two cases $m = n - 1$ and $m \leq n - 2$. While in the latter case we can prove that q is harmonic (therefore $\lambda_* \in \{2, 3, 4, \dots\}$), in the case $m = n - 1$ we prove that q is a solution of the so-called ‘‘Signorini problem’’ (see for instance [AC04, ACS08]). In particular, when $n = 2$, this allows us to characterize all the possible values of λ_* in dimension 2 (as all global two-dimensional homogeneous solutions are classified). Still, this does not exclude that $\lambda_* = 2$. As shown in Proposition 2.10 this can be excluded in the case $m = n - 1$, while the examples constructed in the Appendix show that λ_* may be equal to 2 if $m \leq n - 2$. To circumvent this difficulty, a key ingredient in our analysis comes from Equation (2.10) which shows that, whenever $\lambda_* = 2$, some strong relation between p_* and q holds. Thus, our goal becomes to prove that this relation cannot hold at ‘‘too many’’ singular points.

In order to estimate the size of the set where $\lambda_* < 3$, we first consider the low-dimension cases $n = 2$ and $n = 3$, and then we develop a Federer-type dimension reduction principle to handle the case $n \geq 4$. Note that the Federer dimension reduction principle is not standard in this setting, the reason being that if x_0 and x_1 are two different singular points, then the blow-ups at such points come from different functions, namely $u - p_{*,x_0}$ and $u - p_{*,x_1}$. Still, we can prove the validity of a dimension reduction-type principle allowing us to conclude that, at most points, $\lambda_* \geq 3$. This proves the main part of the theorem.

Then, to show that Σ_{n-1} is contained in a C^{1,α_\circ} -manifolds we prove that $\lambda_* \geq 2 + \alpha_\circ > 2$ at all points in Σ_{n-1} . Also, the C^{1,\log^ϵ} regularity of Σ_m for $m \leq n - 2$ comes a simple consequence of our analysis combined with Caffarelli’s asymptotic convexity estimate [C77].

Finally, the C^2 regularity in two-dimensions requires a further argument based on a new monotonicity formula of Monneau-type.

The paper is organized as follows.

In Section 2 we introduce some classical monotonicity quantities, as well as some variants of them that will play a crucial role in our analysis. In particular, we prove the validity of a Almgren’s monotonicity-type formula. Then, given a singular free boundary point x_\circ , we investigate the properties of the blow-ups of $u(x_\circ + \cdot) - p_{*,x_\circ}$.

In Section 3 we continue our analysis of the possible homogeneities of the blow-ups and show the validity of a Federer-type reduction principle. These results, combined with the ones from Section 2, allow us to prove Theorem 1.1 in dimensions $n \geq 3$, as well as the $C^{1,1}$ regularity of Σ_1 in dimension $n = 2$. The proof of the C^2 regularity of Σ_1 for $n = 2$ is postponed to Section 4.

In the final Appendix we build solutions exhibiting anomalous points that show the sharpness of Theorem 1.1(b).

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2. NOTATION, MONOTONICITY FORMULAE, AND BLOW-UPS

Let us denote

$$(2.1) \quad \mathcal{M} := \left\{ \text{symmetric } n \times n \text{ nonnegative definite matrices } B \text{ with } \text{tr } B = 1 \right\}$$

and

$$\mathcal{P} := \left\{ p(x) = \frac{1}{2} x \cdot Bx \ : \ B \in \mathcal{M} \right\}.$$

Given a singular free boundary point x_\circ , we denote

$$p_{*,x_\circ}(x) = \lim_{r \rightarrow 0} r^{-2} u(x_\circ + rx)$$

(the existence of this limit is guaranteed by [C98], see also [M03]). Note that $\Delta p_{*,x_\circ} \equiv 1$, hence $p_{*,x_\circ} \in \mathcal{P}$. When $x_\circ = 0$, we will sometimes simplify the notation to p_* .

Throughout the paper we will assume that $u \not\equiv p_*$ in B_1 , as otherwise Theorem 1.1 is trivial.

2.1. Weiss, Monneau, and Almgren frequency formula. In this section we assume that $x_\circ = 0$ is a singular point. The goal of the section is to prove that, for any given $p \in \mathcal{P}$, the Almgren frequency formula

$$\phi(r, w) := \frac{r^{2-n} \int_{B_r} |\nabla w|^2}{r^{1-n} \int_{\partial B_r} w^2}, \quad w := u - p,$$

is monotone nondecreasing in r . (Note that, since by assumption $u \not\equiv p_*$, then $w := u - p \not\equiv 0$ for any $p \in \mathcal{P}$ and $\phi(r, w)$ is well defined.)

To this aim, we first recall the definition of the Weiss function

$$W(r, u) := \frac{1}{r^{n+2}} \int_{B_r} (|\nabla u|^2 + 2u) - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2.$$

Proposition 2.1 (Weiss monotone function [W99]). *If 0 is a singular point then*

$$\frac{d}{dr} W(r, u) \geq 0$$

and

$$W(0^+, u) = \frac{\mathcal{H}^{n-1}(\partial B_1)}{2n(n+2)} = W(r, p) \quad \forall p \in \mathcal{P}, \forall r > 0.$$

To prove the monotonicity of ϕ we will use several times the following observation:

Remark 2.2. Since $\Delta u = \Delta p = 1$ in $\{u > 0\}$, we have

$$w \Delta w = \begin{cases} 0 & \text{in } \{u > 0\} \\ p \Delta p = p \geq 0 & \text{in } \{u = 0\}. \end{cases}$$

A short way to write this is

$$(2.2) \quad w \Delta w = p \chi_{\{u=0\}} \geq 0.$$

We also need the following auxiliary result, that is essentially due to Monneau [M03].

Lemma 2.3. *Let 0 be a singular point, $p \in \mathcal{P}$, and $w := u - p$. Then*

$$(2.3) \quad \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 \geq \frac{2}{r^{n+3}} \int_{\partial B_r} w^2$$

and

$$(2.4) \quad \frac{1}{r^{n+3}} \int_{\partial B_r} w(x \cdot \nabla w - 2w) \geq \frac{1}{r^{n+2}} \int_{B_r} w \Delta w \geq 0$$

for all $r > 0$.

Proof. Since $W(0^+, u) = W(r, p)$ for all $r > 0$ (see Proposition 2.1) and $\Delta p \equiv 1$, we have

$$\begin{aligned} 0 &\leq W(r, u) - W(0^+, u) = W(r, u) - W(r, p) \\ &= \frac{1}{r^{n+2}} \int_{B_r} \left(|\nabla w|^2 + 2\nabla w \cdot \nabla p + 2w \right) - \frac{2}{r^{n+3}} \int_{\partial B_r} \left(w^2 + 2wp \right) \\ &= \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 - \frac{2}{r^{n+3}} \int_{\partial B_r} w^2 + \frac{2}{r^{n+3}} \int_{\partial B_r} w(x \cdot \nabla p - 2p) \\ &= \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 - \frac{2}{r^{n+3}} \int_{\partial B_r} w^2, \end{aligned}$$

where we used that p is 2-homogeneous (hence $x \cdot \nabla p = 2p$). This proves (2.3).

Now, since

$$\frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 = \frac{1}{r^{n+2}} \int_{B_r} -w \Delta w + \frac{1}{r^{n+3}} \int_{\partial B_r} w x \cdot \nabla w,$$

(2.4) follows from (2.3) and (2.2). \square

We can now state and prove the monotonicity of the Almgren frequency function. We remark that the fact that ϕ is monotone for all $p \in \mathcal{P}$ (and not only with $p = p_*$) will be crucial in the proof of Theorem 1.1.

Proposition 2.4 (Almgren frequency formula). *Let 0 be a singular point, $p \in \mathcal{P}$, and $w := u - p$. Then*

$$\frac{d}{dr} \log \phi(r, w) \geq \frac{2}{r} \frac{\left(r^{2-n} \int_{B_r} w \Delta w \right)^2}{r^{2-n} \int_{B_r} |\nabla w|^2 \quad r^{1-n} \int_{\partial B_r} w^2} \geq 0.$$

Proof of Proposition 2.4. Let us introduce the adimensional quantities

$$\begin{aligned} D(r) &:= r^{2-n} \int_{B_r} |\nabla w|^2 = r^2 \int_{B_1} |\nabla w|^2(r \cdot), \\ H(r) &:= r^{1-n} \int_{\partial B_r} w^2 = \int_{\partial B_1} w^2(r \cdot), \end{aligned}$$

so that $\phi = D/H$. By scaling it is enough to compute the derivative of ϕ at $r = 1$ and prove that it is nonnegative.

Using lower indices to denote partial derivatives (so $w_i = \partial_{x_i} w$, $w_{ij} = \partial_{x_i x_j}^2 w$, etc.), we have

$$(2.5) \quad \frac{d}{dr} \log \phi = \frac{D'}{D} - \frac{H'}{H}$$

where

$$(2.6) \quad \begin{aligned} D'(1) &= \sum_{i,j} \int_{B_1} 2w_i x_j w_{ij} + 2D(1) \\ &= \sum_{i,j} \int_{\partial B_1} 2w_i x_j w_j \nu_i - \sum_{i,j} \int_{B_1} 2(w_i x_j)_i w_j + 2D(1) \\ &= 2 \int_{\partial B_1} w_\nu^2 - 2 \int_{B_1} \Delta w (x \cdot \nabla w) - 2 \int_{B_1} |\nabla w|^2 + 2D(1) \\ &= 2 \int_{\partial B_1} w_\nu^2 - 2 \int_{B_1} \Delta w (x \cdot \nabla w) \\ &= 2 \int_{\partial B_1} w_\nu^2 - 2 \int_{B_1 \cap \{u=0\}} (x \cdot \nabla p) \\ &= 2 \int_{\partial B_1} w_\nu^2 - 4 \int_{B_1 \cap \{u=0\}} p. \end{aligned}$$

Here we used that $x \cdot \nabla w|_{\partial B_1} = w_\nu|_{\partial B_1}$ is the outer normal derivative, $\Delta w = -\chi_{\{u=0\}}$, and $x \cdot \nabla p = 2p$ (since p is 2-homogeneous).

On the other hand, recalling (2.2), we have

$$(2.7) \quad \int_{\partial B_1} w w_\nu = \int_{B_1} w \Delta w + \int_{B_1} |\nabla w|^2 = \int_{B_1 \cap \{u=0\}} p + \int_{B_1} |\nabla w|^2,$$

therefore

$$(2.8) \quad H'(1) = 2 \int_{\partial B_1} w w_\nu = 2 \int_{B_1 \cap \{u=0\}} p + 2 \int_{B_1} |\nabla w|^2.$$

Hence, combining (2.6), (2.7), (2.8), and (2.5), and denoting

$$I := \int_{B_1} w \Delta w = \int_{B_1 \cap \{u=0\}} p \geq 0,$$

we obtain

$$\begin{aligned} \frac{d}{dr} \log \phi(1, w) &= 2 \left(\frac{\int_{\partial B_1} w_\nu^2 - 2I}{\int_{B_1} |\nabla w|^2} - \frac{\int_{\partial B_1} w w_\nu}{\int_{\partial B_1} w^2} \right) \\ &= 2 \frac{(\int_{\partial B_1} w_\nu^2 - 2I) \int_{\partial B_1} w^2 - \int_{\partial B_1} w w_\nu (\int_{\partial B_1} w w_\nu - I)}{\int_{B_1} |\nabla w|^2 \int_{\partial B_1} w^2} \\ &= 2 \frac{\left\{ \int_{\partial B_1} w_\nu^2 \int_{\partial B_1} w^2 - (\int_{\partial B_1} w w_\nu)^2 \right\} + I \int_{\partial B_1} w (w_\nu - 2w)}{\int_{B_1} |\nabla w|^2 \int_{\partial B_1} w^2} \end{aligned}$$

Observe that the first term inside the brackets is nonnegative by the Cauchy-Schwartz inequality. Also, recalling (2.4), we have that

$$\int_{\partial B_1} w(w_\nu - 2w) \geq \int_{B_1} w \Delta w = I.$$

Since $I \geq 0$, the result follows. \square

Note that, because $r \mapsto \phi(r, w)$ is monotone nondecreasing, it must have a limit as $r \downarrow 0$. The first observation is that this limit is at least 2.

Lemma 2.5. *Let 0 be a singular point, $p \in \mathcal{P}$, and $w := u - p$. Then $\phi(0^+, w) \geq 2$.*

Proof. It suffices to observe that (2.3) is equivalent to $\phi(r, w) \geq 2$ for all $r > 0$. \square

A first classical consequence of the frequency formula is the following monotonicity formula:

Lemma 2.6. *Let 0 be a singular point, $p \in \mathcal{P}$, and $w := u - p$. Given $\lambda > 0$ denote*

$$H_\lambda(r, w) := \frac{1}{r^{n-1+2\lambda}} \int_{\partial B_r} w^2$$

Then the function $r \mapsto H_\lambda(r, w)$ is monotone nondecreasing for all $0 \leq \lambda \leq \phi(0^+, w)$.

Proof. Denoting

$$w_r(x) := (u - p)(rx).$$

we have

$$\frac{H'_\lambda}{H_\lambda}(r, w) = \frac{2r^{-2\lambda} \int_{\partial B_1} w_r(x) (x \cdot \nabla w(rx)) - 2\lambda r^{-2\lambda-1} \int_{\partial B_1} w_r^2}{r^{-2\lambda} \int_{\partial B_1} w_r^2}.$$

Using that

$$r \int_{\partial B_1} w_r(x) (x \cdot \nabla w(rx)) = \int_{\partial B_1} w_r (x \cdot \nabla w_r) = \int_{B_1} |\nabla w_r|^2 + \int_{B_1} w_r \Delta w_r$$

and that $w_r \Delta w_r \geq 0$ (recall (2.2)), we obtain

$$\frac{H'_\lambda}{H_\lambda}(r, w) \geq \frac{2 \int_{B_1} |\nabla w_r|^2}{r \int_{\partial B_1} w_r^2} - \frac{2\lambda}{r} = \frac{2}{r} (\phi(r, w) - \lambda).$$

Since $\phi(r, w) \geq \phi(0^+, w) \geq \lambda$ (by Proposition 2.4), the result follows. \square

Corollary 2.7 (Monneau monotonicity formula [M03]). *Let 0 be a singular point and let H_λ be as in Lemma 2.6. The function $H_2(r, u - p)$ is monotone nondecreasing in r , for all p in \mathcal{P} .*

Proof. It is a direct consequence of Lemmas 2.6 and 2.5. \square

The following result shows the monotonicity for a modified Weiss function. It is remarkable that the quantity below is monotone for all $\lambda > 0$, independently of the value of the frequency.

Lemma 2.8. *Let 0 be a singular point, $\lambda \geq 0$, and $w := u - p$, where $p \in \mathcal{P}$. Then the function*

$$W_\lambda(r, w) := r^{-2\lambda} \left(r^{2-n} \int_{B_r} |\nabla w|^2 - \lambda r^{1-n} \int_{\partial B_r} w^2 \right)$$

is monotone nondecreasing in r .

Proof. For $0 \leq \lambda \leq \phi(0^+, w)$ we have $W_\lambda = (\phi - \lambda)H_\lambda$, the product of two positive nondecreasing functions (thanks to Proposition 2.4 and Lemma 2.6), hence W_λ is nondecreasing.

The result is more interesting for $\lambda > \phi(0^+, w)$ and it requires a different proof. Indeed, using the notation and calculations from the proof of Proposition 2.4 we have, for $I := \int_{B_1} w \Delta w = \int_{B_1 \cap \{u=0\}} p \geq 0$,

$$\begin{aligned} W'_\lambda(1) &= D'(1) - \lambda H'(1) - 2\lambda(D(1) - \lambda H(1)) \\ &= \left(2 \int_{\partial B_1} w_\nu^2 - 4I \right) - 2\lambda \int_{\partial B_1} w w_\nu - 2\lambda(D(1) - \lambda H(1)) \\ &= \left(2 \int_{\partial B_1} w_\nu^2 - 4I \right) - 2\lambda \int_{\partial B_1} w w_\nu - 2\lambda \left(\int_{\partial B_1} w w_\nu - I \right) + 2\lambda^2 H(1) \\ &= 2 \left(\int_{\partial B_1} w_\nu^2 - 2\lambda \int_{\partial B_1} w w_\nu + \lambda^2 \int_{\partial B_1} w^2 \right) + 2(\lambda - 2)I \\ &= 2 \int_{\partial B_1} (w_\nu - \lambda w)^2 + 2(\lambda - 2)I. \end{aligned}$$

Since $\lambda > \phi(0^+, w) \geq 2$ (by Lemma 2.5), the result follows. \square

As a consequence of this result we can prove that, given $\lambda > \lambda_* = \phi(0^+, u - p_*)$, the function H_λ blows up at 0. This, combined with the monotonicity of H_{λ_*} (see Lemma 2.6), shows that

$$(2.9) \quad r^{2\lambda} \lesssim \int_{\partial B_r} w^2 \lesssim r^{2\lambda_*} \quad \text{for } r \ll 1.$$

Note that while this estimate is classical for harmonic functions (since the frequency function is related to the derivative of H_λ), in our case only an inequality is available (see the proof of Lemma 2.6) and a different argument is needed.

Corollary 2.9. *Let 0 be a singular point, $w := u - p_*$, $\lambda_* := \phi(0^+, w)$, and fix $\lambda > \lambda_*$. Let H_λ be as in as in Lemma 2.6. Then*

$$H_\lambda(r, w) \rightarrow +\infty \quad \text{as } r \downarrow 0.$$

Proof. Assume by contradiction that there exists a sequence $r_k \downarrow 0$ such that $H_\lambda(r_k, w) \leq C$ for some constant C . Then, taking $\mu \in (\lambda_*, \lambda)$, it follows that $H_\mu(r_k, w) \rightarrow 0$. Hence, with the notation of Lemma 2.8, this gives (since $W_\mu \geq -\mu H_\mu$)

$$\liminf_{k \rightarrow \infty} W_\mu(r_k, w) \geq \liminf_{k \rightarrow \infty} -\mu H_\mu(r_k, w) = 0.$$

By the monotonicity of W_μ , this implies that $W_\mu(r, w) \geq 0$ for all $r > 0$, or equivalently

$$r^{2-n} \int_{B_r} |\nabla w|^2 \geq \mu r^{1-n} \int_{\partial B_r} w^2 \quad \forall r > 0.$$

But this means that $\phi(r, w) \geq \mu$ for all $r > 0$, a contradiction to the fact that $\mu > \lambda_*$. \square

2.2. Blow-up analysis. We now start investigating the structure of possible blow-ups.

Proposition 2.10. *Let 0 be a singular point, $w := u - p_*$, and for $r > 0$ small define*

$$w_r(x) := w(rx), \quad \tilde{w}_r := \frac{w_r}{\|w_r\|_{L^2(\partial B_1)}}.$$

Let $L := \{p_* = 0\}$, and $m \in \{0, 1, 2, \dots, n-1\}$ be the dimension of L . Also, let $\lambda_* := \phi(0^+, w)$. Then:

- (a) For $0 \leq m \leq n-2$ we have $\lambda_* \in \{2, 3, 4, 5, \dots\}$. Moreover, for every sequence $r_k \downarrow 0$ there is a subsequence r_{k_ℓ} such that $\tilde{w}_{r_{k_\ell}} \rightharpoonup q$ in $W^{1,2}(B_1)$ as $\ell \rightarrow \infty$, where $q \not\equiv 0$ is a λ_* -homogeneous harmonic polynomial.

In addition, if $\lambda_* = 2$, then in an appropriate coordinate frame it holds

$$(2.10) \quad D^2 p_* = \left(\begin{array}{ccc|c} \mu_1 & & & 0_m^{n-m} \\ & \ddots & & \\ & & \mu_{n-m} & \\ \hline & & & 0_m^m \end{array} \right) \quad \text{and} \quad D^2 q = \left(\begin{array}{ccc|c} t & & & 0_m^{n-m} \\ & \ddots & & \\ & & t & \\ \hline & & & -N \end{array} \right),$$

where $\mu_1, \dots, \mu_{n-m}, t > 0$, $\sum_{i=1}^{n-m} \mu_i = 1$, and N is a symmetric nonnegative definite $m \times m$ matrix with $\text{tr}(N) = (n-m)t$.

- (b) For $m = n-1$ we have $\lambda_* \geq 2 + \alpha_o$, where $\alpha_o > 0$ is a dimensional constant. Moreover, for every sequence $r_k \downarrow 0$ there is a subsequence r_{k_ℓ} such that $\tilde{w}_{r_{k_\ell}} \rightharpoonup q$ in $W^{1,2}(B_1)$, where $q \not\equiv 0$ is a λ_* -homogeneous solution of the Signorini problem (with obstacle 0 on L):

$$(2.11) \quad \Delta q \leq 0 \quad \text{and} \quad q \Delta q = 0 \quad \text{in } \mathbb{R}^n, \quad \Delta q = 0 \quad \text{in } \mathbb{R}^n \setminus L, \quad \text{and} \quad q \geq 0 \quad \text{on } L.$$

To prove Proposition 2.10, we need the following auxiliary lemmas:

Lemma 2.11. *Let \tilde{w}_r be as in Proposition 2.10 and assume that, for some sequence $r_{k_\ell} \downarrow 0$, it holds $\tilde{w}_{r_{k_\ell}} \rightharpoonup q$ in $W^{1,2}(B_1)$. Then*

$$(2.12) \quad \int_{\partial B_1} q(p_* - p) \geq 0 \quad \text{for all } p \in \mathcal{P}.$$

Proof. By the definition of p_* it holds that

$$w_r(x) = (u - p_*)(rx) = o(r^2) \quad \text{as } r \downarrow 0.$$

Let us denote $h_r := \|w_r\|_{L^2(\partial B_1)} = o(r^2)$ and $\varepsilon_r := h_r/r^2 = o(1)$ as $r \downarrow 0$. Note that, by the compactness of the trace operator $W^{1,2}(B_1) \rightarrow L^2(\partial B_1)$, we have $\tilde{w}_{r_{k_\ell}} = w_{r_{k_\ell}}/h_{r_{k_\ell}} \rightarrow q$ in $L^2(\partial B_1)$.

By Corollary 2.7 and the definition of p_* , for any fixed $p \in \mathcal{P}$ we have

$$\int_{\partial B_1} \left(\frac{w_r}{r^2} + p_* - p \right)^2 = \int_{\partial B_1} \left(\frac{u(rx) - p(rx)}{r^2} \right)^2 \downarrow \int_{\partial B_1} (p_* - p)^2 \quad \text{as } r \downarrow 0.$$

Hence, since $r^{-2}w_r = \varepsilon_r \tilde{w}_r$,

$$\int_{\partial B_1} (\varepsilon_r \tilde{w}_r + p_* - p)^2 \geq \int_{\partial B_1} (p_* - p)^2 \quad \forall r > 0, \forall p \in \mathcal{P}.$$

Developing the squares and taking $r = r_{k_\ell}$ we get

$$\varepsilon_{r_{k_\ell}}^2 \int_{\partial B_1} \tilde{w}_{r_{k_\ell}}^2 + 2\varepsilon_{r_{k_\ell}} \int_{\partial B_1} \tilde{w}_{r_{k_\ell}} (p_* - p) \geq 0.$$

Dividing by $\varepsilon_{r_{k_\ell}}$ and letting $\ell \rightarrow \infty$ we obtain (2.12). \square

Lemma 2.12. *Let $p_* \in \mathcal{P}$, and assume that $q \not\equiv 0$ is a 2-homogeneous harmonic polynomial satisfying (2.12). Then, in an appropriate system of coordinates, (2.10) holds.*

Proof. Take $p \in \mathcal{P}$ and define $A := D^2 p_*$, $B := D^2 p$, and $C := D^2 q$. Then, since $x \cdot \nabla q = 2q$ and $\Delta q = 0$, it follows from (2.12) that

$$\begin{aligned} 0 &\leq \int_{\partial B_1} q(p_* - p) = \frac{1}{2} \int_{\partial B_1} q_\nu(p_* - p) = \frac{1}{2} \int_{B_1} \nabla q \cdot \nabla(p_* - p) \\ &= \frac{1}{2} \int_{B_1} Cx \cdot (A - B)x \, dx = c_n \operatorname{tr}(C(A - B)), \end{aligned}$$

for some dimensional constant $c_n > 0$. Hence, since $p \in \mathcal{P}$ was arbitrary, we deduce that (recall (2.1))

$$(2.13) \quad \operatorname{tr}(CA) \geq \operatorname{tr}(CB) \quad \text{for all } B \in \mathcal{M}.$$

To show that this implies (2.10), let $\mathbf{v} \in \mathbb{S}^{n-1}$ be an eigenvector for C corresponding to its largest eigenvalue $\nu_{\max} > 0$, and choose $B := \mathbf{v} \otimes \mathbf{v}$. Then, since $A \geq 0$ and $\operatorname{tr}(A) = 1$, (2.13) yields

$$\nu_{\max} = \operatorname{tr}(\nu_{\max} \operatorname{Id} A) \geq \operatorname{tr}(CA) \geq \operatorname{tr}(CB) = \nu_{\max}.$$

Thus

$$\operatorname{tr}([\nu_{\max} \operatorname{Id} - C]A) = 0,$$

and because both A and $\nu_{\max} \operatorname{Id} - C$ are symmetric and nonnegative definite, we deduce that the kernels of these two matrices decompose orthogonally \mathbb{R}^n . In addition, if we set $L = \{p_* = 0\} = \ker(A)$, then $(\nu_{\max} \operatorname{Id} - C)|_{L^\perp} \equiv 0$. Thanks to this fact and recalling that $\operatorname{tr}(C) = 0$ (since q is harmonic), the result follows easily. \square

We can now prove Proposition 2.10.

Proof of Proposition 2.10. For the sake of clarity, we divide the proof into several steps.

- *Step 1.* We note that $\{\tilde{w}_r\}$ is precompact. Indeed, by Proposition 2.4 we have

$$\int_{\partial B_1} \tilde{w}_r^2 = 1 \quad \text{and} \quad \int_{B_1} |\nabla \tilde{w}_r|^2 = \phi(r) \leq \phi(1) < \infty.$$

This yields uniform bounds $\|\tilde{w}_r\|_{W^{1,2}(B_1)} \leq C$ for all $r \in (0, 1)$. As a consequence, given a sequence $r_k \downarrow 0$ there is a subsequence $r_{k_\ell} \downarrow 0$ such that

$$\tilde{w}_{r_{k_\ell}} \rightharpoonup q \quad \text{in } W^{1,2}(B_1).$$

In particular, by the compactness of the trace operator $W^{1,2}(B_1) \rightarrow L^2(\partial B_1)$, it follows that

$$\|q\|_{L^2(\partial B_1)} = 1.$$

- *Step 2.* We prove (a). So, we assume $m \leq n-2$ and we consider q a possible limit of a converging sequence $\tilde{w}_{r_{k_\ell}}$. We want to prove that q is a harmonic homogeneous polynomial.

We first show that q is harmonic. Note that

$$(2.14) \quad \Delta w_r(x) = \Delta u(rx) - \Delta p_*(rx) = -r^2 \chi_{\{u=0\}}(rx) \leq 0,$$

hence $\Delta \tilde{w}_r$ is a nonpositive measure. Note also that the contact set $\{u(r \cdot) = 0\}$ converge in the Hausdorff sense to $L = \{p_* = 0\}$ as $r \rightarrow 0$ (this follows from the uniform convergence of $r^{-2}u(rx)$ to p_* as $r \rightarrow 0$). This implies that q has a distributional Laplacian given by a nonpositive measure supported in L . Since $q \in W^{1,2}(B_1)$ (by Step 1) and L has codimension 2 (and thus it is of zero harmonic capacity) it follows that q must be harmonic.

Let us prove next that q is homogeneous. To this aim we show that

$$(2.15) \quad \lambda_* = \phi(R, q) := \frac{R^{2-n} \int_{\partial B_R} |\nabla q|^2}{R^{1-n} \int_{\partial B_R} q^2} \quad \forall R \in (0, 1].$$

Indeed, by lower semicontinuity of the Dirichlet integral we have

$$\phi(1, q) \leq \liminf_{\ell \rightarrow \infty} \phi(1, \tilde{w}_{r_{k_\ell}}) = \liminf_{\ell \rightarrow \infty} \phi(1, w_{r_{k_\ell}}) = \liminf_{\ell \rightarrow \infty} \phi(r_{k_\ell}, w) = \lambda_*.$$

Also, since q is harmonic, it follows that $R \mapsto \phi(R, q)$ is nondecreasing (this follows from the classical Almgren frequency formula, or equivalently from the proof of Proposition 2.4), thus $\phi(R, q) \leq \lambda_*$ for all $R \in (0, 1]$.

To show the converse inequality we apply Lemma 2.6 to $\tilde{w}_{r_{k_\ell}}$ and let $\ell \rightarrow \infty$ to obtain

$$(2.16) \quad \frac{1}{\rho^{2\lambda_*}} \int_{\partial B_\rho} q^2 \leq \int_{\partial B_1} q^2 = 1.$$

But since q is harmonic (so, in particular, $q\Delta q \equiv 0$) we have

$$\frac{H'_\lambda(R, q)}{H_\lambda(R, q)} = \frac{2}{R}(\phi(R, q) - \lambda)$$

(this is a classical identity that also follows from the proof of Lemma 2.6). Hence, if it was $\phi(R, q) < \lambda_*$ for some $R \in (0, 1)$ then, choosing $\lambda := \phi(R, q)$, we would have that H_λ would be nonincreasing on $(0, R)$. In particular we would find

$$\frac{1}{\rho^{2\lambda}} \int_{\partial B_\rho} q^2 \geq \frac{1}{R^{2\lambda}} \int_{\partial B_R} q^2 > 0 \quad \text{for } \rho \in (0, R),$$

which contradicts (2.16) for ρ small since $\lambda < \lambda_*$. Hence, we proved (2.15).

Note that (2.15) says that the Almgren frequency formula $\phi(R, q)$ is constantly equal to λ_* for all $R \in (0, 1]$. As a classical consequence, q is λ_* -homogeneous.

Hence, since q harmonic, it follows that q is a λ_* -homogeneous harmonic polynomial with $\lambda_* \in \{2, 3, 4, 5, \dots\}$ (recall that $\lambda_* \geq 2$, see Lemma 2.5).

Finally, to complete the proof of (a), it suffices to combine Lemmas 2.11 and 2.12 to obtain that (2.10) holds when $\lambda_* = 2$.

- *Step 3.* We now prove the first part of (b): if $m = n - 1$, then q must be a homogenous solution of the Signorini problem.

Indeed, let $\tilde{w}_{r_{k_\ell}} \rightarrow q$ in $L^2(B_1)$. We first show uniform semiconvexity and Lipschitz estimates that are of independent interest and will be useful later on in the paper. Namely, let us prove the estimate

$$(2.17) \quad \partial_{ee}^2 \tilde{w}_r \geq -C \quad \text{in } B_R, \quad \forall \mathbf{e} \in L \cap \mathbb{S}^{n-1}, \forall R < 1,$$

where $C = C(n, R)$ —in particular C is independent of r .

For this, given a vector $\mathbf{e} \in \mathbb{S}^{n-1}$ and $h > 0$, let

$$\delta_{\mathbf{e}, h}^2 f := \frac{f(\cdot + h\mathbf{e}) + f(\cdot - h\mathbf{e}) - 2f}{h^2}$$

denote a second order incremental quotient. For $\mathbf{e} \in L \cap \mathbb{S}^{n-1}$ we have $\delta_{\mathbf{e}, h}^2 p_* \equiv 0$ (since p_* is constant in the directions of L). Thus, since $\Delta u = 1$ outside of $\{u = 0\}$ and $\Delta u \leq 1$ everywhere,

$$\Delta(\delta_{\mathbf{e}, h}^2 w_r) = \frac{\Delta u(r(\cdot + h\mathbf{e})) + \Delta u(r(\cdot - h\mathbf{e})) - 2\Delta u(r(\cdot))}{h^2} \leq 0 \quad \text{in } B_1 \setminus \{u(r \cdot) = 0\}.$$

On the other hand, since $u \geq 0$ we have

$$\delta_{\mathbf{e}, h}^2 w_r = \delta_{\mathbf{e}, h}^2 u(r \cdot) \geq 0 \quad \text{in } \{u(r \cdot) = 0\}.$$

As a consequence, the negative part of the second order incremental quotient $(\delta_{\mathbf{e}, h}^2 \tilde{w}_r)_-$ is a (nonnegative) subharmonic function, and so is its limit $(\partial_{ee}^2 \tilde{w}_r)_-$ (recall that $u \in C^{1,1}$, hence $\delta_{\mathbf{e}, h}^2 \tilde{w}_r \rightarrow \partial_{ee}^2 \tilde{w}_r$ a.e. as $h \rightarrow 0$).

Therefore, given any radius $R' \in (R, 1)$, by the weak Harnack inequality (see for instance [CC95, Theorem 4.8(2)]) there exists $\epsilon = \epsilon(n) \in (0, 1)$ such that

$$\|(\partial_{ee}^2 \tilde{w}_r)_-\|_{L^\infty(B_R)} \leq C(n, R, R') \left(\int_{B_{R'}} (\partial_{ee} \tilde{w}_r)_-^\epsilon \right)^{1/\epsilon} \leq C(n, R, R') \left(\int_{B_{R'}} |\partial_{ee} \tilde{w}_r|^\epsilon \right)^{1/\epsilon}.$$

Also, by standard interpolation inequalities, the L^ϵ norm (here we use $\epsilon < 1$) can be controlled by the weak L^1 norm, namely

$$\left(\int_{B_{R'}} |\partial_{ee} \tilde{w}_r|^\epsilon \right)^{1/\epsilon} \leq C(n, R') \sup_{t>0} t |\{|\partial_{ee} \tilde{w}_r| > t\} \cap B_{R'}|.$$

Furthermore, by Calderon-Zygmund theory (see for instance [GT01, Equation (9.30)]), the right hand side above is controlled by $\|\Delta \tilde{w}_r\|_{L^1(B_{R''})} + \|\tilde{w}_r\|_{L^1(B_{R''})}$, with $R'' \in (R', 1)$. Finally, since $\Delta \tilde{w}_r \leq 0$, $\|\Delta \tilde{w}_r\|_{L^1(B_{R''})}$ is controlled by the L^1 norm of \tilde{w} inside B_1 : indeed, if χ is a smooth nonnegative cut-off function that is equal to 1 in $B_{R''}$ and vanished outside B_1 , then

$$(2.18) \quad \|\Delta \tilde{w}_r\|_{L^1(B_{R''})} \leq - \int_{B_1} \chi \Delta \tilde{w}_r = - \int_{B_1} \Delta \chi \tilde{w}_r \leq C(n, R'') \int_{B_1} |\tilde{w}_r|.$$

In conclusion, choosing $R' = \frac{2R+1}{3}$ and $R'' = \frac{R+2}{3}$ we obtain

$$\|(\partial_{e'e'}\tilde{w}_r)_-\|_{L^\infty(B_R)} \leq C(n, R)\|\tilde{w}_r\|_{L^1(B_1)} \leq C(n, R)$$

(recall that \tilde{w}_r is uniformly bounded in $W^{1,2}(B_1) \subset L^1(B_1)$, see Step 1), which proves (2.17).

Note that, as a consequence of (2.17), the Laplacian of \tilde{w}_r in the tangential directions is uniformly bounded from below. Since $\Delta\tilde{w}_r \leq 0$ everywhere and L is $(n-1)$ -dimensional, this implies a uniform semiconcavity estimate in the direction orthogonal to L , namely

$$\partial_{e'e'}^2\tilde{w}_r \leq C \quad \text{in } B_R, \quad \text{for } e' \in L^\perp \text{ with } |e'| = 1,$$

where, as before, $R < 1$ and $C = C(n, R)$.

Thanks to the previous semiconvexity and semiconcavity estimates, we deduce in particular a uniform Lipschitz bound:

$$(2.19) \quad |\nabla\tilde{w}_r| \leq C(n, R) \quad \text{in } B_R \quad \forall R < 1.$$

Hence, the convergence $\tilde{w}_{r_{k_\ell}} \rightarrow q$ holds also locally uniformly inside B_1 .

Now, recall that by Proposition 2.4 we have

$$(2.20) \quad \begin{aligned} r\phi'(r, w) &\geq 2\phi(r, w) \frac{\left(r^{2-n} \int_{B_r} w \Delta w\right)^2}{r^{2-n} \int_{B_r} |\nabla w|^2 \ r^{1-n} \int_{\partial B_r} w^2} \\ &= 2 \left(\frac{r^{2-n} \int_{B_r} w \Delta w}{r^{1-n} \int_{\partial B_r} w^2} \right)^2 = 2 \left(\int_{B_1} \tilde{w}_r \Delta \tilde{w}_r \right)^2. \end{aligned}$$

Since

$$\int_{r_{k_\ell}}^{2r_{k_\ell}} r\phi'(r, w) dr \leq 2 \int_{r_{k_\ell}}^{2r_{k_\ell}} \phi'(r, w) dr = 2(\phi(2r_{k_\ell}, w) - \phi(r_{k_\ell}, w)) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty$$

(because $\phi(r, w) \rightarrow \lambda_*$ as $r \rightarrow 0$), using the mean value theorem we may choose $\bar{r}_{k_\ell} \in [r_{k_\ell}, 2r_{k_\ell}]$ such that $\bar{r}_{k_\ell}\phi'(\bar{r}_{k_\ell}, w) \rightarrow 0$ as $\ell \rightarrow \infty$. Hence, thanks to (2.20) and (2.2), we deduce that, for $\rho_\ell := \bar{r}_{k_\ell}/r_{k_\ell} \in [1, 2]$,

$$\int_{B_1} \tilde{w}_{r_{k_\ell}} \Delta \tilde{w}_{r_{k_\ell}} \leq \int_{B_{\rho_\ell}} \tilde{w}_{r_{k_\ell}} \Delta \tilde{w}_{r_{k_\ell}} \rightarrow 0.$$

Since $\Delta\tilde{w}_{r_{k_\ell}} \rightarrow \Delta q$ weakly* as measures inside B_1 , $w_{r_k} \rightarrow q$ strongly in $C_{\text{loc}}^0(B_1)$, and $\tilde{w}_r \Delta \tilde{w}_r \geq 0$, we obtain

$$\int_{B_R} q \Delta q = 0 \quad \forall R < 1,$$

therefore, letting $R \uparrow 1$, $q \Delta q \equiv 0$ inside B_1 .

Now, since

- $\Delta w_r \leq 0$ is supported on $\{u(r \cdot) = 0\}$, that converges to L as $r \downarrow 0$
- $w_r = (u - p_*)(r \cdot) = u(r \cdot) \geq 0$ on L
- $\tilde{w}_{r_{k_\ell}} \rightarrow q$ locally uniformly

in the limit we obtain that $\Delta q \leq 0$, $\Delta q = 0$ outside of L , and $q \geq 0$ on L . This proves that $q \in W^{1,2}(B_1)$ is a solution of the thin obstacle problem (2.11) inside B_1 .

The same argument as the one used in Step 2 for case (a) (which only used that $q\Delta q \equiv 0$) shows that q is λ_* -homogeneous inside B_1 . In particular we can extend q by homogeneity to the whole space, and q satisfies (2.11) in \mathbb{R}^n .

- *Step 4.* We conclude the proof of (b) by showing that $\lambda_* \geq 2 + \alpha_o$ for some dimensional constant $\alpha_o > 0$.

We argue by compactness. Observe that any blow-up q satisfies

$$(2.21) \quad x \cdot \nabla q = \lambda_* q, \quad \int_{\partial B_1} q^2 = 1, \quad \Delta q \leq 0, \quad q\Delta q = 0, \quad q \geq 0 \text{ on } L, \quad q(0) = 0.$$

Also, by Lemma 2.11 we have that (2.12) holds. Now, if we had a sequence of functions $q^{(k)}$ satisfying (2.21) with $\lambda_*^{(k)} \downarrow 2$, then we would find some limiting function $q^{(\infty)}$ satisfying (2.21) with $\lambda_*^{(\infty)} = 2$ and (2.12). Then $q^{(\infty)}$ would be a 2-homogeneous solution of the thin obstacle problem and hence a quadratic harmonic polynomial (see for instance [GP09, Lemma 1.3.4]). Thus, applying Lemma 2.12 with $m = n - 1$ we find that, in an appropriate coordinate system,

$$D^2 p_* = \left(\begin{array}{c|c} 1 & 0_{n-1}^1 \\ \hline 0_1^{n-1} & 0_{n-1}^{n-1} \end{array} \right) \quad \text{and} \quad D^2 q^{(\infty)} = \left(\begin{array}{c|c} t & 0_{n-1}^1 \\ \hline 0_1^{n-1} & -N \end{array} \right)$$

where $N \geq 0$ with $\text{tr}(N) = t > 0$ (since $\|q^{(\infty)}\|_{L^2(\partial B_1)} = 1$). However, since $q^{(\infty)}(0) = 0$ and $q^{(\infty)} \geq 0$ on $L = \{p_* = 0\} = \ker(D^2 p_*)$, we must have $-N \geq 0$, a contradiction. \square

We conclude this section with an interesting observation: the gap between the value of the frequency and 2 controls the decay of the measure of the contact set (recall that $\phi(0^+, w) \geq 2$, see Lemma 2.5).

Proposition 2.13. *Let 0 be a singular point, $w := u - p_*$, and $\lambda_* := \phi(0^+, w)$. Then*

$$\frac{|\{u = 0\} \cap B_r|}{|B_r|} \leq C r^{\lambda_* - 2} \quad \forall r > 0.$$

In addition, the constant $C > 0$ can be chosen uniformly at all singular points in a neighborhood of 0 .

Proof. Let w_r and \tilde{w}_r be defined as in the statement of Proposition 2.10. Since \tilde{w}_r is bounded in $W^{1,2}(B_1)$ (see Step 1 in the proof of Proposition 2.10) and $\Delta w_r \leq 0$ (see (2.14)), we can bound the mass of $\Delta \tilde{w}_k$ inside $B_{1/2}$ by considering a smooth nonnegative cut-off function χ that is equal to 1 in $B_{1/2}$ and vanished outside B_1 , and then argue as in (2.18). In this way we get

$$\int_{B_{1/2}} |\Delta \tilde{w}_r| \leq C \|\tilde{w}_r\|_{L^1(B_1)} \leq C.$$

But since

$$|\Delta \tilde{w}_r| = r^2 \frac{\chi_{\{u(r \cdot) = 0\}}}{\|w_r\|_{L^2(\partial B_1)}}$$

and $\|w_r\|_{L^2(\partial B_1)} \leq Cr^{\lambda^*}$ (see (2.9)), we conclude that

$$r^{2-\lambda^*} \frac{|\{u=0\} \cap B_{r/2}|}{|B_{r/2}|} = r^{2-\lambda^*} \frac{|\{u(r \cdot)=0\} \cap B_{1/2}|}{|B_{1/2}|} \leq C,$$

as desired. \square

Note that the density bound is actually stronger around points corresponding to lower dimensional strata $\{\Sigma_m\}_{1 \leq m \leq n-2}$. Indeed, since $\Delta \tilde{w}_r \leq 0$ and any limit of \tilde{w}_r is harmonic (see Proposition 2.10(a)), it follows that $\int_{B_{1/2}} |\Delta \tilde{w}_r| \rightarrow 0$ as $r \rightarrow 0$, so in this case the constant C appearing in the statement can be replaced by $o(1)$.

Remark 2.14. In the case when $0 \in \Sigma_{n-1}$, we can actually prove a stronger estimate, namely that $\{u=0\} \cap B_r$ is contained in a r^{λ^*-1} -neighborhood of $L = \{p_* = 0\}$. To show this, note that (2.19) implies that

$$(2.22) \quad |\nabla \tilde{w}_r| \leq C \quad \text{in } B_{1/2}, \quad \forall r > 0,$$

or equivalently

$$(2.23) \quad |\nabla u(x) - \nabla p_*(x)| \leq C \frac{\|w_r\|_{L^2(\partial B_1)}}{r} \quad \forall x \in B_{r/2}.$$

Observe now that $\nabla u = 0$ on $\{u=0\}$ (since $u \in C^{1,1}$ and $u \geq 0$). Also, since $p_*(x) = \frac{1}{2}(\mathbf{e} \cdot x)^2$ for some $\mathbf{e} \in \mathbb{S}^{n-1}$,

$$|\nabla p_*(x)| = \text{dist}(x, L) \quad \forall x \in \mathbb{R}^n.$$

Hence, it follows by (2.23) that

$$\text{dist}(x, L) \leq C \frac{\|w_r\|_{L^2(\partial B_1)}}{r} \quad \forall x \in B_{r/2} \cap \{u=0\}.$$

Since $\|w_r\|_{L^2(\partial B_1)} \leq Cr^{\lambda^*}$ (see (2.9)), we conclude that

$$(2.24) \quad \text{dist}(x, L) \leq Cr^{\lambda^*-1} \quad \forall x \in B_{r/2} \cap \{u=0\}.$$

3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. This will require a fine analysis of the possible values of the frequency at singular points. We begin with the simple case $n = 2$.

Lemma 3.1. *Let $n = 2$ and 0 be a singular point in Σ_1 . Then $\lambda_* := \phi(0^+, u - p_*)$ belongs to the set*

$$\{3, 4, 5, 6, \dots\} \cup \left\{ \frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \dots \right\}.$$

In particular $\alpha_o \geq 1$ (here α_o is as in Proposition 2.10(b)).

Proof. From Proposition 2.10(b) we have that any possible blow-up q is a λ_* -homogeneous solutions of the Signorini problem in two dimensions with obstacle 0 on L and $\lambda_* \geq 2 + \alpha_o > 2$. In dimension two, homogeneous solutions to the Signorini problem that are symmetric with respect to L are completely classified via a standard argument by separation of variables, and the set of their possible homogeneities is

$$\{1, 2, 3, 4, 5, \dots\} \cup \left\{ \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \dots \right\}$$

(see for instance [FS17]). In our case q may also have a odd part. However, the odd part is easily seen to be harmonic, hence its possible homogeneity belongs to the set

$$\{1, 2, 3, 4, 5, \dots\}.$$

In conclusion

$$\lambda_* \in \{1, 2, 3, 4, 5, \dots\} \cup \left\{ \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \dots \right\}.$$

Since $\lambda_* > 2$, the lemma follows. \square

As explained in the introduction, our main goal is to prove that the set of points with frequency less than 3 is small. For this, we need to understand what happens when too many singular points accumulate around another singular point. This is the purpose of the next two lemmata: the first concerns the case $m \leq n - 2$, and the second deals with the case $m = n - 1$.

Lemma 3.2. *Let $n \geq 3$ and suppose that 0 is a singular point. Assume that $m := \dim(L) \leq n - 2$, and that there is a sequence of singular points $x_k \rightarrow 0$ and radii $r_k \downarrow 0$ with $|x_k| \leq r_k/2$ such that*

$$\tilde{w}_{r_k} := \frac{(u - p_*)(r_k \cdot)}{\|(u - p_*)(r_k \cdot)\|_{L^2(\partial B_1)}} \rightarrow q \quad \text{in } W^{1,2}(B_1),$$

and $y_k := \frac{x_k}{r_k} \rightarrow y_\infty$. Then $y_\infty \in L$ and $q(y_\infty) = 0$.

Proof. Since $(u - p_*)(r \cdot) = o(r^2)$ and $u(r_k y_k) = u(x_k) = 0$, it follows that $p_*(r_k y_k) = r_k^2 p_*(y_k) = o(r_k^2)$ as $r_k \rightarrow 0$. Therefore $p_*(y_\infty) = 0$, proving that $y_\infty \in L = \{p_* = 0\}$. We now show that $q(y_\infty) = 0$.

Note that, since q is homogeneous (see Proposition 2.10), if $y_\infty = 0$ then the result is trivial. So we can assume that $|y_\infty| > 0$.

We now use that x_k is a singular point for u . Thanks to Lemma 2.5 applied at x_k with $p = p_*$, we know that the frequency of $u(x_k + \cdot) - p_*$ is at least 2, therefore

$$\phi(1/2, u(r_k(y_k + \cdot)) - p_*(r_k \cdot)) \geq 2.$$

(Note that here p_* is the quadratic polynomial of u at 0, not at x_k !) Equivalently, recalling the definition of \tilde{w}_{r_k} , we have

$$(3.1) \quad 2 \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla \tilde{w}_{r_k}(y_k + \cdot) + h_{r_k}^{-1} \nabla (p_*(r_k y_k + r_k \cdot) - p_*(r_k \cdot))|^2}{\int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot) + h_{r_k}^{-1} (p_*(r_k y_k + r_k \cdot) - p_*(r_k \cdot))|^2},$$

where $h_{r_k} := \|(u - p_*)(r_k \cdot)\|_{L^2(\partial B_1)}$. Note that, because p_* is a quadratic polynomial that vanishes on L , we have

$$(3.2) \quad h_{r_k}^{-1} (p_*(r_k y_k + r_k \cdot) - p_*(r_k \cdot)) = c_k + b_k \cdot x$$

for some constant $c_k \in \mathbb{R}$ and some vector $b_k \in \mathbb{R}^n$ with $b_k \perp L$.

We now observe that, since $|y_k| \leq 1/2$ we have $B_{1/2}(y_k) \subseteq B_1$, therefore

$$\int_{B_{1/2}} |\nabla \tilde{w}_{r_k}(y_k + \cdot)|^2 + \int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot)|^2 \leq \|\tilde{w}_{r_k}\|_{W^{1,2}(B_1)}^2 \leq C.$$

We claim that

$$|c_k| \leq C \quad \text{and} \quad |b_k| \leq C, \quad \text{with } C \text{ independent of } k.$$

Indeed, if this was false, dividing by $(|c_k| + |b_k|)^2$ both the numerator and the denominator in (3.1), we would obtain

$$2 \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla(\varepsilon_k(x) + \bar{c}_k + \bar{b}_k \cdot x)|^2}{\int_{\partial B_{1/2}} |\varepsilon_k(x) + \bar{c}_k + \bar{b}_k \cdot x|^2},$$

where $\bar{c}_k := c_k/(|c_k| + |b_k|)$, $\bar{b}_k := b_k/(|c_k| + |b_k|)$, and $\int_{B_{1/2}} |\nabla \varepsilon_k|^2 + \int_{\partial B_{1/2}} \varepsilon_k^2 \rightarrow 0$. Thus, in the limit we would find

$$2 \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla(\bar{c}_\infty + \bar{b}_\infty \cdot x)|^2}{\int_{\partial B_{1/2}} |\bar{c}_\infty + \bar{b}_\infty \cdot x|^2} = \frac{|\bar{b}_\infty|^2}{4n|\bar{c}_\infty|^2 + |\bar{b}_\infty|^2} \leq 1,$$

a contradiction that proves the claim.

Thanks to the claim, up to a subsequence, $c_k \rightarrow c_\infty$ and $b_k \rightarrow b_\infty$ as $k \rightarrow \infty$, with $b_\infty \perp L$. Note now that, since x_k is a singular point, it follows by Corollary 2.7 that, for all $\rho \in (0, 1/2)$,

$$\frac{1}{\rho^4} \int_{\partial B_1} |u(x_k + r_k \rho \cdot) - p_*(r_k \rho \cdot)|^2 \leq 2^4 \int_{\partial B_1} |u(x_k + \frac{r_k}{2} \cdot) - p_*(\frac{r_k}{2} \cdot)|^2,$$

or equivalently, recalling (3.2),

$$(3.3) \quad \frac{1}{\rho^4} \int_{\partial B_\rho} |\tilde{w}_{r_k}(y_k + x) + c_k + b_k \cdot x|^2 \leq 2^4 \int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + x) + c_k + b_k \cdot x|^2.$$

Hence, in the limit (note that $B_\rho(y_\infty) \subset B_1$ for all $\rho \leq 1/2$)

$$(3.4) \quad \frac{1}{\rho^4} \int_{\partial B_\rho} |q(y_\infty + x) + c_\infty + b_\infty \cdot x|^2 \leq 2^4 \int_{\partial B_{1/2}} |q(y_\infty + x) + c_\infty + b_\infty \cdot x|^2 \quad \forall \rho \in (0, 1/2).$$

Since q is a homogeneous harmonic function, $y_\infty \in L$ with $|y_\infty| > 0$, and b_∞ is orthogonal to L , it follows by (3.4) that the gradient of q in the directions of L must vanish at y_∞ , namely $\nabla_L q(y_\infty) = 0$. Hence, by homogeneity we find $q(y_\infty) = 0$, as desired. \square

Lemma 3.3. *Let $n \geq 2$ and suppose that 0 is a singular point with $m := \dim(L) = n - 1$. Assume that there is a sequence of singular points $x_k \rightarrow 0$ with $x_k \in \Sigma_{n-1}$, and let $r_k \downarrow 0$ with $|x_k| \leq r_k/2$. Denote*

$$\lambda_* := \phi(0^+, u - p_*) \quad \text{and} \quad \lambda_{*,x_k} := \phi(0^+, u(x_k + \cdot) - p_{*,x_k}).$$

Suppose that

$$\tilde{w}_{r_k} := \frac{(u - p_*)(r_k \cdot)}{\|(u - p_*)(r_k \cdot)\|_{L^2(\partial B_1)}} \rightarrow q \quad \text{in } W^{1,2}(B_1),$$

and $y_k := \frac{x_k}{r_k} \rightarrow y_\infty$. Also, let $\mathbb{R}e = L^\perp$ with $|e| = 1$, and denote by q^{even} and q^{odd} the even and odd part of q with respect to L , namely

$$q^{\text{even}}(x) = \frac{1}{2} \{q(x) + q(x - 2(e \cdot x)e)\}, \quad q^{\text{odd}}(x) = \frac{1}{2} \{q(x) - q(x - 2(e \cdot x)e)\}.$$

Finally, let α_∞ be as in Proposition 2.10(b).

Then $y_\infty \in L$, and for $\lambda := \inf_k \{\lambda_{*,x_k}\} \geq 2 + \alpha_\circ$ we have

$$(3.5) \quad \rho^{-2\lambda} \int_{\partial B_\rho} q^{\text{even}}(y_\infty + x)^2 \leq 2^{2\lambda} \int_{\partial B_{1/2}} q^{\text{even}}(y_\infty + x)^2$$

In addition, if λ_* is not an integer then

$$(3.6) \quad \rho^{-2\lambda} \int_{\partial B_\rho} q(y_\infty + x)^2 \leq 2^{2\lambda} \int_{\partial B_{1/2}} q(y_\infty + x)^2. \quad \forall \rho \in (0, 1/2).$$

Proof. Let

$$p_{*,x_k}(x) = \lim_{r \downarrow 0} r^{-2} u(x_k + rx)$$

and define the second order harmonic polynomial

$$P_k(x) := \frac{1}{h_{r_k}} (p_{*,x_k}(r_k x) - p_{*,0}(x_k + r_k x)), \quad \text{where } h_{r_k} := \|(u - p_*)(r_k \cdot)\|_{L^2(\partial B_1)}.$$

Since $x_k \in \Sigma_{n-1}$, Proposition 2.10(b) yields

$$(3.7) \quad \phi(r_k/2, u(x_k + \cdot) - p_{*,x_k}) \geq \phi(0^+, u(x_k + \cdot) - p_{*,x_k}) = \lambda_{*,x_k} \geq \lambda \geq 2 + \alpha_\circ > 2,$$

therefore

$$(3.8) \quad 2 + \alpha_\circ \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla u(x_k + r_k \cdot) - \nabla p_{*,x_k}(r_k \cdot)|^2}{\int_{\partial B_{1/2}} |u(x_k + r_k \cdot) - p_{*,x_k}(r_k \cdot)|^2} = \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla \tilde{w}_{r_k}(y_k + \cdot) - \nabla P_k|^2}{\int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot) - P_k|^2}$$

for all k .

We now claim that

$$(3.9) \quad |P_k| \leq C \quad \forall k.$$

Indeed, if the coefficients of P_k are not bounded, then dividing by its maximum in the numerator and the denominator of (3.1) we obtain

$$2 + \alpha_\circ \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla \varepsilon_k - \nabla \bar{P}_k|^2}{\int_{\partial B_{1/2}} |\varepsilon_k - \bar{P}_k|^2},$$

where $\bar{P}_k := P_k/|P_k|$ and $\int_{B_1} |\nabla \varepsilon_k|^2 \rightarrow 0$, thus in the limit we find

$$(3.10) \quad 2 + \alpha_\circ \leq \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla \bar{P}_\infty|^2}{\int_{\partial B_{1/2}} |\bar{P}_\infty|^2}$$

for some quadratic polynomial \bar{P}_∞ . Note now that, since $0, x_k \in \Sigma_{n-1}$, we have

$$p_{*,x_k} = \frac{1}{2} (\mathbf{e}_k \cdot x)^2 \quad \text{for some } \mathbf{e}_k \in \mathbb{S}^{n-1}$$

and

$$p_{*,0} = \frac{1}{2} (\mathbf{e} \cdot x)^2 \quad \text{for } \mathbf{e} \in \mathbb{S}^{n-1} \cap L^\perp.$$

Also, up to replacing \mathbf{e}_k with $-\mathbf{e}_k$ if needed, we have that $\mathbf{e}_k \rightarrow \mathbf{e}$ (since $p_{*,x_k} \rightarrow p_{*,0}$ as $k \rightarrow \infty$). Thus

$$\begin{aligned} P_k(x) &= \frac{1}{h_{r_k}} (p_{*,x_k}(r_k x) - p_{*,0}(x_k + r_k x)) \\ &= \frac{r_k^2}{2h_{r_k}} ((\mathbf{e}_k \cdot x)^2 - (\mathbf{e} \cdot (y_k + x))^2) \\ &= \frac{r_k^2}{2h_{r_k}} ((\mathbf{e}_k \cdot x)^2 - (\mathbf{e} \cdot x)^2 - 2a_k(\mathbf{e} \cdot x) - a_k^2) \end{aligned}$$

where $a_k := (\mathbf{e} \cdot y_k) \rightarrow 0$ (since $y_k \rightarrow y_\infty \in L = \mathbf{e}^\perp$). Thus, since the coefficients of $\bar{P}_k := P_k/|P_k|$ are uniformly bounded and $a_k^2 \ll 2a_k$ we must have $\bar{P}_\infty(0) = 0$ and therefore

$$\bar{P}_\infty(x) = \bar{c}_1(\mathbf{e}' \cdot x)(\mathbf{e} \cdot x) + \bar{c}_2(\mathbf{e} \cdot x),$$

for some constants $\bar{c}_1, \bar{c}_2 \in \mathbb{R}$, where

$$\mathbf{e}' := \lim_{k \rightarrow \infty} \frac{\mathbf{e}_k - \mathbf{e}}{|\mathbf{e}_k - \mathbf{e}|} \in \mathbb{S}^{n-1} \cap L.$$

Now, since $\mathbf{e}' \perp \mathbf{e}$, a direct computation using the formula above yields

$$\frac{1}{2} \frac{\int_{B_{1/2}} |\nabla \bar{P}_\infty|^2}{\int_{\partial B_{1/2}} |\bar{P}_\infty|^2} = \frac{1}{2} \frac{\bar{c}_1^2 \int_{B_{1/2}} (\mathbf{e}' \cdot x)^2 + \bar{c}_1^2 \int_{B_{1/2}} (\mathbf{e} \cdot x)^2 + \bar{c}_2^2 |B_{1/2}|}{\bar{c}_1^2 \int_{\partial B_{1/2}} (\mathbf{e}' \cdot x)^2 (\mathbf{e} \cdot x)^2 + \bar{c}_2^2 \int_{\partial B_{1/2}} (\mathbf{e} \cdot x)^2} = \frac{\frac{1}{2(n+2)} \bar{c}_1^2 + \bar{c}_2^2}{\frac{1}{4(n+2)} \bar{c}_1^2 + \bar{c}_2^2} \leq 2,$$

a contradiction to (3.10). Hence this proves (3.9), and up to a subsequence $P_k \rightarrow P_\infty$ as $k \rightarrow \infty$, where P_∞ is a second order harmonic polynomial. In addition, by the discussion above, P_∞ has the form

$$(3.11) \quad P_\infty(x) = c_1(\mathbf{e}' \cdot x)(\mathbf{e} \cdot x) + c_2(\mathbf{e} \cdot x),$$

where $\mathbf{e}' \perp \mathbf{e}$ and $c_1, c_2 \in \mathbb{R}$.

Now, by Lemma 2.6 applied to $u(x_k + r_k \cdot) - p_{*,x_k}$ and using (3.7), we have

$$\rho^{-2\lambda} \int_{\partial B_\rho} |\tilde{w}_{r_k}(y_k + \cdot) - P_k|^2 \leq 2^{2\lambda} \int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot) - P_k|^2,$$

for all $\rho \in (0, 1/2)$, hence, in the limit,

$$(3.12) \quad \rho^{-2\lambda} \int_{\partial B_\rho} |q(y_\infty + \cdot) - P_\infty|^2 \leq 2^{2\lambda} \int_{\partial B_{1/2}} |q(y_\infty + \cdot) - P_\infty|^2.$$

Since P_∞ is odd with respect to L (see (3.11)), it follows by (3.12) that

$$\begin{aligned} \rho^{-2\lambda} \int_{\partial B_\rho} q^{\text{even}}(y_\infty + \cdot)^2 &= \rho^{-2\lambda} \int_{\partial B_\rho} |(q(y_\infty + \cdot) - P_\infty)^{\text{even}}|^2 \\ &\leq \rho^{-2\lambda} \int_{\partial B_\rho} |q(y_\infty + \cdot) - P_\infty|^2 \\ &\leq 2^{2\lambda} \int_{\partial B_{1/2}} |q(y_\infty + \cdot) - P_\infty|^2, \end{aligned}$$

where we used that $f \mapsto f^{\text{even}}$ is an orthogonal projection in $L^2(\partial B_\rho)$. Hence

$$\rho^{-2\lambda} \int_{\partial B_\rho} q^{\text{even}}(y_\infty + \cdot)^2 \leq C \quad \forall \rho \in (0, 1/2).$$

Noticing that $q^{\text{even}}(y_\infty + \cdot)$ is a solution of Signorini, this bound implies that the frequency of $q^{\text{even}}(y_\infty + \cdot)$ at 0 is at least λ (cp. Corollary 2.9). Thus, arguing as in Lemma 2.6 we deduce that

$$\rho \mapsto \rho^{-2\lambda} \int_{\partial B_\rho} q^{\text{even}}(y_\infty + \cdot)^2$$

is monotone nondecreasing, which proves (3.5).

Assume now that λ_* is not an integer. Since q^{odd} is an odd function (with respect to L) whose Laplacian is concentrated on L then $\Delta q^{\text{odd}} \equiv 0$, which implies that q^{odd} is a harmonic polynomial. On the other hand, since q is λ_* -homogeneous, then so is q^{odd} . Hence, since harmonic polynomials have integer homogeneity, the only possibility is $q^{\text{odd}} \equiv 0$. Thus $q = q^{\text{even}}$, and (3.6) follows from (3.5). \square

For $n \geq 3$ and $m \in \{1, 2, \dots, n-1\}$ we define

$$(3.13) \quad \Sigma_m^a := \{x_\circ \in \Sigma_m : \phi(0^+, u(x_\circ + \cdot) - p_{*,x_\circ}) < 3\}, \quad \Sigma_m^g := \Sigma_m \setminus \Sigma_m^a.$$

We can now give the key lemmas needed to prove Theorem 1.1. We begin by showing that points in Σ_1^a are isolated inside Σ .

Lemma 3.4. *Assume $n \geq 3$. Then Σ_1^a is a discrete set.*

Proof. Assume by contradiction that $0 \in \Sigma_1^a$ and $x_k \rightarrow 0$ is a sequence of singular points. By definition, $0 \in \Sigma_1^a$ means that $\dim(L) = 1$ (where $L := \{p_* = 0\}$) and that $\lambda_* := \phi(0^+, u - p_*) < 3$. Hence, since $n \geq 3$ we have $m = 1 \leq n-2$, thus Proposition 2.10(a) yields $\lambda_* = 2$.

Let $r_k := 2|x_k|$. By Proposition 2.10 and Lemma 3.2 we have (up to extracting a subsequence)

$$\tilde{w}_{r_k} \rightarrow q \quad \text{in } L^2(B_1) \quad \text{and} \quad y_k := \frac{x_k}{r_k} \rightarrow y_\infty \in L \cap \partial B_{1/2}.$$

where q is a 2-homogeneous harmonic polynomial satisfying $q(y_\infty) = 0$. In addition, since $\lambda_* = 2$ we know that (2.10) holds. Namely, in an appropriate coordinate frame (recall that $m = 1$ here) we have

$$(3.14) \quad D^2 p_* = \left(\begin{array}{ccc|c} \mu_1 & & & 0_1^{n-1} \\ & \ddots & & \\ & & \mu_{n-1} & \\ \hline & & & 0 \\ 0_{n-1}^1 & & & \end{array} \right) \quad \text{and} \quad D^2 q = \left(\begin{array}{ccc|c} t & & & 0_1^{n-1} \\ & \ddots & & \\ & & t & \\ \hline & & & -(n-1)t \\ 0_{n-1}^1 & & & \end{array} \right),$$

where $\mu_1, \dots, \mu_{n-1}, t > 0$, $\sum_{i=1}^{n-1} \mu_i = 1$.

Note that, since $|y_\infty| = 1/2$, $q(y_\infty) = 0$, and $y_\infty \in L$, by homogeneity of q we must have $q|_L \equiv 0$. This contradicts the fact that $D^2 q|_{L \otimes L} = -(n-1)t < 0$ (see (3.14)) and concludes the proof. \square

In order to estimate the measure of Σ_m^a for $m \geq 2$ we need to develop a Federer-type dimension reduction argument. As a first step we need the following standard result in geometric measure theory, that we prove for convenience of the reader.

Before stating it, we recall some classical definitions. Given $\beta > 0$ and $\delta \in (0, \infty]$, the Hausdorff premeasures $\mathcal{H}_\delta^\beta(E)$ of a set E are defined as follows:¹

$$(3.15) \quad \mathcal{H}_\delta^\beta(E) := \inf \left\{ \sum_i \text{diam}(E_i)^\beta : E \subset \bigcup_i E_i, \text{diam}(E_i) < \delta \right\}.$$

Then, one defines the β -dimensional Hausdorff measure $\mathcal{H}^\beta(E) := \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^\beta(E)$. We recall that the Hausdorff dimension can be defined in terms of \mathcal{H}_∞^β as follows:

$$(3.16) \quad \dim_{\mathcal{H}}(E) := \inf \{ \beta > 0 : \mathcal{H}_\infty^\beta(E) = 0 \}$$

(this follows from the fact that $\mathcal{H}_\infty^\beta(E) = 0$ if and only if $\mathcal{H}^\beta(E) = 0$, see for instance [Sim83, Section 1.2]).

Lemma 3.5. *Let $E \subset \mathbb{R}^n$ be a set with $\mathcal{H}_\infty^\beta(E) > 0$ for some $\beta \in (0, n]$. Then:*

(a) *For \mathcal{H}^β -almost every point $x_o \in E$, there is a sequence $r_k \downarrow 0$ such that*

$$(3.17) \quad \lim_{k \rightarrow \infty} \frac{\mathcal{H}_\infty^\beta(E \cap B_r(x_o))}{r_k^\beta} \geq c_{n,\beta} > 0,$$

where $c_{n,\beta}$ is a constant depending only on n and β . Let us call these points “density points”.

(b) *Assume that 0 is a “density point”, let $r_k \downarrow 0$ be a sequence along which (3.17) holds, and define the “accumulation set” for E along r_k at 0 as*

$$\mathcal{A} = \mathcal{A}_{E, \{r_k\}} := \left\{ z \in \overline{B_{1/2}} : \exists (z_\ell)_{\ell \geq 1}, (k_\ell)_{\ell \geq 1} \text{ s.t. } z_\ell \in r_{k_\ell}^{-1} E \cap B_{1/2} \text{ and } z_\ell \rightarrow z \right\}.$$

Then

$$\mathcal{H}_\infty^\beta(\mathcal{A}) > 0.$$

Proof. Part (a) of the lemma is a standard property of the Hausdorff (pre)measures, see for instance [Sim83, Theorem 3.6(2)] for a proof. We now prove (b).

Assume that 0 is a density point. Then by (a) we have

$$(3.18) \quad H_\infty^\beta(r_k^{-1} E \cap B_{1/2}) = \frac{H_\infty^\beta(E \cap B_{r_k/2})}{r_k^\beta} \geq 2^{-(\beta+1)} c_{n,\beta} > 0 \quad \text{for all } k \gg 1.$$

Note that the accumulation set \mathcal{A} is a closed. Assume by contradiction that $\mathcal{H}^\beta(\mathcal{A}) = 0$. Then, by definition of \mathcal{H}_∞^β , given any $\varepsilon > 0$ there exists a countable cover of balls $\{\hat{B}_i\}$ such that

$$\mathcal{A} \subset \bigcup_{i \geq 1} \hat{B}_i \quad \text{and} \quad \sum_{i \geq 1} \text{diam}(B_i)^\beta \leq \varepsilon.$$

¹In many textbooks, the definition of \mathcal{H}_δ^β includes a normalization constant chosen so that the Hausdorff measure of dimension k coincides with the standard k -dimensional volume on smooth sets. However such normalization constant is irrelevant for our purposes, so we neglect it.

Since $\mathcal{A} \subset \overline{B_1}$ is compact set, we can find a finite subcover. In particular, there exists $N \in \mathbb{N}$ such that

$$\mathcal{A} \subset \bigcup_{i=1}^N \hat{B}_i \quad \text{and} \quad \sum_{i=1}^N \text{diam}(\hat{B}_i)^\beta \leq \varepsilon.$$

But then, since

$$r_k^{-1}E \cap \overline{B_{1/2}} \subset \bigcup_{i=1}^N \hat{B}_i$$

for k large enough², by definition of \mathcal{H}_∞^β we obtain

$$\mathcal{H}_\infty^\beta(r_k^{-1}E \cap \overline{B_{1/2}}) \leq \varepsilon,$$

a contradiction with (3.18) if ε is small enough. \square

We can now give an appropriate version of Lemma 3.4 for the case $m = \dim(L) \in \{2, \dots, n-2\}$.

Lemma 3.6. *Assume $n \geq 4$ and $m \in \{2, \dots, n-2\}$. Then $\dim_{\mathcal{H}}(\Sigma_m^a) \leq m-1$.*

Proof. Recalling (3.16), we assume by contradiction that $\mathcal{H}_\infty^\beta(\Sigma_m^a) > 0$ for some $\beta > m-1$. By Lemma 3.5(a), there is a point $x_o \in \Sigma_m^a$ and $r_k \downarrow 0$ such that

$$(3.19) \quad r_k^{-\beta} \mathcal{H}_\infty^\beta(\Sigma_m^a \cap B_{r_k}(x_o)) \geq c_{n,\beta} > 0.$$

Assume without loss of generality that $x_o = 0$. Hence, since $0 \in \Sigma_m^a$ and $m \leq n-2$, it follows by (3.13) and Proposition 2.10(a) that

$$\lambda_* := \phi(0^+, u - p_*) = 2$$

Moreover, by Proposition 2.10(a) after replacing r_k by some subsequence r_{k_ℓ} that will still satisfy (3.19) and hence by simplicity we again denote r_k , we will have

$$\tilde{w}_{r_{k_\ell}} \rightarrow q \quad \text{in } L^2(B_1),$$

where q is a 2-homogeneous harmonic polynomial. In addition, since $\lambda_* = 2$, we know that in an appropriate coordinate frame $D^2 p_*$ and $D^2 q$ are given by (2.10). Also, applying Lemma 3.5(b), we deduce that the ‘‘accumulation set’’ $\mathcal{A} = \mathcal{A}_{\Sigma_m^a, \{r_k\}}$ satisfies $H_\infty^\beta(\mathcal{A}) > 0$.

We claim that $\mathcal{A} \subset \overline{B_1} \cap L \cap \{q = 0\}$. Indeed, by definition, a point z belongs to \mathcal{A} if there are sequences of singular points $x_\ell \rightarrow 0$ and of radii $r_{k_\ell} \downarrow 0$ such that $|x_\ell| \leq r_{k_\ell}$ and $x_\ell/r_{k_\ell} \rightarrow z$. Thus $x_\ell/(2r_{k_\ell}) \rightarrow z/2$, and by Lemma 3.2 we obtain $z/2 \in L$ and $q(z/2) = 0$. By homogeneity, this implies that $z \in L \cap \{q = 0\}$ as claimed.

Finally we note that $L \cap \{q = 0\}$ has dimension at most $m-1$. Indeed, if not this would imply that $q \equiv 0$ on L , which would contradict the fact that $\text{tr}(D^2 q|_{L \otimes L}) = -(n-m)t < 0$ (see (2.10)).

Thus $\mathcal{H}^{m-1}(\overline{B_1} \cap L \cap \{q = 0\}) < +\infty$, which yields (since $\beta > m-1$)

$$0 < \mathcal{H}_\infty^\beta(\mathcal{A}) \leq \mathcal{H}_\infty^\beta(\overline{B_1} \cap L \cap \{q = 0\}) = 0,$$

²Otherwise there would be a sequence of points $z_\ell \in r_{k_\ell}^{-1}E \cap \overline{B_{1/2}} \setminus \bigcup_{i=1}^N \hat{B}_i$, and hence their limit z —up to a subsequence— would satisfy at the same time $z \in \mathcal{A}$ and $z \in \overline{B_{1/2}} \setminus \bigcup_{i=1}^N \hat{B}_i$, a contradiction.

contradiction. \square

We now analyze the size of Σ_{n-1}^a . We begin with the case $n = 3$.

Lemma 3.7. *Let $n = 3$. Then Σ_{n-1}^a is a discrete set.*

Proof. Let us assume that $0 \in \Sigma_{n-1}^a$ and that $x_k \rightarrow 0$, where $x_k \in \Sigma_{n-1}$. By Proposition 2.10 and by definition of Σ_{n-1}^a we have

$$\lambda_* := \phi(0^+, u - p_*) \in [2 + \alpha_o, 3).$$

Let $r_k := 2|x_k|$ and note that, by Proposition 2.10(b), we have (up to subsequence)

$$\tilde{w}_{r_k} \rightarrow q \quad \text{and} \quad \frac{x_k}{r_k} \rightarrow z \in \partial B_{1/2}$$

where q is a λ_* -homogeneous solution of the Signorini problem (with zero obstacle on L).

Also, since $\lambda_* < 3$, it follows by Lemma 3.3 that $z \in L$ and

$$\rho^{-2(2+\alpha_o)} \int_{\partial B_\rho(z)} q^2 \leq C(n), \quad \forall \rho \in (0, 1/2)$$

(note that, since $x_k \in \Sigma_{n-1}$, $\inf_k \lambda_{*,x_k} \geq 2 + \alpha_o$ by Proposition 2.10(b)). This implies that q , Dq , and D^2q vanish at z , and that $\lambda^z := \phi(0^+, q(z + \cdot)) \geq 2 + \alpha_o$.

Since q is a solution of Signorini that is homogeneous with respect to the point 0, it is classical fact (this follows from instance from the monotonicity of the frequency function) that a blow-up at $z \in L$,

$$q^z = \lim_j q(z + r_j \cdot) / \|q(z + r_j \cdot)\|_{L^2(\partial B_1)}$$

has translation symmetry in the direction z , and it is λ^z -homogeneous. Thus, since $n = 3$, q^z depends thus only on two variables (equivalently, it has 2-dimensional symmetry). Since homogeneous 2-dimensional solutions of Signorini are completely classified (see the proof of Lemma 3.1) we deduce that

$$\lambda^z \in \{1, 2, 3, 4, 5, \dots\} \cup \left\{ \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \dots \right\}$$

Recalling that $\lambda^z \geq 2 + \alpha_o$, we get $\lambda^z \geq 3$. But then we reach a contradiction since, by monotonicity of the frequency and the fact that the limit as $r \rightarrow +\infty$ of the frequency is independent of the point, we get

$$3 \leq \lambda^z = \phi(0^+, q(z + \cdot)) \leq \phi(+\infty, q(z + \cdot)) = \phi(+\infty, q) = \lambda_* < 3.$$

\square

In order to control the size of Σ_{n-1}^a for $n \geq 4$, we shall use the following result on the Signorini problem:

Theorem 3.8 ([FS17, Theorem 1.3]). *Let $L \subset \mathbb{R}^n$ be a $(n-1)$ -dimensional subspace, and let q be solution of the Signorini problem in \mathbb{R}^n with obstacle 0 on L (see (2.11)). Then, for all z in the contact set $\{q = 0\} \subset L$ it holds*

$$\phi(0^+, q(z + \cdot)) \in \{1, 2, 3, 4, \dots\} \cup \left\{ \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \dots \right\}$$

except for at most a set of Hausdorff dimension $n - 3$.

Lemma 3.9. *Let $n \geq 4$. Then $\dim_{\mathcal{H}}(\Sigma_{n-1}^a) \leq n - 3$.*

Proof. Recalling (3.16), assume by contradiction that $\mathcal{H}_\infty^\beta(\Sigma_{n-1}^a) > 0$ for some $\beta > n - 3$. Then by Lemma 3.5(a) there exists a point $x_o \in \Sigma_{n-1}^a$ and $r_k \downarrow 0$ such that

$$(3.20) \quad r_k^{-\beta} \mathcal{H}_\infty^\beta(\Sigma_{n-1}^a \cap B_{r_k}(x_o)) \geq c_{n,\beta} > 0.$$

Without loss of generality we assume that $x_o = 0$. Then, since $0 \in \Sigma_{n-1}^a$, by (3.13) and Proposition 2.10(b) we have

$$\lambda_* := \phi(0^+, u - p_*) \in [2 + \alpha_o, 3).$$

Moreover, by Proposition 2.10(b) after replacing r_k by some subsequence r_{k_ℓ} that will still satisfy (3.20) and hence by simplicity we again denote r_k , we will have

$$\tilde{w}_{r_k} \rightarrow q \quad \text{in } L^2(B_1),$$

where q is a λ_* -homogeneous solution of the Signorini problem with obstacle 0 on L . Applying Lemma 3.5(b), the ‘‘accumulation set’’ $\mathcal{A} = \mathcal{A}_{\Sigma_{n-1}^a, \{r_k\}}$ satisfies $H_\infty^\beta(\mathcal{A}) > 0$. Set

$$\mathcal{S} =: \{z \in \overline{B}_1 \cap L \cap \{q(z) = 0\} \text{ such that } \phi(0^+, q(z + \cdot)) \geq 2 + \alpha_o\}.$$

Then, by the same argument as in the proof of Lemma 3.6 we deduce that $\mathcal{A} \subset \overline{B}_1 \cap L \cap \{q = 0\}$. Also, since $\lambda_* < 3$, as in the proof of Lemma 3.7 it follows by Lemma 3.3 that $\phi(0^+, q(z + \cdot)) \geq 2 + \alpha_o$ for all $z \in \mathcal{A}$. Hence,

$$\mathcal{A} \subset \mathcal{S}.$$

We now note that, for all $z \in \mathcal{S}$, we have

$$\phi(0^+, q(z + \cdot)) \leq \phi(+\infty, q(z + \cdot)) = \phi(+\infty, q) = \lambda_* < 3$$

(since $0 \in \Sigma_{n-1}^a$). Therefore it follows that

$$\phi(0^+, q(z + \cdot)) \in [2 + \alpha_o, 3) \quad \text{for all } z \in \mathcal{S},$$

and Theorem 3.8 yields $\dim_{\mathcal{H}}(\mathcal{S}) = n - 3$. In particular $\mathcal{H}_\infty^\beta(\mathcal{S}) = 0$ (since $\beta > n - 3$) and we obtain

$$0 < \mathcal{H}_\infty^\beta(\mathcal{A}) \leq \mathcal{H}_\infty^\beta(\mathcal{S}) = 0,$$

a contradiction. □

We will also need the following version of Whitney’s extension theorem (see for instance [Fef09] and the references therein):

Lemma 3.10 (Whitney’s Extension Theorem). *Let $\beta \in (0, 1]$, $\ell \in \mathbb{N}$, $K \subset \mathbb{R}^n$ a compact set, and $f : K \rightarrow \mathbb{R}$ a given mapping. Suppose that for any $x_o \in K$ there exists a polynomial P_{x_o} of degree ℓ such that:*

- (i) $P_{x_o}(x_o) = f(x_o)$;
- (ii) $|D^k P_{x_o}(x) - D^k P_x(x)| \leq C|x - x_o|^{\ell+\beta-k}$ for all $x \in K$ and $k \in \{0, 1, \dots, \ell\}$, where $C > 0$ is independent of x_o .

Then there exists $F : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $C^{\ell,\beta}$ such that

$$F|_K \equiv f \quad \text{and} \quad F(x) = P_{x_o}(x) + O(|x - x_o|^{\ell+\beta}) \quad \forall x_o \in K.$$

We now prove that the set of points with frequency $\geq \lambda$ is contained in a $C^{\lambda-1}$ -manifold. Since the classical argument provided in [PSU12, Theorem 7.9] only shows that the singular set is locally contained in a countable union of manifolds (while here we claim that locally we need only one manifold), we provide the details of the proof.

Lemma 3.11. *Let $n \geq 2$, $m \in \{1, 2, \dots, n-1\}$, and $\lambda > 2$. Let $\ell \in \mathbb{N}$ and $\beta \in (0, 1]$ satisfy $\ell + \beta = \lambda$, and define*

$$S_{m,\lambda} := \{x_o \in \Sigma_m : \phi(0^+, u(x_o + \cdot) - p_{*,x_o}) \geq \lambda\}.$$

Then $S_{m,\lambda}$ locally contained in a m -dimensional manifold of class $C^{\ell-1,\beta}$.

Proof. We prove the result in a neighborhood of the origin.

We begin by recalling that the singular set $\Sigma = \cup_{m=0}^{n-1} \Sigma_m$ is closed (this is a classical fact that follows from the relative openness of the set of regular points, see [C77]). In addition, we note that the monotonicity of the frequency implies that the map

$$\Sigma \ni x_o \mapsto \phi(0^+, u(x_o + \cdot) - p_{*,x_o})$$

is upper semicontinuous, being the monotone decreasing limit (as $r \downarrow 0$) of the continuous functions

$$\Sigma \ni x_o \mapsto \phi(r, u(x_o + \cdot) - p_{*,x_o}), \quad r > 0.$$

Thanks to these facts we deduce that

$$S_\lambda := \{x_o \in \Sigma : \phi(0^+, u(x_o + \cdot) - p_{*,x_o}) \geq \lambda\}$$

is closed. In particular, if we define the compact set $K := S_\lambda \cap \overline{B_{1/4}}$, we have that $\overline{S_{m,\lambda} \cap B_{1/4}} \subset K$.

Now, given $x_o \in K$, we define

$$P_{x_o}(x) := p_{*,x_o}(x - x_o).$$

We want to show that K , $f \equiv 0$, and $\{P_{x_o}\}_{x_o \in K}$ satisfy the assumptions of Lemma 3.10 with ℓ and β as defined above.

Note that, by Lemma 2.6 and the definition of S_λ , for all $x_o \in K$ we have

$$(3.21) \quad \|u(x_o + \rho \cdot) - p_{*,x_o}(\rho \cdot)\|_{L^2(B_1)} \leq (2\rho)^\lambda \|u(x_o + \frac{1}{2} \cdot) - p_{*,x_o}(\frac{1}{2} \cdot)\|_{L^2(\partial B_1)}$$

for all $\rho \in (0, 1/2]$.

Now, given $x_o, x \in K$, set $\rho := |x - x_o|$ (note that $\rho \leq 1/4$), and for simplicity of notation assume that $x_o = 0$. Then it follows from (3.21) applied both at 0 and x that

$$(3.22) \quad \begin{aligned} \|(P_0 - P_x)(\rho \cdot)\|_{L^2(B_1)} &\leq \|u(\rho \cdot) - P_0(\rho \cdot)\|_{L^2(B_1)} + \|u(\rho \cdot) - P_x(\rho \cdot)\|_{L^2(B_1)} \\ &= \|u(\rho \cdot) - p_*(\rho \cdot)\|_{L^2(B_1)} + \|u(\rho \cdot) - p_{*,x}(\rho(\cdot - x/\rho))\|_{L^2(B_1)} \\ &\leq \|u(\rho \cdot) - p_*(\rho \cdot)\|_{L^2(B_1)} + \|u(\rho \cdot) - p_{*,x}(\rho(\cdot - x/\rho))\|_{L^2(B_2(x/\rho))} \\ &= \|u(\rho \cdot) - p_*(\rho \cdot)\|_{L^2(B_1)} + \|u(x + \rho \cdot) - p_{*,x}(\rho \cdot)\|_{L^2(B_2)} \\ &\leq C\rho^\lambda. \end{aligned}$$

In particular, since the norm $\|\cdot\|_{L^2(B_1)}$ is equivalent to the norm $\|\cdot\|_{C^\ell(B_1)}$ on the space of quadratic polynomials, we obtain the existence of a constant $C > 0$ such that

$$|D^k P_{x_o}(x) - D^k P_x(x)| \leq C|x - x_o|^{\ell+\beta-k} \quad \text{fo all } x_o, x \in K \text{ and } k \in \{0, 1, \dots, \ell\}$$

(recall that $\lambda = \ell + \beta$). Since $P_{x_o}(x_o) = 0$ for $x_o \in K$, applying Lemma 3.10 we find a function $F \in C^{\ell, \beta}(\mathbb{R}^n)$ such that

$$F(x) = p_{x_o}(x - x_o) + O(|x - x_o|^{\ell+\beta}) \quad \text{for all } x_o \in K.$$

Therefore

$$S_{m, \lambda} \cap B_{1/4} \subset K \subset \{\nabla F = 0\} = \bigcap_{i=1}^n \{\partial_{x_i} F = 0\}.$$

Now, if $x_o \in S_{m, \ell} \cap B_{1/4}$ then $\dim \ker (D^2 F(x_o)) = \dim \ker (D^2 p_{*, x_o}(0)) = m$. This implies that, up to a change of coordinates, the rank of $D^2_{(x_1, \dots, x_{n-m})} F(x_o)$ is maximal, and we conclude by the Implicit Function Theorem that, in a neighborhood of x_o , $\bigcap_{i=1}^{n-m} \{\partial_{x_i} F = 0\}$ is a m -dimensional manifold of class $C^{\ell-1, \beta}$ that contains $S_{m, \lambda}$. \square

We are now ready to prove Theorem 1.1 (except for the C^2 regularity in dimension 2 that will follow from Theorem 4.7 in the next section).

Proof of Theorem 1.1. We need to prove:

- (a) For $n = 2$, Σ_1 is locally contained in a C^2 curve.
- (b) For $n \geq 3$, Σ_{n-1}^g is locally contained in a $C^{1,1}$ $(n-1)$ -dimensional manifold and Σ_{n-1}^a is a relatively open subset of Σ_{n-1} satisfying $\dim_{\mathcal{H}}(\Sigma_{n-1}^a) \leq n-3$ (the latter set is discrete for $n = 3$).
- (c) For $n \geq 3$, Σ_{n-1} can be locally covered by a C^{1, α_o} $(n-1)$ -dimensional manifold, for some dimensional exponent $\alpha_o > 0$.
- (d) For $n \geq 3$ and $m = 1, \dots, n-2$, Σ_m^g can be locally covered by a $C^{1,1}$ m -dimensional manifold and Σ_m^a is a relatively open subset of Σ_m satisfying $\dim_{\mathcal{H}}(\Sigma_m^a) \leq m-1$ (the latter set is discrete when $m = 1$).
- (e) For $n \geq 3$ and $m = 1, \dots, n-2$, Σ_m can be locally covered by a $C^{1, \log \varepsilon_o}$ m -dimensional manifold, for some dimensional exponent $\varepsilon_o > 0$.

Throughout the proof, we will use the definition of $S_{m, \lambda}$ given in Lemma 3.11.

- *Proof of (a).* By Lemma 3.1 we have that $\Sigma_1 = S_{1,3}$. Thus, applying Lemma 3.11, we obtain that Σ_1 is locally covered by a $C^{1,1}$ curve. To conclude that Σ_1 can be covered by a C^2 curve, we apply Theorem 4.7 from the next section.

- *Proof of (b).* By Lemma 3.9, the Hausdorff dimension of Σ_{n-1}^a is at most $n-3$. Also, by definition we have $\Sigma_{n-1}^g = S_{n-1,3}$, thus Σ^g can be locally covered by a $C^{1,1}$ $(n-1)$ -dimensional manifold, thanks to Lemma 3.11. The fact that Σ_{n-1}^g is relatively closed in Σ_{n-1} is a consequence of the fact that $x_o \mapsto \phi(0^+, u(x_o + \cdot) - p_{*, x_o})$ is upper semicontinuous, as shown in the proof of Lemma 3.11. In the case $n = 3$, Lemma 3.4 gives that Σ_{n-1}^a is a discrete set.

- *Proof of (c).* By Proposition 2.10(b) we have that the whole stratum Σ_{n-1} is contained in $S_{n-1, 2+\alpha_o}$, for some dimensional constant $\alpha_o > 0$. As a consequence, the whole stratum Σ_{n-1} can be covered by a C^{1, α_o} $(n-1)$ -dimensional manifold.

- *Proof of (d).* By Lemma 3.6, for $1 \leq m \leq n-2$ the Hausdorff dimension of Σ_m^a is at most $m-1$ (in the case $m=1$, Lemma 3.6 gives that Σ_1^a is a discrete set). Also, since by definition $\Sigma_m^g = S_{m,3}$, applying again Lemma 3.11 we obtain that Σ_m^g can be locally covered by a $C^{1,1}$ m -dimensional manifold. Finally, as in the proof of (d), the relative closedness of Σ^g follows from the upper semicontinuity of the frequency.

- *Proof of (e).* Let $m \leq n-2$. We claim that the following estimate holds:

$$(3.23) \quad \|u(x_o + r \cdot) - p_{*,x_o}(r \cdot)\|_{L^2(\partial B_1)} \leq Cr^2 \log^{-\varepsilon_o}(1/r) \quad \forall x_o \in \Sigma_m \cap B_{1/2}, \quad \forall r \in (0, 1/2).$$

Observe that it is enough to prove (3.23) at points x_o such that

$$\lambda_{*,x_o} := \phi(0^+, u(x_o + \cdot) - p_{*,x_o}) = 2.$$

Indeed, if $\lambda_{*,x_o} > 2$ then Proposition 2.10(a) yields $\lambda_{*,x_o} \geq 3$, hence (3.23) trivially holds (actually, with a much stronger estimate) thanks to Lemma 2.6. So, without loss of generality, we can assume that $\lambda_{*,x_o} = 2$.

Let $M > 1$ be a large constant to be fixed later. By Caffarelli's asymptotic convexity estimate [C77] (see also [C98, Corollary 5]), we have

$$(3.24) \quad D^2u \geq -C \log^{-\varepsilon_o}(1/r) \text{Id} \quad \text{in } B_r(x_o) \quad \forall x_o \in \Sigma,$$

for some dimensional exponent $\varepsilon_o > 0$. Now, let $a_r := \|r^{-2}u(x_o + r \cdot) - p_{*,x_o}\|_{L^2} = o(1)$ and $L_{x_o} := \{p_{*,x_o} = 0\}$. Thanks to (3.24) we have

$$(3.25) \quad \partial_{ee}(r^{-2}u(x_o + r \cdot) - p_{*,x_o}) = \partial_{ee}(r^{-2}u(x_o + r \cdot)) \geq -C \log^{-\varepsilon_o}(1/r) \quad \text{in } B_1$$

for all $e \in L_{x_o} \cap \mathbb{S}^{n-1}$.

Assume by contradiction that

$$a_{r_k} \geq M \log^{-\varepsilon_o}(1/r_k) \quad \text{for some } r_k \downarrow 0.$$

Then, recalling (3.25), for any $e \in L_{x_o} \cap \mathbb{S}^{n-1}$ we find

$$\partial_{ee} \tilde{w}_{r_k} = \frac{1}{a_k} \partial_{ee}(r^{-2}u(x_o + r \cdot) - p_{*,x_o}) \geq -\frac{C}{M} \quad \text{in } B_1.$$

Thus, since $\tilde{w}_{r_{k_\ell}} \rightarrow q$ in $L^2(B_1)$ for some subsequence r_{k_ℓ} (see Proposition 2.10(a)), we have

$$(3.26) \quad \partial_{ee} q \geq -\frac{C}{M} \quad \text{in } B_1, \quad \forall e \in L_{x_o} \cap \mathbb{S}^{n-1}.$$

In addition, since $\lambda_{*,x_o} = 2$, Proposition 2.10(a) implies that q is a quadratic polynomial satisfying

$$D^2q|_L \leq 0, \quad D^2q|_{L^\perp} \geq 0, \quad \text{tr}(D^2q) = 0, \quad \text{and} \quad \|q\|_{L^2(\partial B_1)} = 1.$$

Thanks to this fact, a simple compactness argument shows that there exists $e' \in L_{x_o} \cap \mathbb{S}^{n-1}$ such that

$$\partial_{e'e'} q \leq -c_1 < 0 \quad \text{in } B_1,$$

for some dimensional constant $c_1 > 0$. This contradicts (3.26) for M sufficiently large, thus establishing (3.23).

Thanks to (3.23), if we define

$$P_{x_o}(x) := p_{*,x_o}(x - x_o) \quad \forall x_o \in \Sigma_m,$$

the argument in the proof of Lemma 3.11 yields

$$(3.27) \quad |D^k P_{x_o}(x) - D^k P_x(x)| \leq C|x-x_o|^{2-k} \log^{-\varepsilon_o}(|x-x_o|) \quad \forall x_o, x \in \Sigma_m \cap \overline{B_{1/2}}, k \in \{0, 1, 2\}.$$

Hence, by Whitney's Extension Theorem (see [Fef09] and the reference therein) and the argument in the proof of Lemma 3.11, we conclude that (3.27) that Σ_m is locally contained in a $C^{1, \log^{\varepsilon_o}}$ m -dimensional manifold. \square

4. ON THIRD ORDER BLOW-UPS

In this section we investigate the uniqueness/continuity of third order blow-ups for points in Σ_m , and prove that Σ_m can be covered by C^2 manifolds, up to a lower dimensional set (see Theorem 4.7 below).

We begin by showing the validity of a third-order almost-monotonicity formula of Monneau-type for all singular points.

Lemma 4.1. *Let 0 be a singular point, assume that $\lambda_* := \phi(0^+, u - p_*) \geq 3$, and let q be a 3-homogeneous harmonic polynomial that vanishes on $L := \{p_* = 0\}$. Set $v := u - p_* - q$, and let H_λ be as in Lemma 2.6. Then*

$$\frac{d}{dr} H_3(r, v) \geq -C \left\| \frac{q^2}{p_*} \right\|_{L^\infty(B_1)},$$

where $C > 0$ is a constant that can be chosen uniformly at all singular points in a neighborhood of 0.

Proof. Set $w := u - p_*$, $w_r(x) := r^{-3}w(rx)$, and $v_r(x) := r^{-3}v(rx) = w_r(x) - q(x)$. Then $H_3(r, v) = H_3(1, v_r)$, and we have

$$(4.1) \quad \frac{d}{dr} H_3(r, v) = \frac{d}{dr} H_3(1, v_r) = \frac{2}{r} \int_{\partial B_1} v_r((v_r)_\nu - 3v_r).$$

We now observe that, because $\lambda_* \geq 3$, it holds

$$W_3(1, w_r) = (\phi(1, w_r) - 3)H_3 \geq 0$$

(here W_λ is as in Lemma 2.8). Also, because q is a 3-homogeneous harmonic polynomial, one easily checks that $W_3(1, q) = 0$. Hence, similarly to the proof of Lemma 2.3, we get

$$\begin{aligned} 0 &\leq W_3(1, w_r) - W_3(1, q) \\ &= \int_{B_1} (|\nabla v_r|^2 + 2\nabla v_r \cdot \nabla q) - 3 \int_{\partial B_1} (v_r^2 + 2v_r q) \\ &= \int_{B_1} |\nabla v_r|^2 - 3 \int_{\partial B_1} v_r^2 + \int_{\partial B_1} v_r(x \cdot \nabla q - 3q) \\ &= \int_{B_1} |\nabla v_r|^2 - 3 \int_{\partial B_1} v_r^2 \\ &= \int_{B_1} -v_r \Delta v_r + \int_{\partial B_1} v_r((v_r)_\nu - 3v_r), \end{aligned}$$

where we used that $\Delta q \equiv 0$ and $x \cdot \nabla q = 3q$. Thus, recalling (4.1) we obtain

$$\frac{d}{dr} H_3(r, v) \geq \frac{2}{r} \int_{B_1} v_r \Delta v_r = \frac{2}{r^{n+5}} \int_{B_r} v \Delta v.$$

Now, since $\Delta v = \Delta u - \Delta p_* - \Delta q = 0$ inside $\{u > 0\}$, we have

$$v \Delta v = (p_* + q) \chi_{\{u=0\}},$$

therefore

$$\frac{d}{dr} H_3(r, v) \geq \frac{2}{r^{n+5}} \int_{B_r} v \Delta v = \frac{2}{r^{n+5}} \int_{B_r \cap \{u=0\}} (p_* + q).$$

Noticing that (since $p_* \geq 0$)

$$p_* + q \geq -\frac{q^2}{2p_*}$$

and that $\frac{q^2}{2p_*}$ is a 4-homogeneous polynomial (this follows from the fact that $q = 0$ on $\{p_* = 0\}$, hence q^2 is divisible by p_*), we conclude that

$$\frac{d}{dr} H_3(r, v) \geq -\frac{1}{r} \int_{B_1 \cap \{u(r \cdot)=0\}} \frac{q^2}{p_*} \geq -\left\| \frac{q^2}{p_*} \right\|_{L^\infty(B_1)} \frac{|\{u(r \cdot) = 0\} \cap B_1|}{r}.$$

Since $\lambda_* \geq 3$, the result follows by Proposition 2.13. \square

In order to apply the previous result, we need to check the size of the points where some third-order blow-up is harmonic and vanishes on $\{p_* = 0\}$. We begin with the case $n = 2$.

Lemma 4.2. *Let $n = 2$, $0 \in \Sigma_1$, and $w := u - p_*$. Assume that there exists a sequence $x_k \in \Sigma_1$ with $x_k \rightarrow 0$ and let $r_k := 2|x_k|$. Then, for some subsequence r_{k_ℓ} we have*

$$\hat{w}_{r_{k_\ell}} := \frac{(u - p_*)(r_{k_\ell} \cdot)}{r_{k_\ell}^3} \rightharpoonup \hat{q} \quad \text{in } W^{1,2}(B_1),$$

where \hat{q} is a 3-homogeneous harmonic polynomial vanishing on $L := \{p_* = 0\}$ and satisfying $\|\hat{q}\|_{L^2(\partial B_1)} = H_3(0^+, w)^{1/2}$.

Proof. Note that if $\phi(0^+, u - p_*) > 3$ then $\|w_r\|_{L^2(B_1)} = o(r^3)$ (see (2.9)), hence $H_3(0^+, w) = 0$ and the result holds with $\hat{q} \equiv 0$. So we can assume that $\phi(0^+, u - p_*) = 3$.

Set

$$\tilde{w}_{r_k} := \frac{w_{r_k}}{\|w_{r_k}\|_{L^2(\partial B_1)}} = \frac{\hat{w}_{r_k}}{H_3(r_k, w)}.$$

By Proposition 2.10(b) after extracting some subsequence r_{k_ℓ} we will have $\tilde{w}_{r_{k_\ell}} \rightharpoonup q$ where q is a 3-homogeneous solution of Signorini —see (2.11)— satisfying $\|q\|_{L^2(\partial B_1)} = 1$. Also, since $r \mapsto H_3(r, w)$ is monotone nondecreasing (see Lemma 2.6) and $\hat{q} \not\equiv 0$, we deduce that

$$\hat{q} = H_3(0^+, w) q.$$

This proves that $\|\hat{q}\|_{L^2(\partial B_1)}^2 = H_3(0^+, w)$.

To conclude the proof it suffices to prove that q is a 3-homogeneous harmonic polynomial vanishing on $L := \{p_* = 0\}$.

Now, applying Lemma 3.3 with $r_k = 2|x_k|$, we deduce that $y_k := \frac{x_k}{r_k} \rightarrow y_\infty \in L \cap \partial B_{1/2}$ and that (thanks to (3.5))

$$\int_{\partial B_\rho} q^{\text{even}}(y_\infty + x)^2 \leq C\rho^6$$

(note that for $n = 2$ we have that $\lambda_{*,x_k} \geq 3$ for all k , see Lemma 3.1). This implies in particular that q^{even} is 3-homogeneous both with respect to 0 and y_∞ , hence it must be one dimensional. Since $\Delta q = 0$ outside L , this implies that q^{even} is affine on each side of L , hence $q^{\text{even}} \equiv 0$ (being q^{even} 3-homogeneous).

This proves that q is odd with respect to L , so q cannot have a singular Laplacian on L . Recalling that $\Delta q = 0$ in $\mathbb{R}^2 \setminus L$, this proves that q is a 3-homogeneous harmonic polynomial. Finally, since $q \geq 0$ on L and q is 3-homogeneous, it must be $q|_L \equiv 0$. \square

The previous Lemma motivates the following

Definition 4.3. We say that a singular point x_\circ belongs to Σ^{3rd} , and write $x_\circ \in \Sigma^{3rd}$, if $\phi(0^+, u(x_\circ + \cdot) - p_{*,x_\circ}) \geq 3$ and there exists some sequence $r_j \downarrow 0$ such that

$$w_{r_j} := \frac{u(x_\circ + r_j \cdot) - p_{*,x_\circ}(r_j \cdot)}{r_j^3} \rightharpoonup \hat{q} \quad \text{in } W^{1,2}(B_1),$$

and \hat{q} is a 3-homogeneous harmonic polynomial vanishing on $\{p_{*,x_\circ} = 0\}$.

Applying the Federer-type reduction argument developed in the previous section and the proof of Lemma 4.2 we obtain the following:

Lemma 4.4. *Let $n \geq 2$, $1 \leq m \leq n - 1$. Then:*

- (i) $\Sigma_1 \setminus \Sigma_1^{3rd}$ consists of isolated points for $n \geq 2$;
- (ii) $\dim_{\mathcal{H}}(\Sigma_m \setminus \Sigma_m^{3rd}) \leq m - 1$ for $2 \leq m \leq n - 1$.

Proof. Since the argument is similar to the ones used in the previous section, we just explain the main steps, leaving the details to the interested reader.

Point (i) for $n = 2$ follows immediately from Lemma 4.2.

The case $n \geq 3$ for $m = 1$ follows instead by Proposition 2.10(a) and Lemma 3.2 after observing that

$$\frac{(u - p_*)(r_{k_\ell} \cdot)}{r_{k_\ell}^3} \rightharpoonup \hat{q} \quad \Rightarrow \quad H(1, \hat{q}) \frac{(u - p_*)(r_{k_\ell} \cdot)}{\|(u - p_*)(r_{k_\ell} \cdot)\|_{L^2(\partial B_1)}} \rightharpoonup \hat{q},$$

cp. proof of Lemma 4.2.

Concerning the case $m = n - 1 \geq 2$, we can use Lemma 3.3 to prove that the conclusion of Lemma 4.2 applies to all points in Σ_{n-1}^g that are density points for Σ_{n-1}^g with respect to the measure \mathcal{H}_∞^β , with $\beta > n - 2$. Thus, as in the proof of Lemma 3.9, Lemma 3.5 implies that the conclusion of Lemma 4.2 applies to all points in Σ_{n-1}^g up to at most a set of Hausdorff dimension $n - 2$. Since $\dim_{\mathcal{H}}(\Sigma_{n-1} \setminus \Sigma_{n-1}^g) = \dim_{\mathcal{H}}(\Sigma_{n-1}^a) \leq n - 3$ (see Theorem 1.1), this proves (ii) when $m = n - 1$.

Analogously, in the case $2 \leq m \leq n - 2$, we can use Lemma 3.2 to prove that the conclusion of Lemma 4.2 applies to all points in Σ_m^g that are density points for Σ_m^g with respect to the measure \mathcal{H}_∞^β , with $\beta > m - 1$. Thus, Lemma 3.5 implies

that the conclusion of Lemma 4.2 applies to all points in Σ_m^g up to at most a set of Hausdorff dimension $m - 1$. Since $\dim_{\mathcal{H}}(\Sigma_m \setminus \Sigma_m^g) = \dim_{\mathcal{H}}(\Sigma_m^a) \leq m - 1$ (see Theorem 1.1), this concludes the proof of (ii). \square

We can also prove the uniqueness and continuity of third-order blow-ups at all points in Σ_m^{3rd} :

Proposition 4.5. *Let $n \geq 2$, $1 \leq m \leq n - 1$, and let $x_\circ \in \Sigma_m^{3rd}$. Then the following limit exists:*

$$(4.2) \quad \frac{u(x_\circ + rx) - p_{*,x_\circ}(rx)}{r^3} \rightharpoonup q_{*,x_\circ}(x) \quad \text{in } W^{1,2}(B_1) \text{ as } r \rightarrow 0,$$

where $q_{*,x_\circ}(x)$ is a 3-homogeneous harmonic polynomial vanishing on $\{p_{*,x_\circ} = 0\}$ and satisfying $\|q_{*,x_\circ}\|_{L^2(\partial B_1)} = H_3(0^+, u(x_\circ + \cdot) - p_{*,x_\circ})$. In addition the above convergence is uniform on compact sets, and the

the map $\Sigma_m^{3rd} \ni x_\circ \mapsto q_{*,x_\circ}$ is continuous.

Proof. Assume $0 \in \Sigma_m^{3rd}$. We first prove the existence of a limit.

Since $0 \in \Sigma_m^{3rd}$, there exists a sequence $r_{k,1} \downarrow 0$ yielding a limit q_1 which is a 3-homogeneous harmonic polynomial vanishing on $\{p_* = 0\}$. Let q_2 be a limit obtained along another sequence $r_{k,2} \downarrow 0$. Up to taking a subsequence of $r_{k,2}$ and relabeling the indices, we can assume that $r_{k,2} \leq r_{k,1}$ for all k . Thus, thanks to Lemma 4.1, we have

$$H_3(r_{k,1}, w - q_1) \geq H_3(r_{k,2}, w - q_1) - C|r_{k,2} - r_{k,1}| \quad \forall k,$$

for some constant C depending on q_1 , so that letting $k \rightarrow \infty$ we obtain

$$0 = \lim_{k \rightarrow \infty} \int_{\partial B_1} (w_{r_{k,1}} - q_1)^2 \geq \lim_{k \rightarrow \infty} \left(\int_{\partial B_1} (w_{r_{k,2}} - q_1)^2 - C|r_{k,2} - r_{k,1}| \right) = \int_{\partial B_1} (q_2 - q_1)^2.$$

This proves the existence of the limit, and that the limit is a 3-homogeneous harmonic polynomial vanishing on $\{p_* = 0\}$.

We now prove the continuity of the map $x_\circ \mapsto q_{*,x_\circ}$ at $0 \in \Sigma_m^{3rd}$. Fix $\varepsilon > 0$, and consider a sequence $x_k \in \Sigma_m^{3rd}$ with $x_k \rightarrow 0$. Thanks to (4.2), there exists a small radius $r_\varepsilon > 0$ such that

$$(4.3) \quad \int_{\partial B_1} \left| \frac{u(r_\varepsilon x) - p_{*,0}(r_\varepsilon x)}{r_\varepsilon^3} - q_{*,0}(x) \right|^2 \leq \varepsilon.$$

Now, let $R_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a rotation that maps the m -dimensional plane $L_k := \{p_{*,x_k} = 0\}$ onto $L_0 := \{p_{*,0} = 0\}$, and note that $R_k \rightarrow \text{Id}$ as $k \rightarrow \infty$ (this follows by the continuity of $\Sigma_m \ni x \mapsto p_{*,x}$). Then, since $q_{*,0} \circ R_k$ vanishes on L_k , we can apply Lemma 4.1 at x_k with $q = q_{*,0} \circ R_k$ to deduce that

$$\begin{aligned} \int_{\partial B_1} |q_{x_k,*} - q_{*,0} \circ R_k|^2 &= \lim_{r \rightarrow 0} \int_{\partial B_1} \left| \frac{u(x_k + rx) - p_{*,x_k}(rx)}{r^3} - q_{*,0} \circ R_k(x) \right|^2 \\ &\leq \int_{\partial B_1} \left| \frac{u(x_k + r_\varepsilon x) - p_{*,x_k}(r_\varepsilon x)}{r_\varepsilon^3} - q_{*,0} \circ R_k(x) \right|^2 + Cr_\varepsilon. \end{aligned}$$

Note that the constant C above is independent of k since, by the continuity of p_{*,x_k} , $p_{*,x_k} \circ R_k^{-1} \geq p_{*,0}/2$ for k large enough, therefore

$$\left\| \frac{(q_{*,0} \circ R_k)^2}{p_{*,x_k}} \right\|_{L^\infty(B_1)} \leq 2 \left\| \frac{q_{*,0}^2}{p_{*,0}} \right\|_{L^\infty(B_1)} \quad \forall k \gg 1.$$

Hence, since $R_k \rightarrow \text{Id}$, letting $k \rightarrow \infty$ and recalling (4.3) we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{\partial B_1} |q_{x_k,*} - q_{*,0}|^2 &\leq \lim_{k \rightarrow \infty} \int_{\partial B_1} \left| \frac{u(x_k + r_\varepsilon x) - p_{*,x_k}(r_\varepsilon x)}{r_\varepsilon^3} - q_{*,0} \circ R_k(x) \right|^2 + Cr_\varepsilon \\ &\leq \varepsilon + Cr_\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the continuity at 0. In addition, arguing as above (using Lemma 4.1) one sees that the convergence in (4.2) is locally uniform with respect to x_o . \square

Remark 4.6. It is important to observe that the above proof shows something stronger: if $x_k \in \Sigma_m^g$ (so their frequency is at least 3, see (3.13)) and $x_k \rightarrow x_o$ with $x_o \in \Sigma_m^{3rd}$, then

$$(4.4) \quad \lim_{k \rightarrow \infty} \int_{\partial B_1} |q_{x_k} - q_{*,0}|^2 = 0$$

whenever q_{x_k} is an arbitrary limit point of $r^{-3}(u(x_k + rx) - p_{*,x_k}(rx))$ as $r \rightarrow 0$. In other words, even if the third order blow-up of $u - p_{*,x_k}$ at x_k may not be unique, any such limit has to converge to $q_{*,0}$ as $x_k \rightarrow x_o$. Indeed, if $\{r_{k,j}\}_{j \geq 1}$ is a sequence converging to 0 such that

$$q_{x_k}(x) = \lim_{j \rightarrow \infty} \frac{u(x_k + r_{k,j}x) - p_{*,x_k}(r_{k,j}x)}{r_{k,j}^3},$$

then Lemma 4.1 applied at x_k with $q = q_{*,0} \circ R_k$ yields (since $r_{k,j} \leq r_\varepsilon$ for $j \gg 1$)

$$\begin{aligned} \int_{\partial B_1} |q_{x_k} - q_{*,0} \circ R_k|^2 &= \lim_{j \rightarrow \infty} \int_{\partial B_1} \left| \frac{u(x_k + r_{k,j}x) - p_{*,x_k}(r_{k,j}x)}{r_{k,j}^3} - q_{*,0} \circ R_k(x) \right|^2 \\ &\leq \int_{\partial B_1} \left| \frac{u(x_k + r_\varepsilon x) - p_{*,x_k}(r_\varepsilon x)}{r_\varepsilon^3} - q_{*,0} \circ R_k(x) \right|^2 + Cr_\varepsilon, \end{aligned}$$

and the result follows as in the proof of Proposition 4.5.

Note also that, when $n = 2$, $q_{*,0}$ is odd with respect to the line $\{p_{*,0} = 0\}$ (see the proof of Lemma 4.2). Hence it follows by (4.4) and the continuity of p_{*,x_k} that

$$(4.5) \quad \lim_{k \rightarrow \infty} \int_{\partial B_1} |q_{x_k}^{\text{odd}} - q_{*,0}|^2 + \int_{\partial B_1} |q_{x_k}^{\text{even}}|^2 = 0,$$

where $q_{x_k}^{\text{odd}}$ (resp. $q_{x_k}^{\text{even}}$) is the odd (resp. even) part of q_{x_k} with respect to $\{p_{*,x_k} = 0\}$. In particular, as in Lemma 4.2, $q_{x_k}^{\text{odd}}$ is a 3-homogeneous harmonic polynomial.

As consequence of the previous results, we obtain the following result about the structure of Σ_{n-1} .

Theorem 4.7. *The following holds:*

($n = 2$) Σ_1 is locally contained in a C^2 curve.

($n \geq 3$) For any $m = 1, \dots, n-1$, the set Σ_m can be covered by a countable family of C^2 m -dimensional manifolds, except for at most a set of Hausdorff dimension $m-1$.

Proof. We start with the case $n = 2$. Let us consider the map

$$(4.6) \quad \Sigma_1 \ni x_o \mapsto q_{x_o}^{\text{odd}},$$

where $q_{x_o}^{\text{odd}}$ is the odd part of q_{x_o} with respect to $\{p_{*,x_o} = 0\}$, and where q_{x_o} is defined as follows:

- if $x_o \in \Sigma_1^{3rd}$, then $q_{x_o} = q_{*,x_o}$ is the third order limit provided by Proposition 4.5;
- if $x_o \in \Sigma_1 \setminus \Sigma_1^{3rd}$, then q_{x_o} is an arbitrary limit point of $r^{-3}(u(x_o + rx) - p_{*,x_o}(rx))$ as $r \rightarrow 0$ (recall that $\Sigma_1 = \Sigma_1^g$ for $n = 2$, see Lemma 3.1).

Fix $R \in (0, 1)$, and for $(r, x_o) \in (0, 1 - R] \times (\Sigma^1 \cap \overline{B}_R)$, let us define the function

$$\mathcal{F}(r, x_o) := r^{-3} \left(\int_{B_r} (u(x_o + \cdot) - p_{*,x_o} - q_{x_o}^{\text{odd}})^2 \right)^{1/2}.$$

Note that, as a consequence of Lemma 4.1, the map $r \mapsto \mathcal{F}(r, x_o)$ is almost monotone, hence the limit as $r \rightarrow 0^+$ exists. Also, for $r > 0$ fixed, the map

$$\Sigma^1 \cap \overline{B}_R \ni x_o \mapsto \mathcal{F}(r, x_o)$$

is continuous as a consequence of (4.4) and (4.5) (recall that the set $\Sigma_1 \setminus \Sigma_1^{3rd}$ consists of isolated points by Lemma 4.4(i), so $\mathcal{F}(r, \cdot)$ is trivially continuous at such points). Thus, as in Lemma 3.11, the almost monotonicity implies that $x_o \mapsto \mathcal{F}(0^+, x_o)$ is upper semicontinuous. Since $\mathcal{F}(0^+, \cdot) = 0$ on Σ_1^{3rd} (by Proposition 4.5) we deduce that, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$(4.7) \quad \mathcal{F}(r, x_o) \leq \varepsilon \quad \forall x_o \in \Sigma_1 \cap \overline{B}_R \quad \text{s.t.} \quad \text{dist}(x_o, \Sigma_1^{3rd}) \leq r_\varepsilon, \quad \forall r \in (0, r_\varepsilon].$$

Now, to any point $x_o \in \Sigma_1$ we associate the third order polynomial

$$P_{x_o}(x) := p_{*,x_o}(x - x_o) + q_{x_o}^{\text{odd}}(x - x_o),$$

and we consider the function $\mathcal{G} : \Sigma_1 \times \Sigma_1 \rightarrow \mathbb{R}$ defined as

$$\mathcal{G}(x_o, x) := \frac{1}{\rho_{x_o, x}^3} \left\| (P_{x_o} - P_x)(\rho_{x_o, x} \cdot) \right\|_{L^2(B_1)}, \quad \rho_{x_o, x} := |x - x_o|.$$

We want to prove that \mathcal{G} is uniformly continuous on $(\Sigma_1 \cap \overline{B}_R) \times (\Sigma_1 \cap \overline{B}_R)$ for any $R \in (0, 1)$.

Observe that, thanks to Lemma 4.4(i), the set

$$O_{r,R} := \{x_o \in \Sigma_1 \cap \overline{B}_R : \text{dist}(x_o, \Sigma_1^{3rd}) \geq r\}$$

is finite for any $r > 0$. In particular, if we define

$$U_{r,R} := \{x_o \in \Sigma_1 \cap \overline{B}_R : \text{dist}(x_o, \Sigma_1^{3rd}) \leq r\},$$

then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ small enough such that

$$\text{dist}(x_1, x_2) > \delta \quad \forall (x_1, x_2) \in (O_{\varepsilon, r_\varepsilon/2} \times O_{\varepsilon, r_\varepsilon/2}) \cup (O_{\varepsilon, r_\varepsilon} \times U_{\varepsilon, r_\varepsilon/2})$$

(here $r_\varepsilon > 0$ is as in (4.7)). Hence it is enough to check the continuity of \mathcal{G} on $U_{\varepsilon, r_\varepsilon} \times U_{\varepsilon, r_\varepsilon}$.

Note that, arguing exactly as in (3.22), it follows that

$$\mathcal{G}(x_o, x) \leq \mathcal{F}(\rho_{x_o, x}, x_o) + \mathcal{F}(2\rho_{x_o, x}, x) \quad \forall x_o, x \in \Sigma_1.$$

In particular, provided $\rho_{x_o, x} = |x - x_o| \leq r_\varepsilon/2$, then it follows by (4.7) that $\mathcal{G}(x_o, x) \leq 2\varepsilon$ whenever $(x_o, x) \in U_{\varepsilon, r_\varepsilon} \times U_{\varepsilon, r_\varepsilon}$, which proves the desired uniform continuity of \mathcal{G} .

Since the norm $\|\cdot\|_{L^2(B_1)}$ is equivalent to the norm $\|\cdot\|_{C^3(B_1)}$ on the space of third order polynomials, the uniform continuity of \mathcal{G} implies that the polynomials P_{x_o} are continuous in the sense of Whitney's Theorem: for any $R \in (0, 1)$ there exists a modulus of continuity ω_R such that

$$|D^k P_{x_o}(x) - D^k P_x(x)| \leq \omega_R(|x - x_o|)|x - x_o|^{3-k} \quad \forall x_o, x \in \Sigma_1 \cap \overline{B}_R, k = 0, 1, 2, 3.$$

Since Σ_1 is closed, the set $\Sigma_1 \cap \overline{B}_R$ is compact, so this allows us to apply the classical Whitney's Theorem to find a map $F \in C^3(\mathbb{R}^2)$ such that

$$F(x) = p_{*, x_o}(x - x_o) + q_{*, x_o}(x - x_o) + o(|x - x_o|^3) \quad \forall x_o \in \Sigma_1^{3rd} \cap \overline{B}_R,$$

and we conclude by the Implicit Function Theorem (see the proof of Lemma 3.11).

Concerning the higher dimensional case, since $\dim_{\mathcal{H}}(\Sigma_m \setminus \Sigma_m^{3rd}) \leq m - 1$ (see Lemma 4.4), for any $j \in \mathbb{N}$ we can find a countable family of balls $\{\hat{B}_i\}$ such that

$$\Sigma_m \setminus \Sigma_m^{3rd} \subset \bigcup_i \hat{B}_i =: \mathcal{O}_j, \quad \text{and} \quad \sum_i \text{diam}(\hat{B}_i)^{m-1+1/j} < \frac{1}{j}.$$

In particular $\mathcal{H}_\infty^{m-1+1/j}(\mathcal{O}_j) < 1/j$ (see (3.15)).

Note that, because of the continuity of the map $\Sigma \ni x_o \mapsto p_{*, x_o}$, the set $\Sigma_{m+1}^+ := \Sigma_{m+1} \cup \dots \cup \Sigma_{n-1}$ is closed, and $\overline{\Sigma}_m \setminus \Sigma_m \subset \Sigma_{m+1}^+$. Hence, since the sets Σ_m and Σ_{m+1}^+ are disjoint, the set

$$\overline{\Sigma}_m \setminus \Sigma_m = \overline{\Sigma}_m \cap \Sigma_{m+1}^+$$

is closed.

Define the sets

$$\mathcal{U}_j := \{x : \text{dist}(x, \overline{\Sigma}_m \setminus \Sigma_m) < 1/j\}, \quad K_j := \Sigma_m \setminus (\mathcal{O}_j \cup \mathcal{U}_j).$$

Then $K_j = \overline{\Sigma}_m \setminus (\mathcal{O}_j \cup \mathcal{U}_j)$, and because the sets \mathcal{O}_j and \mathcal{U}_j are open we deduce that K_j is closed. Noticing that the polynomials $P_{x_o}(x) := p_{*, x_o}(x - x_o) + q_{*, x_o}(x - x_o)$ are continuous with respect to $x_o \in K_j$ (by Proposition 4.5), we can argue as we did above in case $n = 2$ to conclude that K_j can be locally covered by a m -dimensional manifold of class C^2 . Then the result follows by observing that $\cup_j K_j = \Sigma_m \setminus (\cap_j \mathcal{O}_j)$ and $\mathcal{H}_\infty^\beta(\cap_j \mathcal{O}_j) = 0$ for any $\beta > m - 1$, hence $\dim_{\mathcal{H}}(\cap_j \mathcal{O}_j) \leq m - 1$ (see (3.16)). \square

APPENDIX A. EXAMPLES OF $(m - 1)$ -DIMENSIONAL ANOMALOUS SETS Σ_m^a

In this Appendix, for any $1 \leq m \leq n - 2$ we construct an example of solution to the obstacle problem in \mathbb{R}^n for which the anomalous set Σ_m^a is $(m - 1)$ -dimensional. The existence of such examples shows that the assertion $\dim_{\mathcal{H}}(\Sigma_m^a) \leq m - 1$ in Theorem 1.1(b) is optimal.

These solutions are constructed as follows: given $1 \leq m \leq n - 2$ we consider functions

$$(A.1) \quad u(x_1, x_2, \dots, x_n) = u^*(z, r), \quad z := x_m, \quad r := \sqrt{x_{m+1}^2 + x_{m+2}^2 + \dots + x_n^2},$$

which are independent of the first $(m - 1)$ variables and are axially symmetric with respect to the last $m - n$ variables. Then our goal is to find solutions to the obstacle problem which are of the form (A.1) and for which all points in the $(m - 1)$ dimensional affine space

$$\mathcal{Z} := \{x_m = x_{m+1} = \cdots = x_n = 0\}$$

are anomalous points in Σ_m^a . To build these examples, we rely on some ideas introduced in [Y16].

Fix $\phi : [-1, 1] \rightarrow \mathbb{R}$ a nonnegative C^2 function satisfying

$$\phi(0) > 0, \quad \phi(1) = 0, \quad \phi'(z) \leq 0 \quad \forall z \in (0, 1), \quad \text{and} \quad \phi(-z) = \phi(z).$$

Then, for any real number $k > 0$ we consider the solution $u^k \geq 0$ to the obstacle problem in $(-1, 1) \times (0, 1) \subset \mathbb{R}^2$

$$\begin{cases} \operatorname{div}(r^{n-m-1} \nabla u^k) = k r^{n-m-1} \chi_{\{u^k > 0\}} & \text{in } (-1, 1) \times (0, 1), \\ u^k(\pm 1, r) = 0 & r \in (0, 1), \\ u^k(z, 1) = \phi(z) & z \in (-1, 1). \end{cases}$$

In other words, $\frac{1}{k} u^k(x_m, \sqrt{x_{m+1}^2 + \cdots + x_n^2})$ is a solution of the classical obstacle problem in the cylinder $\mathbb{R}^{m-1} \times (-1, 1) \times B_1^{(n-m)}$ satisfying the symmetries in (A.1).

Note that, when we think of this obstacle problem as a two dimensional problem (in the variables z, r), we do not prescribe ‘‘boundary’’ values at $r = 0$ since this line has zero capacity for the operator $\operatorname{div}(r^{n-m-1} \nabla \cdot)$ when $n - m - 1 \geq 1$ (this is equivalent to saying that the set $\{x_{m+1} = \cdots = x_n = 0\}$ has zero harmonic capacity in \mathbb{R}^n).

We now claim that:

- (i) u^k is even z , namely $u^k(z, r) = u^k(-z, r)$;
- (ii) $\partial_z u^k \leq 0$ in $(0, 1) \times (0, 1)$;
- (iii) the set $\{u^k > 0\}$ is convex in the direction z , and is symmetric with respect to $z = 0$.

Indeed, (i) follows from the symmetry of the boundary data (and the uniqueness of solution to the obstacle problem).

To show (ii) we consider the open set $U := (0, 1) \times (0, 1) \setminus \{u^k = 0\}$ and observe that $\partial_z u^k \leq 0$ on ∂U . Indeed:

- $\partial_z u^k = 0$ on $\{z = 0\}$, by symmetry;
- $\partial_z u^k = 0$ on $\partial\{u^k = 0\}$, since u^k is nonnegative and of class $C^{1,1}$;
- $\partial_z u^k \leq 0$ on $\{z = 1\}$, since $u^k(r, 1) = 0$ while $u^k(r, z) \geq 0$ for $z < 1$;
- $\partial_z u^k = \phi' \leq 0$ on $\{r = 1\}$.

As a consequence, since $\operatorname{div}(r^{n-m-1} \nabla(\partial_z u^k)) = 0$ inside U (this follows by differentiating the equation for u with respect to z), we deduce by the maximum principle that $\partial_z u^k \leq 0$ in U . Also, we note that $\partial_z u^k = 0$ in $(0, 1) \times (0, 1) \setminus U$ since $u^k \equiv 0$ there (recall that u^k is nonnegative and of class $C^{1,1}$), proving (ii).

Finally, (iii) is an immediate consequence of (i) and (ii).

We now observe that, for k sufficiently large, the contact set contains a neighborhood of the origin, and hence it must contain a cylindrical neighborhood of $\{r = 0\}$ (thanks to (iii)). On the other hand, for $k \ll 1$ we have $u^k(0) > 0$. Therefore, by

continuity, there exists (a unique) $k_\star > 0$ such that $\partial\{u^{k_\star} > 0\}$ touches tangentially the line $\{r = 0\}$.

Set $u^\star := u^{k_\star}$. Observe that that, with this definition, the function

$$(A.2) \quad u(x) := \frac{1}{k_\star} u^\star(x_m, \sqrt{x_{m+1}^2 + \cdots + x_n^2})$$

has a full $(m-1)$ -dimensional space of singular points on $\mathcal{Z} = \{x_m = x_{m+1} = \cdots = x_n = 0\}$. Also, by the given symmetry, these singular points belong to the stratum Σ_m .

We now prove the following:

Proposition A.1. *Let u be the symmetric solution defined in (A.2). Then the set $\mathcal{Z} \subset \Sigma_m$ consists of anomalous points, that is $\mathcal{Z} \subset \Sigma_m^a$.*

Proof. The proof consists of three steps.

- *Step 1.* We show that u has no other singular points in a neighborhood of \mathcal{Z} (except of course the points in \mathcal{Z}).

To prove this, assume by contradiction that there exists a sequence of singular points $x_\ell \rightarrow 0$. Note that, since u is invariant in the first $m-1$ variable, we can assume that $x_\ell \in \{x_1 = \cdots = x_{m-1} = 0\}$. Then, by the symmetries of u and Lemma 3.2, a blow-up q of $u - p_\star$ at 0 is a homogeneous harmonic polynomial that vanishes on the x_m -axis and that enjoys the same symmetries as u (note that p_\star has the same symmetries as u and vanishes on $r = 0$). Thus, $q = q^\star(z, r)$, where q^\star solves

$$\operatorname{div}(r^{n-m-1} \nabla q^\star) = 0, \quad q^\star(z, 0) = 0 \quad \forall z, \quad \text{and} \quad q^\star \not\equiv 0.$$

Let ℓ be the degree of q . Since q is smooth and $q^\star(z, 0) = 0$, q^\star must be a polynomial of the form $\sum_{1 \leq k \leq \ell/2} a_k z^{\ell-2k} r^{2k}$. Let $k_0 \geq 1$ be the first index such that $a_{k_0} \neq 0$. Then

$$0 = \operatorname{div}(r^{n-m-1} \nabla q^\star) = \left(2k_0[n-m+2(k_0-1)]a_{k_0} z^{\ell-2k_0} + r^2 Q(z, r)\right) r^{n-m-1+2(k_0-1)},$$

where Q is a polynomial of degree $\ell - 2k_0 - 2$. In particular, the terms inside the parenthesis cannot be identically zero, giving the desired contradiction.

As a consequence, there is a neighborhood of \mathcal{Z} which is free of singular points. (except the points in \mathcal{Z}). In particular, as a consequence of [C77, C98], there exists $\varepsilon > 0$ such that $(\partial\{u^\star > 0\} \setminus \{0\}) \cap B_\varepsilon$ is a smooth curve contained inside $(-\varepsilon, \varepsilon) \times (0, \varepsilon) \setminus \{0\}$.

- *Step 2.* We show that, for any $\alpha > 0$ small, there exists $R = R(\alpha) > 0$ small such that the following holds:

$$(A.3) \quad \forall \varrho_\circ \in (0, R), \exists Y_{\varrho_\circ} \in \partial\{u^\star > 0\} \cap B_{2\varrho_\circ} \quad \text{s.t.} \quad Y_{\varrho_\circ} \cdot \mathbf{e}_r \geq \varrho_\circ^{1+\alpha},$$

where $Y_{\varrho_\circ} \cdot \mathbf{e}_r$ denotes the r -component of the point Y_{ϱ_\circ} .

Let $\rho := \sqrt{r^2 + z^2}$ and $\theta := \arctan(z/r) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be polar coordinates in $(0, 1) \times (-1, 1)$. We now state the following fact, whose proof is postponed to the end of the Step 2.

Claim. For any $\alpha > 0$ small, there exists $\delta = \delta(\alpha) > 0$, and a function $\Theta_\alpha : [\delta, \frac{\pi}{2} - \delta] \rightarrow \mathbb{R}$ smooth, such that $S_\alpha(r, z) := \rho^{1+\alpha/4} \Theta_\alpha(\theta)$ satisfies

$$\operatorname{div}(r^{n-m-1} \nabla S_\alpha) = 0 \quad \text{in the cone } \mathcal{C}_\delta := \{(\rho, \theta) \in (0, 1) \times (\delta, \frac{\pi}{2} - \delta)\},$$

and

$$(A.4) \quad \Theta_\alpha(\delta) = \Theta_\alpha\left(\frac{\pi}{2} - \delta\right) = 0, \quad \Theta_\alpha > 0 \quad \text{for } \theta \in \left(\delta, \frac{\pi}{2} - \delta\right).$$

Note that, since 0 is a singular point and recalling the symmetries of u , $u(x) = p_*(x) + o(|x|^2)$ where $p_*(x) = \frac{1}{2(n-m)} \sum_{i=m+1}^n x_i^2$. This implies that $u^*(z, r) \geq \frac{k_*}{4(n-m)} r^2 + o(z^2)$ near the origin. Thus, given $\alpha > 0$, there exists $\eta = \eta(\alpha) > 0$ small enough such that the inclusion

$$(A.5) \quad \mathcal{C}_\delta \cap B_\eta \subset \mathcal{C}_{\delta/2} \cap B_\eta \subset \{u^* > 0\}$$

holds. Note that, up to reducing η , we can assume that $\eta < \varepsilon$, so that (by Step 1) $\partial\{u^* > 0\} \setminus \{0\}$ is a smooth curve inside B_η .

Let us consider the function $h := -\partial_z u^*$, and recall that $h \geq 0$ (by property (iii) in the construction of u^*). Also, differentiating the equation

$$\operatorname{div}(r^{n-m-1} \nabla u^*) = k_* r^{n-m-1} \quad \text{in } \{u^* > 0\}$$

with respect to z , we obtain

$$(A.6) \quad \operatorname{div}(r^{n-m-1} \nabla h) = 0 \quad \text{in } \{u^* > 0\}.$$

Since $h > 0$ inside $\{u^* > 0\}$ (by the strong maximum principle), it follows that

$$(A.7) \quad \partial\{h > 0\} \cap B_\eta \cap \{z > 0\} = \partial\{u^* > 0\} \cap B_\eta \cap \{z > 0\}$$

and $\mathcal{C}_\delta \cap \partial B_\eta \subset \subset \{u^* > 0\}$. Thus there exists a constant $c_0 = c_0(\alpha) > 0$ such that $h \geq c_1 S_\alpha$ on $\mathcal{C}_\delta \cap \partial B_\eta$. In addition $h \geq 0 = S_\alpha$ on $\partial \mathcal{C}_\delta \cap B_\eta$. Hence, since h and S_α solve the same equation, it follows by the maximum principle that $h \geq S_\alpha$ inside $\mathcal{C}_\delta \cap B_\eta$. Recalling (A.4), this implies that

$$(A.8) \quad h \geq c_1 \rho^{1+\alpha/4} \quad \text{in } \{(\rho, \theta) \in (0, \eta) \times [2\delta, \frac{\pi}{2} - 2\delta]\}$$

for some $c_1 = c_1(\alpha) > 0$.

We now want to find a lower bound on the normal derivative of h at points on $\partial\{u^* > 0\}$. For this we use a Hopf-type argument, constructing suitable barriers for our operator $\operatorname{div}(r^{n-m-1} \nabla \cdot)$. These are given by the family of functions

$$S_b^H(r, z) := 2(r - b) - (n - m - 1)(z - 1)^2, \quad \text{where } b \in \mathbb{R}.$$

Note that

$$\operatorname{div}(r^{n-m-1} \nabla S_b^H) = 0 \quad \text{in } \{r > 0\},$$

and $S_b^H < 0$ outside of the parabolic region $\mathcal{P}_b := \{b + \frac{n-m-1}{2}(z-1)^2 \leq r\}$.

Now, given $0 < \varrho_\circ \ll \eta$ we consider the rescaled function

$$h_{\varrho_\circ} := \varrho_\circ^{-1-\alpha/2} h(\varrho_\circ \cdot)$$

and we note that (thanks to (A.8))

$$(A.9) \quad h_{\varrho_\circ}\left(\rho, \frac{\pi}{2} - 2\delta\right) \geq c_1 \varrho_\circ^{-\alpha/4} \rho^{1+\alpha/4} \quad \forall \rho \in (0, \frac{\eta}{\varrho_\circ}).$$

Consider now our barriers S_b^H . For $b > 1/2$ the parabolic region \mathcal{P}_b is contained inside the set $\{\theta < \frac{\pi}{2} - 2\delta\}$. In particular $S_b^H < 0$ inside the cone $\hat{\mathcal{C}}_{2\delta} := \{\theta \in (\frac{\pi}{2} - 2\delta, \frac{\pi}{2})\}$. Then, we start decreasing b until the first value b_\circ such that $\partial \mathcal{P}_{b_\circ}$ touches $\partial\{h_{\varrho_\circ} > 0\}$. Note that, thanks to (A.7) and Step 1, $b_\circ > 0$ and the contact point will happen for some $\tilde{Y}_{\varrho_\circ} \in \{r > 0\}$.

Since $h_{\varrho_o} \geq 0 = S_{b_o}^H$ on $\partial\mathcal{P}_{b_o} \cap \hat{\mathcal{C}}_{2\delta}$ and $h_{\varrho_o} \geq c_2(\alpha)\varrho_o^{-\alpha/4} \geq S_{b_o}^H$ on $\mathcal{P}_{b_o} \cap \partial\hat{\mathcal{C}}_{2\delta}$ for ϱ_o sufficiently small (see (A.9)), it follows by the maximum principle that $h_{\varrho_o} \geq S_{b_o}^H$ inside $\mathcal{P}_{b_o} \cap \hat{\mathcal{C}}_{2\delta}$. Hence, since both h_{ϱ_o} and $S_{b_o}^H$ vanish at \tilde{Y}_{ϱ_o} , we deduce that

$$(A.10) \quad \partial_\nu h_{\varrho_o}(\tilde{Y}_{\varrho_o}) \geq \partial_\nu S_{b_o}^H(\tilde{Y}_{\varrho_o}) \geq c_2 > 0,$$

where ν is the unit inwards normal to $\partial\{h_{\varrho_o} > 0\}$, and $c_2 = c_2(\alpha)$ is independent of ϱ_o (here we use $b_o \leq 1/2$). Observe also that, provided δ is small enough, $\mathcal{P}_{b_o} \cap \hat{\mathcal{C}}_{2\delta} \subset B_2$.

Rescaling (A.10), we obtain that for all $\varrho_o \in (0, \eta)$ small enough there is a point $Y_{\varrho_o} \in \partial\{u^* > 0\} \cap (0, 1) \times (0, 1)$ such that,

$$(A.11) \quad |Y_{\varrho_o}| \leq 2\varrho_o \quad \text{and} \quad \partial_\nu h(Y_{\varrho_o}) \geq c_2\varrho_o^{\alpha/2},$$

where ν denotes the unit inwards normal to $\{u^* > 0\}$ (see (A.7)).

To conclude the proof we observe that differentiating the equation

$$\operatorname{div}(r^{n-m-1}\nabla u^*) = k^* r \chi_{\{u^* > 0\}}$$

with respect to z we obtain $-\operatorname{div}(r^{n-m-1}\nabla h) = k^* r^{n-m-1}\nu_z \mathcal{H}^1|_{\partial\{u^* > 0\}}$, thus

$$-r^{n-m-1}\partial_\nu h = k^* r^{n-m-1}\nu_z \quad \text{on } \partial\{u^* > 0\}$$

where $\nu_z = \nu \cdot \mathbf{e}_z$ denotes the z -component of ν . Recalling (A.11), this proves that

$$(A.12) \quad \nu_z(Y_{\varrho_o}) \leq -\frac{c_2}{k^*}\varrho_o^{\alpha/4} \quad \text{and} \quad |Y_{\varrho_o}| \leq 2|\varrho_o|.$$

Note that (A.12) already implies that $\partial\{u^* > 0\}$ cannot be a $C^{1,\alpha}$ curve at 0. We now prove the more precise estimate (A.3).

Since $b_o > 0$ and the rescaled parabolic cap

$$\left\{ (\varrho_o z, \varrho_o r) : b_o + \frac{n-m-1}{2}(z-1)^2 \leq r \leq 1 \right\}$$

touches $\partial\{u^* = 0\}$ at $Y_{\varrho_o} = (r_{\varrho_o}, z_{\varrho_o})$, it follows that

$$(A.13) \quad \frac{n-m-1}{2\varrho_o}(z_{\varrho_o} - \varrho_o)^2 < \varrho_o b_o + \frac{n-m-1}{2\varrho_o}(z_{\varrho_o} - \varrho_o)^2 = r_o$$

and that

$$(A.14) \quad \nu(Y_{\varrho_o}) = \frac{\left(1, \frac{n-m-1}{\varrho_o}(z_{\varrho_o} - \varrho_o)\right)}{\sqrt{1 + \left(\frac{n-m-1}{\varrho_o}(z_{\varrho_o} - \varrho_o)\right)^2}}.$$

Combining (A.13) with (A.12), we get

$$\frac{\frac{n-m-1}{\varrho_o}(z_{\varrho_o} - \varrho_o)}{\sqrt{1 + \left(\frac{n-m-1}{\varrho_o}(z_{\varrho_o} - \varrho_o)\right)^2}} \leq -\frac{c_2}{k^*}\varrho_o^{\alpha/4} \quad \Rightarrow \quad \frac{n-m-1}{\varrho_o}(z_{\varrho_o} - \varrho_o) \leq -\frac{c_2}{k^*}\varrho_o^{\alpha/4}.$$

Recalling (A.13), this yields

$$\frac{1}{2(n-m-1)}\left(\frac{c_2}{k^*}\right)^2 \varrho_o^{1+\alpha/2} \leq r_o = Y_{\varrho_o} \cdot \mathbf{e}_r.$$

Since $\varrho_o^{1+\alpha} \ll \varrho_o^{1+\alpha/2}$, this proves (A.3).

We conclude Step 2 proving the claim.

Proof of Claim. The linear function $z = \rho \sin \theta$ satisfies $\operatorname{div}(r^{n-m-1} \nabla z) = 0$. Also $\Theta_0(\theta) = \sin \theta$ is positive on $(0, \pi/2)$ and satisfies $\Theta_0(0) = 0$, hence Θ_0 must be the minimizer of the Rayleigh quotient

$$\min \left\{ \frac{\int_0^{\pi/2} (\cos \theta)^{n-m-1} (\Theta')^2}{\int_0^{\pi/2} (\cos \theta)^{n-m-1} (\Theta)^2} : \Theta(0) = 0 \right\} = n - m$$

(equivalently, recalling (A.1), the restriction to the harmonic function x_m to $\mathbb{R}^{m-1} \times \mathbb{S}^{n-m} \subset \mathbb{R}^{m-1} \times \mathbb{R}^{n-m+1}$ is a minimizer of the Rayleigh quotient for the classical Dirichlet integral on $\mathbb{S}^{n-m} \cap \{x_m \geq 0\}$).

Since $n - m - 1 \geq 1$ (this is where we crucially use this assumption), we have

$$\lim_{\delta \downarrow 0} \min \left\{ \frac{\int_0^{\pi/2-\delta} (\cos \theta)^{n-m-1} (\Theta')^2}{\int_0^{\pi/2-\delta} (\cos \theta)^{n-m-1} (\Theta)^2} : \Theta(\delta) = \Theta(\pi/2 - \delta) = 0 \right\} \downarrow n - m$$

(this is equivalent to saying that the north pole in $\mathbb{S}^{n-m} \cap \{x_m \geq 0\}$ has harmonic capacity 0, so the boundary condition $\Theta(\pi/2 - \delta) = 0$ disappears in the limit $\delta \rightarrow 0$). Thus, by continuity, for any $\alpha > 0$ small there exists $\delta = \delta(\alpha) > 0$ small enough such that the previous Rayleigh quotient in $(\delta, \pi/2 - \delta)$ will give the value

$$\mu_\alpha := (n - m + \alpha/4)(1 + \alpha/4).$$

This implies that if we denote by Θ_α the first eigenfunction (i.e., the function attaining the minimal quotient value μ_α), then $-((\cos \theta)^{n-m-1} \Theta'_\alpha)' = \mu_\alpha (\cos \theta)^{n-m-1} \Theta_\alpha$, and it follows by a direct computation (or by classical spectral theory) that $S_\alpha(r, z) := \rho^{1+\alpha/4} \Theta_\alpha(\theta)$ satisfies $\operatorname{div}(r^{n-m-1} \nabla S_\alpha) = 0$, as desired. Finally, the strict positivity of Θ_α inside $(\delta, \frac{\pi}{2} - \delta)$ is a classical property of the first eigenfunction.

- *Step 3.* We conclude the proof of the proposition by showing that (A.3) is incompatible with $0 \in \Sigma_m^g$. Recall that, by definition, 0 belongs to Σ_m^g (resp. Σ_m^a) if $\phi(0^+, u - p_*) \geq 3$ (resp. $\phi(0^+, u - p_*) < 3$), see (3.13).

Assume by contradiction that $\phi(0^+, u - p_*) \geq 3$. Then by Lemma 2.6 we have

$$\int_{B_R} |u - p_*| \leq \left(\int_{B_R} (u - p_*)^2 \right)^{1/2} \leq CR^3 \quad \forall R > 0.$$

We now note that

$$\Delta(u - p_*) = -\chi_{\{u=0\}} \leq 0,$$

so it follows by the mean value formula for superharmonic functions that

$$(A.15) \quad u - p_* \geq -CR^3 \quad \text{in } B_R.$$

Recalling that $p_* = p_*(z, r) = \frac{1}{2(n-m)} r^2$ in the variables (z, r) , it follows by (A.15) that

$$-\frac{1}{n-m} r^2 \geq -C(r^2 + z^2)^{3/2} \quad \text{on } \partial\{u = 0\},$$

therefore

$$r \leq C|(z, r)|^{3/2} \quad \text{on } \partial\{u = 0\},$$

which clearly contradicts (A.3) if we choose $\alpha < 1/2$. As a consequence 0 (and by symmetry all points on \mathcal{Z}) must belong to Σ_m^a . \square

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